TOPIC 2 • VECTORS

Lesson 2.1: Representing and Modeling with Vector Quantities

Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Which vector has a magnitude of 10?

   a. $\langle 2,5 \rangle$

   b. $\langle -4,6 \rangle$

   c. $\langle -6,8 \rangle$

   d. $\langle 5,-5 \rangle$

2. Which vector could have initial point $P(1, -4)$ and terminal point $Q(3, -1)$?

   a. $\langle 2,3 \rangle$

   b. $\langle -2,-3 \rangle$

   c. $\langle 4,-5 \rangle$

   d. $\langle -4,5 \rangle$

Common Core State Standards

SMP
1 ✓ 2 ✓
3 ✓ 4 ✓
5 ✓ 6 ✓
7 ✓ 8 ✓

N–VM.1 (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\boldsymbol{v}$, $|\boldsymbol{v}|$, $||\boldsymbol{v}||$, v).

N–VM.2 (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

N–VM.3 (+) Solve problems involving velocity and other quantities that can be represented by vectors.

Essential Questions

1. What does it mean for a quantity to have both magnitude and direction?
2. Where do vectors occur in real life?

WORDS TO KNOW

components of a vector for a given vector $\vec{v} = \langle a, b \rangle$, the x-component is a and the y-component is b

directed line segment a line segment $\overrightarrow{PQ}$ directed from point P to point Q; note that because the line segment is directed, $\overrightarrow{QP}$ points in the opposite direction, thus the order of letters is important

direction the way in which a vector points; may be specified by an angle, a slope, or a pair of vector components

displacement vector a vector that represents a distance traveled in the x- and y-directions; the magnitude of a displacement vector is the shortest distance from the initial point to the terminal point

distance formula a formula that states the distance between points (x_1, y_1) and (x_2, y_2) is equal to $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

force an influence vector that represents a directed push or pull on an object; the strength of a force vector is its magnitude because it has both magnitude and direction

initial point the point at which a vector begins; the "tail" of a vector

SWB p. 39

magnitude — the length of a vector, denoted by $\|\mathbf{v}\|$; the length of vector $\vec{v} = \langle a, b \rangle$ is $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$

scalar — a quantity, usually a constant; a numerical quantity without an associated direction

speed — the magnitude of an object's velocity vector

terminal point — the point at which a vector ends; the "head" of a vector

vector — a quantity that has both magnitude and direction

velocity vector — a vector that represents the motion of an object

Recommended Resources

- PatrickJMT: Just Math Tutorials. "Finding the Components of a Vector, Ex 1."

  http://www.walch.com/rr/06001

  This video explains the process of finding the components of a vector when given the magnitude and direction angle.

- Paul's Online Math Notes. "Vectors—the Basics."

  http://www.walch.com/rr/06002

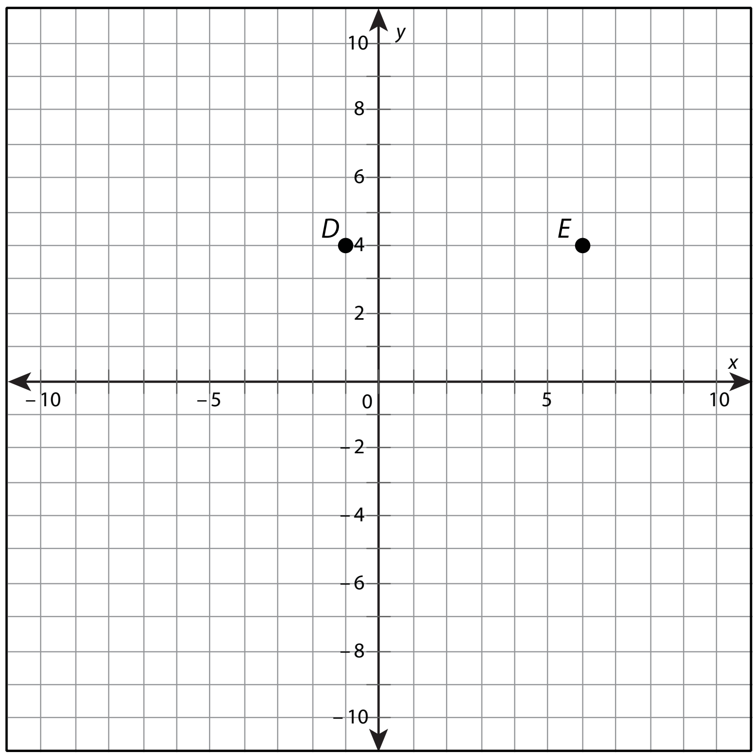
  This site explains vectors, their notation, and how to find their components when given the endpoints and direction.

TOPIC 2 • VECTORS

Lesson 2.1: Representing and Modeling with Vector Quantities

Warm-Up 2.1

Aaron drew a map of his town on a coordinate plane. On the map, each unit represents 1 mile. Brianna's house is located at point *B* (–2, 1), Carlos's house is located at point *C* (2, –1), and David's and Emily's houses are located at points *D* and *E*, as shown.



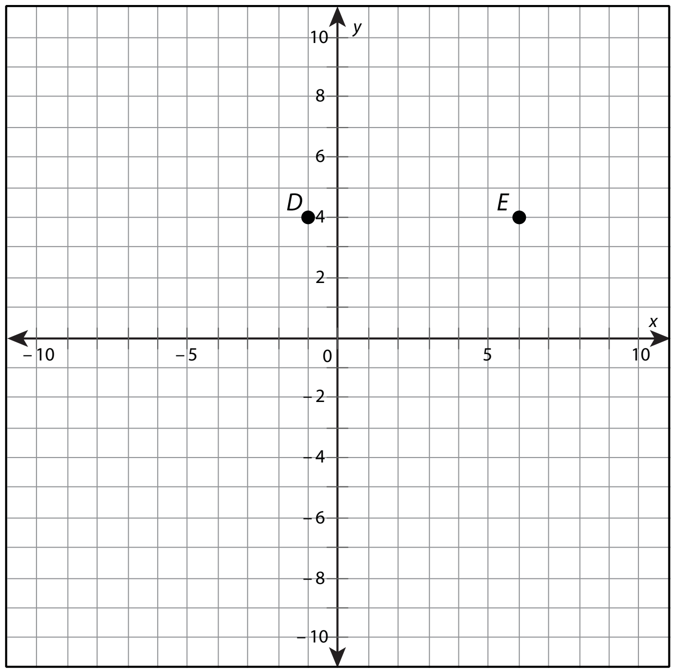
1. Draw points *B* and *C* on the map, and find the coordinates of points *D* and *E*.

2. To walk to David's house from his own house, Aaron walks 4 miles west and 2 miles north. Find the coordinates of Aaron's house, and draw and label point *A*.

3. Find the distances from Aaron's house to each of his friends' houses. Which friend lives closest to Aaron? Which friend lives farthest from Aaron? Round your answers to the nearest tenth of a mile.

Warm-Up 2.1 Debrief

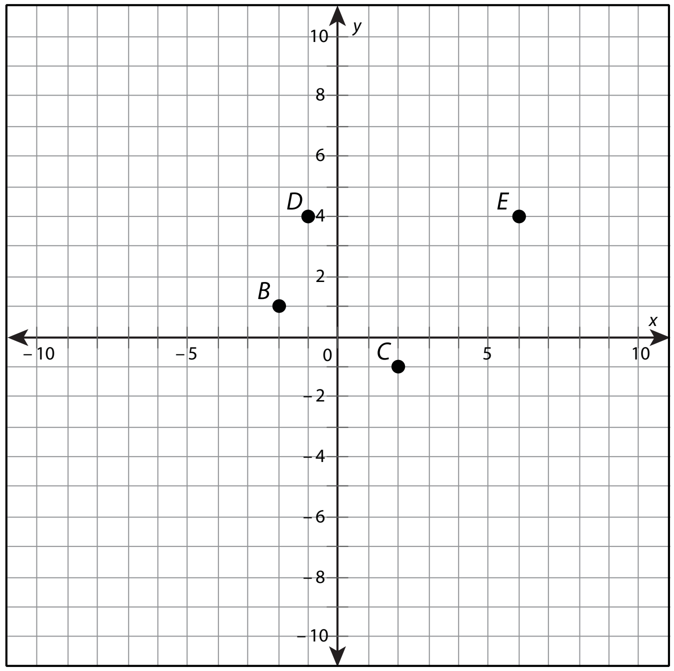
Aaron drew a map of his town on a coordinate plane. On the map, each unit represents 1 mile. Brianna's house is located at point *B* (–2, 1), Carlos's house is located at point *C* (2, –1), and David's and Emily's houses are located at points *D* and *E*, as shown.



1. Draw points *B* and *C* on the map, and find the coordinates of points *D* and *E*.

   Points *B* and *C* have the coordinates (–2, 1) and (2, –1), respectively. Plot each of the points on the coordinate plane.

   As indicated on the given coordinate plane, the location of points *D* and *E* are *D* (–1, 4) and *E* (6, 4), respectively.



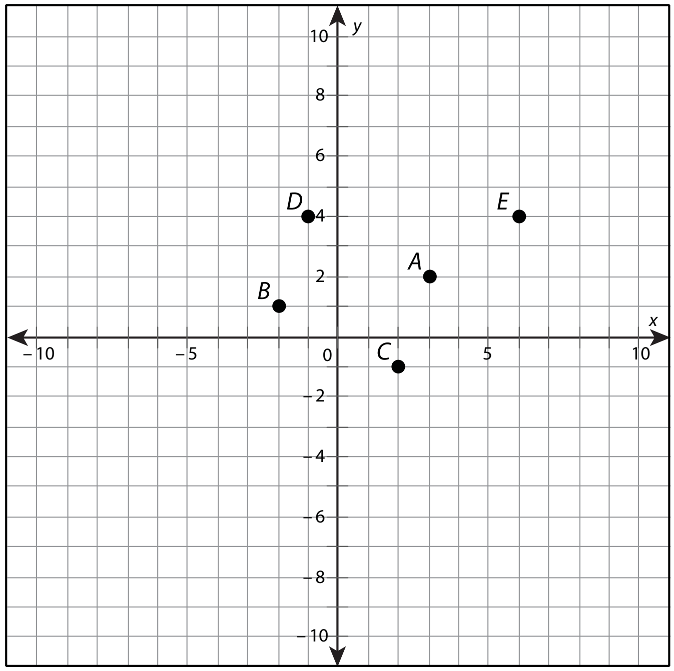
2. To walk to David's house from his own house, Aaron walks 4 miles west and 2 miles north. Find the coordinates of Aaron's house, and draw and label point *A*.

   Since David's house is 4 miles west and 2 miles north of Aaron's house, to find the coordinates of Aaron's house, start at David's house and "travel backward."

   The opposite of 4 miles west is 4 miles east, so add 4 to the *x*-coordinate: $-1 + 4 = 3$.

   The opposite of 2 miles north is 2 miles south, so subtract 2 from the *y*-coordinate: $4 - 2 = 2$.

   Aaron's house is located at point *A* (3, 2) on the map.



3. Find the distances from Aaron's house to each of his friends' houses. Which friend lives closest to Aaron? Which friend lives farthest from Aaron? Round your answers to the nearest tenth of a mile.

   Use the distance formula, $d=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$, to determine the distance from Aaron's house to the house of each of his friends.

   To find the distance from Aaron's house to Brianna's house, let (x_1, y_1) be (3, 2) and (x_2, y_2) be (–2, 1).

$d=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$	Distance formula
$d=\sqrt{\left[(-2)-(3)\right]^2+(1)-(2)^2}$	Substitute (3, 2) for (x_1, y_1) and (–2, 1) for (x_2, y_2).
$d=\sqrt{(-5)^2+(-1)^2}$	Simplify.
$d=\sqrt{25+1}$	
$d=\sqrt{26}$	
$d\approx 5.1$	

   The distance from Aaron's house to Brianna's house is $\sqrt{26}$ or approximately 5.1 miles.

To find the distance from Aaron's house to Carlos's house, let (x_1, y_1) be (3, 2) and (x_2, y_2) be (2, –1).

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	Distance formula
$d=\sqrt{[(2)-(3)]^2+[(-1)-(2)]^2}$	Substitute (3, 2) for (x_1, y_1) and (2, –1) for (x_2, y_2).
$d=\sqrt{(-1)^2+(-3)^2}$	Simplify.
$d=\sqrt{1+9}$	
$d=\sqrt{10}$	
$d\approx 3.2$	

The distance from Aaron's house to Carlos's house is $\sqrt{10}$ or approximately 3.2 miles.

To find the distance from Aaron's house to David's house, let (x_1, y_1) be (3, 2) and (x_2, y_2) be (–1, 4).

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	Distance formula
$d=\sqrt{[(-1)-(3)]^2+[(4)-(2)]^2}$	Substitute (3, 2) for (x_1, y_1) and (–1, 4) for (x_2, y_2).
$d=\sqrt{(-4)^2+(2)^2}$	Simplify.
$d=\sqrt{16+4}$	
$d=\sqrt{20}$	
$d=2\sqrt{5}$	
$d\approx 4.5$	

The distance from Aaron's house to David's house is $2\sqrt{5}$ or approximately 4.5 miles.

To find the distance from Aaron's house to Emily's house, let (x_1, y_1) be (3, 2) and (x_2, y_2) be (6, 4).

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	Distance formula
$d=\sqrt{[(6)-(3)]^2+[(4)-(2)]^2}$	Substitute (3, 2) for (x_1, y_1) and (6, 4) for (x_2, y_2).
$d=\sqrt{(3)^2+(2)^2}$	Simplify.
$d=\sqrt{9+4}$	
$d=\sqrt{13}$	
$d\approx 3.6$	

The distance from Aaron's house to Emily's house is $\sqrt{13}$ or approximately 3.6 miles.

Brianna lives 5.1 miles from Aaron, Carlos lives 3.2 miles from Aaron, David lives 4.5 miles from Aaron, and Emily lives 3.6 miles from Aaron. The shortest distance is 3.2 miles, and the longest distance is 5.1 miles. Therefore, Carlos lives the closest to Aaron, and Brianna lives the farthest from Aaron.

Connection to the Lesson

- Students will find the magnitude of a vector by using the distance formula to find the distance between two points.
- Students will interpret directions for *x*- and *y*-coordinates in order to go from one point to another.
- Students will determine and compare the magnitudes of vectors and investigate the relationship between a vector's components and the coordinates of its initial and terminal points.

Prerequisite Skills

This lesson requires the use of the following skills:

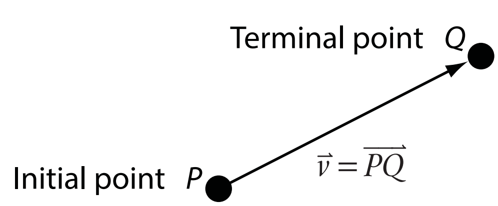
- graphing points on a coordinate plane
- applying the Pythagorean Theorem
- applying the distance formula
- simplifying radicals

Introduction

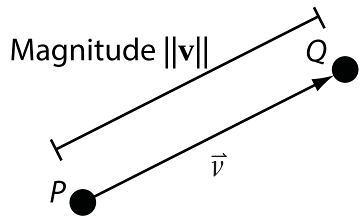
Vectors are quantities that have both magnitude and direction, and they are used to describe the movement of objects. When you say a car travels "northeast at 60 mph" you are describing the car's motion with a vector, because the motion has both magnitude (length, which is used to represent the speed of the car) and direction. In this case, the vector represents the speed and direction of an object's motion, and is called a velocity vector. Vectors are also commonly used to represent the displacement of a moved object, or the forces acting on an object.

Key Concepts

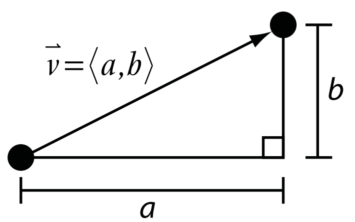
- A vector, $\vec{v}$, is a quantity that has both magnitude and direction. In contrast, a quantity that has only magnitude is a scalar. A **scalar** is a numerical quantity without an associated direction.
- Graphically, a vector is represented by a **directed line segment**, which is a line segment $\overrightarrow{PQ}$ that is directed or started from an **initial point**, P, to an ending or **terminal point**, Q. The initial point is also referred to as the tail of the vector and the terminal point is also referred to as the head of the vector. The vector may also be referred to as vector $\overrightarrow{PQ}$.
- Note that the order of the letters is important when naming a vector. Because it is a *directed* line segment, as indicated by the arrow over the letters, the vector $\overrightarrow{QP}$ points in the opposite direction of $\overrightarrow{PQ}$.



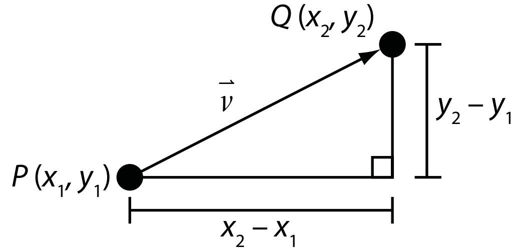
- The **magnitude** of vector $\vec{v}$ is the length of this directed line segment and is denoted $\|\mathbf{v}\|$.



- Algebraically, a vector, $\vec{v}$, is represented by specifying its **components** *a* and *b* in the *x*- and *y*-directions, respectively.
- The magnitude of a vector $\vec{v}=\langle a,b\rangle$ corresponds to the length of the hypotenuse of a right triangle with legs of length *a* and *b*. The magnitude can be found using the Pythagorean Theorem: $\|\mathbf{v}\|=\sqrt{a^2+b^2}$.



- A vector can also be specified by the coordinates of its initial and terminal points. If $\vec{v}=\overrightarrow{PQ}$ has an initial point $P(x_1, y_1)$ and a terminal point $Q(x_2, y_2)$, then the vector may be written in component form as $\vec{v}=\langle a,b\rangle$, where $a = x_2 - x_1$ and $b = y_2 - y_1$.
- The magnitude of $\vec{v}=\overrightarrow{PQ}$ from $P(x_1, y_1)$ to $Q(x_2, y_2)$ can be found using the **distance formula** such that $\|\mathbf{v}\|=\|\mathbf{PQ}\|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.



- In addition to its magnitude $\|\mathbf{v}\|$, a vector $\vec{v}$ also has a **direction**, which can be described in several different ways. One way to describe direction is to specify the components of the vectors themselves, another way is to determine the slope of the line that contains the vector, and a third way is to note the angle that the vector makes with respect to a reference ray.

- Together, the components of $\vec{v} = \langle a, b \rangle$ specify the direction in which the vector points. In fact, the ratio $a : b$ of the components of $\vec{v}$ is the same for all vectors that point in the direction of $\vec{v}$.
- An equivalent way to describe the direction of vector $\vec{v} = \langle a, b \rangle$ is to specify the slope, $\frac{b}{a}$, of the line which contains $\vec{v}$, and whether the vector points left or right (or up/down) along this line. As with the ratio $a : b$, the quotient $\frac{b}{a}$ is the same for all vectors that point in the direction of $\vec{v}$.
- Finally, you can describe the direction of vector $\vec{v}$ by specifying the angle that $\vec{v}$ makes with respect to a reference ray. A common reference ray is the positive *x*-axis, from which the angle to $\vec{v}$ is measured in the counterclockwise direction.
- Here are some examples of vectors with common real-world applications:
    - A **displacement vector** represents the path of an object moved in a straight line from one point to another; in other words, the distance traveled in the *x*- and *y*-directions. The magnitude of a displacement vector is the distance the object was moved.
    - A **velocity vector** represents the speed and direction of a moving object. The magnitude of a velocity vector is the **speed** at which the object moves.
    - **Force** is a vector that represents a push or pull on an object. The magnitude of a force vector is the strength of the force, or how hard of a push or pull the object receives.

Common Errors/Misconceptions

- subtracting the terminal point from the initial point instead of the other way around
- miscalculating slope by placing the change in *x* over the change in *y* instead of the other way around
- incorrectly calculating the subtraction of the *x*-coordinates and/or the *y*-coordinates when calculating slope

Guided Practice 2.1

Example 1

Find the magnitude of the vector $\vec{v} = \langle 2, -3 \rangle$ and describe the vector's direction.

1. Determine the components *a* and *b* of $\vec{v}$.

   For a given vector $\vec{v} = \langle a, b \rangle$, the *x*-component is *a* and the *y*-component is *b*.

   The *x*-component of $\vec{v}$ is the first number in the ordered pair, so *a* = 2. Similarly, the *y*-component of $\vec{v}$ is the second number in the ordered pair, so *b* = –3.

2. Use the Pythagorean Theorem to find the magnitude of $\vec{v}$.

   Because the components of $\vec{v}$ are the lengths of the legs of a right triangle with hypotenuse $\vec{v}$, the magnitude can be found using the Pythagorean Theorem.

   $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$ Magnitude formula derived from the Pythagorean Theorem

   $\|\mathbf{v}\| = \sqrt{(2)^2 + (-3)^2}$ Substitute 2 for *a* and –3 for *b*.

   $\|\mathbf{v}\| = \sqrt{4 + 9}$ Simplify.

   $\|\mathbf{v}\| = \sqrt{13}$

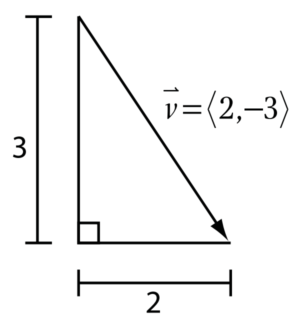
   The magnitude of $\vec{v}$ is $\sqrt{13}$.

3. Describe the direction of $\vec{v}$.

   One way to describe the direction of vector $\vec{v} = \langle a, b \rangle$ is to specify the slope, $\frac{b}{a}$, of the line which contains $\vec{v}$, and whether the vector points left or right (or up/down) along this line.

   For this vector, $a = 2$ and $b = -3$, so the slope $\frac{b}{a}$ is $-\frac{3}{2}$.

   Vector $\vec{v}$ measures 3 units vertically and 2 units horizontally, and points down and to the right along a line with a slope of $-\frac{3}{2}$. We know that the vector points down because the y-component of the vector ($b = -3$) is negative. We know that the vector points to the right because the x-component of the vector ($a = 2$) is positive.



Example 2

Find the magnitude and describe the direction of the vector $\vec{v} = \overrightarrow{PQ}$ from P (7, –2) to Q (1, 1).

1. Determine the component form of $\vec{v}$.

   The x-component of $\vec{v}$ is the difference in the x-values.

   $a = x_2 - x_1$

   $a = 1 - 7$

   $a = -6$

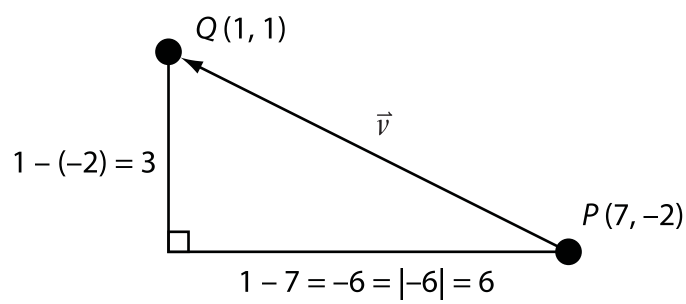
   The y-component of $\vec{v}$ is the difference of the y-values.

   $b = y_2 - y_1$

   $b = 1 - (-2)$

   $b = 3$

   The components are –6 and 3; thus, the component form is $\vec{v} = \langle -6, 3 \rangle$.



2. Find the magnitude of $\vec{v}$.

   Use either the distance formula or the Pythagorean Theorem to find the magnitude of $\vec{v}$.

   Now that you have the components of $\vec{v}$, you can find the magnitude, $\|\mathbf{v}\|$, using the Pythagorean Theorem.

$\|\mathbf{v}\|=\sqrt{a^2+b^2}$	Magnitude formula derived from the Pythagorean Theorem
$\|\mathbf{v}\|=\sqrt{(-6)^2+(3)^2}$	Substitute –6 for *a* and 3 for *b*.
$\|\mathbf{v}\|=\sqrt{36+9}$	Simplify.
$\|\mathbf{v}\|=\sqrt{45}$	
$\|\mathbf{v}\|=3\sqrt{5}$	

   The magnitude of $\vec{v}$ is $3\sqrt{5}$.

   Alternatively, you can find the magnitude $\|\mathbf{v}\|$ directly from the coordinates of *P* and *Q* using the distance formula.

$\|\mathbf{v}\|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	Magnitude formula derived from the distance formula
$\|\mathbf{v}\|=\sqrt{[(1)-(7)]^2+[(1)-(-2)]^2}$	Substitute (7, –2) for (x_1, y_1) and (1, 1) for (x_2, y_2).
$\|\mathbf{v}\|=\sqrt{(-6)^2+3^2}$	Simplify.
$\|\mathbf{v}\|=\sqrt{45}$	
$\|\mathbf{v}\|=3\sqrt{5}$	

   Again, the magnitude of $\vec{v}$ is $3\sqrt{5}$.

3. Describe the direction of $\vec{v}$.

   One way to describe the direction of a vector $\vec{v}$ is to specify its components, $\vec{v} = \langle a, b \rangle$. Here, the component form is $\vec{v} = \langle -6, 3 \rangle$, meaning vector $\vec{v}$ measures 3 units up for every 6 units left, so the ratio of the components is –2 : 1.

   Another way to describe the direction of a vector $\vec{v}$ is to specify the slope of the line that contains $\vec{v}$, and indicate whether $\vec{v}$ points up or down and left or right along this line. $\vec{v} = \langle -6, 3 \rangle$ points up (since the *y*-component is positive) and to the left (since the *x*-component is negative) along a line with a slope of $-\frac{1}{2}$.

Example 3

The vector $\vec{v} = \langle -5, 3 \rangle$ has an initial point *P* and a terminal point *Q*. Given the coordinates of one endpoint, find the coordinates of the other endpoint. Then graph each instance of the vector $\vec{v}$:

a. Given *P* (0, 0), find the coordinates of *Q*.

b. Given *P* (6, –2), find the coordinates of *Q*.

c. Given *Q* (0, 0), find the coordinates of *P*.

d. Given *Q* (–3, 7), find the coordinates of *P*.

1. Use the given coordinates of *P* and the components of $\vec{v}$ to find the coordinates of *Q*.

   The components of a vector $\vec{v}$ tell you how the coordinates change from the initial point *P* to the terminal point *Q*. That is, vector $\vec{v} = \langle a, b \rangle$ extends from *P* (x_1, y_1) to *Q* $(x_1 + a, y_1 + b)$.

   a. From initial point *P* (0, 0), vector $\vec{v} = \langle -5, 3 \rangle$ extends 5 units to the left and 3 units up to the point *Q*. Therefore, *Q* (0 – 5, 0 + 3) = *Q* (–5, 3).

   b. From initial point *P* (6, –2), vector $\vec{v} = \langle -5, 3 \rangle$ extends 5 units to the left and 3 units up to the point *Q*. Therefore, *Q* (6 – 5, –2 + 3) = *Q* (1, 1).

2. Use the coordinates of Q and the components of $\vec{v}$ to find the coordinates of P.

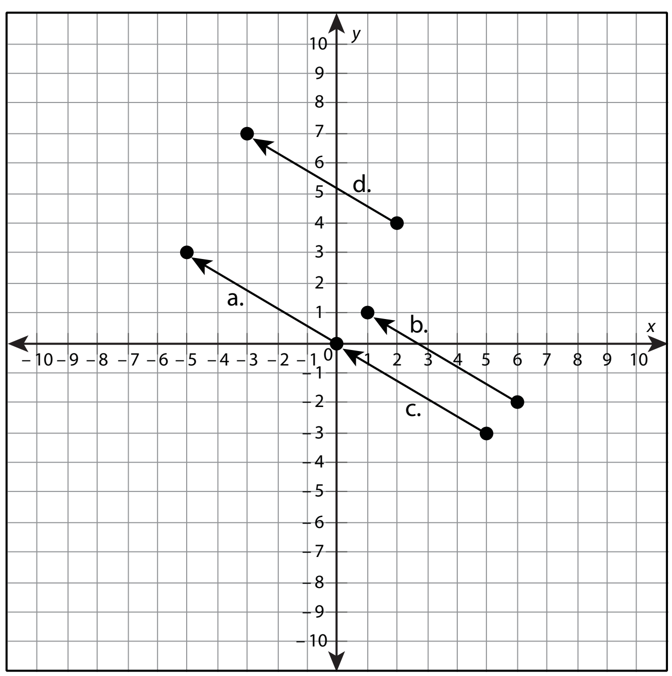
   From step 1, vector $\vec{v} = \langle a, b \rangle$ extends from $P(x_1, y_1)$ to $Q(x_1 + a, y_1 + b)$. Rewritten in terms of the coordinates of Q, this is equivalent to stating vector $\vec{v} = \langle a, b \rangle$ extends from $P(x_2 - a, y_2 - b)$ to $Q(x_2, y_2)$.

   c. Ending at terminal point $Q(0, 0)$, vector $\vec{v} = \langle -5, 3 \rangle$ begins 5 units to the right and 3 units down at the point P. Therefore, $P(0 - (-5), 0 - 3) = P(5, -3)$.

   d. Ending at terminal point $Q(-3, 7)$, vector $\vec{v} = \langle -5, 3 \rangle$ begins 5 units to the right and 3 units down at the point P. Therefore, $P(-3 - (-5), 7 - 3) = P(2, 4)$.

3. Graph each instance of the vector $\vec{v}$.

   On a coordinate plane, plot each pair of endpoints from parts a–d. Then connect each point P to its corresponding point Q with an arrow pointing from P to Q.



Example 4

An airplane leaves airport A and flies in a straight line for 2 hours at a constant speed to airport B. On a map with a scale of 1 cm : 100 km, the airports' coordinates are A (3, 1) and B (11, 7). Find vectors $\vec{d}$ and $\vec{v}$ that represent the displacement and velocity of the airplane, respectively. Then interpret the magnitude of each vector.

1. Find the displacement vector, $\vec{d}$.

   Airport A located at (3, 1) is the initial point for displacement vector $\vec{d}$ and airport B located at (11, 7) is the terminal point. You can find the components of $\vec{d}$ by subtracting the coordinates of point A from the coordinates of point B.

$\vec{d}=\langle x_2-x_1, x_2-x_1\rangle$	Subtract the initial point from the terminal point.
$\vec{d}=\langle (11)-(3),(7)-(1)\rangle$	Substitute (3, 1) for (x_1, y_1) and (11, 7) for (x_2, y_2).
$\vec{d}=\langle 8,6\rangle$	Simplify.

   Thus, in component form, the displacement vector is $\vec{d}=\langle 8,6\rangle$ in centimeters on the map, or equivalently $\vec{d}=\langle 800,600\rangle$ in kilometers.

2. Find the velocity vector $\vec{v}$.

   Since the airplane travels in a straight line, the velocity vector points in the same direction as the displacement vector, and is therefore a scalar multiple of $\vec{d}$. The airplane travels $\vec{d}$ in 2 hours at velocity $\vec{v}$, so $\vec{d}=2\vec{v}$ (a vector version of $d = rt$). You can find the components of $\vec{v}$ by dividing each component of $\vec{d}$ by 2 hours.

$\vec{v}=\left\langle \frac{800\text{ km}}{2\text{ hr}}, \frac{600\text{ km}}{2\text{ hr}}\right\rangle$	Divide the components by 2 hours.
$\vec{v}=\langle 400, 300\rangle\frac{\text{km}}{\text{hr}}$	Simplify.

   In component form, $\vec{v}=\langle 400,300\rangle\frac{\text{km}}{\text{hr}}$.

3. Interpret the magnitude of each vector.

The magnitude of the displacement vector $\vec{d} = \langle 800, 600 \rangle$ km represents the distance between the two airports, and is found using the Pythagorean Theorem.

$\lVert \mathbf{d} \rVert = \sqrt{a^2 + b^2}$	Magnitude formula derived from the Pythagorean Theorem
$\lVert \mathbf{d} \rVert = \sqrt{(800)^2 + (600)^2}$	Substitute 800 for *a* and 600 for *b*.
$\lVert \mathbf{d} \rVert = \sqrt{640{,}000 + 360{,}000}$	Simplify.
$\lVert \mathbf{d} \rVert = \sqrt{1{,}000{,}000}$	
$\lVert \mathbf{d} \rVert = 1000$	

Therefore, the magnitude of the displacement vector $\vec{d}$ is 1,000 km/hr, so the distance from airport *A* to airport *B* is 1,000 km.

The magnitude of the velocity vector $\vec{v} = \langle 400, 300 \rangle \dfrac{\text{km}}{\text{hr}}$ represents the speed at which the airplane travels, and is found using the Pythagorean Theorem.

$\lVert \mathbf{v} \rVert = \sqrt{a^2 + b^2}$	Magnitude formula derived from the Pythagorean Theorem
$\lVert \mathbf{v} \rVert = \sqrt{(400)^2 + (300)^2}$	Substitute 400 for *a* and 300 for *b*.
$\lVert \mathbf{v} \rVert = \sqrt{160{,}000 + 90{,}000}$	Simplify.
$\lVert \mathbf{v} \rVert = \sqrt{250{,}000}$	
$\lVert \mathbf{v} \rVert = 500$	

The magnitude of $\vec{v}$ is 500 km/hr, which is the speed at which the airplane flies.

Scaffolded Practice 2.1

Example 1

Find the magnitude of the vector $\vec{v} = \langle 2, -3 \rangle$ and describe the vector's direction.

1. Determine the components a and b of $\vec{v}$.

2. Use the Pythagorean Theorem to find the magnitude of $\vec{v}$.

3. Describe the direction of $\vec{v}$.

continued

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Example 2

Find the magnitude and describe the direction of the vector $\vec{v} = \overrightarrow{PQ}$ from *P* (7, –2) to *Q* (1, 1).

Example 3

The vector $\vec{v} = \langle -5, 3 \rangle$ has an initial point *P* and a terminal point *Q*. Given the coordinates of one endpoint, find the coordinates of the other endpoint. Then graph each instance of the vector $\vec{v}$:

a. Given *P* (0, 0), find the coordinates of *Q*.

b. Given *P* (6, –2), find the coordinates of *Q*.

c. Given *Q* (0, 0), find the coordinates of *P*.

d. Given *Q* (–3, 7), find the coordinates of *P*.

Example 4

An airplane leaves airport *A* and flies in a straight line for 2 hours at a constant speed to airport *B*. On a map with a scale of 1 cm : 100 km, the airports' coordinates are *A* (3, 1) and *B* (11, 7). Find vectors $\vec{d}$ and $\vec{v}$ that represent the displacement and velocity of the airplane, respectively. Then interpret the magnitude of each vector.

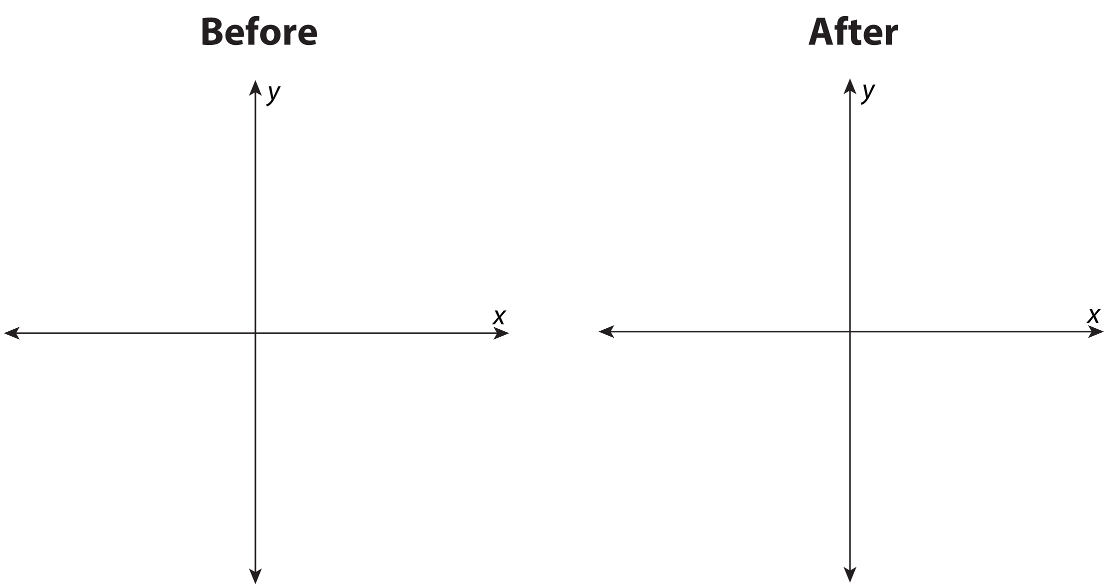
Problem-Based Task 2.1: When Two Cars Collide

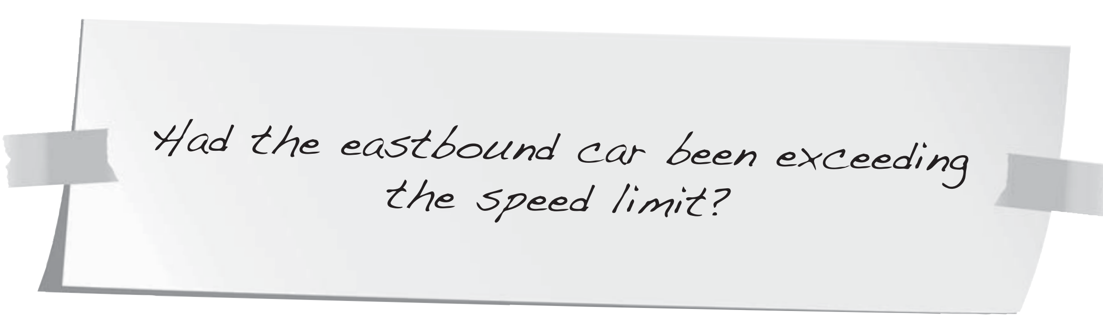
SMP
1 ✓ 2 ✓
3 ✓ 4 ✓
5 ✓ 6 ✓
7 ✓ 8 ✓

Officer Smith was driving north, approaching an intersection with a perpendicular road. The light was green, so neither she nor the car in front of her slowed. As the car in front of her entered the intersection, it was hit from the left by a second car driving east. Upon impact, the two cars stuck together and slid off the road at an angle, with their original velocity vectors forming the components of the post-collision velocity vector. By analyzing the tire tread marks on the pavement, Officer Smith determined that immediately after impact, the pair of cars slid away from the intersection at 30 mph.

Officer Smith knew the eastbound car was at fault, having seen it run a red light without slowing down, but she wondered whether its driver should also get a speeding ticket. She knew that she could get a reasonable estimate of the car's speed by ignoring friction during the collision and assuming that momentum (mass times velocity) would be conserved. She also noted that the two cars were of similar models and therefore had approximately the same mass. The northbound car in front of her had been driving exactly 35 mph, the speed limit on both roads. Had the eastbound car been exceeding the speed limit? Explain your reasoning.

The *x*- and *y*-axes below represent the intersection of the two roads, before and after the accident. Use them as needed to draw the vectors and complete the problem.





Problem-Based Task 2.1: When Two Cars Collide

Coaching

a. How were the cars arranged and moving just before the collision? On the "before" axes, sketch the cars and velocity vectors just before the collision. Let x be the unknown magnitude of the eastbound velocity vector.

b. How were the cars arranged and moving just after the collision? On the "after" axes, sketch the cars and velocity vector just after the collision.

c. How are your before and after sketches related? Sketch a triangle that shows how the cars' original velocity vectors are related to their joint velocity vector after the collision. Remember, momentum (mass times velocity) is conserved here, and both cars (that is, twice the mass) are moving at 30 mph after the collision.

d. At what speed was the eastbound car driving prior to the collision?

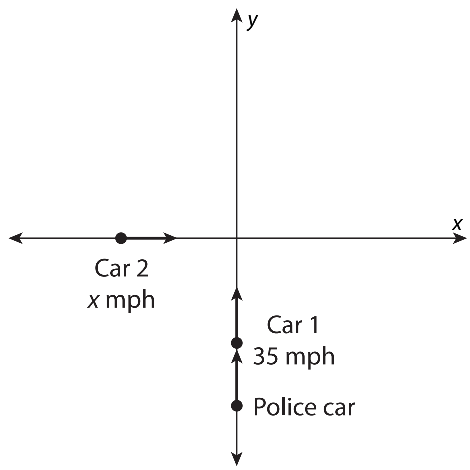
e. Had the eastbound car been speeding when it hit the northbound car? Explain your reasoning.

Problem-Based Task 2.1: When Two Cars Collide

Coaching Sample Responses

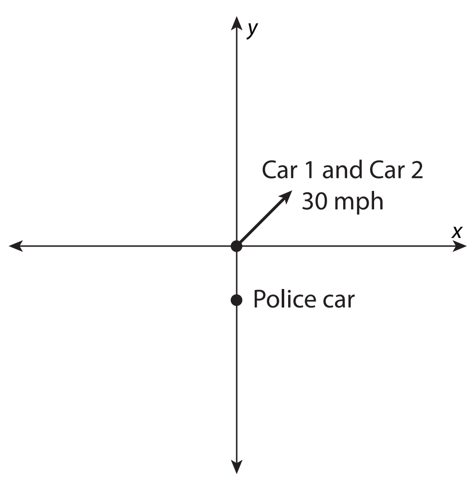
a. How were the cars arranged and moving just before the collision? On the "before" axes, sketch the cars and velocity vectors just before the collision. Let x be the unknown magnitude of the eastbound velocity vector.

Just before entering the intersection, the northbound car was driving north at 35 mph, and the police car was immediately behind this car. The eastbound car was driving east at an unknown speed of x mph.



b. How were the cars arranged and moving just after the collision? On the "after" axes, sketch the cars and velocity vector just after the collision.

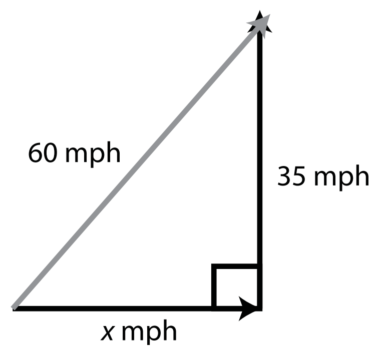
After the collision, the two cars stuck together and moved off the intersection at an angle. Since the components of the velocity point east and north, the cars moved along the new joint velocity vector into the upper right quadrant of the intersection.



c. How are your before and after sketches related? Sketch a triangle that shows how the cars' original velocity vectors are related to their joint velocity vector after the collision. Remember, momentum (mass times velocity) is conserved here, and both cars (that is, twice the mass) are moving at 30 mph after the collision.

The initial momenta of the cars form the components of the post-collision momentum vector. Because the cars have the same mass, all sides of the triangle will have a factor of mass in them. For convenience, divide everything by mass.

Since there are two cars moving together after the collision, they have twice the mass. The component in the *y* direction is 35 mph, the speed of the northbound car. The component in the *x* direction is *x* mph, the unknown speed of the eastbound car. The hypotenuse of the triangle formed by the two component velocities is twice the final velocity vector of the two-car system after the collision.



d. At what speed was the eastbound car driving prior to the collision?

The magnitudes of the three velocity vectors are related by the Pythagorean Theorem.

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2}$$

$$2(30) = \sqrt{(x)^2 + (35)^2}$$

$$60^2 = x^2 + 35^2$$

$$x^2 = 60^2 - 35^2 = 2375$$

$$x = \sqrt{2375} \approx 48.7$$

The speed of the eastbound car was approximately 48.7 mph.

e. Had the eastbound car been speeding when it hit the northbound car? Explain your reasoning.

Yes, the eastbound car was speeding. Even with Officer Smith's slight approximations, the calculated speed of 48.7 mph is so far above the speed limit of 35 mph that she can confidently conclude the eastbound car was speeding. The driver will get ticketed not only for running the red light, but also for speeding.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

Practice 2.1: Representing and Modeling with Vector Quantities

Use your knowledge of vectors to complete each problem.

1. Find the magnitude and describe the direction of the vector $\vec{v} = \langle -2, -5 \rangle$.

2. Find the value of a such that $a < 0$ in a vector of the form $\vec{v} = \langle a, b \rangle$, if $b = 10$ and $\|\mathbf{v}\| = 26$.

3. Find the components and magnitude of a vector of the form $\vec{v} = \overrightarrow{PQ}$ from P (3, –2) to Q (0, 4). Describe the direction of the vector.

4. Find the coordinates of the terminal point Q for vector $\vec{v} = \langle 7, -1 \rangle$ with initial point P (–9, –4).

5. Find all possible values of the missing y-coordinate of point Q (2, y), if vector $\vec{v} = \overrightarrow{PQ}$ has magnitude $\|\mathbf{v}\| = 5$ and initial point P (6, 1).

continued

6. Janelle walked 6 blocks north and then walked another 3 blocks west. Write her displacement vector $\vec{d}$ in component form $\vec{d} = \langle a,b \rangle$ and interpret the magnitude.

7. A car is driving on a highway with the velocity vector $\vec{v} = \langle 33,56 \rangle$, where each component is given in miles per hour. The car leaves point P (25, 20) at time $t = 0$. Find the speed of the car, and the location Q of the car after 2 hours.

8. Two birds left their nest and flew to a bird feeder. The male flew in a straight line from the nest to the bird feeder, while the female first flew south for 2 yards, then turned and flew due east to the bird feeder. If the male flew 8 yards, find the distance the female flew on the second leg of her flight. Round your answer to the nearest tenth of a yard.

9. Luke rode his bike 15 miles due south. This trip took him 1.5 hours. Find the magnitudes of vectors $\vec{d}$ and $\vec{v}$ that represent Luke's displacement in miles and velocity in miles per hour.

10. A car drives in a straight line from Price, UT, to Monticello, UT. The trip takes 3 hours. On a map with a scale of 1 cm : 10 km, the coordinates of Price and Monticello are P (8, 31) and M (29, 10), respectively. Find the magnitudes of vectors $\vec{d}$ and $\vec{v}$ that represent the displacement (in km) and velocity (in km/hr) of the car.

Progress Assessment

Circle the letter of the best answer.

1. What is the magnitude of the vector $\vec{v} = \langle 8, -5 \rangle$?

   a. $2\sqrt{10}$

   b. $\sqrt{89}$

   c. $\sqrt{105}$

   d. $4\sqrt{10}$

2. What are the coordinates of the initial point *P* if the terminal point of vector $\vec{v} = \langle -3, -5 \rangle$ is *Q* (7, 2)?

   a. *P* (–8, 9)

   b. *P* (–4, –7)

   c. *P* (4, –3)

   d. *P* (10, 7)

Use what you have learned about vectors to complete all parts of problem 3.

3. A pilot flies a plane with a velocity vector of $\vec{v} = \langle 98, 112 \rangle$, where each component is given in miles per hour. The plane took off from point *P* (14, 32) at time $t = 0$.

   a. Determine the speed of the plane. Round your answer to the nearest hundredth.

   b. Find the location of the terminal point *Q* of the plane after 2 hours.

Pre-Assessment

Circle the letter of the best answer.

1. What is the sum of the vectors $\langle 6,4\rangle$ and $\langle -5,3\rangle$?
   a. $\langle 1,7\rangle$
   b. $\langle 11,1\rangle$
   c. $\langle 9,-9\rangle$
   d. $\langle 10,-2\rangle$

2. Given vectors $\vec{u}=\langle -1,3\rangle$ and $\vec{v}=\langle -4,5\rangle$, what vector is represented by the expression $3\vec{u}-2\vec{v}$?
   a. $\langle -5,8\rangle$
   b. $\langle -9,22\rangle$
   c. $\langle 5,-1\rangle$
   d. $\langle -11,19\rangle$

Common Core State Standards

N–VM.4 (+) Add and subtract vectors.

N–VM.5 (+) Multiply a vector by a scalar.

SMP
1 ✓ 2 ✓
3 ✓ 4 ✓
5 ✓ 6 ✓
7 ✓ 8 ✓

Essential Questions

1. In what ways can two vectors be added or subtracted to obtain a new vector?
2. What happens when a vector is multiplied by a scalar, and what is the significance of this in real-world situations?

WORDS TO KNOW

commutative An operation on two objects is commutative if changing the order of the objects does not change the result.

component-wise addition addition of vectors performed such that the *x*-components are added to obtain the new *x*-component, and the *y*-components are added to obtain the new *y*-component

component-wise scalar multiplication multiplication of vectors performed such that the *x*-component and *y*-component of a vector are each multiplied by the scalar to obtain the new *x*-component and new *y*-component

head-to-tail method a way to add two vectors; given two vectors, place the head of one vector at the tail of the other vector, then draw a third vector that connects the tail of the first vector to the head of the second vector. The third vector is the resultant vector, or the sum.

Parallelogram Rule a method for vector addition; when the initial points of two vectors $\vec{u}$ and $\vec{v}$ are aligned, they define a parallelogram whose diagonal represents the vector sum $\vec{u}+\vec{v}$

resultant vector the result of vector addition; adding vectors $\vec{u}$ and $\vec{v}$ yields the resultant vector $\vec{u}+\vec{v}$

unit vector a vector with magnitude 1. Given any vector $\vec{v}$, the scalar multiple $\frac{\vec{v}}{\|\mathbf{v}\|}$ is the unique unit vector that points in the same direction as $\vec{v}$.

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Instruction

Recommended Resources

- Khan Academy. "Multiplying a vector by a scalar."

  http://www.walch.com/rr/06003

  This video shows how to multiply a vector by a scalar. Matrices are used to represent the vectors.

- Math Planet. "Vectors."

  http://www.walch.com/rr/02015

  This site provides an illustrated introduction to vectors and vector notation, followed by a video lesson.

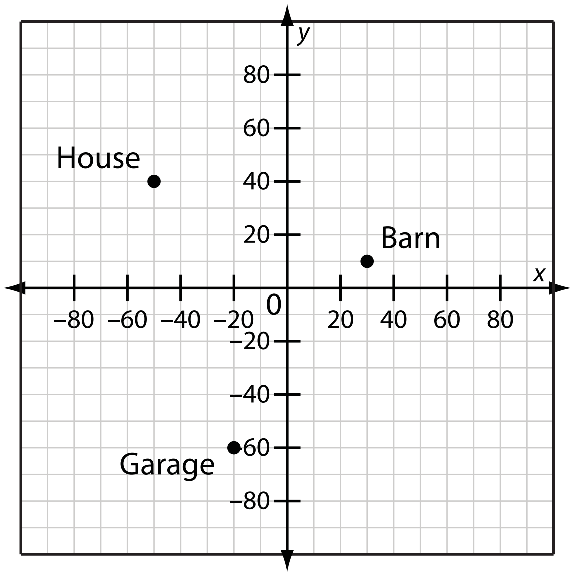
- PhET Interactive Simulations, University of Colorado. "Vector Addition."

  http://www.walch.com/rr/02016

  This interactive simulation of vector addition allows users to construct the sum of any number of vectors with moveable endpoints, and view the corresponding vector components in a variety of representations.

Warm-Up 2.2

Sara's property includes a house, a garage, and a barn, spaced out in meters as shown on the graph. Sara walked from her house to the garage and then from the garage to the barn.



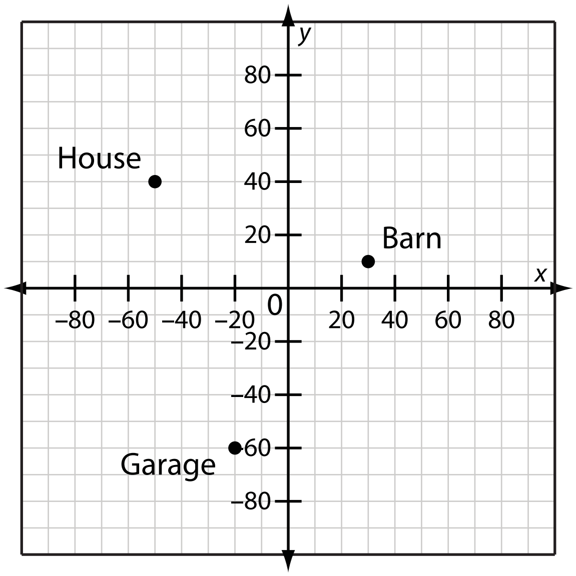
1. Two vectors can represent the path that Sara walks. Draw each vector on the graph, and find the coordinates of each vector in component form, $\vec{v} = \langle a, b \rangle$. Let $\vec{u}$ represent the walk from her house to the garage and $\vec{v}$ represent her walk from the garage to the barn.

2. How far did Sara walk in total? Round to the nearest tenth of a meter.

3. What is the difference between the distance Sara walked and the distance she would have walked if she had walked directly from the house to the barn? Draw the vector representing this direct path on the graph.

Warm-Up 2.2 Debrief

Sara's property includes a house, a garage, and a barn, spaced out in meters as shown on the graph. Sara walked from her house to the garage and then from the garage to the barn.



1. Two vectors can represent the path that Sara walks. Draw each vector on the graph, and find the coordinates of each vector in component form, $\vec{v}=\langle a,b\rangle$. Let $\vec{u}$ represent the walk from her house to the garage and $\vec{v}$ represent her walk from the garage to the barn.

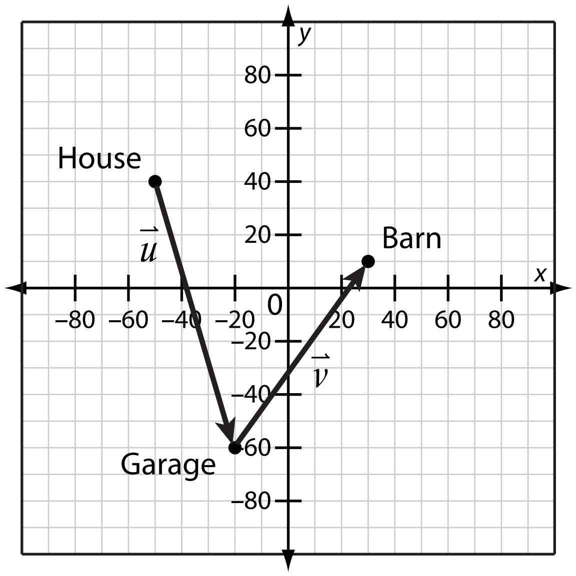
   Recall that the component form for a given vector is $\vec{v}=\langle a,b\rangle$, where the *x*-component is *a* and the *y*-component is *b*.

   Sara begins at her house and walks toward the garage. Create a vector with the initial point at her house and the terminal point at the garage.

   By counting units on the coordinate plane, we find that the garage is 30 meters east of the house, in the positive *x*-direction, and 100 meters south of the house, in the negative *y*-direction. The components are 30 and –100; thus, the component form is $\vec{u}=\langle 30,-100\rangle$.

   Sara then walks from the garage to the barn. Create a vector with the initial point at the garage and the terminal point at the barn.

The barn is 50 units east of the garage, in the positive *x*-direction, and 70 units north of the garage, in the positive *y*-direction. The components are 50 and 70; thus, the component form is $\vec{v} = \langle 50, 70 \rangle$.



Alternatively, the components of $\vec{u}$ and $\vec{v}$ can be found directly from the coordinates of the house, garage, and barn. The graph shows that the house is located at point (–50, 40), the garage is located at point (–20, –60), and the barn is located at point (30, 10).

Vector $\vec{u}$, from the house to the garage, has initial point (–50, 40) and terminal point (–20, –60).

The *x*-component of $\vec{u}$ is the difference in the *x*-values:

$a = x_2 - x_1$

$a = (-20) - (-50)$

$a = 30$ meters

The *y*-component of $\vec{u}$ is the difference of the *y*-values:

$b = y_2 - y_1$

$b = (-60) - (40)$

$b = -100$ meters

The components are 30 and –100. Therefore, as we found earlier, the component form is $\vec{u} = \langle 30, -100 \rangle$.

Vector $\vec{v}$, from the house to the garage, has initial point (–20, –60) and terminal point (30, 10).

The *x*-component of $\vec{v}$ is the difference in the *x*-values:

$a = x_2 - x_1$

$a = (30) - (-20)$

$a = 50$ meters

The *y*-component of $\vec{v}$ is the difference of the *y*-values:

$b = y_2 - y_1$

$b = (10) - (-60)$

$b = 70$ meters

The components are 50 and 70. Therefore, as we found earlier, the component form is $\vec{v} = \langle 50, 70 \rangle$.

2. How far did Sara walk in total? Round to the nearest tenth of a meter.

Sara followed the vector $\vec{u} = \langle 30, -100 \rangle$ from the house to the garage, then she followed the vector $\vec{v} = \langle 50, 70 \rangle$ from the garage to the barn. The total distance Sara walked is the sum of the lengths of these two vectors.

The distance from the house to the garage is the magnitude $\|\mathbf{u}\|$, which can be found using the Pythagorean Theorem, applied as the magnitude formula, $\|\mathbf{u}\| = \sqrt{a^2 + b^2}$. Note that since magnitude is always positive, we find the positive square root.

Recall that vectors are written in the form $\langle a, b \rangle$. For $\vec{u} = \langle 30, -100 \rangle$, let $a = 30$ and $b = -100$.

$\|\mathbf{u}\| = \sqrt{a^2 + b^2}$	Magnitude formula for $\|\mathbf{u}\|$
$\|\mathbf{u}\| = \sqrt{(30)^2 + (-100)^2}$	Substitute 30 for *a* and –100 for *b*.
$\|\mathbf{u}\| = \sqrt{900 + 10{,}000}$	Apply the exponents.
$\|\mathbf{u}\| = \sqrt{10{,}900}$	Add.
$\|\mathbf{u}\| = 10\sqrt{109}$	Simplify.
$\|\mathbf{u}\| \approx 104.4$	Approximate to the nearest tenth.

Sara walked approximately 104.4 meters from the house to the garage.

The distance from the garage to the barn is the magnitude $\|\mathbf{v}\|$, which can be found in the same manner. For $\vec{v}=\langle 50,70\rangle$, let a = 50 and b = 70.

$\|\mathbf{v}\|=\sqrt{a^2+b^2}$ Magnitude formula for $\|\mathbf{v}\|$

$\|\mathbf{v}\|=\sqrt{(50)^2+(70)^2}$ Substitute 50 for a and 70 for b.

$\|\mathbf{v}\|=\sqrt{2500+4900}$ Apply the exponents.

$\|\mathbf{v}\|=\sqrt{7400}$ Add.

$\|\mathbf{v}\|=10\sqrt{74}$ Simplify.

$\|\mathbf{v}\|\approx 86.0$ Approximate to the nearest tenth.

Sara walked approximately 86 meters from the garage to the barn.

Therefore, the total distance Sara walked, from the house to the garage to the barn, is $\|\mathbf{u}\|+\|\mathbf{v}\|\approx 104.4+86.0\approx 190.4$ meters.

3. What is the difference between the distance Sara walked and the distance she would have walked if she had walked directly from the house to the barn? Draw the vector representing this direct path on the graph.

Walking directly from the house to the barn, Sara would have followed a different vector with the initial point at the house and the terminal point at the barn; let's call this vector $\vec{w}$. The garage is 80 meters east of the house, in the positive x-direction, and 30 meters south of the house, in the negative y-direction. The components are 80 and –30, so the component form is $\vec{w}=\langle 80,-30\rangle$.

The distance from the house to the barn is the magnitude $\|\mathbf{w}\|$, which can be found using the Pythagorean Theorem, applied as the magnitude formula. For $\vec{w}=\langle 80,-30\rangle$, let a = 80 and b = –30.

$\|\mathbf{w}\|=\sqrt{a^2+b^2}$ Magnitude formula for $\|\mathbf{w}\|$

$\|\mathbf{w}\|=\sqrt{(80)^2+(-30)^2}$ Substitute 80 for a and –30 for b.

$\|\mathbf{w}\|=\sqrt{6400+900}$ Apply the exponents.

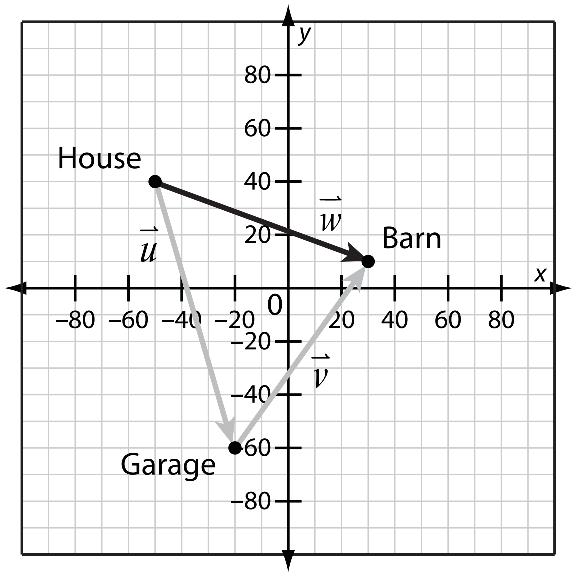
$\|\mathbf{w}\|=\sqrt{7300}$ Add.

$\|\mathbf{w}\|=10\sqrt{73}$ Simplify.

$\|\mathbf{w}\|\approx 85.4$ Approximate to the nearest tenth.

Sara would have walked approximately 85.4 meters directly from the house to the barn. This is 105 meters shorter than the 190.4-meter path that Sara actually followed.

The graph with the vector $\vec{w}$ drawn should resemble the following.



Connection to the Lesson

- Students will find the sum of two vectors and compare the magnitude of the sum to the sum of the original vectors' magnitudes.
- Students will identify the components of a vector from the coordinates of the initial and terminal points and use these components to determine the vector's magnitude.
- Students will determine the result of two vectors traveling in sequence, and find the straight-line distance from the initial point to the terminal point of the path followed.

Prerequisite Skills

This lesson requires the use of the following skills:

- recognizing vector notation, magnitude, and direction
- graphing points and vectors on a coordinate plane
- applying the distance formula
- applying the Pythagorean Theorem

Introduction

Operations on vectors arise in many contexts, from navigating an airplane in a crosswind to manipulating computer graphics in an animated movie. Vectors can be added or subtracted to produce a new vector, the **resultant vector**. Vectors can also be scaled or multiplied by a constant called a scalar.

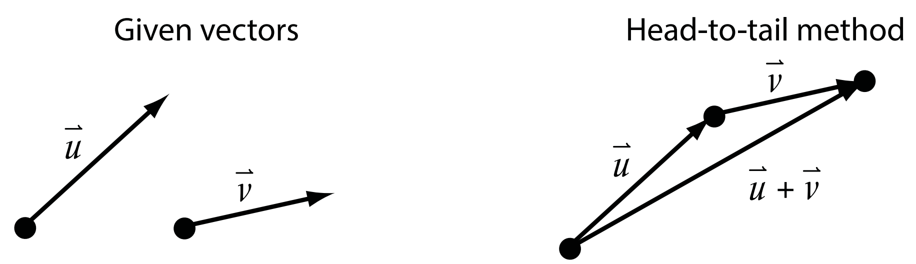
Key Concepts

- Two vectors $\vec{u}$ and $\vec{v}$ can be added by using the **head-to-tail method**.
- When using this method, all vectors must be drawn to scale with the given angle or slope.
- Align the tail of the second vector (the initial point of $\vec{v}$) to the head of the first vector (the terminal point of $\vec{u}$).
- The vector drawn from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$ is equal to the vector sum, $\vec{u}+\vec{v}$.

Method

Head-to-Tail Method

Given two vectors $\vec{u}$ and $\vec{v}$, place the tail of $\vec{v}$ at the head of $\vec{u}$. Then, connect the tail of $\vec{u}$ with the head of $\vec{v}$ to determine $\vec{u}+\vec{v}$.

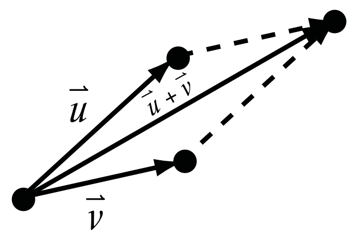


- The order in which the two vectors are added does not affect the resultant vector. That is, vector addition is **commutative**: $\vec{u}+\vec{v}=\vec{v}+\vec{u}$.
- Another method for adding two vectors is called the **Parallelogram Rule**. Align the initial points of vectors $\vec{u}$ and $\vec{v}$, and place another copy of each vector to form a parallelogram. The vector sum $\vec{u}+\vec{v}$ is the diagonal from the initial point of $\vec{u}$ and $\vec{v}$ that extends to the opposite vertex of the parallelogram.

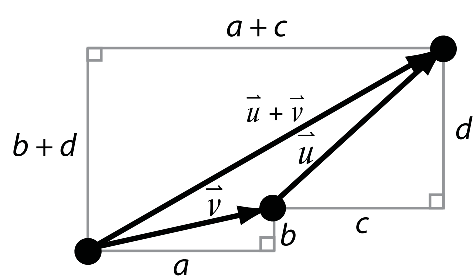
Rule

Parallelogram Rule

When the initial points of two vectors $\vec{u}$ and $\vec{v}$ are aligned, they define a parallelogram whose diagonal represents the vector sum $\vec{u}+\vec{v}$.

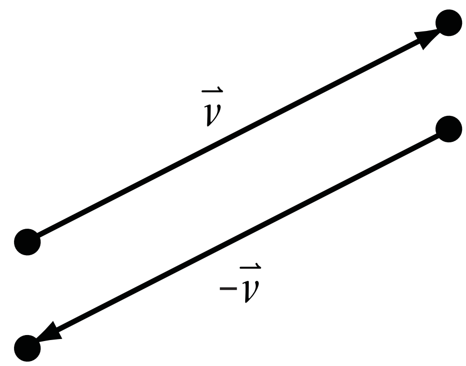


- Vectors can also be added by finding the sum of the components. The sum of the vectors $\vec{u}=\langle a,b\rangle$ and $\vec{v}=\langle c,d\rangle$ using **component-wise addition** is the resultant vector $\vec{u}+\vec{v}=\langle a+c,b+d\rangle$, where the *x*-components of $\vec{u}$ and $\vec{v}$ are added to result in the new *x*-component, and the *y*-components of $\vec{u}$ and $\vec{v}$ are added to result in the new *y*-component.

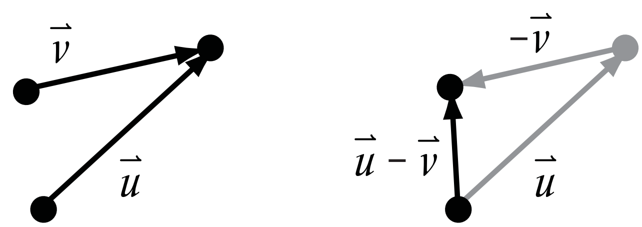


- Recall that with real numbers, subtracting a number is equivalent to adding the opposite number. For example, the difference 7 – 5 is equivalent to the sum 7 + (–5).

- A similar concept applies to vector subtraction. Given any vector $\vec{v}$, its opposite vector is the vector with the same magnitude that points in the opposite direction.

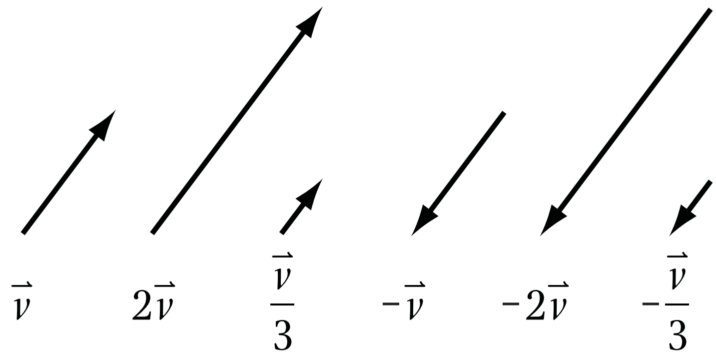


- To find a vector difference, add the opposite vector: $\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$.
- To subtract vector $\vec{v}$ from vector $\vec{u}$ graphically, draw vector $-\vec{v}$ so that it is the same size as $\vec{v}$ but pointing in the opposite direction, then add $\vec{u}+(-\vec{v})$.



- Component-wise, the opposite vector is obtained by changing the sign of each component. For example, the opposite vector of $\vec{v}=\langle 2,-1\rangle$ is $-\vec{v}=\langle -2,1\rangle$.
- In component form, the difference of the vectors $\vec{u}=\langle a,b\rangle$ and $\vec{v}=\langle c,d\rangle$ is $\vec{u}-\vec{v}=\langle a-c,b-d\rangle$, where the *x*-components of $\vec{u}$ and $\vec{v}$ are subtracted to result in the new *x*-component, and the *y*-components of $\vec{u}$ and $\vec{v}$ are subtracted to result in the new *y*-component.
- As with real numbers, vector subtraction is not commutative.
- Another way to create new vectors from original vectors is with scalar multiplication. Recall that a scalar is a numerical quantity without an associated direction. It is a number that is multiplied by a vector.

- Graphically, a scalar multiple of a vector lengthens or shortens the vector, but does not change its slope. When the scalar is positive, the direction of the vector remains the same. When the scalar is negative, the direction of the vector is reversed.



- **Component-wise scalar multiplication** is performed algebraically. To multiply a vector by a scalar *c*, multiply each individual component by *c*; that is, $c \bullet \langle a,b \rangle = \langle c \bullet a, c \bullet b \rangle$. For example, $2\langle 1,-3 \rangle = \langle 2,-6 \rangle$.

- A **unit vector** is a vector with a magnitude of 1. To find the components of a unit vector, divide the components by the vector's magnitude. Given any vector $\vec{v}$, $\frac{\vec{v}}{\|\mathbf{v}\|}$ is the unique unit vector that points in the same direction as $\vec{v}$.

- The magnitude of a scalar multiple of a vector is the absolute value of the scalar times the original magnitude: $\|c\mathbf{v}\| = |c| \bullet \|\mathbf{v}\|$.

Common Errors/Misconceptions

- trying to multiply the components of a vector instead of multiplying by the scalar quantity
- adding vector components by pairs rather than component-wise
- expressing the magnitude of a negated vector as $-\|\mathbf{v}\|$ instead of $\|-\mathbf{v}\| = \|\mathbf{v}\|$

Guided Practice 2.2

Example 1

Given vectors $\vec{u}=\langle -4,6\rangle$ and $\vec{v}=\langle 8,-1\rangle$, find the vector sum $\vec{u}+\vec{v}$ in three ways:

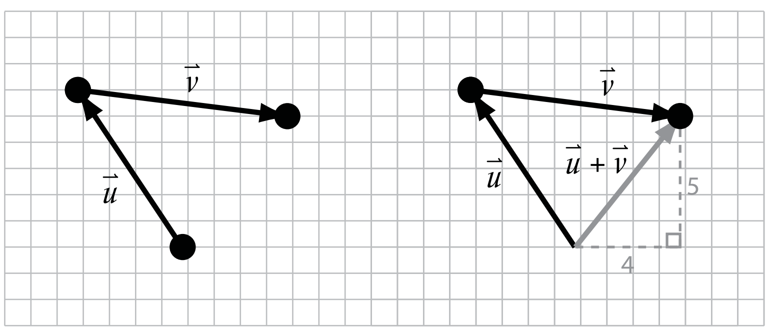
a. by placing them head-to-tail

b. by using the Parallelogram Rule

c. by using component-wise addition

1. Find the vector sum $\vec{u}+\vec{v}$ by placing $\vec{u}$ and $\vec{v}$ head-to-tail.

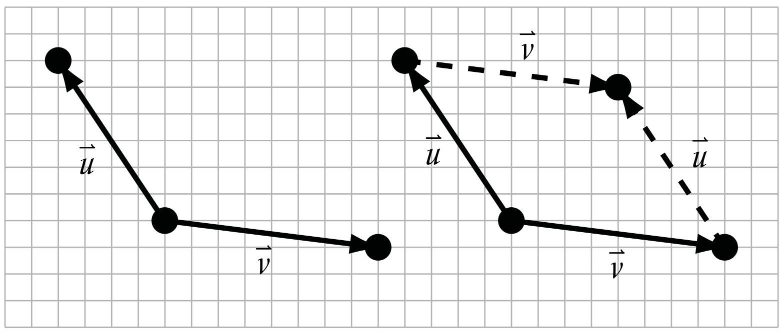
   To add $\vec{u}$ and $\vec{v}$ using the head-to-tail method, draw the vectors so that the terminal point of $\vec{u}$ is aligned with the initial point of $\vec{v}$. To plot $\vec{u}=\langle -4,6\rangle$, choose and plot any point in the coordinate plane. This represents the initial point of $\vec{u}$. From the initial point, move 4 units to the left and 6 units up and plot the terminal point of $\vec{u}$. Use this point as the initial point of $\vec{v}$. Since $\vec{v}$ is given by $\vec{v}=\langle 8,-1\rangle$, move 8 units to the right and 1 unit down to plot the terminal point of $\vec{v}$. The vector from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$ is the vector sum $\vec{u}+\vec{v}$.



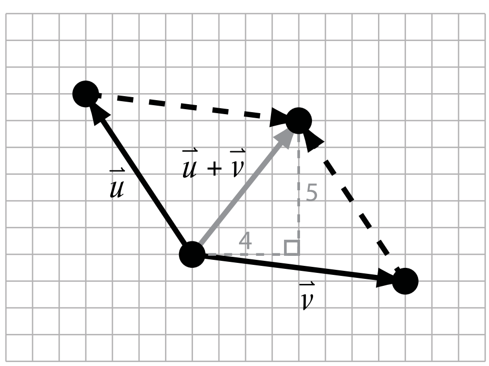
   You can find the components of $\vec{u}+\vec{v}$ from the graph. The terminal point of $\vec{u}+\vec{v}$ is 4 units to the right of its initial point, so the *x*-component of $\vec{u}+\vec{v}$ is 4. The terminal point of $\vec{u}+\vec{v}$ is 5 units above its initial point, so the *y*-component of $\vec{u}+\vec{v}$ is 5. Thus in component form, $\vec{u}+\vec{v}=\langle 4,5\rangle$.

2. Find the vector sum $\vec{u}+\vec{v}$ using the Parallelogram Rule.

   To add vectors $\vec{u}$ and $\vec{v}$ using the Parallelogram Rule, draw $\vec{u}$ and $\vec{v}$ so that the initial point of $\vec{u}$ is aligned with the initial point of $\vec{v}$. Draw the vectors so that the terminal points represent the components given for each vector. Now draw a copy of $\vec{u}$ with its initial point at the terminal point of the first $\vec{v}$, and draw a copy of $\vec{v}$ with its initial point at the terminal point of the first $\vec{u}$, as shown by the dashed vectors in the following figure.



   Together, the four vectors form a parallelogram. The diagonal of the parallelogram that contains the initial point of $\vec{u}$ and $\vec{v}$ is the vector sum $\vec{u}+\vec{v}$. Note that the other diagonal does *not* represent the vector sum since it does not contain the initial point. From the figure, the terminal point of $\vec{u}+\vec{v}$ is 4 units to the right of its initial point, so the *x*-component of $\vec{u}+\vec{v}$ is 4. The terminal point of $\vec{u}+\vec{v}$ is 5 units above its initial point, so the *y*-component of $\vec{u}+\vec{v}$ is 5. Thus in component form, $\vec{u}+\vec{v}=\langle 4,5\rangle$.



3. Find the vector sum $\vec{u}+\vec{v}$ using component-wise addition.

To add $\vec{u}=\langle -4,6\rangle$ and $\vec{v}=\langle 8,-1\rangle$ component-wise, first add the *x*-components to get the new *x*-component, and then add the *y*-components to get the new *y*-component. Note that $\vec{u}=\langle a,b\rangle$ and $\vec{v}=\langle c,d\rangle$.

$\vec{u}+\vec{v}=\langle -4,6\rangle+\langle 8,-1\rangle$	Write the vectors being added in component form.
$\vec{u}+\vec{v}=\langle a+c,b+d\rangle$	Formula for component-wise vector addition
$\vec{u}+\vec{v}=\langle (-4)+(8),(6)+(-1)\rangle$	Substitute –4 for *a*, 6 for *b*, 8 for *c*, and –1 for *d* into the formula.
$\vec{u}+\vec{v}=\langle 4,5\rangle$	Simplify.

The vector sum $\vec{u}+\vec{v}$ is $\langle 4,5\rangle$.

Example 2

Given vectors $\vec{m}=\langle -5,-3\rangle$ and $\vec{n}=\langle -9,2\rangle$, find the vector difference $\vec{m}-\vec{n}$ in three ways:

a. by placing them head-to-tail

b. by using the Parallelogram Rule

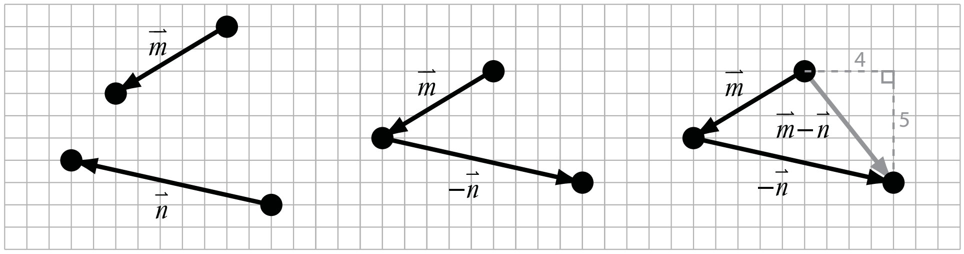
c. by using component-wise addition

1. Find the vector difference $\vec{m}-\vec{n}$ by placing them head-to-tail.

Vector subtraction is essentially vector addition. To subtract $\vec{n}$ from $\vec{m}$ head-to-tail, draw the vectors $\vec{m}$ and $-\vec{n}$ so the terminal point of $\vec{m}$ aligns with the initial point of $-\vec{n}$. First, determine the components of $-\vec{n}$. Since $-\vec{n}$ can be found by multiplying $\vec{n}=\langle -9,2\rangle$ by –1, $-\vec{n}=\langle 9,-2\rangle$.

Plot the initial point of $\vec{m}$ on the coordinate plane. Move 5 units to the left and 3 units down to plot its terminal point. From this terminal point, move 9 units to the right and 2 units down to plot the terminal point of $-\vec{n}$. The vector from the initial point of $\vec{m}$ to the terminal point of $-\vec{n}$ is the vector difference $\vec{m}-\vec{n}$.

(*continued*)

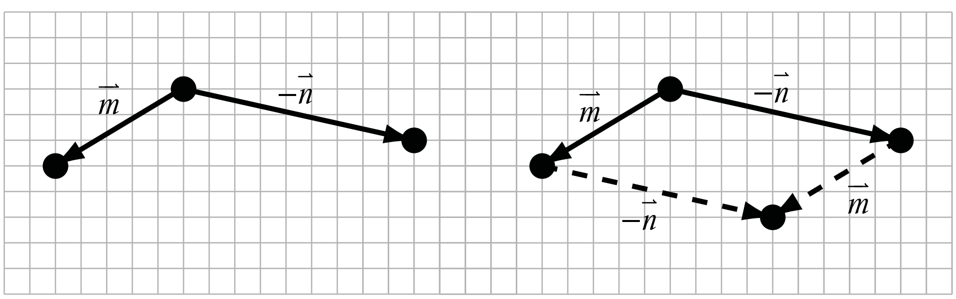


You can find the components of $\vec{m}-\vec{n}$ from the graph. The terminal point of $\vec{m}-\vec{n}$ is 4 units to the right of its initial point, so the x-component of $\vec{m}-\vec{n}$ is 4. The terminal point of $\vec{m}-\vec{n}$ is 5 units below its initial point, so the y-component of $\vec{m}-\vec{n}$ is –5. Thus in component form, $\vec{m}-\vec{n}=\langle 4,-5\rangle$.

Recall that vector subtraction is *not* commutative: $\vec{m}-\vec{n}\neq\vec{n}-\vec{m}$.

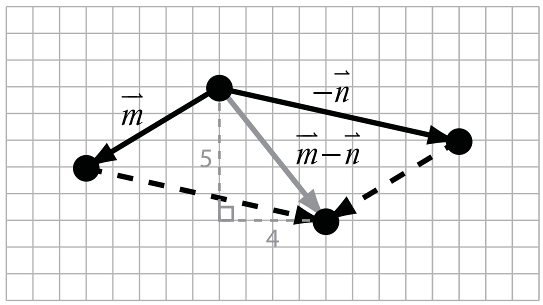
2. Find the vector difference $\vec{m}-\vec{n}$ using the Parallelogram Rule.

   To subtract vector $\vec{n}$ from vector $\vec{m}$ using the Parallelogram Rule, draw $\vec{m}$ and $-\vec{n}$, the opposite vector of $\vec{n}$, so that the initial point of $\vec{m}$ is aligned with the initial point of $-\vec{n}$. Now draw a copy of $\vec{m}$ with its initial point at the terminal point of the first $-\vec{n}$, and draw a copy of $-\vec{n}$ with its initial point at the terminal point of the first $\vec{m}$, as shown by the dashed vectors in the figure.



(*continued*)

Together, the four vectors form a parallelogram. The diagonal of the parallelogram that contains the initial point of $\vec{m}$ and $-\vec{n}$ is the vector difference $\vec{m}-\vec{n}$. From the figure, the terminal point of $\vec{m}-\vec{n}$ is 4 units to the right of its initial point, so the *x*-component of $\vec{m}-\vec{n}$ is 4. The terminal point of $\vec{m}-\vec{n}$ is 5 units below its initial point, so the *y*-component of $\vec{m}-\vec{n}$ is 5. Thus, in component form, $\vec{m}-\vec{n}=\langle 4,-5\rangle$.



3. Find the vector difference $\vec{m}-\vec{n}$ using component-wise addition.

To subtract $\vec{n}=\langle -9,2\rangle$ from $\vec{m}=\langle -5,-3\rangle$ component-wise, subtract the *x*-components to get the new *x*-component and subtract the *y*-components to get the new *y*-component. Note that $\vec{m}=\langle a,b\rangle$ and $\vec{n}=\langle c,d\rangle$.

$\vec{m}-\vec{n}=\langle -5,-3\rangle-\langle -9,2\rangle$	Write the vectors being subtracted in component form.
$\vec{m}-\vec{n}=\langle a-c,b-d\rangle$	Formula for component-wise vector subtraction
$\vec{m}-\vec{n}=\langle (-5)-(-9),(-3)-(2)\rangle$	Substitute –5 for *a*, –3 for *b*, –9 for *c*, and 2 for *d* into the formula.
$\vec{m}-\vec{n}=\langle 4,-5\rangle$	Simplify.

The vector difference $\vec{m}-\vec{n}$ is $\langle 4,-5\rangle$.

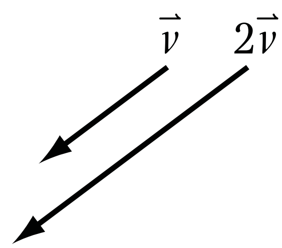
Example 3

Draw and find the components of each of these scalar multiples of $\vec{v}=\langle -4,-3\rangle$: $2\vec{v}$, $-\frac{1}{2}\vec{v}$, and $\frac{\vec{v}}{\|\mathbf{v}\|}$.

1. Draw the scalar multiple $2\vec{v}$.

   Multiplying a vector by a scalar multiple *c* changes the length of the vector by a factor of *c*. If *c* is positive, the direction of the vector remains unchanged, and if *c* is negative, the direction of the vector is reversed.

   The scalar multiple corresponding to vector $2\vec{v}$ is 2, so the length of vector $2\vec{v}$ is twice the length of vector $\vec{v}$. Since 2 is positive, $2\vec{v}$ points in the same direction as $\vec{v}$.



2. Find the components of the scalar multiple $2\vec{v}$.

   To multiply a vector by a scalar multiple *c*, multiply each component by *c*; that is, $c\bullet\langle a,b\rangle=\langle c\bullet a,c\bullet b\rangle$.

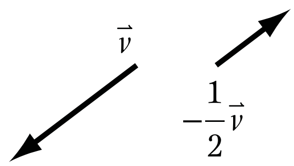
   To find the components of the vector $2\vec{v}$, multiply each component of $\vec{v}=\langle -4,-3\rangle$ by the scalar 2.

$\vec{v}=\langle -4,-3\rangle$	Vector $\vec{v}$ in component form
$2\vec{v}=2\bullet\langle -4,-3\rangle$	Multiply both sides by the scalar, 2.
$2\vec{v}=\langle 2\bullet(-4),2\bullet(-3)\rangle$	Multiply each component by 2.
$2\vec{v}=\langle -8,-6\rangle$	Simplify.

   Thus in component form, $2\vec{v}=\langle -8,-6\rangle$.

3. Draw the scalar multiple $-\frac{1}{2}\vec{v}$.

The scalar multiple for vector $-\frac{1}{2}\vec{v}$ is $-\frac{1}{2}$, so the length of vector $-\frac{1}{2}\vec{v}$ is $\frac{1}{2}$ of the length of vector $\vec{v}$. Since $-\frac{1}{2}$ is negative, $-\frac{1}{2}\vec{v}$ points in the *opposite* direction from $\vec{v}$.



4. Find the components of the scalar multiple $-\frac{1}{2}\vec{v}$.

To find the components of the vector $-\frac{1}{2}\vec{v}$, multiply each component of $\vec{v}=\langle -4,-3\rangle$ by the scalar $-\frac{1}{2}$.

$\vec{v}=\langle -4,-3\rangle$	Vector $\vec{v}$ in component form
$-\frac{1}{2}\vec{v}=-\frac{1}{2}\bullet\langle -4,-3\rangle$	Multiply both sides by the scalar, $-\frac{1}{2}$.
$-\frac{1}{2}\vec{v}=\left\langle -\frac{1}{2}\bullet(-4),-\frac{1}{2}\bullet(-3)\right\rangle$	Multiply each component by $-\frac{1}{2}$.
$-\frac{1}{2}\vec{v}=\left\langle 2,\frac{3}{2}\right\rangle$	Simplify.

Thus in component form, $-\frac{1}{2}\vec{v}=\left\langle 2,\frac{3}{2}\right\rangle$.

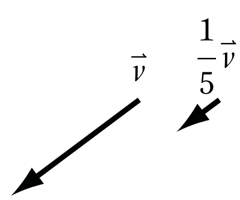
5. Draw the unit vector that points in the same direction as $\vec{v}$, $\frac{\vec{v}}{\|\mathbf{v}\|}$.

The vector $\frac{\vec{v}}{\|\mathbf{v}\|}$ is the unit vector that points in the same direction as $\vec{v}$. The scalar multiple corresponding to vector $\frac{\vec{v}}{\|\mathbf{v}\|}$ is $\frac{1}{\|\mathbf{v}\|}$, so you first need to find $\|\mathbf{v}\|$, the magnitude of $\vec{v}$. The magnitude $\|\mathbf{v}\|$ can be found using the Pythagorean Theorem.

$\|\mathbf{v}\|=\sqrt{a^2+b^2}$	Magnitude formula derived from the Pythagorean Theorem
$\|\mathbf{v}\|=\sqrt{(-4)^2+(-3)^2}$	Substitute –4 for *a* and –3 for *b*.
$\|\mathbf{v}\|=\sqrt{16+9}$	Simplify.
$\|\mathbf{v}\|=\sqrt{25}$	Add.
$\|\mathbf{v}\|=5$	Take the positive square root.

The magnitude of the vector $\vec{v}$ is $\|\mathbf{v}\|=5$.

The length of vector $\frac{\vec{v}}{\|\mathbf{v}\|}$ is therefore $\frac{1}{\|\mathbf{v}\|}=\frac{1}{5}$ of the length of $\vec{v}$.



6. Find the components of the unit vector $\frac{\vec{v}}{\|\mathbf{v}\|}$.

To find the components of the vector $\frac{\vec{v}}{\|\mathbf{v}\|}=\frac{1}{5}\vec{v}$, multiply each component of $\vec{v}=\langle -4,-3\rangle$ by the scalar $\frac{1}{5}$.

$\frac{\vec{v}}{\|\mathbf{v}\|}=\frac{1}{5}\vec{v}$ Formula for the length of vector $\frac{\vec{v}}{\|\mathbf{v}\|}$

$\frac{\vec{v}}{\|\mathbf{v}\|}=\frac{1}{5}\bullet\langle -4,3\rangle$ Write vector $\vec{v}$ in component form on the right side of the equation.

$\frac{\vec{v}}{\|\mathbf{v}\|}=\left\langle \frac{1}{5}\bullet(-4),\frac{1}{5}\bullet 3\right\rangle$ Multiply each component by $\frac{1}{5}$.

$\frac{\vec{v}}{\|\mathbf{v}\|}=\left\langle -\frac{4}{5},\frac{3}{5}\right\rangle$ Simplify.

Thus in component form, the unit vector $\frac{\vec{v}}{\|\mathbf{v}\|}=\left\langle -\frac{4}{5},\frac{3}{5}\right\rangle$. ✓

Example 4

Given $\vec{u}=\langle 7,-2\rangle$, $\vec{v}=\langle -1,3\rangle$, and $\vec{w}=\langle -6,-4\rangle$, find the components of each combination of vectors:

a. $2\vec{u}+3\vec{v}$

b. $-2\vec{u}+4\vec{v}-2\vec{w}$

1. Find the components of $2\vec{u}+3\vec{v}$, which is the sum of scalar multiples of two vectors.

   To find the components of the vector $2\vec{u}+3\vec{v}$, begin by finding the components of the scalar multiples $2\vec{u}$ and $3\vec{v}$.

   Find the components of the vector $2\vec{u}$.

$\vec{u}=\langle 7,-2\rangle$	Vector $\vec{u}$ in component form
$2\vec{u}=2\bullet\langle 7,-2\rangle$	Multiply both sides by the scalar, 2.
$2\vec{u}=\langle 2\bullet 7,2\bullet(-2)\rangle$	Multiply each component by 2.
$2\vec{u}=\langle 14,-4\rangle$	Simplify.

   The components of the vector $2\vec{u}$ are $\langle 14,-4\rangle$.

   Find the components of the vector $3\vec{v}$.

$\vec{v}=\langle -1,3\rangle$	Vector $\vec{v}$ in component form
$3\vec{v}=3\bullet\langle -1,3\rangle$	Multiply both sides by the scalar, 3.
$3\vec{v}=\langle 3\bullet(-1),3\bullet 3\rangle$	Multiply each component by 3.
$3\vec{v}=\langle -3,9\rangle$	Simplify.

   The components of the vector $3\vec{v}$ are $\langle -3,9\rangle$.

   Find the components of the vector sum $2\vec{u}+3\vec{v}$.

   To add vectors $2\vec{u}$ and $3\vec{v}$, use the formula for component-wise vector addition, $\vec{u}+\vec{v}=\langle a+c,b+d\rangle$, where $2\vec{u}=\langle a,b\rangle$ and $3\vec{v}=\langle c,d\rangle$.

$2\vec{u}+3\vec{v}=\langle 14,-4\rangle+\langle -3,9\rangle$	Write the vectors being added in component form.
$2\vec{u}+3\vec{v}=\langle 14+(-3),-4+9\rangle$	Rewrite using component-wise addition.
$2\vec{u}+3\vec{v}=\langle 11,5\rangle$	Simplify.

   In component form, $2\vec{u}+3\vec{v}=\langle 11,5\rangle$.

2. Find the components of $-2\vec{u}+4\vec{v}-2\vec{w}$, which is a combination of scalar multiples of more than two vectors.

To find the components of the vector $-2\vec{u}+4\vec{v}-2\vec{w}$, begin by finding the components of the scalar multiples $-2\vec{u}$, $4\vec{v}$, and $-2\vec{w}$.

Find the components of the vector $-2\vec{u}$.

$\vec{u}=\langle 7,-2\rangle$	Vector $\vec{u}$ in component form
$-2\vec{u}=-2\bullet\langle 7,-2\rangle$	Multiply both sides by the scalar, –2.
$-2\vec{u}=\langle -2\bullet 7,-2\bullet(-2)\rangle$	Multiply each component by –2.
$-2\vec{u}=\langle -14,4\rangle$	Simplify.

The components of the vector $-2\vec{u}$ are $\langle -14,4\rangle$.

Find the components of the vector $4\vec{v}$.

$\vec{v}=\langle -1,3\rangle$	Vector $\vec{v}$ in component form
$4\vec{v}=4\bullet\langle -1,3\rangle$	Multiply both sides by the scalar, 4.
$4\vec{v}=\langle 4\bullet(-1),4\bullet 3\rangle$	Multiply each component by 4.
$4\vec{v}=\langle -4,12\rangle$	Simplify.

The components of the vector $4\vec{v}$ are $\langle -4,12\rangle$.

Find the components of the vector $-2\vec{w}$.

$\vec{w}=\langle -6,-4\rangle$	Vector $\vec{w}$ in component form
$-2\vec{w}=-2\bullet\langle -6,-4\rangle$	Multiply both sides by the scalar, –2.
$-2\vec{w}=\langle -2\bullet(-6),-2\bullet(-4)\rangle$	Multiply each component by –2.
$-2\vec{w}=\langle 12,8\rangle$	Simplify.

The components of the vector $-2\vec{w}$ are $\langle 12,8\rangle$.

(*continued*)

Find the components of the vector combination $-2\vec{u}+4\vec{v}-2\vec{w}$.

$-2\vec{u}+4\vec{v}-2\vec{w}=\langle -14,4\rangle+\langle -4,12\rangle+\langle 12,8\rangle$	Write the vectors being added in component form.
$-2\vec{u}+4\vec{v}-2\vec{w}=\langle -14+(-4)+12,\ 4+12+8\rangle$	Rewrite using component-wise addition.
$-2\vec{u}+4\vec{v}-2\vec{w}=\langle -6,24\rangle$	Simplify.

In component form, $-2\vec{u}+4\vec{v}-2\vec{w}=\langle -6,24\rangle$.

Example 5

When an airplane is flown in a crosswind, the actual velocity of the airplane is the vector sum of the velocity of the crosswind and the velocity at which the pilot steers the airplane. If an airplane flies in a crosswind that has a velocity of $\vec{w}=\langle 24,32\rangle$ miles per hour, and the pilot wants the actual velocity of the airplane to be $\vec{v}=\langle 240,450\rangle$ miles per hour, with what velocity $\vec{u}$ should the pilot fly the airplane? What are the corresponding speeds of each velocity vector?

1. Write a vector equation that relates the unknown vector $\vec{u}$ to the known vectors $\vec{v}$ and $\vec{w}$.

   The actual velocity $\vec{v}$ of the airplane is the vector sum of the velocity $\vec{u}$ at which the airplane is flown, and the crosswind velocity $\vec{w}$. Therefore, $\vec{v}=\vec{u}+\vec{w}$.

2. Solve the vector equation for the unknown vector $\vec{u}$.

   Solve for $\vec{u}$.

$\vec{v}=\vec{u}+\vec{w}$	Equation determined from the previous step
$\vec{v}-\vec{w}=\vec{u}+\vec{w}-\vec{w}$	Subtract vector $\vec{w}$ from both sides.
$\vec{v}-\vec{w}=\vec{u}$	Simplify.

   Therefore, $\vec{v}-\vec{w}=\vec{u}$.

3. Find the vector difference $\vec{v} - \vec{w}$.

To subtract vector $\vec{w}$ from vector $\vec{v}$, add the opposite vector, $-\vec{w}$.

$\vec{v} - \vec{w} = \vec{v} + (-\vec{w})$	Add $\vec{v}$ and $-\vec{w}$.
$\vec{v} - \vec{w} = \langle 240, 450\rangle + \langle -24, -32\rangle$	Write vectors $\vec{v}$ and $-\vec{w}$ in component form on the right side of the equation.
$\vec{v} - \vec{w} = \langle 240 + (-24),\ 450 + (-32)\rangle$	Rewrite using component-wise vector addition.
$\vec{v} - \vec{w} = \langle 216, 418\rangle$	Simplify.

The vector difference $\vec{v} - \vec{w}$ is $\langle 216, 418\rangle$.

4. Determine the velocity $\vec{u}$ at which the pilot should fly the airplane.

The vector $\vec{u}$ is related to the desired actual velocity $\vec{v}$ and the crosswind velocity $\vec{w}$ by the equation $\vec{u} = \vec{v} - \vec{w}$, which is equivalent to $\vec{v} - \vec{w} = \langle 216, 418\rangle$. Therefore, the pilot should fly the airplane with a velocity of $\vec{u} = \langle 216, 418\rangle$.

5. Find the corresponding speed of each velocity vector.

The speed of a moving object is the magnitude of its velocity vector, which can be calculated using the Pythagorean Theorem.

Find $\|\mathbf{v}\|$, which represents the desired actual speed.

$\|\mathbf{v}\| = \sqrt{a^2 + b^2}$	Magnitude formula for $\|\mathbf{v}\|$
$\|\mathbf{v}\| = \sqrt{(240)^2 + (450)^2}$	Substitute 240 for *a* and 450 for *b*.
$\|\mathbf{v}\| = \sqrt{57{,}600 + 202{,}500}$	Apply the exponents.
$\|\mathbf{v}\| = \sqrt{260{,}100}$	Add.
$\|\mathbf{v}\| = 510$	Simplify.

Thus, the desired actual speed is 510 mph.

(*continued*)

Find $\|\mathbf{w}\|$, which represents the crosswind speed.

$\|\mathbf{w}\|=\sqrt{a^2+b^2}$	Magnitude formula for $\|\mathbf{w}\|$
$\|\mathbf{w}\|=\sqrt{(24)^2+(32)^2}$	Substitute 24 for a and 32 for b.
$\|\mathbf{w}\|=\sqrt{576+1024}$	Apply the exponents.
$\|\mathbf{w}\|=\sqrt{1600}$	Add.
$\|\mathbf{w}\|=40$	Simplify.

Therefore, the crosswind speed is 40 mph.

Find $\|\mathbf{u}\|$, which represents the speed at which the pilot should steer the plane.

$\|\mathbf{u}\|=\sqrt{a^2+b^2}$	Magnitude formula for $\|\mathbf{u}\|$
$\|\mathbf{u}\|=\sqrt{(216)^2+(418)^2}$	Substitute 216 for a and 418 for b.
$\|\mathbf{u}\|=\sqrt{46{,}656+174{,}724}$	Apply the exponents.
$\|\mathbf{u}\|=\sqrt{221{,}380}$	Add.
$\|\mathbf{u}\|=2\sqrt{55{,}345}$	Simplify.
$\|\mathbf{u}\|\approx 470.5$	Approximate to the nearest tenth.

Based on this result, the pilot should fly the plane at an approximate speed of 470.5 mph.

Scaffolded Practice 2.2

Example 1

Given vectors $\vec{u} = \langle -4, 6 \rangle$ and $\vec{v} = \langle 8, -1 \rangle$, find the vector sum $\vec{u} + \vec{v}$ in three ways:

a. by placing them head-to-tail

b. by using the Parallelogram Rule

c. by using component-wise addition

1. Find the vector sum $\vec{u} + \vec{v}$ by placing $\vec{u}$ and $\vec{v}$ head-to-tail.

2. Find the vector sum $\vec{u} + \vec{v}$ using the Parallelogram Rule.

3. Find the vector sum $\vec{u} + \vec{v}$ using component-wise addition.

continued

Lesson 2.2: Performing Operations on Vectors

Example 2

Given vectors $\vec{m}=\langle -5,-3\rangle$ and $\vec{n}=\langle -9,2\rangle$, find the vector difference $\vec{m}-\vec{n}$ in three ways:

a. by placing them head-to-tail

b. by using the Parallelogram Rule

c. by using component-wise addition

Example 3

Draw and find the components of each of these scalar multiples of $\vec{v}=\langle -4,-3\rangle$: $2\vec{v}$, $-\frac{1}{2}\vec{v}$, and $\frac{\vec{v}}{\|\mathbf{v}\|}$.

Example 4

Given $\vec{u}=\langle 7,-2\rangle$, $\vec{v}=\langle -1,3\rangle$, and $\vec{w}=\langle -6,-4\rangle$, find the components of each combination of vectors:

a. $2\vec{u}+3\vec{v}$

b. $-2\vec{u}+4\vec{v}-2\vec{w}$

Example 5

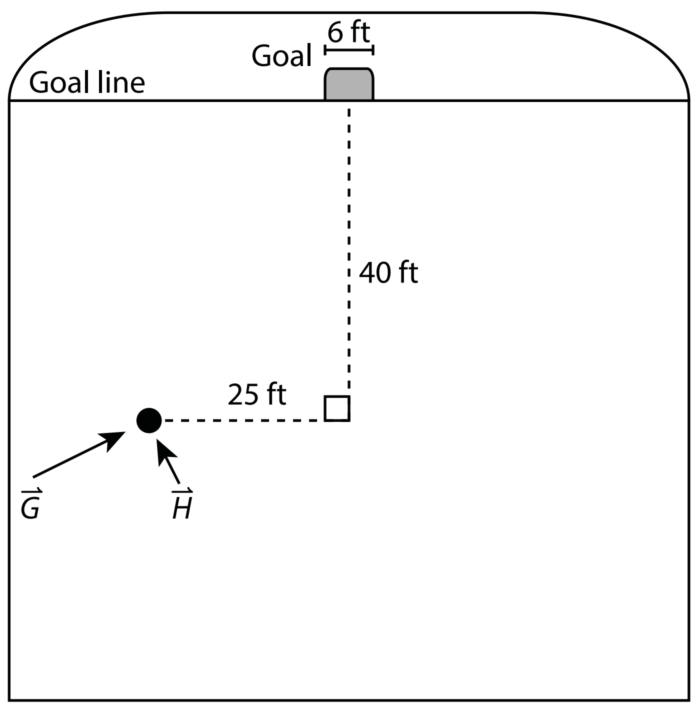
When an airplane is flown in a crosswind, the actual velocity of the airplane is the vector sum of the velocity of the crosswind and the velocity at which the pilot steers the airplane. If an airplane flies in a crosswind that has a velocity of $\vec{w}=\langle 24,32\rangle$ miles per hour, and the pilot wants the actual velocity of the airplane to be $\vec{v}=\langle 240,450\rangle$ miles per hour, with what velocity $\vec{u}$ should the pilot fly the airplane? What are the corresponding speeds of each velocity vector?

Problem-Based Task 2.2: Hitting Hockey Goals

SMP
1 ✓ 2 ✓
3 ✓ 4 ✓
5 ✓ 6 ✓
7 ✓ 8 ✓

When a hockey player hits a puck that is initially at rest on the ice, the puck moves in the direction of the force with which the puck is hit. Sometimes, two players strike a hockey puck at the same time. When this happens, the direction in which the hockey puck moves is determined by the total force acting on the hockey puck, which is the vector sum of the individual forces applied by each player.

The hockey puck shown in the figure is initially at rest and then simultaneously hit by two players, Gustavo and Hector, at a point in the game when the goalie has been pulled from the goal. The players hit the hockey puck with force vectors $\vec{G} = \langle 4,2 \rangle$ and $\vec{H} = \langle -1,2 \rangle$, measured in pounds of force. Find the unit vector that points in the direction of the hockey puck's motion. Will the hockey puck enter the goal if it continues to move in this direction? *Note*: The *x*- and *y*-distances of 25 feet and 40 feet shown in the diagram indicate the distances to the very center of the 6-foot-wide goal.



Will the hockey puck enter the goal if it continues to move in this direction?

Problem-Based Task 2.2: Hitting Hockey Goals

Coaching

a. What are the components of the resultant force $\vec{G}+\vec{H}$?

b. What is the magnitude of the resultant force $\vec{G}+\vec{H}$?

c. What is the unit vector in the direction of the resultant force $\vec{G}+\vec{H}$?

d. Before the hockey puck is hit, how far is it from the goal line?

e. When the hockey puck reaches the goal line, how far to the right or left will it have moved?

f. Will the puck enter the goal? Explain.

Problem-Based Task 2.2: Hitting Hockey Goals

Coaching Sample Responses

a. What are the components of the resultant force $\vec{G}+\vec{H}$?

The components of the vector sum $\vec{G}+\vec{H}$ can be found by adding the vectors $\vec{G}$ and $\vec{H}$ component-wise.

$$\vec{G}+\vec{H}=\langle 4,2\rangle+\langle -1,2\rangle$$

$$\vec{G}+\vec{H}=\langle 4+(-1),\ 2+2\rangle$$

$$\vec{G}+\vec{H}=\langle 3,4\rangle$$

In component form, $\vec{G}+\vec{H}=\langle 3,4\rangle$.

b. What is the magnitude of the resultant force $\vec{G}+\vec{H}$?

The magnitude $\|\mathbf{G}+\mathbf{H}\|$ can be found using the Pythagorean Theorem.

$$\|\mathbf{G}+\mathbf{H}\|=\sqrt{a^2+b^2}$$

$$\|\mathbf{G}+\mathbf{H}\|=\sqrt{(3)^2+(4)^2}$$

$$\|\mathbf{G}+\mathbf{H}\|=\sqrt{9+16}=\sqrt{25}=5$$

The magnitude of $\vec{G}+\vec{H}$, $\|\mathbf{G}+\mathbf{H}\|$, is 5.

c. What is the unit vector in the direction of the resultant force $\vec{G}+\vec{H}$?

Use the component form of $\vec{G}+\vec{H}$, $\langle 3,4\rangle$, and the magnitude for $\|\mathbf{G}+\mathbf{H}\|$, 5, to find the unit vector $\frac{\vec{G}+\vec{H}}{\|\mathbf{G}+\mathbf{H}\|}$. Recall that dividing by 5 is the same as multiplying by $\frac{1}{5}$.

$$\frac{\vec{G}+\vec{H}}{\|\mathbf{G}+\mathbf{H}\|}=\frac{1}{5}\bullet\left(\vec{G}+\vec{H}\right)$$

$$\frac{\vec{G}+\vec{H}}{\|\mathbf{G}+\mathbf{H}\|}=\frac{1}{5}\bullet\langle 3,4\rangle$$

$$\frac{\vec{G}+\vec{H}}{\|\mathbf{G}+\mathbf{H}\|}=\left\langle \frac{1}{5}\bullet 3,\ \frac{1}{5}\bullet 4\right\rangle=\left\langle \frac{3}{5},\frac{4}{5}\right\rangle$$

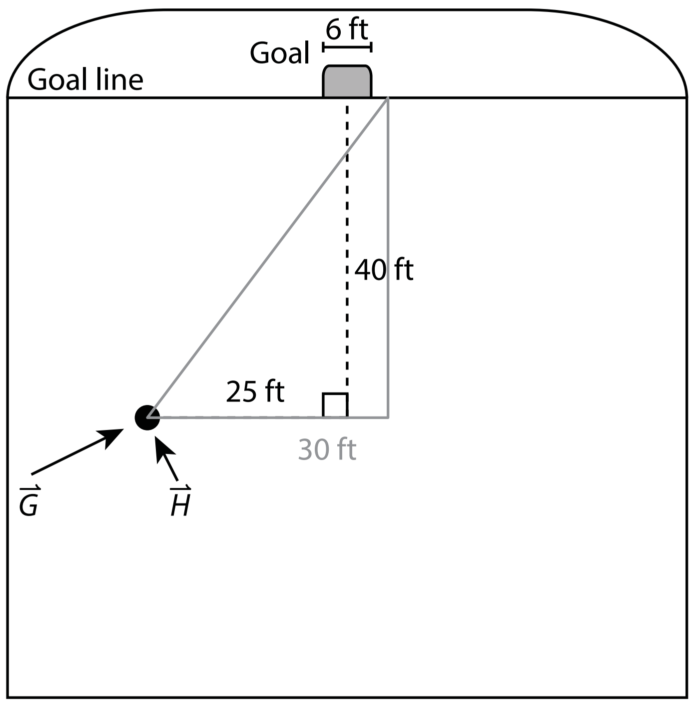
Thus, the hockey puck moves in the direction given by the unit vector $\frac{\vec{G}+\vec{H}}{\|\mathbf{G}+\mathbf{H}\|}=\left\langle \frac{3}{5},\frac{4}{5}\right\rangle$.

d. Before the hockey puck is hit, how far is it from the goal line?

The goal line stretches across the entire width of the rink, so the shortest distance from the puck to the goal line is the *y*-distance as shown in the diagram. Therefore, the *y*-distance from the puck to the goal line is 40 feet.

e. When the hockey puck reaches the goal line, how far to the right or left will it have moved?

To find the *x*-component of the vector in the direction of $\vec{G}+\vec{H}$ that has a *y*-component of 40 feet, note that the ratio of the components of the force is 3 : 4 since $\vec{G}+\vec{H}=\langle 3,4\rangle$. In order to maintain this 3 : 4 ratio, then if the *y*-component is 40 feet, the *x*-component must be 30 feet.



f. Will the puck enter the goal? Explain.

No, it will not enter the goal. The puck begins 25 feet to the left of the center of the goal, and crosses the goal line 30 feet to the right of this. Thus, the puck crosses the line 5 feet to the right of the center of the goal. Since the goal is 6 feet wide, it only extends 3 feet to the right of its center. Therefore, the puck misses the goal by about 2 feet.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

Practice 2.2: Performing Operations on Vectors

For problems 1 and 2, find the vector sum or difference by the method indicated.

1. Given $\vec{u}=\langle 5,3\rangle$ and $\vec{v}=\langle -4,2\rangle$, find the vector difference $\vec{u}-\vec{v}$ by graphing.

2. Given $\vec{u}=\langle 9,-3\rangle$ and $\vec{v}=\langle -5,7\rangle$, find the vector sum $\vec{u}+\vec{v}$ using components.

For problems 3 and 4, find the scalar multiple of the vector.

3. Given $\vec{v}=\langle -1,-3\rangle$, find $3\vec{v}$.

4. Given $\vec{w}=\langle 25,-40\rangle$, find $-\frac{3}{5}\vec{w}$.

For problems 5 and 6, find the given vector combination if $\vec{u}=\langle -3,8\rangle$, $\vec{v}=\langle 7,-2\rangle$, and $\vec{w}=\langle -4,-3\rangle$.

5. $\vec{u}+\vec{v}-\vec{w}$

6. $3\vec{u}-3\vec{v}-4\vec{w}$

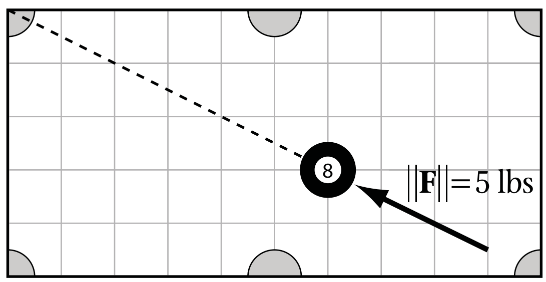
For problems 7–10, use your knowledge of vectors to complete each problem.

7. A pilot flies an airplane with a velocity of $\vec{u}=\langle -372,420\rangle$, but the actual velocity of the airplane is $\vec{v}=\langle -330,476\rangle$. The actual velocity $\vec{v}$ of the airplane is the vector sum of the velocity $\vec{u}$ at which the airplane is flown, and the crosswind velocity $\vec{w}$. Find the velocity and speed of the crosswind in which the airplane flies.

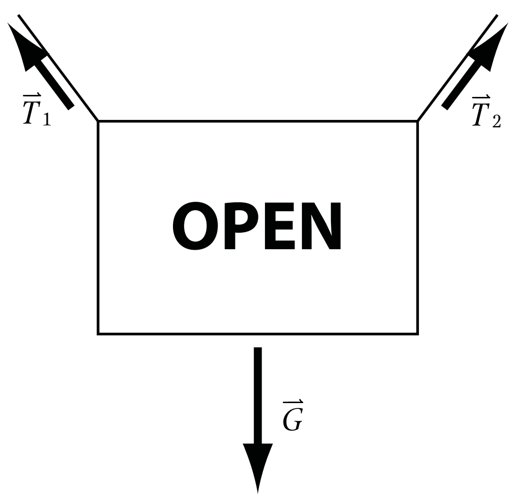
continued

8. Timothy wants to row his boat across a river that flows south. The velocity of the river's current is $\vec{u}=\langle 0,-5\rangle$ feet per second. He rows the boat with a velocity of $\vec{v}=\langle 4,2\rangle$ feet per second. With what velocity does his boat actually travel? If the river is 400 feet wide, how far downriver will the current have carried Timothy when he reaches the far side?

9. A billiard ball is hit from the position shown in the diagram, with a force $\vec{F}$ of magnitude $\|\mathbf{F}\|=5$ pounds. The ball travels in the direction of $\vec{F}$, into the upper left pocket of the table. Find the components of the unit vector $\vec{u}$ in the direction of the ball's motion, and then find the components of the force $\vec{F}$.



10. A sign in a store window hangs from two chains, as shown in the diagram. The force of gravity pulls downward on the sign with a force vector of $\vec{G}=\langle 0,-4\rangle$ pounds. The force of tension in each chain pulls the sign in the direction shown, so that $\vec{T}_1+\vec{T}_2=-\vec{G}$ and each chain supports half the weight of the sign. Find the components of the tension vectors $\vec{T}_1$ and $\vec{T}_2$.



Progress Assessment

Circle the letter of the best answer.

1. What is the vector difference of $\vec{u}=\langle -1,4\rangle$ and $\vec{v}=\langle -6,5\rangle$?
   a. $\langle 5,-1\rangle$
   b. $\langle -7,9\rangle$
   c. $\langle -5,-1\rangle$
   d. $\langle 6,20\rangle$

2. Given $\vec{v}=\langle -6,10\rangle$, what is the scalar multiple $\frac{3}{2}\vec{v}$?
   a. $\langle -2,5\rangle$
   b. $\langle -6,15\rangle$
   c. $\langle -9,5\rangle$
   d. $\langle -9,15\rangle$

3. What vector is represented by $3\vec{u}-2\vec{v}-2\vec{w}$, if $\vec{u}=\langle 2,-1\rangle$, $\vec{v}=\langle -2,4\rangle$, and $\vec{w}=\langle 5,-3\rangle$?
   a. $\langle 0,-1\rangle$
   b. $\langle 0,-5\rangle$
   c. $\langle 12,-1\rangle$
   d. $\langle 12,-5\rangle$