Lesson 2.1: Representing a System of Linear Equations as a Single Matrix Equation

Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Which matrix equation correctly represents the system $\begin{cases} y+9x=3z-2 \\ 3z=-8x+1 \\ 4y+7-z=6x \end{cases}$?

a. $\begin{bmatrix} 1 & 9 & 3 \\ 3 & -8 & 1 \\ 4 & 7 & -1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix}$

b. $\begin{bmatrix} 9 & 0 & -3 \\ 8 & 0 & 3 \\ 4 & 6 & -1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$

c. $\begin{bmatrix} 9 & 1 & -3 \\ 8 & 0 & 3 \\ -6 & 4 & -1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -7 \end{bmatrix}$

d. $\begin{bmatrix} 9 & 1 & 3 \\ -8 & 0 & 3 \\ 6 & 4 & -1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$

2. What is the inverse of the matrix $\begin{bmatrix} 10 & 7 \\ -4 & -3 \end{bmatrix}$?

a. $\begin{bmatrix} -1.5 & -3.5 \\ 2 & 5 \end{bmatrix}$

b. $\begin{bmatrix} -1.5 & -2 \\ 3.5 & 5 \end{bmatrix}$

c. $\begin{bmatrix} 1.5 & 2 \\ -3.5 & -5 \end{bmatrix}$

d. $\begin{bmatrix} 1.5 & 3.5 \\ -2 & -5 \end{bmatrix}$

Lesson 2.1: Representing a System of Linear Equations as a Single Matrix Equation

Instruction

Common Core State Standard

A–REI.8 (+) Represent a system of linear equations as a single matrix equation in a vector variable.

SMP
1 ✓ 2 ✓
3 4 ✓
5 6 ✓
7 ✓ 8 ✓

Essential Questions

1. How can a system of equations be written as a matrix equation?
2. How can the inverse of a matrix be found?

WORDS TO KNOW

determinant a specific value that is associated with a square matrix and has multiple applications

identity matrix a square matrix that has ones along the main diagonal and zeros everywhere else. For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. When a matrix is multiplied by an identity matrix, the original matrix does not change.

inverse of a matrix a matrix that when multiplied by the original matrix produces the identity matrix. The inverse of matrix A is denoted A^{-1}.

matrix an ordered arrangement of numbers or expressions in rows and columns. The plural of *matrix* is *matrices*.

matrix equation an equation in which a variable stands for a matrix

square matrix a matrix with the same number of rows and columns

vector a quantity having both direction and magnitude

Recommended Resource

- Khan Academy. "Matrix Equations and Systems."

  http://www.walch.com/rr/06020

  This video explains how to set up and solve a system of equations with matrices.

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TOPIC 2 • USING MATRICES TO SOLVE SYSTEMS OF EQUATIONS

Lesson 2.1: Representing a System of Linear Equations as a Single Matrix Equation

Warm-Up 2.1

A test with 26 problems is worth 100 total points. The test consists of multiple-choice problems that are worth 2 points each and short-answer problems that are worth 6 points each.

1. Write a system of equations that describes this situation. Let x represent the number of multiple-choice problems and y represent the number of short-answer problems.

2. How many multiple-choice problems are on the test?

3. How many short-answer problems are on the test?

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Warm-Up 2.1 Debrief

A test with 26 problems is worth 100 total points. The test consists of multiple-choice problems that are worth 2 points each and short-answer problems that are worth 6 points each.

1. Write a system of equations that describes this situation. Let x represent the number of multiple-choice problems and y represent the number of short-answer problems.

   The total number of multiple-choice and short-answer problems is 26. Therefore, $x + y = 26$.

   Since each multiple-choice problem is worth 2 points, the total points from multiple-choice problems can be represented as $2x$. Since each short-answer problem is worth 6 points, the total points from short-answer problems can be represented as $6y$. The total number of points on the test is 100. Thus, $2x + 6y = 100$.

   A system of equations that describes this situation is $\begin{cases} x+y=26 \\ 2x+6y=100 \end{cases}$.

2. How many multiple-choice problems are on the test?

   This system of equations can be solved either by substitution or elimination. Since neither variable in $x + y = 26$ has a coefficient, substitution is probably easier.

   Solve the equation $x + y = 26$ for y by subtracting x from both sides.

   $y = 26 - x$

   Now substitute $(26 - x)$ for y in $2x + 6y = 100$ and solve for x.

$2x + 6y = 100$	Second equation in the system
$2x + 6(26 - x) = 100$	Substitute $(26 - x)$ for y.
$2x + 156 - 6x = 100$	Distribute.
$-4x + 156 = 100$	Combine like terms.
$-4x = -56$	Subtract 156 from both sides.
$x = 14$	Divide both sides by –4.

   The variable x represents the number of multiple-choice problems on the test.

   There are 14 multiple-choice problems on the test.

3. How many short-answer problems are on the test?

   The variable y represents the number of short-answer problems on the test.

   To find y, substitute 14 for x in $y = 26 - x$ and solve.

$y = 26 - x$	First equation solved for y
$y = 26 - (14)$	Substitute 14 for x.
$y = 12$	Simplify.

   There are 12 short-answer problems on the test.

Connection to the Lesson

- Students will write systems of equations.
- Students will rewrite systems of equations in matrix form.

Prerequisite Skills

This lesson requires the use of the following skills:

- understanding the basic properties of matrices
- determining the order of matrices
- multiplying a matrix by a scalar
- multiplying matrices
- writing systems of equations from context

Introduction

Systems of equations can be solved using substitution or elimination, but the more variables a system has, the harder it is to use these algebraic methods. An alternative method involves using matrices to solve systems. Conveniently, this latter method can be used no matter how many variables are involved in the system. However, understanding how to rewrite a system of equations as a matrix equation is a crucial first step in mastering this skill.

Key Concepts

Writing a System of Equations as a Matrix Equation

- A **matrix** is an ordered arrangement of numbers or expressions in rows and columns. The plural of matrix is matrices.
- The **determinant** of a matrix is a specific value that is associated with a **square matrix**, a matrix with the same number of rows and columns. The determinant has multiple applications. The determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is equal to $ad - bc$.
- A system of equations can be rewritten as a **matrix equation,** an equation in which a variable stands for a matrix.
- The system will take the form $AX = B$, where A is a matrix created from the coefficients of the variables, X is a vector matrix composed of the variables themselves, and B is a matrix composed of the results of the equations. A **vector** is a quantity that has both magnitude and direction.
- The first step to writing a system of equations in the matrix form $AX = B$ is to arrange all equations of the system into the form $ax + by = c$. In other words, the variables must be in alphabetical order on the left side of the equation, and the constant term must be on the right side of the equation.

- Next, write the equations as a matrix equation of the form $AX = B$. The coefficients of the variables stay in order as they form the A matrix, the variables form the X vector matrix, and the results (the constant terms) form the B matrix.

- For example, the system $\begin{cases} 3x-4y=8 \\ 2x+5y=-1 \end{cases}$ becomes $\begin{bmatrix} 3 & -4 \\ 2 & 5 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$, where $\begin{bmatrix} 3 & -4 \\ 2 & 5 \end{bmatrix}$ is matrix A, $\begin{bmatrix} x \\ y \end{bmatrix}$ is vector matrix X, and $\begin{bmatrix} 8 \\ -1 \end{bmatrix}$ is matrix B.

Finding the Inverse of a Matrix Using a Formula

- The **inverse of a matrix** is the matrix that when multiplied by the original matrix produces the identity matrix.

- An **identity matrix** is a square matrix that has ones along the main diagonal and zeros everywhere else. An example of an identity matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. When a matrix is multiplied by an identity matrix, the value of the original matrix does not change.

- The inverse of matrix A is denoted as A^{-1}.

- The formula for the inverse of a 2×2 matrix is $A^{-1} = \frac{1}{\det A} \bullet \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Notice that within the matrix, a and d are switched and the opposites of b and c are taken. Then the reciprocal of the determinant is multiplied by the new matrix.

- Note that if the determinant of a matrix is 0, it does not have an inverse. Given that $A^{-1} = \frac{1}{\det A} \bullet \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, if the determinant is 0 then $A^{-1} = \frac{1}{0} \bullet \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Since $\frac{1}{0}$ is undefined, no inverse exists.

Common Errors/Misconceptions

- not arranging the equations correctly before writing the system as a matrix equation
- writing the terms of a matrix equation in an incorrect order
- switching the b and c entries when finding the determinant of a 2×2 matrix

Guided Practice 2.1

Example 1

Write the following system of equations as a matrix equation.

$$\begin{cases} 5y+3x-1=12z \\ 2x+10y=9 \\ 5-7y=-6z+4x \end{cases}$$

1. Arrange each equation so that the variables are on the left side of the equation in alphabetical order and the constant terms are on the right.

   Rearrange the first equation in the system, $5y + 3x - 1 = 12z$.

$5y + 3x - 1 = 12z$	First equation
$5y + 3x - 1 - 12z = 0$	Subtract $12z$ from both sides.
$5y + 3x - 12z = 1$	Add 1 to both sides.
$3x + 5y - 12z = 1$	Rearrange the variable terms in alphabetical order.

   Thus, the first equation can be rewritten as $3x + 5y - 12z = 1$.

   The second equation, $2x + 10y = 9$, is already in the correct form.

   Rearrange the third equation in the system, $5 - 7y = -6z + 4x$.

$5 - 7y = -6z + 4x$	Third equation
$5 - 7y + 6z = 4x$	Add $6z$ to both sides.
$5 - 7y + 6z - 4x = 0$	Subtract $4x$ from both sides.
$-7y + 6z - 4x = -5$	Subtract 5 from both sides.
$-4x - 7y + 6z = -5$	Rearrange the variable terms in alphabetical order.

   Therefore, the third equation can be rewritten as $-4x - 7y + 6z = -5$.

   The rewritten system of equations is now $\begin{cases} 3x+5y-12z=1 \\ 2x+10y=9 \\ -4x-7y+6z=-5 \end{cases}$.

2. Create the matrix, *A*, from the coefficients of the variables.

   Keep the coefficients of the variables in their rearranged order to create the *A* matrix.

   The first row of the matrix will come from the first equation: $3x + 5y - 12z = 1$. The coefficients of *x*, *y*, and *z* are 3, 5, and –12, respectively. Therefore, the entries of the first row of the matrix are 3, 5, and –12:

   $$\begin{bmatrix} 3 & 5 & -12 \\ & & \\ & & \end{bmatrix}$$

   The second row of the matrix will come from the second equation: $2x + 10y = 9$. Since there is no *z* term, the coefficient of *z* is understood to be 0. Therefore, the entries of the second row of the matrix are 2, 10, and 0:

   $$\begin{bmatrix} 3 & 5 & -12 \\ 2 & 10 & 0 \\ & & \end{bmatrix}$$

   The third row of the matrix will composed of the coefficients of the variables in the third equation: $-4x - 7y + 6z = -5$. Therefore, the entries of the third row are –4, –7, and 6.

   The completed matrix, *A*, is $\begin{bmatrix} 3 & 5 & -12 \\ 2 & 10 & 0 \\ -4 & -7 & 6 \end{bmatrix}$.

3. Create the vector matrix, *X*, from the variables.

   The vector matrix, *X*, is composed of the variables themselves and always has just one column.

   The variables *x*, *y*, and *z* must be written in order vertically.

   The vector matrix, *X*, is $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

4. Write the matrix, B, from the results of the equations.

   The matrix, B, is also just one column and is composed of the results, or the constant terms, of the equations.

   In the rearranged system $\begin{cases} 3x+5y-12z=1 \\ 2x+10y=9 \\ -4x-7y+6z=-5 \end{cases}$, the result of the first equation is 1, the result of the second equation is 9, and the result of the last equation is –5. Write these numbers in order vertically to create the B matrix.

   The completed matrix, B, is $\begin{bmatrix} 1 \\ 9 \\ -5 \end{bmatrix}$.

5. Write the system in the form of a matrix equation, $AX = B$.

   Combine the matrices to form the matrix equation $AX = B$.

   The matrix equation is $\begin{bmatrix} 3 & 5 & -12 \\ 2 & 10 & 0 \\ -4 & -7 & 6 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ -5 \end{bmatrix}$. ✓

Example 2

Show that $\begin{bmatrix} 6 & -2 & 3 \\ 1 & 0 & -4 \\ 5 & 9 & 2 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ -3 \end{bmatrix}$ is equivalent to the system $\begin{cases} 6x-2y+3z=8 \\ x-4z=10 \\ 5x+9y+2z=-3 \end{cases}$.

1. Multiply matrix A by vector matrix X.

   The given matrix equation is in the form $AX = B$, where the first matrix is A, the second matrix is vector matrix X, and the third matrix is B.

   Since the first matrix has 3 rows and the second matrix has 3 columns, these matrices can be multiplied.

   Recall that to find the first entry of the product, the first entry in the first row of the first matrix is multiplied by the first entry in the first column of the second matrix. Then the second entry in the first row of the first matrix is multiplied by the second entry in the first column of the second matrix. Then the products are added together, as shown.

$$\begin{bmatrix} 6 & -2 & 3 \\ 1 & 0 & -4 \\ 5 & 9 & 2 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \bullet x + (-2) \bullet y + 3 \bullet z \\ 1 \bullet x + 0 \bullet y + (-4) \bullet z \\ 5 \bullet x + 9 \bullet y + 2 \bullet z \end{bmatrix} = \begin{bmatrix} 6x-2y+3z \\ x-4z \\ 5x+9y+2z \end{bmatrix}$$

2. Use the result to rewrite the original matrix equation.

Since $\begin{bmatrix} 6 & -2 & 3 \\ 1 & 0 & -4 \\ 5 & 9 & 2 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6x-2y+3z \\ x-4z \\ 5x+9y+2z \end{bmatrix}$,

then, by substitution, the original equation

$\begin{bmatrix} 6 & -2 & 3 \\ 1 & 0 & -4 \\ 5 & 9 & 2 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ -3 \end{bmatrix}$ can be rewritten as

$\begin{bmatrix} 6x-2y+3z \\ x-4z \\ 5x+9y+2z \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ -3 \end{bmatrix}$.

3. Write each set of corresponding entries as an equation.

When two matrices are equal, their corresponding entries are equal.

Since $\begin{bmatrix} 6x-2y+3z \\ x-4z \\ 5x+9y+2z \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ -3 \end{bmatrix}$, each set of corresponding entries is equal—the first row of the first matrix is equal to the first row of the second matrix, and so on. Thus, $\begin{cases} 6x-2y+3z=8 \\ x-4z=10 \\ 5x+9y+2z=-3 \end{cases}$.

This result matches the system of equations from the problem statement.

Therefore, the matrix equation $\begin{bmatrix} 6 & -2 & 3 \\ 1 & 0 & -4 \\ 5 & 9 & 2 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ -3 \end{bmatrix}$ is equivalent to $\begin{cases} 6x-2y+3z=8 \\ x-4z=10 \\ 5x+9y+2z=-3 \end{cases}$.

✓

Example 3

Use the formula $A^{-1}=\frac{1}{\det A}\bullet\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ to find the inverse of $A=\begin{bmatrix} -6 & -7 \\ 4 & 5 \end{bmatrix}$.

1. Find the determinant of the matrix.

   The determinant of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is equal to $ad - bc$.

   The values for a, b, c, and d from the matrix $A=\begin{bmatrix} -6 & -7 \\ 4 & 5 \end{bmatrix}$ are $a = -6$, $b = -7$, $c = 4$, and $d = 5$.

$ad - bc$	Formula for the determinant
$= (-6)(5) - (-7)(4)$	Substitute –6 for a, –7 for b, 4 for c, and 5 for d.
$= -30 - (-28)$	Multiply.
$= -2$	Simplify.

   The determinant, det A, is –2.

2. Write the inverse of $A=\begin{bmatrix} -6 & -7 \\ 4 & 5 \end{bmatrix}$.

The formula for the inverse of a 2×2 matrix is

$$A^{-1}=\frac{1}{\det A}\bullet\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Rearrange the matrix according to the formula and substitute in the determinant, –2.

Notice that within the matrix, *a* and *d* are switched and the opposites of *b* and *c* are taken.

$$A^{-1}=\frac{1}{\det A}\bullet\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ Formula for writing the inverse

$$A^{-1}=\frac{1}{(-2)}\bullet\begin{bmatrix} (5) & -(-7) \\ -(4) & (-6) \end{bmatrix}$$ Substitute –2 for the determinant of A, –6 for a, –7 for b, 4 for c, and 5 for d.

$$A^{-1}=-\frac{1}{2}\bullet\begin{bmatrix} 5 & 7 \\ -4 & -6 \end{bmatrix}$$ Simplify.

$$A^{-1}=\begin{bmatrix} \left(-\frac{1}{2}\right)\bullet 5 & \left(-\frac{1}{2}\right)\bullet 7 \\ \left(-\frac{1}{2}\right)\bullet -4 & \left(-\frac{1}{2}\right)\bullet -6 \end{bmatrix}$$ Multiply each entry of the matrix by $-\frac{1}{2}$.

$$A^{-1}=\begin{bmatrix} -\frac{5}{2} & -\frac{7}{2} \\ 2 & 3 \end{bmatrix}$$ Simplify.

The inverse of $\begin{bmatrix} -6 & -7 \\ 4 & 5 \end{bmatrix}$ is $\begin{bmatrix} -\frac{5}{2} & -\frac{7}{2} \\ 2 & 3 \end{bmatrix}$.

Scaffolded Practice 2.1

Example 1

Write the following system of equations as a matrix equation.

$$\begin{cases} 5y+3x-1=12z \\ 2x+10y=9 \\ 5-7y=-6z+4x \end{cases}$$

1. Arrange each equation so that the variables are on the left side of the equation in alphabetical order and the constant terms are on the right.

2. Create the matrix, *A,* from the coefficients of the variables.

3. Create the vector matrix, *X*, from the variables.

4. Write the matrix, *B*, from the results of the equations.

5. Write the system in the form of a matrix equation, $AX = B$.

continued

Example 2

Show that $\begin{bmatrix} 6 & -2 & 3 \\ 1 & 0 & -4 \\ 5 & 9 & 2 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ -3 \end{bmatrix}$ is equivalent to the system $\begin{cases} 6x - 2y + 3z = 8 \\ x - 4z = 10 \\ 5x + 9y + 2z = -3 \end{cases}$.

Example 3

Use the formula $A^{-1} = \dfrac{1}{\det A} \bullet \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ to find the inverse of $A = \begin{bmatrix} -6 & -7 \\ 4 & 5 \end{bmatrix}$.

Name: Date:

TOPIC 2 • USING MATRICES TO SOLVE SYSTEMS OF EQUATIONS

Lesson 2.1: Representing a System of Linear Equations as a Single Matrix Equation

Problem-Based Task 2.1: Encryption System

SMP	
1 ✓	2 ✓
3	4 ✓
5	6 ✓
7 ✓	8 ✓

A set of numbers, such as a debit card's personal identification number (PIN), can be encrypted by multiplying the original matrix by an encoding matrix. To decrypt the encrypted matrix, the encrypted matrix must be multiplied by the inverse of the encoding matrix. Max is a security expert for a bank, and is testing a new encryption system. He receives the encrypted matrix $\begin{bmatrix} -15 & 4 \\ -59 & 16 \end{bmatrix}$, and he knows the encoding matrix is $\begin{bmatrix} -4 & 1 \\ -7 & 2 \end{bmatrix}$. What matrix represents the decrypted PIN? Verify the encrypted matrix is correct by multiplying the original PIN by the encoding matrix.

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Problem-Based Task 2.1: Encryption System

Coaching

a. How can the decrypted PIN be found?

b. What is the inverse of the encoding matrix?

c. What matrix represents the decrypted PIN?

d. Since the Commutative Property of Multiplication does not apply to matrix multiplication, how can you ensure that the matrices were multiplied in the correct order? Explain your plan and then use it to verify your results.

Problem-Based Task 2.1: Encryption System

Coaching Sample Responses

a. How can the decrypted PIN be found?

First the inverse of the encoding matrix must be found. Then the encrypted matrix must be multiplied by the inverse of the encoding matrix.

b. What is the inverse of the encoding matrix?

The encoding matrix is $\begin{bmatrix} -4 & 1 \\ -7 & 2 \end{bmatrix}$. The inverse can be found by using the formula

$A^{-1} = \dfrac{1}{\det A} \bullet \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

First, find the determinant using the formula $ad - bc$.

Substitute –4 for a, 1 for b, –7 for c, and 2 for d:

$ad - bc = (-4)(2) - (1)(-7) = -8 - (-7) = -1$

The determinant is –1.

Use the formula to write the inverse matrix. Substitute –1 for the determinant, switch –4 and 2, take the opposite of 1 and –7, and then multiply:

$$A^{-1} = \frac{1}{\det A} \bullet \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(-1)} \bullet \begin{bmatrix} (2) & -(1) \\ -(-7) & (-4) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -7 & 4 \end{bmatrix}$$

The inverse of the encoding matrix is $\begin{bmatrix} -2 & 1 \\ -7 & 4 \end{bmatrix}$.

c. What matrix represents the decrypted PIN?

To find the decrypted PIN, multiply the encrypted matrix, $\begin{bmatrix} -15 & 4 \\ -59 & 16 \end{bmatrix}$, by the inverse of the encoding matrix, $\begin{bmatrix} -2 & 1 \\ -7 & 4 \end{bmatrix}$.

$$\begin{bmatrix} -15 & 4 \\ -59 & 16 \end{bmatrix} \bullet \begin{bmatrix} -2 & 1 \\ -7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -15(-2)+4(-7) & -15(1)+4(4) \\ -59(-2)+16(-7) & -59(1)+16(4) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 6 & 5 \end{bmatrix}$$

The matrix $\begin{bmatrix} 2 & 1 \\ 6 & 5 \end{bmatrix}$ represents the decrypted PIN.

d. Since the Commutative Property of Multiplication does not apply to matrix multiplication, how can you ensure that the matrices were multiplied in the correct order? Explain your plan and then use it to verify your results.

The encrypted matrix, $\begin{bmatrix} -15 & 4 \\ -59 & 16 \end{bmatrix}$, was created by multiplying the original PIN by the encoding matrix, $\begin{bmatrix} -4 & 1 \\ -7 & 2 \end{bmatrix}$. To check that the multiplication was done in the correct order, multiply the decrypted PIN, $\begin{bmatrix} 2 & 1 \\ 6 & 5 \end{bmatrix}$, by the encoding matrix, $\begin{bmatrix} -4 & 1 \\ -7 & 2 \end{bmatrix}$. If the product is the encrypted matrix, $\begin{bmatrix} -15 & 4 \\ -59 & 16 \end{bmatrix}$, then the decrypted PIN has been found correctly.

$$\begin{bmatrix} 2 & 1 \\ 6 & 5 \end{bmatrix} \bullet \begin{bmatrix} -4 & 1 \\ -7 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2(-4)+1(-7) & 2(1)+1(2) \\ 6(-4)+5(-7) & 6(1)+5(2) \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 4 \\ -59 & 16 \end{bmatrix}$$

The results match; therefore, the decrypted PIN $\begin{bmatrix} 2 & 1 \\ 6 & 5 \end{bmatrix}$ was found correctly.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

Practice 2.1: Representing a System of Linear Equations as a Single Matrix Equation

For problems 1–4, write each system of equations as a matrix equation.

1. $\begin{cases} 5x=3y-7 \\ 2+y=9x \end{cases}$

2. $\begin{cases} z=2x-3y+7 \\ y+9x=11 \\ 5z-2x+6y+10=0 \end{cases}$

3. $\begin{cases} 3a+7b-c=6 \\ -5b+7c=6a+1 \\ 10c-3=9a \end{cases}$

4. $\begin{cases} 2x+7y-z=5 \\ 3x-1=4z+9y \\ 8y=2x+5z \end{cases}$

For problems 5–7, find the inverse of the given matrix. Write entries as decimals rounded to the nearest tenth if needed.

5. $\begin{bmatrix} 8 & 5 \\ -6 & -4 \end{bmatrix}$

6. $\begin{bmatrix} -2 & 3 \\ 3 & -7 \end{bmatrix}$

7. $\begin{bmatrix} -1 & -4 \\ 2 & 6 \end{bmatrix}$

For problems 8–10, an encrypted matrix for a debit card PIN is given. An encrypted matrix can be decrypted if it is multiplied by the inverse of the encoding matrix. If the encoding matrix is $\begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$, what is the original PIN?

8. $\begin{bmatrix} 5 & -6 \\ 13 & -17 \end{bmatrix}$

9. $\begin{bmatrix} 4 & -5 \\ -1 & 2 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 3 \\ -3 & 5 \end{bmatrix}$

Progress Assessment

Circle the letter of the best answer.

1. Which matrix equation correctly represents the system $\begin{cases} 4x-4=3y+9z \\ 5y+z=10 \\ y=-3x+7z+12 \end{cases}$?

a. $\begin{bmatrix} 4 & -4 & 3 \\ 5 & 1 & 0 \\ 1 & -3 & 7 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \\ 12 \end{bmatrix}$

c. $\begin{bmatrix} 4 & -3 & -9 \\ 5 & 1 & 0 \\ 1 & 3 & -7 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 12 \end{bmatrix}$

b. $\begin{bmatrix} 4 & -3 & -9 \\ 0 & 5 & 1 \\ 3 & 1 & -7 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 12 \end{bmatrix}$

d. $\begin{bmatrix} 4 & 3 & 9 \\ 0 & 5 & 1 \\ -3 & 1 & 7 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 10 \\ 12 \end{bmatrix}$

2. Which system of equations is equivalent to the matrix equation $\begin{bmatrix} 6 & 1 \\ -3 & 5 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ -8 \end{bmatrix}$?

a. $\begin{cases} -3x+5=-8y \\ 6y+x=9 \end{cases}$

c. $\begin{cases} 6y+x=9 \\ -3x+5y=-8 \end{cases}$

b. $\begin{cases} y=-6x+9 \\ 8-3x=-5y \end{cases}$

d. $\begin{cases} 9=6x+y \\ 3x+5y=-8 \end{cases}$

Use what you have learned about matrices to complete all parts of the following problem.

3. Samantha spent all $200 of her work bonus on new shirts and dresses. She bought 6 items of clothing. If the shirts she bought cost $25 each, and the dresses she bought cost $50 each, how many shirts and dresses did she buy? (Assume there is no sales tax to consider.)

   a. Write a system of equations that describes this scenario.

   b. What matrix equation can be written from the system of equations?

Pre-Assessment

Circle the letter of the best answer.

1. What is the inverse of the matrix $\begin{bmatrix} 7 & -2 & -6 \\ 3 & 5 & 9 \\ 0 & 2 & -1 \end{bmatrix}$?

a. $\begin{bmatrix} -1 & \frac{1}{2} & 0 \\ \frac{1}{9} & \frac{1}{5} & \frac{1}{3} \\ -\frac{1}{6} & -\frac{1}{2} & \frac{1}{7} \end{bmatrix}$

c. $\begin{bmatrix} \frac{1}{7} & -\frac{1}{2} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{9} \\ 0 & \frac{1}{2} & -1 \end{bmatrix}$

b. $\begin{bmatrix} -\frac{12}{203} & \frac{81}{203} & \frac{23}{203} \\ \frac{2}{29} & \frac{1}{29} & \frac{2}{29} \\ -\frac{3}{203} & -\frac{6}{203} & -\frac{41}{203} \end{bmatrix}$

d. $\begin{bmatrix} \frac{23}{203} & \frac{2}{29} & -\frac{12}{203} \\ -\frac{3}{203} & \frac{1}{29} & \frac{81}{203} \\ -\frac{6}{203} & \frac{2}{29} & -\frac{41}{203} \end{bmatrix}$

2. What does y equal in the solution to this system of equations?

$$\begin{cases} y+2x=7z+43 \\ 4y+2z-9x=-50 \\ 5x+6y-8z-80=0 \end{cases}$$

a. $y = -4$

b. $y = -1$

c. $y = 3$

d. $y = 6$

Common Core State Standard

A–REI.9 (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater).

SMP	
1 ✓	2 ✓
3	4 ✓
5 ✓	6 ✓
7 ✓	8 ✓

Essential Questions

1. What process is used to solve a matrix equation?
2. When would using a matrix be an easier way to solve a system of equations than by graphing or using algebra?

WORDS TO KNOW

Commutative Property of Multiplication The order in which quantities are multiplied does not affect the product. For example, $a \bullet b = b \bullet a$.

inverse of a matrix a matrix that when multiplied by the original matrix produces the identity matrix. The inverse of matrix A is denoted A^{-1}.

system of equations a set of equations with the same unknowns

Recommended Resources

- MathIsFun.com. "Solving Systems of Linear Equations Using Matrices."

  http://www.walch.com/rr/02012

  This site reviews how to solve a three-variable equation using the inverse matrix. Interactive practice problems are included.

- SparkNotes. "Solving Using Matrices and Row Reduction."

  http://www.walch.com/rr/02013

  This site shows an alternate way of solving matrix equations without a calculator.

Lesson 2.2: Finding the Inverse of a Matrix and Using It to Solve a System of Equations

Warm-Up 2.2

The drama department and the football team are both selling cookies for a fund-raiser. The drama team sells 20 packs of sugar cookies and 18 packs of snickerdoodles for a total of 416 cookies. The football team sells a total of 394 cookies, which includes 25 packs of sugar cookies and 12 packs of snickerdoodles.

1. What system of equations describes this situation?

2. How many cookies are in each pack of sugar cookies?

3. How many cookies are in each pack of snickerdoodles?

Warm-Up 2.2 Debrief

The drama department and the football team are both selling cookies for a fund-raiser. The drama team sells 20 packs of sugar cookies and 18 packs of snickerdoodles for a total of 416 cookies. The football team sells a total of 394 cookies, which includes 25 packs of sugar cookies and 12 packs of snickerdoodles.

1. What system of equations describes this situation?

   Let x represent the number of cookies in a pack of sugar cookies and y represent the number of cookies in a pack of snickerdoodles.

   Since the drama team sold 20 packs of sugar cookies, the number of sugar cookies they sold is equal to $20x$ (the number of packs times the number of cookies in each pack). Since they sold 18 packs of snickerdoodles, the number of snickerdoodles they sold is equal to $18y$. They sold 416 cookies total; therefore, the sum of the number of sugar cookies they sold ($20x$) and the number of snickerdoodles they sold ($18y$) is equal to 416. Thus, $20x + 18y = 416$.

   Since the football team sold 25 packs of sugar cookies, the number of sugar cookies they sold is equal to $25x$. They sold 12 packs of snickerdoodles, so the number of snickerdoodles they sold is equal to $12y$. They sold 394 cookies total; therefore, the sum of the number of sugar cookies they sold ($25x$) and the number of snickerdoodles they sold ($12y$) is equal to 394. Thus, $25x + 12y = 394$.

   The system of equations $\begin{cases} 20x+18y=416 \\ 25x+12y=394 \end{cases}$ describes this situation.

2. How many cookies are in each pack of sugar cookies?

   The variable x represents the number of cookies in a pack of sugar cookies. To find its value, solve the system of equations, $\begin{cases} 20x+18y=416 \\ 25x+12y=394 \end{cases}$.

   Since all the variables have coefficients, elimination is probably the best method to solve this system.

   To find the value of x, eliminate the y terms by manipulating their coefficients to be opposites of each other. The opposite terms will sum to 0, canceling each other out.

   The least common multiple of 18 and 12 is 36. If the first equation is multiplied by 2 and the second equation is multiplied by –3, the coefficients of the y terms will be opposites: 36 and –36.

   Multiply both sides of the first equation by 2:

$20x + 18y = 416$	First equation in the system
$2 \bullet (20x + 18y) = 2 \bullet 416$	Multiply both sides by 2.
$40x + 36y = 832$	Distribute and multiply.

Multiply both sides of the second equation by –3.

$25x + 12y = 394$	Second equation in the system
$-3 \bullet (25x + 12y) = -3 \bullet 394$	Multiply both sides by –3.
$-75x - 36y = -1182$	Distribute and multiply.

Now, combine the two resulting equations by adding each column:

$$\begin{array}{r} 40x+36y= \quad 832 \\ +\left(-75x-36y=-1182\right) \\ \hline -35x+\ 0 \quad =-350 \end{array}$$

Solve the resulting equation for x:

$-35x + 0 = -350$	Resulting equation
$-35x = -350$	Simplify.
$x = 10$	Divide both sides by –35.

Since x represents the number of cookies in a pack of sugar cookies, there are 10 cookies in each pack of sugar cookies.

3. How many cookies are in each pack of snickerdoodles?

The variable y represents the number of cookies in a pack of snickerdoodles. To find the value of y, substitute the value of x, 10, back into one of the original equations and solve for y.

$20x + 18y = 416$	First equation in the system
$20(10) + 18y = 416$	Substitute 10 for x.
$200 + 18y = 416$	Multiply.
$18y = 216$	Subtract 200 from both sides.
$y = 12$	Divide both sides by 18.

There are 12 cookies in each pack of snickerdoodles.

Connection to the Lesson

- Students will use matrices to solve systems of equations.

Prerequisite Skills

This lesson requires the use of the following skills:

- understanding the basic properties of matrices
- determining the order of matrices
- multiplying a matrix by a scalar
- multiplying matrices
- writing systems of equations as a matrix equation
- finding the inverse of a 2×2 matrix

Introduction

Imagine trying to use substitution or elimination to solve a 5-equation system with 5 different variables. Though it can be done, the process would be laborious and inefficient. Fortunately, we can use matrices to make solving such complex systems much more simple. Once a system has been written as a matrix equation, the equation can be solved by simply multiplying the inverse of matrix *A* by both sides of the equation.

Key Concepts

Finding the Inverse

- Not all matrices have an inverse. If the determinant of a matrix is 0, the matrix does not have an inverse. Therefore, when finding the inverse of a matrix, it is important to calculate the determinant of the matrix first.
- The inverse of a 2×2 matrix can be found by using the formula $A^{-1} = \frac{1}{\det A} \bullet \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- The inverse of a matrix can also be found by using a graphing calculator.
- Follow the directions appropriate to your calculator model to find the inverse of matrix *A*.

On a TI-83/84:

Step 1: To enter matrix A, press [2ND][MATRIX]. Arrow over to EDIT and press [ENTER]. Enter the dimensions of the matrix (first the number of rows, then the number of columns), pressing [ENTER] after each number; i.e., for a 3×3 matrix, press [3][ENTER][3][ENTER].

Step 2: Input each entry for matrix A, pressing [ENTER] after each.

Step 3: Go back to the home screen by pressing [2ND][MODE].

Step 4: To calculate the inverse, press [2ND][MATRIX][ENTER][x^{-1}] [ENTER]. Arrow to the right to see the full matrix if it is large.

Step 5: To view the entries in fraction form, press [MATH][ENTER][ENTER].

On a TI-Nspire:

Step 1: From the home screen, open a new blank document. To enter matrix A, press [menu] and select 7: Matrix & Vector, 1: Create, and 1: Matrix. Enter the number of rows and columns. Press [enter].

Step 2: Input each entry of matrix A, using the arrow keys to navigate between fields. Press [enter].

Step 3: To calculate the inverse, press [ctrl][ ]. Navigate to the exponent icon [ ] and press [enter]. Type [(–)][1] as the exponent and press [enter].

Solving a System of Equations Using a Matrix

- To solve a **system of equations** using a matrix, first write the system of equations as a matrix equation of the form $AX = B$. Then find the inverse of matrix A. Finally, multiply A^{-1} (the inverse of matrix A) by both sides of the equation. This will result in the solution.
- Note that since the **Commutative Property of Multiplication** (that is, $a \bullet b = b \bullet a$) does not apply to matrix multiplication, A^{-1} must be the first factor when multiplied by both sides of the equation: $A^{-1}AX = A^{-1}B$.

Common Errors/Misconceptions

- reversing the order of matrix multiplication when solving equations
- making errors while using the formula for finding the inverse of a 2×2 matrix
- neglecting to pay attention to sign changes and other details when multiplying matrices

Guided Practice 2.2

Example 1

Determine if the matrix $\begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix}$ has an inverse.

1. Find the determinant of the matrix.

   The determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is equal to $ad - bc$.

$ad - bc$	Formula for the determinant
$= (-8)(-2) - (4)(4)$	Substitute –8 for a, 4 for b, 4 for c, and –2 for d.
$= 16 - 16$	Multiply.
$= 0$	Subtract.

   The determinant of the matrix $\begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix}$ is 0.

2. Determine if the matrix has an inverse.

   A matrix has an inverse as long as its determinant is not 0.

   Since the determinant of the matrix $\begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix}$ is 0, it does not have an inverse.

Example 2

Use a calculator to find the inverse of the matrix $\begin{bmatrix} 6 & -5 & 2 \\ 1 & -4 & 10 \\ 5 & 8 & 1 \end{bmatrix}$. Leave entries as fractions if necessary.

1. Find the inverse of the matrix.

   Enter the given matrix into the calculator and find its inverse using the directions specific to your calculator model.

   On a TI-83/84:

   Step 1: To enter matrix *A*, press [2ND][MATRIX]. Arrow over to EDIT and press [ENTER]. Enter the dimensions of the matrix: [3][ENTER][3][ENTER].

   Step 2: Input each entry of the matrix, pressing [ENTER] after each.

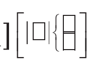
   Step 3: Go back to the home screen by pressing [2ND][MODE].

   Step 4: To calculate the inverse, press [2ND][MATRIX][ENTER][x^{-1}] [ENTER]. Arrow to the right to see the full matrix if it is large.

   On a TI-Nspire:

   Step 1: From the home screen, open a new blank document. To enter matrix *A*, press [menu] and select 7: Matrix & Vector, 1: Create, and 1: Matrix. Enter the number of rows and columns. Press [enter].

   Step 2: Input each entry of matrix *A*, using the arrow keys to navigate between the fields. Press [enter].

   Step 3: To calculate the inverse, press [ctrl]. Navigate to the exponent icon [$\square^{\square}$] and press [enter]. Type [(–)][1] as the exponent and press [enter].

   (*continued*)

Depending on your calculator model, the inverse of the original matrix will be displayed in either decimal or fraction form:

$$\begin{bmatrix} 0.1212121212 & -0.0303030303 & 0.0606060606 \\ -0.0707070707 & 0.0057720058 & 0.0836940837 \\ -0.0404040404 & 0.1053391053 & 0.0274170274 \end{bmatrix}$$

$$\begin{bmatrix} \frac{4}{33} & -\frac{1}{33} & \frac{2}{33} \\ -\frac{7}{99} & \frac{4}{693} & \frac{58}{693} \\ -\frac{4}{99} & \frac{73}{693} & \frac{19}{693} \end{bmatrix}$$

2. If necessary, convert the decimals to fractions.

   On a TI-Nspire, the entries should be listed as fractions by default.

   On a TI-83/84, convert the decimals to fractions:

   On a TI-83/84:

   Step 5: To view the entries in fraction form, press [MATH][ENTER][ENTER].

   The matrix should now appear on the screen in fraction form:

$$\begin{bmatrix} \frac{4}{33} & -\frac{1}{33} & \frac{2}{33} \\ -\frac{7}{99} & \frac{4}{693} & \frac{58}{693} \\ -\frac{4}{99} & \frac{73}{693} & \frac{19}{693} \end{bmatrix}$$

The inverse of $\begin{bmatrix} 6 & -5 & 2 \\ 1 & -4 & 10 \\ 5 & 8 & 1 \end{bmatrix}$ is $\begin{bmatrix} \frac{4}{33} & -\frac{1}{33} & \frac{2}{33} \\ -\frac{7}{99} & \frac{4}{693} & \frac{58}{693} \\ -\frac{4}{99} & \frac{73}{693} & \frac{19}{693} \end{bmatrix}$.

Example 3

Use matrices to solve the following system of equations without using a calculator.

$$\begin{cases} 8x+8=4y \\ 3y=-5x+28 \end{cases}$$

1. Arrange the equations so that the variables are on the left in alphabetical order and the constant terms are on the right.

   Rearrange the first equation in the system, $8x + 8 = 4y$.

$8x + 8 = 4y$	First equation
$8x + 8 - 4y = 0$	Subtract $4y$ from both sides.
$8x - 4y = -8$	Subtract 8 from both sides.

   Thus, the first equation can be rewritten as $8x - 4y = -8$.

   Rearrange the second equation in the system, $3y = -5x + 28$.

$3y = -5x + 28$	Second equation
$3y + 5x = 28$	Add $5x$ to both sides.
$5x + 3y = 28$	Rearrange the variable terms in alphabetical order.

   Therefore, the second equation can be rewritten as $5x + 3y = 28$.

   The rewritten system of equations is $\begin{cases} 8x-4y=-8 \\ 5x+3y=28 \end{cases}$.

2. Write the system as a matrix equation of the form $AX = B$.

   Keep the coefficients of the variables in their rearranged order to create matrix A: $\begin{bmatrix} 8 & -4 \\ 5 & 3 \end{bmatrix}$.

   Create vector matrix X from the variables: $\begin{bmatrix} x \\ y \end{bmatrix}$.

   Create matrix B from the results on the right side of the equations (the constant terms): $\begin{bmatrix} -8 \\ 28 \end{bmatrix}$.

   Combine the matrices to form the matrix equation $AX = B$:

   $$\begin{bmatrix} 8 & -4 \\ 5 & 3 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ 28 \end{bmatrix}$$

   The matrix equation formed from the system $\begin{cases} 8x - 4y = -8 \\ 5x + 3y = 28 \end{cases}$ is $\begin{bmatrix} 8 & -4 \\ 5 & 3 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ 28 \end{bmatrix}$.

3. Find the inverse of matrix A.

   Matrix A is a 2×2 matrix, so its inverse can be found by using the formula $A^{-1} = \dfrac{1}{\det A} \bullet \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

   First, find the determinant of the matrix.

   The determinant of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is equal to $ad - bc$; therefore, the determinant of the matrix $\begin{bmatrix} 8 & -4 \\ 5 & 3 \end{bmatrix}$ is $ad - bc = (8)(3) - (-4)(5) = 24 - (-20) = 44$.

   Since the determinant is not 0, we know that an inverse exists for matrix A.

(continued)

Substitute the determinant and the values for *a*, *b*, *c*, and *d* into the formula for the inverse. Notice that within the matrix, *a* and *d* are switched and the opposites of *b* and *c* are taken.

$$A^{-1}=\frac{1}{\det A}\bullet\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ Formula for writing the inverse

$$A^{-1}=\frac{1}{(44)}\bullet\begin{bmatrix} (3) & -(-4) \\ -(5) & (8) \end{bmatrix}$$ Substitute 44 for det *A*, 8 for *a*, –4 for *b*, 5 for *c*, and 3 for *d*.

$$A^{-1}=\frac{1}{44}\bullet\begin{bmatrix} 3 & 4 \\ -5 & 8 \end{bmatrix}$$ Simplify.

For the purpose of solving a system of equations, it is often easier to leave the scalar outside the matrix and wait to multiply it until later.

The inverse of matrix *A* is $\frac{1}{44}\bullet\begin{bmatrix} 3 & 4 \\ -5 & 8 \end{bmatrix}$.

4. Multiply both sides of the original matrix equation by A^{-1} following the formula $A^{-1}AX = A^{-1}B$.

   The inverse of matrix *A* is $A^{-1}=\frac{1}{44}\bullet\begin{bmatrix} 3 & 4 \\ -5 & 8 \end{bmatrix}$.

   Multiply both sides of the original matrix equation,

   $\begin{bmatrix} 8 & -4 \\ 5 & 3 \end{bmatrix}\bullet\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -8 \\ 28 \end{bmatrix}$, by A^{-1}, as shown:

$$\frac{1}{44}\bullet\begin{bmatrix} 3 & 4 \\ -5 & 8 \end{bmatrix}\bullet\begin{bmatrix} 8 & -4 \\ 5 & 3 \end{bmatrix}\bullet\begin{bmatrix} x \\ y \end{bmatrix}=\frac{1}{44}\bullet\begin{bmatrix} 3 & 4 \\ -5 & 8 \end{bmatrix}\bullet\begin{bmatrix} -8 \\ 28 \end{bmatrix}$$

   Since the Commutative Property of Multiplication does not apply to matrix multiplication, the placement of the matrix is important. Notice that A^{-1} is listed as the first factor on both sides of the equation following the formula $A^{-1}AX = A^{-1}B$.

5. Multiply the matrices using matrix multiplication.

To simplify the equation, first use matrix multiplication to multiply the matrices on each side.

$$\frac{1}{44}\bullet\begin{bmatrix}3 & 4\\-5 & 8\end{bmatrix}\bullet\begin{bmatrix}8 & -4\\5 & 3\end{bmatrix}\bullet\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{44}\bullet\begin{bmatrix}3 & 4\\-5 & 8\end{bmatrix}\bullet\begin{bmatrix}-8\\28\end{bmatrix}$$

Start with the matrices on the left side of the equation:

$$\begin{bmatrix}3 & 4\\-5 & 8\end{bmatrix}\bullet\begin{bmatrix}8 & -4\\5 & 3\end{bmatrix}=\begin{bmatrix}3(8)+4(5) & 3(-4)+4(3)\\-5(8)+8(5) & -5(-4)+8(3)\end{bmatrix}=\begin{bmatrix}44 & 0\\0 & 44\end{bmatrix}$$

Then multiply the matrices on the right side of the equation:

$$\begin{bmatrix}3 & 4\\-5 & 8\end{bmatrix}\bullet\begin{bmatrix}-8\\28\end{bmatrix}=\begin{bmatrix}3(-8)+4(28)\\-5(-8)+8(28)\end{bmatrix}=\begin{bmatrix}88\\264\end{bmatrix}$$

The rewritten matrix equation is $\frac{1}{44}\bullet\begin{bmatrix}44 & 0\\0 & 44\end{bmatrix}\bullet\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{44}\bullet\begin{bmatrix}88\\264\end{bmatrix}$.

6. Simplify the rewritten matrix equation.

On each side of the equation, multiply $\frac{1}{44}$ by the matrix.

$$\frac{1}{44}\bullet\begin{bmatrix}44 & 0\\0 & 44\end{bmatrix}\bullet\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{44}\bullet\begin{bmatrix}88\\264\end{bmatrix}$$

Rewritten matrix equation

$$\begin{bmatrix}\frac{1}{44}\bullet 44 & \frac{1}{44}\bullet 0\\\frac{1}{44}\bullet 0 & \frac{1}{44}\bullet 44\end{bmatrix}\bullet\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{1}{44}\bullet 88\\\frac{1}{44}\bullet 264\end{bmatrix}$$

Multiply $\frac{1}{44}$ by each entry within the matrix.

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\bullet\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2\\6\end{bmatrix}$$

Simplify.

(continued)

The identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is produced since the matrix on the left side was multiplied by its inverse. When the identity matrix is multiplied by $\begin{bmatrix} x \\ y \end{bmatrix}$, the latter matrix does not change.

Therefore, the matrix equation can be rewritten as $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$.

When the matrix equations are multiplied by the inverse matrix, the result is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$.

7. State the solution to the system of equations.

From the equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $x = 2$ and $y = 6$. Or, written as an ordered pair, the solution is (2, 6).

This answer can be checked by substituting the values for x and y back into the original system of equations.

Example 4

Use matrices and a calculator to solve the following system of equations:

$$\begin{cases} 3x+z=-20-5y \\ 9z-24=-6x \\ -10z-4y+2x-2=0 \end{cases}$$

1. Arrange the equations so that the variables are on the left in alphabetical order and the constant terms are on the right.

   Rearrange the first equation in the system, $3x + z = -20 - 5y$.

$3x + z = -20 - 5y$	First equation
$3x + z + 5y = -20$	Add $5y$ to both sides.
$3x + 5y + z = -20$	Rearrange the variable terms in alphabetical order.

   Thus, the first equation can be rewritten as $3x + 5y + z = -20$.

   Rearrange the second equation in the system, $9z - 24 = -6x$.

$9z - 24 = -6x$	Second equation
$9z - 24 + 6x = 0$	Add $6x$ to both sides.
$9z + 6x = 24$	Add 24 to both sides.
$6x + 9z = 24$	Rearrange the variable terms in alphabetical order.

   Therefore, the second equation can be rewritten as $6x + 9z = 24$.

   Rearrange the third equation in the system, $-10z - 4y + 2x - 2 = 0$.

$-10z - 4y + 2x - 2 = 0$	Third equation
$-10z - 4y + 2x = 2$	Add 2 to both sides.
$2x - 4y - 10z = 2$	Rearrange the variable terms in alphabetical order.

   The third equation can be rewritten as $2x - 4y - 10z = 2$.

   The rewritten system of equations is $\begin{cases} 3x+5y+z=-20 \\ 6x+9z=24 \\ 2x-4y-10z=2 \end{cases}$.

2. Write the system as a matrix equation of the form $AX = B$.

Keep the coefficients of the variables in their rearranged order to create matrix A. Recall that variables with no coefficient (such as z) have an understood coefficient of 1 and that variables that are not listed (such as the missing y term in the second equation) have an understood coefficient of 0: $\begin{bmatrix} 3 & 5 & 1 \\ 6 & 0 & 9 \\ 2 & -4 & -10 \end{bmatrix}$.

Create vector matrix X from the variables: $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Create matrix B from the results on the right side of the equations (the constant terms): $\begin{bmatrix} -20 \\ 24 \\ 2 \end{bmatrix}$.

Combine the matrices to form the matrix equation $AX = B$:

$$\begin{bmatrix} 3 & 5 & 1 \\ 6 & 0 & 9 \\ 2 & -4 & -10 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -20 \\ 24 \\ 2 \end{bmatrix}$$

The matrix equation formed from the system $\begin{cases} 3x+5y+z=-20 \\ 6x+9z=24 \\ 2x-4y-10z=2 \end{cases}$ is

$$\begin{bmatrix} 3 & 5 & 1 \\ 6 & 0 & 9 \\ 2 & -4 & -10 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -20 \\ 24 \\ 2 \end{bmatrix}.$$

3. Find the inverse of matrix A.

   Use a graphing calculator to find the inverse of the matrix

$$\begin{bmatrix} 3 & 5 & 1 \\ 6 & 0 & 9 \\ 2 & -4 & -10 \end{bmatrix}.$$

   Note: You will need to store the matrix and its inverse on your calculator for future use. The following calculator directions mirror those given in Example 2, but also include steps for storing the matrix and its inverse.

   On a TI-83/84:

   Step 1: To enter matrix A, press [2ND][MATRIX]. Arrow over to EDIT and press [ENTER]. Enter the dimensions of the matrix: [3][ENTER][3][ENTER].

   Step 2: Input each entry of the matrix, pressing [ENTER] after each. The matrix will automatically be stored as matrix "A."

   Step 3: Go back to the home screen by pressing [2ND][MODE].

   Step 4: To calculate the inverse, press [2ND][MATRIX][ENTER] [x^{-1}][ENTER]. Arrow to the right to see the full matrix if it is large.

   Step 5: To store the inverse matrix, press [STO>][2ND][MATRIX]. Choose a letter for the matrix; for example, to choose D, press [4][ENTER]. This matrix is now stored as matrix "D."

(*continued*)

On a TI-Nspire:

Step 1: From the home screen, open a new blank document. To enter matrix *A*, press [menu] and select 7: Matrix & Vector, 1: Create, and 1: Matrix. Enter the number of rows and columns. Press [enter].

Step 2: Input each entry of matrix *A*, using the arrow keys to navigate between fields. Press [enter].

Step 3: To store matrix *A*, press [ctrl][sto→] and choose a variable such as [A]. Press [enter] to store it as matrix "*a*."

Step 4: To calculate the inverse, press [ctrl][▯|▯{▯]. Navigate to the exponent icon $[\square^{\square}]$ and press [enter]. Type [(–)][1] as the exponent and press [enter].

Step 5: To store the inverse matrix, press [ctrl][sto→] and choose a variable such as [D]. Press [enter] to store it as matrix "*d*."

Depending on your calculator model, the inverse will appear in decimal or fraction form, as shown. Convert to fraction form if necessary by following the calculator directions provided in Example 2.

$$\begin{bmatrix} 0.0759493671 & 0.0970464135 & 0.0949367089 \\ 0.164556962 & -0.0675105485 & -0.0443037975 \\ -0.0506329114 & 0.0464135021 & -0.0632911392 \end{bmatrix}$$

$$\begin{bmatrix} \frac{6}{79} & \frac{23}{237} & \frac{15}{158} \\ \frac{13}{79} & -\frac{16}{237} & -\frac{7}{158} \\ -\frac{4}{79} & \frac{11}{237} & -\frac{5}{79} \end{bmatrix}$$

The inverse of matrix *A* is $\begin{bmatrix} \frac{6}{79} & \frac{23}{237} & \frac{15}{158} \\ \frac{13}{79} & -\frac{16}{237} & -\frac{7}{158} \\ -\frac{4}{79} & \frac{11}{237} & -\frac{5}{79} \end{bmatrix}$.

4. Multiply both sides of the equation by A^{-1} following the formula $A^{-1}AX = A^{-1}B$.

The inverse of matrix A is $A^{-1} = \begin{bmatrix} \frac{6}{79} & \frac{23}{237} & \frac{15}{158} \\ \frac{13}{79} & -\frac{16}{237} & -\frac{7}{158} \\ -\frac{4}{79} & \frac{11}{237} & -\frac{5}{79} \end{bmatrix}$.

The original matrix equation is $\begin{bmatrix} 3 & 5 & 1 \\ 6 & 0 & 9 \\ 2 & -4 & -10 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -20 \\ 24 \\ 2 \end{bmatrix}$.

Multiply both sides of the equation by A^{-1}, as shown:

$$\begin{bmatrix} \frac{6}{79} & \frac{23}{237} & \frac{15}{158} \\ \frac{13}{79} & -\frac{16}{237} & -\frac{7}{158} \\ -\frac{4}{79} & \frac{11}{237} & -\frac{5}{79} \end{bmatrix} \bullet \begin{bmatrix} 3 & 5 & 1 \\ 6 & 0 & 9 \\ 2 & -4 & -10 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

$$\begin{bmatrix} \frac{6}{79} & \frac{23}{237} & \frac{15}{158} \\ \frac{13}{79} & -\frac{16}{237} & -\frac{7}{158} \\ -\frac{4}{79} & \frac{11}{237} & -\frac{5}{79} \end{bmatrix} \bullet \begin{bmatrix} -20 \\ 24 \\ 2 \end{bmatrix}$$

Since the Commutative Property of Multiplication does not apply to matrix multiplication, the placement of the matrix is important. Notice that A^{-1} is listed as the first factor on both sides of the equation following the formula $A^{-1}AX = A^{-1}B$.

5. Multiply the matrices in the equation using matrix multiplication.

First, use the information stored on your graphing calculator to multiply matrix A^{-1} by matrix A:

$$\begin{bmatrix} \frac{6}{79} & \frac{23}{237} & \frac{15}{158} \\ \frac{13}{79} & -\frac{16}{237} & -\frac{7}{158} \\ -\frac{4}{79} & \frac{11}{237} & -\frac{5}{79} \end{bmatrix} \bullet \begin{bmatrix} 3 & 5 & 1 \\ 6 & 0 & 9 \\ 2 & -4 & -10 \end{bmatrix}$$

On a TI-83/84:

Step 1: Recall that matrix A was previously stored as matrix "A" and the inverse matrix, A^{-1}, was previously stored as matrix "D."

Step 2: Choose matrix "A" by pressing [2ND][MATRIX][1].

Step 3: Press [×].

Step 4: Choose the previously stored inverse matrix "D" by pressing [2ND][MATRIX][4]. Press [ENTER].

On a TI-Nspire:

Step 1: Recall that matrix A was previously stored as matrix "*a*" and the inverse matrix, A^{-1}, was previously stored as matrix "*d*."

Step 2: Press [A][×][D], then press [enter].

The result on either calculator will be the identity matrix, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Thus, on the left side of the matrix equation,

$$\begin{bmatrix} \frac{6}{79} & \frac{23}{237} & \frac{15}{158} \\ \frac{13}{79} & -\frac{16}{237} & -\frac{7}{158} \\ -\frac{4}{79} & \frac{11}{237} & -\frac{5}{79} \end{bmatrix} \bullet \begin{bmatrix} 3 & 5 & 1 \\ 6 & 0 & 9 \\ 2 & -4 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(*continued*)

The equation is now

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{6}{79} & \frac{23}{237} & \frac{15}{158} \\ \frac{13}{79} & -\frac{16}{237} & -\frac{7}{158} \\ -\frac{4}{79} & \frac{11}{237} & -\frac{5}{79} \end{bmatrix} \bullet \begin{bmatrix} -20 \\ 24 \\ 2 \end{bmatrix}.$$

On the left side of the equation, matrix A was multiplied by its inverse, which produced the identity matrix, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. When the identity matrix is multiplied by the vector matrix X, $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, the vector matrix does not change.

Therefore, the equation is now

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{6}{79} & \frac{23}{237} & \frac{15}{158} \\ \frac{13}{79} & -\frac{16}{237} & -\frac{7}{158} \\ -\frac{4}{79} & \frac{11}{237} & -\frac{5}{79} \end{bmatrix} \bullet \begin{bmatrix} -20 \\ 24 \\ 2 \end{bmatrix}.$$

Next, use the inverse matrix information stored on your calculator to multiply matrix A^{-1} by matrix B (the right side of the equation):

$$\begin{bmatrix} \frac{6}{79} & \frac{23}{237} & \frac{15}{158} \\ \frac{13}{79} & -\frac{16}{237} & -\frac{7}{158} \\ -\frac{4}{79} & \frac{11}{237} & -\frac{5}{79} \end{bmatrix} \bullet \begin{bmatrix} -20 \\ 24 \\ 2 \end{bmatrix}$$

(continued)

On a TI-83/84:

Step 1: Recall that the inverse matrix, A^{-1}, was previously stored as matrix "D."

Step 2: To input matrix *B*, press [2ND][MATRIX], arrow over to EDIT, then press [2] to edit matrix "B." Enter the dimensions of matrix *B*: [3][ENTER][1][ENTER]. Input each entry of the matrix, pressing [ENTER] after each. The matrix is now automatically stored as matrix "B."

Step 3: Go back to the home screen by pressing [2ND][MODE].

Step 4: Choose matrix "D" by pressing [2ND][MATRIX][4].

Step 5: Press [×].

Step 6: Choose matrix "B" by pressing [2ND][MATRIX][2]. Press [ENTER].

On a TI-Nspire:

Step 1: Recall that the inverse matrix, A^{-1}, was previously stored as matrix "*d*."

Step 2: To input matrix *B*, press [menu] and select 7: Matrix & Vector, 1: Create, and 1: Matrix. Enter the number of rows [3] and columns [1]. Press [enter]. Input each entry of matrix *B*, using the arrow keys to navigate between fields. Press [enter]. To store matrix *B*, press [ctrl][sto→] then [B][enter] to store it as matrix "*b*."

Step 3: Press [D][×][B][enter].

The result on either calculator will be $\begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$.

(*continued*)

Thus, on the right side of the matrix equation,

$$\begin{bmatrix} \frac{6}{79} & \frac{23}{237} & \frac{15}{158} \\ \frac{13}{79} & -\frac{16}{237} & -\frac{7}{158} \\ -\frac{4}{79} & \frac{11}{237} & -\frac{5}{79} \end{bmatrix} \bullet \begin{bmatrix} -20 \\ 24 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}.$$

The equation is now $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$.

6. State the solution to the system of equations.

From the equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$, $x = 1$, $y = -5$, and $z = 2$. Or, written as a set of coordinates, the solution is (1, –5, 2).

This answer can be checked by substituting the values for x, y, and z back into the original system of equations.

Scaffolded Practice 2.2

Example 1

Determine if the matrix $\begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix}$ has an inverse.

1. Find the determinant of the matrix.

2. Determine if the matrix has an inverse.

continued

Example 2

Use a calculator to find the inverse of the matrix $\begin{bmatrix} 6 & -5 & 2 \\ 1 & -4 & 10 \\ 5 & 8 & 1 \end{bmatrix}$. Leave entries as fractions if necessary.

Example 3

Use matrices to solve the following system of equations without using a calculator.

$$\begin{cases} 8x+8=4y \\ 3y=-5x+28 \end{cases}$$

Example 4

Use matrices and a calculator to solve the following system of equations:

$$\begin{cases} 3x+z=-20-5y \\ 9z-24=-6x \\ -10z-4y+2x-2=0 \end{cases}$$

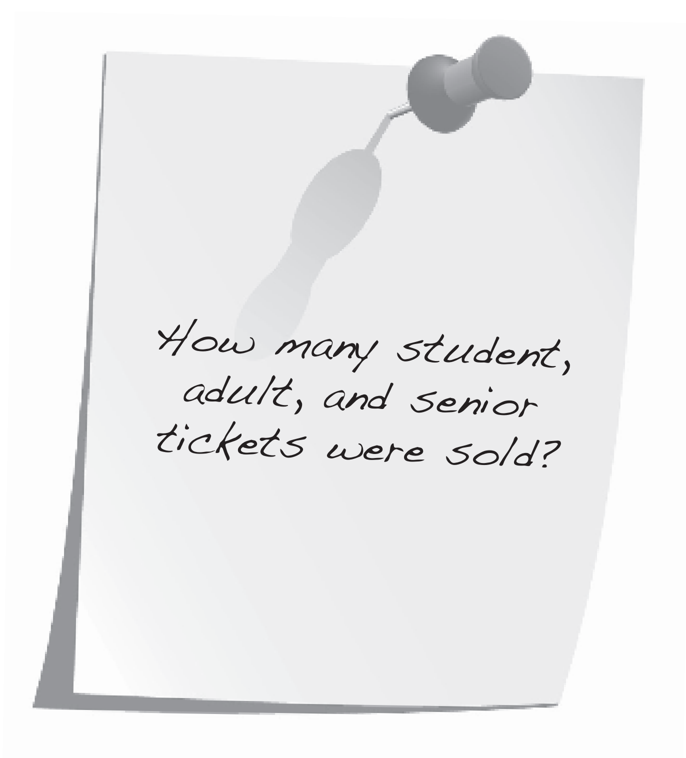
TOPIC 2 • USING MATRICES TO SOLVE SYSTEMS OF EQUATIONS

Lesson 2.2: Finding the Inverse of a Matrix and Using It to Solve a System of Equations

Problem-Based Task 2.2: Sold Out!

SMP	
1 ✓	2 ✓
3	4 ✓
5 ✓	6 ✓
7 ✓	8 ✓

The school's drama department just sold out the tickets for their upcoming play. The department sold 300 tickets for a total revenue of $2,080. Student tickets cost $5, adult tickets cost $10, and senior tickets were discounted to $8 each. Twice as many student tickets were sold as adult tickets. How many student, adult, and senior tickets were sold?



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Problem-Based Task 2.2: Sold Out!

Coaching

a. A total of 300 tickets were sold, at three different price levels. What equation can be written to represent this situation?

b. The total revenue from the tickets was $2,080; student tickets cost $5, adult tickets cost $10, and senior tickets were discounted to $8 each. What equation can be written to represent this situation?

c. What equation can be written from the fact that twice as many student tickets were sold as adult tickets?

d. Write the results of parts a–c as a system of equations. Then, rewrite the system and use it to create a matrix equation of the form $AX = B$.

e. What is the inverse of matrix A?

f. What is the solution to the matrix equation?

g. How many student, adult, and senior tickets were sold?

Problem-Based Task 2.2: Sold Out!

Coaching Sample Responses

a. A total of 300 tickets were sold, at three different price levels. What equation can be written to represent this situation?

Let x represent the number of student tickets sold, y represent the number of adult tickets sold, and z represent the number of senior tickets sold.

Since the total number of tickets sold is 300, the sum of x, y, and z must be 300.

Therefore, $x + y + z = 300$.

b. The total revenue from the tickets was $2,080; student tickets cost $5, adult tickets cost $10, and senior tickets were discounted to $8 each. What equation can be written to represent this situation?

Student tickets cost $5 each, and x represents the number of student tickets sold. Thus, 5 times x or $5x$ represents the sales from student tickets.

Adult tickets cost $10 each, and y represents the number of adult tickets sold. Thus, 10 times y or $10y$ represents the sales from adult tickets.

Senior tickets cost $8 each, and z represents the number of senior tickets sold. Thus, 8 times z or $8z$ represents the sales from senior tickets.

Since the total revenue was $2,080, the sum of the costs of the various tickets ($5x$, $10y$, and $8z$) must be $2,080.

Thus, $5x + 10y + 8z = 2080$.

c. What equation can be written from the fact that twice as many student tickets were sold as adult tickets?

The variable x represents the number of student tickets and y is the number of adult tickets.

Twice as many student tickets were sold. That means that the number of adult tickets (y) would have to be multiplied by 2 in order to equal the number of student tickets (x).

Therefore, $2y = x$.

d. Write the results of parts a–c as a system of equations. Then, rewrite the system and use it to create a matrix equation of the form $AX = B$.

The system of equations is $\begin{cases} x+y+z=300 \\ 5x+10y+8z=2080 \\ 2y=x \end{cases}$.

To create a matrix equation, these equations must first be written so that the variable terms are on the left in alphabetical order and the constant terms are on the right. The first two equations are already arranged this way. The last equation must be rearranged.

Rearrange $2y = x$ by subtracting x from both sides and rearranging the terms:

$2y - x = 0$

$-x + 2y = 0$

The rewritten system of equations is $\begin{cases} x+y+z=300 \\ 5x+10y+8z=2080 \\ -x+2y=0 \end{cases}$.

Write this system as a matrix equation of the form $AX = B$. Recall that matrix A is formed from the coefficients of the variables, matrix X is formed from the variables themselves, and matrix B is formed from the constant terms.

The matrix equation is $\begin{bmatrix} 1 & 1 & 1 \\ 5 & 10 & 8 \\ -1 & 2 & 0 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 300 \\ 2080 \\ 0 \end{bmatrix}$.

e. What is the inverse of matrix A?

Use a calculator to find the inverse of matrix A: $\begin{bmatrix} 1 & 1 & 1 \\ 5 & 10 & 8 \\ -1 & 2 & 0 \end{bmatrix}$.

The inverse of matrix A is $A^{-1} = \begin{bmatrix} 4 & -\frac{1}{2} & \frac{1}{2} \\ 2 & -\frac{1}{4} & \frac{3}{4} \\ -5 & \frac{3}{4} & -\frac{5}{4} \end{bmatrix}$.

f. What is the solution to the matrix equation?

Multiply both sides of the equation by A^{-1} following the formula $A^{-1}AX = A^{-1}B$.

The inverse of matrix A is $A^{-1} = \begin{bmatrix} 4 & -\frac{1}{2} & \frac{1}{2} \\ 2 & -\frac{1}{4} & \frac{3}{4} \\ -5 & \frac{3}{4} & -\frac{5}{4} \end{bmatrix}$.

The original matrix equation is $\begin{bmatrix} 1 & 1 & 1 \\ 5 & 10 & 8 \\ -1 & 2 & 0 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 300 \\ 2080 \\ 0 \end{bmatrix}$. Multiply both sides by A^{-1}:

$$\begin{bmatrix} 4 & -\frac{1}{2} & \frac{1}{2} \\ 2 & -\frac{1}{4} & \frac{3}{4} \\ -5 & \frac{3}{4} & -\frac{5}{4} \end{bmatrix} \bullet \begin{bmatrix} 1 & 1 & 1 \\ 5 & 10 & 8 \\ -1 & 2 & 0 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & -\frac{1}{2} & \frac{1}{2} \\ 2 & -\frac{1}{4} & \frac{3}{4} \\ -5 & \frac{3}{4} & -\frac{5}{4} \end{bmatrix} \bullet \begin{bmatrix} 300 \\ 2080 \\ 0 \end{bmatrix}$$

Multiply the matrices in the equation using matrix multiplication.

First, use a calculator to multiply $\begin{bmatrix} 4 & -\frac{1}{2} & \frac{1}{2} \\ 2 & -\frac{1}{4} & \frac{3}{4} \\ -5 & \frac{3}{4} & -\frac{5}{4} \end{bmatrix} \bullet \begin{bmatrix} 1 & 1 & 1 \\ 5 & 10 & 8 \\ -1 & 2 & 0 \end{bmatrix}$.

The result is the identity matrix, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Therefore, $\begin{bmatrix} 4 & -\frac{1}{2} & \frac{1}{2} \\ 2 & -\frac{1}{4} & \frac{3}{4} \\ -5 & \frac{3}{4} & -\frac{5}{4} \end{bmatrix} \bullet \begin{bmatrix} 1 & 1 & 1 \\ 5 & 10 & 8 \\ -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The equation is now $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & -\frac{1}{2} & \frac{1}{2} \\ 2 & -\frac{1}{4} & \frac{3}{4} \\ -5 & \frac{3}{4} & -\frac{5}{4} \end{bmatrix} \bullet \begin{bmatrix} 300 \\ 2080 \\ 0 \end{bmatrix}$.

On the left side of the equation, matrix A was multiplied by its inverse, which produced the identity matrix, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. When the identity matrix is multiplied by the vector matrix $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, the vector matrix does not change.

Therefore, the equation is now $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & -\frac{1}{2} & \frac{1}{2} \\ 2 & -\frac{1}{4} & \frac{3}{4} \\ -5 & \frac{3}{4} & -\frac{5}{4} \end{bmatrix} \bullet \begin{bmatrix} 300 \\ 2080 \\ 0 \end{bmatrix}$.

Next, use a calculator to multiply $\begin{bmatrix} 4 & -\frac{1}{2} & \frac{1}{2} \\ 2 & -\frac{1}{4} & \frac{3}{4} \\ -5 & \frac{3}{4} & -\frac{5}{4} \end{bmatrix} \bullet \begin{bmatrix} 300 \\ 2080 \\ 0 \end{bmatrix}$.

The result is $\begin{bmatrix} 160 \\ 80 \\ 60 \end{bmatrix}$.

Therefore, $\begin{bmatrix} 4 & -\frac{1}{2} & \frac{1}{2} \\ 2 & -\frac{1}{4} & \frac{3}{4} \\ -5 & \frac{3}{4} & -\frac{5}{4} \end{bmatrix} \bullet \begin{bmatrix} 300 \\ 2080 \\ 0 \end{bmatrix} = \begin{bmatrix} 160 \\ 80 \\ 60 \end{bmatrix}$.

The matrix equation is now $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 160 \\ 80 \\ 60 \end{bmatrix}$.

g. How many student, adult, and senior tickets were sold?

From the equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 160 \\ 80 \\ 60 \end{bmatrix}$, $x = 160$, $y = 80$, and $z = 60$.

Recall that x represents the number of student tickets, y represents the number of adult tickets, and z represents the number of senior tickets.

Thus, the drama department sold 160 student tickets, 80 adult tickets, and 60 senior tickets.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

Practice 2.2: Finding the Inverse of a Matrix and Using It to Solve a System of Equations

For problems 1 and 2, use a calculator to find the inverse of each matrix. Leave entries as fractions if necessary.

1. $\begin{bmatrix} 3 & 7 & -11 \\ 5 & -8 & -2 \\ 0 & 6 & 5 \end{bmatrix}$

2. $\begin{bmatrix} 8 & 3 & -1 \\ 5 & 4 & 2 \\ -7 & 6 & 1 \end{bmatrix}$

For problems 3 and 4, use matrices to solve each system of equations without using a calculator. Write the solution as a set of coordinates.

3. $\begin{cases} 2x+7y-13=0 \\ 3x=5y-27 \end{cases}$

4. $\begin{cases} x-13=8y \\ 2y=7x-37 \end{cases}$

continued

For problems 5–8, use matrices and a calculator to solve each system of equations. Write the solution as a set of coordinates.

5. $\begin{cases} 3x+7y-z=26 \\ 5y-2x-6z+26=0 \\ 8y+3z=64-7x \end{cases}$

6. $\begin{cases} 5x+7z-29=0 \\ 3y+7=2x-z \\ 2+3x=5y+8z \end{cases}$

7. $\begin{cases} -4z+5x+10y=-58 \\ 6y+11+z=0 \\ 5z=2y+3x+41 \end{cases}$

8. $\begin{cases} 3y+2z+46=5x \\ 2x+y+7z+25=0 \\ 2y+z=7x-51 \end{cases}$

For problems 9 and 10, write a system of equations that describes each situation. Then use matrices to find the answer.

9. Han's little brother set up a cookie and lemonade stand in which he sold cookies for $0.50 and cups of lemonade for $0.75. Last week, his brother sold 50 items for a total of $32.25 in sales. How many of each item did Han's brother sell?

10. Annaliese babysits for two families during the summer. Two weeks ago, she made a total of $255 by babysitting 15 hours for the Daniels family and 12 hours for the Garcia family. Last week, she made $262 babysitting 18 hours for the Daniels family and 10 hours for the Garcia family. How much does each family pay Annaliese per hour of babysitting?

Progress Assessment

Circle the letter of the best answer.

1. What is the inverse of $\begin{bmatrix} 6 & -2 \\ -9 & 3 \end{bmatrix}$?

   a. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

   b. $\begin{bmatrix} \frac{1}{12} & \frac{1}{18} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix}$

   c. $\begin{bmatrix} \frac{1}{6} & -\frac{1}{18} \\ -\frac{1}{4} & \frac{1}{12} \end{bmatrix}$

   d. There is no inverse.

2. Use matrices to determine the value of z in the solution to the system of equations,

   $$\begin{cases} 3x+2y+16=5z \\ 7y+6x=-16 \\ y-21=5x+6z \end{cases}.$$

   a. –5

   b. –3

   c. 1

   d. 2

Use your knowledge of matrices and systems of equations to complete all parts of the following problem.

3. For breakfast yesterday, Anthony ate 5 medium strawberries and 8 ounces of yogurt for a total of 116 calories. Today, he ate 20 medium strawberries and 5 ounces of yogurt for a total of 140 calories.

   a. Write a system of equations that describes this situation.

   b. What matrix equation can be written from the system of equations?

   c. How many calories are in each strawberry?

   d. How many calories are in each ounce of yogurt?

Problem Solving with Matrices

Instruction

Common Core State Standards

A–REI.8 (+) Represent a system of linear equations as a single matrix equation in a vector variable.

A–REI.9 (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimensions 3×3 or greater).

Activity Overview

Background

Systems of equations can be used to analyze and solve many real-world situations. In this extension activity, students solve a complex system using matrix operations. This will lead students to a better understanding of how and why matrix operations can be used to solve a system and what a solution to a system of equations means.

Introduction

Introduce the problem by discussing ways of solving systems of equations, including graphing, substitution, elimination, as well as using matrix operations. Often students get mired in the computations when solving a system and lose the understanding of what a solution means.

Systems of equations are used to solve many types of real-world problems including optimization problems, mixture problems, and population problems. Today computers do most of the work when solving systems, but it is critical to have an understanding of how a solution is obtained and what it means in the context of the situation.

Computers can do the computation, but they can't do the analysis. A human is needed to understand the problem and plan the solution as well as to write the code for the computer to use.

Implementation Suggestions

- Students should work individually. Because the systems they will be solving have three variables and three equations, students may use a graphing calculator to perform the matrix calculations.
- In preparation for the activity, have students use one of the many systems-of-equations solvers available online to solve any 2×2 system. Discovery Education offers one such solver.

  Discovery Education. "Solve a Simultaneous Set of Two Linear Equations."

  http://www.walch.com/rr/06021
- Have students work in groups to solve a simple system such as $\begin{cases} x + y = 2 \\ 5x - 2y = 24 \end{cases}$ using matrices to review the steps before tackling more difficult problems.

Monitoring/Facilitating the Activity

Ask questions and prompt student thinking so that they:

- understand what the solution to the system means
- understand how to write a system to solve the given problem
- realize that solving a system is an iterative process, no matter what algebraic method is used
- understand how to use a graphing calculator to solve a system

Debriefing the Extension Activity

- Upon completion of the activity, students should check their answers with an online 3×3 system-of-equations solver, such as this one by Symbolab:

  Symbolab. "System of Equations Calculator."

  http://www.walch.com/rr/06022
- Discuss how students determined the equations that comprised each system. Were there different ways to write the system, or just one way?
- Discuss how changing the parameters of each problem changed the system. Were the new systems "solvable" without computation? Why or why not?

Suggestions for Extending the Activity

- Students are familiar with how to solve a 2×2 system by graphing and understand what consistent/inconsistent and independent/dependent mean with respect to two variables. Ask them to consider how to graph a linear system of three variables, and what consistent/inconsistent and independent/dependent mean with respect to three variables.
- Students with knowledge of 3-D software can create 3-D examples of solutions to a linear system in three variables.

Recommended Resources

- Math Warehouse. "Systems of 3 Variable Equations."

  http://www.walch.com/rr/06023

  This site describes what solutions of 3×3 systems look like graphically.
- Shmoop.com. "Systems of Linear Equations: In the Real World."

  http://www.walch.com/rr/06024

  This site provides examples of real-world systems.

Problem Solving with Matrices

Part 1

Jason hopes to market and sell the best chicken chowder in the world. The three main ingredients of his chowder are chicken, corn, and milk. Jason wants each can of chowder to have 66 grams of protein, 94.5 grams of carbohydrates, and 21 grams of fat. The table shows the quantities of these nutrients in grams found in each serving of chicken, corn, and milk. (Assume that the rest of the ingredients have negligible amounts of protein, carbohydrates, and fat.)

Nutrients (in grams per serving)

Ingredient	Protein	Carbohydrates	Fat
Chicken	30	35	4
Corn	3	16	3
Milk	9	13	6

1. Write a system of equations that describes this situation.

2. What matrix equation can be used to solve the system?

3. Use your matrix equation and a calculator to solve the system. Interpret the solution in the context of the problem.

4. Jason has decided to create a version of the chowder using squash instead of corn. Each serving of squash has 1.5 grams of protein, 8 grams of carbohydrates, and 1.5 grams of fat. Determine how these new parameters change the solution without solving a new system of equations.

5. Jason decides to change the recipe for his original chicken-corn chowder so that each serving has 33 grams of protein, 47.25 grams of carbohydrates and 10.5 grams of fat. Determine how this information changes the solution to problem 3 without solving a new system of equations.

continued

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Part 2

A ballpark manager is ordering cases of candy to stock the concession stand. She orders a total of 340 cases for $632. Candy bars cost $2.25 per case, lollipops cost $1.20 per case, and sour candies cost $1.90 per case. She orders 30 more cases of candy bars than sour candies.

1. Write a system of equations that describes this situation.

2. What matrix equation can be used to solve the system?

3. Use your matrix equation and a calculator to solve the system. Interpret the solution in the context of the problem.

4. The manager decides to order 680 cases for a total of $1,264. She orders 60 more cases of candy bars than sour candies. Determine how these new parameters change the solution without solving a new system of equations.

5. The distributor is offering a pre-season special: all prices are 50% off for the first order. Therefore, the manager would like to go back to spending her original budget of $632, but double the number of cases to 680. She still would like 30 more cases of candy bars than sour candies. What new system can be solved to determine the number of cases of each type of candy she should buy? Determine the number of cases she should order for each type of candy. Can this answer be found without solving the system? Explain.