Common Core State Standards

Mathematics III

Integrated Pathway

Teacher Resource

Program Overview

WALCH

INTEGRATED MATH

This program was developed and reviewed by experienced math educators who have both academic and professional backgrounds in mathematics. This ensures: freedom from mathematical errors, grade level appropriateness, freedom from bias, and freedom from unnecessary language complexity.

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PROGRAM OVERVIEW

Introduction to the Program

Introduction

The *Common Core State Standards Integrated Pathway: Mathematics III Program* is a complete set of materials developed around the Common Core State Standards (CCSS), the overview of the Integrated Pathway for the Common Core State Mathematics Standards, and the Mathematics III content map found in Appendix A of the Common Core State Standards. Topics are built around accessible core curricula, ensuring that the *CCSS Integrated Pathway: Mathematics III Program* is useful for striving students and diverse classrooms.

This program realizes the benefits of exploratory and investigative learning and employs a variety of instructional models to meet the learning needs of students with a range of abilities.

The *CCSS Integrated Pathway: Mathematics III Program* includes components that support problem-based learning, instruct and coach as needed, provide practice, and assess students' skills. Instructional tools and strategies are embedded throughout.

The program includes:

- More than 150 hours of lessons, addressing the six units of CCSS IP: Mathematics III
- Essential Questions for each instructional topic
- Vocabulary
- Instruction and Guided Practice
- Problem-based Tasks and Coaching questions
- Step-by-step graphing calculator instructions for the TI-Nspire and the TI-83/84
- Station activities to promote collaborative learning and problem-solving skills

Purpose of Materials

The *CCSS Integrated Pathway: Mathematics III Program* has been organized to coordinate with the CCSS Integrated Pathway: Mathematics III content map and specifications from Appendix A of the Common Core State Standards.

Each lesson includes activities that offer opportunities for exploration and investigation. These activities incorporate concept and skill development and guided practice, then move on to the application of new skills and concepts in problem-solving situations. Throughout the lessons and activities, problems are contextualized to enhance rigor and relevance.

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PROGRAM OVERVIEW
Introduction to the Program

This program includes all the topics addressed in the CCSS Integrated Pathway: Mathematics III content map. These include:

- Inferences and Conclusions from Data
- Polynomial Relationships
- Rational and Radical Relationships
- Trigonometry of General Triangles and Trigonometric Functions
- Mathematical Modeling of Inverse, Logarithmic, and Trigonometric Functions
- Mathematical Modeling and Choosing a Model

The eight Mathematical Practices described in the Common Core are infused throughout:

- CCSS.MP.1: Make sense of problems and persevere in solving them.
- CCSS.MP.2: Reason abstractly and quantitatively.
- CCSS.MP.3: Construct viable arguments and critique the reasoning of others.
- CCSS.MP.4: Model with mathematics.
- CCSS.MP.5: Use appropriate tools strategically.
- CCSS.MP.6: Attend to precision.
- CCSS.MP.7: Look for and make use of structure.
- CCSS.MP.8: Look for and express regularity in repeated reasoning.

Structure of the Teacher Resource

The *CCSS Integrated Pathway: Mathematics III Program* is provided in an online, digital format, and/or in hard copy (Program Overview and six unit volumes). The materials, either online or hard copy, are completely reproducible. Online materials can be provided in your Learning Management System (such as Canvas or Schoolology) or in Walch's proprietary system, WalchConnect. The nested folder organization in WalchConnect allows you to access the materials quickly and easily. The digital format also facilitates printing and copying student pages and/or making assignments online.

The Program Overview is the first section. This section helps you to navigate the materials, offers a collection of graphic organizers and suggested strategies for their use, and shows the correlation between the Common Core State Standards and the CCSS Integrated Pathway: Mathematics III content map found in Appendix A of the Common Core State Standards.

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The remaining materials focus on content, knowledge, and application of the six units in the CCSS Integrated Pathway Mathematics III curriculum: Inferences and Conclusions from Data; Polynomial Relationships; Rational and Radical Relationships; Trigonometry of General Triangles and Trigonometric Functions; Mathematical Modeling of Inverse, Logarithmic, and Trigonometric Functions; and Mathematical Modeling and Choosing a Model. The units in the *CCSS Integrated Pathway: Mathematics III Program* are designed to be flexible so that you can mix and match activities as the needs of your students and your instructional style dictate.

The Station Activities correspond to the content in the units and provide students with the opportunity to apply concepts and skills, while you have a chance to circulate, observe, speak to individuals and small groups, and informally assess and plan.

Each lesson begins with a pre-assessment and ends with a progress assessment. These allow you to assess students' progress as you move from lesson to lesson, enabling you to gauge how well students have understood the material and to differentiate as appropriate.

***Please note*:** Throughout the Teacher Resource, page references are provided on the lower, inner corner of some pages. These references indicate the corresponding page(s) in the Student Workbook (SWB) and/or Student Resource Book (SRB) as appropriate. These page references have been included here to facilitate assigning materials to students.

Glossary

The Glossary contains vocabulary terms and formulas from throughout the program, organized alphabetically. Each listing provides the term and the definition in both English and Spanish. The listings include the lesson number(s) where the terms can be found in the Words to Know.

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Correspondence to Standards for Mathematical Practice

How Do Walch Integrated Mathematics Resources Address the CCSS 8 Standards for Mathematical Practice?

Walch's programs for the *CCSS Integrated Pathway Mathematics I, II,* and *III* courses employ a problem-based model of instruction that supports and reinforces the 8 Standards for Mathematical Practice. Although the following table focuses on Problem-Based Tasks, Walch's full programs also include hundreds of additional problems in warm-ups and practices. The Implementation Guides for selected PBTs highlight SMPs to focus on during implementation and discussion.

CCSS Standards for Mathematical Practice		Relevant Attributes of Walch Integrated Math Resources
1	Make sense of problems and persevere in solving them.	Each sub-lesson is built around a Problem-Based Task (PBT) that requires students to "make sense of problems and persevere in solving them."
2	Reason abstractly and quantitatively.	Each PBT uses a meaningful real-world context that requires students to reason both abstractly about the situation/relationships and quantitatively about the values representing the elements and relationships.
3	Construct viable arguments and critique the reasoning of others.	Since the PBT provides opportunities for multiple problem-solving approaches and varied solutions, students are required to construct viable arguments to support their approach and answer. This, in turn, provides other students the opportunity to analyze and critique their classmates' reasoning.
4	Model with mathematics.	Each PBT represents a real-world situation and requires students to model it with mathematics.
5	Use appropriate tools strategically.	PBTs require students to make choices about using appropriate tools, such as calculators, spreadsheets, graph paper, manipulatives, protractors, and compasses. The tasks do not prescribe specific tools, but instead provide opportunities for their use.
6	Attend to precision.	The real-world contexts of the PBTs require students to be precise in their solutions, both in the ways that the solutions are stated, labeled, and explained, and in the degree of precision necessary given the context (e.g., tripling chili for a crowd vs. machining a part for an airplane engine).
7	Look for and make use of structure.	The PBTs present students with complicated scenarios that must be analyzed to discern patterns and significant mathematical features.
8	Look for and express regularity in repeated reasoning.	PBTs require multiple steps, providing opportunities for students to note repeated calculations, monitor their process, and continually evaluate reasonableness of intermediate results before arriving at a solution.

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Correspondence to CCSS Publishers' Criteria

How Do Walch Integrated Mathematics Resources Address the High School Publishers' Criteria for CCSS Mathematics?

Walch's programs for the *CCSS Integrated Pathway Mathematics I, II,* and *III* courses were designed to the specifications of Appendix A and the Publishers' Criteria. Focus, Coherence, and Rigor represent the fundamental assumptions of the resources.

Publishers' Criteria	Relevant Attributes of Walch Integrated Math Resources
FOCUS Focus strongly where the standards focus.	Each course is built around the units specified in the CCSS for Mathematics Appendix A. Units are organized around important topics in mathematics and reflect the focus of the standards. Units range from 22 sub-lessons for Math I Unit 2, "Linear and Exponential Relationships," which address 21 Common Core standards, to three sub-lessons for Math I Unit 6, "Connecting Algebra and Geometry Through Coordinates," which address only three Common Core standards.
COHERENCE Think across grades/courses, and link to major topics in each course.	Walch Integrated Math programs include two features contributing to coherence. Each sub-lesson identifies necessary prerequisite skills (our forthcoming Support Supplement will point to where they were introduced, including in elementary or middle school, and provide instructional materials for remediation). Warm-Ups tie to these prerequisites in order to activate prior knowledge. Also, the major topics of each course are mapped forward to show where they are revisited and expanded upon in future courses.
RIGOR In major topics, pursue with equal intensity • conceptual understanding, • procedural skill and fluency, and • applications.	The instructional components of Walch's Integrated Math programs guide students from conceptual understanding and procedural skill developed through modeling and guided practice to the application opportunities of the PBT. The resources move students on to additional practice to refine procedural skill and fluency, and later to station activities to crystalize conceptual understanding.

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Correspondence to NCTM *Principles to Actions* Teaching Practices

How Do Walch Integrated Mathematics Resources Address the NCTM *Principles to Actions* Mathematics Teaching Practices?

Walch's programs for the *CCSS Integrated Pathway Mathematics I, II,* and *III* courses were designed by experienced educators and curriculum developers, informed by best-practice research, and refined through an iterative process of implementation and feedback. Together with professional development, these materials support and sustain good teaching practices.

NCTM Mathematics Teaching Practices	Relevant Attributes of Walch Integrated Math Resources
Establish mathematics goals to focus learning.	Each sub-lesson in Walch's programs addresses specified standards which can be used as goals to focus learning. Essential Questions offer further focus.
Implement tasks that promote reasoning and problem solving.	Each sub-lesson in Walch's programs is built around a Problem-Based Task (PBT), set in a meaningful real-world context and designed to promote reasoning and problem solving. The courses include dozens of PBTs as well as warm-up and practice problems.
Use and connect mathematical representations.	Walch's High School Math programs make frequent use of, and connections among and between, equations, tables, and graphs. PBTs often require students to use and connect two or more of these representations, and the representations are modeled through guided practice.
Facilitate meaningful mathematical discourse.	Several features of the programs support mathematical discourse, including warm-up debriefs with connections to upcoming lessons, implementation guides and optional coaching questions for the PBTs, and discussion guides for Station Activities. Explanations of PBT solutions are another opportunity for discourse. Please note: Mathematical discourse is an important topic for professional development, in conjunction with implementation of these materials.
Pose purposeful questions.	The implementation guides, coaching questions and discussion guides provide samples of purposeful questions. Note that this is another important topic for professional development.
Build procedural fluency from conceptual understanding.	The programs develop conceptual understanding through modeling, guided practice, and application, and then provide additional opportunities to practice and develop fluency.
Support productive struggle in learning mathematics.	The PBTs require "productive struggle;" implementation guides include suggestions for facilitation and monitoring, and coaching questions provide an option for additional support as appropriate, allowing students to proceed through the task and ensuring that the struggle remains productive rather than too frustrating.
Elicit and use evidence of student thinking.	Various discussions and PBTs require students to display their thinking. Implementation guides offer specific prompts and suggestions for eliciting and responding to student thinking. Professional development supports teachers in using that evidence to respond in instructionally appropriate ways.

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Unit Structure

All of the instructional units have common features. Each unit begins with a list of all the standards addressed in the lessons, and a list of one or more conceptual activities. Each lesson begins with a pre-assessment, followed by the list of standards addressed in the lesson; Essential Questions; vocabulary (titled "Words to Know"); and a list of recommended websites to be used as additional resources.

Each sub-lesson begins with a list of identified prerequisite skills that students need to have mastered in order to be successful with the new material in the upcoming sub-lesson. This is followed by an introduction, key concepts, common errors/misconceptions, guided practice examples, a problem-based task with coaching questions and sample responses, a closure activity, and practice. Each lesson ends with a progress assessment to evaluate students' learning.

All of the components are described below and on the following pages for your reference.

Standards Correlations and Connections to Future Courses

In this section, you'll find a comprehensive list of the Common Core State Standard(s) addressed in each sub-lesson, followed by a map detailing where and how future courses will build upon the topics introduced in the unit.

Conceptual Activities

Conceptual understanding serves as the foundation on which to build deeper understanding of mathematics. In an effort to build conceptual understanding of mathematical ideas and to provide more than procedural fluency and application, links to interactive open education and Desmos resources are included. (*Note*: These website links will be monitored and repaired or replaced as necessary.)

Pre-Assessment

This can be used to gauge students' prior knowledge and to inform instructional planning.

Common Core State Standards for the Lesson

All standards that are addressed in the entire lesson are listed.

Essential Questions

These are intended to guide students' thinking as they proceed through the lesson. By the end of each lesson, students should be able to respond to the questions.

Words to Know

Vocabulary terms and formulas are provided as background information for instruction or to review key concepts that are addressed in the lesson.

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Recommended Resources

This is a list of websites that can be used as additional resources. Some websites are games; others provide additional examples and/or explanations. The links for these resources are live in the PDF version of the Teacher Resource. (*Note*: These website links will be monitored and repaired or replaced as necessary.)

Common Core State Standards for the Sub-Lesson

When lessons are broken down into sub-lessons, the specific standard or standards that are addressed are presented at the beginning of the instructional portion of the lesson.

Warm-Up

Each warm-up takes approximately 5 minutes and addresses either prerequisite and critical-thinking skills or previously taught math concepts.

Warm-Up Debrief

Each debrief provides the answers to the warm-up questions, and offers suggestions for situations in which students might have difficulties. A section titled Connection to the Lesson is also included in the debrief to help answer students' questions about the relevance of the particular warm-up activity to the upcoming instruction. Warm-Ups with debriefs are also provided in PPT presentations with short video clips to enhance engagement.

Identified Prerequisite Skills

Presented at the beginning of each sub-lesson, this list cites the skills necessary to be successful with the new material.

Introduction

This brief paragraph gives a description of the concepts about to be presented and often contains some Words to Know.

Key Concepts

Provided in bulleted form, this instruction highlights the important ideas and/or processes for meeting the standard.

Common Errors/Misconceptions

This is a list of the common errors students make when applying Key Concepts. This list suggests what to watch for when students arrive at an incorrect answer or are struggling with solving the problems.

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Guided Practice

This section provides step-by-step examples of applying the Key Concepts. The three to five examples are intended to aid during initial instruction, but are also for individuals needing additional instruction and/or for use during review and test preparation. GeoGebra applets that model selected Guided Practice examples are provided on the flash drive that accompanies this program.

Problem-Based Task

This activity can serve as the centerpiece of a problem-based lesson, or it can be used to walk students through the application of the standard, prior to traditional instruction or at the end of instruction. The task makes use of critical-thinking skills.

Problem-Based Task Implementation Guides (Appendix)

This instructional overview, found in an appendix at the end of each unit, highlights connections between the task and the sub-lesson's key concepts and SMPs. It also offers suggestions for facilitating and monitoring, and provides alternative solutions.

Optional Problem-Based Task Coaching Questions

These questions scaffold the task and guide students to solving the problem(s) presented in the task. They should be used at the discretion of the teacher for students requiring additional support.

Problem-Based Task Coaching Questions Sample Responses

These are the answers and suggested appropriate responses to the coaching questions. In some cases answers may vary, but a sample answer is given for each question.

Recommended Closure Activity

Students are given the opportunity to synthesize and reflect on the lesson through a journal entry or discussion of one or more of the Essential Questions.

Practice

Each sub-lesson includes practice problems to support students' achievement of the learning objectives. These sheets are written for the student. They can be used in any combination of teacher-led instruction, cooperative learning, or independent application of knowledge.

Progress Assessment

Each lesson ends with 10 multiple-choice questions, as well as one extended-response question that incorporates critical thinking and writing components. This can be used to document the extent to which students grasp the concepts and skills addressed during instruction.

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Unit Assessment

Each unit ends with 12 multiple-choice questions and three extended-response questions that incorporate critical thinking and writing components. This can be used to document the extent to which students grasped the concepts and skills of each unit.

Answer Key

Answers for all of the Warm-Ups, Assessments, and Practice problems from the Teacher Resource and all of the problems from the Student Resource Book are provided at the end of each unit. (Student editions include odd answers for the exercises in the student book.)

Graphing Calculators

Step-by-step instructions for using a TI-Nspire and a TI-83/84 are provided whenever graphing calculators are referenced.

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Standards Correlations

Each lesson in this Integrated Pathway: Mathematics III program was written specifically to address the Common Core State Standards. Each lesson lists the standards covered in all the sub-lessons, and each sub-lesson lists the standards addressed in that particular section. In this section, you'll find a comprehensive list mapping the sub-lessons to the CCSS.

Guide to Common Core State Standards Annotation

As you use this program, you will come across a symbol included with the Common Core standards for some of the lessons and activities. These symbols are explained below.

Symbol: ⋆

Denotes: Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (⋆).

From http://www.walch.com/CCSS/00003

Symbol: (+)

Denotes: College and Career Readiness Standards

Advanced mathematics standards that are required in higher-level courses such as advanced statistics may also be included in lower-level courses. These additional standards are denoted by (+). According to the Common Core State Standards Initiative, "the evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6–8."

From http://www.walch.com/CCSS/00004

Connections to Future Courses

This section provides a map between topics introduced in each unit of this course and subsequent courses where each topic is revisited and built upon.

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CCSS INTEGRATED PATHWAY: MATHEMATICS III STANDARDS CORRELATIONS

Unit 1: Inferences and Conclusions from Data				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Using the Normal Curve**			
	1.1.1	Normal Distributions and the 68–95–99.7 Rule	S–ID.4★	U1-6
	1.1.2	Standard Normal Calculations	S–ID.4★	U1-33
	1.1.3	Assessing Normality	S–ID.4★	U1-60
Lesson 2	**Populations Versus Random Samples and Random Sampling**			
	1.2.1	Differences Between Populations and Samples	S–IC.1★	U1-99
	1.2.2	Simple Random Sampling	S–IC.2★	U1-125
	1.2.3	Other Methods of Random Sampling	S–IC.2★	U1-156
Lesson 3	**Surveys, Experiments, and Observational Studies**			
	1.3.1	Identifying Surveys, Experiments, and Observational Studies	S–IC.3★	U1-193
	1.3.2	Designing Surveys, Experiments, and Observational Studies	S–IC.3★	U1-211
Lesson 4	**Estimating Sample Proportions and Sample Means**			
	1.4.1	Estimating Sample Proportions	S–IC.4★	U1-240
	1.4.2	The Binomial Distribution	S–IC.4★	U1-260
	1.4.3	Estimating Sample Means	S–IC.4★	U1-287
	1.4.4	Estimating with Confidence	S–IC.4★	U1-300
Lesson 5	**Comparing Treatments and Reading Reports**			
	1.5.1	Evaluating Treatments	S–IC.5★	U1-328
	1.5.2	Designing and Simulating Treatments	S–IC.5★	U1-355
	1.5.3	Reading Reports	S–IC.6★	U1-374
Lesson 6	**Making and Analyzing Decisions**			
	1.6.1	Making Decisions	S–MD.6★ (+)	U1-399
	1.6.2	Analyzing Decisions	S–MD.7★ (+)	U1-418

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Unit 2A: Polynomial Relationships				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Polynomial Structures and Operating with Polynomials**			
	2A.1.1	Structures of Expressions	A–SSE.1a★	U2A-4
	2A.1.2	Adding and Subtracting Polynomials	A–APR.1	U2A-16
	2A.1.3	Multiplying Polynomials	A–APR.1	U2A-26
Lesson 2	**Proving Identities**			
	2A.2.1	Polynomial Identities	A–SSE.1b★ A–SSE.2 A–APR.4	U2A-43
	2A.2.2	Complex Polynomial Identities	N–CN.8 (+) A–SSE.1b★ A–SSE.2 A–APR.4	U2A-59
	2A.2.3	The Binomial Theorem	A–SSE.1a★ A–SSE.1b★ A–SSE.2 A–APR.4 A–APR.5 (+)	U2A-72
Lesson 3	**Graphing Polynomial Functions**			
	2A.3.1	Describing End Behavior and Turns	F–IF.7c★	U2A-96
	2A.3.2	The Remainder Theorem	A–APR.2	U2A-115
	2A.3.3	Finding Zeros	A–APR.3 N–CN.9 (+) F–IF.7c★	U2A-135
	2A.3.4	The Rational Root Theorem	A–APR.3	U2A-153

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Lesson	Sub-lesson number	Title	Standard(s)	Pages
Lesson 4	**Solving Systems of Equations with Polynomials**			
	2A.4.1	Solving Systems of Equations Graphically	A–REI.11★	U2A-178
Lesson 5	**Geometric Series**			
	2A.5.1	Geometric Sequences	A–SSE.4★	U2A-217
	2A.5.2	Sum of a Finite Geometric Series	A–SSE.4★	U2A-239
	2A.5.3	Sum of an Infinite Geometric Series	A–SSE.4★	U2A-267

CCSS INTEGRATED PATHWAY: MATHEMATICS III STANDARDS CORRELATIONS

Unit 2B: Rational and Radical Relationships				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Operating with Rational Expressions**			
	2B.1.1	Structures of Rational Expressions	A–SSE.1a★ A–SSE.1b★ A–SSE.2	U2B-5
	2B.1.2	Adding and Subtracting Rational Expressions	A–APR.7 (+) A–SSE.2	U2B-23
	2B.1.3	Multiplying Rational Expressions	A–APR.7 (+) A–SSE.2	U2B-45
	2B.1.4	Dividing Rational Expressions	A–APR.6 A–APR.7 (+) A–SSE.2	U2B-61
Lesson 2	**Solving Rational and Radical Equations**			
	2B.2.1	Solving Rational Equations	A–REI.2	U2B-90
	2B.2.2	Solving Radical Equations	A–REI.2	U2B-110
	2B.2.3	Solving Systems of Equations	A–REI.11★	U2B-130

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Unit 3: Trigonometry of General Triangles and Trigonometric Functions				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Radians and the Unit Circle**			
	3.1.1	Radians	F–TF.1 F–TF.2	U3-5
	3.1.2	The Unit Circle	F–TF.2	U3-21
	3.1.3	Special Angles in the Unit Circle	F–TF.2	U3-45
	3.1.4	Evaluating Trigonometric Functions	F–TF.2	U3-69
Lesson 2	**Trigonometry of General Angles**			
	3.2.1	Proving the Law of Sines	G–SRT.9 (+) G–SRT.10 (+)	U3-99
	3.2.2	Proving the Law of Cosines	G–SRT.10 (+)	U3-123
	3.2.3	Applying the Laws of Sines and Cosines	G–SRT.11 (+)	U3-142
Lesson 3	**Graphs of Trigonometric Functions**			
	3.3.1	Periodic Phenomena and Amplitude, Frequency, and Midline	F–TF.5★	U3-167
	3.3.2	Using Trigonometric Functions to Model Periodic Phenomena	F–TF.5★	U3-185

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Unit 4A: Mathematical Modeling of Inverse, Logarithmic, and Trigonometric Functions				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Inverses of Functions**			
	4A.1.1	Determining Inverses of Quadratic Functions	F–BF.4a	U4A-5
	4A.1.2	Determining Inverses of Other Functions	F–BF.4a	U4A-36
Lesson 2	**Modeling Logarithmic Functions**			
	4A.2.1	Logarithmic Functions as Inverses	F–BF.4a F–LE.4★	U4A-74
	4A.2.2	Common Logarithms	F–IF.8 F–LE.4★	U4A-96
	4A.2.3	Natural Logarithms	F–IF.8 F–LE.4★	U4A-114
	4A.2.4	Graphing Logarithmic Functions	F–IF.7e★	U4A-136
	4A.2.5	Interpreting Logarithmic Models	F–IF.4★ F–IF.5★ F–IF.6★	U4A-161
Lesson 3	**Modeling Trigonometric Functions**			
	4A.3.1	Graphing the Sine Function	F–IF.7e★	U4A-187
	4A.3.2	Graphing the Cosine Function	F–IF.7e★	U4A-230

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Unit 4B: Mathematical Modeling and Choosing a Model				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Creating Equations**			
	4B.1.1	Creating Equations in One Variable	A–CED.1★	U4B-7
	4B.1.2	Representing and Interpreting Constraints	A–CED.3★	U4B-36
	4B.1.3	Rearranging Formulas	A–CED.4★	U4B-56
Lesson 2	**Transforming a Model and Combining Functions**			
	4B.2.1	Transformations of Parent Graphs	F–BF.3	U4B-90
	4B.2.2	Recognizing Odd and Even Functions	F–BF.3	U4B-116
	4B.2.3	Combining Functions	F–BF.1b★	U4B-131
Lesson 3	**Comparing Properties Within and Between Functions**			
	4B.3.1	Reading and Identifying Key Features of Real-World SituationGraphs	F–IF.4★ F–IF.5★ F–IF.6★	U4B-155
	4B.3.2	Calculating Average Rates of Change	F–IF.6★	U4B-188
	4B.3.3	Comparing Functions	F–IF.6★ F–IF.9	U4B-210

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Lesson	Sub-lesson number	Title	Standard(s)	Pages
Lesson 4	**Choosing a Model**			
	4B.4.1	Linear, Exponential, and Quadratic Functions	A–CED.2★ F–IF.4★ F–IF.5★ F–BF.3	U4B-246
	4B.4.2	Piecewise, Step, and Absolute Value Functions	F–IF.4★ F–IF.5★ F–IF.7b★ F–BF.3	U4B-270
	4B.4.3	Square Root and Cube Root Functions	F–IF.4★ F–IF.5★ F–IF.7b★ F–BF.3	U4B-293
Lesson 5	**Geometric Modeling**			
	4B.5.1	Two-Dimensional Cross Sections of Three-Dimensional Objects	G–GMD.4 G–MG.1★	U4B-326
	4B.5.2	Density	G–MG.2★	U4B-344
	4B.5.3	Design	G–MG.3★	U4B-364

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TOPICS FROM FUTURE COURSES: CCSS INTEGRATED PATHWAY MATHEMATICS III

Math III	Topic introduced	Course/unit where addressed	How addressed
Unit 1	Methods of gathering data for statistical analysis; making and analyzing decisions and making inferences about sample means and proportions; normal and approximately normal distributions	AP Statistics or Math IV	calculating the expected value of distributions other than normal and approximately normal distributions is applied to making inferences and decisions, including decision-making based on payoffs calculated from expected value
Unit 2A	Interpreting structures of higher-order polynomial expressions; systems of equations, including linear-polynomial, polynomial-polynomial, and absolute value-polynomial systems; sequences as infinite and finite series	Precalculus, Calculus, or Math IV	extended to systems of equations, including larger, more complex systems (e.g., 3×3 and larger systems) using matrices
Unit 2B	Rational expressions; solving rational and radical equations; linear-rational and linear-radical systems	Precalculus, Calculus, or Math IV	solving systems that can be extended to larger, more complex systems (e.g., 3×3 and larger) using vectors and matrices; graphing more complex rational functions and exploring features of the graphs
Unit 3	Calculating the sine, cosine, and tangent for any triangle using the laws of sines and cosines; trigonometric ratios, including angles measured in radians; the unit circle	Precalculus, Calculus, or Math IV	representing and operating with vectors using trigonometric ratios, trigonometric functions, and complex numbers; evaluating special angles in right triangles using the unit circle and concepts of periodicity

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TOPICS FROM FUTURE COURSES: CCSS INTEGRATED PATHWAY MATHEMATICS III

Math III	Topic introduced	Course/unit where addressed	How addressed
Unit 4A	Finding the inverses of quadratic and other functions, including rational and trigonometric functions; making the connection between exponential functions and their inverses, logarithmic functions, as the method for solving exponential equations	Precalculus, Calculus, or Math IV	extended to complex logarithms and verifying inverses using compositions of functions
	Exploring the intervals where trigonometric functions are increasing and decreasing	Precalculus, Calculus, or Math IV	creating inverses of trigonometric functions
Unit 4B	Using knowledge of function families to generate models that fit real-world phenomena	Precalculus, Calculus, or Math IV	exploring odd and even functions with respect to trigonometric functions; studying and graphing more complex logarithmic and trigonometric functions

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Conceptual Activities

Use these interactive open education and/or Desmos resources to build conceptual understanding of mathematical ideas. (*Note*: Activity links will be monitored and repaired or replaced as necessary.)

Unit 1

- Illustrative Mathematics. "The Titanic 3."

  http://www.walch.com/ca/10005

  This task poses an open-ended question that forces students to think about how they can answer it. Students will analyze data given in the two-way table to find meaningful probabilities and increase their understanding of conditional probability and independence.

- Illustrative Mathematics. "Should We Send Out a Certificate?"

  http://www.walch.com/ca/10006

  This task allows students to practice calculating normal distributions and further encourages them to draw conclusions from their results based on the properties of normal distributions. Students will communicate their findings in a narrative form within the context of the problem rather that reporting a simple computed number.

Unit 2A

- Desmos. "Constructing Polynomials."

  http://www.walch.com/ca/01054

  In this activity, students will consider properties of polynomial functions such as end behavior, leading terms, and properties of roots. They will explore connections between those properties and the factored forms of the equations of the polynomials.

- Desmos. "Polygraph: Polynomial Functions."

  http://www.walch.com/ca/01055

  This activity is designed to spark vocabulary-rich conversations about polynomial functions. Key vocabulary terms that may appear in student questions include *degree*, *roots*, *end behavior*, *limit*, *quadrant*, *axis*, *increasing*, *decreasing*, *maximum*, *minimum*, *extrema*, *concave up*, and *concave down*.

- Desmos. "Polynomial Equation Challenges."

  http://www.walch.com/ca/01056

  In this activity, students will create polynomial equations (of degree 2, 3, and 4) to match given zeros and points. Students will explore how the factored form of the equations relates to the zeros and the order of those zeros.

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Unit 2B

- Desmos. "Marbleslides: Rationals."

  http://www.walch.com/ca/01057

  In this activity, students will transform rational functions to send marbles through stars.

- Desmos. "Polygraph: Rational Functions."

  http://www.walch.com/ca/01058

  This activity is designed to spark vocabulary-rich conversations about rational functions. Key vocabulary terms that may appear in student questions include *asymptote*, *vertical*, *horizontal*, *quadrant*, *axis*, *increasing*, *decreasing*, *discontinuity*, and *hole*.

Unit 3

- Desmos. "Burning Daylight."

  http://www.walch.com/ca/01059

  In this activity, students use sinusoids to model daylight data for two U.S. cities. They predict which city has more total daylight during a given year, and then use their model to calculate an answer to that question.

- Desmos. "Graphing the Sine Function Using Amplitude, Period, and Vertical Translation."

  http://www.walch.com/ca/01060

  Students will build a visual understanding of amplitude, period, and phase shift in this introduction to trigonometric graphing. They will use this understanding to find models for given graphs of the sine function.

- Desmos. "Marbleslides: Periodics."

  http://www.walch.com/ca/01061

  In this activity, students will transform periodic functions to send marbles through stars.

- Desmos. "Polygraph: Sinusoids."

  http://www.walch.com/ca/01062

  This activity is designed to spark vocabulary-rich conversations about sinusoids. Key vocabulary terms that may appear in student questions include *amplitude*, *periods*, *maximum*, *minimum*, and *shift*.

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- Desmos. "Polygraph: Sinusoids with Vertical Transformations."

  http://www.walch.com/ca/01063

  This activity is designed to spark vocabulary-rich conversations about vertical transformations of sinusoids. Key vocabulary terms that may appear in student questions include *translation*, *dilation*, *amplitude*, *midline*, and *sinusoidal axis*.

- Desmos. "Trigonometric Graphing: Introduction to Amplitude and Vertical Shift."

  http://www.walch.com/ca/01064

  In this activity, students will informally explore range, midline, and amplitude of trigonometric functions. They'll use what they learn about the relationships to write equations of sine and cosine graphs.

Unit 4A

- Desmos. "Polygraph: Exponential & Logarithmic Functions."

  http://www.walch.com/ca/01045

  This activity is designed to spark vocabulary-rich conversations about exponential and logarithmic functions. Key vocabulary terms that may appear in student questions include *exponential*, *asymptote*, *logarithmic*, and *quadrant*.

Unit 4B

- Desmos. "Card Sort: Exponentials."

  http://www.walch.com/ca/01044

  In this activity, students practice what they've learned about exponential functions by matching equations to properties of the graphs the functions will produce. Students will then use their knowledge of transforming exponential functions to pair equations with graphs.

- Desmos. "Polygraph: Exponential Functions."

  http://www.walch.com/ca/01046

  This activity is designed to spark vocabulary-rich conversations about exponential functions. Key vocabulary terms that may appear in student questions include *increasing*, *decreasing*, *asymptote*, *quadrant*, and *axis*.

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- Desmos. "Polygraph: Parent Functions."

  http://www.walch.com/ca/01051

  This activity is designed to spark vocabulary-rich conversations about graphs of parent functions. Key vocabulary terms that may appear in student questions include *increasing*, *decreasing*, *linear*, *quadratic*, *cubic*, *absolute value*, *exponential*, *logarithmic*, *rational*, *radical*, *axis*, *intercept*, and *coordinate*.

- Desmos. "Polygraph: Twelve Functions."

  http://www.walch.com/ca/01053

  This activity is designed to spark vocabulary-rich conversations about various functions. Key vocabulary terms that may appear in student questions include *linear*, *quadratic*, *exponential*, *cubic*, *absolute value*, *rational*, *radical*, *sinusoid*, and *step*.

- Desmos. "Writing Rules: Linear, Quadratic, and Exponential."

  http://www.walch.com/ca/01047

  In this activity, students have an opportunity to deepen their understanding of linear, quadratic, and exponential functions by making connections between their tables, graphs, and equations.

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Station Activities Guide

Introduction

Each unit includes a collection of station-based activities to provide students with opportunities to practice and apply the mathematical skills and concepts they are learning. You may use these activities in addition to the instructional lessons, or, especially if the pre-test or other formative assessment results suggest it, instead of direct instruction in areas where students have the basic concepts but need practice. The debriefing discussions after each set of activities provide an important opportunity to help students reflect on their experiences and synthesize their thinking. Debriefing also provides an additional opportunity for ongoing, informal assessment to guide instructional planning.

Implementation Guide

The following guidelines will help you prepare for and use the activity sets in this section.

Setting Up the Stations

Each activity set consists of four or five stations. Set up each station at a desk, or at several desks pushed together, with enough chairs for a small group of students. Place a card with the number of the station on the desk. Each station should also contain the materials specified in the teacher's notes, and a stack of student activity sheets (one copy per student). Place the required materials (as listed) at each station.

When a group of students arrives at a station, each student should take one of the activity sheets to record the group's work. Although students should work together to develop one set of answers for the entire group, each student should record the answers on his or her own activity sheet. This helps keep students engaged in the activity and gives each student a record of the activity for future reference.

Forming Groups of Students

All activity sets consist of four or five stations. You might divide the class into four or five groups by having students count off from 1 to 4 or 5. If you have a large class and want to have students working in small groups, you might set up two identical sets of stations, labeled A and B. In this way, the class can be divided into eight groups, with each group of students rotating through the "A" stations or "B" stations.

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Assigning Roles to Students

Students often work most productively in groups when each student has an assigned role. You may want to assign roles to students when they are assigned to groups and change the roles occasionally. Some possible roles are as follows:

- Reader—reads the steps of the activity aloud
- Facilitator—makes sure that each student in the group has a chance to speak and pose questions; also makes sure that each student agrees on each answer before it is written down
- Materials Manager—handles the materials at the station and makes sure the materials are put back in place at the end of the activity
- Timekeeper—tracks the group's progress to ensure that the activity is completed in the allotted time
- Spokesperson—speaks for the group during the debriefing session after the activities

Timing the Activities

The activities in this section are designed to take approximately 10 minutes per station. Therefore, you might plan on having groups change stations every 10 minutes, with a two-minute interval for moving from one station to the next. It is helpful to give students a "5-minute warning" before it is time to change stations.

Since each activity set consists of four or five stations, the above time frame means that it will take about 50 to 60 minutes for groups to work through all stations.

Guidelines for Students

Before starting the first activity set, you may want to review the following "ground rules" with students. You might also post the rules in the classroom.

- All students in a group should agree on each answer before it is written down. If there is a disagreement within the group, discuss it with one another.
- You can ask your teacher a question only if everyone in the group has the same question.
- If you finish early, work together to write problems of your own that are similar to the ones on the activity sheet.
- Leave the station exactly as you found it. All materials should be in the same place and in the same condition as when you arrived.

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Debriefing the Activities

After each group has rotated through every station, bring students together for a brief class discussion. At this time, you might have the groups' spokespersons pose any questions they had about the activities. Before responding, ask if students in other groups encountered the same difficulty or if they have a response to the question. The class discussion is also a good time to reinforce the essential ideas of the activities. The questions that are provided in the teacher's notes for each activity set can serve as a guide to initiating this type of discussion.

You may want to collect the student activity sheets before beginning the class discussion. However, it can be beneficial to collect the sheets afterward so that students can refer to them during the discussion. This also gives students a chance to revisit and refine their work based on the debriefing session. If you run out of time to hold class discussions, you might want to have students journal about their experiences and follow up with a class discussion the next day.

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Digital Enhancements Guide

Introduction

With the *Common Core State Standards Integrated Pathway: Mathematics III Program*, you have access to the following digital components, described here with guidelines and suggestions for implementation.

Digital Warm-Ups

Each of the warm-ups in this collection includes a video clip to enhance student engagement. Hard copies of all warm-ups are also included in the Teacher Resource. These optional versions of the warm-ups and debriefs run on PowerPoint. (*Please note*: Computers may render PowerPoint images differently. For best viewing and display, use a PowerPoint Viewer and adjust your settings to optimize images and text.)

We suggest that the video clip be played as students enter the class. The green "play" button on the first PowerPoint slide will link your computer to a browser and then to the video, so you will need an Internet connection. Make sure that you use the "full screen" option for display and adjust the volume appropriately. The video clips have been selected to catch students' attention by addressing the context (not the content) of the warm-up problem in an engaging way.

After the video, each warm-up takes approximately 5 minutes and addresses critical-thinking skills and/or previously taught math concepts that are prerequisites for the new content in the lesson.

The warm-ups include debriefs that reveal the answers to the warm-up questions. The answers slide into view and can be used to facilitate discussion.

In the notes section, you will find the URL for the video and "Connections to the Lesson," which describes concepts students will glean or skills they will need in the upcoming lesson. The "Connections" help transition from the warm-up to instruction.

GeoGebra Applets

In addition to the hard copy materials found in the Teacher Resource, two interactive GeoGebra applets are provided for most lessons. The applets model the mathematics in two of the Guided Practice examples. Links to all of the applets can be found on the flash drive for this program. With an Internet connection, simply click on the link identified for the lesson and example.

Please adjust your view to maximize the image. Each applet illustrates the specific problem addressed in the Guided Practice example. The applets allow you to walk through the solution by visually demonstrating the steps, such as defining points and drawing lines. Variable components of the applets (usually fill-in boxes or sliders) allow you to substitute different values in order to explore

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the mathematics. For example, "What happens to the line when we increase the amount of time?" or "What if we cut the number of students in half?" This experimentation and discussion supports development of conceptual understanding.

GeoGebra for PC/MAC

GeoGebra is not required for using the applets, but can be downloaded for free for further exploration at the following link:

http://www.geogebra.org/cms/en/download

GeoGebra Applet Troubleshooting

If you are experiencing any difficulty in using the applets in your browser, please visit the following link for our troubleshooting document.

http://www.walch.com/applethelp

ExamView Assessment Banks

These are available with sufficient items to compile up to 5 versions of each pre-, progress, and unit assessment in the program. ExamView items may also be used to generate additional worksheets for practice.

For information about ExamView, please contact Customer Service at (800) 341-6094 or customerservice@walch.com.

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Standards for Mathematical Practice Implementation Guide

Introduction

The eight Standards for Mathematical Practice describe features of lesson design, teaching pedagogy, and student actions that will lead to a true conceptual understanding of the mathematics standards. The Walch lessons, practice problems, and Problem-Based Tasks lend themselves to teaching through this framework. When the Walch resources are combined with high-level questioning and engaging teacher decisions in the classroom, it will lead to high-level math instruction and student achievement.

Here is a brief description of the SMPs and how they can be applied in the classroom:

CCSS.MP.1: Make sense of problems and persevere in solving them.

Students will read, interpret, and understand complicated mathematical and real-world problems, and they will be willing to try multiple methods with the ultimate goal of determining the correct answer. Strategies such as annotation and student discourse can lead to improvement on this standard. Presenting students with higher-level problems is essential to ensuring students achieve maximum understanding. Teacher prompts that can enhance this standard include:

- What is the problem asking you to solve?
- What are some (other) strategies you could use to solve this problem?
- Compare your answer with a classmate's answer. Who is correct? Why?

CCSS.MP.2: Reason abstractly and quantitatively.

Mathematical reasoning with numbers and variables is essential to understanding the connections among the standards. Students must be able to discover and formalize general rules using numbers and variables, and apply them to determine numerical quantities in other situations. Teacher prompts that can enhance this standard include:

- Substitute realistic numbers into the situation.
- What operation/strategy would you use?
- Will your strategy work for any number?
- For which categories of numbers (negative integers, all real numbers, etc.) will your strategy work?

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CCSS.MP.3: Construct viable arguments and critique the reasoning of others.

Many students are most concerned with the "what" aspects of mathematics, i.e. "what" do we do or "what" is the answer. However, math educators must develop the "why" of mathematics. Students must learn to question algorithms, challenge answers, and justify their reasoning in order to truly understand the concepts behind their answers. Teacher prompts that can enhance this standard include:

- How did you determine your answer?
- Why did you choose that strategy?
- Defend your answer based on a real-world situation.

CCSS.MP.4: Model with mathematics.

An important goal of mathematics instruction is for students to be able to apply mathematics to the world around them. Students should be able to link a real problem to a mathematical concept, identify quantities that are modeled well with mathematics, and use mathematics to find a solution. Emphasizing this standard will help students represent and interpret information using physical, visual, and abstract models. Encourage students to use any or all of their learning experiences to gain a deep and flexible understanding of mathematics. Teacher prompts that can enhance this standard include:

- Can you represent this situation with a visual model?
- How will it help you solve the problem?
- What information is needed to solve this problem?
- Is there another way to solve this problem?
- While working to solve this problem, what do you notice/wonder?

CCSS.MP.5: Use appropriate tools strategically.

There are many available tools suitable for mathematics, such as calculators, manipulatives, formulas, rulers, computers, and developed mathematical strategies. Choosing and using the correct tool to work through a problem is an important skill for mathematicians. Teacher prompts that can enhance this standard include:

- Can you graph this equation in the calculator to see a relationship?
- What formula or strategy might help you determine the answer to this question?
- How can you represent the situation using handheld tools (rulers, protractors, etc.) to determine an answer?

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CCSS.MP.6: Attend to precision.

When using mathematics to solve problems, an answer can be considered correct only if it is sufficiently precise and accurate for the situation to which it pertains. When applying mathematics, it is vital to clearly define the question, the reasoning, the answer, and the explanation. Vocabulary, units, numerical responses, and pictures must be represented precisely in questions and answers to ensure that the mathematical solutions represent the true answer to a question. Teacher prompts that can enhance this standard include:

- What does your answer represent in a real-world context?
- Is your answer reasonable based on your initial estimate?
- What units of measure help describe your numerical answer?

CCSS.MP.7: Look for and make use of structure.

Structure, whether geometric, algebraic, statistical, or numerical, is an important aspect of mathematical reasoning that students often overlook. Teachers often explicitly refer to geometric and other visual structures as explanations of mathematical concepts, but algebraic and numerical structures can often be just as important in analyzing and interpreting mathematical situations. These structures yield clues as to the meaning of expressions, equations, graphs, and other representations. As students interpret these structures, they will gain a greater understanding of the mathematical concepts. Teacher prompts that can enhance this standard include:

- What do the characteristics of the graph tell us about the situation?
- What do each of the variables and numbers in the equation/formula represent?
- How are these situations the same and different based on their representations?

CCSS.MP.8: Look for and express regularity in repeated reasoning.

Just as patterns appear in real life, patterns appear throughout the subject of mathematics. Recognizing and applying these patterns, and applying the reasoning contained within, is one of the most important skills teachers can instill in their students. Rather than teaching isolated algorithms to determine answers, have students discover relationships, create their own algorithms, and apply the reasoning to other situations. These skills can be applied throughout their education and will enrich their lives after high school. Teacher prompts that can enhance this standard include:

- What relationship do you notice in the graph/table/numbers?
- Why did you choose to use this process to solve this word problem/equation?
- How can you apply this process in other situations?

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z-scores Table

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

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z-scores Table

z	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00
–3.4	0.0002	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
–3.3	0.0003	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0005	0.0005	0.0005
–3.2	0.0005	0.0005	0.0005	0.0006	0.0006	0.0006	0.0006	0.0006	0.0007	0.0007
–3.1	0.0007	0.0007	0.0008	0.0008	0.0008	0.0008	0.0009	0.0009	0.0009	0.0010
–3.0	0.0010	0.0010	0.0011	0.0011	0.0011	0.0012	0.0012	0.0013	0.0013	0.0013
–2.9	0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019
–2.8	0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026
–2.7	0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035
–2.6	0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047
–2.5	0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062
–2.4	0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082
–2.3	0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107
–2.2	0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139
–2.1	0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179
–2.0	0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228
–1.9	0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287
–1.8	0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359
–1.7	0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446
–1.6	0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548
–1.5	0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668
–1.4	0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793	0.0808
–1.3	0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951	0.0968
–1.2	0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131	0.1151
–1.1	0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335	0.1357
–1.0	0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562	0.1587
–0.9	0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814	0.1841
–0.8	0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090	0.2119
–0.7	0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389	0.2420
–0.6	0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709	0.2743
–0.5	0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050	0.3085
–0.4	0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409	0.3446
–0.3	0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783	0.3821
–0.2	0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168	0.4207
–0.1	0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562	0.4602
–0.0	0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960	0.5000

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t-distribution Table

One-tailed	0.50	0.25	0.20	0.15	0.10	0.05	0.025	0.01	0.005	0.001	0.0005
Two-tailed	1.00	0.50	0.40	0.30	0.20	0.10	0.05	0.02	0.01	0.002	0.001
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.000	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
Confidence level	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%

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Suggested Pacing Guide

Overview

The sub-lessons in Walch Integrated Math are designed to be covered over one, two, or three days, depending on the depth and complexity of the topic. The pacing guide that follows provides *suggested* pacing, but this should be viewed as a flexible guideline only. If students need an extra day to solidify understanding, a one-day lesson may be spread over two days, or a two-day lesson may be spread over three days. Conversely, if students grasp a concept more quickly than suggested by the pacing guide, it is fine to shorten the duration of the lesson and move on to the next topic.

Each sub-lesson features several components. The estimated time for each component is given in parentheses. Components include: a Warm-Up, Instruction (including an Introduction, Key Concepts, and Guided Practice), a Problem-Based Task, and Practice.

Please note: PowerPoint versions of each sub-lesson's Warm-Up and Instruction are provided. Warm-Up PowerPoints include a video clip and the Warm-Up Debrief. Instructional PowerPoints feature the Introduction, Key Concepts, and selected Guided Practice Examples with links to corresponding GeoGebra applets. Estimated times reflect presentation of these PowerPoint versions and applets.

Guided Practice, Scaffolded Practice, and Practice

For each Guided Practice in the Teacher Resource (TRB), corresponding Scaffolded Practice worksheets are provided in the Student Workbook (SWB). Practice problems may be assigned from the SWB and/or the parallel Student Resource (SRB) worksheet for in-class work or homework assignments. The SRB worksheets are designed to closely resemble the format of the SWB worksheets, but are composed of different problems for additional practice.

Please note: When a sub-lesson is spread across more than one class period, the Guided Practice Examples will also be spread across the same number of class periods. Assign Practice problems that correspond to the Guided Practice Example(s) you select for each day, to ensure appropriate prior instruction.

Suggested Progression of One-, Two-, and Three-Day Sub-Lessons

One Day

- Warm-Up PowerPoint and Debrief (10 minutes)
- Introduction and Key Concepts (10 minutes)
- One or two Guided/Scaffolded Practice Examples, including PowerPoint applet presentation as appropriate (15 minutes)
- Problem-Based Task and discussion (15 minutes)
- Homework: Practice from the SWB or SRB

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Two Days

Day 1

- Warm-Up PowerPoint and Debrief (10 minutes)
- Introduction and Key Concepts (15 minutes)
- Two or three Guided/Scaffolded Practice Examples, including PowerPoint applet presentation as appropriate (15 minutes)
- In-class practice: selected problems from the SWB Practice (10 minutes)
- Homework: selected problems from the SRB Practice

Day 2

- Review Practice problems (10 minutes)
- One or two remaining Guided/Scaffolded Practice Examples, including PowerPoint applet presentation as appropriate (10 minutes)
- Problem-Based Task and discussion (20 minutes)
- In-class practice: the remainder of the SWB Practice (10 minutes)
- Homework: the remainder of the SRB Practice

Three Days

Day 1

- Warm-Up PowerPoint and Debrief (10 minutes)
- Introduction and Key Concepts (25 minutes)
- One or two Guided/Scaffolded Practice Examples, including PowerPoint applet presentation as appropriate (15 minutes)
- Homework: selected problems from the SRB Practice

Day 2

- Review Practice problems (15 minutes)
- Two or three remaining Guided/Scaffolded Practice Examples, including PowerPoint applet presentation as appropriate (15 minutes)
- Problem-Based Task and discussion (20 minutes)
- Homework: the remainder of the SRB Practice

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Day 3

- Review Practice problems (15 minutes)
- In-class practice: selected problems from the SWB Practice (35 minutes)
- Homework: the remainder of the SWB Practice

Assessments and Station Activities

Each lesson includes a Pre-Assessment and a Progress Assessment, and each unit concludes with a Unit Assessment. Units also feature sets of Station Activities to complement instruction.

Pre-Assessments are short, multiple-choice assessments with five problems, designed to evaluate prior knowledge of the upcoming lesson. These are brief and should not factor into a student's grade. Pre-Assessments are often given on the same day as the preceding lesson's Progress Assessment, to prepare for the upcoming lesson.

Progress Assessments include 10 multiple-choice problems and one extended-response problem. Progress Assessments may not take an entire class period. The additional time may be used to review before the assessment, or to begin the next lesson after the assessment.

Unit Assessments include 12 multiple-choice problems and three extended-response problems, and generally require a full class period.

Station Activities generally require a full class period for the students to rotate through each station and then to engage in a class discussion at the end.

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Unit 1: Inferences and Conclusions from Data (42 Days)

Unit Overview: Students learn about using the normal curve, as well as about populations versus random samples and random sampling. Then, they learn about strategies for collecting data, including surveys, experiments, and observational studies. Students estimate sample proportions and sample means and develop tools for comparing treatments and reading reports. Finally, they look at making and analyzing decisions with data.

Lesson 1: Using the Normal Curve

Students explore the concept of probability distribution, and especially the normal distribution and uniform distribution. They are also introduced to standard deviation. Next, students look more closely at normal distributions and learn how to find the *z*-score. Finally, students learn three methods for determining whether a given situation follows a normal distribution.

Day	Area of study/content	Pages	Standard(s)
Day 1 Day 2	Lesson 1 Pre-Assessment 1.1.1: Normal Distributions and the 68–95–99.7 Rule	**Pre-Assessment** TRB: U1-1 **Instruction/Practice** TRB: U1-6 SRB: U1-5 SWB: 11	S–ID.4★
Day 3 Day 4	1.1.2: Standard Normal Calculations	TRB: U1-33 SRB: U1-23 SWB: 21	S–ID.4★
Day 5	1.1.3: Assessing Normality	TRB: U1-60 SRB: U1-40 SWB: 31	S–ID.4★
Day 6	Station Activities Set 1: *z*-scores	TRB: U1-465 SWB: 203	S–ID.4★
Day 7	Lesson 1 Progress Assessment Lesson 2 Pre-Assessment	TRB: U1-91 TRB: U1-94	

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Lesson 2: Populations Versus Random Samples and Random Sampling

This lesson begins with an in-depth look at the differences between populations and samples, as well as how to determine the reliability and validity of a sample. Then, students explore simple random sampling and sampling bias. The lesson ends with a look at several other methods of sampling, such as cluster sampling, systematic sampling, and stratified sampling.

Day	Area of study/content	Pages	Standard(s)
Day 8 Day 9	1.2.1: Differences Between Populations and Samples	TRB: U1-99 SRB: U1-67 SWB: 45	S–IC.1★
Day 10 Day 11	1.2.2: Simple Random Sampling	TRB: U1-125 SRB: U1-82 SWB: 59	S–IC.2★
Day 12 Day 13	1.2.3: Other Methods of Random Sampling	TRB: U1-156 SRB: U1-102 SWB: 73	S–IC.2★
Day 14	Lesson 2 Progress Assessment Lesson 3 Pre-Assessment	TRB: U1-184 TRB: U1-188	

Lesson 3: Surveys, Experiments, and Observational Studies

This lesson focuses on various ways of collecting data. First, students look at the diverse ways that data can be collected. Then, they investigate more closely the steps needed to design a study so that the outcomes are reliable and relevant to the topic being studied.

Day	Area of study/content	Pages	Standard(s)
Day 15 Day 16	1.3.1: Identifying Surveys, Experiments, and Observational Studies	TRB: U1-193 SRB: U1-125 SWB: 91	S–IC.3★
Day 17 Day 18	1.3.2: Designing Surveys, Experiments, and Observational Studies	TRB: U1-211 SRB: U1-136 SWB: 101	S–IC.3★
Day 19	Lesson 3 Progress Assessment Lesson 4 Pre-Assessment	TRB: U1-230 TRB: U1-233	

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Lesson 4: Estimating Sample Proportions and Sample Means

Students learn about sample proportions and the standard error for the proportion. They explore binomial experiments and binomial distributions, as well as the addition rule for mutually exclusive events. Then, they study sample means and their relationship to population means, and take a look at the margin of error and the confidence level. Finally, students investigate how to calculate the probability that a given value falls within a specified interval.

Day	Area of study/content	Pages	Standard(s)
Day 20 Day 21	1.4.1: Estimating Sample Proportions	TRB: U1-240 SRB: U1-152 SWB: 111	S–IC.4★
Day 22 Day 23	1.4.2: The Binomial Distribution	TRB: U1-260 SRB: U1-160 SWB: 121	S–IC.4★
Day 24 Day 25	1.4.3: Estimating Sample Means	TRB: U1-287 SRB: U1-179 SWB: 131	S–IC.4★
Day 26 Day 27	1.4.4: Estimating with Confidence	TRB: U1-300 SRB: U1-185 SWB: 141	S–IC.4★
Day 28	Station Activities Set 2: Distributions and Estimating with Confidence	TRB: U1-476 SWB: 209	S–ID.4★ S–IC.4★
Day 29	Lesson 4 Progress Assessment Lesson 5 Pre-Assessment	TRB: U1-320 TRB: U1-323	

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Lesson 5: Comparing Treatments and Reading Reports

Students learn what a treatment is and how to evaluate the confidence level of a given treatment. They explore how to use a *t*-test to establish the statistical significance of a set of data. Next, they investigate the concept of simulations and how to design an accurate simulation. Finally, students learn how to question the validity of the conclusions and assumptions made based on reported data.

Day	Area of study/content	Pages	Standard(s)
Day 30 Day 31	1.5.1: Evaluating Treatments	TRB: U1-328 SRB: U1-200 SWB: 151	S–IC.5★
Day 32 Day 33	1.5.2: Designing and Simulating Treatments	TRB: U1-355 SRB: U1-219 SWB: 161	S–IC.5★
Day 34 Day 35	1.5.3: Reading Reports	TRB: U1-374 SRB: U1-230 SWB: 171	S–IC.6★
Day 36	Lesson 5 Progress Assessment Lesson 6 Pre-Assessment	TRB: U1-391 TRB: U1-394	

Lesson 6: Making and Analyzing Decisions

Students explore ways of making decisions based on the probability of certain events occurring, and they learn how to calculate the expected value. They then focus on analyzing the accuracy and fairness of situations and outcomes.

Day	Area of study/content	Pages	Standard(s)
Day 37 Day 38	1.6.1: Making Decisions	TRB: U1-399 SRB: U1-243 SWB: 181	S–MD.6★ (+)
Day 39 Day 40	1.6.2: Analyzing Decisions	TRB: U1-418 SRB: U1-256 SWB: 193	S–MD.7★ (+)
Day 41	Lesson 6 Progress Assessment	TRB: U1-442	
Unit Assessment			
Day 42	Unit Assessment	TRB: U1-445	

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Unit 2A: Polynomial Relationships (34 Days)

Unit Overview: Students begin by exploring polynomial structures and operating with polynomials. Then, they learn how to prove identities, graph polynomial functions, and solve systems of equations with polynomials. Finally, students learn about geometric series.

Lesson 1: Polynomial Structures and Operating with Polynomials

Students review the vocabulary used when discussing polynomials. They practice adding, subtracting, and multiplying polynomials and simplifying the results.

Day	Area of study/content	Pages	Standard(s)
Day 1	Lesson 1 Pre-Assessment 2A.1.1: Structures of Expressions	**Pre-Assessment** TRB: U2A-1 **Instruction/Practice** TRB: U2A-4 SRB: U2A-3 SWB: 7	A–SSE.1a★
Day 2	2A.1.2: Adding and Subtracting Polynomials	TRB: U2A-16 SRB: U2A-10 SWB: 17	A–APR.1
Day 3 Day 4	2A.1.3: Multiplying Polynomials	TRB: U2A-26 SRB: U2A-15 SWB: 27	A–APR.1
Day 5	Lesson 1 Progress Assessment Lesson 2 Pre-Assessment	TRB: U2A-37 TRB: U2A-39	

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Lesson 2: Proving Identities

Students begin this lesson by exploring common polynomial identities, such as the difference of two squares and the square of sums. Then they expand their investigation to include complex numbers in the polynomial identities. They end the lesson with an exploration of the Binomial Theorem and a look at its usefulness when finding binomial expansions.

Day	Area of study/content	Pages	Standard(s)
Day 6 Day 7	2A.2.1: Polynomial Identities	TRB: U2A-43 SRB: U2A-24 SWB: 37	A–SSE.1b★ A–SSE.2 A–APR.4
Day 8 Day 9	2A.2.2: Complex Polynomial Identities	TRB: U2A-59 SRB: U2A-33 SWB: 47	N–CN.8 (+) A–SSE.1b★ A–SSE.2 A–APR.4
Day 10 Day 11	2A.2.3: The Binomial Theorem	TRB: U2A-72 SRB: U2A-38 SWB: 57	A–SSE.1a★ A–SSE.1b★ A–SSE.2 A–APR.4 A–APR.5 (+)
Day 12	Lesson 2 Progress Assessment Lesson 3 Pre-Assessment	TRB: U2A-88 TRB: U2A-90	

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Lesson 3: Graphing Polynomial Functions

Students begin this lesson by analyzing the end behavior and number of turns of the graphs of polynomial functions. Then they learn synthetic division of polynomials, and apply this to the Remainder Theorem. Their study of polynomials continues with an exploration of how to find the zeros of a polynomial function. Lastly, students learn the Rational Root Theorem as a means of locating the zeros of polynomials that are difficult to factor.

Day	Area of study/content	Pages	Standard(s)
Day 13 Day 14	2A.3.1: Describing End Behavior and Turns	TRB: U2A-96 SRB: U2A-52 SWB: 67	F–IF.7c★
Day 15 Day 16	2A.3.2: The Remainder Theorem	TRB: U2A-115 SRB: U2A-66 SWB: 77	A–APR.2
Day 17 Day 18	2A.3.3: Finding Zeros	TRB: U2A-135 SRB: U2A-79 SWB: 89	A–APR.3 N–CN.9 (+) F–IF.7c★
Day 19 Day 20	2A.3.4: The Rational Root Theorem	TRB: U2A-153 SRB: U2A-89 SWB: 99	A–APR.3
Day 21	Station Activities Set 1: Polynomial Functions	TRB: U2A-305 SWB: 151	N–CN.9 (+) A–SSE.2 A–APR.2 A–APR.3 F–IF.2 F–IF.7c★
Day 22	Lesson 3 Progress Assessment Lesson 4 Pre-Assessment	TRB: U2A-169 TRB: U2A-173	

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Lesson 4: Solving Systems of Equations with Polynomials

This brief lesson focuses on how to find the solution(s) to a system of equations graphically when at least one of the equations is a polynomial. The systems of equations in this lesson also include absolute value equations and exponential equations.

Day	Area of study/content	Pages	Standard(s)
Day 23 Day 24	2A.4.1: Solving Systems of Equations Graphically	TRB: U2A-178 SRB: U2A-98 SWB: 109	A–REI.11★
Day 25	Lesson 4 Progress Assessment Lesson 5 Pre-Assessment	TRB: U2A-206 TRB: U2A-212	

Lesson 5: Geometric Series

This lesson provides an in-depth look at geometric series, beginning with the associated vocabulary. Students learn the difference between recursive and explicit formulas. Next, students explore how to find the sum of a finite geometric series, followed by how to find the sum of an infinite geometric series.

Day	Area of study/content	Pages	Standard(s)
Day 26 Day 27	2A.5.1: Geometric Sequences	TRB: U2A-217 SRB: U2A-121 SWB: 121	A–SSE.4★
Day 28 Day 29	2A.5.2: Sum of a Finite Geometric Series	TRB: U2A-239 SRB: U2A-136 SWB: 131	A–SSE.4★
Day 30 Day 31	2A.5.3: Sum of an Infinite Geometric Series	TRB: U2A-267 SRB: U2A-152 SWB: 141	A–SSE.4★
Day 32	Station Activities Set 2: Sequences and Series	TRB: U2A-318 SWB: 157	A–SSE.4★
Day 33	Lesson 5 Progress Assessment	TRB: U2A-287	
Unit Assessment			
Day 34	Unit Assessment	TRB: U2A-290	

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Unit 2B: Rational and Radical Relationships (18 Days)

Unit Overview: Students work with operations involving rational expressions. Then, they learn to solve rational and radical equations.

Lesson 1: Operating with Rational Expressions

Students learn the definition of a rational expression and how to simplify rational expressions. Then, they focus on adding and subtracting rational expressions, followed by multiplying and dividing them.

Day	Area of study/content	Pages	Standard(s)
Day 1	Lesson 1 Pre-Assessment 2B.1.1: Structures of Rational Expressions	**Pre-Assessment** TRB: U2B-1 **Instruction/Practice** TRB: U2B-5 SRB: U2B-4 SWB: 7	A–SSE.1a★ A–SSE.1b★ A–SSE.2
Day 2 Day 3	2B.1.2: Adding and Subtracting Rational Expressions	TRB: U2B-23 SRB: U2B-13 SWB: 17	A–APR.7 (+) A–SSE.2
Day 4 Day 5	2B.1.3: Multiplying Rational Expressions	TRB: U2B-45 SRB: U2B-27 SWB: 27	A–APR.7 (+) A–SSE.2
Day 6 Day 7	2B.1.4: Dividing Rational Expressions	TRB: U2B-61 SRB: U2B-35 SWB: 37	A–APR.6 A–APR.7 (+) A–SSE.2
Day 8	Lesson 1 Progress Assessment Lesson 2 Pre-Assessment	TRB: U2B-83 TRB: U2B-86	

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Lesson 2: Solving Rational and Radical Equations

Students first learn how to solve rational equations, and then how to solve radical equations. The lesson ends with students solving systems of equations algebraically, including polynomial, radical, and rational equations.

Day	Area of study/content	Pages	Standard(s)
Day 9 Day 10	2B.2.1: Solving Rational Equations	TRB: U2B-90 SRB: U2B-53 SWB: 47	A–REI.2
Day 11	Station Activities Set 1: Rational Expressions and Equations	TRB: U2B-171 SWB: 77	A–SSE.2 A–APR.6 A–APR.7 (+) A–REI.2
Day 12 Day 13	2B.2.2: Solving Radical Equations	TRB: U2B-110 SRB: U2B-65 SWB: 57	A–REI.2
Day 14 Day 15	2B.2.3: Solving Systems of Equations	TRB: U2B-130 SRB: U2B-78 SWB: 67	A–REI.11★
Day 16	Station Activities Set 2: Solving Systems of Equations	TRB: U2B-188 SWB: 83	A–REI.11★
Day 17	Lesson 2 Progress Assessment	TRB: U2B-158	
Unit/Comprehensive Assessment			
Day 18	Unit Assessment (covering Unit 2B) and/or Comprehensive Unit Assessment (covering Units 2A and 2B)	**Unit Assessment** TRB: U2B-161 **Comprehensive Unit Assessment** TRB: U2-CA1	

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Unit 3: Trigonometry of General Triangles and Trigonometric Functions (24 Days)

Unit Overview: Students begin the unit with a study of radians and the unit circle. Then, they explore the trigonometry of general angles, including the Law of Sines and the Law of Cosines. Finally, students move on to modeling periodic phenomena with the graphs of the sine and cosine functions.

Lesson 1: Radians and the Unit Circle

Students explore the meaning of radian measure, as well as the relationship between radians, radius, and arc length measurements on a circle. They also learn to convert from radians to degrees and vice versa. Then, students learn about the unit circle, how to calculate distances using the three basic trigonometric functions, and how to find reference angles. This is followed by the exploration of the special angles on the unit circle. Finally, students learn how to evaluate trigonometric functions with the help of the unit circle.

Day	Area of study/content	Pages	Standard(s)
Day 1	Lesson 1 Pre-Assessment 3.1.1: Radians	**Pre-Assessment** TRB: U3-1 **Instruction/Practice** TRB: U3-5 SRB: U3-4 SWB: 7	F–TF.1 F–TF.2
Day 2 Day 3	3.1.2: The Unit Circle	TRB: U3-21 SRB: U3-13 SWB: 17	F–TF.2
Day 4 Day 5	3.1.3: Special Angles in the Unit Circle	TRB: U3-45 SRB: U3-29 SWB: 29	F–TF.2
Day 6 Day 7	3.1.4: Evaluating Trigonometric Functions	TRB: U3-69 SRB: U3-43 SWB: 39	F–TF.2
Day 8	Station Activities Set 1: Trigonometric Functions	TRB: U3-219 SWB: 107	F–TF.2
Day 9	Lesson 1 Progress Assessment Lesson 2 Pre-Assessment	TRB: U3-91 TRB: U3-94	

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Lesson 2: Trigonometry of General Angles

Students discover how to find the measures of missing side lengths and angle measurements if the given triangle is *not* a right triangle, by using the Law of Sines and the Law of Cosines. After learning these two useful formulas, students use the Law of Sines and the Law of Cosines to solve problems.

Day	Area of study/content	Pages	Standard(s)
Day 10 Day 11	3.2.1: Proving the Law of Sines	TRB: U3-99 SRB: U3-61 SWB: 49	G–SRT.9 (+) G–SRT.10 (+)
Day 12 Day 13	3.2.2: Proving the Law of Cosines	TRB: U3-123 SRB: U3-76 SWB: 63	G–SRT.10 (+)
Day 14 Day 15	3.2.3: Applying the Laws of Sines and Cosines	TRB: U3-142 SRB: U3-89 SWB: 75	G–SRT.11 (+)
Day 16	Station Activities Set 2: The Laws of Sines and Cosines	TRB: U3-236 SWB: 117	G–SRT.9 (+) G–SRT.10 (+) G–SRT.11 (+)
Day 17	Lesson 2 Progress Assessment Lesson 3 Pre-Assessment	TRB: U3-160 TRB: U3-163	

Lesson 3: Graphs of Trigonometric Functions

Students begin studying the periodic nature of trigonometric functions by looking at the graphs of the sine, cosine, and tangent functions. This includes analyzing the period, frequency, amplitude, and midline of these functions.

Day	Area of study/content	Pages	Standard(s)
Day 18 Day 19	3.3.1: Periodic Phenomena and Amplitude, Frequency, and Midline	TRB: U3-167 SRB: U3-103 SWB: 85	F–TF.5★
Day 20 Day 21 Day 22	3.3.2: Using Trigonometric Functions to Model Periodic Phenomena	TRB: U3-185 SRB: U3-114 SWB: 95	F–TF.5★
Day 23	Lesson 3 Progress Assessment	TRB: U3-204	
Unit Assessment			
Day 24	Unit Assessment	TRB: U3-208	

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Unit 4A: Mathematical Modeling of Inverse, Logarithmic, and Trigonometric Functions (23 Days)

Unit Overview: Students begin by learning about the inverses of quadratics and other functions. This leads to learning about graphing and interpreting logarithmic functions and models. Students continue exploring and applying the graphs of the sine and cosine functions learned previously.

Lesson 1: Inverses of Functions

Students learn how to find the inverse of a quadratic function, and then how to find the inverse of other functions, such as rational and radical functions.

Day	Area of study/content	Pages	Standard(s)
Day 1 Day 2	Lesson 1 Pre-Assessment 4A.1.1: Determining Inverses of Quadratic Functions	**Pre-Assessment** TRB: U4A-1 **Instruction/Practice** TRB: U4A-5 SRB: U4A-3 SWB: 7	F–BF.4a
Day 3 Day 4	4A.1.2: Determining Inverses of Other Functions	TRB: U4A-36 SRB: U4A-26 SWB: 17	F–BF.4a
Day 5	Station Activities Set 1: Inverse Functions	TRB: U4A-291 SWB: 105	F–BF.4a
Day 6	Lesson 1 Progress Assessment Lesson 2 Pre-Assessment	TRB: U4A-65 TRB: U4A-69	

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Lesson 2: Modeling Logarithmic Functions

Students are introduced to logarithmic functions, and they explore logarithmic functions as inverses of exponential functions. Next, students investigate common logarithms and natural logarithms. This is followed by a look at the graphs of logarithmic functions. Finally, students learn how analyze and interpret situations that can be modeled with logarithmic functions.

Day	Area of study/content	Pages	Standard(s)
Day 7 Day 8	4A.2.1: Logarithmic Functions as Inverses	TRB: U4A-74 SRB: U4A-48 SWB: 27	F–BF.4a F–LE.4★
Day 9 Day 10	4A.2.2: Common Logarithms	TRB: U4A-96 SRB: U4A-62 SWB: 37	F–IF.8 F–LE.4★
Day 11 Day 12	4A.2.3: Natural Logarithms	TRB: U4A-114 SRB: U4A-71 SWB: 47	F–IF.8 F–LE.4★
Day 13 Day 14	4A.2.4: Graphing Logarithmic Functions	TRB: U4A-136 SRB: U4A-83 SWB: 57	F–IF.7e★
Day 15 Day 16	4A.2.5: Interpreting Logarithmic Models	TRB: U4A-161 SRB: U4A-102 SWB: 69	F–IF.4★ F–IF.5★ F–IF.6★
Day 17	Lesson 2 Progress Assessment Lesson 3 Pre-Assessment	TRB: U4A-179 TRB: U4A-183	

Lesson 3: Modeling Trigonometric Functions

Students return to the analysis of the sine and cosine functions they studied previously, this time with a closer investigation into the translation of trigonometric functions.

Day	Area of study/content	Pages	Standard(s)
Day 18 Day 19	4A.3.1: Graphing the Sine Function	TRB: U4A-187 SRB: U4A-116 SWB: 81	F–IF.7e★
Day 20 Day 21	4A.3.2: Graphing the Cosine Function	TRB: U4A-230 SRB: U4A-146 SWB: 93	F–IF.7e★
Day 22	Lesson 3 Progress Assessment	TRB: U4A-268	
Unit Assessment			
Day 23	Unit Assessment	TRB: U4A-271	

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Unit 4B: Mathematical Modeling and Choosing a Model (38 Days)

Unit Overview: Students revisit the process of creating equations in one variable and explore creating constraints and rearranging formulas. They then learn about transforming models and combining functions. Students review various kinds of functions, including linear, exponential, quadratic, piecewise, step, absolute value, square root, and cube root functions, all with an eye to choosing a model for a real-world situation. Finally, students consider geometric models, including two-dimensional cross sections of three-dimensional objects.

Lesson 1: Creating Equations

Students begin by reviewing vocabulary associated with equations and inequalities, and they practice writing equations to represent various scenarios. Next, they take a closer look at the constraints that exist for some situations and some equations, and learn how to find restricted domains. Finally, students apply their equation-solving skills to formulas, when they solve formulas for a given variable.

Day	Area of study/content	Pages	Standard(s)
Day 1 Day 2	Lesson 1 Pre-Assessment 4B.1.1: Creating Equations in One Variable	**Pre-Assessment** TRB: U4B-1 **Instruction/Practice** TRB: U4B-7 SRB: U4B-5 SWB: 7	A–CED.1★
Day 3 Day 4	4B.1.2: Representing and Interpreting Constraints	TRB: U4B-36 SRB: U4B-25 SWB: 17	A–CED.3★
Day 5 Day 6	4B.1.3: Rearranging Formulas	TRB: U4B-56 SRB: U4B-37 SWB: 27	A–CED.4★
Day 7	Lesson 1 Progress Assessment Lesson 2 Pre-Assessment	TRB: U4B-82 TRB: U4B-86	

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Lesson 2: Transforming a Model and Combining Functions

Students take an in-depth look at transformations of functions. Next, they analyze functions both graphically and algebraically to determine whether the functions are even, odd, or neither. Finally, students learn how to combine two functions by adding, subtracting, multiplying, and dividing them, as well as by substituting one function into the other (composition of functions).

Day	Area of study/content	Pages	Standard(s)
Day 8 Day 9	4B.2.1: Transformations of Parent Graphs	TRB: U4B-90 SRB: U4B-57 SWB: 37	F–BF.3
Day 10 Day 11	4B.2.2: Recognizing Odd and Even Functions	TRB: U4B-116 SRB: U4B-74 SWB: 49	F–BF.3
Day 12 Day 13	4B.2.3: Combining Functions	TRB: U4B-131 SRB: U4B-81 SWB: 59	F–BF.1b★
Day 14	Lesson 2 Progress Assessment Lesson 3 Pre-Assessment	TRB: U4B-145 TRB: U4B-148	

Lesson 3: Comparing Properties Within and Between Functions

Students compare the features and the limitations of a real-world mathematical situation and the "pure" mathematical model that can represent it, with a goal of finding the most accurate mathematical model possible. Next, students learn to calculate the average rate of change of nonlinear functions. The lesson ends with an analysis of how to compare two functions that represent a real-world scenario.

Day	Area of study/content	Pages	Standard(s)
Day 15 Day 16	4B.3.1: Reading and Identifying Key Features of Real-World Situation Graphs	TRB: U4B-155 SRB: U4B-91 SWB: 69	F–IF.4★ F–IF.5★ F–IF.6★
Day 17 Day 18	4B.3.2: Calculating Average Rates of Change	TRB: U4B-188 SRB: U4B-115 SWB: 83	F–IF.6★
Day 19 Day 20	4B.3.3: Comparing Functions	TRB: U4B-210 SRB: U4B-131 SWB: 93	F–IF.6★ F–IF.9
Day 21	Lesson 3 Progress Assessment Lesson 4 Pre-Assessment	TRB: U4B-233 TRB: U4B-241	

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Lesson 4: Choosing a Model

Students learn how to choose the best model (linear, exponential, or quadratic) for a given situation by analyzing the given data. Then, they analyze more models, including piecewise functions, step functions, and absolute value functions. Students discover which of these functions apply to different types of situations. Finally, they look at situations that are best modeled with square root and cube root functions.

Day	Area of study/content	Pages	Standard(s)
Day 22 Day 23	4B.4.1: Linear, Exponential, and Quadratic Functions	TRB: U4B-246 SRB: U4B-148 SWB: 103	A–CED.2★ F–IF.4★ F–IF.5★ F–BF.3
Day 24 Day 25	4B.4.2: Piecewise, Step, and Absolute Value Functions	TRB: U4B-270 SRB: U4B-163 SWB: 115	F–IF.4★ F–IF.5★ F–IF.7b★ F–BF.3
Day 26 Day 27	4B.4.3: Square Root and Cube Root Functions	TRB: U4B-293 SRB: U4B-178 SWB: 129	F–IF.4★ F–IF.5★ F–IF.7b★ F–BF.3
Day 28	Station Activities Set 1: Choosing a Model	TRB: U4B-413 SWB: 179	F–IF.4★ F–IF.5★ F–IF.7b★ F–BF.3
Day 29	Lesson 4 Progress Assessment Lesson 5 Pre-Assessment	TRB: U4B-315 TRB: U4B-321	

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Lesson 5: Geometric Modeling

Students find ways to model geometric situations, first by looking at cross sections of three-dimensional objects, and how these cross-sectional areas can be used to model a given problem. The modeling continues as students explore the concept of density, and how the density of a quantity can be compared to a basic unit of area or volume to solve real-world problems. Finally, students explore the role of mathematics in the design of products and services.

Day	Area of study/content	Pages	Standard(s)
Day 30 Day 31	4B.5.1: Two-Dimensional Cross Sections of Three-Dimensional Objects	TRB: U4B-326 SRB: U4B-197 SWB: 141	G–GMD.4 G–MG.1★
Day 32 Day 33	4B.5.2: Density	TRB: U4B-344 SRB: U4B-209 SWB: 153	G–MG.2★
Day 34 Day 35	4B.5.3: Design	TRB: U4B-364 SRB: U4B-223 SWB: 165	G–MG.3★
Day 36	Station Activities Set 2: Geometric Modeling	TRB: U4B-431 SWB: 187	G–GMD.4 G–MG.1★ G–MG.2★ G–MG.3★
Day 37	Lesson 5 Progress Assessment	TRB: U4B-387	
Unit/Comprehensive Assessment			
Day 38	Unit Assessment (covering Unit 4B) and/or Comprehensive Unit Assessment (covering Units 4A and 4B)	**Unit Assessment** TRB: U4B-392 **Comprehensive Unit Assessment** TRB: U4-CA1	

End-of-Course Assessment	
Day 39	End-of-Course Assessment Please go to walch.com/updates and register to receive End-of-Course Assessments and other updates and additions as they become available.

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Graphic Organizers

Overview

Graphic organizers can be a versatile tool in your classroom. Organizers offer an easy, straightforward way to visually present a wide range of material. Research suggests that graphic organizers support learning in the classroom for all levels of learners. Gifted students, students on grade level, and students with learning difficulties all benefit from the use of graphic organizers. They reduce the cognitive demand on students by helping them access information quickly and easily. Using graphic organizers, learners can understand content more clearly and can take concise notes. Ultimately, learners find it easier to retain and apply what they've learned.

Graphic organizers help foster higher-level thinking skills. They help students identify main ideas and details in their reading. They make it easier for students to see patterns such as cause and effect, comparing and contrasting, and chronological order. Organizers also help students master critical-thinking skills by asking them to recall, evaluate, synthesize, analyze, and apply what they've learned. Research suggests that graphic organizers contribute to better test scores because they help students understand relationships between key ideas, and enable them to be more focused as they study.

Types of Graphic Organizers

There are four main purposes for using graphic organizers in mathematics and a variety of tools within each category:

Purpose 1: Organizing, Categorizing, and Classifying	**Purpose 2: Problem Solving**	**Purpose 3: Understanding Mathematical Information**	**Purpose 4: Communicating Mathematical Information**
Tables Flowcharts Webs Venn Diagrams	Number Lines Geometric Drawings Factor Trees Attribute Tables Cause and Effect Maps Coordinate Plane Probability Trees	Frayer Model Semantic Map/ Concept Map Compare-and-Contrast Diagram	Line Graphs Bar Charts

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Tables

A table is simply a grid with rows and columns. Tables are useful because information stored in a table is easy to find—much easier than the same information embedded in text.

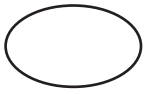
Usually, a table has a row (horizontal) for each item being listed. The columns (vertical) provide places for details about the listed items—the things they have in common. The places where the rows and columns meet are called cells. In each cell, we write information that fits both the topic of the row (the thing being listed) and the topic of the column (the aspect being examined). To create a table, we make rows and columns to fit the number of items and attributes.

Flowcharts

Flowcharts are graphic organizers that show the steps in a process. Flowcharts can be very simple—just a series of boxes with one step in each box. However, there is also a more formal type of flowchart. These flowcharts use special symbols to show different things, such as starting and stopping points, or points where decisions must be made. These symbols make flowcharts especially useful for showing complicated processes.

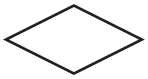
Each step in a flowchart is written in a box. The boxes are connected by arrows to show the sequence of steps. The boxes aren't all rectangular; different shapes are used to indicate different actions. The shapes and symbols are a kind of visual shorthand. Whenever a certain symbol is used, it always has the same meaning.

- Circles and ovals show starting and stopping points. They often contain the words start or stop. The "start" circle or oval has no arrows in and one arrow out. The "stop" circle or oval has one arrow in and no arrows out.



- Arrows show the direction in which the process is moving.

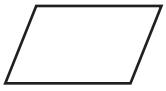
- Diamonds show points where a decision must be made or a question must be answered. The question can usually be answered either "yes" or "no."



- Rectangles and squares show steps where a process or an operation takes place.



- Parallelograms show input or output, such as writing or printing a result or solution.



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Webs

Webs are graphic organizers that help take notes, identify important ideas, and show relationships between and among pieces of information. In a web, the main idea is written in the center circle. Details are recorded in other circles with lines to connect related topics. Circles or lines can be added or deleted as necessary.

Number Lines

In its simplest form, a number line is any line that uses equally spaced marks to show numbers. Number lines are used to visualize equalities and inequalities, positive and negative numbers, and measurements of all kinds. They can "map" math problems, especially ones that involve negative numbers or distances.

Geometric Drawings

A geometric drawing is a representation on paper (or some other surface) of a geometric figure. The geometric drawings we make can never be as perfect as the geometric figures they represent, but as long as they are reasonably accurate, they can help us visualize the figures. In fact, it's often impossible to solve a geometry problem without making a drawing.

Factor Trees

There are several ways to find factors. One that helps to visually keep track of all the factors is called a factor tree. This is a diagram with a tree-like shape. It uses "branches" to show the factors of a number.

All whole numbers other than 1 can be written as the product of factors. A prime number is a number that has only two factors, itself and 1. An example of a prime number is 13. Its only factors are 13 and 1. A composite number is a number that has more than two factors. An example of a composite number is 6. Its factors include 6, 3, 2, and 1. Prime factors are factors that are also prime numbers. The greatest common factor (GCF) of two numbers is the largest number that is a factor of both numbers.

Coordinate Plane

This is the plane determined by a horizontal number line, called the x-axis, and a vertical number line, called the y-axis, intersecting at a point called the origin. A coordinate plane can be used to illustrate locations and relationships using ordered pairs of numbers.

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Venn Diagrams

A set is a list of objects in no particular order. Items in a set can be numbers, but they can also be letters or words. Venn diagrams are a visual way of showing how sets of things can include one another, overlap, or be distinct from one another.

Venn diagrams are often used to compare and contrast things. But they are also a useful tool to sort and classify information. You can use Venn diagrams to take notes on material that shows relationships between things or ideas. You can also use them to solve certain types of word problems. When a word problem names two or three different categories and asks you how many items fall into each category, a Venn diagram can be a useful problem-solving tool.

A Venn diagram begins with a rectangle representing the universal set. Then each set in the problem is represented by a circle. Circles can be separate, overlapping, or one within another. When two circles overlap, it means that the two sets intersect. Some members of one set are also members of the other set.

Venn Diagrams AND Compare-and-Contrast Diagrams

The Venn diagram is an organizing device for planning comparisons and contrasts. A completed Venn diagram helps students categorize and organize similarities and differences, and provides a blueprint for a comparison-and-contrast exercise. The compare-and-contrast diagram provides a structure to identify or list similarities and differences between two objects.

Attribute Tables

To solve logic problems, you need a way to keep track of the subjects and which attributes they have or don't have. An attribute table can help. This is a table with a row for each subject in the problem, and a column for each attribute. The rows and columns meet to form cells. Because the attributes in logic problems are usually exclusive, you can use Xs or check marks (✔) to show which attribute belongs to which subject.

Cause and Effect Maps

Cause and effect maps help you work through information to make sense of it. Write each cause in the oval. Write all its effects in the boxes. Add or delete ovals and boxes as needed.

Frayer Model

The Frayer Model is a word categorization activity that helps learners to develop their understanding of concepts. Using this model, students provide a definition, list characteristics, and provide examples and non-examples of the concept.

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Semantic Map

A semantic word map allows students to conceptually explore their knowledge of a new term or concept by mapping it with other related words, concepts, or phrases that are similar in meaning. Semantic maps portray the schematic relations that compose a concept. It assumes that there are multiple relations between a concept and the knowledge that is associated with the concept.

Line Graphs

Line graphs are often used to show how things change over time. They clearly show trends in data and can let you make predictions about future trends, too. Line graphs use two number lines, one horizontal and one vertical. The horizontal number line is called the *x*-axis. The vertical line is called the *y*-axis. The *x*-axis often shows the passage of time. The *y*-axis often shows a quantity of some kind, such as height, speed, cost, and so forth.

Bar Charts

Bar charts are useful when you want to compare things or to show how one thing changes over time. They are a good way to show overall trends. Bar charts use horizontal or vertical bars to represent data. Longer bars represent higher values. Different colors can be used to show different variables. When you look at a bar chart, it's easy to see which element has the greatest value—the one with the longest bar.

Bar charts have an *x*-axis (horizontal) and a *y*-axis (vertical). If the graph is being used to show how something changes over time, the *x*-axis has numbers for the time period. If the graph is being used to compare things, the *x*-axis shows which things are being compared. The *y*-axis has numbers that show how much of each thing there is.

Probability Trees

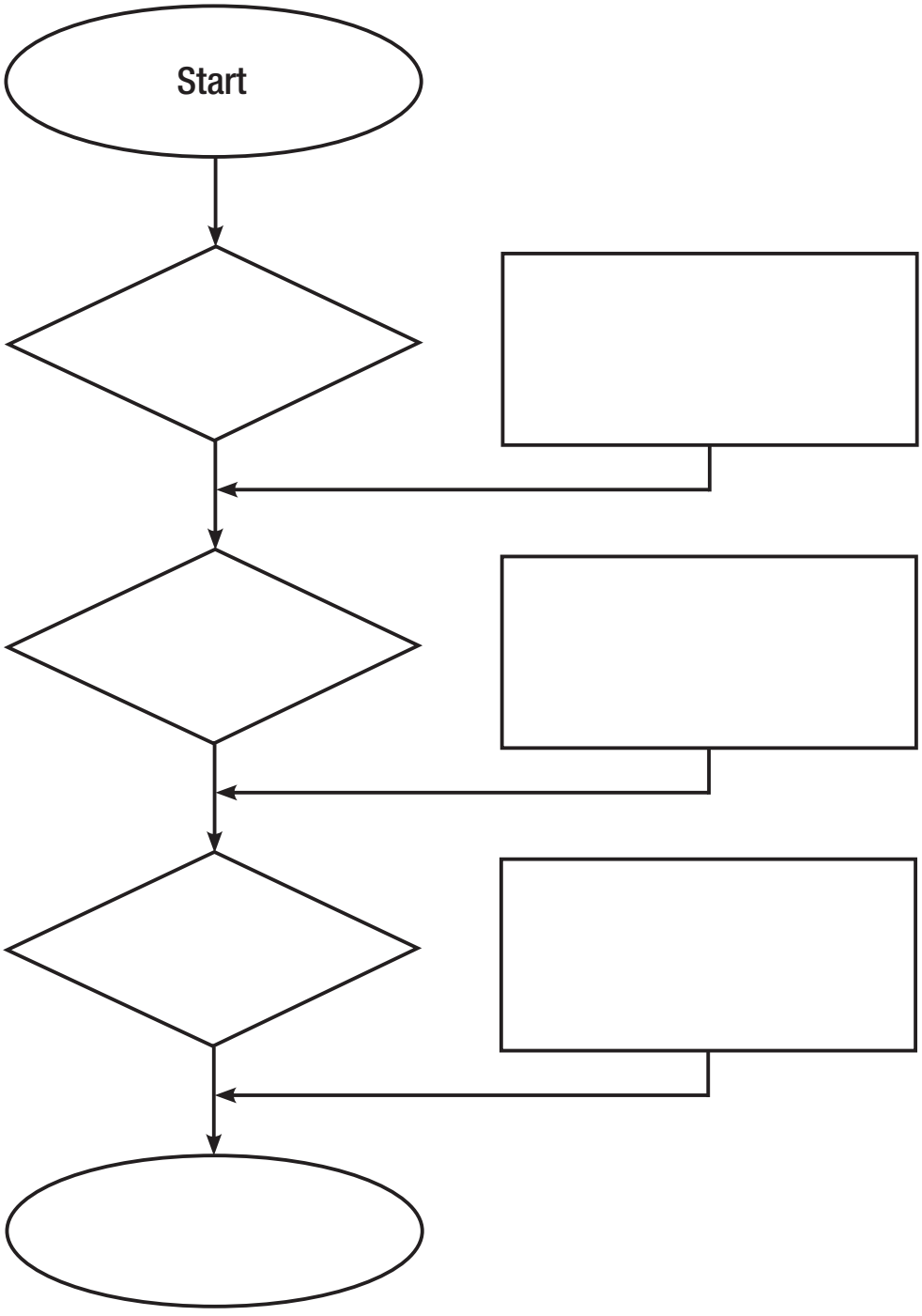
When we have probability problems with many possible outcomes, or events that depend on one another, probability trees can help. Probability trees show all the possible outcomes of an event. Whenever a problem calls for figuring out how many possible outcomes there are, and the probability that any one of them will happen, a probability tree can be useful.

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Table

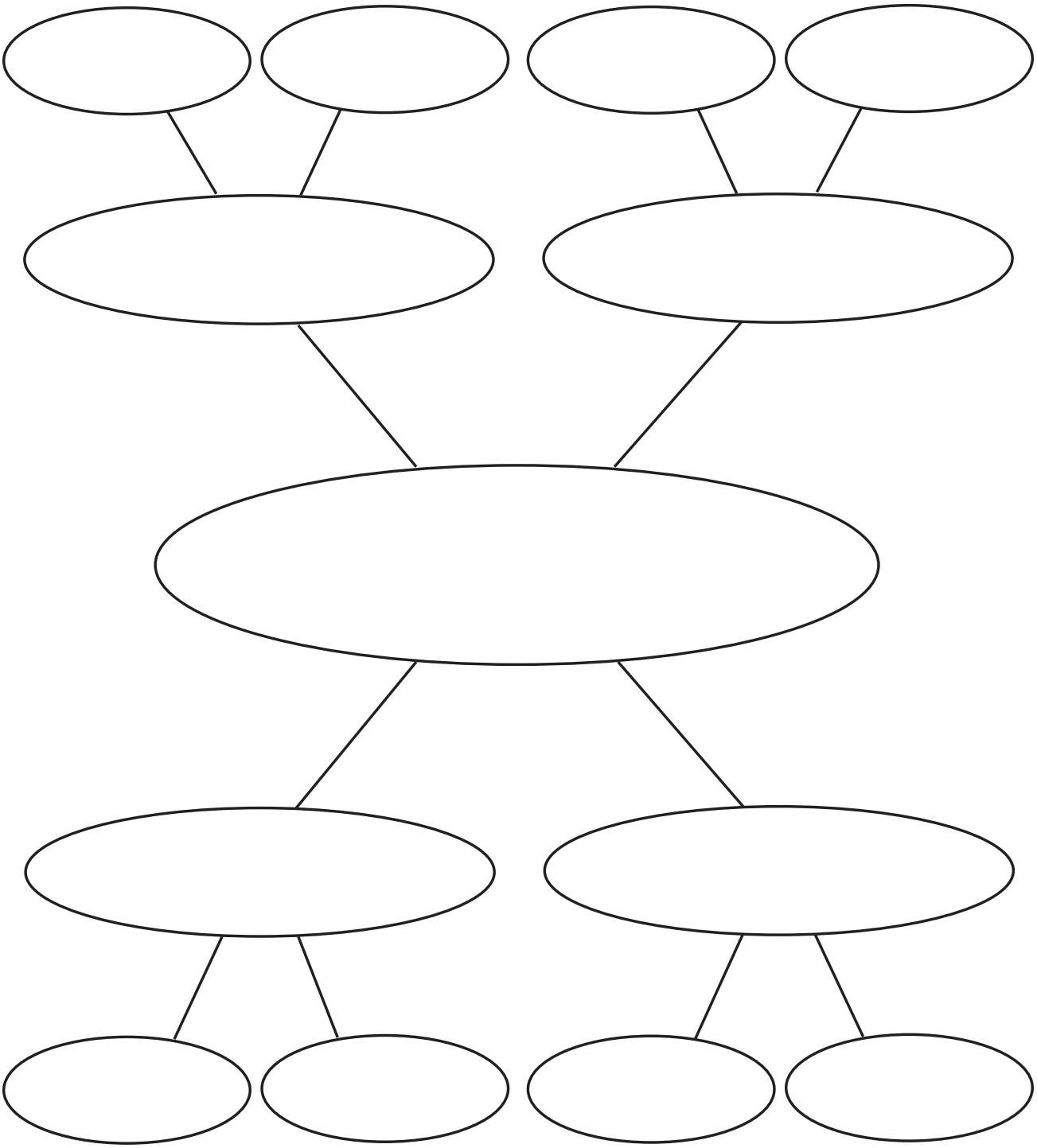
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Flowchart



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Web



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Number Line



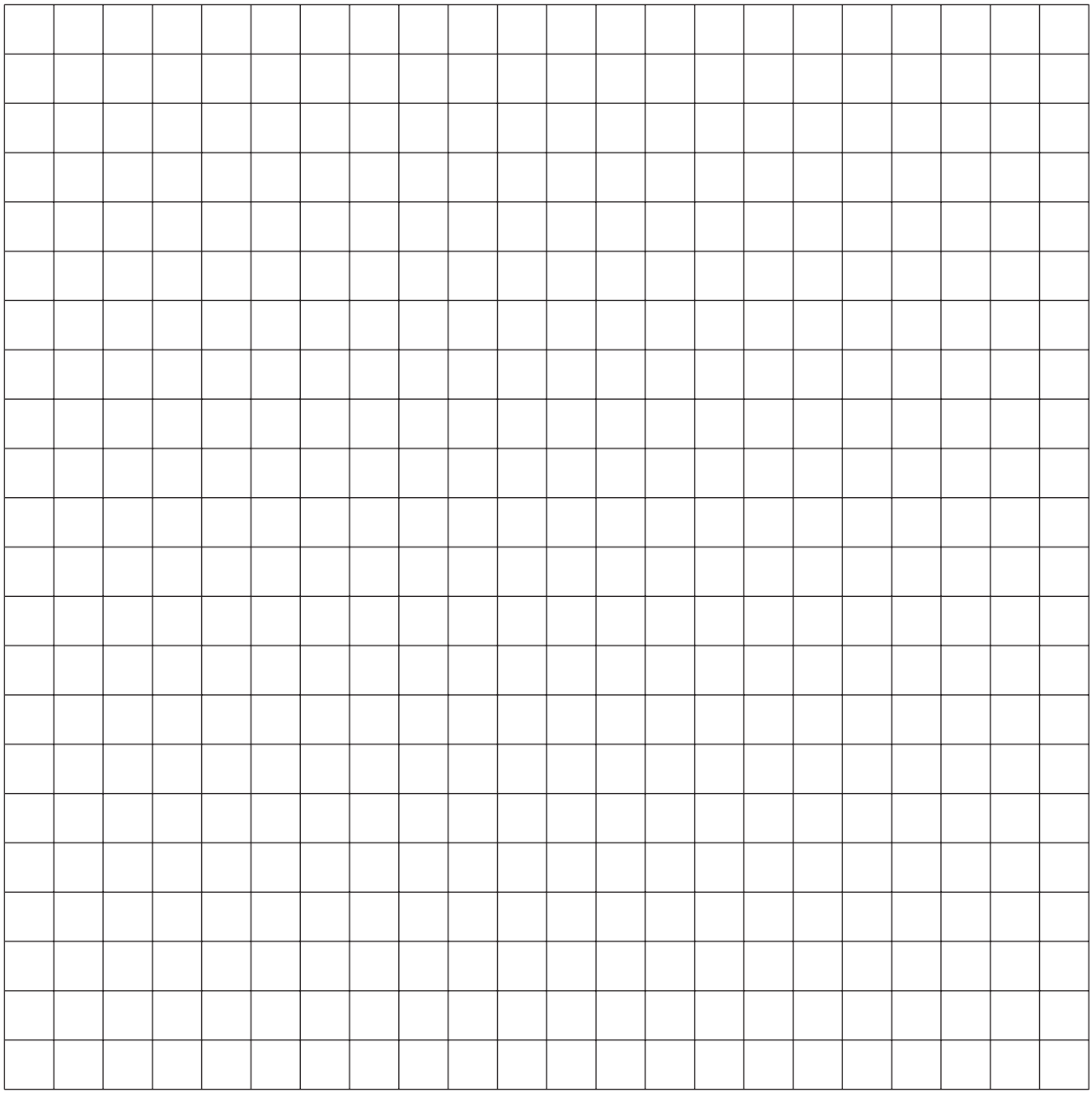
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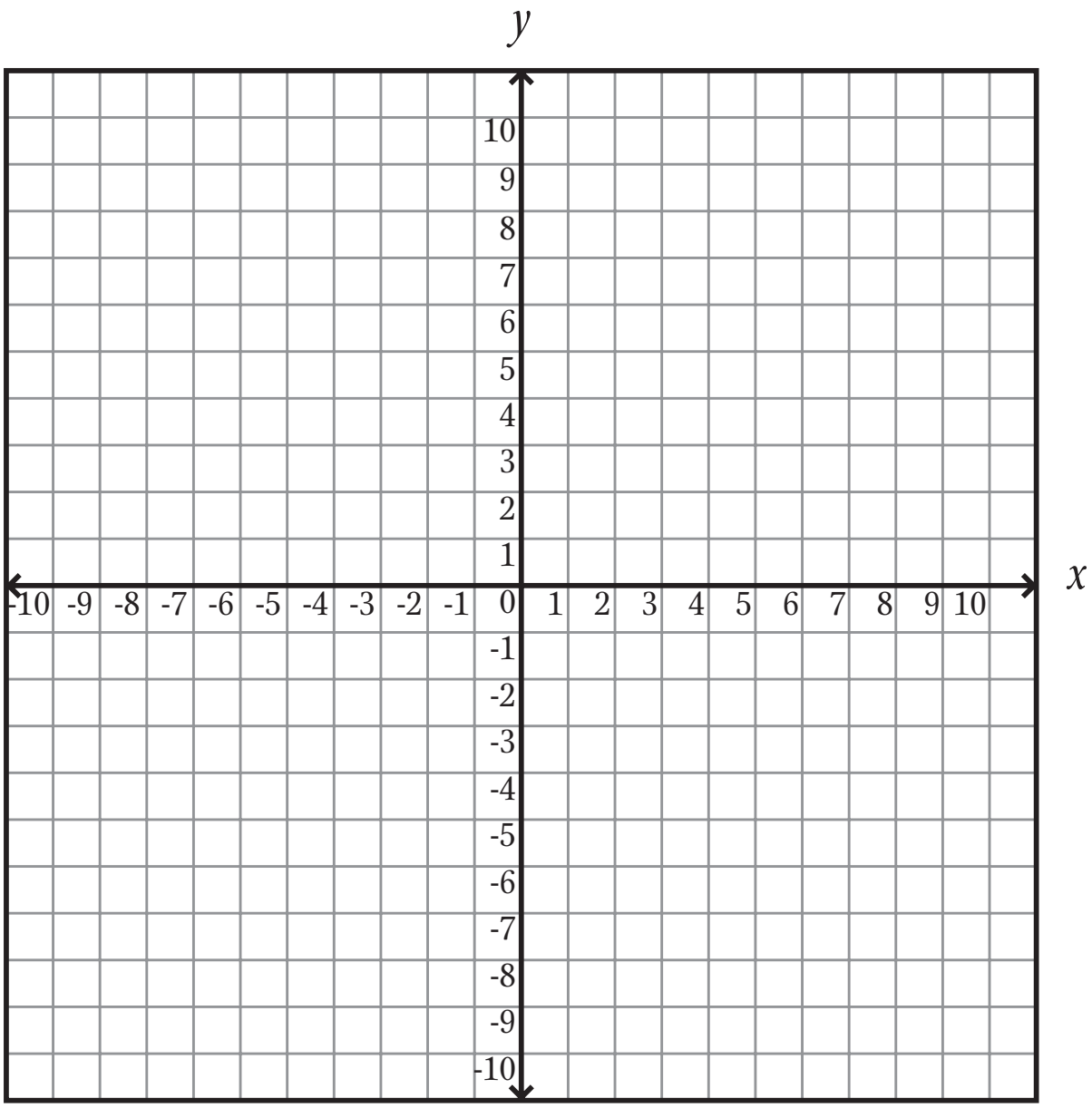
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Geometric Drawing



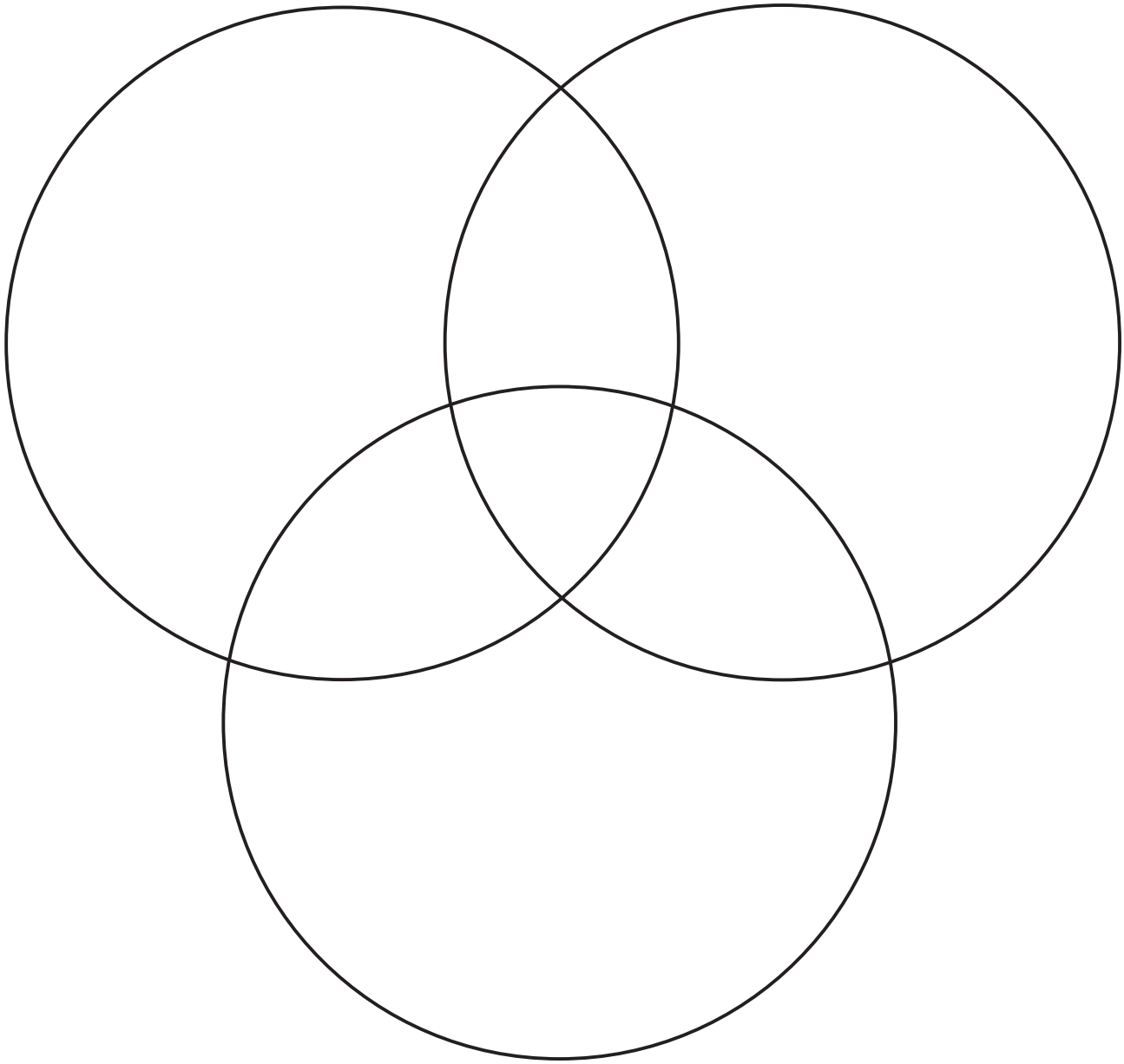
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Coordinate Plane



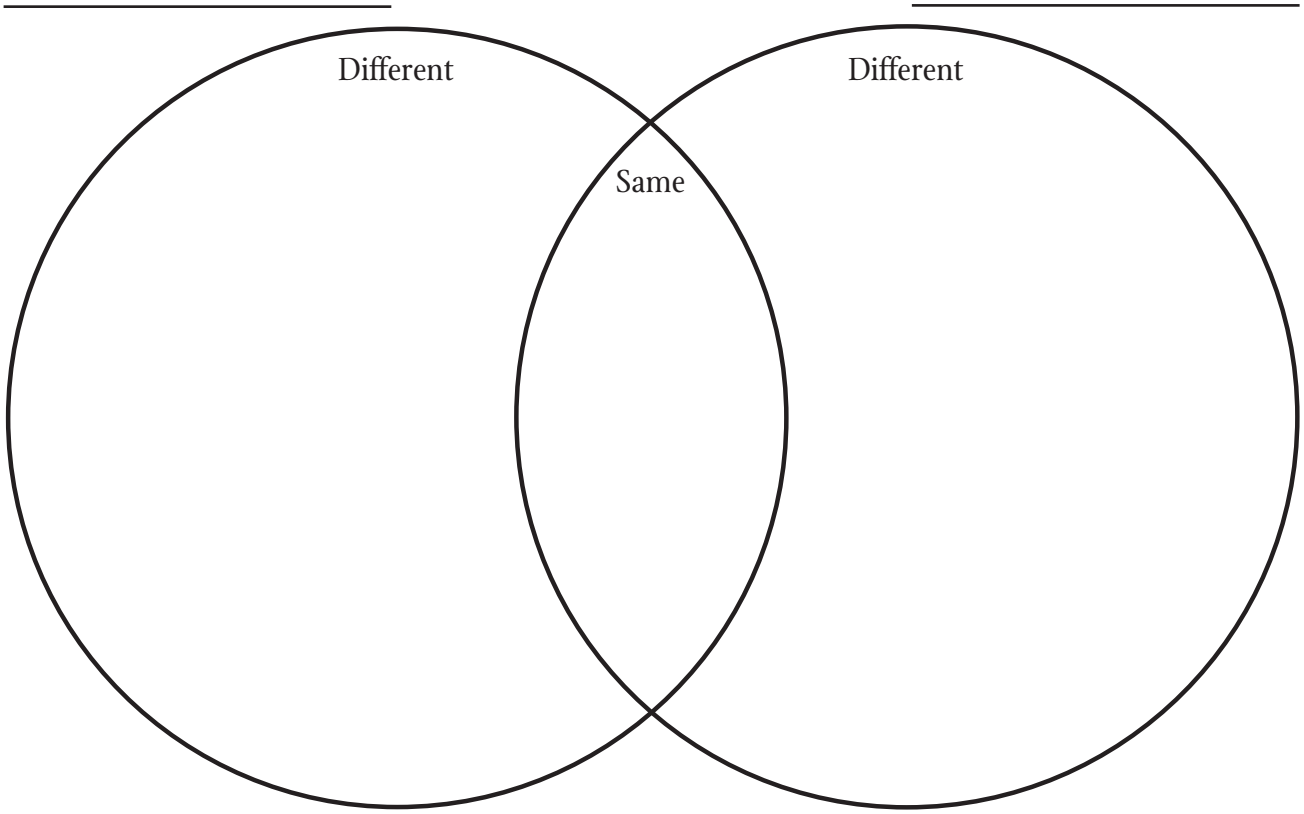
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Venn Diagram



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Venn Diagram



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Compare-and-Contrast Diagram

Item 1 _____ Item 2 _____

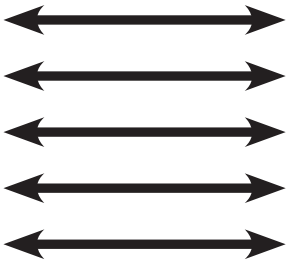
How Alike?

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How Different?

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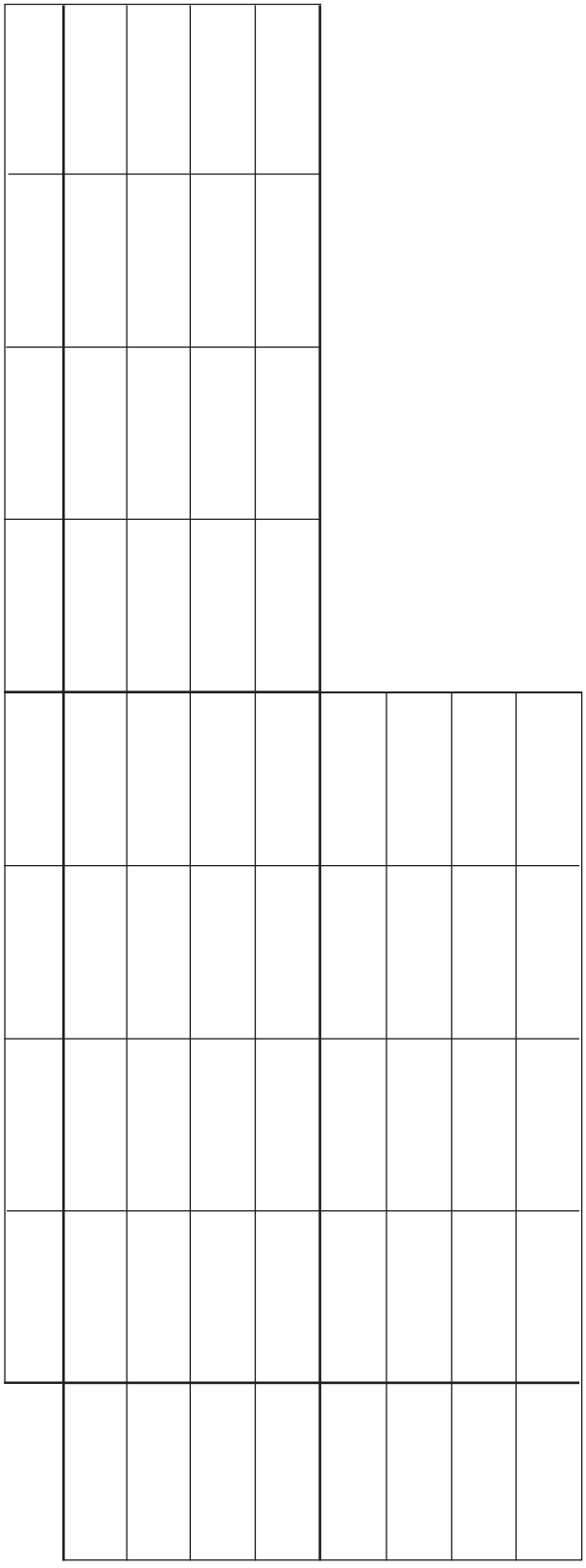
With Regard To



_____
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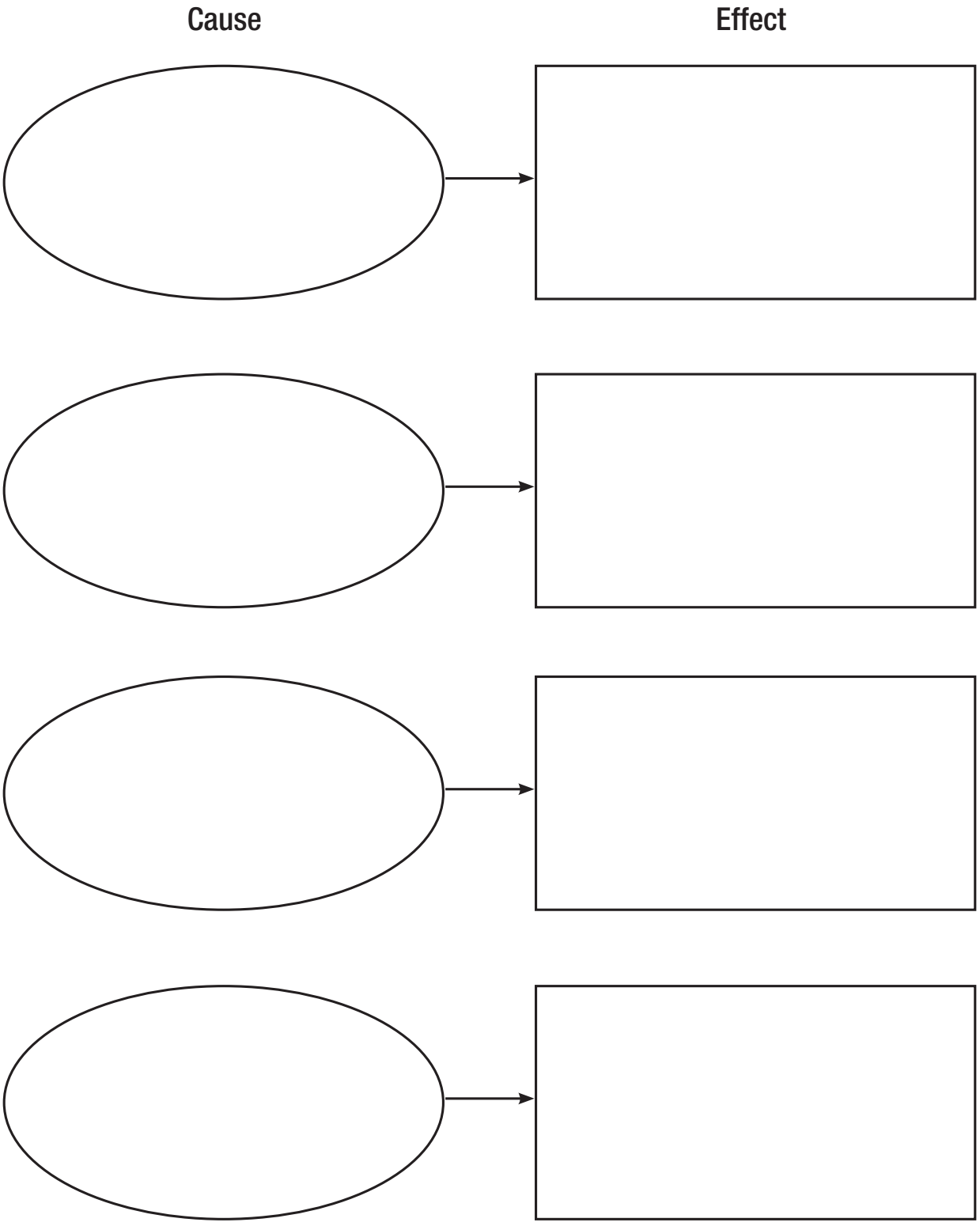
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Attribute Table



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Cause and Effect Map



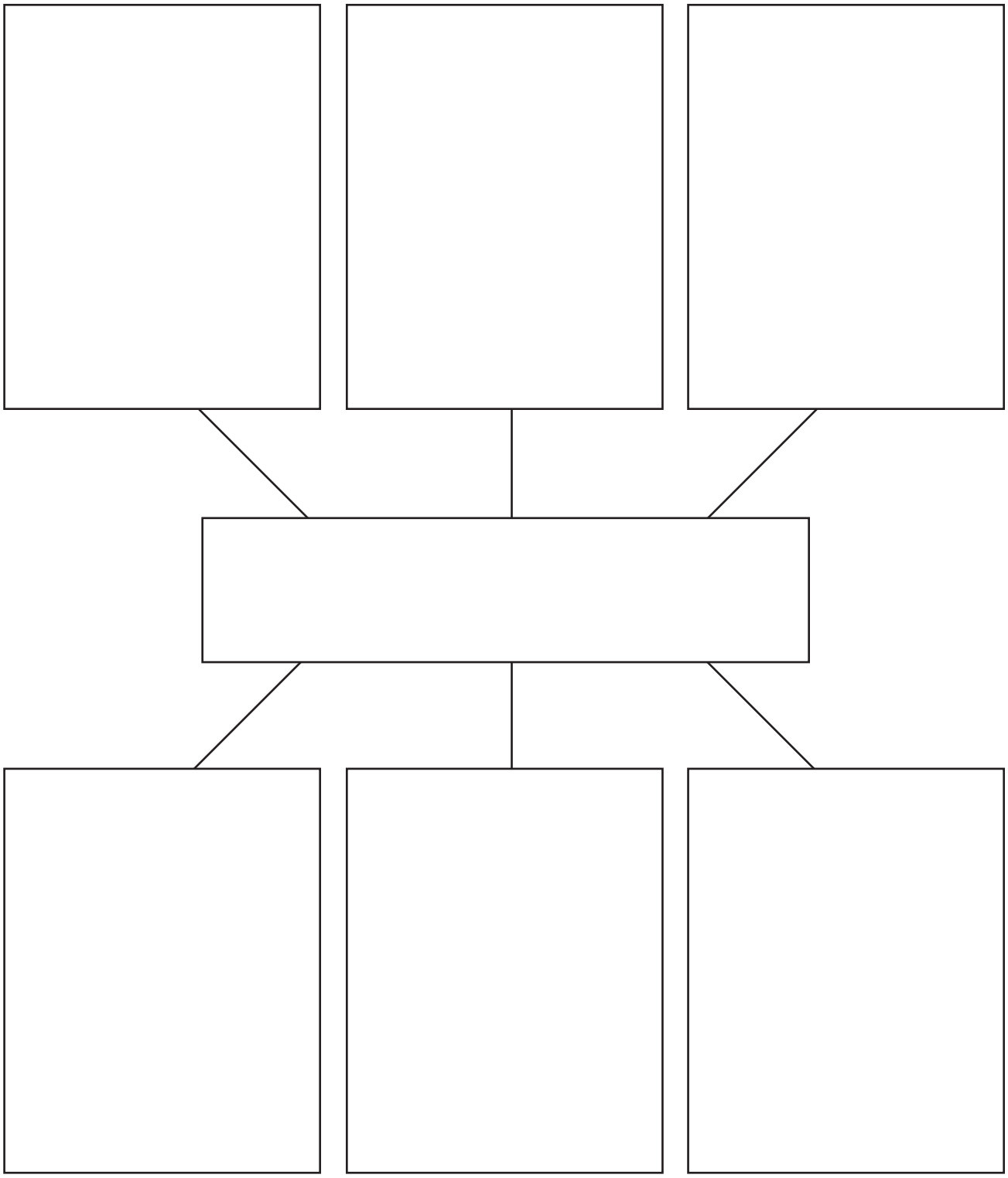
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Frayer Model

Definition	Characteristics
WORD	
Examples from Life	Non-Examples

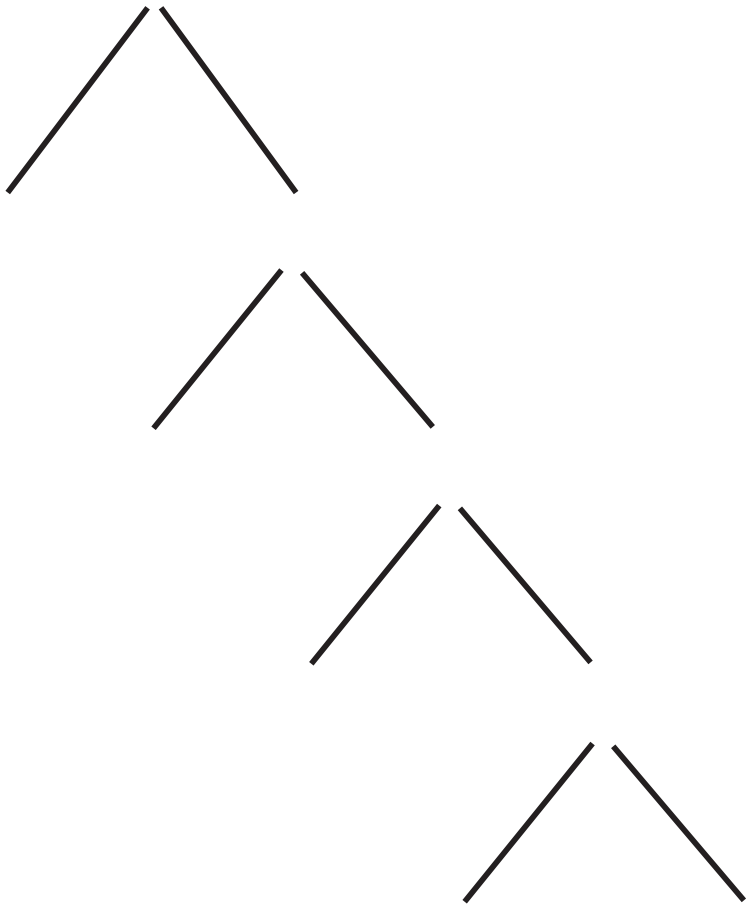
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Semantic Map/Concept Map



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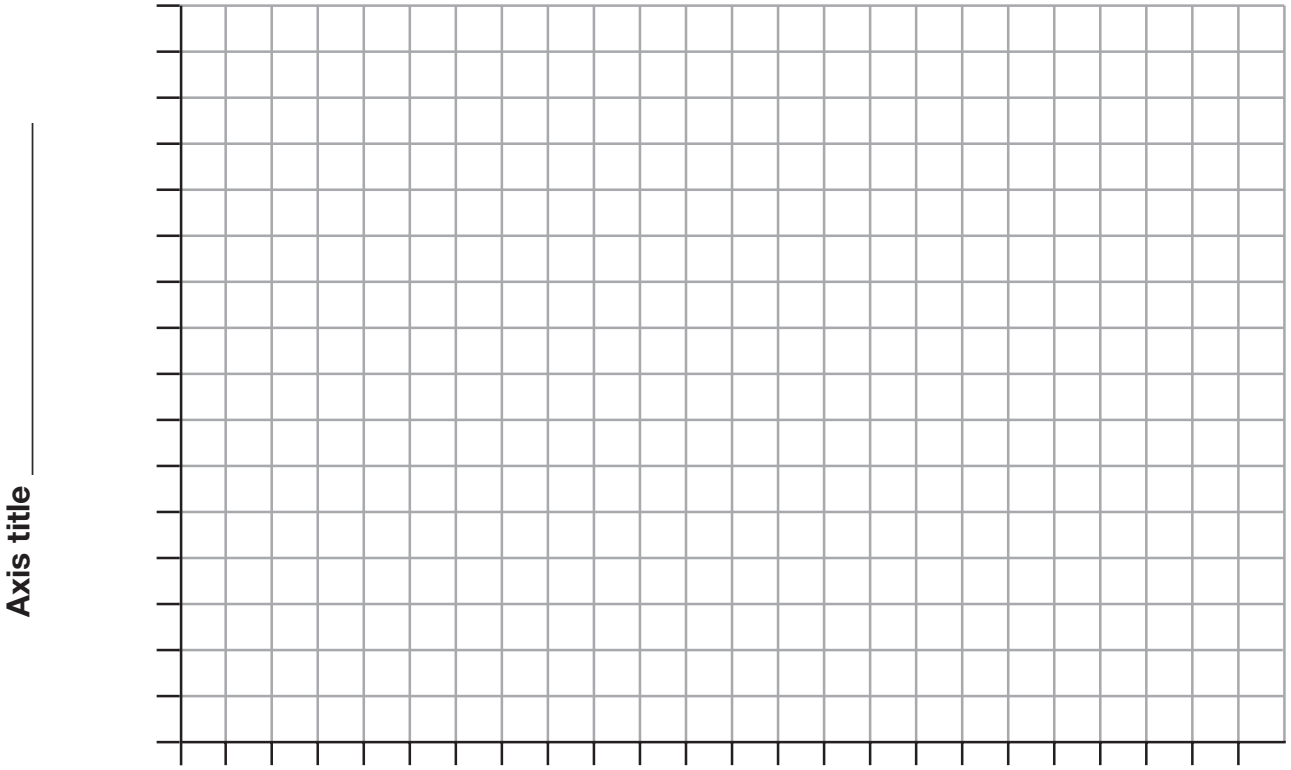
Factor Tree



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Line Graph

Graph title _____

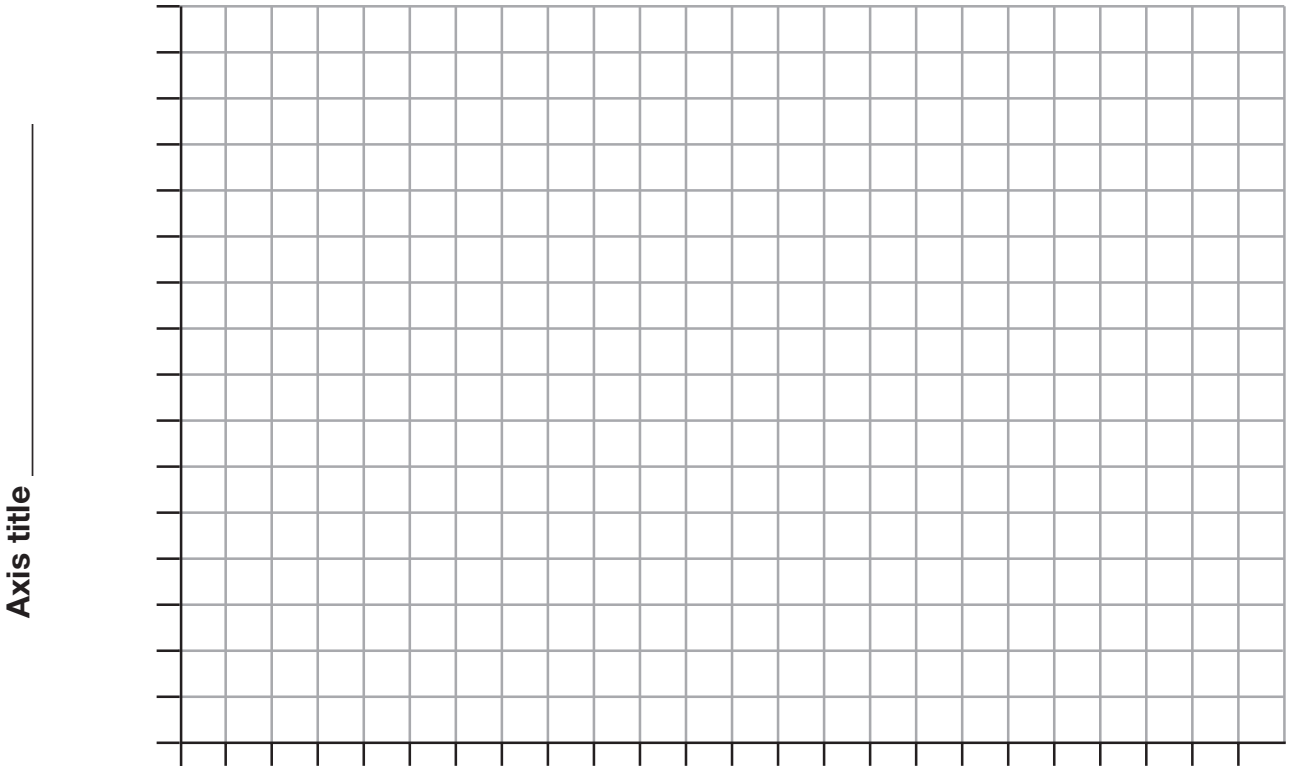


Axis title _____

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Bar Chart/Histogram

Graph title _____



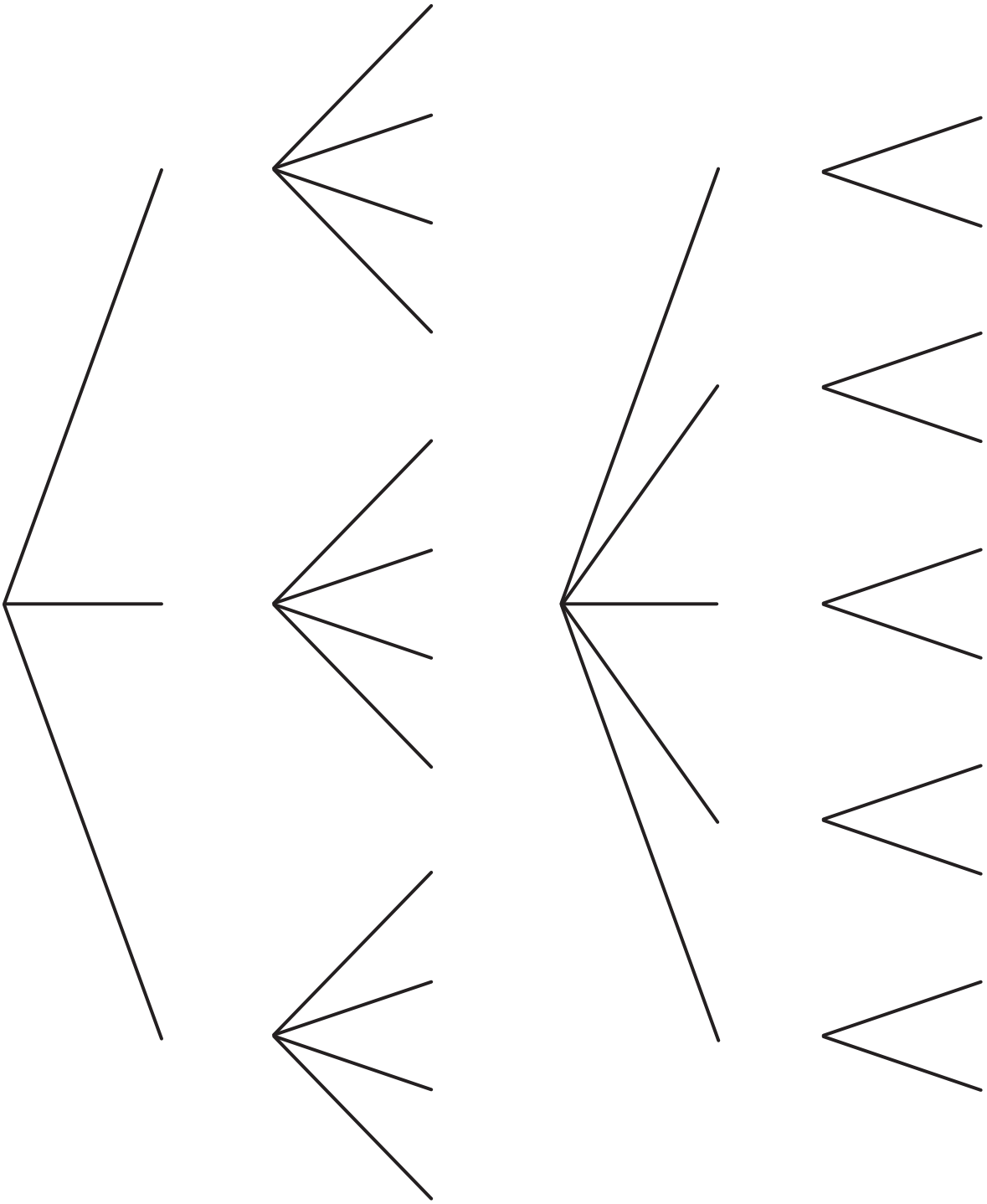
KEY

☐
☐
☐
☐

Axis title _____

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Probability Trees



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ALGEBRA

Symbols	
$\approx$	Approximately equal to
$\neq$	Is not equal to
$\lvert a \rvert$	Absolute value of a
$\sqrt{a}$	Square root of a
∞	Infinity
[	Inclusive on the lower bound
]	Inclusive on the upper bound
(	Non-inclusive on the lower bound
)	Non-inclusive on the upper bound
Σ	Sigma
Δ	Delta

General	
(x, y)	Ordered pair
$(x, 0)$	x-intercept
$(0, y)$	y-intercept

Linear Equations	
$m = \frac{y_2 - y_1}{x_2 - x_1}$	Slope
$y = mx + b$	Slope-intercept form
$ax + by = c$	General form
$y - y_1 = m(x - x_1)$	Point-slope form

Exponential Equations	
$A = P\left(1 + \frac{r}{n}\right)^{nt}$	Compounded interest formula
Compounded...	n (number of times per year)
Yearly/annually	1
Semiannually	2
Quarterly	4
Monthly	12
Weekly	52
Daily	365

Exponential Functions	
$1 + r$	Growth factor
$1 - r$	Decay factor
$f(t) = a(1+r)^t$	Exponential growth function
$f(t) = a(1-r)^t$	Exponential decay function
$f(x) = ab^x$	Exponential function in general form
$y = ab^{\frac{x}{t}}$	Exponential equation

Binomial Theorem

$$\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \bullet a^{n-k}b^k = 1a^nb^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n-1)}{1 \bullet 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \bullet 2 \bullet 3}a^{n-3}b^3 + \cdots + 1a^0b^n$$

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Functions	
$f(x)$	Function notation, "f of x"
$f^{-1}(x)$	Inverse function notation
$f(x) = mx + b$	Linear function
$f(x) = b^x + k$	Exponential function
$f(x) = ax^2 + b^x + c$	Quadratic function
$(f + g)(x) = f(x) + g(x)$	Addition
$(f - g)(x) = f(x) - g(x)$	Subtraction
$(f \bullet g)(x) = f(x) \bullet g(x)$	Multiplication
$(f \circ g)(x) = f(g(x))$	Composition
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	Division
$\frac{f(b) - f(a)}{b - a}$	Average rate of change
$r = \frac{\Delta f(x)}{\Delta g(x)}$	Concise rate of change
$f(-x) = -f(x)$	Odd function
$f(-x) = f(x)$	Even function
$f(x) = \lfloor x \rfloor$	Floor/greatest integer function
$f(x) = \lceil x \rceil$	Ceiling/least integer function
$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$	Polynomial function
$f(x) = a\sqrt[3]{(x - h)} + k$	Cube root function
$f(x) = a\sqrt[n]{(x - h)} + k$	Radical function
$f(x) = a\lvert x - h\rvert + k$	Absolute value function
$f(x) = \frac{p(x)}{q(x)};\ \ q(x) \neq 0$	Rational function
$\log_a bx = c$	Logarithmic function

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Quadratic Functions and Equations	
$x = \frac{-b}{2a}$	Axis of symmetry
$x = \frac{p+q}{2}$	Axis of symmetry using the midpoint of the x-intercepts
$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$	Vertex
$f(x) = ax^2 + bx + c$	General form
$f(x) = a(x-h)^2 + k$	Vertex form
$f(x) = a(x-p)(x-q)$	Factored/intercept form
$b^2 - 4ac$	Discriminant
$x^2 + bx + \left(\frac{b}{2}\right)^2$	Perfect square trinomial
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Quadratic formula

Properties of Exponents	
Property	General rule
Zero Exponent	$a^0 = 1$
Negative Exponent	$b^{-\frac{m}{n}} = \frac{1}{b^{\frac{m}{n}}}$
Product of Powers	$a^m \bullet a^n = a^{m+n}$
Quotient of Powers	$\frac{a^m}{a^n} = a^{m-n}$
Power of a Power	$\left(b^m\right)^n = b^{mn}$
Power of a Product	$(bc)^n = b^n c^n$
Power of a Quotient	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Multiplication of Complex Conjugates
$(a + bi)(a - bi) = a^2 + b^2$

Common Polynomial Identities	
$(a+b)^2 = a^2 + 2ab + b^2$	Square of Sums
$(a-b)^2 = a^2 - 2ab + b^2$	Square of Differences
$a^2 - b^2 = (a+b)(a-b)$	Difference of Two Squares
$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$	Sum of Two Cubes
$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$	Difference of Two Cubes

Radicals to Rational Exponents
$\sqrt[n]{a} = a^{\frac{1}{n}}$
$\sqrt[n]{x^m} = x^{\frac{m}{n}}$

Imaginary Numbers
$i = \sqrt{-1}$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$

Logarithmic Functions	
e	Base of a natural logarithm
$\log_a b = \frac{\log b}{\log a}$	Change of base formula
$\frac{2\pi}{b}$	Period
$-\frac{b}{a}$	Phase shift

Properties of Radicals
$\sqrt{ab} = \sqrt{a} \bullet \sqrt{b}$
$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

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Series and Sequences	
$r = \frac{a_n}{a_{n-1}}$	Common ratio
$a_n = a_1 \bullet r^{n-1}$	Explicit formula for a geometric sequence
$\sum_{k=1}^{n} a_1 r^{k-1}$	Finite geometric series
$\sum_{k=1}^{\infty} a_1 r^{k-1}$	Infinite geometric series
$a_n = a_{n-1} \bullet r$	Recursive formula for a geometric sequence
$S_n = \frac{a_1(1-r^n)}{1-r}$	Sum formula for a finite geometric series
$S_n = \frac{a_1}{1-r}$	Sum formula for an infinite geometric series
$P = \sum_{k=1}^{n} A\left(\frac{1}{1+i}\right)^{k-1}$	Amortization loan formula

Properties of Logarithms	
Product property	$\log_a (x \bullet y) = \log_a x + \log_b y$
Quotient property	$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
Power property	$\log_a x^y = y \bullet \log_a x$

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STATISTICS AND DATA ANALYSIS

Symbols	
$\varnothing$	Empty/null set
$\cap$	Intersection, "and"
$\cup$	Union, "or"
$\subset$	Subset
$\overline{A}$	Complement of Set A
!	Factorial
${}_nC_r$	Combination
${}_nP_r$	Permutation
μ	Population mean
$\overline{x}$	Sample mean
σ	Standard deviation of a population
s	Standard deviation of a sample
$\hat{p}$	Sample proportion
SEM	Standard error of the mean
SEP	Standard error of the proportion
MOE	Margin of error
CI	Confidence interval
df	Degrees of freedom

Empirical Rule/68–95–99.7 Rule
$\mu \pm 1\sigma \approx 68\%$
$\mu \pm 2\sigma \approx 95\%$
$\mu \pm 3\sigma \approx 99.7\%$

Common Critical Values

Confidence level	99%	98%	96%	95%	90%	80%	50%
Critical value (z_c)	2.58	2.33	2.05	1.96	1.645	1.28	0.6745

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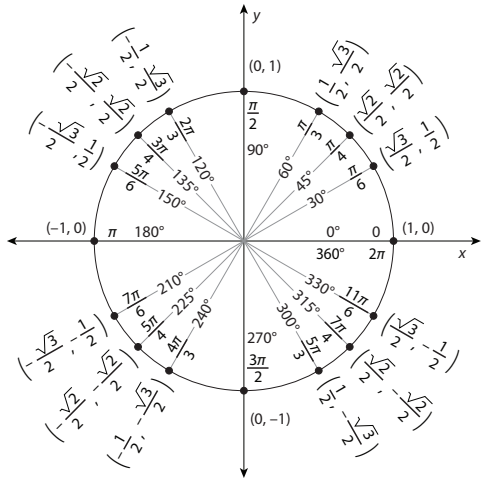
Formulas	
$\mu = \frac{x_1 + x_2 + \cdots + x_n}{n}$	Mean of a population
$\overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$	Mean of a sample
$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n}}$	Standard deviation of a population
$s = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n}}$	Standard deviation of a sample
$z = \frac{x - \mu}{\sigma}$	*z*-score
$\hat{p} = \frac{p}{n}$	Sample proportion
$\text{SEM} = \frac{s}{\sqrt{n}}$	Standard error of the mean
$\text{SEP} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$	Standard error of the proportion
$\text{MOE} = \pm z_c \frac{s}{\sqrt{n}}$	Margin of error of a sample mean
$\text{MOE} = \pm z_c \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$	Margin of error for a sample proportion
$\text{CI} = \hat{p} \pm z_c \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$	Confidence interval for a sample population with proportion $\hat{p}$
$\text{CI} = \overline{x} \pm z_c \frac{s}{\sqrt{n}}$	Confidence interval for a sample population with mean $\overline{x}$
$t = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$	*t*-value for two sets of sample data
$t = \frac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}}$	*t*-value for sample data and population
$df = \frac{n_1 - 1 + n_2 - 1}{2}$	Degrees of freedom

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Rules and Equations	
$P(E)=\dfrac{\#\text{ of outcomes in } E}{\#\text{ of outcomes in sample space}}$	Probability of event E
$P(A\cup B)=P(A)+P(B)-P(A\cap B)$	Addition rule
$P(\overline{A})=1-P(A)$	Complement rule
$P\left(B\middle\vert A\right)=\dfrac{P(A\cap B)}{P(A)}$	Conditional probability
$E(X)=p_1P(X_1)+p_2P(X_2)+p_3P(X_3)$	Expected value
$P(A\cap B)=P(A)\bullet P\left(B\middle\vert A\right)$	Multiplication rule
$P(A\cap B)=P(A)\bullet P(B)$	Multiplication rule if A and B are independent
${}_nC_r=\dfrac{n!}{(n-r)!r!}$	Combination
${}_nP_r=\dfrac{n!}{(n-r)!}$	Permutation
$n!=n\bullet(n-1)\bullet(n-2)\bullet\cdots\bullet 1$	Factorial
$P=\begin{pmatrix} n \\ x \end{pmatrix}p^xq^{n-x}$	Binomial probability distribution

GEOMETRY

Unit Circle



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Symbols	
$\overset{\frown}{ABC}$	Major arc length
$\overset{\frown}{AB}$	Minor arc length
$\angle$	Angle
$\odot$	Circle
$\cong$	Congruent
$\overleftrightarrow{PQ}$	Line
$\overline{PQ}$	Line segment
$\overrightarrow{PQ}$	Ray
$\parallel$	Parallel
$\perp$	Perpendicular
$\bullet$	Point
$\triangle$	Triangle
$\square$	Parallelogram
A'	Prime
$^\circ$	Degrees
θ	Theta
ϕ	Phi
π	Pi
ρ	Rho

Trigonometric Ratios		
$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$	$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$	$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$
$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$	$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$	$\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$

Volume	
$V = lwh$	Rectangular prism
$V = Bh$	Prism
$V = \frac{1}{3}\pi r^2 h$	Cone
$V = \frac{1}{3}Bh$	Pyramid
$V = \pi r^2 h$	Cylinder
$V = \frac{4}{3}\pi r^3$	Sphere

Trigonometric Identities
$\sin\theta = \cos(90° - \theta)$
$\cos\theta = \sin(90° - \theta)$
$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$\csc\theta = \frac{1}{\sin\theta}$
$\sec\theta = \frac{1}{\cos\theta}$
$\cot\theta = \frac{1}{\tan\theta}$
$\cot\theta = \frac{\cos\theta}{\sin\theta}$
$\sin^2\theta + \cos^2\theta = 1$

Pythagorean Theorem
$a^2 + b^2 = c^2$

Area	
$A = lw$	Rectangle
$A = \frac{1}{2}bh$	Triangle
$A = \frac{1}{2}ab\sin C$	Triangle with unknown height
$A = \pi r^2$	Circle
$A = \frac{1}{2}(b_1 + b_2)h$	Trapezoid

Distance Formula
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Pi Defined
$\pi = \frac{\text{circumference}}{\text{diameter}} = \frac{\text{circumference}}{2 \bullet \text{radius}}$

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Circumference of a Circle	
$C = 2\pi r$	Circumference given the radius
$C = \pi d$	Circumference given the diameter

Converting Between Degrees and Radians
$\dfrac{\text{radian measure}}{\pi} = \dfrac{\text{degree measure}}{180}$

Laws of Sines and Cosines	
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$	Law of Sines
$c^2 = a^2 + b^2 - 2ab \cos C$	Law of Cosines

Inverse Trigonometric Functions
Arcsin $\theta = \sin^{-1}\theta$
Arccos $\theta = \cos^{-1}\theta$
Arctan $\theta = \tan^{-1}\theta$

Arc Length	
$s = \theta r$	Arc length (θ in radians)

Density		
$\text{Density}_{\text{Area}} = \dfrac{\text{mass or quantity}}{\text{number of } \textit{square} \text{ units}}$	or $\rho_A = \dfrac{m}{A}$	Area density
$\text{Density}_{\text{Volume}} = \dfrac{\text{mass or quantity}}{\text{number of } \textit{cubic} \text{ units}}$	or $\rho = \dfrac{m}{V}$	Volume density

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MEASUREMENTS

Length
Metric
1 kilometer (km) = 1000 meters (m)
1 meter (m) = 100 centimeters (cm)
1 centimeter (cm) = 10 millimeters (mm)
Customary
1 mile (mi) = 1760 yards (yd)
1 mile (mi) = 5280 feet (ft)
1 yard (yd) = 3 feet (ft)
1 foot (ft) = 12 inches (in)

Volume and Capacity
Metric
1 liter (L) = 1000 milliliters (mL)
Customary
1 gallon (gal) = 4 quarts (qt)
1 quart (qt) = 2 pints (pt)
1 pint (pt) = 2 cups (c)
1 cup (c) = 8 fluid ounces (fl oz)

Weight and Mass
Metric
1 kilogram (kg) = 1000 grams (g)
1 gram (g) = 1000 milligrams (mg)
1 metric ton (MT) = 1000 kilograms
Customary
1 ton (T) = 2000 pounds (lb)
1 pound (lb) = 16 ounces (oz)

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English	Unit/Lesson	Español
	#	
68–95–99.7 rule a rule that states percentages of data under the normal curve are as follows: $\mu \pm 1\sigma \approx 68\%$, $\mu \pm 2\sigma \approx 95\%$, and $\mu \pm 3\sigma \approx 99.7\%$; also known as the *Empirical Rule*	1.1	**regla 68–95–99,7** regla que establece los siguientes porcentajes de datos bajo la curva normal: $\mu \pm 1\sigma \approx 68\%$, $\mu \pm 2\sigma \approx 95\%$ y $\mu \pm 3\sigma \approx 99,7\%$; también se la conoce como *Regla Empírica*
	A	
absolute value equation an equation of the form $y = \lvert ax + b \rvert + c$, where x is the independent variable, y is the dependent variable, and a, b, and c are real numbers	2A.4	**ecuación de valor absoluto** ecuación de la forma $y = \lvert ax + b \rvert + c$, donde x es la variable independiente, y es la variable dependiente y a, b y c son números reales
absolute value function a function of the form $f(x) = \lvert ax + b \rvert + c$, where x is the independent variable and a, b, and c are real numbers	4B.4	**función de valor absoluto** función de la forma $f(x) = \lvert ax + b \rvert + c$, donde x es la variable independiente a, b y c son números reales
addition rule for mutually exclusive events If events A and B are mutually exclusive, then the probability that A or B will occur is the sum of the probability of each event; $P(A \text{ or } B) = P(A) + P(B)$.	1.4	**regla de adición para eventos mutuamente excluyentes** si los eventos A y B son mutuamente excluyentes, la probabilidad de que A o B suceda es la suma de la probabilidad de cada evento; $P(A \text{ or } B) = P(A) + P(B)$.
alternative hypothesis any hypothesis that differs from the null hypothesis; that is, a statement that indicates there is a difference in the data from two treatments; represented by H_a	1.5	**hipótesis alternativa** toda hipótesis que difiera de la hipótesis nula; es decir, una afirmación que indica que existe una diferencia en los datos de dos tratamientos; representada por H_a
altitude the perpendicular line from a vertex of a figure to its opposite side; height	3.2	**altitud** línea perpendicular desde el vértice de una figura hasta su lado opuesto; altura
ambiguous case a situation wherein the Law of Sines produces two possible answers. This only occurs when the lengths of two sides and the measure of the non-included angle are given (SSA).	3.2	**caso ambiguo** situación en la cual la Ley de Senos produce dos respuestas posibles. Esto solo ocurre cuando están dadas las longitudes de los dos lados y la medida del ángulo no incluido (SSA).

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amplitude the coefficient a or c of the sine or cosine term in a function of the form $f(x) = a \sin bx$ or $g(x) = c \cos dx$; on a graph of the cosine or sine function, the vertical distance from the y-coordinate of the maximum point on the graph to the midline of the cosine or sine curve	3.3 4A.3	**amplitud** el coeficiente a o c del término de seno o coseno en una función de la forma $f(x) = a \sin bx$ o $g(x) = c \cos dx$; en un gráfico de la función seno o coseno, la distancia vertical desde la coordenada y del punto máximo en la gráfica hasta la línea media de la curva de seno o coseno
arccosine the inverse of the cosine function, written $\cos^{-1}\theta$ or $\arccos\theta$	3.2	**arcocoseno** inversa de la función coseno; se expresa $\cos^{-1}\theta$ o $\arccos\theta$
arc length the distance between the endpoints of an arc; written as $d(\overset{\frown}{ABC})$ or $m\overset{\frown}{AC}$	3.1	**longitud de arco** distancia entre los puntos extremos de un arco; se expresa como $d(\overset{\frown}{ABC})$ o $m\overset{\frown}{AC}$
arcsine the inverse of the sine function, written $\sin^{-1}\theta$ or $\arcsin\theta$	3.2	**arcoseno** inversa de la función seno; se expresa $\text{sen}^{-1}\theta$ o $\text{arcsen}\theta$
argument the result of raising the base of a logarithm to the power of the logarithm, so that b is the argument of the logarithm $\log_a b = c$; the term $cx + d$ in a cosine or sine function of the form $f(x) = a + b \sin(cx + d)$ or $g(x) = a + b \cos(cx + d)$	4A.2 4A.3	**argumento** el resultado de elevar la base de un logaritmo a la potencia del logaritmo, de manera que b es el argumento del logaritmo $\log_a b = c$; el término $cx + d$ en una función coseno o seno de la forma $f(x) = a + b \sin(cx + d)$ o $g(x) = a + b \cos(cx + d)$
asymptote a line that a function gets closer and closer to as one of the variables increases or decreases without bound	4B.3	**asíntota** una línea que una función se acerca cada vez más cerca de una de las variables aumenta o disminuye sin límite
average rate of change the ratio of the difference of output values to the difference of the corresponding input values: $\frac{f(b)-f(a)}{b-a}$; a measure of how a quantity changes over some interval	4B.3	**tasa de cambio promedio** proporción de la diferencia de valores de salida a la diferencia de valores correspondientes de entrada: $\frac{f(b)-f(a)}{b-a}$; medida de cuánto cambia una cantidad en cierto intervalo
axis of rotation a line about which a plane figure can be rotated in three-dimensional space to create a solid figure, such as a diameter or a symmetry line	4B.5	**eje de rotación** línea alrededor de la cual puede girar la figura de un plano en un espacio de tres dimensiones para crear una figura sólida, como un diámetro o una línea de simetría

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	B	
base the quantity that is being raised to an exponent in an exponential expression; in a^x, a is the base; or, the quantity that is raised to an exponent which is the value of the logarithm, such as 2 in the equation $\log_2 g(x) = 3 - x$	4A.2	**base** cantidad que es elevada a un exponente en una expresión exponencial; en a^x, a es la base; o, la cantidad que se eleva a un exponente que es el valor del logaritmo, tal que 2 en la ecuación $\log_2 g(x) = 3 - x$
bias leaning toward one result over another; having a lack of neutrality	1.2 1.3 1.5	**sesgo** inclinación por un resultado sobre otro; carecer de neutralidad
biased sample a sample in which some members of the population have a better chance of inclusion in the sample than others	1.2	**muestra sesgada** muestra en la cual algunos miembros de la población tienen una mayor posibilidad de ser incluidos en la muestra que otros
binomial experiment an experiment in which there are a fixed number of trials, each trial is independent of the others, there are only two possible outcomes (success or failure), and the probability of each outcome is constant from trial to trial	1.4	**experimento binomial** experimento en el que existe un número fijo de pruebas, cada prueba es independiente de las demás, existen dos resultados posibles (éxito o fracaso) y la probabilidad de cada resultado es constante de prueba a prueba
binomial probability distribution formula the distribution of the probability, P, of exactly x successes out of n trials, if the probability of success is p and the probability of failure is q; given by the formula $P = \binom{n}{x} p^x q^{n-x}$	1.4	**fórmula de distribución binomial de probabilidad** la distribución de la probabilidad, P, de exactamente x éxitos entre n pruebas, si la probabilidad de éxito es p y la probabilidad de fracaso es q; dada por la fórmula $P = \binom{n}{x} p^x q^{n-x}$

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Binomial Theorem a theorem stating that a binomial $(a + b)^n$ can be expanded using the formula $\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \bullet a^{n-k}b^k = 1a^nb^0 + \frac{n}{1} \bullet a^{n-1}b^1 + \frac{n(n-1)}{1 \bullet 2} \bullet a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \bullet 2 \bullet 3} \bullet a^{n-3}b^3 + \cdots + 1a^0b^n$	2A.2	**Teorema del Binomio** teorema que expresa que un binomio $(a + b)^n$ puede ampliarse utilizando la fórmula $\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \bullet a^{n-k}b^k = 1a^nb^0 + \frac{n}{1} \bullet a^{n-1}b^1 + \frac{n(n-1)}{1 \bullet 2} \bullet a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \bullet 2 \bullet 3} \bullet a^{n-3}b^3 + \cdots + 1a^0b^n$
boundary condition a constraint or limit on a function or domain value based on real-world conditions or restraints in the problem or its solution	4B.3	**condición de contorno** restricción o límite en un valor de dominio o función basado en condiciones del mundo real o restricciones en el problema o su solución
C		
central angle an angle with its vertex at the center of a circle	3.1	**ángulo central** ángulo con su vértice en el centro de un círculo
chance variation a measure showing how precisely a sample reflects the population, with smaller sampling errors resulting from large samples and/or when the data clusters closely around the mean; also called *sampling error*	1.2	**variación aleatoria** medida que muestra cómo una muestra refleja con precisión la población, con errores de muestreo más pequeños que resultan de muestras grandes y/o cuando los datos se agrupan estrechamente alrededor de la media; también llamada *error de muestreo*
closure a system is closed, or shows closure, under an operation if the result of the operation is within the system	2A.1	**cierre** un sistema es cerrado, o tiene cierre, en una operación si el resultado de la misma está dentro del sistema
cluster sample a sample in which naturally occurring groups of population members are chosen for a sample	1.2	**muestreo en grupos** muestra en la cual se eligen para una muestra grupos naturalmente ya formados de miembros de la población
coefficient the number multiplied by a variable in an algebraic expression	2A.1	**coeficiente** número multiplicado por una variable en una expresión algebraica

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combination a subset of a group of objects taken from a larger group of objects; the order of the objects does not matter, and objects may be repeated. A combination of size r from a group of n objects can be represented using the notation ${}_nC_r$, where ${}_nC_r = \frac{n!}{(n-r)!r!}$.	1.2	**combinación** subconjunto de un grupo de objetos tomado de un grupo más grande de objetos; el orden de los objetos no importa y los objetos pueden repetirse. Se puede representar una combinación de tamaño r de un grupo de n objetos con la notación ${}_nC_r$, donde ${}_nC_r = \frac{n!}{(n-r)!r!}$.
combination of functions the process of combining two or more functions using the operations of addition, subtraction, multiplication, or division to create a new function	4B.2	**combinación de funciones** proceso de combinar dos o más funciones utilizando las operaciones de adición, sustracción, multiplicación o división para crear una nueva función
common denominator a quantity that is a shared multiple of the denominators of two or more fractions	2B.1 2B.2	**denominador común** cantidad que es un múltiplo compartido de los denominadores de dos o más fracciones
common logarithm a base-10 logarithm which is usually written without the number 10, such as $\log x = \log_{10} x$	4A.2	**logaritmo común** logaritmo de base 10 que se escribe normalmente sin el número 10, como $\log x = \log_{10} x$
common ratio the ratio of a term in a geometric sequence to the previous term in that sequence; indicated by the variable r and given by the formula $r = \frac{a_n}{a_{n-1}}$	2A.5	**relación común** la relación de un término en una secuencia geométrica con el término anterior en esa secuencia; indicada por la variable r y dada por la fórmula $r = \frac{a_n}{a_{n-1}}$
complex conjugate the complex number that when multiplied by another complex number produces a value that is wholly real; the complex conjugate of $a + bi$ is $a - bi$	2A.2 2A.3	**conjugado de número complejo** número complejo que cuando se multiplica por otro número complejo produce un valor totalmente real; el conjugado complejo de $a + bi$ es $a - bi$

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Complex Conjugate Theorem Let $p(x)$ be a polynomial with real coefficients. If $a + bi$ is a root of the equation $p(x) = 0$, where a and b are real and $b \neq 0$, then $a - bi$ is also a root of the equation.	2A.3	**Teorema de Complejo Conjugado** suponiendo que $p(x)$ es un polinomio con coeficientes reales, si $a + bi$ es una raíz de la ecuación $p(x) = 0$, donde a y b son reales y $b \neq 0$, entonces $a - bi$ es también una raíz de la ecuación.
complex number a number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit	2A.2	**número complejo** número en la forma $a + bi$, donde a y b son números reales e i es la unidad imaginaria
composition of functions the process of substituting one function for the independent variable of another function to create a new function	4B.2	**composición de funciones** proceso de sustituir una función por la variable independiente de otra función para crear una función nueva
conditional probability of *B* given *A* the probability that event B occurs, given that event A has already occurred. If A and B are two events from a sample space with $P(A) \neq 0$, then the conditional probability of B given A, denoted $P(B \mid A)$, has two equivalent expressions: $P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\text{number of outcomes of } (A \text{ and } B)}{\text{all outcomes in } A}$	1.6	**probabilidad condicional de *B* dado *A*** probabilidad de que el evento B se produzca, dado que el evento A ya se ha producido. Si A y B son dos eventos de un espacio de muestra con $P(A) \neq 0$, entonces la probabilidad condicional de B dado A, indicado $P(B \mid A)$, tiene dos expresiones equivalentes: $P(B \mid A) = \frac{P(A \text{ y } B)}{P(A)} = \frac{\text{cantidad de resultados de (A y B)}}{\text{todos los resultados en A}}$
confidence interval an interval of numbers within which it can be claimed that repeated samples will result in the calculated parameter; generally calculated using the estimate plus or minus the margin of error	1.4	**intervalo de confianza** intervalo de números dentro del cual se puede afirmar que las muestras repetidas tendrán como resultado el parámetro calculado; generalmente se calcula usando la estimación más o menos el margen de error
confidence level the probability that a parameter's value can be found in a specified interval; also called *level of confidence*	1.4 1.5	**nivel de confianza** probabilidad de que se pueda encontrar el valor de un parámetro en un intervalo específico; también llamado *grado de confianza*

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confounding variable an ignored or unknown variable that influences the result of an experiment, survey, or study	1.3 1.5	**variable de confusión** una variable ignorada o desconocida que influye sobre el resultado de un experimento, encuesta o estudio
congruent having the same shape, size, lines, and angles; the symbol for representing congruency between figures is ≅	4B.5	**congruente** tener la misma forma, tamaño, rectas y ángulos; el símbolo para representar la congruencia entre números es ≅
constant term a term whose value does not change	2A.1	**término constante** término cuyo valor no cambia
constraint a limit or restriction on the domain, range, and/or solutions of a mathematical or real-world problem	4B.1	**restricción** límite en el dominio, rango y/o soluciones de un problema matemático o del mundo real
continuous data a set of values for which there is at least one value between any two given values	1.1	**datos continuos** conjunto de valores para el que existe al menos un valor entre dos valores dados
continuous distribution the graphed set of values, a curve, in a continuous data set	1.1	**distribución continua** conjunto de valores representado gráficamente, una curva, en un conjunto de datos continuos
continuous function a function that does not have a break in its graph across a specified domain	4B.4	**función continua** función que no tiene una interrupción en su curva a lo largo de un dominio específico
control group the group of participants in a study who are not subjected to the treatment, action, or process being studied in the experiment, in order to form a comparison with participants who are subjected to it	1.3	**grupo de control** grupo de participantes en un estudio que no están sujetos al tratamiento, acción o proceso que está en estudio en el experimento con el fin de establecer una comparación con participantes que sí lo están.
convenience sample a sample in which members are chosen to minimize the time, effort, or expense involved in sampling	1.2	**muestreo de conveniencia** muestreo en el cual se eligen los miembros para minimizar el tiempo, esfuerzo o gasto involucrado en este proceso

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converge to approach a finite limit. If the sequence of the partial sums of a series approaches the value of a given number (the limit), then the entire series converges to that limit; that is, the series has a sum. An infinite series converges when the absolute value of the common ratio r is less than 1 $(\lvert r\rvert<1)$.	2A.5	**converger** acercarse a un límite finito. Si la secuencia de las sumas parciales de una serie se acerca al valor de un número dado (el límite), entonces la serie completa converge hacia ese límite; es decir, la serie tiene una suma. Una serie infinita converge cuando el valor absoluto de la relación común r es menor que 1 $(\lvert r\rvert<1)$.
correlation a measure of the power of the association between exactly two quantifiable variables	1.5	**correlación** una medida de la potencia de la asociación entre exactamente dos variables cuantificables
cosecant the reciprocal of sine, $\csc\theta=\frac{1}{\sin\theta}$; the cosecant of $\theta=\csc\theta=\frac{\text{length of hypotenuse}}{\text{length of opposite side}}$	3.1	**cosecante** la recíproca del seno, $\csc\theta=\frac{1}{\sin\theta}$; la cosecante de $\theta=\csc\theta=\frac{\text{largo de la hipotenusa}}{\text{largo del lado opuesto}}$
cosine a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the side adjacent to the length of the hypotenuse; the cosine of $\theta=\cos\theta=\frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$	3.1 3.2 4A.3	**coseno** función trigonométrica de un ángulo agudo en un triángulo rectángulo que es la proporción de la longitud de lado adyacente a la longitud de la hipotenusa; el coseno de $\theta=\cos\theta=\frac{\text{longitud del lado adyacente}}{\text{longitud de la hipotenusa}}$
cosine function a trigonometric function of the form $f(x)=a\cos[b(x-c)]+d$, in which a, b, c, and d are constants and x is a variable defined in radians over the domain $(-\infty,\infty)$	3.3 4A.3	**función del coseno** función trigonométrica de la forma $f(x)=a\cos[b(x-c)]+d$, donde a, b, c y d son constantes y x es una variable definida en radianes a lo largo del dominio $(-\infty,\infty)$

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cotangent the reciprocal of tangent, $\cot\theta = \frac{1}{\tan\theta}$; the cotangent of θ = cot $\theta = \frac{\text{length of adjacent side}}{\text{length of opposite side}}$	3.1	**cotangente** recíproco de la tangente, $\cot\theta = \frac{1}{\tan\theta}$; la cotangente de θ = cot $\theta = \frac{\text{longitud del lado adyacente}}{\text{longitud del lado opuesto}}$
coterminal angles angles that, when drawn in standard position, share the same terminal side	3.1	**ángulos coterminales** ángulos que, cuando están trazados en una posición estándar, comparten el mismo lado terminal
critical value a measure of the number of standards of error to be added to or subtracted from the mean in order to achieve the desired confidence level; also known as z_c-*value*	1.4	**valor crítico** medida de la cantidad de estándares de error que se suma o se resta de la media para lograr el nivel de confianza deseado; también conocido como *valor* z_c
cross section the plane figure formed by the intersection of a plane with a solid figure, where the plane is at a right angle to the surface of the solid figure	4B.5	**sección transversal** figura del plano formada por la intersección de un plano con una figura sólida, donde el plano está en un ángulo recto a la superficie de la figura sólida
cube root function a function that contains the cube root of a variable. The general form is $f(x) = a\sqrt[3]{(x-h)-k}$, where a, h, and k are real numbers.	4B.4	**función raíz cúbica** función que contiene la raíz cúbica de una variable. La forma general es $y = a\sqrt[3]{(x-h)}+k$, donde a, h, y k son números reales.
cycle the smallest representation of a cosine or sine function graph as defined over a restricted domain; equal to one repetition of the period of a function	3.3 4A.3	**ciclo** la representación más pequeña de una gráfica de la función coseno o seno definida a través de un dominio restringido; igual a una repetición del período de una función
	D	
data numbers in context	1.3	**datos** números en contexto
data fitting the process of assigning a rule, usually an equation or formula, to a collection of data points as a method of predicting the values of new dependent variables that result from new independent-variable values	4B.1	**ajuste de datos** proceso de asignar una regla, normalmente una ecuación o fórmula, a una colección de puntos de datos como un método de predecir los valores de nuevas variables dependientes que vienen de nuevos valores de variables independientes

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data point a point (x, y) on a two-dimensional coordinate plane that represents the value of an independent variable (x) that results in a specific dependent variable value (y). The term also refers to solutions for an equation or inequality in one variable that originate from a real-world situation. A data point is also called an *ordered pair*.	4B.1	**punto de datos** un punto (x, y) en un plano de coordenadas de dos dimensiones que representa el valor de una variable independiente (x) que da como resultado un valor específico de variable dependiente (y). El término también se refiere a las soluciones para una ecuación o desigualdad en una variable que proviene de una situación del mundo real. Un punto de datos se denomina también un *par ordenado*.
degree a unit used to measure angles. One degree represents $\frac{1}{360}$ of a full rotation.	3.1 4A.3	**grado** unidad usada para medir ángulos. Un grado representa $\frac{1}{360}$ de una rotación completa.
degree of a one-variable polynomial the greatest exponent of the variable in a polynomial	2A.1	**grado de un polinomio de una variable** el mayor exponente de la variable en un polinomio
degrees of freedom (*df*) the number of data values that are free to vary in the final calculation of a statistic; that is, values that can change or move without violating the constraints on the data	1.5	**grados de libertad (*df*)** la cantidad de valores de datos que varían libremente en el cálculo final de una estadística; es decir, los valores que pueden cambiar o moverse sin violar las restricciones en los datos
***delta* (Δ)** a Greek letter commonly used to represent the change in a value	4B.3	***delta* (Δ)** letra griega utilizada comúnmente para representar el cambio en un valor
denominator the value located below the line of a rational expression or fraction; the divisor	2B.1	**denominador** el valor ubicado debajo de la línea de una expresión racional o fracción; el divisor
density the amount, number, or other quantity per unit of area or volume of some substance or population being studied	4B.5	**densidad** la cantidad, el número u otra cantidad por unidad de área o volumen de alguna sustancia o población que está siendo estudiada
dependent system a system of equations that intersect at every point	2A.4 2B.2	**sistema dependiente** sistema de ecuaciones que se cruzan en cada punto

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dependent variable labeled on the *y*-axis; the quantity that is based on the input values of the independent variable; the output variable of a function	4B.1	**variable dependiente** designada en el eje de *y*; cantidad que se basa en los valores de entrada de la variable independiente; variable de salida de una función
depressed polynomial the result of dividing a polynomial by one of its binomial factors	2A.3	**polinomio deprimido** el resultado de dividir un polinomio por uno de sus factores binómicos
descending order polynomials ordered by the power of the variables, with the largest power listed first and the constant last	2A.1	**orden descendente** polinomios ordenados según la potencia de las variables, con la potencia más grande indicada primero y la constante, última
desirable outcome the data sought or hoped for, represented by *p*; also known as *favorable outcome* or *success*	1.4	**resultado deseado** datos buscados o esperados, representado por *p*; también conocido como *resultado favorable* o *éxito*
discontinuous function a function that has a break, hole, or jump in the graph	4B.4	**función discontinua** una función que tiene un descanso, agujero, o salto en el gráfico
discrete data a set of values with gaps between successive values	1.1	**datos discretos** conjunto de valores con interrupciones entre valores sucesivos
discrete function a function in which every element of the domain is individually separate and distinct	2A.5 4B.4	**función discreta** función en la cual cada elemento del dominio está individualmente separado y distinguible
diverge to *not* approach a finite limit. If a series does not have a sum (that is, the sequence of its partial sums does not approach a given number), then the series diverges. An infinite series diverges when the absolute value of the common ratio *r* is greater than 1 $(\lvert r\rvert > 1)$.	2A.5	**divergir** *no* acercarse a un límite finito. Si una serie no tiene una suma (es decir, la secuencia de sus sumas parciales no se acerca a un número dado), entonces la serie diverge. Una serie infinita diverge cuando el valor absoluto de la relación común *r* es mayor que 1 $(\lvert r\rvert > 1)$.
domain the set of all input values (*x*-values) that satisfy the given function without restriction	4A.1 4B.1	**dominio** conjunto de todos los valores de entrada (valores de *x*) que satisfacen la función dada sin restricciones
double-blind study a study in which neither the researcher nor the participants know who has been subjected to the treatment, action, or process being studied, and who is in a control group	1.3	**estudio doble-ciego** estudio en el cual ni el investigador ni los participantes saben quién se sometió al tratamiento, acción o proceso que está siendo estudiado y quién está en un grupo de control

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	E	
e an irrational number with an approximate value of 2.71828; *e* is the base of the natural logarithm (ln *x* or $\log_e x$)	4A.2	***e*** número irracional con un valor aproximado de 2,71828; *e* es la base del logaritmo natural (In *x* o $\log_e x$)
empirical probability the number of times an event actually occurs divided by the total number of trials, given by the formula $P(E)=\frac{\text{number of occurrences of the event}}{\text{total number of trials}}$; also called *experimental probability*	1.6	**probabilidad empírica** cantidad de veces que se produce un evento dividido por la cantidad total de pruebas, dada por la fórmula $P(E)=\frac{\text{cantidad de ocurrencias del evento}}{\text{cantidad total de pruebas}}$; también se denomina *probabilidad experimental*
Empirical Rule a rule that states percentages of data under the normal curve are as follows: $\mu \pm 1\sigma \approx 68\%$, $\mu \pm 2\sigma \approx 95\%$, and $\mu \pm 3\sigma \approx 99.7\%$; also known as the *68–95–99.7 rule*	1.1	**Regla Empírica** regla que establece los siguientes porcentajes de datos bajo la curva normal: $\mu \pm 1\sigma \approx 68\%$, $\mu \pm 2\sigma \approx 95\%$ y $\mu \pm 3\sigma \approx 99{,}7\%$; también se conoce como *la regla 68–95–99,7*
empty set a set that has no elements, denoted by ∅; the solution to a system of equations with no intersection points, denoted by { }	2A.4 2B.2	**conjunto vacío** un conjunto que no tiene elementos, indicado por ∅; la solución a un sistema de ecuaciones sin puntos de intersección, indicado por { }
end behavior the behavior of the graph as *x* approaches positive or negative infinity	2A.3	**comportamiento final** el comportamiento del gráfico a medida que *x* se acerca al infinito positivo o negativo
even-degree polynomial function a polynomial function in which the highest exponent is an even number. Both ends of the graph of an even-degree polynomial function will extend in the same direction, either upward or downward.	2A.3	**función polinómica de grado par** función polinómica en la cual el exponente mayor es un número par. Ambos extremos del gráfico de una función polinómica de grado par se extenderán en la misma dirección, hacia arriba o hacia abajo.
even function a function that, when evaluated for $-x$, results in a function that is the same as the original function; $f(-x) = f(x)$	4B.2	**función par** función que, cuando se la evalúa para $-x$, tiene como resultado una función que es igual a la original; $f(-x) = f(x)$

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expected value an estimate of value that is determined by finding the product of a total value and a probability of a given event; symbolized by $E(X)$	1.6	**valor esperado** estimación de valor que se determina al encontrar el producto de un valor total y una probabilidad de un evento dado; simbolizado por $E(X)$
experimental probability the number of times an event actually occurs divided by the total number of trials, given by the formula $P(E) = \frac{\text{number of occurrences of the event}}{\text{total number of trials}}$; also called *empirical probability*	1.6	**probabilidad experimental** cantidad de veces que se produce un evento dividido por la cantidad total de pruebas, dada por la fórmula $P(E) = \frac{\text{cantidad de ocurrencias del evento}}{\text{cantidad total de pruebas}}$; también se denomina *probabilidad empírica*
experiment a process or action that has observable results	1.3	**experimento** proceso o acción con resultados observables
explicit formula for a geometric sequence a formula used to find any term in a sequence. The formula is $a_n = a_1 \bullet r^{n-1}$, where n is a positive integer that represents the number of terms in the sequence, r is the common ratio, a_1 is the value of the first term in the sequence, and a_n is the value of the nth term of the sequence.	2A.5	**fórmula explícita para una secuencia geométrica** fórmula usada para encontrar algún término en una secuencia. La fórmula es $a_n = a_1 \bullet r^{n-1}$, donde n es un número entero positivo que representa la cantidad de términos en la secuencia, r es la relación común, a_1 es el valor del primer término de la secuencia y a_n es el valor del término n de la secuencia.
exponential equation an equation of the form $y = ab^x$, where x is the independent variable, y is the dependent variable, and a and b are real numbers	2A.4 4B.1	**ecuación exponencial** ecuación de la forma $y = ab^x$, en la que x es la variable independiente, y es la variable dependiente, y a y b son números reales
exponential expression an expressionthat contains a base raised to a power/ exponent	2A.1	**expresión exponencial** expresión que incluye una base elevada a una potencia o exponente
exponential function a function of the form $f(x) = ab^{cx}$, in which a, b, and c are constants; a function that has a variable in the exponent, such as $f(x) = 0.4 \bullet 6^{-0.2x}$	4A.2 4B.4	**función exponencial** función de la fórmula $f(x) = ab^{cx}$ en la cual a, b y c son constantes; una función que tiene una variable en el exponente, tal como $f(x) = 0{,}4 \bullet 6^{-0{,}2x}$

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extraneous solution a solution of an equation that arises during the solving process, but which is not a solution of the original equation	2B.2	**solución extraña** solución de una ecuación que surge durante el proceso de resolución pero que no es una solución de la ecuación original
extrema the minima or maxima of a function	4B.3	**extremos** los mínimos o máximos de una función
	F	
factor one of two or more numbers or expressions that when multiplied produce a given product	2A.1	**factor** uno de dos o más números o expresiones que cuando se multiplican generan un producto determinado
factorial the product of an integer and all preceding positive integers, represented using a ! symbol; $n! = n \bullet (n-1) \bullet (n-2) \bullet \cdots \bullet 1$. For example, 5! = 5 • 4 • 3 • 2 • 1. By definition, 0! = 1.	1.2 1.4 2A.2	**factorial** producto de un entero y todos los enteros positivos anteriores, que se representa con el símbolo !; $n! = n \bullet (n-1) \bullet (n-2) \bullet \ldots \bullet 1$. Por ejemplo, 5! = 5 • 4 • 3 • 2 • 1. Por definición, 0! = 1.
factor of a polynomial any polynomial that divides evenly into the function $p(x)$	2A.3	**factor de un polinomio** todo polinomio que divide de manera uniforme en la función $p(x)$
Factor Theorem The binomial $x - a$ is a factor of the polynomial $p(x)$ if and only if $p(a) = 0$.	2A.3	**Teorema del Factor** el binomio $x - a$ es un factor del polinomio $p(x)$ si y solo si $p(a) = 0$.
failure the occurrence of an event that was not sought out or wanted, represented by q; also known as *undesirable outcome* or *unfavorable outcome*	1.4	**fracaso** ocurrencia de un evento que no fue buscado ni deseado, representado por q; también conocido como *resultado no deseado* o *resultado desfavorable*
fair describes a situation or game in which all of the possible outcomes have an equal chance of occurring	1.6	**equitativo** describe una situación o juego en el cual todos los resultados posibles tienen igual probabilidad de producirse
false negative result a determination that an experiment has produced an incorrect negative result	1.6	**resultado falso negativo** determinación de que un experimento ha producido un resultado negativo incorrecto
false positive result a determination that an experiment has produced an incorrect positive result	1.6	**resultado falso positivo** determinación de que un experimento ha producido un resultado positivo incorrecto

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family of functions a set of functions whose graphs have the same general shape as their parent function. The parent function is the function with a simple algebraic rule that represents the family of functions.	4B.2	**familia de funciones** conjunto de funciones cuyas gráficas tienen la misma forma general que su función raíz. La función raíz es la función con una regla algebraica simple que representa la familia de funciones.
favorable outcome the data sought or hoped for, represented by *p*; also known as *desirable outcome* or *success*	1.4	**resultado favorable** datos buscados o esperados, representados por *p*; también conocido como *resultado deseado* o *éxito*
finite limited in number	2A.5	**finito** limitado en número
finite geometric series a series that has a limited or definite number of terms; can be written as $\sum_{k=1}^{n} a_1 r^{k-1}$, where *n* is a positive integer that represents the number of terms in the series, a_1 is the first term, *r* is the common ratio, and *k* is the number of the term	2A.5	**serie geométrica finita** serie que tiene una cantidad limitada o definida de términos; puede ser escrita como $\sum_{k=1}^{n} a_1 r^{k-1}$, donde *n* es un número entero positivo que representa la cantidad de términos en la serie, a_1 es el primer término, *r* es la relación común y *k* es el número del término
formula a mathematical statement of the relationship between two or more variables	4B.1	**fórmula** afirmación matemática de la relación entre dos o más variables
fraction a ratio of two expressions or quantities	2B.1	**fracción** relación de dos expresiones o cantidades
frequency of a periodic function the reciprocal of the period for a periodic function; indicates how often the function repeats	3.3	**frecuencia de una función periódica** recíproca del período para una función periódica; indica con que frecuencia se repite la función
function a relation in which every element of the domain is paired with exactly one element of the range; that is, for every value of *x*, there is exactly one value of *y*	4A.1	**función** relación en la que cada elemento del dominio se empareja con un único elemento del rango; es decir, para cada valor de *x*, existe exactamente un valor de *y*

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Fundamental Theorem of Algebra If $p(x)$ is a polynomial function of degree $n \geq 1$ with complex coefficients, then the related equation $p(x) = 0$ has at least one complex solution (root).	2A.3	**Teorema fundamental del álgebra** Si $P(x)$ es una función polinómica de grado $n \geq 1$ con coeficientes complejos, entonces la ecuación relacionada $P(x) = 0$ tiene al menos una solución compleja (raíz).
	G	
geometric sequence an ordered list of terms in which each new term is the product of the preceding term and a common ratio, r. For a sequence to be geometric, r cannot be equal to 1 ($r \neq 1$).	2A.5	**secuencia geométrica** lista ordenada de términos en la cual cada término nuevo es el producto del término anterior y una relación común, r. Para que una secuencia sea geométrica, r no puede ser igual a 1 ($r \neq 1$).
geometric series the sum of a specified number of terms from a geometric sequence	2A.5	**serie geométrica** suma de una cantidad específica de términos de una secuencia geométrica
	H	
half plane a region containing all points on one side of a boundary, which is a line or curve that continues in both directions infinitely. The line or curve may or may not be included in the region. A half plane can be used to represent a solution to an inequality statement.	4B.1	**semiplano** región que contiene todos los puntos en un lado de un límite, que es una línea recta o curva que continúa en ambas direcciones de manera infinita. La línea recta o curva puede o no estar incluida en la región. Un semiplano puede utilizarse para representar una solución a una afirmación de desigualdad.
hypothesis a statement that you are trying to prove or disprove	1.5	**hipótesis** afirmación que usted intenta probar o desaprobar
hypothesis testing assessing data in order to determine whether the data supports (or fails to support) the hypothesis as it relates to a parameter of the population	1.5	**prueba de hipótesis** evaluación de datos para determinar si los datos respaldan (o no respaldan) la hipótesis mientras se relaciona con un parámetro de la población

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	I	
imaginary number any number of the form *bi*, where *b* is a real number, $i=\sqrt{-1}$, and $b \neq 0$	2A.2	**número imaginario** cualquier número de la forma *bi*, en el que *b* es un número real, $i=\sqrt{-1}$, y $b \neq 0$
imaginary unit, *i* the letter *i*, used to represent the non-real value $i=\sqrt{-1}$	2A.2	**unidad imaginaria, *i*** la letra *i*, utilizada para representar el valor no real $i=\sqrt{-1}$
included angle the angle between two sides	3.2	**ángulo incluido** ángulo entre dos lados
independent system a system of equations with a finite number of points of intersection	2A.4	**sistema independiente** sistema de ecuaciones con una cantidad finita de puntos de intersección
independent variable labeled on the *x*-axis; the quantity that changes based on values chosen; the input variable of a function	4B.1	**variable independiente** designada en el eje *x*; cantidad que cambia según los valores seleccionados; variable de entrada de una función
inequality a mathematical statement that compares the value of an expression in one independent variable to the value of a dependent variable using the comparison symbols >, <, ≥, and ≤	4B.1	**desigualdad** afirmación matemática que compara el valor de una expresión en una variable independiente con el valor de una variable dependiente usando los símbolos de comparación >, <, ≥ y ≤
inference a conclusion reached upon the basis of evidence and reasoning	1.2	**inferencia** conclusión alcanzada sobre la base de evidencia y razonamiento
infinite geometric series a series that has an unlimited number of terms $(n=\infty)$; can be written as $\sum_{k=1}^{\infty} a_1 r^{k-1}$, where a_1 is the first term, *r* is the common ratio, and *k* is the number of the term	2A.5	**serie geométrica infinita** serie que tiene una cantidad ilimitada de términos $(n=\infty)$; se puede escribir como $\sum_{k=1}^{\infty} a_1 r^{k-1}$, donde a_1 es el primer término, *r* es la relación común y *k* es el número del término
infinite having no limit; represented by the infinity symbol, ∞	2A.5	**infinito** no tiene límite; representado por el símbolo de infinitud ∞
initial condition a constraint or limit on a function or domain value that exists in the form of a *y*-intercept or other starting-point mathematical restraint in a real-world problem or solution	4B.3	**condición inicial** restricción o límite en el valor de una función o de un dominio que existe en la forma de un intercepto de *y* u otra restricción matemática del punto de inicio en un problema o solución del mundo real

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initial side the stationary ray of an angle from which the measurement of the angle starts	3.1	**lado inicial** rayo fijo de un ángulo desde el cual comienza la medición del ángulo
integer a number that is not a fraction or decimal	2A.3	**entero** un número que no es una fracción ni un decimal
Integral Zero Theorem If the coefficients of a polynomial function are integers such that $a_n = 1$ and $a_0 \neq 0$, then any rational zeros of the function must be factors of a_0.	2A.3	**Teorema Integral Cero** si los coeficientes de una función polinómica son números enteros tales como $a_n = 1$ y $a_0 \neq 0$, entonces todos los ceros racionales de la función deben ser factores de a_0.
interval a set of values between a lower bound and an upper bound	1.1	**intervalo** conjunto de valores entre un límite inferior y un límite superior
inverse function the function that may result from switching the x- and y-variables in a given function; the inverse of $f(x)$ is written as $f^{-1}(x)$	4A.1	**función inversa** función que se produce como resultado de cambiar las variables x y y en una función determinada; la inversa de $f(x)$ se expresa como $f^{-1}(x)$
inverse relation a relation $g(x)$ such that $g(f(x)) = x$ and $f(g(y)) = y$ where $f(x)$ is a function	4A.1	**relación inversa** una relación $g(x)$ tal que $g(f(x)) = x$ y $f(g(y)) = y$ donde $f(x)$ es una función
Irrational Root Theorem If a polynomial $p(x)$ has rational coefficients and $a + b\sqrt{c}$ is a root of the polynomial equation $p(x) = 0$, where a and b are rational and $\sqrt{c}$ is irrational, then $a - b\sqrt{c}$ is also a root of $p(x) = 0$.	2A.3	**Teorema de la Raíz Irracional** Si un polinomio $p(x)$ tiene coeficientes racionales y $a + b\sqrt{c}$ es una raíz de la ecuación de polinomios $p(x) = 0$, donde a y b son racionales y $\sqrt{c}$ es irracional, entonces $a - b\sqrt{c}$ es también una raíz de $p(x) = 0$.
	L	
Law of Cosines a formula for any triangle which states $c^2 = a^2 + b^2 - 2ab\cos C$, where C is the included angle in between sides a and b, and c is the nonadjacent side across from $\angle C$	3.2	**Ley de Cosenos** fórmula para todo triángulo que establece $c^2 = a^2 + b^2 - 2ab\cos C$, donde C es el ángulo incluido entre los lados a y b, y c es el lado no adyacente u opuesto $\angle C$

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Law of Sines a formula for any triangle which states $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$, where a represents the measure of the side opposite $\angle A$, b represents the measure of the side opposite $\angle B$, and c represents the measure of the side opposite $\angle C$	3.2	**Ley de Senos** fórmula para todo triángulo que establece $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$, donde a representa la medida del lado opuesto $\angle A$, b representa la medida del lado opuesto $\angle B$ y c representa la medida del lado opuesto $\angle C$
leading coefficient the coefficient of the term with the highest power	2A.1	**coeficiente líder** coeficiente del término con la mayor potencia
least common denominator (LCD) the least common multiple of the denominators of two or more fractions	2B.1 2B.2	**denominador común mínimo (DCM)** múltiplo común mínimo de los denominadores de dos o más fracciones
level of confidence the probability that a parameter's value can be found in a specified interval; also called *confidence level*	1.4 1.5	**grado de confianza** probabilidad de que se pueda encontrar el valor de un parámetro en un intervalo específico; también llamado *nivel de confianza*
like terms terms that contain the same variables raised to the same power	2A.1	**términos semejantes** términos que contienen las mismas variables elevadas a la misma potencia
linear equation an equation that can be written in the form $ax + by = c$, where a, b, and c are constants; can also be written as $y = mx + b$, in which m is the slope and b is the y-intercept. The graph of a linear equation is a straight line; its solutions are the infinite set of points on the line.	2A.4 4B.1	**ecuación lineal** ecuación que puede expresarse en la forma $ax + by = c$, donde a, b y c son constantes; también puede escribirse como $y = mx + b$, en donde m es la pendiente y b es el intercepto de y. El gráfico de una ecuación lineal es una línea recta; sus soluciones son el conjunto infinito de puntos en la línea.
linear function a function that can be written in the form $ax + by = c$, where a, b, and c are constants; can also be written as $f(x) = mx + b$, in which m is the slope and b is the y-intercept. The graph of a linear function is a straight line; its solutions are the infinite set of points on the line.	4B.4	**función lineal** función que puede expresarse en la forma $ax + by = c$, donde a, b y c son constantes; también puede escribirse como $f(x) = mx + b$, donde m es la pendiente y b es el intercepto de y. El gráfico de una función lineal es una línea recta; sus soluciones son el conjunto infinito de puntos en la línea.

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local maximum the greatest value of a function for a particular interval of the function; also known as a *relative maximum*	2A.3 4B.3	**máximo local** el mayor valor de una función para un intervalo específico de la función; también conocido como *máximo relativo*
local minimum the least value of a function for a particular interval of the function; also known as a *relative minimum*	2A.3 4B.3	**mínimo local** el menor valor de una función para un intervalo específico de la función; también conocido como *mínimo relativo*
logarithmic equation an equation of the form $y = \log_a x$, which is an inverse of the exponential equation $y = a^x$ if $x > 0$	4B.1	**ecuación logarítmica** ecuación de la forma $y = \log_a x$, que es una inversa de la ecuación exponencial $y = a^x$ si $x > 0$
logarithmic function the inverse of an exponential function; for the exponential function $g(x) = 5^x$, the inverse logarithmic function is $x = \log_5 g(x)$	4A.2	**función logarítmica** la inversa de una función exponencial; para la función exponencial $g(x) = 5^x$, la función logarítmica inversa es $x = \log_5 g(x)$
	M	
margin of error the quantity that represents the level of confidence in a calculated parameter, abbreviated MOE. The margin of error can be calculated by multiplying the critical value by the standard deviation, if known, or by the SEM.	1.4	**margen de error** cantidad que representa el nivel de confianza en un parámetro calculado, abreviado MOE. El margen de error puede calcularse multiplicando el valor crítico por la desviación estándar, si se conoce, o por el SEM.
maximum the greatest value or highest point of a function	3.3	**máximo** el mayor valor o punto más alto de una función
mean a measure of center in a set of numerical data, computed by adding the values in a data set and then dividing the sum by the number of values in the data set; denoted as the Greek lowercase letter *mu*, μ; given by the formula $\mu = \frac{x_1 + x_2 + \ldots + x_n}{n}$, where each *x*-value is a data point and *n* is the total number of data points in the set	1.1	**media** medida del centro en un conjunto de datos numéricos, calculada al sumar los valores en un conjunto de datos y luego al dividir la suma por el número de valores en el conjunto de datos; indicada con la letra griega minúscula *mu*, μ; dada por la fórmula $\mu = \frac{x_1 + x_2 + \ldots + x_n}{n}$, donde cada valor de x es un punto de datos y *n* es la cantidad total de puntos de datos en el conjunto

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measurement bias bias that occurs when the tool used to measure the data is not accurate, current, or consistent	1.5	**sesgo de medición** sesgo que se produce cuando la herramienta utilizada para medir los datos no es exacta, actual o constante
median the middle-most value of a data set; 50% of the data is less than this value, and 50% is greater than it	1.1	**mediana** valor máximo-medio de un conjunto de datos; el 50 % de los datos es menor que este valor y el otro 50 % es mayor que él
midline in a cosine function or sine function of the form $f(x) = a + \sin x$ or $g(x) = a + \cos x$, a horizontal line of the form $y = a$ that bisects the vertical distance on a graph between the minimum and maximum function values	3.3 4A.3	**línea media** en una función del coseno o en una función del seno de la forma $f(x) = a + \sin x$ o $g(x) = a + \cos x$, una línea horizontal de la forma $y = a$ que divide en dos la distancia vertical en un gráfico entre los valores de funciones mínimos y máximos
minimum the least value or lowest point of a function	3.3	**mínimo** el menor valor o el punto más bajo de una función
***mu,* μ** a Greek letter used to represent mean	1.1	***mu,* μ** letra griega usada para representar la media
multiplicity (of a zero) the number of times a zero of a polynomial function occurs	2A.3	**multiplicidad (de un cero)** la cantidad de veces que sucede cero en una función polinómica
mutually exclusive events events that have no outcomes in common. If A and B are mutually exclusive events, then they cannot both occur.	1.4	**eventos mutuamente excluyent es** eventos que no tienen resultados en común. Si A y B son eventos mutuamente excluyentes, entonces no pueden producirse ambos.
	N	
natural logarithm a logarithm whose base is the irrational number e; usually written in the form "ln," which means "$\log_e$."	4A.2	**logaritmo natural** logaritmo cuya base es el número irracional e; escrito normalmente en la forma "ln", que significa "$\log_e$".
negatively skewed a distribution in which there is a "tail" of isolated, spread-out data points to the left of the median. "Tail" describes the visual appearance of the data points in a histogram. Data that is negatively skewed is also called *skewed to the left*.	1.1	**sesgado negativamente** distribución en la cual existe una "cola" de puntos de datos aislados y esparcidos a la izquierda de la mediana. La "cola" describe la apariencia visual de los puntos de datos en un histograma. Los datos que están sesgados negativamente también se denominan *sesgados a la izquierda*.

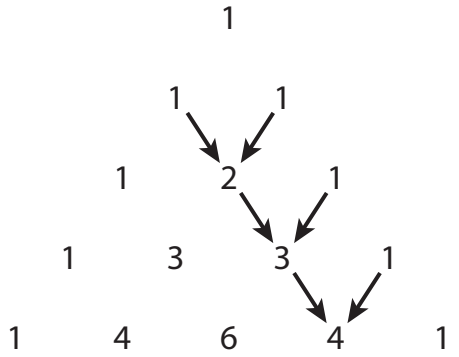
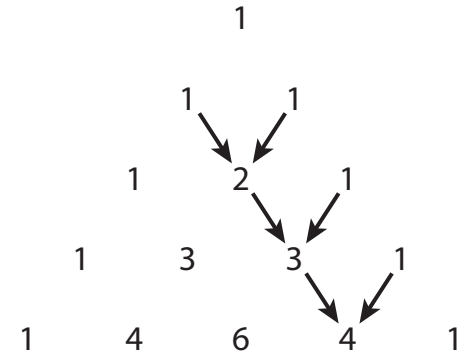
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neutral not biased or skewed toward one side or another; regarding surveys, neutral refers to phrasing questions in a way that does not lead the response toward one particular answer or side of an issue	1.3	**neutral** no sesgado hacia un lado u otro; respecto de las encuestas, neutral se refiere a la formulación de preguntas de una manera que no conduzca la respuesta hacia una respuesta o lado específico de un tema
nonresponse bias bias that occurs when the respondents to a survey have different characteristics than nonrespondents, causing the population that does not respond to be underrepresented in the survey's results	1.5	**sesgo sin respuesta** sesgo que se produce cuando los encuestados de una encuesta tienen características diferentes de los no encuestados, dando pie a que la población que no responde sea subrepresentada en los resultados de la encuesta
normal curve a symmetrical curve representing the normal distribution	1.1	**curva normal** curva simétrica que representa la distribución normal
normal distribution a set of values that are continuous, are symmetric to a mean, and have higher frequencies in intervals close to the mean than equal-sized intervals away from the mean	1.1	**distribución normal** conjunto de valores que son continuos, simétricos a una media y tienen frecuencias más altas en intervalos cercanos a la media que los intervalos de igual tamaño lejos de la media
null hypothesis the statement or idea that will be tested, represented by H_0; generally characterized by the concept that there is no relationship between the data sets, or that the treatment has no effect on the data	1.5	**hipótesis nula** afirmación o idea que será probada, representada por H_0; caracterizada generalmente por el concepto de que no hay relación entre los conjuntos de datos o que el tratamiento no tiene efecto en los datos
numerator the value located above the line of a rational expression or fraction; the dividend	2B.1	**numerador** el valor ubicado por encima de la línea de una expresión racional o fracción; el dividendo
	O	
oblique triangle a triangle that does not contain a right angle	3.2	**triángulo oblicuo** triángulo que no contiene un ángulo recto
observational study a study in which all data, including observations and measurements, are recorded in a way that does not change the subject that is being measured or studied	1.3	**estudio observacional** estudio en el cual todos los datos, incluyendo las observaciones y las mediciones, están registrados de tal manera que no cambian el objeto que está siendo medido o estudiado

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odd-degree polynomial function a polynomial function in which the highest exponent is an odd number. One end of the graph of an odd-degree polynomial function will extend upward and the other end will extend downward.	2A.3	**función polinómica de grado impar** función polinómica en la cual el exponente mayor es un número impar. Un extremo del gráfico de una función polinómica de grado impar se extenderá hacia arriba y el otro extremo se extenderá hacia abajo.
odd function a function that, when evaluated for $-x$, results in a function that is the opposite of the original function; $f(-x) = -f(x)$	4B.2	**función impar** función que, cuando se evalúa para $-x$, tiene como resultado una función que es lo opuesto a la función original; $f(-x) = -f(x)$
one-tailed test a *t*-test performed on a set of data to determine if the data could belong in one of the tails of the bell-shaped distribution curve; with this test, the area under only one tail of the distribution is considered	1.5	**prueba de una cola o unilateral** una prueba *t* realizada en un conjunto de datos para determinar si estos podrían pertenecer a una de las colas de la curva de distribución con forma de campana; con esta prueba, solo se considera el área debajo de una cola de la distribución
one-to-one correspondence the feature of a function whereby each value in the domain corresponds to a unique function value; that is, if $x = a$ and $x = b$, the two points would be $(a, f(a))$ and $(b, f(b))$, and if $a \neq b$, then $f(a) \neq f(b)$ for a function to exhibit one-to-one correspondence	4A.1	**correspondencia uno a uno** la característica de una función por la cual cada valor del dominio corresponde a un único valor de función; es decir, si $x = a$ y $x = b$, los dos puntos serían $(a, f(a))$ y $(b, f(b))$, y si $a \neq b$, entonces $f(a) \neq f(b)$ para que una función exhiba una correspondencia uno a uno
ordered pair a point (x, y) on a two-dimensional coordinate plane that represents the value of an independent variable (x) that results in a specific dependent variable value (y). The term also refers to solutions for an equation or inequality in one variable that originate from a real-world situation. An ordered pair is also called a *data point*.	4B.1	**par ordenado** un punto (x, y) en un plano de coordenadas de dos dimensiones que representa el valor de una variable independiente (x) que da como resultado un valor específico de variable dependiente (y). El término también se refiere a las soluciones para una ecuación o desigualdad en una variable que proviene de una situación del mundo real. Un par ordenado también se denomina un *punto de datos*.
outcome the observable result of an experiment	1.3	**resultado** producto observable de un experimento
outlier a value far above or below other values of a distribution	1.1	**valor atípico** valor muy por encima o muy por debajo de otros valores de una distribución

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	P	
parameter numerical value(s) representing the data in a set, including proportion, mean, and variance	1.2 1.4 1.5	**parámetro** valores numéricos que representan los datos en un conjunto, incluyendo la proporción, la media y la varianza
parent function a function with a simple algebraic rule that represents a family of functions. The graphs of the functions in the family have the same general shape as the parent function.	4A.3 4B.1 4B.2	**función principal** función con una regla algebraica simple que representa una familia de funciones. Los gráficos de las funciones en la familia tienen la misma forma general que la función principal.
partial sum the sum of part of a series	2A.5	**suma parcial** la suma de la parte de una serie
Pascal's Triangle a triangle displaying a pattern of numbers in which the terms in additional rows are found by adding pairs of terms in previous rows, so that any given term is the sum of the two terms directly above it. The number 1 is the top number of the triangle, and is also the first and last number of each row.  	2A.2	**triángulo de Pascal** triángulo que muestra un patrón de números en el cual los términos se encuentran en filas adicionales al agregar pares de términos en filas anteriores, de manera que cualquier término dado es la suma de los dos términos directamente por encima de este. El número 1 es el número superior del triángulo y también es el primero y el último de cada fila. 
period in a cosine or sine function graph, the horizontal distance from a maximum to a maximum or from a minimum to a minimum; one repetition of the period of a function is called a *cycle*	3.3 4A.3	**período** en una curva de la función del seno o coseno, distancia horizontal desde un máximo a un máximo o desde un mínimo a un mínimo; una repetición del período de una función se llama *ciclo*

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periodic function a function whose values repeat at regular intervals	3.3 4B.3	**función periódica** función cuyos valores se repiten a intervalos regulares
periodic phenomena real-life situations that repeat at regular intervals and can be represented by a periodic function	3.3	**fenómenos periódicos** situaciones de la vida real que se repiten a intervalos regulares y se pueden representar mediante una función periódica
phase shift on a cosine or sine function graph, the horizontal distance by which the curve of a parent function is shifted by the addition of a constant or other expression in the argument of the function	4A.3	**cambio de fase** en una gráfica de la función del seno o del coseno, distancia horizontal por la cual la curva de una función raíz se cambia por la adición de una constante u otra expresión en el argumento de la función
piecewise function a function that is defined by two or more expressions on separate portions of the domain	4B.4	**función por partes** función definida por dos o más expresiones en porciones separadas del dominio
placebo a substance that is used as a control in testing new medications; the substance has no medicinal effect on the subject	1.3	**placebo** sustancia que se utiliza como control en las pruebas de medicamentos nuevos; la sustancia no tiene efecto medicinal sobre el sujeto
plane a flat, two-dimensional figure without depth that has at least three non-collinear points and extends infinitely in all directions	4B.5	**plano** figura plana, bidimensional, sin profundidad, que tiene al menos tres puntos no colineales y se extiende infinitamente en todas direcciones
plane figure a two-dimensional shape on a plane	4B.5	**figura plana** forma bidimensional sobre un plano
point(s) of intersection in a graphed system of equations, the ordered pair(s) where graphed functions intersect on a coordinate plane	2A.4 2B.2	**punto(s) de intersección** en un sistema de ecuaciones graficado, par(es) ordenado(s) donde las funciones de la gráfica se cruzan sobre un plano de coordenadas
polyhedron a three-dimensional object that has faces made of polygons	4B.5	**poliedro** objeto tridimensional que tiene caras compuestas por polígonos
polynomial an expression that contains variables, numeric quantities, or both, where variables are raised to integer powers greater than or equal to 0	2A.1	**polinomio** expresión que contiene variables, cantidades numéricas o ambas en donde las variables se elevan a potencias de números enteros mayores o iguales a 0

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polynomial equation an equation of the general form $y = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$, where a_1 is a rational number, $a_n \neq 0$, and n is a nonnegative integer and the highest degree of the polynomial	2A.4	**ecuación polinómica** ecuación de la forma general $y = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$, donde a_1 es un número racional, $a_n \neq 0$ y n es un número entero no negativo y el grado más alto del polinomio
polynomial function a function of the general form $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$, where a_1 is a rational number, $a_n \neq 0$, and n is a nonnegative integer and the highest degree of the polynomial	2A.1 2A.3	**función polinómica** función de la forma general $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$, donde a_1 es un número racional, $a_n \neq 0$ y n es un número entero no negativo y el grado más alto del polinomio
polynomial identity a true equation that is often generalized so it can apply to more than one example	2A.2	**identidad del polinomio** ecuación verdadera que suele generalizarse para que pueda aplicarse a más de un ejemplo
population all of the people, objects, or phenomena of interest in an investigation; the entire data set	1.4	**población** todas las personas, los objetos o fenómenos de interés en una investigación; el conjunto completo de datos
population all of the people, objects, or phenomena of interest in an investigation	1.1 1.2 1.4	**población** todas las personas, los objetos o fenómenos de interés en una investigación
population average the sum of all quantities in a population, divided by the total number of quantities in the population; typically represented by μ; also known as *population mean*	1.4	**promedio de la población** suma de todas las cantidades de una población, dividida por el número total de cantidades de la población; representada normalmente por μ; también se conoce como *media poblacional*
population mean the sum of all quantities in a population, divided by the total number of quantities in the population; typically represented by μ; also known as *population average*	1.4	**media poblacional** suma de todas las cantidades de una población, dividida por el número total de cantidades de la población; representada normalmente μ; también se conoce como *promedio de la población*

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positively skewed a distribution in which there is a "tail" of isolated, spread-out data points to the right of the median. "Tail" describes the visual appearance of the data points in a histogram. Data that is positively skewed is also called *skewed to the right*.	1.1	**sesgado positivamente** distribución en la cual existe una "cola" de puntos de datos aislados esparcidos hacia la derecha de la mediana. La "cola" describe la apariencia de los puntos de datos en un histograma. Los datos que están positivamente sesgados también se denominan *sesgados a la derecha*.
power the quantity that shows the number of times the base is being multiplied by itself in an exponential expression, such as x in the logarithmic function $x = \log_5 g(x)$ and its exponential function, $g(x) = 5^x$	4A.2	**potencia** cantidad que muestra el número de veces que se multiplica la base por sí misma en una expresión exponencial, tal que x en la función logarítmica $x = \log_5 g(x)$ y su función exponencial $g(x) = 5^x$
probability distribution the values of a random variable with associated probabilities	1.1	**distribución de probabilidad** los valores de una variable aleatoria con probabilidades asociadas
proportion a statement of equality between two ratios	2B.2	**proporción** afirmación de igualdad entre dos relaciones
***p*-value** a number between 0 and 1 that determines whether to accept or reject the null hypothesis	1.5	**valor *p*** número entre 0 y 1 que determina si se acepta o se rechaza la hipótesis nula
	Q	
quadratic equation an equation that can be written in the form $ax^2 + bx + c = 0$, where x is the variable, a, b, and c are constants, and $a \neq 0$	2A.4 2B.1 4B.1	**ecuación cuadrática** ecuación que se puede expresar en la forma $ax^2 + bx + c = 0$, donde x es la variable, a, b, y c son constantes, y $a \neq 0$
quadratic expression an algebraic expression that can be written in the form $ax^2 + bx + c$, where x is the variable, a, b, and c are constants, and $a \neq 0$	2B.1	**expresión cuadrática** expresión algebraica que se puede expresar en la forma $ax^2 + bx + c$, donde x es la variable, a, b, y c son constantes, y $a \neq 0$

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quadratic formula a formula that states the solutions of a quadratic equation of the form $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. A quadratic equation in this form can have no real solutions, one real solution, or two real solutions.	2B.1	**fórmula cuadrática** fórmula que establece que las soluciones de una ecuación cuadrática de la forma $ax^2 + bx + c = 0$ están dadas por $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Una ecuación cuadrática en esta forma tener ningún solución real, o tener una solución real, o dos soluciones reales.
quadratic function a function defined by a second-degree expression of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants. The graph of any quadratic function is a parabola.	4A.1 4B.4	**función cuadrática** función definida por una expresión de segundo grado de la forma $f(x) = ax^2 + bx + c$, donde $a \neq 0$ y a, b y c son constantes. La representación gráfica de toda función cuadrática es una parábola.
	R	
radian the measure of the central angle that intercepts an arc equal in length to the radius of the circle; π radians = 180°	3.1 4A.3	**radián** medida del ángulo central que intercepta un arco de longitud igual al radio del círculo; π radianes = 180°
radical equation an algebraic equation in which at least one term includes a radical expression	2B.2	**ecuación radical** ecuación algebraica en la cual al menos un término incluye una expresión radical
radical expression an expression containing a root	2B.2	**expresión radical** expresión que contiene una raíz
random the designation of a group or sample that has been formed without following any kind of pattern and without bias. Each group member has been selected without having more of a chance than any other group member of being chosen.	1.3	**aleatorio** designación de un grupo o muestra que se formó sin seguir ninguna clase de patrón y sin sesgo. Cada miembro del grupo se seleccionó sin tener más probabilidades de ser elegido que cualquier otro miembro del grupo.

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randomization the selection of a group, subgroup, or sample without following a pattern, so that the probability of any item in the set being generated is equal; the process used to ensure that a sample best represents the population	1.3	**aleatorización** selección de un grupo, subgrupo o muestra sin seguir un patrón, de manera que la probabilidad de cualquier elemento en el conjunto que está siendo generado sea igual; proceso utilizado para asegurar que la muestra sea la que mejor represente a la población
random number generator a tool used to select a number without following a pattern, where the probability of generating any number in the set is equal	1.2 1.6	**generador de números aleatorios** herramienta utilizada para seleccionar un número sin seguir un patrón, donde la probabilidad de generar cualquier número en el conjunto es igual
random sample a subset or portion of a population or set that has been selected without bias, with each item in the population or set having the same chance of being found in the sample	1.2 1.4	**muestra aleatoria** subconjunto o porción de población o conjunto que ha sido seleccionado sin sesgo, con cada elemento de la población o conjunto con la misma probabilidad de encontrarse en la muestra
random variable a variable whose numerical value changes depending on each outcome in a sample space; the values of a random variable are associated with chance variation	1.1	**variable aleatoria** variable cuyo valor numérico cambia según cada resultado en un espacio de muestra; los valores de una variable aleatoria están asociados con una variación al azar
range the set of all outputs of a function; the set of *y*-values that are valid for the function	4A.1 4B.1	**rango** conjunto de todas las salidas de una función; conjunto de valores de *y* que son válidos para la función
rate of change a ratio that describes how much one quantity changes with respect to the change in another quantity; also known as the slope of a line	4B.3	**tasa de cambio** proporción que describe cuánto cambia una cantidad con respecto al cambio de otra cantidad; también se la conoce como pendiente de una recta
ratio the relation between two quantities; can be expressed in words, fractions, decimals, or as a percentage	2B.2	**proporción** relación entre dos cantidades; puede expresarse en palabras, fracciones, decimales o como porcentaje
rational equation an algebraic equation that contains at least one rational expression	2B.2	**ecuación racional** una ecuación algebraica que contiene al menos una expresión racional

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rational expression an expression made of the ratio of two polynomials, in which a variable appears in the denominator	2B.1	**expresión racional** expresión que resulta de la relación de dos polinomios, en la cual una variable aparece en el denominador
rational number any number that can be written as $\frac{m}{n}$, where m and n are integers and $n \neq 0$; any number that can be written as a decimal that ends or repeats	2B.2	**números racionales** números que pueden expresarse como $\frac{m}{n}$, en los que m y n son enteros y $n \neq 0$; cualquier número que puede escribirse como decimal finito o periódico
Rational Root Theorem If the polynomial $p(x)$ has integer coefficients, then every rational root of the polynomial equation $p(x) = 0$ can be written in the form $\frac{p}{q}$, where p is a factor of the constant term $p(x)$ and q is a factor of the leading coefficient of $p(x)$.	2A.3	**Teorema de la Raíz Racional** Si el polinomio $p(x)$ tiene coeficientes enteros, entonces toda raíz racional de la ecuación polinómica $p(x) = 0$ puede escribirse en la forma $\frac{p}{q}$, donde p es un factor del término constante $p(x)$ y q es un factor del coeficiente principal de $p(x)$.
reciprocal a number that, when multiplied by the original number, has a product of 1	2B.1	**recíproco** número que multiplicado por el número original tiene producto 1
recursive formula for a geometric sequence a formula used to find the next term in a sequence. The formula is $a_n = a_{n-1} \bullet r$, where n is a positive integer that represents the number of terms in the sequence and r is the common ratio.	2A.5	**fórmula recursiva para una secuencia geométrica** fórmula utilizada para encontrar el término siguiente en una secuencia. La fórmula es $a_n = a_{n-1} \bullet r$, donde n es un número entero positivo que representa la cantidad de términos en la secuencia y r es la relación común.
reference angle the acute angle that the terminal side makes with the x-axis. The sine, cosine, and tangent of the reference angle are the same as that of the original angle (except for the sign, which is based on the quadrant in which the terminal side is located).	3.1	**ángulo de referencia** el ángulo agudo que forma el lado terminal con el eje x. El seno, el coseno y la tangente del ángulo de referencia son iguales a los del ángulo original (con excepción del signo, que se basa en el cuadrante en el que se ubica el lado terminal).

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regular polyhedron a polyhedron with faces that are all congruent regular polygons; the angles created by the intersecting faces are congruent, and the cross sections are similar figures	4B.5	**poliedro regular** poliedro cuyas caras son todas polígonos regulares congruentes; los ángulos creados por las caras que se cruzan son congruentes y las secciones transversales son figuras similares
relation a relationship between two variables in which at least one value of the domain or independent variable, x, is matched with one or more values of the dependent or range variable, y	4A.1	**relación** relación entre dos variables en la que al menos un valor del dominio o variable independiente, x, concuerda con uno o más valores de la variable de rango o dependiente, y
relative maximum the greatest value of a function for a particular interval of the function; also known as a *local maximum*	2A.3 4B.3	**máximo relativo** el mayor valor de una función para un intervalo en particular de la función; también conocido como *máximo local*
relative minimum the least value of a function for a particular interval of the function; also known as a *local minimum*	2A.3 4B.3	**mínimo relativo** el menor valor de una función para un intervalo en particular de la función; también conocido como *mínimo local*
reliability the degree to which a study or experiment performed many times would have similar results	1.2	**confiabilidad** grado en el cual un estudio o experimento realizado varias veces tendría resultados similares
Remainder Theorem For a polynomial $p(x)$ and a number a, dividing $p(x)$ by $x - a$ results in a remainder of $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.	2A.3	**Teorema del Resto** para un polinomio $p(x)$ y un número a, dividiendo $p(x)$ por $x - a$ resulta un resto de $p(a)$, entonces $p(a) = 0$ si y solo si $(x - a)$ es un factor de $p(x)$.
repeated root a polynomial function with a root that occurs more than once	2A.3	**raíz repetida** función polinómica con una raíz que aparece más de una vez
representative sample a sample in which the characteristics of the people, objects, or items in the sample are similar to the characteristics of the population	1.2	**muestra representativa** muestra en la cual las características de las personas, los objetos o elementos en ella son similares a las características de la población
response bias bias that occurs when responses by those surveyed have been influenced in some manner	1.5	**sesgo de respuesta** sesgo que se produce cuando las respuestas de los encuestados fueron influenciadas de alguna manera
restricted domain a subset of a function's defined domain	4B.1	**dominio restringido** subconjunto del dominio definido de una función
***rho* (ρ)** a lowercase Greek letter commonly used to represent density	4B.5	***rho* (ρ)** letra griega minúscula comúnmente utilizada para representar densidad

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root the *x*-intercept of a function; also known as *zero*	2A.3	**raíz** intercepto de *x* de una función; también conocida como *cero*
rotation in three dimensions, a transformation in which a plane figure is moved about one of its sides, a fixed point, or a line that is not located in the plane of the figure, such that a solid figure is produced	4B.5	**rotación** en tres dimensiones, transformación en la cual una figura plana se mueve sobre uno de sus lados, un punto fijo o una línea que no está ubicada en el plano de la figura, de manera que se produce una figura sólida
	S	
sample a subset of the population	1.1 1.2	**muestra** subconjunto de la población
sample average the sum of all quantities in a sample divided by the total number of quantities in the sample, typically represented by $\overline{x}$; also known as *sample mean*	1.4	**promedio de la muestra** suma de todas las cantidades en una muestra dividida por el número total de cantidades en la muestra, normalmente representada por $\overline{x}$; también se conoce como *media de la muestra*
sample mean the sum of all quantities in a sample divided by the total number of quantities in the sample, typically represented by $\overline{x}$; also known as *sample average*	1.4	**media de la muestra** suma de todas las cantidades en una muestra dividida por el número total de cantidades en la muestra, normalmente representada por $\overline{x}$; también se conoce como *promedio de la muestra*
sample population a portion of the population; the number of elements or observations in a sample population is represented by *n*	1.4	**población de la muestra** porción de la población; la cantidad de elementos u observaciones en una población de muestra se representa por *n*
sample proportion the fraction of favorable results *p* from a sample population *n*; conventionally represented by $\hat{p}$, which is pronounced "p hat." The formula for the sample proportion is $\hat{p}=\frac{p}{n}$, where *p* is the number of favorable outcomes and *n* is the number of elements or observations in the sample population.	1.4	**proporción de la muestra** fracción de los resultados favorables *p* de una población de muestra *n*; convencionalmente representada por $\hat{p}$, que se pronuncia "p hat". La fórmula para la proporción de la muestra es $\hat{p}=\frac{p}{n}$, donde *p* es la cantidad de resultados favorables y *n* es la cantidad de elementos u observaciones en la población de la muestra.

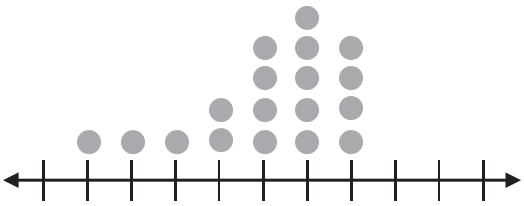
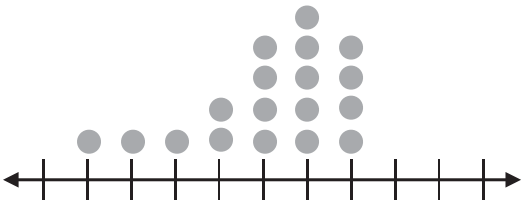
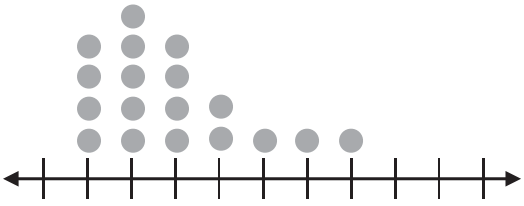
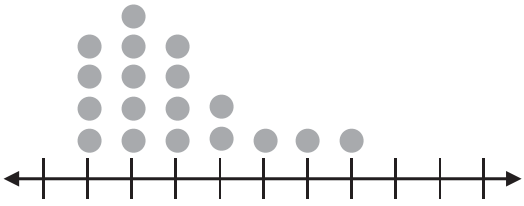
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sample survey a survey carried out using a sampling method so that only a portion of the population is surveyed rather than the whole population	1.3	**encuesta de muestra** encuesta realizada utilizando un método de muestreo para encuestar solo una porción de la población en lugar de toda la población.
sampling bias errors in estimation caused by flawed (non-representative) sample selection	1.2	**sesgo de muestreo** errores de cálculo ocasionados por una selección defectuosa (no representativa) de la muestra
sampling error a measure showing how precisely a sample reflects the population, with smaller sampling errors resulting from large samples and/or when the data clusters closely around the mean; also called *chance variation*	1.2	**error de muestreo** medición que demuestra qué tan precisamente refleja una muestra a una población, con pequeños errores de muestreo ocasionados por muestras grandes y/o cuando los datos se agrupan estrechamente alrededor de la media; también se llama *variación aleatoria*
scale the numbers representing the interval of a variable and the increments into which it is subdivided; usually includes the interval endpoints and the increments of the basic unit of the variable	4B.4	**escala** números que representan el intervalo de una variable y los incrementos en los cuales esta se subdivide; generalmente incluye los extremos del intervalo y los incrementos de la unidad básica de la variable
secant the reciprocal of cosine, $\sec\theta = \frac{1}{\cos\theta}$; the secant of $\theta = \sec\theta = \frac{\text{length of hypotenuse}}{\text{length of adjacent side}}$	3.1	**secante** recíproca del coseno, $\sec\theta = \frac{1}{\cos\theta}$; la secante de $\theta = \sec\theta = \frac{\text{largo de la hipotenusa}}{\text{largo del lado adyacente}}$
sequence an ordered list of numbers or elements	2A.5	**secuencia** lista ordenada de números o elementos
series the sum of the terms of a sequence	2A.5	**serie** suma de los términos de una secuencia
***sigma* (lowercase), σ** a Greek letter used to represent standard deviation	1.1	***sigma* (minúscula) o σ** letra griega utilizada para representar la desviación estándar
***sigma* (uppercase), Σ** a Greek letter used to represent the summation of values	1.1 2A.2 2A.5	***sigma* (mayúscula) o Σ** letra griega utilizada para representar la sumatoria de valores

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similar figures two figures that are the same shape but not necessarily the same size; the symbol for representing similarity between figures is $\sim$.	4B.5	**figuras similares** dos figuras que tienen la misma forma pero no necesariamente el mismo tamaño; el símbolo que representa la similitud entre figuras es $\sim$.
simple random sample a sample in which any combination of a given number of individuals in the population has an equal chance of selection	1.2	**muestra aleatoria simple** muestra en la cual cualquier combinación de una cantidad dada de individuos de la población tiene iguales posibilidades de selección
simulation a set of data that models an event that could happen in real life	1.5	**simulación** conjunto de datos que imita un evento que podría suceder en la vida real
sine a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the hypotenuse; the sine of $\theta = \sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$	3.1 3.2 4A.3	**seno** función trigonométrica de un ángulo agudo en un triángulo rectángulo que es la proporción de la longitud del lado opuesto a la longitud de la hipotenusa; sen de $\theta = \text{sen}\,\theta = \frac{\text{longitud del lado opuesto}}{\text{longitud de la hipotenusa}}$
sine curve a curve with a constant amplitude and period, which are given by a sine or cosine function; also called a *sine wave* or *sinusoid*	3.3	**curva del seno** curva con amplitud y período constantes que están dados por una función seno o coseno; también se denomina *onda de seno* o *sinusoide*
sine function a trigonometric function of the form $f(x) = a \sin [b(x - c)] + d$, in which a, b, c, and d are constants and x is a variable defined in radians over the domain $(-\infty,\infty)$	3.3 4A.3	**función seno** función trigonométrica de la forma $f(x) = a \sin [b(x - c)] + d$, en la cual a, b, c y d son constantes y x es una variable expresada en radianes sobre el dominio $(-\infty,\infty)$
sine wave a curve with a constant amplitude and period given by a sine or cosine function; also called a *sine curve* or *sinusoid*	3.3	**onda senoidal** curva con amplitud y período constantes dados por una función seno o coseno; también se denomina *curva del seno* o *sinusoide*
sinusoid a curve with a constant amplitude and period given by a sine or cosine function; also called a *sine curve* or *sine wave*	3.3	**sinusoide** curva con amplitud o período constantes dados por una función seno o coseno; también se denomina *curva del seno* u *onda senoidal*

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skewed to the left a distribution in which there is a "tail" of isolated, spread-out data points to the left of the median. "Tail" describes the visual appearance of the data points in a histogram. Data that is skewed to the left is also called *negatively skewed*. Example: 	1.1	**sesgado a la izquierda** distribución en la cual existe una "cola" de puntos de datos aislados extendidos hacia la izquierda de la mediana. La "cola" describe la apariencia de los puntos de datos en un histograma. Los datos sesgados a la izquierda también se denominan *negativamente sesgados*. Ejemplo: 
skew to distort or bias, as in data	1.3	**sesgar** distorsionar o afectar, como en el caso de los datos
skewed to the right a distribution in which there is a "tail" of isolated, spread-out data points to the right of the median. "Tail" describes the visual appearance of the data points in a histogram. Data that is skewed to the right is also called *positively skewed*. Example: 	1.1	**sesgado a la derecha** distribución en la cual existe una "cola" de puntos de datos aislados extendidos hacia la derecha de la mediana. La "cola" describe la apariencia de los puntos de datos en un histograma. Los datos sesgados a la derecha también se denominan *positivamente sesgados*. Ejemplo: 
solid figure a three-dimensional object that has length, width, and height (depth)	4B.5	**figura sólida** objeto tridimensional que tiene largo, ancho y altura (profundidad)
solution set the set of ordered pairs that represent all of the solutions to an equation or a system of equations	2B.2 4B.1	**conjunto de soluciones** conjunto de pares ordenados que representa todas las soluciones para una ecuación o sistema de ecuaciones
spread refers to how data is spread out with respect to the mean; sometimes called *variability*	1.4	**dispersión** forma en que los datos se esparcen con respecto a la media; algunas veces se denomina *variabilidad*

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square root function a function that contains a square root of a variable. The general form is $f(x)=\sqrt{ax^2+bx+c}$, where *a*, *b*, and *c* are real numbers.	4B.4	**función raíz cuadrada** función que contiene la raíz cuadrada de una variable. La forma general es $f(x)=\sqrt{ax^2+bx+c}$, donde *a*, *b* y *c* son números reales.
standard deviation how much the data in a given set is spread out, represented by *s* or σ. The standard deviation of a sample can be found using the following formula: $s=\sqrt{\frac{\sum\left(x_i-\overline{x}\right)^2}{n-1}}$. The standard deviation of a population can be found using the following formula: $\sigma=\sqrt{\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}}$.	1.1 1.4	**desviación estándar** cuánto se extienden los datos en un conjunto dado, representada por *s* o σ. Se puede calcular la desviación estándar de una muestra utilizando la siguiente fórmula: $s=\sqrt{\frac{\sum\left(x_i-\overline{x}\right)^2}{n-1}}$. Se puede calcular la desviación estándar de una población utilizando la siguiente fórmula: $\sigma=\sqrt{\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}}$.
standard error of the mean the variability of the mean of a sample; given by $\text{SEM}=\frac{s}{\sqrt{n}}$, where *s* represents the standard deviation and *n* is the number of elements or observations in the sample population	1.4	**error estándar de la media** variabilidad de la media de una muestra; dado por $\text{SEM}=\frac{s}{\sqrt{n}}$, donde *s* representa la desviación estándar y *n* la cantidad de elementos u observaciones en la población de la muestra

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standard error of the proportion the variability of the measure of the proportion of a sample, abbreviated SEP. The standard error (*SEP*) of a sample proportion $\hat{p}$ is given by the formula $SEP = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, where $\hat{p}$ is the sample proportion determined by the sample and n is the number of elements or observations in the sample population.	1.4	**error estándar de la proporción** variabilidad de la medida de la proporción de una muestra, abreviada SEP. El error estándar (*SEP*) de una proporción de la muestra $\hat{p}$ está dado por la fórmula $SEP = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, donde $\hat{p}$ es la proporción de la muestra determinada por la muestra y n representa la cantidad de elementos u observaciones en la población de la muestra.
standard normal distribution a normal distribution that has a mean of 0 and a standard deviation of 1; data following a standard normal distribution forms a normal curve when graphed	1.1	**distribución normal estándar** distribución normal que tiene una media de 0 y una desviación estándar de 1; los datos que siguen una distribución normal estándar forman una curva normal al graficarse
standard position (of an angle) a position in which the vertex of the angle is at the origin of the coordinate plane and is the center of the unit circle. The angle's initial side is located along the positive x-axis and the terminal side may be in any location.	3.1	**posición estándar (de un ángulo)** posición en la cual el vértice del ángulo está en el origen del plano de coordenadas y es el centro del círculo unitario. El lado inicial del ángulo está ubicado a lo largo del eje positivo x y el lado terminal puede estar en cualquier ubicación.
statistical significance a measure used to determine whether the outcome of an experiment is a result of the treatment being applied, as opposed to random chance	1.5	**relevancia estadística** medida utilizada para determinar si el resultado de un experimento es el resultado del tratamiento aplicado, en oposición al resultado producto del azar

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statistics a branch of mathematics focusing on how to collect, organize, analyze, and interpret information from data gathered; numbers used to summarize, describe, or represent sets of data	1.2 1.3	**estadística** rama de la matemática enfocada en la manera de recabar, organizar, analizar e interpretar la información proveniente de los datos reunidos; números utilizados para resumir, describir o representar conjuntos de datos
step function a function that is a series of disconnected constant functions	4B.4	**función escalonada** función que es una serie de funciones constantes desconectadas
stratified sample a sample chosen by first dividing a population into subgroups of people or objects that share relevant characteristics, then randomly selecting members of each subgroup for the sample	1.2	**muestra estratificada** muestra escogida dividiendo primero una población en subgrupos de personas u objetos que comparten características relevantes, luego seleccionando al azar miembros de cada subgrupo para la muestra
subtended arc the section of an arc formed by a central angle that passes through the circle, thus creating the endpoints of the arc	3.1	**arco subtendido** sección de un arco formada por un ángulo central que pasa por el círculo, creando así los puntos extremos del arco
success the data sought or hoped for, represented by *p*; also known as *desirable outcome* or *favorable outcome*	1.4	**éxito** datos buscados o esperados, representados por *p*; también conocido como *resultado deseado* o *resultado favorable*
sum formula for a finite geometric series $S_n = \frac{a_1(1-r^n)}{1-r}$, where S_n is the sum, a_1 is the first term, r is the common ratio, and n is the number of terms	2A.5	**fórmula de suma para una serie geométrica finita** $S_n = \frac{a_1(1-r^n)}{1-r}$ donde S_n es la suma, a_1 es el primer término, r es la relación común y n es la cantidad de términos
sum formula for an infinite geometric series $S_n = \frac{a_1}{1-r}$, where S_n is the sum, a_1 is the first term, and r is the common ratio	2A.5	**fórmula de suma para una serie geométrica infinita** $S_n = \frac{a_1}{1-r}$, donde S_n es la suma, a_1 es el primer término y r es la relación común

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summation notation a symbolic way to represent a series (the sum of a sequence) using the uppercase Greek letter *sigma*, Σ	1.1 2A.5	**notación sumatoria** forma simbólica de representar una serie (la suma de una secuencia) utilizando la letra griega mayúscula *sigma*, Σ
survey a study of particular qualities or attributes of items or people of interest to a researcher	1.3	**encuesta** estudio de las cualidades o atributos particulares de elementos o personas de interés para un investigador
symmetric distribution a data distribution in which a line can be drawn so that the left and right sides are mirror images of each other	1.1	**distribución simétrica** distribución de datos en la cual se puede trazar una línea de manera que los lados derecho e izquierdo sean imágenes especulares entre sí
symmetry of a function the property whereby a function exhibits the same behavior (e.g., graph shape, function values, etc.) for specific domain values and their opposites	4B.3	**simetría de una función** propiedad por la cual una función exhibe el mismo comportamiento (por ej., forma de la gráfica, valores de la función, etc.) para valores específicos del dominio y sus opuestos
synthetic division a shorthand way of dividing a polynomial by a linear binomial	2A.3	**división sintética** forma abreviada de dividir un polinomio por un binomio lineal
synthetic substitution the process of using synthetic division to evaluate a function by using only the coefficients	2A.3	**sustitución sintética** proceso que utiliza la división sintética para evaluar una función utilizando solo los coeficientes
systematic sample a sample drawn by selecting people or objects from a list, chart, or grouping at a uniform interval; for example, selecting every fourth person	1.2	**muestra sistemática** la muestra se obtiene mediante la selección de personas u objetos a partir de una lista, una tabla o mediante la agrupación a intervalos regulares; por ej., eligiendo una de cada cuatro personas
system of equations a set of equations with the same unknowns	2A.4 2B.2	**sistema de ecuaciones** conjunto de ecuaciones con las mismas incógnitas

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	T	
tangent a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the adjacent side; the tangent of $\theta = \tan\theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$	3.1 3.2	**tangente** función trigonométrica de un ángulo agudo en un triángulo rectángulo que es la proporción de la longitud del lado opuesto a la longitud del lado adyacente; tangente de $\theta = \tan\theta = \frac{\text{longitud del lado opuesto}}{\text{longitud del lado adyacente}}$
term an element in a sequence. In the sequence $\{a_1, a_2, a_3, ..., a_n\}$, a_1 is the first term, a_2 is the second term, a_3 is the third term, and a_n is the nth term; a number, a variable, or the product of a number and variable(s)	2A.1 2A.5	**término** elemento de una secuencia. En la secuencia $\{a_1, a_2, a_3, ..., a_n\}$, a_1 es el primer término, a_2 es el segundo término, a_3 es el tercer término y a_n es el término n; un número, una variable o el producto de un número y variable(s)
terminal side for an angle in standard position, the movable ray of an angle that can be in any location and which determines the measure of the angle	3.1	**lado terminal** para un ángulo en posición estándar, el rayo móvil de un ángulo que puede estar en cualquier ubicación y que determina la medida del ángulo
theoretical probability the probability that an outcome will occur as determined through reasoning or calculation, given by the formula $P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}$	1.6	**probabilidad teórica** probabilidad de que un resultado se produzca como se determinó mediante razonamiento o cálculo, dado por la fórmula $P(E) = \frac{\text{cantidad de resultados en } E}{\text{cantidad de resultados en el espacio de muestreo}}$
***theta* (θ)** a Greek letter commonly used to refer to unknown angle measures	3.1	***theta* (θ)** letra griega generalmente utilizada para referirse a las medidas de un ángulo desconocido
translation in three dimensions, the horizontal or vertical movement of a plane figure in a direction that is not in the plane of the figure, such that a solid figure is produced	4B.5	**traslación** en tres dimensiones, movimiento horizontal o vertical de una figura plana en una dirección que no está en el plano de la figura, de manera que se produzca una figura sólida

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treatment the process or intervention provided to the population being observed	1.5	**tratamiento** proceso o intervención efectuada sobre la población que está siendo observada
trial each individual event or selection in an experiment or treatment	1.4 1.5 1.6	**ensayo** cada evento o selección individual en un experimento o tratamiento
true negative result a determination that an experiment has produced a correct negative result	1.6	**resultado negativo verdadero** determinación de que un experimento ha producido un resultado negativo correcto
true positive result a determination that an experiment has produced a correct positive result	1.6	**resultado positivo verdadero** determinación de que un experimento ha producido un resultado positivo correcto
***t*-test** a procedure to establish the statistical significance of a set of data using the mean, standard deviation, and degrees of freedom for the sample or population	1.5	**prueba *t*** procedimiento para establecer la relevancia estadística de un conjunto de datos utilizando la media, la desviación estándar y los grados de libertad para la muestra o población
turning point a point where the graph of the function changes direction, from sloping upward to sloping downward or vice versa	2A.3	**punto de inflexión** punto en el cual la gráfica de función cambia de dirección, de una inclinación o pendiente ascendente a una descendente o viceversa
***t*-value** the result of a *t*-test	1.5	**valor *t*** resultado de una prueba *t*
two-tailed test a *t*-test performed on a set of data to determine if the data could belong in either of the tails of the bell-shaped distribution curve; with this test, the area under both tails of the distribution is considered	1.5	**prueba de dos colas o prueba bilateral** prueba *t* realizada sobre un conjunto de datos para determinar si esos datos podrían pertenecer a alguna de las colas de una curva de distribución en forma de campana; con esta prueba, se tiene en cuenta el área bajo ambas colas de la distribución
	U	
undesirable outcome the data not sought or hoped for, represented by *q*; also known as *unfavorable outcome* or *failure*	1.4	**resultado no deseado** datos no buscados o esperados, representados por *q*; también conocido como *resultado desfavorable* o *fracaso*
unfavorable outcome the data not sought or hoped for, represented by *q*; also known as *undesirable outcome* or *failure*	1.4	**resultado desfavorable** datos no buscados o esperados, representados por *q*; también conocido como *resultado no deseado* o *fracaso*

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uniform distribution a set of values that are continuous, are symmetric to a mean, and have equal frequencies corresponding to any two equally sized intervals. In other words, the values are spread out uniformly throughout the distribution.	1.1	**distribución uniforme** conjunto de valores que son continuos, simétricos respecto de la media y tienen frecuencias iguales que corresponden a cualquiera de dos intervalos del mismo tamaño. En otras palabras, los valores se extienden uniformemente en la distribución.
unit circle a circle with a radius of 1 unit. The center of the circle is located at the origin of the coordinate plane.	3.1 4A.3	**círculo unitario** círculo con un radio de una unidad. El centro del círculo está ubicado en el origen del plano de coordenadas.
	V	
validity the degree to which the results obtained from a sample measure what they are intended to measure	1.2	**validez** el grado en el cual los resultados obtenidos de una muestra miden lo que se pretende que midan
variability refers to how data is spread out with respect to the mean; sometimes called *spread*	1.4	**variabilidad** hace referencia al modo en que se distribuyen los datos respecto de la media; algunas veces se denomina *dispersión*
vertex a point at which two or more lines meet	3.2	**vértice** punto en el que se encuentran dos o más líneas
voluntary response bias bias that occurs when the sample is not representative of the population due to the sample having the option of responding to the survey	1.5	**sesgo de respuesta voluntaria** sesgo que se produce cuando la muestra no es representativa de la población debido a que en la muestra existe la opción de responder a la encuesta (la respuesta es optativa)
	Z	
z_c-value a measure of the number of standards of error to be added or subtracted from the mean in order to achieve the desired confidence level; also known as *critical value*	1.4	**valor-z_c** medida de la cantidad de estándares de error a sumar o restar de la media para alcanzar el nivel de confianza deseado; conocido también como *valor crítico*
zero the *x*-intercept of a function; also known as *root*	2A.3	**cero** el intercepto de *x* de una función; también se conoce como *raíz*
***z*-score** the number of standard deviations that a score lies above or below the mean; given by the formula $z = \frac{x - \mu}{\sigma}$	1.1	**puntuación *z*** cantidad de desviaciones estándar por encima o por debajo de la media que presenta la muestra; dada por la fórmula $z = \frac{x - \mu}{\sigma}$

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Common Core State Standards

Mathematics III

Integrated Pathway

Teacher Resource

Unit 1

WALCH

INTEGRATED MATH

This program was developed and reviewed by experienced math educators who have both academic and professional backgrounds in mathematics. This ensures: freedom from mathematical errors, grade level appropriateness, freedom from bias, and freedom from unnecessary language complexity.

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Standards Correlations

Each lesson in this Integrated Pathway: Mathematics III program was written specifically to address the Common Core State Standards. Each lesson lists the standards covered in all the sub-lessons, and each sub-lesson lists the standards addressed in that particular section. In this section, you'll find a comprehensive list mapping the sub-lessons to the CCSS.

Guide to Common Core State Standards Annotation

As you use this program, you will come across a symbol included with the Common Core standards for some of the lessons and activities. These symbols are explained below.

Symbol: ⋆

Denotes: Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (⋆).

From http://www.walch.com/CCSS/00003

Symbol: (+)

Denotes: College and Career Readiness Standards

Advanced mathematics standards that are required in higher-level courses such as advanced statistics may also be included in lower-level courses. These additional standards are denoted by (+). According to the Common Core State Standards Initiative, "the evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6–8."

From http://www.walch.com/CCSS/00004

Connections to Future Courses

This section provides a map between topics introduced in each unit of this course and subsequent courses where each topic is revisited and built upon.

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CCSS INTEGRATED PATHWAY: MATHEMATICS III STANDARDS CORRELATIONS

Unit 1: Inferences and Conclusions from Data				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Using the Normal Curve**			
	1.1.1	Normal Distributions and the 68–95–99.7 Rule	S–ID.4★	U1-6
	1.1.2	Standard Normal Calculations	S–ID.4★	U1-33
	1.1.3	Assessing Normality	S–ID.4★	U1-60
Lesson 2	**Populations Versus Random Samples and Random Sampling**			
	1.2.1	Differences Between Populations and Samples	S–IC.1★	U1-99
	1.2.2	Simple Random Sampling	S–IC.2★	U1-125
	1.2.3	Other Methods of Random Sampling	S–IC.2★	U1-156
Lesson 3	**Surveys, Experiments, and Observational Studies**			
	1.3.1	Identifying Surveys, Experiments, and Observational Studies	S–IC.3★	U1-193
	1.3.2	Designing Surveys, Experiments, and Observational Studies	S–IC.3★	U1-211
Lesson 4	**Estimating Sample Proportions and Sample Means**			
	1.4.1	Estimating Sample Proportions	S–IC.4★	U1-240
	1.4.2	The Binomial Distribution	S–IC.4★	U1-260
	1.4.3	Estimating Sample Means	S–IC.4★	U1-287
	1.4.4	Estimating with Confidence	S–IC.4★	U1-300
Lesson 5	**Comparing Treatments and Reading Reports**			
	1.5.1	Evaluating Treatments	S–IC.5★	U1-328
	1.5.2	Designing and Simulating Treatments	S–IC.5★	U1-355
	1.5.3	Reading Reports	S–IC.6★	U1-374
Lesson 6	**Making and Analyzing Decisions**			
	1.6.1	Making Decisions	S–MD.6★ (+)	U1-399
	1.6.2	Analyzing Decisions	S–MD.7★ (+)	U1-418

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TOPICS FROM FUTURE COURSES: CCSS INTEGRATED PATHWAY MATHEMATICS III

Math III	Topic introduced	Course/unit where addressed	How addressed
Unit 1	Methods of gathering data for statistical analysis; making and analyzing decisions and making inferences about sample means and proportions; normal and approximately normal distributions	AP Statistics or Math IV	calculating the expected value of distributions other than normal and approximately normal distributions is applied to making inferences and decisions, including decision-making based on payoffs calculated from expected value

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z-scores Table

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

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z-scores Table

z	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00
–3.4	0.0002	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
–3.3	0.0003	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0005	0.0005	0.0005
–3.2	0.0005	0.0005	0.0005	0.0006	0.0006	0.0006	0.0006	0.0006	0.0007	0.0007
–3.1	0.0007	0.0007	0.0008	0.0008	0.0008	0.0008	0.0009	0.0009	0.0009	0.0010
–3.0	0.0010	0.0010	0.0011	0.0011	0.0011	0.0012	0.0012	0.0013	0.0013	0.0013
–2.9	0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019
–2.8	0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026
–2.7	0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035
–2.6	0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047
–2.5	0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062
–2.4	0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082
–2.3	0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107
–2.2	0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139
–2.1	0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179
–2.0	0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228
–1.9	0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287
–1.8	0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359
–1.7	0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446
–1.6	0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548
–1.5	0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668
–1.4	0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793	0.0808
–1.3	0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951	0.0968
–1.2	0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131	0.1151
–1.1	0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335	0.1357
–1.0	0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562	0.1587
–0.9	0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814	0.1841
–0.8	0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090	0.2119
–0.7	0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389	0.2420
–0.6	0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709	0.2743
–0.5	0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050	0.3085
–0.4	0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409	0.3446
–0.3	0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783	0.3821
–0.2	0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168	0.4207
–0.1	0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562	0.4602
–0.0	0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960	0.5000

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t-distribution Table

One-tailed	0.50	0.25	0.20	0.15	0.10	0.05	0.025	0.01	0.005	0.001	0.0005
Two-tailed	1.00	0.50	0.40	0.30	0.20	0.10	0.05	0.02	0.01	0.002	0.001
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.000	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
Confidence level	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%

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Conceptual Activities

Use these interactive open education and/or Desmos resources to build conceptual understanding of mathematical ideas. (*Note*: Activity links will be monitored and repaired or replaced as necessary.)

- Illustrative Mathematics. "The Titanic 3."

  http://www.walch.com/ca/10005

  This task poses an open-ended question that forces students to think about how they can answer it. Students will analyze data given in the two-way table to find meaningful probabilities and increase their understanding of conditional probability and independence.

- Illustrative Mathematics. "Should We Send Out a Certificate?"

  http://www.walch.com/ca/10006

  This task allows students to practice calculating normal distributions and further encourages them to draw conclusions from their results based on the properties of normal distributions. Students will communicate their findings in a narrative form within the context of the problem rather that reporting a simple computed number.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Which of the following is an example of a continuous distribution?
   a. the sum of two six-sided dice
   b. the speeds of vehicles passing by a mile marker on a highway
   c. the number of births per year in a local hospital
   d. the number of runs scored by a high school softball team

2. Which phrase best describes a *z*-score?
   a. a measure of the variation in a set of numbers
   b. a measure of the average in a set of numbers
   c. a measure of the most common values in a set of numbers
   d. a measure of the number of standard deviations a value lies above or below the mean

3. What percent of the values in a normal distribution lie above the mean?
   a. 25%
   b. 34%
   c. 50%
   d. 95%

4. A tire manufacturer has determined that, before needing to be replaced, one type of tire it produces has a normal distribution with a mean of 48,000 miles and a standard deviation of 6,000 miles. What percent of this type of tire had distances between 42,000 and 54,000 miles before replacement?
   a. 34%
   b. 50%
   c. 68%
   d. 95%

5. Which set of characteristics does not correspond to a normal distribution?
   a. the mean, median, and mode of the distribution are equal
   b. the distribution is symmetric with a high concentration of values close to the mean
   c. the probabilities associated with any two intervals of equal width are the same
   d. approximately 95% of the values in the distribution lie within two standard deviations of the mean

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Lesson 1: Using the Normal Curve

Instruction

Common Core State Standard

SMP
1 ✓ 2 ✓
3 ✓ 4 ✓
5 ✓ 6 ✓
7 8 ✓

S–ID.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.★

Essential Questions

1. How is discrete data different from continuous data?
2. How can you tell if a set of values is normally distributed?
3. How can the standard normal distribution be used with a normal distribution that has a different mean and standard deviation?
4. Why is it a mistake to use the standard normal distribution to make decisions about data that are not normally distributed?

WORDS TO KNOW

68–95–99.7 rule a rule that states percentages of data under the normal curve are as follows: $\mu \pm 1\sigma \approx 68\%$, $\mu \pm 2\sigma \approx 95\%$, and $\mu \pm 3\sigma \approx 99.7\%$; also known as the *Empirical Rule*

continuous data a set of values for which there is at least one value between any two given values

continuous distribution the graphed set of values, a curve, in a continuous data set

discrete data a set of values with gaps between successive values

Empirical Rule a rule that states percentages of data under the normal curve are as follows: $\mu \pm 1\sigma \approx 68\%$, $\mu \pm 2\sigma \approx 95\%$, and $\mu \pm 3\sigma \approx 99.7\%$; also known as the *68–95–99.7 rule*

interval a set of values between a lower bound and an upper bound

mean a measure of center in a set of numerical data, computed by adding the values in a data set and then dividing the sum by the number of values in the data set; denoted as the Greek lowercase letter *mu*, μ; given by the formula $\mu = \frac{x_1 + x_2 + \ldots + x_n}{n}$, where each x-value is a data point and n is the total number of data points in the set

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Instruction

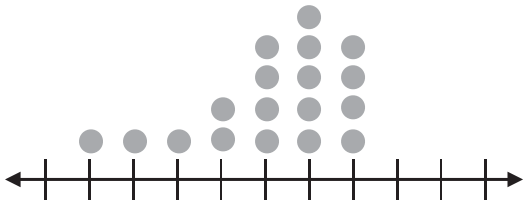
median	the middle-most value of a data set; 50% of the data is less than this value, and 50% is greater than it
***mu*, μ**	a Greek letter used to represent mean
negatively skewed	a distribution in which there is a "tail" of isolated, spread-out data points to the left of the median. "Tail" describes the visual appearance of the data points in a histogram. Data that is negatively skewed is also called *skewed to the left*.
normal curve	a symmetrical curve representing the normal distribution
normal distribution	a set of values that are continuous, are symmetric to a mean, and have higher frequencies in intervals close to the mean than equal-sized intervals away from the mean
outlier	a value far above or below other values of a distribution
population	all of the people, objects, or phenomena of interest in an investigation
positively skewed	a distribution in which there is a "tail" of isolated, spread-out data points to the right of the median. "Tail" describes the visual appearance of the data points in a histogram. Data that is positively skewed is also called *skewed to the right*.
probability distribution	the values of a random variable with associated probabilities
random variable	a variable whose numerical value changes depending on each outcome in a sample space; the values of a random variable are associated with chance variation
sample	a subset of the population
***sigma* (lowercase), σ**	a Greek letter used to represent standard deviation
***sigma* (uppercase), Σ**	a Greek letter used to represent the summation of values

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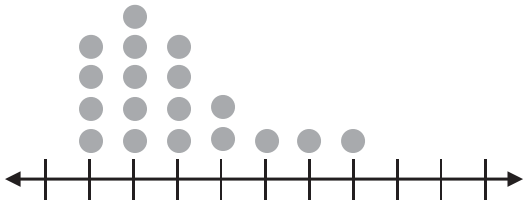
skewed to the left

a distribution in which there is a "tail" of isolated, spread-out data points to the left of the median. "Tail" describes the visual appearance of the data points in a histogram. Data that is skewed to the left is also called *negatively skewed*. Example:



skewed to the right

a distribution in which there is a "tail" of isolated, spread-out data points to the right of the median. "Tail" describes the visual appearance of the data points in a histogram. Data that is skewed to the right is also called *positively skewed*. Example:



standard deviation

how much the data in a given set is spread out, represented by s or σ. The standard deviation of a sample can be found using the following formula: $s=\sqrt{\frac{\sum\left(x_i-\overline{x}\right)^2}{n-1}}$. The standard deviation of a population can be found using the following formula: $\sigma=\sqrt{\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}}$.

standard normal distribution

a normal distribution that has a mean of 0 and a standard deviation of 1; data following a standard normal distribution forms a normal curve when graphed

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summation notation	a symbolic way to represent a series (the sum of a sequence) using the uppercase Greek letter *sigma*, Σ
symmetric distribution	a data distribution in which a line can be drawn so that the left and right sides are mirror images of each other
uniform distribution	a set of values that are continuous, are symmetric to a mean, and have equal frequencies corresponding to any two equally sized intervals. In other words, the values are spread out uniformly throughout the distribution.
***z*-score**	the number of standard deviations that a score lies above or below the mean; given by the formula $z = \frac{x - \mu}{\sigma}$

Recommended Resources

- Measuring Usability. "*Z*-score to Percentile Calculator."

  http://www.walch.com/rr/00176

  Users can enter a *z*-score into this online calculator to find the percentage of the area under the normal curve that is associated with that *z*-score. The site displays the area associated with the score and the area of 100%, with visuals of each area of interest. Users may choose one-sided or two-sided calculations. This site also links to a calculator for converting percentiles to *z*-scores, as well as an interactive graph of a standard normal curve.

- SkyMark. "Normal Test Plot."

  http://www.walch.com/rr/00177

  This site offers a brief description of one method of creating a normal test plot, and then shows examples of what to look for in the plot to determine if the plot represents normally distributed data. The examples include skewed data.

- Texas A&M University Department of Statistics. "Empirical Rule Demonstration."

  http://www.walch.com/rr/00178

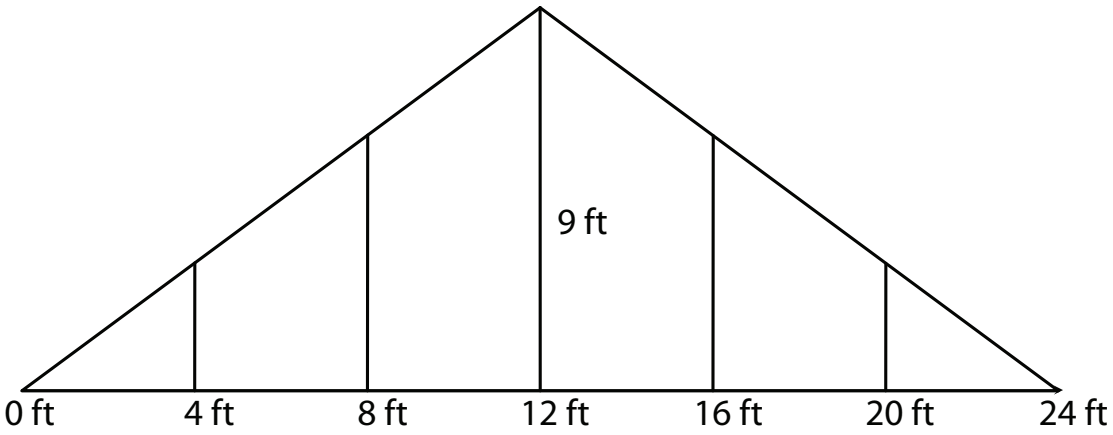
  This applet displays a standard normal distribution, with the area shaded under the curve from –1 to +1 standard deviations. Users can input new values for the mean and standard deviation to change the curve. A slider allows users to manipulate the shaded area; the applet will recalculate the standard deviation for the shaded area as the slider moves. The applet requires Java software to run.

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Lesson 1.1.1: Normal Distributions and the 68–95–99.7 Rule

Warm-Up 1.1.1

The following diagram shows a cross section of a roof with support columns. The cross section is an isosceles triangle, with a base of 24 feet and a height of 9 feet. Five columns are placed at 4-foot intervals to support the roof. Each of the columns is perpendicular to the base of the cross section. Use this information to answer the questions that follow.



1. Find the area of the cross section of the roof. Use the formula $A = \frac{1}{2}bh$.

2. What percent of the area of the cross section lies to the right of the centerline?

3. Find the heights of the support columns at the interval marking 8 feet and at the interval marking 16 feet.

4. What percent of the area of the cross section lies between the support columns at 8 feet and 16 feet?

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Lesson 1.1.1: Normal Distributions and the 68–95–99.7 Rule

Common Core State Standard

S–ID.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.★

Warm-Up 1.1.1 Debrief

1. Find the area of the cross section of the roof. Use the formula $A = \frac{1}{2}bh$.

   The cross section forms a triangle with a base of 24 feet and a height of 9 feet. The area of a triangle is equal to one-half the base times the height.

   $A = \frac{1}{2}bh$ Formula for the area of a triangle

   $A = \frac{1}{2}(24)(9)$ Substitute known values.

   $A = 108$ Multiply.

   The area of the cross section is 108 ft^2.

2. What percent of the area of the cross section lies to the right of the centerline?

   The cross section is symmetric to the centerline, so the area to the right of the centerline is equal to the area to the left of the centerline. Fifty percent of the area of the cross section lies to the right of the centerline.

3. Find the heights of the support columns at the interval marking 8 feet and at the interval marking 16 feet.

   We have two options for finding each height.

   Method 1

   Determine the slope, which is $\frac{\text{rise}}{\text{run}}$, of one side of the roof.

   The height of the cross section is 9 feet, and the length from the centerline of the cross section to the edge is 12 feet, as can be determined from the diagram.

   The slope of one side of the roof is $\frac{9}{12}$ or $\frac{3}{4}$.

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Instruction

Since the support column at the interval marking 8 feet is a horizontal distance of 8 feet from the left edge of the roof (the run), the height of this column can be found by solving the slope equation for the rise.

$\text{slope} = \frac{\text{rise}}{\text{run}}$ Formula for slope

$\left(\frac{3}{4}\right) = \frac{\text{rise}}{(8)}$ Substitute known values.

$\frac{3}{4}(8) = \text{rise} = 6$ Multiply.

The support column at this interval is 6 feet tall.

By symmetry, the height of the support column at the interval marking 16 feet is the same as the height at the interval marking 8 feet: 6 feet.

Method 2

All of the right triangles formed by a support column, a portion of the base of the roof, and a portion of the sloped section of the roof are similar triangles. Thus, the height of the column at 8 feet is proportional to the height of the column at the centerline. Set up a proportion to determine the missing height.

$\frac{9}{12} = \frac{x}{8}$ Proportion using known values

$x = \frac{9(8)}{12} = \frac{72}{12} = 6$ Solve for x.

The support columns at the interval marking 8 feet and at the interval marking 16 feet are each 6 feet tall.

Again by symmetry, the height of the support column at the interval marking 16 feet is the same as the height at the interval marking 8 feet: 6 feet.

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Instruction

4. What percent of the area of the cross section lies between the support columns at 8 feet and 16 feet?

   There are various methods for determining area. One such method is to find the area using the formula for a trapezoid. The region between the centerline and the support column at 8 feet forms a trapezoid, with base lengths of 6 feet and 9 feet and a height of 4 feet.

   The area of a trapezoid can be found using the formula $A=\frac{1}{2}(b_1+b_2)h$.

   $A=\frac{1}{2}(b_1+b_2)h$ Formula for the area of a trapezoid

   $A=\frac{1}{2}(6+9)4$ Substitute known values.

   $A = 30$ Simplify.

   The area is 30 ft^2.

   By symmetry, the area of the cross section between the centerline and the support column at 16 feet is also 30 square feet. Thus, to find the area of the region between the support columns at the intervals marked as 8 feet and 16 feet, add 30 ft^2 and 30 ft^2.

   30 + 30 = 60

   The total area is 60 ft^2.

   In order to find the percent of the area between the support columns, calculate what percent 60 ft^2 makes up of the total area of the cross section, 108 ft^2.

   To find this percent, divide 60 by 108.

   $$\frac{60}{108}=0.55\overline{5}=55\frac{5}{9}\%$$

   The area between the support columns is $55\frac{5}{9}\%$ of the total area.

Connection to the Lesson

- Students will be finding proportions of values under the normal curve for specified intervals.
- Students will be working with area as it relates to the probability in a normal distribution.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- determining the area of rectangles, triangles, and trapezoids (6.G.1)
- calculating probabilities using ratios (7.SP.7)
- calculating the mean of a distribution of numbers (6.SP.5c)
- recognizing the mean as a balancing point (6.SP.5d)
- distinguishing between measures of center and measures of variation (6.SP.3)

Introduction

Probability distributions are useful in making decisions in many areas of life, including business and scientific research. The normal distribution is one of many types of probability distributions, and perhaps the one most widely used. Learning how to use the properties of normal distributions will be a valuable asset in many careers and subjects, including economics, education, finance, medicine, psychology, and sports.

Understanding a data set requires finding four key components:

- the overall shape of the distribution
- a measure of central tendency or average
- a measure of variation
- a measure of population or sample size

The first three components are used in determining proportions and probabilities associated with values in normal distributions. The two main classes of data are discrete and continuous. We will begin by focusing on continuous distributions, particularly the normal distribution.

Key Concepts

- Understanding a data set, and how an individual value relates to the data set, requires information about the overall shape of the distribution as well as measures of center, measures of variation, and population (or sample) size. There are two types of data: discrete and continuous.
- **Discrete data** refers to a set of values with gaps between successive values.

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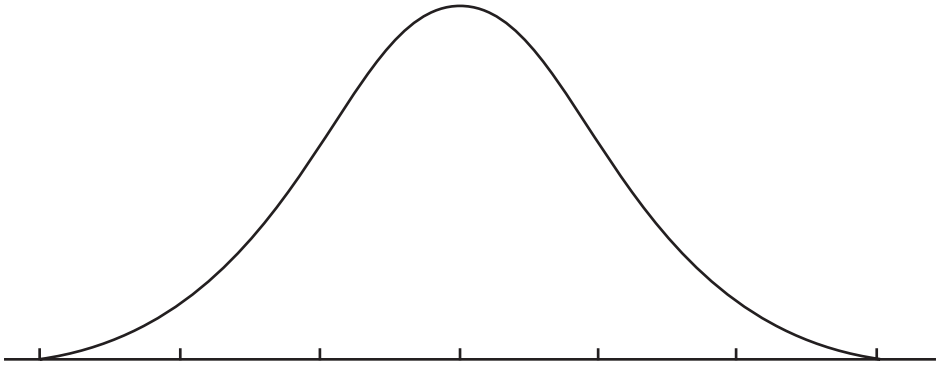
Instruction

- For example, if you hire a bus with 65 seats for a field trip, but 82 people sign up to go on the field trip, you need more seats. You would increase the number of buses from one bus to two buses, rather than from one bus to a fraction of a second bus.
- When using discrete data, we can assign probabilities to individual values. For example, the probability of rolling a 6 on a fair die is $\frac{1}{6}$.
- In contrast, **continuous data** is a set of values for which there is at least one value between any two given values—there are no gaps. For example, if a car accelerates from 30 miles per hour to 40 miles per hour, the car passes through every speed between 30 and 40 miles per hour. It does not skip instantly from 30 miles per hour to 40.
- When using continuous data, we need to assign probabilities to an interval or range of values.
- For continuous data, the probability of an exact value is essentially 0, so we must assign a range or an interval of interest to calculate probability. For example, a car will accelerate through a series of speeds in miles per hour, including an infinite number of decimals. Because there are an infinite number of values between the starting speed and the desired speed, the probability of determining an exact speed is essentially 0.
- An **interval** is a range or a set of values that starts with a specified value, ends with a specified value, and includes every value in between. The starting and ending values are the limits, or boundaries, of the interval.
- In other words, an interval is a set of values between a lower bound and an upper bound. The size of the interval depends on the situation being observed.
- The probability that a randomly selected student from a given high school is exactly 64 inches tall is effectively 0, since methods of measuring are not completely precise. Measuring tapes and rulers can vary slightly, and when we take measurements, we often round to the nearest quarter inch or eighth of an inch; it is impossible to determine a person's height to the exact decimal place. However, we can determine the probability that a student's height falls *between* two values, such as 63.5 and 64.5 inches, since this interval includes all of the infinite decimal values between these two heights.
- To determine the probability of an outcome using continuous data, we use the proportion of the area under the normal curve associated with the distribution of that data.

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Instruction

- A **normal curve** is a symmetrical curve representing the normal distribution.



- A **probability distribution** is a graph of the values of a random variable with associated probabilities.
- A **random variable** is a variable with a numerical value that changes depending on each outcome in a sample space. A random variable can take on different values, and the value that a random variable takes is associated with chance.
- The area under a probability distribution is equal to 1; that is, 100% of all possible data values within the interval are represented under the curve.
- A **continuous distribution** is a graphed set of values (a curve) in a continuous data set.
- We will examine two types of continuous distributions: uniform and normal.

Continuous Uniform Distributions

- A **uniform distribution** is a set of values that are continuous, are symmetric to a mean, and have equal frequencies corresponding to any two equally sized intervals.
- In other words, the values are spread out *uniformly* throughout the distribution.
- To determine the probability of an outcome using a uniform distribution, we calculate the ratio of the width of the interval of interest for the given outcome to the overall width of the distribution:

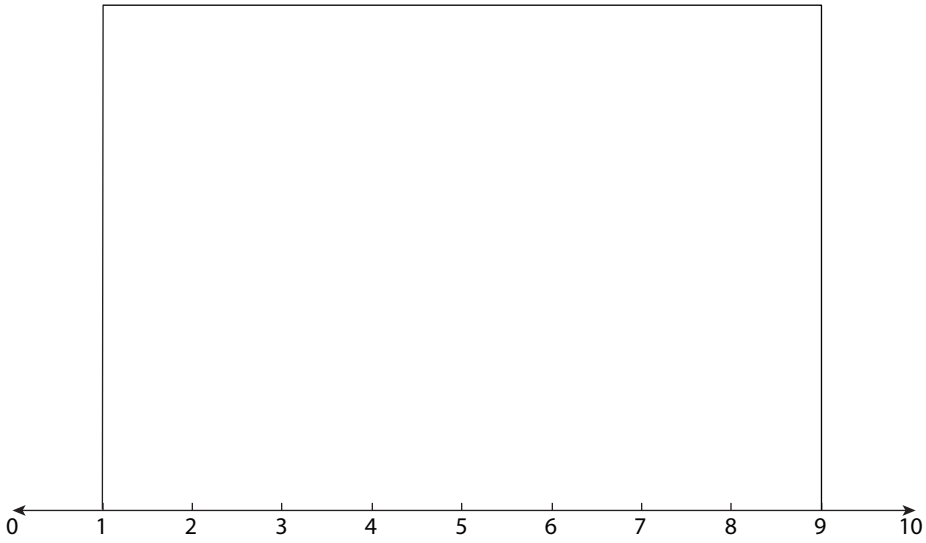
$$\frac{\text{width of the interval of interest}}{\text{total width of the interval of distribution}}$$

- The result of this proportion is equal to the probability of the outcome.

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Instruction

- In the uniform distribution that follows, the data values are spread evenly from 1 to 9:



Continuous Normal Distributions

- Another type of a continuous distribution is a normal distribution.
- A **normal distribution** is a set of values that are continuous, are symmetric to the mean, and have higher frequencies in intervals close to the mean than equal-sized intervals away from the mean. When graphed, data following a normal distribution forms a normal curve.
- Normal distributions are symmetric to the mean. This means that 50% of the data is to the right of the mean and 50% of the data is to the left of the mean.
- The **mean** is a measure of center in a set of numerical data, computed by adding the values in a data set and then dividing the sum by the number of values in the data set.
- The mean is denoted by the Greek lowercase letter ***mu***, μ.
- The Greek letter μ is also used when reporting the mean of a population.
- A **population** is made up of all of the people, objects, or phenomena of interest in an investigation. A **sample** is a subset of the population—that is, a smaller portion that represents the whole population.

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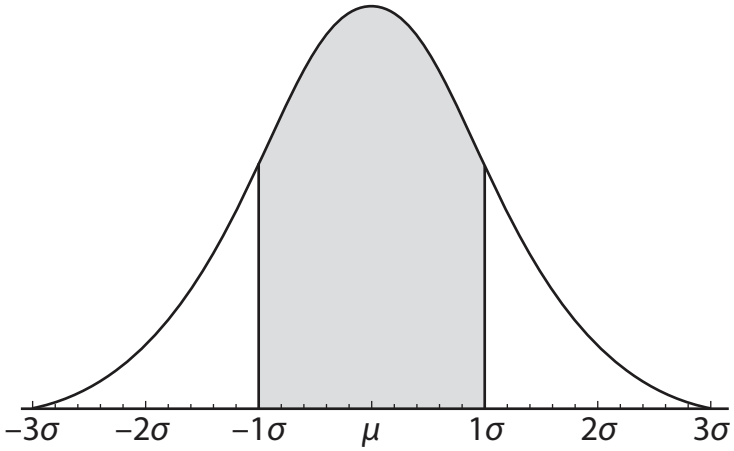
Instruction

- The **standard deviation** is a measure of average variation about a mean.
- Technically, the standard deviation is the square root of the average squared difference from the mean, and is denoted by the lowercase Greek letter ***sigma***, σ.

Steps to Find the Standard Deviation
1. Calculate the difference between the mean and each number in the data set.
2. Square each difference.
3. Find the mean of the squared differences.
4. Take the square root of the resulting number.

- Approximately 68% of the values in a normal distribution are within one standard deviation of the mean. Written as an equation, this is $\mu \pm 1\sigma \approx 68\%$. In other words, the mean, μ, plus or minus the standard deviation σ times 1 is approximately equal to 68% of the values in the distribution.
- In the graph that follows, the shading represents these 68% of values that fall within one standard deviation of the mean.

Data Within One Standard Deviation of the Mean



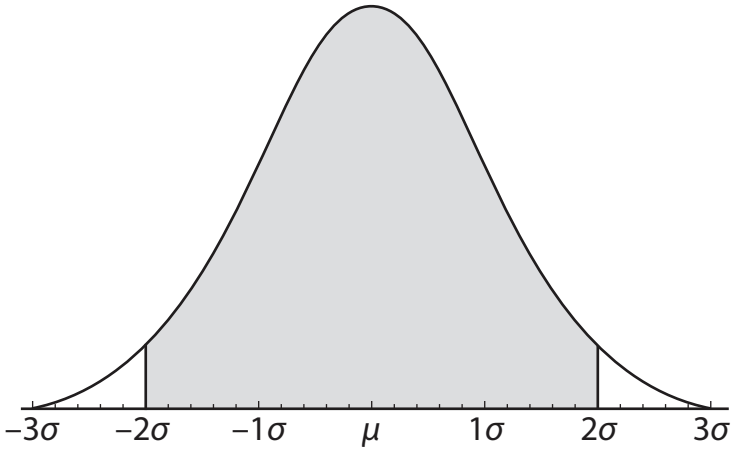
$\mu \pm 1\sigma \approx 68\%$

- Approximately 95% of the values in a normal distribution are within two standard deviations of the mean, as shown by the shading in the graph that follows.

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Instruction

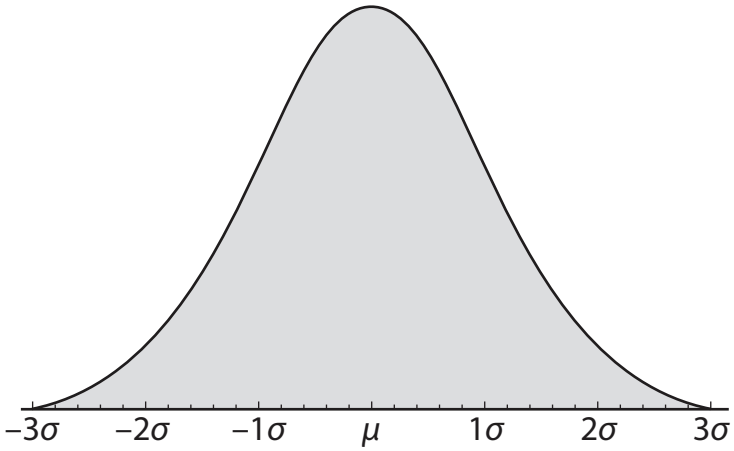
Data Within Two Standard Deviations of the Mean



$\mu \pm 2\sigma \approx 95\%$

- Approximately 99.7% of the values in a normal distribution are within three standard deviations of the mean, as shaded in the following graph.

Data Within Three Standard Deviations of the Mean



$\mu \pm 3\sigma \approx 99.7\%$

- These percentages of data under the normal curve ($\mu \pm 1\sigma \approx 68\%$, $\mu \pm 2\sigma \approx 95\%$, and $\mu \pm 3\sigma \approx 99.7\%$) follow what is called the **68–95–99.7 rule**. This rule is also known as the **Empirical Rule**.
- The **standard normal distribution** has a mean of 0 and a standard deviation of 1. A normal curve is often referred to as a bell curve, since its shape resembles the shape of a bell. Normal distribution curves are a common tool for teachers who want to analyze how their students performed on a test. If a test is "fair," you can expect a handful of students to do very well or very poorly, with most scores being near average—a normal curve. If the curve is shifted strongly toward the lower or higher ends of the scores, then the test was too hard or too easy.

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Instruction

Common Errors/Misconceptions

- applying the 68–95–99.7 rule to distributions that are not normally distributed
- assuming that all normal distributions have a mean of 0 and/or a standard deviation of 1
- not applying symmetry in a normal distribution to calculate probabilities

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Instruction

Guided Practice 1.1.1

Example 1

Find the proportion of values between 0 and 1 in a uniform distribution that has an interval of –3 to +3.

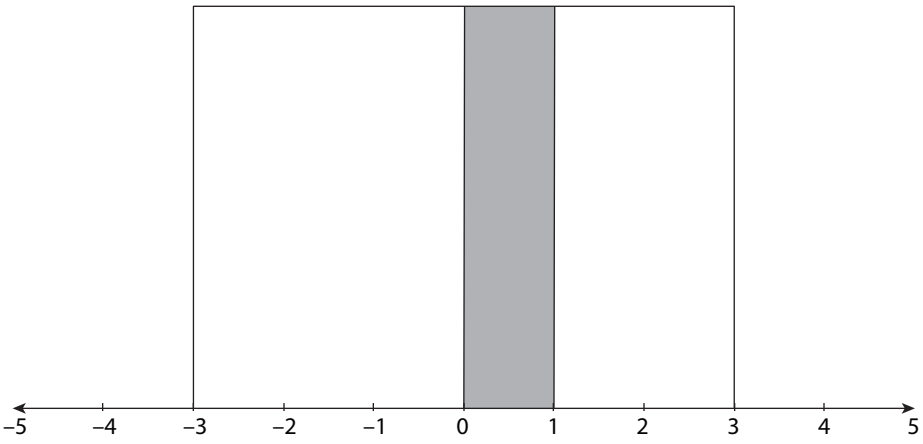
1. Sketch a uniform distribution and shade the area of the interval of interest.

   Start by drawing a number line. Be sure to include values on either side of the given interval. In this case, choose values greater than +3 and less than –3.

   A uniform distribution looks like a rectangle because each value in the continuous distribution has an equal probability.

   Draw a box that spans from –3 to +3 to show the distribution of the interval.

   Shade the region from 0 to 1.



2. Determine the width of the interval of interest.

   The interval of interest is between 0 and 1. We can see from the drawing of the uniform distribution that the width of this interval is 1.

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3. Determine the total width of the distribution.

   The total width of the distribution is determined by calculating the absolute value of the difference of the endpoints of the interval.

   The endpoints are at +3 and –3.

   $|3-(-3)|=|6|=6$

   The width of the distribution is 6.

4. Determine the proportion of values found in the interval of interest.

   The proportion of values between 0 and 1 is equal to the width of the interval from 0 to 1 divided by the width of the interval from –3 to +3.

   For distributions, the proportion of values should be written as a decimal.

   $$\frac{\text{width of the interval of interest}}{\text{total width of the interval of distribution}}=\frac{1}{6}=0.1\overline{6}$$

   The proportion of values is $0.1\overline{6}$.

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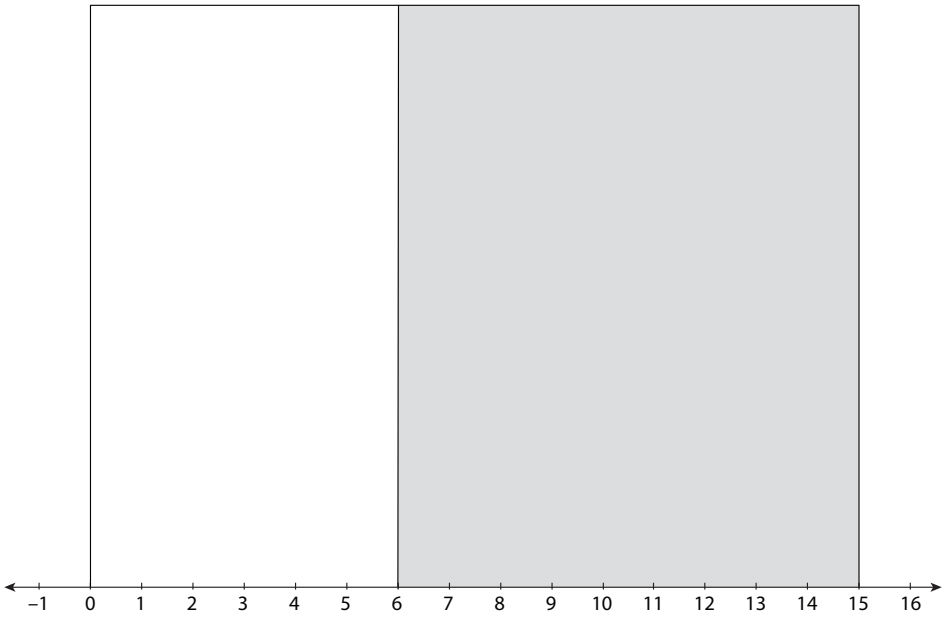
Example 2

Madison needs to ride a shuttle bus to reach an airport terminal. Shuttle buses arrive every 15 minutes, and the arrival times for buses are uniformly distributed. What is the probability that Madison will need to wait more than 6 minutes for the bus?

1. Sketch a uniform distribution and shade the area of the interval of interest.

   Start by drawing a number line.

   The interval of the distribution goes from 0 minutes to 15 minutes, and the interval of interest is from 6 to 15. Shade the region between 6 and 15.



2. Determine the total width of the distribution.

   We can see that the total width of the distribution is 15 minutes.

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Instruction

3. Determine the width of the interval of interest.

   Find the absolute value of the difference of the endpoints of the interval of interest.

   $|15-6|=|9|=9$

   The width of the interval of interest is 9 minutes.

4. Determine the proportion of the area of the interval of interest to the total area of the distribution.

   Create a ratio comparing the area that corresponds to arrival times between 6 and 15 minutes to the area of the total time frame of 15 minutes between buses.

   The proportion of the area of interest to the total area of the distribution is equal to the area of interest divided by the total area of the distribution.

   $$\frac{\text{width of the interval of interest}}{\text{total width of the interval of distribution}}=\frac{9}{15}=\frac{3}{5}=0.6$$

   The proportion of the area of interest to the total area of the distribution is 0.6.

5. Interpret the proportion in terms of the context of the problem.

   The probability that Madison will wait more than 6 minutes for the bus is 0.6.

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Example 3

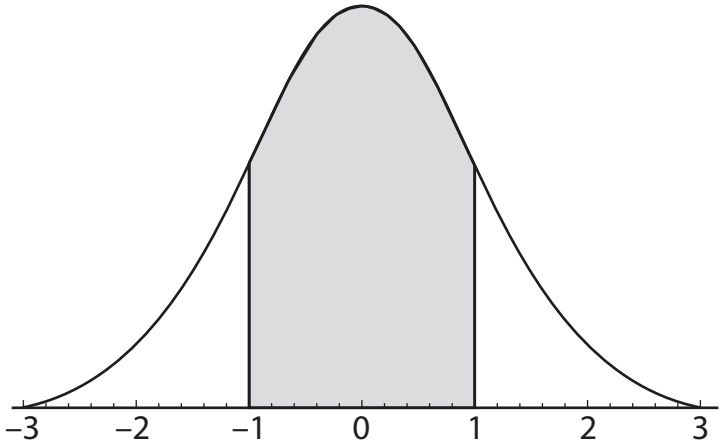
Temperatures in a carefully controlled room are normally distributed throughout the day, with a mean of 0° Celsius and a standard deviation of 1° Celsius. Shane randomly selects a time of day to enter the room. What is the probability that the temperature will be between –1° and +1° Celsius?

1. Sketch a normal curve and shade the area of the interval of interest.

   A normal curve is a bell-shaped curve, with its midpoint at the mean. In this problem, the mean is 0 and the standard deviation is 1.

   Start by drawing a number line. Be sure to include the range of values –3 to 3.

   Shade the region from –1 to 1.



2. Determine the proportion of the area of interest to the total area.

   The problem statement says the standard deviation is 1°. From the 68–95–99.7 rule, we know that $\mu \pm 1\sigma \approx 68\%$, and that describes our area of interest. Therefore, the proportion is 68%, or 0.68.

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Instruction

3. Interpret the proportion in terms of the context of the problem.

   The proportion of the area of interest is equal to the probability. Therefore, the probability that Shane will walk into the room and the temperature will be between –1°C and +1°C is 0.68.

   You can use a graphing calculator to verify this probability.

   On a TI-83/84:

   Step 1: Press [2ND][VARS] to bring up the distribution menu.

   Step 2: Arrow down to 2: normalcdf. Press [ENTER].

   Step 3: Enter the following values for the lower bound, upper bound, mean (μ), and standard deviation (σ). Press [,] after typing each value. Lower: [(–)][1]; upper: [1]; μ: [0]; σ: [1].

   Step 4: Press [ENTER] to calculate the probability.

   On a TI-Nspire:

   Step 1: Press the [home] key.

   Step 2: Arrow over to the spreadsheet icon and press [enter].

   Step 3: Press the [menu] key. Arrow down to 4: Statistics, then arrow right to bring up the sub-menu. Arrow down to 2: Distributions and press [enter].

   Step 4: Arrow down to 2: Normal Cdf. Press [enter].

   Step 5: Enter the values for the lower bound, upper bound, mean (μ), and standard deviation (σ), using the [tab] key to navigate between fields. Lower Bound: [(–)][1]; Upper Bound: [1]; μ; [0]; σ: [1]. Tab down to "OK" and press [enter].

   Step 6: The values entered will appear in the spreadsheet. Press [enter] again to calculate the probability.

   The calculator verifies that the probability is 0.68.



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Instruction

Example 4

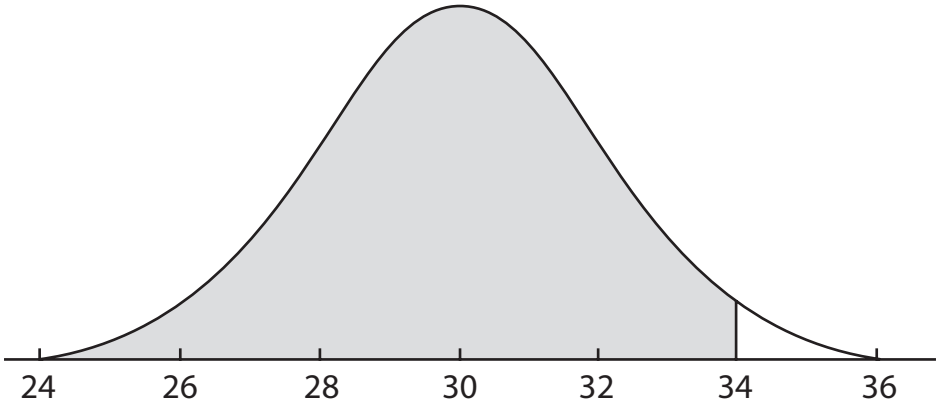
The scores of a particular college admission test are normally distributed, with a mean score of 30 and a standard deviation of 2. Erin scored a 34 on her test. If possible, determine the percent of test-takers whom Erin outperformed on the test.

1. Sketch a normal curve and shade the area of the interval of interest.

   To sketch the normal curve, follow the procedures shown in Example 3.

   We want to know how many test-takers had scores lower than Erin's.

   Erin scored a 34; therefore, the area of interest is the area to the left of 34.



2. Determine how many standard deviations away from the mean Erin's score is.

   From the problem statement, we know that Erin scored a 34, the mean is 30, and the standard deviation is 2. Erin's score is greater than the mean.

   Also, we can determine that Erin scored two standard deviations above the mean.

   $\mu + 1\sigma = 30 + 1(2) = 32$

   $\mu + 2\sigma = 30 + 2(2) = 34$ Erin's score

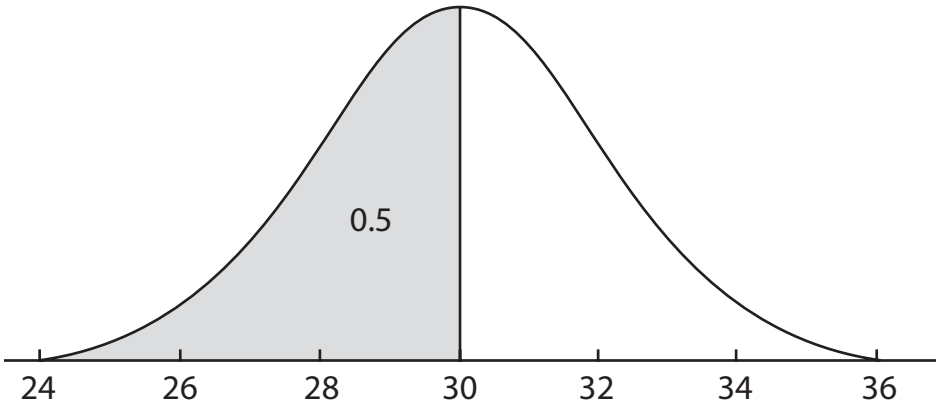
   $\mu + 3\sigma = 30 + 2(3) = 36$

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Instruction

3. Use symmetry and the 68–95–99.7 rule to determine the area of interest.

We know that the data in a normal curve is symmetrical about the mean. Since the area under the curve is equal to 1, the area to the left of the mean is 0.5, as shaded in the graph below.



Erin's score is above the mean; therefore, we need to determine the area between the mean and Erin's score and add it to the area below the mean to find the total area of interest.

Recall that the 68–95–99.7 rule states the percentages of data under the normal curve are as follows: $\mu \pm 1\sigma \approx 68\%$, $\mu \pm 2\sigma \approx 95\%$, and $\mu \pm 3\sigma \approx 99.7\%$. We know that $\mu \pm 2\sigma = 95\%$. We have already accounted for the area to the left of the mean, which includes from the mean down to -2σ. Since we found that Erin's score is two standard deviations from the mean, we need to determine the area from the mean up to $+2\sigma$.

Since data is symmetric about the mean, we know that half of the area encompassed between $\mu \pm 2\sigma$ is above the mean. Therefore, divide 0.95 by 2.

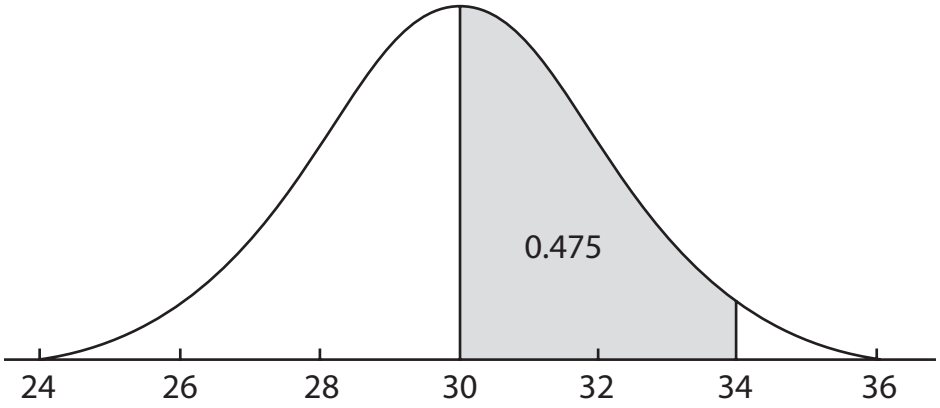
$$\frac{0.95}{2} = 0.475$$

(*continued*)

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The following graph shows the shaded area of interest to the right of the mean up until Erin's score of 34.



Add the two areas together to get the total area below 2σ, which is equal to Erin's score of 34.

$0.50 + 0.475 = 0.975$

The total area of interest for this data is 0.975.

A graphing calculator can also be used to calculate the area of interest.

On a TI-83/84:

Step 1: Press [2ND][VARS] to bring up the distribution menu.

Step 2: Arrow down to 2: normalcdf. Press [ENTER].

Step 3: Enter the following values for the lower bound, upper bound, mean (μ), and standard deviation (σ). Press [,] after typing each value. Lower: [(–)][99]; upper: [34]; μ: [30]; σ: [2].

Step 4: Press [ENTER] to calculate the area of interest.

(*continued*)

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On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the spreadsheet icon and press [enter].

Step 3: Press the [menu] key. Arrow down to 4: Statistics, then arrow right to bring up the sub-menu. Arrow down to 2: Distributions and press [enter].

Step 4: Arrow down to 2: Normal Cdf. Press [enter].

Step 5: Enter the values for the lower bound, upper bound, mean (μ), and standard deviation (σ), using the [tab] key to navigate between fields. Lower Bound: [(–)][99]; Upper Bound: [34]; μ: [30]; σ: [2]. Tab down to "OK" and press [enter].

Step 6: The values entered will appear in the spreadsheet. Press [enter] again to calculate the probability.

The result from the graphing calculator verifies the area of interest is 0.975.

4. Interpret the proportion in terms of the context of the problem.

   Convert the area of interest to a percent.

   0.975 = 97.5%

   Erin outperformed 97.5% of the students who also took the exam.

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Problem-Based Task 1.1.1: Lily's Lemonade Stand

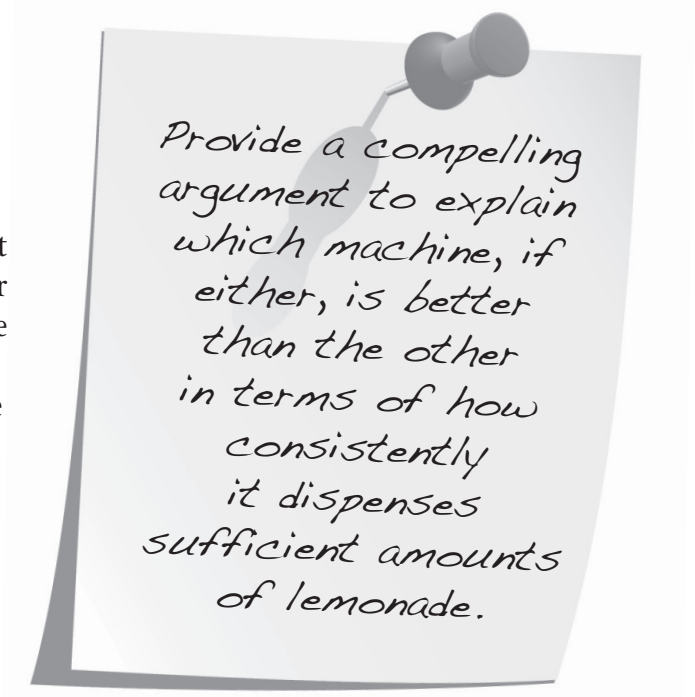
SMP	
1 ✓	2
3 ✓	4 ✓
5	6
7	8

Lily is setting up an automated lemonade stand to earn money for college. She bought two machines that fill cups automatically after customers deposit money. When the machines were delivered, Lily found that they were both set to dispense an average serving size of 8.10 fluid ounces, slightly greater than the 8 ounces that Lily had already printed on her advertising. The owner's manual says that the machines may sometimes dispense slightly more or less than the set amount. Lily's profits will suffer if the machines always dispense more than what she's charging for, but if she lowers the setting to exactly 8 ounces, some customers will get less than they're paying for. She needs to determine how much she can lower the setting and still make sure that customers are consistently getting at least 8 ounces of lemonade. After collecting samples from each machine, Lily came up with the following estimates:

- Machine A dispenses a mean of 8.10 fluid ounces with a standard deviation of 0.10 fluid ounces.
- Machine B dispenses a mean of 8.10 fluid ounces with a standard deviation of 0.05 fluid ounces.
- The amount of lemonade that each machine dispenses is normally distributed.

By adjusting the settings on the machine, Lily can change the mean amount of lemonade dispensed per cup. The standard deviation will stay the same.

Provide a compelling argument to explain which machine, if either, is better than the other in terms of how consistently it dispenses sufficient amounts of lemonade. Include compliance with advertising claims and Lily's cost to keep the machines filled with lemonade in your argument. Then determine how Lily could change the setting on the machine that doesn't perform as well so that 97.5% of her customers will receive at least 8 fluid ounces of lemonade. Show or explain your reasoning.



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Problem-Based Task 1.1.1: Lily's Lemonade Stand

Coaching

a. Is the expense of keeping the machines filled a concern in determining which machine is better?

b. How many standard deviations above the advertised amount does Machine A dispense per serving?

c. What percent of cups dispensed by Machine A will contain at least 8 fluid ounces?

d. How many standard deviations above the advertised amount does Machine B dispense per serving?

e. What percent of cups dispensed by Machine B will contain at least 8 fluid ounces?

f. Based on your answers from parts b–e, provide a compelling argument to explain which machine, if either, is better. Include compliance with advertising claims and the cost of lemonade in your argument.

g. What does the setting of the less reliable machine need to be so that its mean for ounces per serving is two standard deviations above the advertised amount of ounces per serving?

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Instruction

Problem-Based Task 1.1.1: Lily's Lemonade Stand

Coaching Sample Responses

a. Is the expense of keeping the machines filled a concern in determining which machine is better?

No. Both machines dispense a mean of 8.10 fluid ounces per serving. On average, both machines will use up the same amount of lemonade (unless adjustments are made).

b. How many standard deviations above the advertised amount does Machine A dispense per serving?

Machine A has a standard deviation of 0.10 fluid ounces and dispenses a mean of 8.10 fluid ounces per cup. The advertised amount is 8 fluid ounces per cup. 8.10 – 0.10 = 8, so Machine A is one standard deviation above the advertised amount.

c. What percent of cups dispensed by Machine A will contain at least 8 fluid ounces?

The area of interest is the area to the right of -1σ, since 8 fluid ounces is one standard deviation below the mean. From -1σ to the mean is half of the area from -1σ to $+1\sigma$, so 68/2 = 34%. Then the area to the right of the mean is 50%. Add the two areas together to get the total area.

50 + 34 = 84

Approximately 84% of the cups dispensed by Machine A will contain at least 8 fluid ounces of lemonade.

d. How many standard deviations above the advertised amount does Machine B dispense per serving?

On average, Machine B dispenses an amount of lemonade that is two standard deviations above the advertised amount.

e. What percent of cups dispensed by Machine B contain at least 8 fluid ounces?

The standard deviation for Machine B is 0.05 fluid ounces. Approximately 97.5% of the cups dispensed by Machine B will contain at least 8 fluid ounces, since 8 fluid ounces is two standard deviations below the mean of 8.10 fluid ounces.

Calculate the area of interest by breaking it up into two smaller known parts. The area to the left of the mean is the area between two standard deviations divided by 2. The area of $\mu \pm 2\sigma = 0.95$ or 95%, so $\frac{95}{2} = 47.5\%$. The area to the right of the mean is 50%. Add the two areas together for the total area.

47.5 + 50 = 97.5

Approximately 97.5% of the cups dispensed by Machine B contain at least 8 fluid ounces of lemonade.

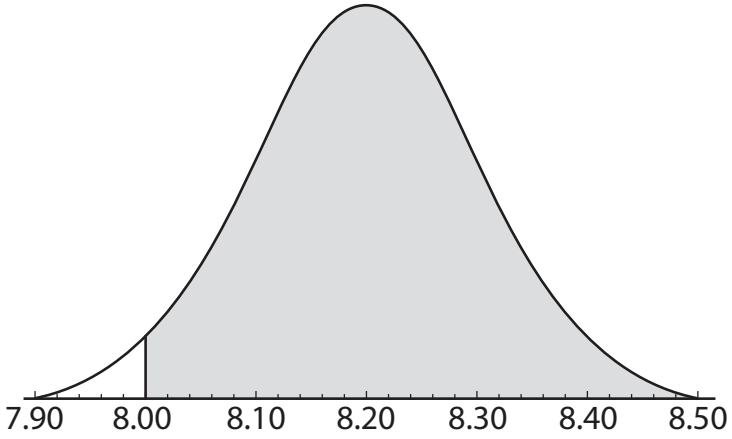
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f. Based on your answers from parts b–e, provide a compelling argument to explain which machine, if either, is better. Include compliance with advertising claims and the cost of lemonade in your argument.

Machine B is better because 97.5% of the cups it dispenses contain at least 8 fluid ounces of lemonade as Lily's advertisements claim, while only 84% of the cups from Machine A contain at least 8 fluid ounces of lemonade.

g. What does the setting of the less reliable machine need to be so that its mean for ounces per serving is two standard deviations above the advertised amount of ounces per serving?

The standard deviation of Machine A is 0.10 fluid ounces. In order for its mean to be two standard deviations above the advertised amount of 8 ounces per serving, the setting for Machine A needs to be 8.20, because 8.00 + 2(0.10) = 8.20.



Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.1.1: Normal Distributions and the 68–95–99.7 Rule

Use the information below to solve problems 1 and 2.

> The mean gas mileage for cars driven by the students at Chillville High School is 28.0 miles per gallon, and the standard deviation is 4.0 miles per gallon. Assume that the gas mileages are normally distributed.

1. What percent of the cars driven by the students at Chillville have gas mileages between 24.0 and 32.0 miles per gallon?

2. What percent of the cars driven by the students at Chillville have gas mileages greater than 20.0 miles per gallon?

Use the information below to solve problems 3 and 4.

> The response times for a certain ambulance company are normally distributed, with a mean of 12.5 minutes. Ninety-five percent of the response times are between 10 and 15 minutes.

3. What is the standard deviation of the response times?

4. What percent of the response times are longer than 15 minutes?

continued

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Use the information below to solve problems 5 and 6.

The Soaking Sojourn ride at the WattaWatta Water Park is an 18-minute ride through man-made rapids and waterfalls. While the ride is in full operation, riding times for passengers are uniformly distributed between 0 and 18 minutes. Suppose an electrical problem leads to a temporary stoppage of the ride.

5. What percent of the riders had been on the ride for less than 2 minutes when the stoppage occurred?

6. What percent of the riders had been on the ride between 10 and 15 minutes when the stoppage occurred?

Use the information below to solve problems 7 and 8.

A quality control inspector for a bagel shop periodically checks the caloric content of the bagels. The inspector has determined that the multi-grain bagels have a mean of 300 calories and a standard deviation of 10 calories. The inspector has determined that the calories are normally distributed.

7. What percent of the multi-grain bagels have a caloric content that is within two standard deviations of the mean?

8. What percent of the multi-grain bagels have between 290 and 320 calories?

Use the information below to solve problems 9 and 10.

Real estate prices in the coastal town of Rockland have a mean of $240,000 and a standard deviation of $150,000. Many of the properties are two- and three-bedroom cottages in the $100,000 to $150,000 price range, but there are several ocean-view homes with prices well over $1 million.

9. Why is it a mistake to apply the properties of a normal distribution to the real estate prices in Rockland?

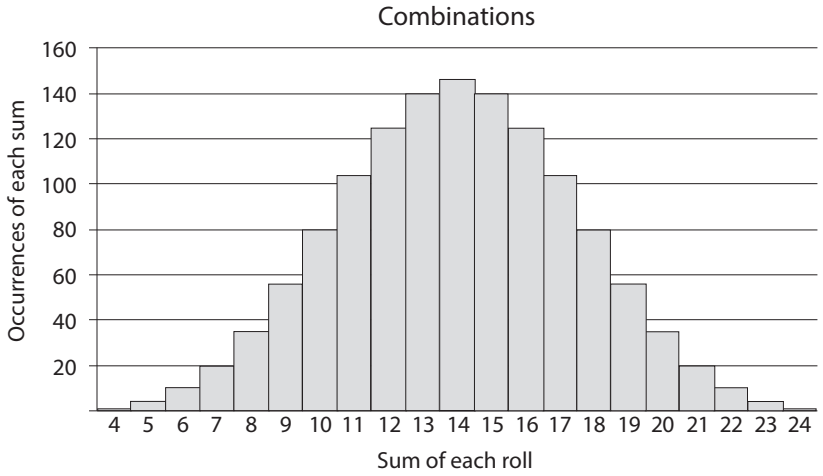
10. Use a compelling mathematical argument to show that the real estate prices in Rockland are not normally distributed.

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Lesson 1.1.2: Standard Normal Calculations

Warm-Up 1.1.2

Richie is studying the probabilities associated with rolling four six-sided dice. Richie listed all 1,296 (6^4) possible combinations of dice rolls and recorded their sums. Richie's results are shown in the histogram and table that follow. Use the information to answer the questions.



Probabilities of Sums of Four Dice

Sum of each roll, x	Combinations (number of occurrences of each sum)	Probability of each roll, $P(x)$
4	1	0.0008
5	4	0.0031
6	10	0.0077
7	20	0.0154
8	35	0.0270
9	56	0.0432
10	80	0.0617
11	104	0.0802
12	125	0.0965
13	140	0.1080
14	146	0.1127
15	140	0.1080
16	125	0.0965
17	104	0.0802
18	80	0.0617
19	56	0.0432
20	35	0.0270
21	20	0.0154
22	10	0.0077
23	4	0.0031
24	1	0.0008
Total	1,296	1.000

continued

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1. Estimate the mean of the sums.

2. Why is it more likely to roll a sum of 10 than a sum of 5?

3. What is the probability of rolling a sum greater than 20?

4. What is the probability of rolling a sum between 10 and 15 inclusive?

5. Describe the similarities and differences between the distribution of the sums of four six-sided dice and a normal distribution.

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Lesson 1.1.2: Standard Normal Calculations

Common Core State Standard

S–ID.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.★

Warm-Up 1.1.2 Debrief

1. Estimate the mean of the sums.

   The mean is the balancing point of the distribution. Since this distribution is symmetric, the mean is 14—the value at the center of the distribution.

2. Why is it more likely to roll a sum of 10 than a sum of 5?

   There are many more combinations of 10 than there are of 5. From the middle column of the table, we can see that the sum of 5 can be obtained only 4 ways. There are 80 ways to roll a sum of 10.

3. What is the probability of rolling a sum greater than 20?

   To determine the probability of rolling a sum greater than 20, we can add the probabilities associated with rolls of 21, 22, 23, and 24, shown in the table:

   $0.0154 + 0.0077 + 0.0031 + 0.0008 = 0.0270$

   We can also work directly with the combinations for 21, 22, 23, and 24. We can find the probability of rolling a sum greater than 20 by adding up these combinations, then dividing the sum by the total number of combinations.

   The number of combinations associated with rolls of 21, 22, 23, and 24 are as follows: 20, 10, 4, and 1.

   $20 + 10 + 4 + 1 = 35$

   Divide 35 by the total number of combinations (1,296).

   $35 \div 1296 \approx 0.0270$

   The probability of rolling a sum greater than 20 is approximately 0.0270.

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4. What is the probability of rolling a sum between 10 and 15 inclusive?

   To determine the probability of rolling a sum between 10 and 15 inclusive, we can add the probabilities associated with the rolls of 10, 11, 12, 13, 14, and 15.

   $0.0617 + 0.0802 + 0.0965 + 0.1080 + 0.1127 + 0.1080 = 0.5671$

   We can also add up the combinations for 10, 11, 12, 13, 14, and 15, then divide the sum by the total number of combinations.

   The number of combinations associated with rolls of 10, 11, 12, 13, 14, and 15 are as follows: 80, 104, 125, 140, 146, and 140.

   $80 + 104 + 125 + 140 + 146 + 140 = 735$

   Divide this result by the number of combinations.

   $735 \div 1296 \approx 0.5671$

   The probability of rolling a sum between 10 and 15 is approximately 0.5671.

5. Describe the similarities and differences between the distribution of the sums of four six-sided dice and a normal distribution.

   The overall shape of the histogram has three important similarities to the normal distribution:

   - The distribution is symmetric.
   - Higher probabilities are associated with values near the mean than away from the mean.
   - There are inflection points on either side of the mean at which the graph changes from concave upward to concave downward (on the left) or concave downward to concave upward (on the right).

   There are two key differences:

   - The distribution of the sums of the dice is discrete, while the normal distribution is continuous. We cannot roll a sum of 11.5 or 11.75—we jump from 11 to 12.
   - Also, the distribution of sums begins at 4 and ends at 24, so that the probability of rolling any sum less than 4 or greater than 24 is 0. Even though the theoretical values greater than three standard deviations from the mean in a normal distribution are small, they are still greater than 0.

Connection to the Lesson

- Students will continue their study of normally distributed data including investigating the properties of the normal distribution, such as symmetry and clustering of data near the mean.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- recognizing the relationship between probabilities and area under a curve (S–ID.4)
- finding the mean and standard deviation of a distribution of numbers (6.SP.5c)
- distinguishing between measures of center and variation (6.SP.3)

Introduction

Previous lessons demonstrated the use of the standard normal distribution. While distributions with a mean of 0 and a standard deviation of 1 are rare in the real world, there is a formula that allows us to use the properties of a standard normal distribution for any normally distributed data. With this formula, we can generate a number called a *z*-score to use with our data. This makes the normal distribution a powerful tool for analyzing a wide variety of situations in business and industry as well as the physical and social sciences.

Using and understanding *z*-scores requires a deeper understanding of standard deviation. In the previous sub-lesson, we found the standard deviations of small data sets. In this lesson, we will explore how to use *z*-scores and graphing calculators to evaluate large data sets.

Key Concepts

- Recall that a population is all of the people or things of interest in a given study, and that a sample is a subset (or smaller portion) of the population.
- Samples are used when it is impractical or inefficient to measure an entire population. Sample statistics are often used to estimate measures of the population (parameters).
- The mean of a sample is the sum of the data points in the sample divided by the number of data points, and is denoted by the Greek letter *mu*, μ.
- The mean is given by the formula $\mu = \frac{x_1 + x_2 + \ldots + x_n}{n}$, where each *x*-value is a data point and *n* is the total number of data points in the set.
- From a visual perspective, the mean is the balancing point of a distribution.
- The mean of a symmetric distribution is also the median of the distribution.
- A **symmetric distribution** is a distribution of data in which a line can be drawn so that the left and right sides are mirror images of each other.
- The **median** is the middle value in a list of numbers.
- Both the mean and median are at the center of a symmetric distribution.

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Instruction

- The standard deviation of a distribution is a measure of variation.
- Another way to think of standard deviation is "average distance from the mean." The formula for the standard deviation is given by $\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}}$, where σ (the lowercase Greek letter *sigma*) represents the standard deviation, x_i is a data point, and $\sum_{i=1}^{n}$ means to take the sum from 1 to n data points.
- **Summation notation** is used in the formula for calculating standard deviation; it is a symbolic way to represent the sum of a sequence.
- Summation notation uses the uppercase version of the Greek letter ***sigma***, Σ.
- After calculating the standard deviation, σ, you can use this value to calculate a *z*-score.
- A ***z*-score** measures the number of standard deviations that a given score lies above or below the mean. For example, if a value is three standard deviations above the mean, its *z*-score is 3.
- A positive *z*-score corresponds to an individual score that lies above the mean, while a negative *z*-score corresponds to an individual score that lies below the mean.
- By using *z*-scores, probabilities associated with the standard normal distribution (mean = 0, standard deviation = 1) can be used for any non-standard normal distribution (mean ≠ 0, standard deviation ≠ 1).
- The formula for calculating the *z*-score is given by $z = \frac{x - \mu}{\sigma}$, where z is the *z*-score, x is the data point, μ is the mean, and σ is the standard deviation.
- *z*-scores can be looked up in a table to determine the associated area or probability.
- The numerical value of a *z*-score can be rounded to the nearest hundredth.
- Graphing calculators can greatly simplify the process of finding statistics and probabilities associated with normal distributions.

Common Errors/Misconceptions

- calculating and applying a *z*-score to a distribution that is not normally distributed
- using the area to the left of the *z*-score when the area to the right of the *z*-score is the area of interest and vice versa
- misreading the table with the associated probability

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Instruction

Guided Practice 1.1.2

Example 1

In the 2012 Olympics, the mean finishing time for the men's 100-meter dash finals was 10.10 seconds and the standard deviation was 0.72 second. Usain Bolt won the gold medal, with a time of 9.63 seconds. Assume a normal distribution. What was Usain Bolt's *z*-score?

1. Write the known information about the distribution.

   Let x represent Usain Bolt's time in seconds.

   $\mu = 10.10$

   $\sigma = 0.72$

   $x = 9.63$

2. Substitute these values into the formula for calculating *z*-scores.

   The *z*-score formula is $z = \frac{x - \mu}{\sigma}$.

   $$z = \frac{x - \mu}{\sigma} = \frac{9.63 - 10.10}{0.72} \approx -0.65$$

   Usain Bolt's *z*-score for the race was –0.65. Therefore, his time was 0.65 standard deviations below the mean.

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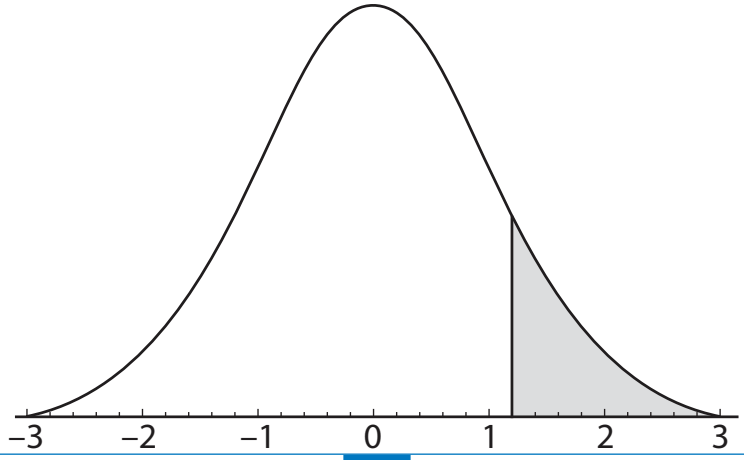
Example 2

What percent of the values in a normal distribution are more than 1.2 standard deviations above the mean?

1. Sketch a normal curve and shade the area that corresponds to the given information.

   Start by drawing a number line. Be sure to include the range of values –3 to 3.

   Create a vertical line at 1.2. Shade the region to the right of 1.2.



2. Use a table of *z*-scores or a graphing calculator to determine the shaded area.

   A *z*-score table can be used to determine the area.

   Since the area of interest is 1.2 standard deviations above the mean and greater, we need to look up the area associated with a *z*-score of 1.2.

(*continued*)

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Instruction

The following table contains *z*-scores for values around 1.2σ.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319

To find the area to the left of 1.2, locate 1.2 in the left-hand column of the *z*-score table, then locate the remaining digit 0 as 0.00 in the top row. The entry opposite 1.2 and under 0.00 is 0.8849; therefore, the area to the left of a *z*-score of 1.2 is 0.8849 or 88.49%.

We are interested in the area to the right of the *z*-score. Therefore, subtract the area found in the table from the total area under the normal distribution, 1.

$1 - 0.8849 = 0.1151$

The area greater than 1.2 standard deviations under the normal curve is about 0.1151 or 11.51%.

(*continued*)

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Alternately, you can use a graphing calculator to determine the area of the shaded region.

Note: The lower bound is 1.2, but the upper bound is infinity, so any large positive integer will work as the upper bound value. Use 100 as the upper bound. Since this problem is based on standard deviations under the standard normal distribution, the mean = 0 and the standard deviation = 1.

On a TI-83/84:

Step 1: Press [2ND][VARS] to bring up the distribution menu.

Step 2: Arrow down to 2: normalcdf. Press [ENTER].

Step 3: Enter the following values for the lower bound, upper bound, mean (μ), and standard deviation (σ). Press [,] after typing each value. Lower: [1.2]; upper: [100]; μ: [0]; σ: [1].

Step 4: Press [ENTER] to calculate the area of the shaded region.

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the spreadsheet icon and press [enter].

Step 3: Press the [menu] key. Arrow down to 4: Statistics, then arrow right to bring up the sub-menu. Arrow down to 2: Distributions and press [enter].

Step 4: Arrow down to 2: Normal Cdf. Press [enter].

Step 5: Enter the values for the lower bound, upper bound, mean (μ), and standard deviation (σ), using the [tab] key to navigate between fields. Lower Bound: [1.2]; Upper Bound: [100]; μ; [0]; σ: [1]. Tab down to "OK" and press [enter].

Step 6: The values entered will appear in the spreadsheet. Press [enter] again to calculate the area of the shaded region.

The area returned on either calculator is about 0.1151 or 11.51%.



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Instruction

Example 3

If a population of human body temperatures is normally distributed with a mean of 98.2°F and a standard deviation of 0.7°F, estimate the percent of temperatures between 98.0°F and 99.0°F.

1. Calculate the *z*-scores associated with the bounds of the given interval.

   Use the formula for *z*-scores, $z = \frac{x - \mu}{\sigma}$.

   Determine the known values. Let x_1 represent the lower bound, and x_2 represent the upper bound.

   $x_1 = 98.0$

   $x_2 = 99.0$

   $\mu = 98.2$

   Substitute values into the formula to find the *z*-score for the lower bound (z_1), then for the upper bound (z_2).

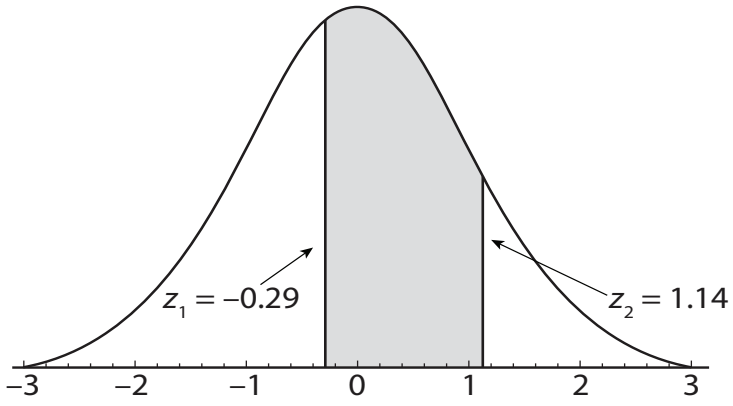
   lower bound = $z_1 = \frac{x_1 - \mu}{\sigma} = \frac{(98.0) - (98.2)}{(0.7)} = -0.29$

   upper bound = $z_2 = \frac{x_1 - \mu}{\sigma} = \frac{(99.0) - (98.2)}{(0.7)} = 1.14$

2. Sketch a normal curve and shade the area of interest.

   Start by drawing a number line. Be sure to include the range of values –3 to 3.

   Create vertical lines at –0.29 and 1.14. Shade the region between –0.29 and 1.14.



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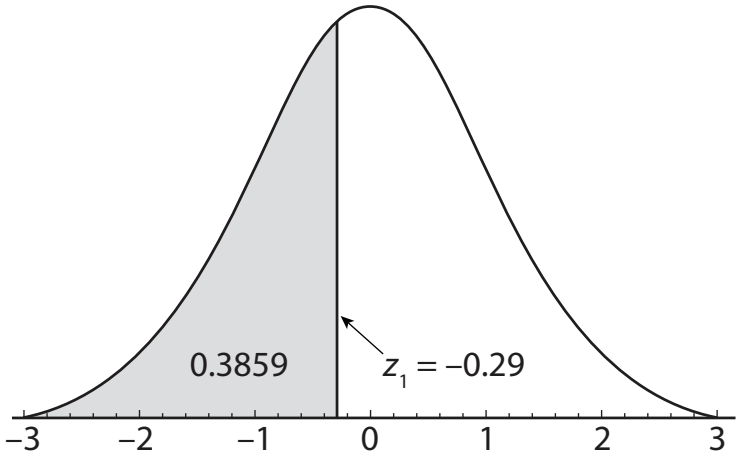
Instruction

3. Use a table of *z*-scores or a graphing calculator to find the value of the area of interest.

   A *z*-score table can be used to determine the number value of the area of the shaded region.

   To find the area to the left of z_1, –0.29, locate –0.2 in the left-hand column of the *z*-score table, then locate the remaining digit 9 as 0.09 in the top row. The entry opposite –0.2 and under 0.09 is 0.3859; therefore, the area to the left of a *z*-score of –0.29 is 0.3859 or 38.59%.

   The area to the left of z_1 is 0.3859 and corresponds to the shaded area in the following graph:



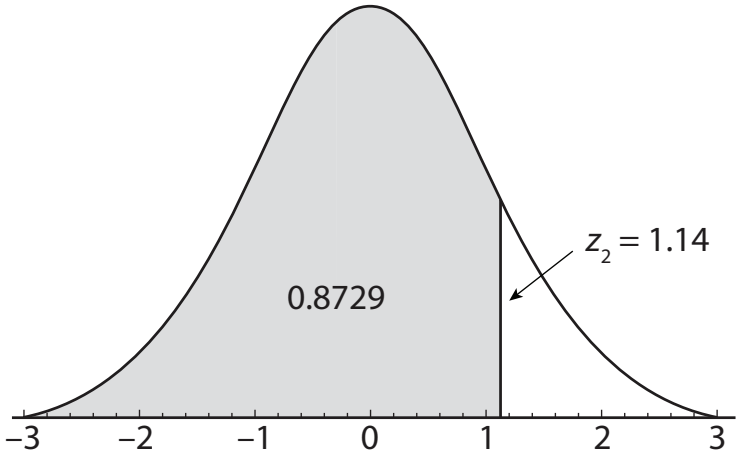
   To find the area to the left of z_2, 1.14, locate 1.1 in the left-hand column of the *z*-score table, then locate the remaining digit 4 as 0.04 in the top row. The entry opposite 1.1 and under 0.04 is 0.8729; therefore, the area to the left of a *z*-score of 1.14 is 0.8729 or 97.29%.

(*continued*)

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Instruction

The area to the left of z_2 is 0.8729 and corresponds to the shaded area in the following graph:



Subtract the area of z_1 from the area of z_2 to calculate the area of the interval of interest.

$0.8729 - 0.3859 = 0.4870$

Follow the calculator directions described in Example 2 to determine the area of the shaded region. Use these values as identified in the problem:

lower bound: 98

upper bound: 99

μ: 98.2

σ: 0.7

The calculated area of the interval of interest is 0.485902 or, rounded, 0.486.

Either using a table or a calculator gives an area of about 0.486 or 0.487. The difference is due to rounding in the table. Either value is correct.

4. Interpret the results in terms of the context of the problem.

   The result means that about 48.7% of the temperatures will be between the given interval of 98°F and 99°F.

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Instruction

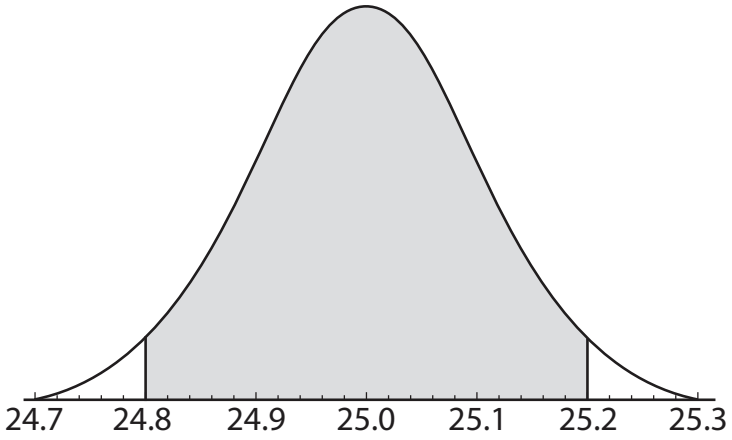
Example 4

The manufacturing specifications for nails produced at a machine shop require a minimum length of 24.8 centimeters and a maximum length of 25.2 centimeters. The operator of the machine shop adjusts the nail-making machine so that the machine produces nails with a mean length of 25.0 centimeters. What standard deviation is required for 95% of the nails to meet manufacturing specifications? Assume the lengths of nails produced by the machine are normally distributed.

1. Sketch the normal curve and the area of interest.

   Start by drawing a number line. The curve should account for nails that are too small or large to meet the requirements, so the intervals shown on the curve should start somewhere less than 24.7 and somewhere more than 25.2.

   Create vertical lines at 24.8 and 25.2. Shade the region between 24.8 and 25.2.



2. Determine the *z*-scores for the boundaries of the interval of interest.

   First, we need to determine the percentage of the area that is outside the area of interest.

   We know that the area of interest is comprised of 95% of the nails. This leaves 5% of the area to be shown in the tails of the curve. Since data in a normal distribution is symmetric about the mean, half of the 5% area that is not shaded is in the left tail and half is in the right tail.

   Half of 5% is 2.5% or 0.025, so each tail has an area of 0.025. Use this value when consulting the *z*-score table.

(*continued*)

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Instruction

We only need to find the *z*-score for the left tail in order to be able to use the *z*-score formula to calculate the standard deviation. In the negative *z*-score values section of the table that follows, look for an area that is close in value to 0.025, and then find the corresponding *z*-score.

Once you find the area 0.025, look at the value in the left-most column, –1.9. Then look up from 0.025 to the topmost value, 0.06, to arrive at the answer of –1.96.

The *z*-score associated with an area of 0.025 is –1.96.

Compare this result to that found using a graphing calculator.

On a TI-83/84:

Step 1: Press [2ND][VARS] to bring up the distribution menu.

Step 2: Arrow down to 3: invNORM(. Press [ENTER].

Step 3: Enter values for the area, μ, and σ. Press [ENTER] after typing each value to navigate between fields.

Step 4: Press [ENTER] three times to calculate the *z*-score.

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the spreadsheet icon and press [enter].

Step 3: Press the [menu] key. Arrow down to 4: Statistics, then arrow right to bring up the sub-menu. Arrow down to 2: Distributions and press [enter].

Step 4: Arrow down to 3: Inverse Normal. Press [enter].

Step 5: Enter values for the area, μ, and σ, using the [tab] key to navigate between fields. Tab down to "OK" and press [enter].

Step 6: The values entered will appear in the spreadsheet. Press [enter] again to calculate the *z*-score.

The calculated value is –1.95996, which rounds to –1.96, the *z*-score found using the table.

The *z*-score corresponding to the lower bound of 24.8 is –1.96. By symmetry, the *z*-score corresponding to the upper bound (25.2) is 1.96.

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Instruction

3. Use the *z*-score formula and the lower boundary of the area of interest to calculate the standard deviation.

   Substitute the known values into the formula, $z=\frac{x-\mu}{\sigma}$. Let x represent the lower bound and z represent the z-score for the lower bound.

   Known values:

   $z = -1.96$

   $x = 24.8$

   $\mu = 25$

   $z=\frac{x-\mu}{\sigma}$ — z-score formula

   $(-1.96)=\frac{(24.8)-(25)}{\sigma}$ — Substitute the values into the formula.

   $-1.96=\frac{-0.2}{\sigma}$ — Simplify.

   $-1.96\sigma = -0.2$ — Multiply both sides by σ.

   $\sigma=\frac{-0.2}{-1.96}$ — Divide both sides by –1.96.

   $\sigma \approx 0.10$

   The standard deviation required to produce 95% of the nails within the acceptable range is approximately 0.10.

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Instruction

Example 5

Find the mean and standard deviation of the positive single-digit even numbers (2, 4, 6, and 8). Treat this set as a population.

1. Find the mean of the data set.

   The mean is the balancing point of a distribution. To compute the mean, add all the x-values of the data set and divide them by the number of x-values in the set.

   There are 4 values in this data set: 2, 4, 6, and 8.

   $\mu = \frac{x_1 + x_2 + \ldots + x_n}{n}$ Equation to find the mean of a data set

   $\mu = \frac{2+4+6+8}{4}$ Substitute the given x-values; substitute 4 for n.

   $\mu = \frac{20}{4} = 5$ Simplify.

   The mean is 5 ($\mu = 5$).

2. Calculate the standard deviation using the standard deviation formula.

   The standard deviation is the square root of the average squared difference from the mean. The formula for standard deviation is $\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}}$, where σ represents the standard deviation, x_i is a data point, and $\sum_{i=1}^{n}$ means to take the sum from 1 to n data points.

   Since there are 4 numbers in the data set, $n = 4$.

(*continued*)

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Instruction

To organize the information, make a table and sum the column of $(x_i - \mu)^2$.

x_i	$x_i - \mu$	$(x_i - \mu)^2$
2	–3	9
4	–1	1
6	1	1
8	3	9
$\sum$		20

Substitute the values into the standard deviation formula.

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}} = \sqrt{\frac{20}{4}} = \sqrt{5} \approx 2.23607$$

The standard deviation is approximately 2.23607 ($\sigma \approx 2.23607$).

A graphing calculator can also be used to find the mean and standard deviation of the data set.

On a TI-83/84:

Step 1: Press [STAT] to bring up the statistics menu. The first option, 1: Edit, will already be highlighted. Press [ENTER].

Step 2: Arrow up to L1 and press [CLEAR], then [ENTER], to clear the list. Repeat this process to clear L2 and L3 if needed.

Step 3: From L1, press the down arrow to move your cursor into the list. Enter each number from the data set, pressing [ENTER] after each number to navigate down to the next blank spot in the list.

Step 4: Press [STAT]. Arrow over to the CALC menu. The first option, 1–Var Stats, will already be highlighted. Press [ENTER]. This brings up the 1–Var Stats menu.

Step 5: In the menu, "L1" should be displayed next to "List." Press [2ND][1] if not.

Step 6: Press [ENTER] three times to evaluate the data set. This will display a list of calculated values for the set. The mean will be listed to the right of "$\bar{x} =$". (Note that $\bar{x}$ is another way to represent μ.) The standard deviation will be listed to the right of "$\sigma x =$".

(*continued*)

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Instruction

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the spreadsheet icon and press [enter].

Step 3: The cursor will be in the first cell of the first column. Enter each number from the data set, pressing [enter] after each number to navigate down to the next blank cell.

Step 4: Arrow up to the topmost cell of the column, labeled "A." Name the column "values" using the letters on your keypad. Press [enter].

Step 5: Press the [menu] key. Arrow down to 4: Statistics, then arrow right to bring up the sub-menu. The first option, 1: Stat Calculations, will be highlighted. Arrow right to bring up the next sub-menu, where option 1: One-Variable Statistics, will be highlighted. Press [enter].

Step 6: Type [1] and press [enter] if the number of lists in the field is blank. Press [enter] two times to evaluate the data set. This will bring you back to the spreadsheet, where columns B and C will be populated with the titles and values for each calculation. Note that the mean is represented by $\bar{x}$ instead of μ. Use the arrow key to scroll down the rows of the spreadsheet to find the standard deviation, listed to the right of "$\sigma x := \sigma_n x$...".

Each calculator yields a mean of 5 and a standard deviation of approximately 2.23607.



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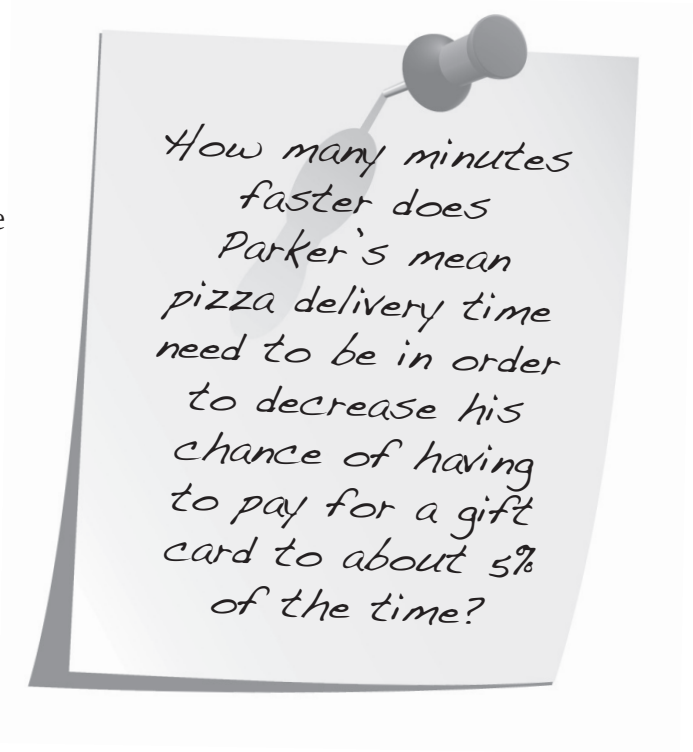
Problem-Based Task 1.1.2: Parker's Pizza Delivery

SMP	
1 ✓	2 ✓
3	4
5	6
7	8 ✓

Parker earns money for college by delivering pizzas for his father's pizza restaurant. Each driver has to log the time it takes to deliver every order. Starting next week, Parker's father is going to send customers a $20 gift card for any pizza delivery that takes more than 30 minutes, and the cost of the card will be deducted from the delivery driver's paycheck. Parker wants to analyze his delivery history to determine the probability that he'll have to pay for gift cards. He decides to use the times for his last 40 deliveries to determine his mean delivery time. Parker's delivery times, rounded to the nearest minute, are shown in the table.

Times in Minutes for 40 Deliveries				
17	12	22	16	30
22	30	19	28	30
33	19	17	25	17
21	12	26	21	19
27	24	15	32	26
31	28	23	26	32
21	31	22	22	25
23	22	31	21	22

What is the probability that Parker will be required to pay for a gift card? How many minutes faster does Parker's mean pizza delivery time need to be in order to decrease his chance of having to pay for a gift card to about 5% of the time? Assume the same standard deviation for Parker's current mean and his reduced mean.



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Problem-Based Task 1.1.2: Parker's Pizza Delivery

Coaching

a. What is the mean of Parker's last 40 delivery times?

b. What is the standard deviation of the delivery times?

c. What *z*-score is associated with a delivery time of 30 minutes?

d. What percent of the values in a normal distribution are more than this number (the *z*-score calculated in part c) of standard deviations above the mean?

e. What is the probability that Parker will have to pay for a gift card?

f. What is the desired *z*-score for an area of interest that corresponds to a 5% probability of having to issue a gift card?

g. What formula can you use to calculate the desired mean?

h. What is the desired mean?

i. How many minutes faster is the desired mean compared to Parker's actual mean?

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Instruction

Problem-Based Task 1.1.2: Parker's Pizza Delivery

Coaching Sample Responses

a. What is the mean of Parker's last 40 delivery times?

Use the formula $\mu = \frac{x_1 + x_2 + \ldots + x_n}{n}$ or a graphing calculator to calculate the mean. The result of either method is a mean time of 23.5 minutes.

b. What is the standard deviation of the delivery times?

Use the formula to calculate the standard deviation, or use a graphing calculator.

Recall the formula for standard deviation is $\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}}$, where x_i is a data point, and $\sum_{i=1}^{n}$ means to take the sum from 1 to n data points.

To organize the information, make a table to keep track of values.

x_i	$x_i - \mu$	$(x_i - \mu)^2$	x_i	$x_i - \mu$	$(x_i - \mu)^2$
17	–6.5	42.25	15	–8.5	72.25
22	–1.5	2.25	23	–0.5	0.25
33	9.5	90.25	22	–1.5	2.25
21	–2.5	6.25	31	7.5	56.25
27	3.5	12.25	16	–7.5	56.25
31	7.5	56.25	28	4.5	20.25
21	–2.5	6.25	25	1.5	2.25
23	–0.5	0.25	21	–2.5	6.25
12	–11.5	132.25	32	8.5	72.25
30	6.5	42.25	26	2.5	6.25
19	–4.5	20.25	22	–1.5	2.25
12	–11.5	132.25	21	–2.5	6.25
24	0.5	0.25	30	6.5	42.25
28	4.5	20.25	30	6.5	42.25
31	7.5	56.25	17	–6.5	42.25
22	–1.5	2.25	19	–4.5	20.25
22	–1.5	2.25	26	2.5	6.25
19	–4.5	20.25	32	8.5	72.25
17	–6.5	42.25	25	1.5	2.25
26	2.5	6.25	22	–1.5	2.25

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Sum all the values for $(x_i - \mu)^2$. The sum is 1,226.

Substitute 1,226 into the numerator of the formula for standard deviation. Since there are a total of 40 delivery times in the set, $n = 40$.

$$\sqrt{\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}} = \sqrt{\frac{1226}{40}} = \sqrt{30.65} \approx 5.5$$

The standard deviation is approximately 5.5.

A graphing calculator will return a similar result. Round the answer to the nearest tenth.

c. What *z*-score is associated with a delivery time of 30 minutes?

Use the formula for calculating the *z*-score: $z = \frac{x-\mu}{\sigma}$. From the problem scenario, we know that $x = 30$ and $\mu = 23.5$.

$$z = \frac{x-\mu}{\sigma} = \frac{30-23.5}{5.5} = 1.18$$

The *z*-score is 1.18.

d. What percent of the values in a normal distribution are more than this number (the *z*-score calculated in part c) of standard deviations above the mean?

Approximately 11.9% of the values in a standard normal distribution are more than 1.18 standard deviations above the mean. This comes from looking up the area in the *z*-scores table related to the *z*-score of 1.18. The area to the left of the *z*-scores is given by 0.8810. However, we are interested in the area to the right of the *z*-score. Therefore, subtract the area given in the table from 1, the value of a normal distribution.

1 – 0.8810 = 0.119 = 11.9%

e. What is the probability that Parker will have to pay for a gift card?

The probability is equal to the area of interest. Therefore, Parker will need to provide gift cards approximately 11.9% of the time.

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Instruction

f. What is the desired *z*-score for an area of interest that corresponds to a 5% probability of having to issue a gift card?

Use a table of *z*-scores to look up the desired area that corresponds to 5% or 0.05. Look in the negative *z*-scores, because we are looking for the area to the left and we will apply symmetry to obtain the positive *z*-score.

An area of about 0.05 corresponds to a *z*-score of –1.65 or –1.64 (depending on rounding of values). Each *z*-score is 0.0005 units away from the desired 0.05 area.

$z_{-1.65} = 0.0495$

$z_{-1.64} = 0.0505$

For the rest of these calculations, we will use a *z*-score of –1.65.

The positive *z*-score that corresponds to the same amount of area but to the right of the mean of interest is +1.65. Verify this by finding the *z*-score of +1.65, finding the corresponding area, and subtracting that area from 1.

The corresponding area for a *z*-score of +1.65 is 0.9505.

$1 - 0.9505 = 0.0495$

g. What formula can you use to calculate the desired mean?

Use the *z*-score formula given by $z = \frac{x - \mu}{\sigma}$.

h. What is the desired mean?

Use the formula from the previous step to determine the desired mean.

$$z = \frac{x - \mu}{\sigma}$$
$$1.65 = \frac{30 - \mu}{5.5}$$
$$9.075 = 30 - \mu$$
$$-20.925 = -\mu$$
$$\mu = 21$$

The desired mean time for delivering pizzas is about 21 minutes.

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i. How many minutes faster is the desired mean compared to Parker's actual mean?

Parker's actual average delivery time is 23.5 minutes. Subtract the desired mean time from this amount.

$23.5 - 21 = 2.5$

Parker's desired mean is 2.5 minutes faster than his actual mean.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.1.2: Standard Normal Calculations

Use the information below to solve problems 1 and 2.

> The mean score on the verbal section of a particular state's high school exit exam in 2011 was 497, and the standard deviation was 114. Nefani scored a 620 on the test. Assume that the scores are normally distributed.

1. What was Nefani's z-score?

2. What percent of students who took the test in 2011 scored lower than Nefani on the verbal section?

Use the information below to solve problems 3–5.

> A factory produces plastic cell phone cases. To fit properly, each case must have a width between 53.5 and 54.5 millimeters. The quality control manager for the factory collects a random sample of 100 cases and determines that the widths are normally distributed, with a mean width of 54.2 millimeters and a standard deviation of 0.3 millimeter.

3. What percent of the cell phone cases meet manufacturing specifications?

4. Suppose the production line is adjusted so that the mean width is decreased to 54.0 millimeters and the standard deviation remains at 0.3 millimeter. What percent of cell phone cases will meet manufacturing specifications?

5. Suppose that the mean width of the cell phone cases is 54.0 millimeters, and management would like 95% of the cases to meet manufacturing specifications. What standard deviation is required?

continued

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Use the information below to solve problems 6 and 7.

The wait times for a table at a particular restaurant are normally distributed, with a mean of 25 minutes. Seventy-five percent of the parties who dine there wait less than 30 minutes for a table.

6. What is the standard deviation of wait times at the restaurant?

7. What percent of the parties wait for more than 15 minutes?

Use the information below to solve problems 8–10.

A marketing firm examines the ages of patrons who attend the Saturday matinee at a local movie theater. The ages of 40 people are listed below. Assume that the ages of movie patrons at the Saturday matinee are normally distributed.

Ages of Randomly Selected Movie Patrons at a Saturday Matinee				
31	30	35	37	30
51	40	44	37	23
33	44	36	40	30
39	30	32	41	43
52	40	37	40	37
24	33	28	29	33
27	28	30	35	33
39	23	50	38	38

8. Find the z-score for a 24-year-old patron who attends the matinee.

9. What percent of the patrons are older than 24?

10. Estimate the percent of patrons in the population who are between 40 and 50 years old.

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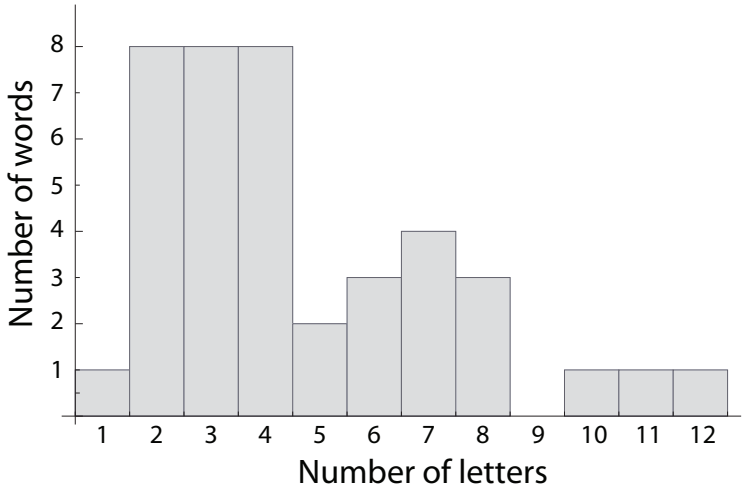
Lesson 1.1.3: Assessing Normality

Warm-Up 1.1.3

Domenic decided to research the word lengths in his favorite novel for his statistics class. He used a sampling technique to select 40 words from the book. Then, he created a table of the sample words and the number of letters in each word, and used it to create a histogram. Use Domenic's table and histogram to solve the problems that follow.

Sample word	Number of letters	Sample word	Number of letters	Sample word	Number of letters
a	1	the	3	seemed	6
he	2	was	3	vermin	6
he	2	was	3	outside	7
it	2	call	4	shaking	7
no	2	door	4	sliding	7
of	2	hope	4	tonight	7
to	2	like	4	attempts	8
to	2	roof	4	distinct	8
to	2	room	4	greatest	8
few	3	said	4	forcefully	10
lay	3	with	4	extremities	11
one	3	clips	5	anticipation	12
Jon	3	skill	5		
she	3	Arthur	6		

Word Lengths Histogram



continued

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1. What percent of words in the sample have more than 10 letters?

2. Estimate the number of 3-letter words used in the book. Why do you think there are so many 3-letter words?

3. Find the mean and median word length for the sample.

4. Find the standard deviation of the word lengths in the sample.

5. Are the word lengths in this book symmetric? Justify your answer.

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Instruction

Lesson 1.1.3: Assessing Normality

Common Core State Standard

S–ID.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.★

Warm-Up 1.1.3 Debrief

1. What percent of words in the sample have more than 10 letters?

   There are 40 words in the sample, and 2 of them have more than 10 letters.

   $$\frac{2}{40} = 0.05 = 5\%$$

   5% of the words in the sample have more than 10 letters.

2. Estimate the number of 3-letter words used in the book. Why do you think there are so many 3-letter words?

   Assuming the sample words are representative of the words in the book, we can use the percent of 3-letter words in the sample to estimate the percent of 3-letter words in the population (all the words in the book).

   $$\frac{8}{40} = 0.2 = 20\%$$

   20% of the words in the book are estimated to be 3-letter words. There are several possible reasons for why there are so many 3-letter words. For example, the sample includes words such as *the* and *was* that are commonly used in the English language. There are also several 3-letter pronouns and prepositions.

3. Find the mean and median word length for the sample.

   The mean is the sum of the data points divided by the number of data points.

   $$\mu = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

   The mean of this data set is 4.65 letters per word.

   The median is the middle number, or the average of the two numbers closest to the middle, when the numbers are arranged in order from lowest to highest (or highest to lowest).

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Instruction

Since there are 40 words, the two numbers closest to the middle are the twentieth and twenty-first words in the list. Each of these words (*hope* and *like*) has 4 letters, so the median is 4 letters per word.

4. Find the standard deviation of the word lengths in the sample.

   The formula for calculating standard deviation is $\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}}$.

   Use a calculator to find the standard deviation of the word lengths in the sample.

   $\sigma \approx 2.7$ words

5. Are word lengths in this book symmetric? Justify your answer.

   No. The word lengths are not symmetric because the percentage of data that lies above the mean is different from the percentage of data that lies below the mean. In this sample, $\frac{25}{40} = 62.5\%$ of the word lengths in the sample are below the mean, while only 37.5% are above the mean. We expect something closer to a 50–50 split for symmetric distributions.

Connection to the Lesson

- The warm-up data set reviews proportion, mean, median, and standard deviation, which are used in the lesson.
- Students will be analyzing histograms to determine if a data set is normally distributed.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- constructing histograms and analyzing properties such as symmetry and clustering from histograms (S–ID.1)
- using a calculator to find mean, median, and standard deviation (no standard)
- calculating *z*-scores (S–ID.4)
- plotting points in a coordinate plane (5.G.1)
- comparing and contrasting proportions in a sample to probabilities in a standard normal distribution (S–IC.2)

Introduction

Previous lessons have demonstrated that the normal distribution provides a useful model for many situations in business and industry, as well as in the physical and social sciences. Determining whether or not it is appropriate to use normal distributions in calculating probabilities is an important skill to learn, and one that will be discussed in this lesson.

There are many methods to assess a data set for normality. Some can be calculated without a great deal of effort, while others require advanced techniques and sophisticated software. Here, we will focus on three useful methods:

- Rules of thumb using the properties of the standard normal distribution (including symmetry and the 68–95–99.7 rule).
- Visual inspection of histograms for symmetry, clustering of values, and outliers.
- Use of normal probability plots.

With advances in technology, it is now more efficient to calculate probabilities based on normal distributions. With our new understanding of a few important concepts, we will be ready to conduct research that was formerly reserved for a small percentage of people in society.

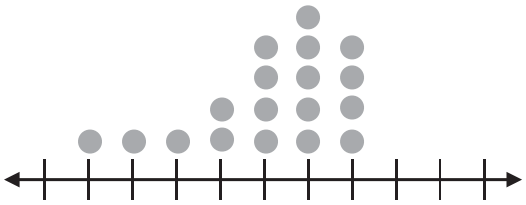
Key Concepts

- Although the normal distribution has a wide range of useful applications, it is crucial to assess a distribution for normality before using the probabilities associated with normal distributions.

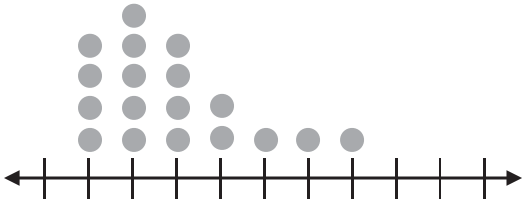
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Instruction

- Assessing a distribution for normality requires evaluating the distribution's four key components: a sample or population size, a sketch of the overall shape of the distribution, a measure of average (or central tendency), and a measure of variation.
- It is difficult to assess normality in a distribution without a proper sample size. When possible, a sample with more than 30 items should be used.
- **Outliers** are values far above or below other values of a distribution.
- The use of mean and standard deviation is inappropriate for distributions with outliers. Probabilities based on normal distributions are unreliable for data sets that contain outliers.
- Some outliers, like those caused by mistakes in data entry, can be eliminated from a data set before a statistical analysis is performed.
- Other outliers must be considered on a case-by-case basis.
- Histograms and other graphs provide more efficient methods to assess the normality of a distribution.
- If a histogram is approximately symmetric with a concentration of values near the mean, then using a normal distribution is reasonable (assuming there are no outliers).
- If a histogram has most of its weight on the right side of the graph with a long "tail" of isolated, spread-out data points to the left of the median, the distribution is said to be **skewed to the left**, or **negatively skewed**:



- In a negatively skewed distribution, the mean is often, but not always, less than the median.
- If a histogram has most of its weight on the left side of the graph with a long tail on the right side of the graph, the distribution is said to be **skewed to the right**, or **positively skewed**:



- In a positively skewed distribution, the mean is often, but not always, greater than the median.

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Instruction

- Histograms should contain between 5 and 20 categories of data, including categories with frequencies of 0.
- Recall that the 68–95–99.7 rule, also known as the Empirical Rule, states percentages of data under the normal curve are as follows: $\mu \pm 1\sigma \approx 68\%$, $\mu \pm 2\sigma \approx 95\%$, and $\mu \pm 3\sigma \approx 99.7\%$.
- The 68–95–99.7 rule can also be used for a quick assessment of normality. For example, in a sample with less than 100 items, obtaining a *z*-score below –3.0 or above +3.0 indicates possible outliers or skew.
- Graphing calculators and computers can be used to construct normal probability plots, which are a more advanced system for assessing normality.
- In a normal probability plot, the *z*-scores in a data set are paired with their corresponding *x*-values.
- If the points in the normal plot are approximately linear with no systematic pattern of values above and below the line of best fit, then it is reasonable to assume that the data set is normally distributed.

Common Errors/Misconceptions

- treating a data set that has outliers as if it were a normal distribution
- removing outliers without justification
- adhering too strictly to the rules of thumb for assessing normality
- deeming a distribution as normal when it is actually skewed left or right

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Guided Practice 1.1.3

Example 1

The following frequency table shows the cholesterol levels in milligrams per deciliter (mg/dL) of 100 randomly selected high school students. The mean cholesterol level in the sample is 165 mg/dL and the standard deviation is 20 mg/dL. Analyze the frequency table using the 68–95–99.7 rule to decide if cholesterol levels in the population are normally distributed.

Cholesterol level (mg/dL)	Number of students
105.0–124.5	2
125.0–144.5	15
145.0–164.5	34
165.0–184.5	36
185.0–204.5	11
205.0–224.5	2
Total	100

1. Determine the percent of students with cholesterol levels within one standard deviation of the mean.

   The mean is 165 mg/dL and the standard deviation is 20. The lower bound of the interval in question is 165 – 20 = 145 mg/dL. The upper bound of the interval is 165 + 20 = 185 mg/dL. Values from 145 to 185 are within one standard deviation of the mean.

   There are 34 values in the class from 145 to 164.5, and 36 values in the class from 165 to 184.5. There are a total of 34 + 36 = 70 values in the interval from 145 to 185. Since there are 100 values in the data set, the percent of values is $\frac{70}{100} = 0.7 = 70\%$.

   The percent of students in the sample that have a cholesterol level within one standard deviation of the mean is 70%. This is close to the 68% figure in a normal distribution.

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Instruction

2. Determine the percent of students with cholesterol levels within two standard deviations of the mean.

   Since the mean is 165 and the standard deviation is 20, the lower bound is 165 – 2(20) = 165 – 40 = 125 mg/dL. The upper bound is 165 + 2(20) = 165 + 40 = 205. This means that the interval between cholesterol levels of 125 and 205 mg/dL is within two standard deviations from the mean.

   There are 15, 34, 36, and 11 values in the categories from 125.0 to 144.5, 145.0 to 164.5, 165.0 to 184.5, and 185.0 to 204.5, respectively. Adding these values, we find that there are 15 + 34 + 36 + 11 = 96 values within two standard deviations of the mean.

   The percent of students in the sample that have a cholesterol level within two standard deviations of the mean is 96%. This is close to the 95% figure in a normal distribution.

3. Determine the percent of students with cholesterol levels within three standard deviations of the mean.

   Since the mean is 165 and the standard deviation is 20, the lower bound is 165 – 3(20) = 165 – 60 = 105 mg/dL. The upper bound is 165 + 3(20) = 165 + 60 = 225 mg/dL.

   There are no values in the table less than the lower bound or greater than the upper bound.

   All, or 100%, of the students in the sample have a cholesterol level between 105 and 225 (three standard deviations of the mean). This is close to the 99.7% figure in a normal distribution.

4. Use your findings to determine whether the data is normally distributed.

   Since the data set is from a sample, minor differences from the proportions in the sample and the proportions that correspond to a normal distribution are acceptable.

   We cannot be sure that cholesterol levels are normally distributed, but it seems reasonable to assume that they are for this population. Based on the sample, the normal distribution provides a useful model for analyzing cholesterol levels in this population.

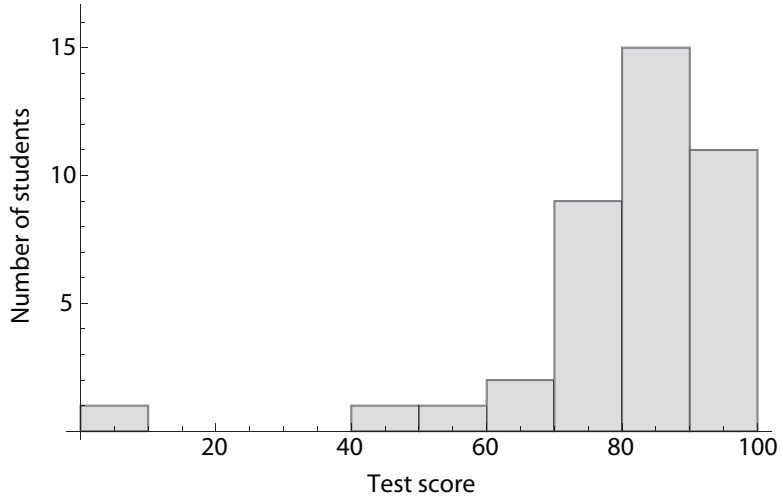
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Instruction

Example 2

In order to constantly improve instruction, Mr. Hoople keeps careful records on how his students perform on exams. The histogram below displays the grades of 40 students on a recent United States history test. The table next to it summarizes some of the characteristics of the data. Use the properties of a normal distribution to determine if a normal distribution is an appropriate model for the grades on this test.

Recent U.S. History Test Scores



Summary statistics	
n	40
μ	80.5
Median	85.0
σ	18.1
Minimum	0
Maximum	98

1. Analyze the histogram for symmetry and concentration of values.

   The histogram is asymmetric; there is a skew to the left (or a negative skew). The mean is 85.0 – 80.5 = 4.5 less than the median. Also, there appears to be a higher concentration of values above the mean (80.5) than below the mean.

2. Examine the distribution for outliers and evaluate their significance, if any outliers exist.

   There is one negative outlier (0) on this test. There may be outside factors that affected this student's performance on the test, such as illness or lack of preparation.

3. Determine whether a normal distribution is an appropriate model for this data.

   Because of the outlier, the normal distribution is not an appropriate model for this population.

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Instruction

Example 3

Rent at the Cedar Creek apartment complex includes all utilities, including water. The operations manager at the complex monitors the daily water usage of its residents. The following table shows water usage, in gallons, for residents of 36 apartments. To better assess the data, the manager sorted the values from lowest to highest. Does the data show an approximate normal distribution?

Daily Water Usage per Apartment (in Gallons)			
181	290	344	379
210	294	345	380
211	303	345	388
224	304	350	391
239	306	353	401
247	307	355	405
267	329	361	414
270	332	362	426
290	336	378	431

1. Determine the number of categories.

   Generally, there are between 5 and 9 categories in a histogram. The data set contains 36 data points.

   First, calculate the range of data.

   range = maximum value – minimum value

   range = 431 – 181 = 250

   Since there are 36 data points, either 5 or 6 categories would be appropriate. We will start with the choice of 6 categories, $c = 6$.

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Instruction

2. Determine the category width.

   Each category should have the same width. Therefore, divide the total range of the data by the number of desired categories.

$$\text{category width} = \frac{\text{range}}{c} = \frac{250}{6} \approx 41.67$$

   For convenience, we will use a category width of 40 gallons and begin the first category with the lowest value, 180 gallons.

3. Construct a frequency table.

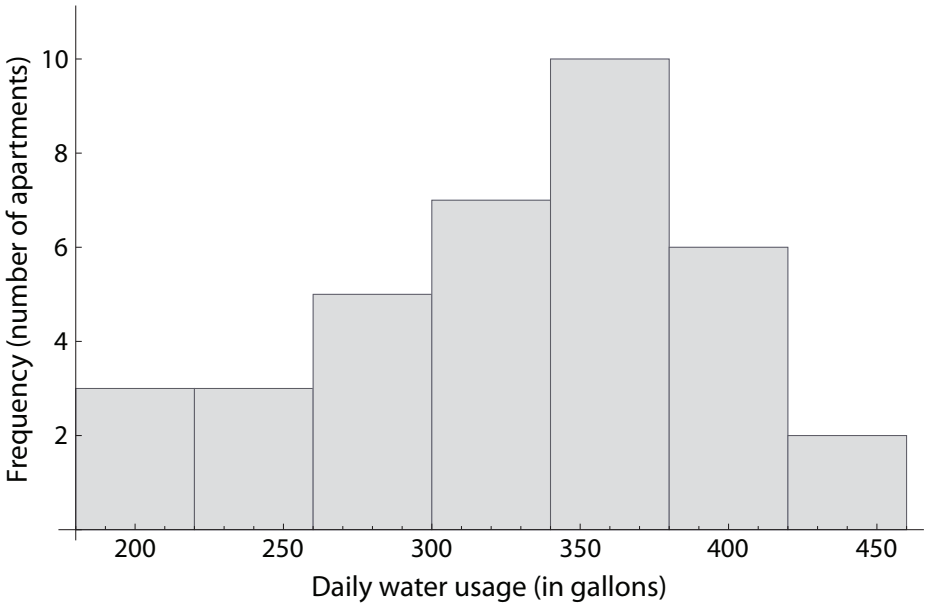
Category (daily water usage in gallons)	Frequency (number of apartments)
180–219	3
220–259	3
260–299	5
300–339	7
340–379	10
380–419	6
420–459	2
Total	36

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Instruction

4. Construct a histogram from the frequency table.

   The horizontal axis is used for the unit of study (in this case, daily water usage). The vertical axis is used for the frequency (the number of apartments) corresponding to each category.



5. Describe the overall shape of the distribution.

   The distribution has a slight negative skew. The highest concentrations of values are between 250 and 420 gallons of water since these are the four categories with the highest frequencies. There are no outliers in the data set.

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Instruction

6. Draw conclusions.

   As with most statistical analyses, use your judgment about whether to assume normality here. Think about the context of the problem and what the calculations would be used for. Will the calculations be used to make a decision that could have serious results? Or do you need to get a rough idea of the calculations to inform a decision that is not life-impacting?

   Apartments with more water usage could have more people living in them, but without knowing how many residents are in each apartment, it's difficult to tell for sure. Or, they could have a washing machine, dishwasher, or other appliance that uses a large amount of water. Without other data, it is not possible to make these claims.

   Without knowing the context of how the data will be used, the safest conclusion is that we cannot assume a normal distribution here since the data is slightly skewed.

   However, with more information about how the data will be used, in some cases, it would be safe to assume normality since the data is only slightly skewed and has no outliers. Careful judgment is required.



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Instruction

Example 4

Use a graphing calculator to construct a normal probability plot of the following values. Do the data appear to come from a normal distribution?

{1, 2, 4, 8, 16, 32}

1. Use a graphing calculator or computer software to obtain a normal probability plot.

   Different graphing calculators and computer software will produce different graphs; however, the following directions can be used with TI-83/84 or TI-Nspire calculators.

   On a TI-83/84:

   Step 1: Press [STAT] to bring up the statistics menu. The first option, 1: Edit, will already be highlighted. Press [ENTER].

   Step 2: Arrow up to L1 and press [CLEAR], then [ENTER], to clear the list. Repeat this process to clear L2 and L3 if needed.

   Step 3: From L1, press the down arrow to move your cursor into the list. Enter each number from the data set, pressing [ENTER] after each number to navigate down to the next blank spot in the list.

   Step 4: Press [Y=]. Press [CLEAR] to delete any equations.

   Step 5: Set the viewing window by pressing [WINDOW]. Enter the following values, using the arrow keys to navigate between fields and [CLEAR] to delete any existing values: Xmin = 0, Xmax = 35, Xscl = 5, Ymin = –3, Ymax = 3, Yscl = 1, and Xres = 1.

   Step 6: Press [2ND][Y=] to bring up the STAT PLOTS menu.

   Step 7: The first option, Plot 1, will already be highlighted. Press [ENTER].

   Step 8: Under Plot 1, press [ENTER] to select "On" if it isn't selected already. Arrow down to "Type," then arrow right to the normal probability plot icon (the last of the six icons shown) and press [ENTER].

   Step 9: Press [GRAPH].

(*continued*)

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Instruction

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the spreadsheet icon and press [enter].

Step 3: The cursor will be in the first cell of the first column. Enter each number from the data set, pressing [enter] after each number to navigate down to the next blank cell.

Step 4: Arrow up to the topmost cell of the column, labeled "A." Name the column "exp1" using the letters and numbers on your keypad. Press [enter].

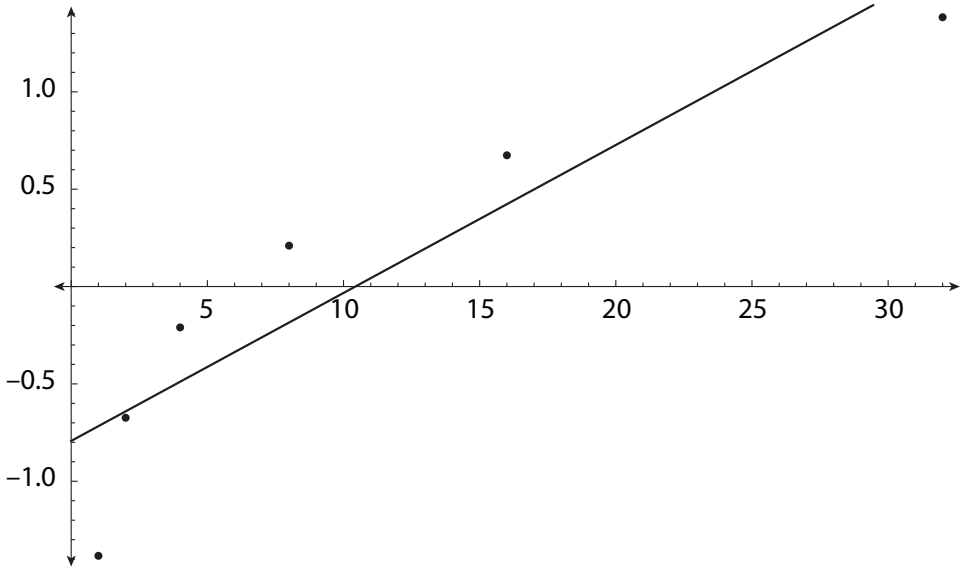
Step 5: Press the [home] key. Arrow over to the data and statistics icon and press [enter].

Step 6: Press the [menu] key. Arrow down to 2: Plot Properties, then arrow right to bring up the sub-menu. Arrow down to 4: Add X Variable, if it isn't already highlighted. Press [enter].

Step 7: Arrow down to {...}exp1 if it isn't already highlighted. Press [enter]. This will graph the data values along an *x*-axis.

Step 8: Press [menu]. The first option, 1: Plot Type, will be highlighted. Arrow right to bring up the next sub-menu. Arrow down to 4: Normal Probability Plot. Press [enter].

Your graph should show the general shape of the plot as follows.



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Instruction

2. Analyze the graph to determine whether it follows a normal distribution.

   Do the points lie close to a straight line? If the data lies close to the line, is roughly linear, and does not deviate from the line of best fit with any systematic pattern, then the data can be assumed to be normally distributed. If any of these criteria are not met, then normality cannot be assumed.

   The data does not lie close to the line; the data is not roughly linear. The data seems to curve about the line, which suggests a pattern. Therefore, normality cannot be assumed. The normal distribution is not an appropriate model for this data set.



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Instruction

Example 5

The following table lists the ages of United States presidents at the time of their inauguration. Use this information and a graphing calculator to provide a thorough description of the data set.

President	Age	President	Age	President	Age
George Washington	57	Abraham Lincoln	52	Herbert Hoover	54
John Adams	61	Andrew Johnson	56	Franklin Roosevelt	51
Thomas Jefferson	57	Ulysses Grant	46	Harry Truman	60
James Madison	57	Rutherford Hayes	54	Dwight Eisenhower	62
James Monroe	58	James Garfield	49	John Kennedy	43
John Quincy Adams	57	Chester Arthur	51	Lyndon Johnson	55
Andrew Jackson	61	Grover Cleveland	47	Richard Nixon	56
Martin Van Buren	54	Benjamin Harrison	55	Gerald Ford	61
William Harrison	68	Grover Cleveland	55	Jimmy Carter	52
John Tyler	51	William McKinley	54	Ronald Reagan	69
James Polk	49	Theodore Roosevelt	42	George H. W. Bush	64
Zachary Taylor	64	William Taft	51	Bill Clinton	46
Millard Fillmore	50	Woodrow Wilson	56	George W. Bush	54
Franklin Pierce	48	Warren Harding	55	Barack Obama	47
James Buchanan	65	Calvin Coolidge	51		

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Instruction

1. Note the size of the population or sample.

   There have been 44 United States presidents. *Note*: Grover Cleveland is listed twice because he was elected to nonconsecutive terms.

2. Show the overall shape of the distribution.

   Use a histogram. Use the following directions to create a histogram on your graphing calculator.

   On a TI-83/84:

   Step 1: Press [STAT] to bring up the statistics menu. The first option, 1: Edit, will already be highlighted. Press [ENTER].

   Step 2: Arrow up to L1 and press [CLEAR], then [ENTER], to clear the list. Repeat this process to clear L2 and L3 if needed.

   Step 3: From L1, press the down arrow to move your cursor into the list. Enter each number from the data set, pressing [ENTER] after each number to navigate down to the next blank spot in the list.

   Step 4: Press [Y=]. Press [CLEAR] to delete any equations.

   Step 5: Press [2ND][Y=] to bring up the STAT PLOTS menu.

   Step 6: The first option, Plot 1, will already be highlighted. Press [ENTER].

   Step 7: Under Plot 1, select "On" if it isn't selected already. Arrow down to "Type," then arrow right to the histogram icon (the third of the six icons shown) and press [enter].

   Step 8: Set the viewing window. Press [WINDOW]. Enter the following values: Xmin = 42, Xmax = 70, Xscl = 4, Ymin = 0, Ymax = 10, Yscl = 1, and Xres = 1.

   Step 9: Press [GRAPH].

(*continued*)

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Instruction

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the spreadsheet icon and press [enter].

Step 3: Enter each number from the data set into the first column, pressing [enter] after each number to navigate down to the next blank cell.

Step 4: Arrow up to the topmost cell of the column, labeled "A." Name the column "age" using the letters and numbers on your keypad. Press [enter].

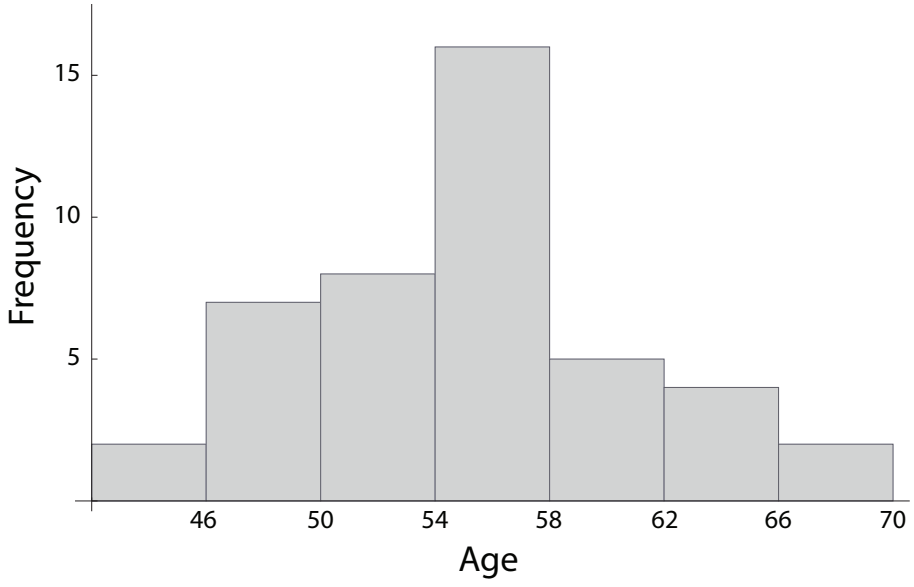
Step 5: Press the [home] key. Arrow over to the data and statistics icon and press [enter].

Step 6: Press the [menu] key. Arrow down to 2: Plot Properties, then arrow right to bring up the sub-menu. Arrow down to 4: Add X Variable, if it isn't already highlighted. Press [enter].

Step 7: Arrow down to {...}age if it isn't already highlighted. Press [enter]. This will graph the data values along an *x*-axis.

Step 8: Press [menu]. The first option, 1: Plot Type, will be highlighted. Arrow right to bring up the next sub-menu. Arrow down to 3: Histogram. Press [enter].

Your graph should show the general shape of the histogram as follows:



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Instruction

3. Evaluate the overall shape of the distribution to determine whether it could follow a normal distribution.

   The distribution is approximately symmetric, with a high concentration of ages near the mean and a lower concentration of ages away from the mean. There are no severe outliers in either direction. The normal distribution could be an appropriate model for this data set. Therefore, continue to analyze the data to determine whether it represents a normal distribution.

4. Create a normal probability plot for the data set.

   Use the data you've already entered into your graphing calculator to create the plot.

   On a TI-83/84:

   Step 1: Press [2ND][Y=] to bring up the STAT PLOTS menu.

   Step 2: Press [ENTER] twice to bring up Plot 1. Arrow down then right to the normal probability plot icon and press [ENTER].

   Step 3: Press [WINDOW]. Adjust the following values: Ymin = –3 and Ymax = 3.

   Step 4: Press [GRAPH].

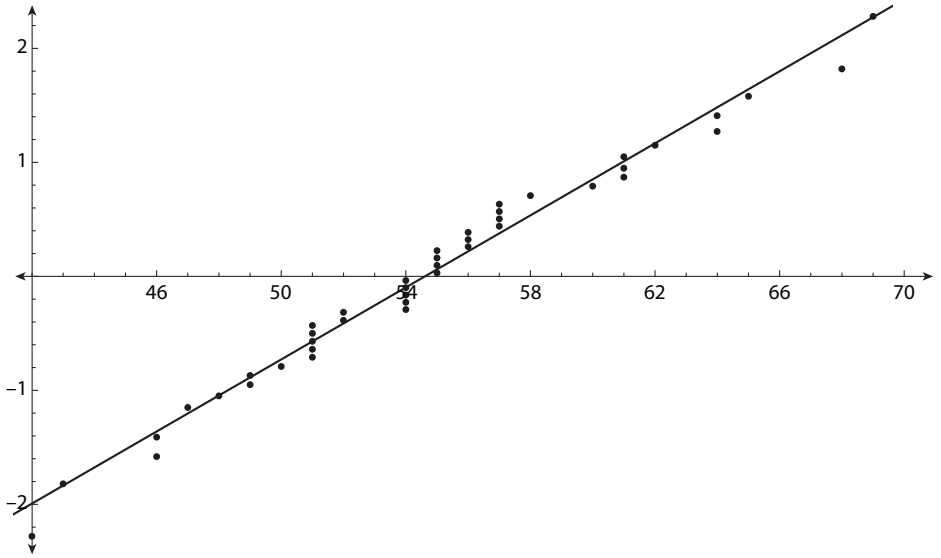
   On a TI-Nspire:

   Step 1: Starting at the screen that shows the histogram created in step 2, press [menu]. Select 1: Plot Type, and press [enter]. Arrow down to 4: Normal Probability Plot. Press [enter].

(*continued*)

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Your graph should show the general shape of the plot as follows:



The normal probability plot follows the line of best fit fairly closely and is roughly linear, but it does have a bit of a systematic pattern of deviation.

5. Draw a conclusion.

   Based on the roughly symmetric histogram and the normal probability plot, a normal distribution can be applied to this data set.

6. Calculate measures of center and spread, and summarize the results.

   Since the distribution is roughly normal, we can use mean and standard deviation to describe the center and spread of the data.

   Use the directions appropriate to your calculator model to calculate the measures of center and spread.

   On a TI-83/84:

   Step 1: Press [STAT]. Arrow over to the CALC menu. The first option, 1–Var Stats, will already be highlighted. Press [ENTER].

   Step 2: In the menu, "L1" should be displayed next to "List." Press [2ND][1] if not.

   Step 3: Press [ENTER] three times to evaluate the data set. This will display a list of calculated values for the set. The mean will be listed to the right of "$\overline{x}$ =". The standard deviation will be listed to the right of "σx =".

(*continued*)

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Instruction

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the spreadsheet icon and press [enter].

Step 3: Use the [ctrl] key and left arrow key on the navigation pad to return to the spreadsheet page containing the "age" data previously entered.

Step 4: Press [menu]. Arrow down to 4: Statistics, then arrow right to bring up the sub-menu. At 1: Stat Calculations, arrow right to the sub-menu. The first option, 1: One-Variable Statistics, will be highlighted. [Press enter].

Step 5: Type [1] and press [enter] if the number of lists in the field is blank. Press [enter] two times to evaluate the data set. This will bring you back to the spreadsheet, where columns B and C will be populated with the titles and values for each calculation. Use the arrow key to scroll down the rows of the spreadsheet and find the measures of center and spread. Note that the mean is represented by $\overline{x}$ instead of μ.

The relevant statistics are:

$\overline{x}$	54.6591	Round to 54.7 years.
σ_x	6.18629	Round to 6.2 years.
n	44	There are 44 presidents in the population.
Median	54.5	The median age is 54.5 years. *Note*: This is extremely close to the mean.



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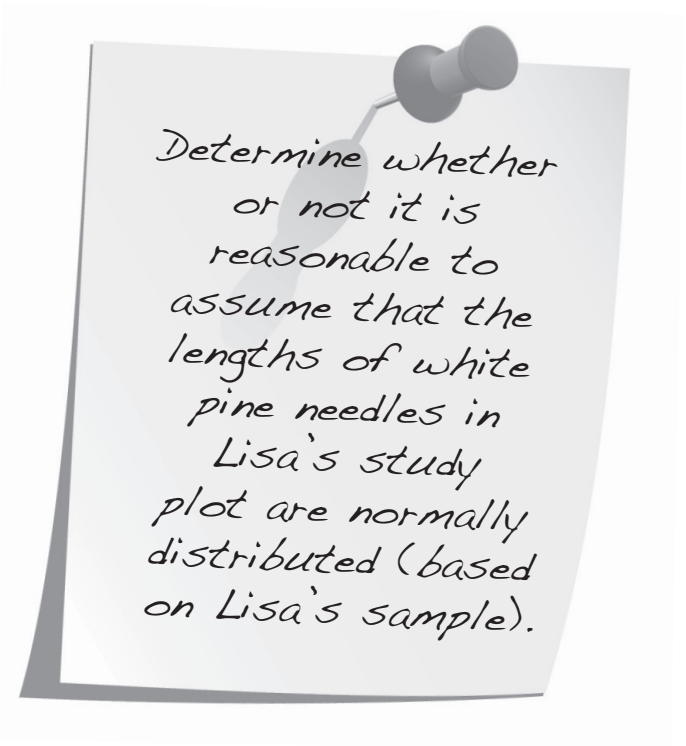
Problem-Based Task 1.1.3: White Pines

SMP	
1	2
3 ✓	4
5 ✓	6 ✓
7	8

Lisa is conducting research on white pine trees for her graduate degree in environmental science. She would like to establish a baseline for several measures, such as needle length, so that she can make comparisons in future years. The lengths of the first sample of white pine needles in her study plot are listed in the table below:

Lengths of White Pine Needles in Centimeters					
7.4	7.7	7.9	7.7	8.4	7.5
8.1	7.1	7.6	8.6	7.5	6.5
7.6	7.3	7.1	7.7	7.5	6.6
7.5	7.2	7.8	8.5	7.6	7.0
7.6	7.3	8.2	7.7	7.5	7.0

Using a graphing calculator or software, determine whether or not it is reasonable to assume that the lengths of white pine needles in Lisa's study plot are normally distributed (based on Lisa's sample). Provide a thorough description of Lisa's sample.



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Problem-Based Task 1.1.3: White Pines

Coaching

a. Create a histogram of the data.

b. Are there outliers in the sample that would rule out use of a normal distribution, or make the use of the mean and standard deviation inappropriate?

c. Is the sample distribution approximately symmetric?

d. Is there a higher concentration of values nearer the mean than farther away from the mean?

e. What is the normal probability plot of the data?

f. Do the points in the normal probability plot lie reasonably close to a straight line?

g. Are there systematic patterns of points above and below the line?

h. What conclusions can you draw?

i. What are the four key components in the proper description of a data set?

j. What is the size of the sample?

k. Describe the histogram and the probability plot of the data.

l. What are the measures of center and spread that are appropriate for this data set?

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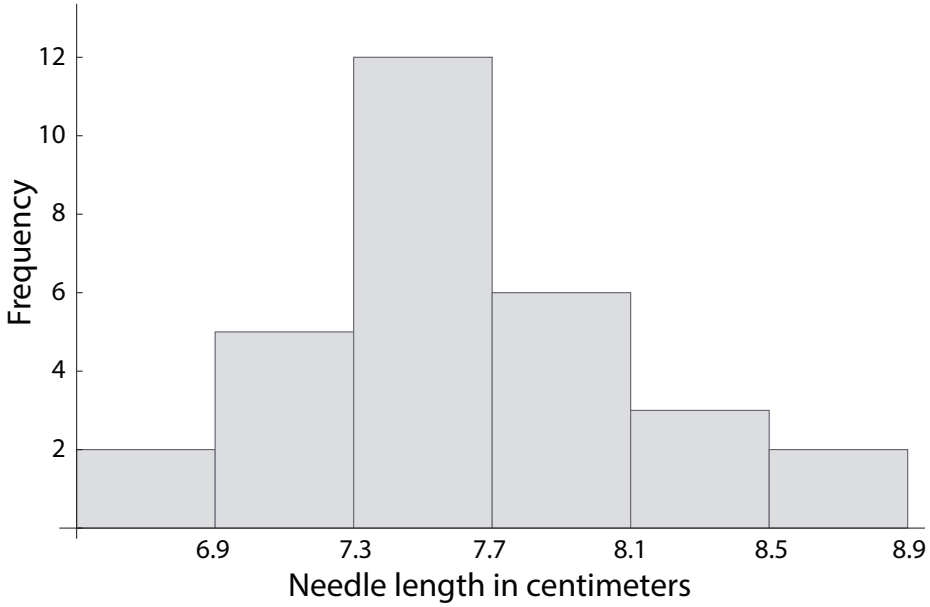
Instruction

Problem-Based Task 1.1.3: White Pines

Coaching Sample Responses

a. Create a histogram of the data.

Use a calculator or graphing software to create a histogram of the data.



b. Are there outliers in the sample that would rule out use of a normal distribution, or make the use of the mean and standard deviation inappropriate?

No. There are no outliers in the sample.

c. Is the sample distribution approximately symmetric?

As the histogram shows, the distribution is approximately symmetric.

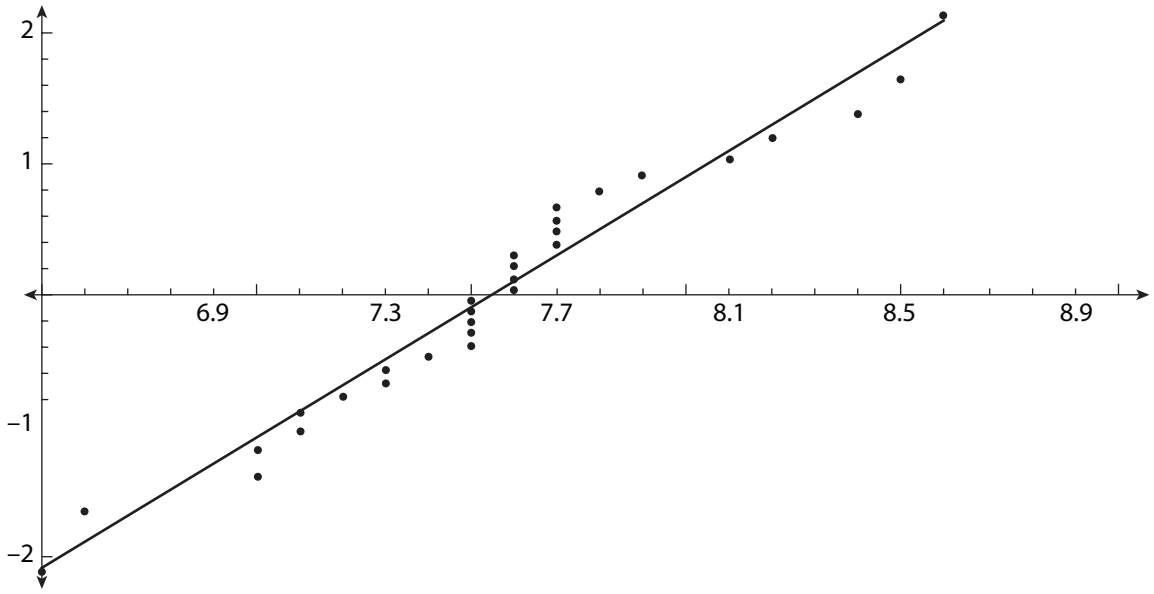
d. Is there a higher concentration of values nearer the mean than farther away from the mean?

Yes; needle lengths are clustered near the mean.

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e. What is the normal probability plot of the data?

Use a calculator or graphing software to create a normal probability plot of the data.



f. Do the points in the normal probability plot lie reasonably close to a straight line?

As shown in the figure, the points in the normal probability plot lie reasonably close to a straight line.

g. Are there systematic patterns of points above and below the line?

Notice that there are a number of consecutive points below the line for pine needles between 7.0 and 7.5 centimeters and above the line for pine needles between 7.5 and 8.0 centimeters. However, the near linearity of the normal probability plot suggests a population that is approximately normal.

h. What conclusions can you draw?

Based on the roughly symmetric histogram and roughly linear normal probability plot, we can conclude that a normal distribution is an adequate model for this sample.

i. What are the four key components in the proper description of a data set?

The four key components are a sample or population size, a sketch of the overall shape of the distribution, a measure of average (or central tendency), and a measure of variation.

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j. What is the size of the sample?

The sample size is 30 ($n = 30$).

k. Describe the histogram and the probability plot of the data.

The histogram is roughly symmetric with values clustered around the mean. The normal probability plot shows slight deviations from the line of best fit, but is overall roughly linear. Therefore, we can assume a normal distribution of the sample.

l. What are the measures of center and spread that are appropriate for this data set?

Since the data is assumed to be normal, the mean is an appropriate measure of center and the standard deviation is an appropriate measure of variation.

Use a calculator or graphing software to determine the mean and standard deviation. The mean is about 7.6 centimeters, with a standard deviation of about 0.5 centimeter.

Recommended Closure Activity

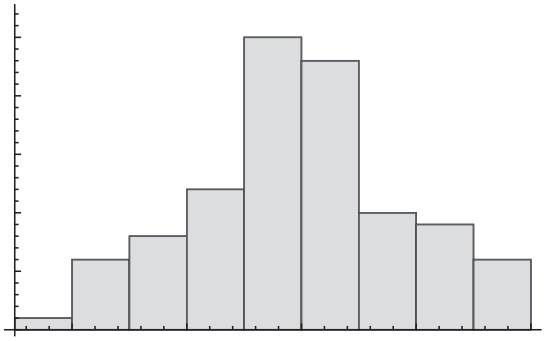
Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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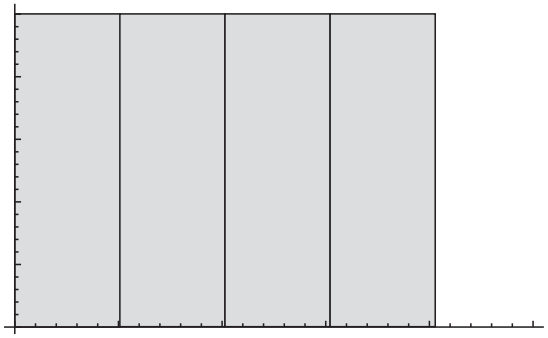
Practice 1.1.3: Assessing Normality

Use the provided histograms to solve problems 1–3.

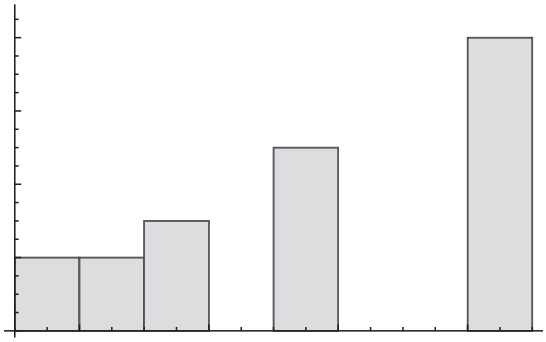
Histogram A



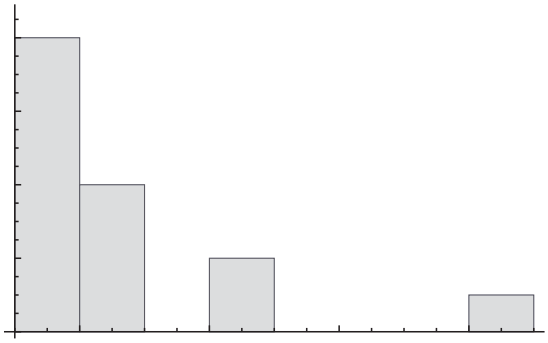
Histogram C



Histogram B



Histogram D



1. Which histograms, if any, are normal or approximately normal?

2. Which histograms, if any, are skewed to the right?

3. Which histograms, if any, have a mean that is less than the median?

continued

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The table below lists the positions and weekly salaries for the 16 employees of the Down-in-the-Dirt Landscaping Company. Use the information to solve problems 4–6.

Position	Weekly salary	Position	Weekly salary	Position	Weekly Salary
Apprentice	$320	Laborer	$490	Supervisor	$600
Apprentice	$320	Laborer	$490	Supervisor	$600
Apprentice	$330	Laborer	$500	Supervisor	$600
Laborer	$480	Laborer	$500	Company president	$1,500
Laborer	$480	Laborer	$500		
Laborer	$490	Laborer	$500		

4. Identify any outliers. Give a possible reason for the existence of an outlier or outliers and decide whether the outlier(s) should be eliminated.

5. What percent of the employees at Down-in-the-Dirt make more than the mean salary?

6. Is the normal distribution an appropriate model for these salaries? Justify your answer.

Use the information below to solve the problems that follow.

Mike's job is to analyze food products for nutritional value. Recently, Mike determined the grams of sugar in samples of 12-ounce soft drinks sold at a local convenience store. The sugar content of 30 cans of soft drinks is shown in the following table.

Grams of Sugar per Can					
27.5	27.9	26.2	30.2	27.1	23.6
26.7	25.1	24.3	28.9	24.9	28.1
26.7	28.3	24.8	25.8	27.4	27.0
27.6	26.9	26.4	27.5	28.4	29.2
28.1	27.7	26.2	27.0	27.3	24.1

7. What percent of cans have a sugar content within one standard deviation of the mean?

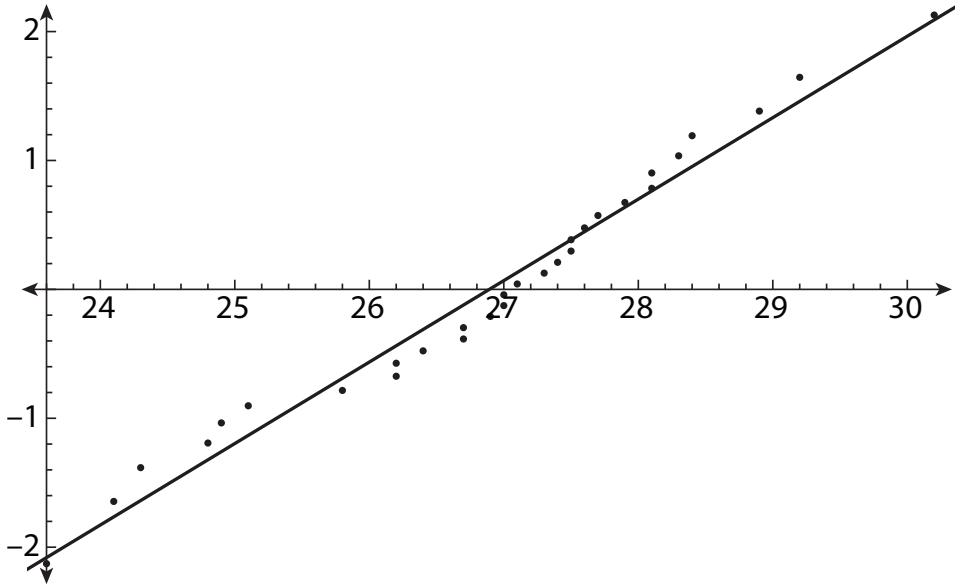
continued

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8. What percent of cans have a sugar content within two standard deviations of the mean?

9. What percent of cans have a sugar content within three standard deviations of the mean?

Mike used his soda data to create a normal probability plot, shown below. Use the plot to solve problem 10.



10. Is it reasonable to assume that the sugar content in the population from which these cans were selected is normally distributed? Explain your answer.

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Progress Assessment

Circle the letter of the best answer.

1. Which of the following measures involves continuous data?

   a. the number of students checking their e-mail in a given class

   b. the heights of students in a given class

   c. the sums of two dice

   d. the scores on a writing rubric

2. Lindsay earned a 98 on her final exam for biology. The mean score was 78.2 and the standard deviation was 8.4. What was Lindsay's *z*-score?

   a. 2.36

   b. –1.18

   c. 86.6

   d. 99

3. The mean height of a population of men is 70.8 inches and the standard deviation is 3.2 inches. What proportion of these men can walk through a 6-foot-tall doorway without having to duck?

   a. 16%

   b. 35%

   c. 65%

   d. 84%

4. After the first round of fitness training, the director of a police academy plans to eliminate the bottom 20% of candidates based on individual fitness scores. Fitness scores were normally distributed, with a mean of 72.0 and a standard deviation of 18.5. What fitness score does a candidate need to stay in the academy?

   a. 35.0

   b. 56.4

   c. 72.0

   d. 90.5

5. What percent of the values in a normal distribution lie within one standard deviation of the mean?

   a. 34%

   b. 50%

   c. 68%

   d. 95%

continued

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Assessment

6. Water consumption by the residents in a local community is normally distributed, with a mean of 320 gallons per day. Ninety-five percent of the residents use between 200 and 440 gallons per day. What is the standard deviation of water consumption in the population?

   a. 60 gallons per day
   b. 120 gallons per day
   c. 240 gallons per day
   d. 640 gallons per day

7. A researcher studying how often humpback whales surface for air reported that the mean time between surfacing was 10 minutes and the standard deviation was 5 minutes. The researcher found that the times between surfacing were approximately normal. Which of the following is true about this sample of whales?

   a. Most of the humpback whales will surface between 5 and 10 minutes.
   b. Fewer than 10% of the whales will surface in less than 15 minutes.
   c. No humpback whale can hold its breath for more than 25 minutes.
   d. Most of the humpback whales surface in less than 12 minutes.

8. Which of the following is often true about a distribution that is positively skewed?

   a. The mean is greater than the median.
   b. The mean is less than the median.
   c. The mean and the median are approximately equal.
   d. The distribution is symmetric.

9. How does a uniform distribution differ from a normal distribution?

   a. A uniform distribution is symmetric, but a normal distribution is not.
   b. A uniform distribution has a higher concentration of values close to the mean, but a normal distribution does not.
   c. A normal distribution is symmetric, but a uniform distribution is not.
   d. A normal distribution has a high concentration of values close to the mean, but a uniform distribution does not.

continued

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Assessment

10. Chompy Oats cereal comes in boxes labeled as having 16 ounces of cereal, but the machinery used to fill the boxes isn't always precise. The tables that follow list the weights of the cereal in 10 randomly selected boxes and the summary statistics for the sample boxes.

Weights (ounces)	
16.3	15.4
15.7	16.2
15.8	18.2
16.2	16.4
15.7	15.7

Summary statistics	
Mean weight	16.16 ounces
Standard deviation	0.79 ounce
Sample size	10 boxes

Which of the following statements is *not* supported by the data?

a. The median weight of a cereal box in the sample is equal to 16 ounces.

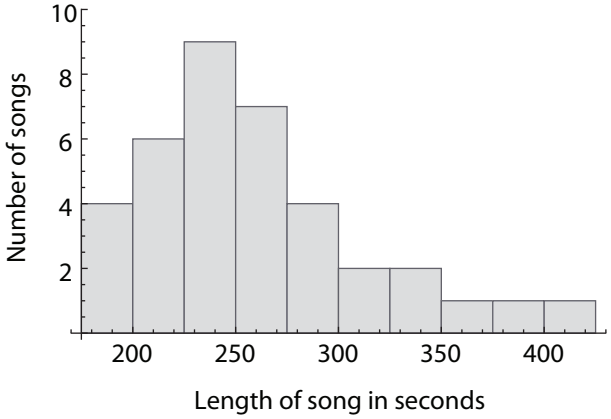
b. The mean weight of a cereal box in the sample is greater than the advertised amount.

c. Of customers who purchase Chompy Oats, 68% will receive between 15.37 and 16.96 ounces of cereal.

d. A larger sample is needed to determine if at least 95% of the boxes of Chompy Oats contain at least 16 ounces.

Use the given information to complete all parts of the problem that follows.

11. A benefit album for the victims of Japan's 2011 earthquake is comprised of 37 songs from popular artists around the world. The histogram and table of summary statistics below are based on the length of each song in seconds.



Summary statistics	
Mean	258.0 seconds
Median	247.7 seconds
Standard deviation	54.3 seconds
Minimum	185 seconds
Maximum	405 seconds
Number of songs	37

a. Determine the minimum and maximum *z*-scores in the distribution.

b. Estimate the percent of songs that are between 200 and 300 seconds long.

c. Determine if the song lengths are normally distributed based on the information in the histogram and table. Use a compelling argument to support your answer.

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Pre-Assessment

Circle the letter of the best answer.

1. Why are samples used in statistics?
   a. Sample statistics are generally more accurate than population parameters.
   b. Statistics taken from samples are less biased than population parameters.
   c. It is often impractical to analyze all members of a population.
   d. There is more variation in statistics than in the parameters that statistics are used to estimate.

2. How does increasing a sample's size affect statistics?
   a. Increasing the sample size decreases the value of the measures of average and variation of a population.
   b. Increasing the sample size increases the value of the measures of average and variation of a population.
   c. Increasing the sample size increases the variability of estimates of average and variation of a population.
   d. Increasing the sample size increases the reliability of measures of average and variation of a population.

3. What is a simple random sample?
   a. A wide variety of members of a population are carefully chosen for the sample.
   b. Each combination of members of a given sample size has an equal chance of being chosen for the sample.
   c. A sample is chosen from the population without much consideration of the elements chosen.
   d. The population is divided into groups and members of the sample are randomly selected from each group.

continued

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4. What is sampling bias?

   a. Some members of the population have a better chance of selection than others.

   b. All members of the population have an equal chance of selection.

   c. The sample size is too small to be unbiased.

   d. The sample statistics are less than or greater than the population parameters.

5. What is a stratified sample?

   a. All members of the population have an equal chance of selection.

   b. A population is divided into groups and members of the sample are randomly selected from each group.

   c. A population is divided into groups and only some of the groups are chosen for the sample.

   d. Each member of the population is assigned a number and every *n*th member is chosen for the sample.

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Lesson 2: Populations Versus Random Samples and Random Sampling

Instruction

Common Core State Standards

SMP
1 ✓ 2 ✓
3 ✓ 4 ✓
5 ✓ 6 ✓
7 ✓ 8 ✓

S–IC.1 Understand statistics as a process for making inferences about population parameters based on a random sample from that population.★

S–IC.2 Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?*★

Essential Questions

1. How is a sample different from a population?
2. Why are samples used in research?
3. How and why are samples used in research?
4. What are the advantages and disadvantages of using a simple random sample compared to using other methods of sampling?

WORDS TO KNOW

bias	leaning toward one result over another; having a lack of neutrality
biased sample	a sample in which some members of the population have a better chance of inclusion in the sample than others
chance variation	a measure showing how precisely a sample reflects the population, with smaller sampling errors resulting from large samples and/or when the data clusters closely around the mean; also called *sampling error*
cluster sample	a sample in which naturally occurring groups of population members are chosen for a sample

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Instruction

combination	a subset of a group of objects taken from a larger group of objects; the order of the objects does not matter, and objects may be repeated. A combination of size r from a group of n objects can be represented using the notation ${}_nC_r$, where ${}_nC_r = \frac{n!}{(n-r)!r!}$.
convenience sample	a sample in which members are chosen to minimize the time, effort, or expense involved in sampling
factorial	the product of an integer and all preceding positive integers, represented using a ! symbol; $n! = n \bullet (n-1) \bullet (n-2) \bullet \cdots \bullet 1$. For example, 5! = 5 • 4 • 3 • 2 • 1. By definition, 0! = 1.
inference	a conclusion reached upon the basis of evidence and reasoning
parameter	numerical value(s) representing the data in a set, including proportion, mean, and variance
population	all of the people, objects, or phenomena of interest in an investigation
random number generator	a tool used to select a number without following a pattern, where the probability of generating any number in the set is equal
random sample	a subset or portion of a population or set that has been selected without bias, with each item in the population or set having the same chance of being found in the sample
reliability	the degree to which a study or experiment performed many times would have similar results
representative sample	a sample in which the characteristics of the people, objects, or items in the sample are similar to the characteristics of the population
sample	a subset of the population
sampling bias	errors in estimation caused by flawed (non-representative) sample selection

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sampling error	a measure showing how precisely a sample reflects the population, with smaller sampling errors resulting from large samples and/or when the data clusters closely around the mean; also called *chance variation*
simple random sample	a sample in which any combination of a given number of individuals in the population has an equal chance of selection
statistics	numbers used to summarize, describe, or represent sets of data
stratified sample	a sample chosen by first dividing a population into subgroups of people or objects that share relevant characteristics, then randomly selecting members of each subgroup for the sample
systematic sample	a sample drawn by selecting people or objects from a list, chart, or grouping at a uniform interval; for example, selecting every fourth person
validity	the degree to which the results obtained from a sample measure what they are intended to measure

Recommended Resources

- eMathZone. "Simple Random Sampling."

  http://www.walch.com/rr/00179

  This site provides a summary of simple random sampling, explains how it differs from random sampling, and describes methods for selecting a simple random sample.

- Stat Trek. "Simulation of Random Events."

  http://www.walch.com/rr/00180

  This tutorial explains how to conduct a simulation of random events to mirror real-world outcomes and provides a link to a random number generator.

- Stat Trek. "Survey Sampling Methods."

  http://www.walch.com/rr/00181

  This website describes and gives examples of probability and non-probability sampling methods, followed by a sample problem with multiple-choice answers and a solution with explanation.

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Lesson 1.2.1: Differences Between Populations and Samples

Warm-Up 1.2.1

Mrs. Kittle teaches a class in composition and literature. She is conducting a portfolio review with her students, asking them to discuss some of the essays they've written over the past semester. Hannah submitted 6 essays this semester, and received the following grades:

100 95 90 85 80 60

Hannah has been asked to select 2 essays to discuss with Mrs. Kittle.

1. How many possible combinations of 2 essays can be chosen from the 6 that Hannah submitted?

2. What is the lowest possible average grade that Hannah can have in a sample of 2 essays?

3. What is the highest average grade that Hannah can have in a sample of 2 essays?

4. Suppose Hannah was asked to discuss 4 essays instead. What is the lowest possible average grade that Hannah can have in a sample of 4 essays?

5. What is the highest possible average grade that Hannah can have in a sample of 4 essays?

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Instruction

Lesson 1.2.1: Differences Between Populations and Samples

Common Core State Standard

S–IC.1 Understand statistics as a process for making inferences about population parameters based on a random sample from that population.★

Warm-Up 1.2.1 Debrief

1. How many possible combinations of 2 essays can be chosen from the 6 that Hannah submitted?

   Determining the number of ways to select 2 essays from the total of 6 essays submitted means finding a combination. In combinations, the order in which the items are chosen does not impact the result.

   The general formula for calculating a combination is $_nC_r = \frac{n!}{(n-r)!r!}$, where n is the total number of items from which to choose and r is the number of items actually chosen. Hannah has 6 papers total, so $n = 6$. She is choosing 2 papers to discuss, so $r = 2$.

   $_nC_r = \frac{n!}{(n-r)!r!}$ Formula for calculating a combination

   $_{(6)}C_{(2)} = \frac{(6)!}{[(6)-(2)]!(2)!}$ Substitute 6 for n and 2 for r.

   $_6C_2 = \frac{6!}{4!2!}$ Simplify.

   $_6C_2 = \frac{6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}{(4 \bullet 3 \bullet 2 \bullet 1)(2 \bullet 1)}$ Apply the factorial.

   $_6C_2 = \frac{720}{(24)(2)}$ Simplify.

   $_6C_2 = \frac{720}{48}$

   $_6C_2 = 15$

   There are 15 ways Hannah can choose 2 out of the 6 essays.

2. What is the lowest possible average grade that Hannah can have in a sample of 2 essays?

   To find the lowest possible sample average, select the two essays with the lowest scores and find the average.

   The two lowest scores are 60 and 80.

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Instruction

Recall that an average can be found by adding the values and then dividing the sum by the number of values in the set.

$$\text{average} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$ Formula for calculating an average

$$\text{average} = \frac{(80) + (60)}{(2)}$$ Substitute the known values.

average = 70 Simplify.

The lowest possible sample average in a sample of two of Hannah's essays is 70.

3. What is the highest possible average grade that Hannah can have in a sample of 2 essays?

   To find the highest possible sample average, select the two essays with the highest scores and find the average.

   The two highest scores are 100 and 95.

$$\text{average} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$ Formula for calculating an average

$$\text{average} = \frac{(100) + (95)}{(2)}$$ Substitute the known values.

average = 97.5 Simplify.

   The highest possible sample average in a sample of two of Hannah's essays is 97.5.

4. Suppose Hannah was asked to discuss 4 essays instead. What is the lowest possible average grade that Hannah can have in a sample of 4 essays?

   To find the lowest possible sample average, select the four essays with the lowest scores and find the average.

   The four lowest scores are 60, 80, 85, and 90.

$$\text{average} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$ Formula for calculating an average

$$\text{average} = \frac{(60) + (80) + (85) + (90)}{(4)}$$ Substitute the known values.

average = 78.75 Simplify.

   The lowest possible sample average in a sample of four of Hannah's essays is 78.75.

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5. What is the highest possible average grade that Hannah can have in a sample of 4 essays?

   To find the highest possible sample average, select the four essays with the highest scores and find the average.

   The four highest scores are 100, 95, 90, and 85.

$$\text{average} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$ Formula for calculating an average

$$\text{average} = \frac{(100) + (95) + (90) + (85)}{(4)}$$ Substitute the known values.

average = 92.5 Simplify.

   The highest possible sample average in a sample of four of Hannah's essays is 92.5.

Connection to the Lesson

- Students have the opportunity to see that a number of possible samples exist even for a small population.
- Students are introduced to the idea of variability in sampling. Students begin to understand that the variability of most measures is reduced by increasing the sample size.
- Students begin to recognize that different samples from the same population will have different averages.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- being able to find the number of combinations of a given size *r* that can be chosen from a set with *n* items (S–CP.9)
- calculating the mean and standard deviation of a data set using a graphing calculator (S–ID.2)

Introduction

In medicine, business, sports, science, and other fields, important decisions are based on statistical information drawn from samples. A sample is a subset of the population. The wise selection of samples often determines the success of those who use the information. It is possible that one sample is more reliable to predict an election or justify a new medical procedure, while other samples are simply not reliable. Some conclusions based on statistical samples are little more than guesses, and some are reckless conclusions in life-or-death matters; in many cases, it all comes down to whether the sample selected is genuinely random.

Key Concepts

- The word **statistics** has two different but related meanings.
- On a basic level, a statistic is a measure of a sample that is used to estimate a corresponding measure of a population (all of the people, objects, or phenomena of interest in an investigation). A statistic is a number used to summarize, describe, or represent something about a sample drawn from a larger population; the statistic allows us to make predictions about that population. A measure of the population that we are interested in is a **parameter**, a numerical value that represents the data in a set.
- We use different notation for sample statistics and population parameters. For example, the symbol for the mean of a population is μ, the Greek letter *mu*, whereas the symbol for the mean of a sample population is $\bar{x}$, pronounced "x bar." The symbol for the standard deviation of a population is σ, the lowercase version of the Greek letter *sigma*; the symbol for the standard deviation of a sample population is *s*.

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Instruction

- Though the formulas for the mean of a population and the mean of a sample population are essentially the same, the formula for the standard deviation of a sample population is slightly different from the formula for the standard deviation of a population.
  - For a population, the formula is $\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}}$, with σ representing the standard deviation of the population and μ representing the mean of the population.
  - For a sample, the formula is $s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}}$, with s representing the standard deviation of the sample and $\overline{x}$ representing the mean of the sample.
- When using a graphing calculator to find standard deviations of data sets, it is important to recognize whether the data set is a population or a sample so that the proper measure of standard deviation is selected.
- On a higher level, the field of statistics concerns the science and mathematics of describing and making inferences about a population from a sample.
- An **inference** is a conclusion reached upon the basis of evidence and reasoning.
- How well a statistic computed from a sample describes a population depends greatly upon the quality of the sampling method(s) used.
- First, the sample must be representative of the population. A **representative sample** is a sample in which the characteristics of the people, objects, or items in the sample are similar to the characteristics of the population.
- Samples that represent a population well can provide valuable information about that population. In research, it may be impractical to gather information about an entire population because of time, money, availability, privacy, and many other issues. In these cases, representative sampling may provide researchers with an efficient way to gather information and make decisions.
- In addition to the need for sampling to be representative, it must also produce reliable measures.
- **Reliability** refers to the degree to which a study or experiment performed many times would have similar results.

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Instruction

- When small samples are used, there is often great variability and little consistency among the statistics that are found.
- By increasing sample size, the variability in many sample statistics (such as means, standard deviations, and proportions) can be reduced, resulting in improved reliability and greater consistency of results.
- Statistical reasoning often involves making decisions based on limited information. In particular, when a population of interest is too large or expensive to study, a carefully chosen sample is used.
- One of the most important things that a researcher should understand about a population is the amount of **chance variation**, or **sampling error**, that is present in the measures of interest in that population.
- Chance variation is a measure showing how precisely a sample reflects the population, with smaller sampling errors resulting from large samples and/or when the data clusters closely around the mean.
- If a population is small enough, then parameters (such as measures of average, variation, or proportions) can be measured directly. There is no need for sampling in these cases. For example, if a teacher wants to know the mean grade for a recent test, he can calculate the mean of the entire class.
- If a population is large, or if it is impractical to measure all members of a population, then estimates are made from samples. The accuracy and reliability of the estimates depends on the quality of the sampling procedures used.
- In general, estimates of a population based on data from large samples are more reliable than estimates from small samples.
- In estimating the mean of a population, a sample size greater than 30 is recommended. In some cases, the sample size is much larger.
- In estimating proportions, a larger sample is desirable.
- **Validity** is the degree to which the results obtained from a sample measure what they are intended to measure.
- The validity of inferences made about a population depends greatly on the amount of **bias**, or lack of neutrality, in sampling procedures.

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Instruction

- A **biased sample** is a sample in which some members of the population have a better chance of inclusion in the sample than others.
- An estimate made using a sample that is biased is likely to be inaccurate even if a large sample is used. For example, if a publisher wants to determine the percent of readers who prefer printed books to e-books, interviewing 100 people shopping at a bookstore may yield biased results, since those people are more likely to be deliberately seeking out printed books instead of e-books for a variety of reasons (they prefer printed books, they don't own e-readers, they lack Internet access, etc.).
- The use of a **random number generator** can be helpful in selecting samples. A random number generator is a tool used to select a number without following a pattern, where the probability of generating any number in the set is equal.

Common Errors/Misconceptions

- not recognizing that results from an experiment or an observational study with a small sample size are unreliable
- not recognizing that samples that are biased can lead to misleading results even if numerical calculations are accurate
- not understanding that some of the variation in samples can be attributed to chance variation/sampling error

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Guided Practice 1.2.1

Example 1

Adam rolled a six-sided die 4 times and obtained the following results: 5, 5, 3, and 4. He computed the mean of the 4 rolls and used the result to estimate the mean of the population. Identify the parameter, sample, and statistic of interest in this situation. Calculate the identified statistic.

1. Identify the parameter in this situation.

   The parameter is the theoretical mean of all rolls of the six-sided die.

2. Identify the sample in this situation.

   The sample is the 4 rolls of the six-sided die.

3. Identify the statistic of interest in this situation.

   The statistic of interest is the mean of Adam's 4 rolls.

4. Calculate the identified statistic.

   Use the formula $\bar{x} = \frac{\sum x_i}{n} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$ to calculate the mean of Adam's 4 rolls.

$\bar{x} = \frac{\sum x_i}{n} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$	Formula for calculating the mean of a sample
$\bar{x} = \frac{(5)+(5)+(3)+(4)}{(4)}$	Substitute the value of each roll for x and 4 for n, the number of rolls.
$\bar{x} = \frac{17}{4}$	Simplify.
$\bar{x} = 4.25$	

The mean value of Adam's four rolls is 4.25. This value can be used to estimate the mean value of any number of rolls of a six-sided die.

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Example 2

High levels of blood glucose are a strong predictor for developing diabetes. Blood glucose is typically tested after fasting overnight, and the test result is called a fasting glucose level. A doctor wants to determine the percentage of his patients who have high glucose levels. He reviewed the glucose test results for 25 patients to determine how many of them had a fasting glucose level greater than 100 mg/dL (milligrams per deciliter). He recorded each patient's fasting glucose level in a table as follows.

Patient glucose levels in mg/dL				
99.9	105.4	131.8	79.7	66.6
116.7	111.5	98.1	86.4	76.4
105.8	107.0	95.7	87.6	99.1
75.4	106.2	87.6	89.2	72.4
58.9	86.8	66.0	53.6	88.1

Identify the population, parameter, sample, and statistic of interest in this situation, and then calculate the percent of patients in the sample with a fasting glucose level above 100 mg/dL.

1. Identify the population in this situation.

   The population is all patients of this doctor.

2. Identify the parameter in this situation.

   The parameter is the percent of patients with a fasting glucose level greater than 100 mg/dL.

3. Identify the sample in this situation.

   The sample is the 25 patients whose blood tests the doctor reviewed.

4. Identify the statistic of interest in this situation.

   The statistic of interest is the percent of patients in the sample with a fasting glucose level greater than 100 mg/dL.

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5. Calculate the statistic of interest.

   To calculate the percent of patients in the sample with a fasting glucose level greater than 100 mg/dL, use the fraction $\frac{x}{n}$, where x represents the number of patients with a fasting glucose level greater than 100 mg/dL and n represents the number of patients in the sample.

   From the table, it can be seen that 7 of the values are greater than 100, so $x = 7$. The total number of patients in the sample is 25, so $n = 25$.

$\frac{x}{n}$	Fraction
$\frac{(7)}{(25)} = 0.28$	Substitute 7 for x and 25 for n, and then solve.

   Of the patients in the sample, 0.28 or 28% had a fasting glucose level greater than 100 mg/dL.

   Note: It is important to recognize that this may be an inaccurate estimate because the patients in the sample may not be representative of the entire population of the doctor's patients.



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Example 3

Data collected by the National Climatic Data Center from 1971 to 2000 was used to determine the average total yearly precipitation for each state. The following table shows the mean yearly precipitation for a random sample of 10 states and each state's ranking in relation to the rest of the states, where a ranking that's closer to 1 indicates a higher mean yearly precipitation. Use the sample data to estimate the total rainfall in all 50 states for the 30-year period from 1971 to 2000. Identify the population, parameter, sample, and statistic of interest in this situation.

Ranking	State	Mean yearly precipitation (in inches)
5	Florida	54.5
8	Arkansas	50.6
12	Kentucky	48.9
28	Ohio	39.1
35	Kansas	28.9
38	Nebraska	23.6
39	Alaska	22.5
41	South Dakota	20.1
43	North Dakota	17.8
46	New Mexico	14.6

1. Identify the population in this situation.

   The population is all 50 states.

2. Identify the parameter in this situation.

   The parameter is the total rainfall from 1971 to 2000.

3. Identify the sample in this situation.

   The sample is the 10 randomly selected states.

4. Identify the statistic of interest in this situation.

   The statistic of interest is the mean yearly precipitation for the sample.

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5. Calculate the statistic of interest.

To calculate the mean yearly precipitation for the sample, first find the total mean yearly precipitation in the sample states.

To do this, find the sum of the mean yearly precipitation of each state.

mean yearly precipitation = 22.5 + 50.6 + 54.5 + 28.9 + 48.9 + 23.6 + 14.6 + 17.8 + 39.1 + 20.1 = 320.6

The total mean yearly precipitation for the 10 sample states is 320.6 inches.

Next, use this value to estimate the total precipitation in all 50 states for 1 year.

Create a proportion, as shown; then, solve it for the unknown value.

$$\frac{\text{sample mean yearly precipitation}}{\text{sample size}} = \frac{\text{population mean yearly precipitation}}{\text{population size}}$$

$\frac{(320.6)}{(10)} = \frac{x}{(50)}$ Substitute known values.

$10x = 50(320.6)$ Cross-multiply to solve for x.

$10x = 16{,}030$ Simplify.

$x = 1603$

Based on the data from 1971 to 2000, the estimated total precipitation in all 50 states for 1 year during this time frame is 1,603 inches per year.

Use this value to estimate the total precipitation in all 50 states for this period of 30 years.

Multiply the precipitation in all 50 states for 1 year by 30.

1603(30) = 48,090

The estimated total rainfall in all 50 states for the 30-year period from 1971 to 2000 is 48,090 inches.



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Example 4

For her math project, Stephanie wants to estimate the mean and standard deviation of the points scored by the home and away teams in the National Basketball Association. She randomly selects one home game and one away game for each of 16 NBA teams during the 2012 season and records their scores in a table.

Selected NBA game scores in 2012			
Home score	**Away score**	**Home score**	**Away score**
101	109	106	112
104	94	83	82
95	104	95	113
122	108	106	91
96	107	103	83
101	97	106	85
97	81	128	96
87	94	103	111

Use a graphing calculator to estimate the mean and standard deviation of the points scored by the home and away teams in the NBA. Identify the population, parameters, sample, and statistics of interest.

1. Identify the population.

   The population is all NBA games.

2. Identify the parameters.

   There are four parameters in the population: the mean points scored by the home team; the mean points scored by the away team; the standard deviation of points scored by the home team; and the standard deviation of points scored by the away team.

3. Identify the sample.

   The sample is the 2 games per team selected for 16 teams.

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4. Identify the statistics of interest.

   There are four statistics of interest in the sample: the mean points scored by the home team; the mean points scored by the away team; the standard deviation of points scored by the home team; and the standard deviation of points scored by the away team.

5. Use a graphing calculator to find the mean and standard deviation of the home and away scores.

   Follow the steps specific to your calculator model to find the mean and standard deviation.

   On a TI-83/84:

   Step 1: Press [STAT] to bring up the statistics menu. The first option, 1: Edit, will already be highlighted. Press [ENTER].

   Step 2: Arrow up to L1 and press [CLEAR], then [ENTER], to clear the list. Repeat this process to clear L2 and L3 if needed.

   Step 3: From L1, press the down arrow to move your cursor into the list. Enter each of the home scores from the table into L1, pressing [ENTER] after each number to navigate down to the next blank spot in the list.

   Step 4: Arrow over to L2 and enter the away scores as listed in the table.

   Step 5: To calculate the mean and standard deviation of the home scores (L1), press [STAT]. Arrow over to the CALC menu. The first option, 1–Var Stats, will already be highlighted. Press [ENTER]. This brings up the 1–Var Stats menu.

   Step 6: In the menu, "L1" should be displayed next to "List." Press [2ND][1] if not. This will enter "L1."

   Step 7: Press [ENTER] three times to evaluate the data set. The mean of the sample, 102.0625, will be listed to the right of $\bar{x} =$ and the standard deviation of the sample, 11.1861149, will be listed to the right of $S_x =$.

(*continued*)

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Step 8: To calculate the mean and standard deviation of the away scores, press [STAT], arrow over to the CALC menu, and press 1: 1–Var Stats. When prompted, press [2ND][1] to enter L2 next to "List."

Step 9: Press [ENTER] three times to evaluate the data set. The mean of the sample, 97.9375, will be listed to the right of $\bar{x} =$ and the standard deviation of the sample, 11.41033888, will be listed to the right of $S_x =$.

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the spreadsheet icon, the fourth icon from the left, and press [enter].

Step 3: To clear the lists in your calculator, arrow up to the topmost cell of the table to highlight the entire column, then press [menu]. Use the arrow key to choose 3: Data, then 4: Clear Data, then press [enter]. Repeat for each column as necessary.

Step 4: Arrow up to the topmost cell of the first column, labeled "A." Name the column "home" using the letters on your keypad. Press [enter].

Step 5: Arrow down to the first cell of the column. Enter each of the home scores from the table in the home column, pressing [enter] after each number to navigate down to the next blank cell.

Step 6: Arrow up to the topmost cell of the second column, labeled "B." Name the column "away" using the letters on your keypad. Press [enter].

Step 7: Arrow down to the first cell of the column and enter each of the away scores from the table, pressing [enter] after each number.

Step 8: To calculate the mean and standard deviation of both data sets, press [menu], arrow down to 4: Statistics, then arrow to 1: Stat Calculations, then 1: One-Variable Statistics. Press [enter]. When prompted, enter 2 for Num of Lists, tab to "OK," and then press [enter].

(*continued*)

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Step 9: At X1 List, enter A using your keypad to select the data in column A. Tab to the X2 List, then enter B to select the data in column B. Tab to 1st Result Column and enter C. Tab down to "OK" and press [enter] to evaluate the data sets. This will bring you back to the spreadsheet, where column C will be populated with the title of each calculation, and columns D and E will list the values for each data set. Use the arrow key to scroll through the rows of the spreadsheet to find the rows for the sample mean and sample standard deviation. The sample means will be listed to the right of $\bar{x}$, and the sample standard deviations will be listed to the right of "$sx := s_{n-1}x$". For the home scores, the sample mean is 102.0625 and the sample standard deviation is 11.1861149. For the away scores, the sample mean is 97.9375, and the sample standard deviation is 11.41033888.

Rounded to the nearest tenth, the mean of the sample of home scores is approximately 102.1. The standard deviation of the sample of the home scores is approximately 11.2.

The mean of the sample of the away scores is approximately 97.9. The standard deviation of the sample of the away scores is approximately 11.4.

These sample statistics can be used to estimate the population parameters.



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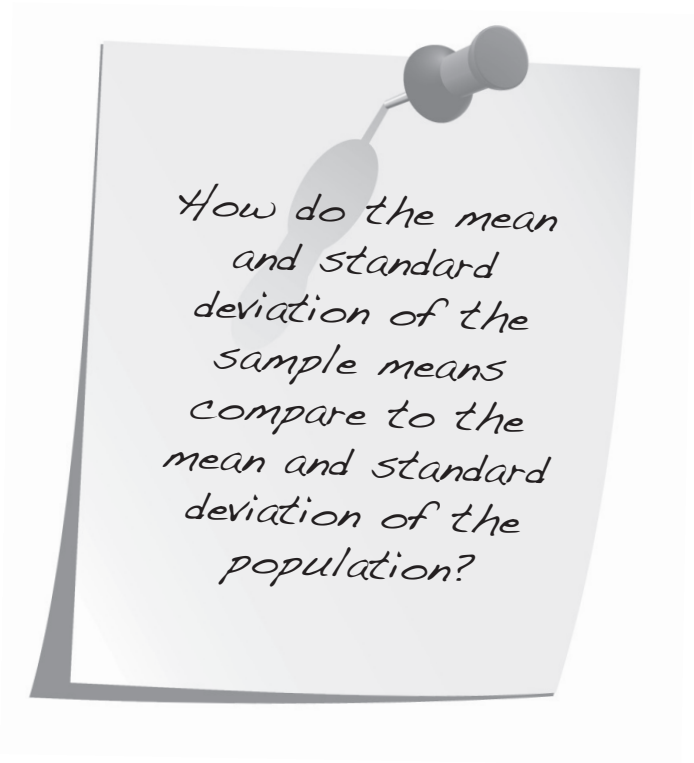
Problem-Based Task 1.2.1: Song Requests

SMP	
1 ✓	2 ✓
3	4
5	6
7 ✓	8

The manager of a radio station tracked the songs most requested by listeners for the years 2007 through 2012. Her data is listed in the table below. The most popular song for each year is labeled with a letter.

Year	Song	Number of requests (in thousands)
2007	A	2.7
2008	B	3.4
2009	C	4.8
2010	D	4.4
2011	E	5.8
2012	F	6.8

Consider the 6 listed songs a population. Let all possible samples of size 3 be the sample. How do the mean and standard deviation of the sample means compare to the mean and standard deviation of the population?



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Problem-Based Task 1.2.1: Song Requests

Coaching

a. What are the mean and the standard deviation of the population?

b. How many combinations of 3 items (songs) are there in a group of 6 items?

c. What does each combination represent?

d. What are the possible combinations? Use the letter labels to make it easier to identify the samples.

e. What is the mean of each sample found in part d?

f. How would you determine which of these sample means is the best estimate of the population mean? The worst estimate?

g. What are the mean and standard deviation for the entire list of sample means found in part e?

h. How does the mean of the list of sample means compare to the mean of the population?

i. How does the standard deviation of the list of sample means compare to the standard deviation of the population?

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Instruction

Problem-Based Task 1.2.1: Song Requests

Coaching Sample Responses

a. What are the mean and the standard deviation of the population?

The mean and standard deviation of the population can be found using a graphing calculator. Follow the calculator directions that are appropriate to your calculator model.

The mean of the population is approximately 4.65.

The standard deviation of the population is approximately 1.38.

b. How many combinations of 3 items (songs) are there in a group of 6 items?

The general formula for calculating a combination is $_nC_r = \frac{n!}{(n-r)!r!}$, where n is the total number of items from which to choose and r is equal to the number of items actually chosen.

In this scenario, $n = 6$ and $r = 3$.

$$_nC_r = \frac{n!}{(n-r)!r!}$$

$$_{(6)}C_{(3)} = \frac{(6)!}{[(6)-(3)]!(3)!}$$

$$_6C_3 = \frac{6!}{3!3!}$$

$$_6C_3 = 20$$

There are 20 possible combinations of 3 songs from the group of 6 songs.

c. What does each combination represent?

Each of these combinations represents a separate sample.

d. What are the possible combinations? Use the letter labels to make it easier to identify the samples.

Recall that with a combination, the order of the songs does not matter, so ABC is the same as ACB.

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Create a table to organize the 20 samples.

Possible song combinations (samples)				
ABC	ACD	ADF	BCF	CDE
ABD	ACE	AEF	BDE	CDF
ABE	ACF	BCD	BDF	CEF
ABF	ADE	BCE	BEF	DEF

e. What is the mean of each sample found in part d?

Find the mean of each sample by adding the number of requests for each song, then dividing by 3.

Organize the results in a table. All song request figures are in thousands.

Sample combination	Number of requests for the first song in the sample	Number of requests for the second song in the sample	Number of requests for the third song in the sample	Sample mean
ABC	2.7	3.4	4.8	3.63
ABD	2.7	3.4	4.4	3.50
ABE	2.7	3.4	5.8	3.97
ABF	2.7	3.4	6.8	4.30
ACD	2.7	4.8	4.4	3.97
ACE	2.7	4.8	5.8	4.43
ACF	2.7	4.8	6.8	4.77
ADE	2.7	4.4	5.8	4.30
ADF	2.7	4.4	6.8	4.63
AEF	2.7	5.8	6.8	5.10
BCD	3.4	4.8	4.4	4.20
BCE	3.4	4.8	5.8	4.67
BCF	3.4	4.8	6.8	5.00
BDE	3.4	4.4	5.8	4.53
BDF	3.4	4.4	6.8	4.87
BEF	3.4	5.8	6.8	5.33
CDE	4.8	4.4	5.8	5.00
CDF	4.8	4.4	6.8	5.33
CEF	4.8	5.8	6.8	5.80
DEF	4.4	5.8	6.8	5.67

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f. How would you determine which of these sample means is the best estimate of the population mean? The worst estimate?

Begin by finding the difference between the population mean, 4.65, and each sample mean.

To find which sample mean is the best estimate of the population mean, find the absolute value of the differences between each sample mean and the population mean, then choose the lowest value.

To find which sample mean is the worst estimate of the population mean, find the absolute value of the differences between each sample mean and then population mean, then choose the highest value.

Organize the results in a table. *Note*: Differences between your calculations and the values in the following table are due to rounding.

Sample combination	Sample mean	Population mean – sample mean	Absolute value of the difference in means
ABC	3.63	–1.02	1.02
ABD	**3.50**	**–1.15**	**1.15**
ABE	3.97	–0.68	0.68
ABF	4.30	–0.35	0.35
ACD	3.97	–0.68	0.68
ACE	4.43	–0.22	0.22
ACF	4.77	0.14	0.14
ADE	4.30	–0.35	0.35
ADF	**4.63**	**–0.02**	**0.02**
AEF	5.10	0.47	0.47
BCD	4.20	–0.45	0.45
BCE	**4.67**	**0.04**	**0.04**
BCF	5.00	0.37	0.37
BDE	4.53	–0.12	0.12
BDF	4.87	0.24	0.24
BEF	5.33	0.70	0.70
CDE	5.00	0.37	0.37
CDF	5.33	0.70	0.70
CEF	**5.80**	**1.17**	**1.17**
DEF	5.67	1.04	1.04

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It can be seen from the table that the samples with the lowest absolute values for the difference in means are ADF and BCE. These two samples are the best estimates of the population mean.

ABD and CEF have the highest absolute values for the difference in means. These two samples are the worst estimates of the population mean.

g. What are the mean and standard deviation for the entire list of sample means found in part e?

Use a graphing calculator to enter the sample means as if they were individual scores, then find the mean and standard deviation of the list. Treat this as a population. Follow the calculator directions that are appropriate to your calculator model.

The mean of the list of 20 sample means is approximately 4.65.

The standard deviation for the list of sample means is 0.62.

h. How does the mean of the list of sample means compare to the mean of the population?

The mean of the list of sample means (4.65) is equal to the population mean (4.65).

i. How does the standard deviation of the list of sample means compare to the standard deviation of the population?

The standard deviation of the list of sample means (0.62) is less than the standard deviation for the population of individual songs (1.38).

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.2.1: Differences Between Populations and Samples

For problems 1–3, choose the best response.

1. Which statement explains why a state government would use population parameters (the number of votes cast in the entire state) rather than samples from each county to determine the outcome of an election for governor?

   a. Modern technology makes it quick and easy to count votes.

   b. A sample only represents a portion of the entire population. A gubernatorial election is too important to decide based on estimates from sample statistics.

   c. It takes much longer to count the votes in a sample than in a population.

   d. Not every eligible person votes.

2. Which statement explains why sample statistics are used by the media to make predictions prior to presidential elections?

   a. Percentages are difficult to compute with large numbers.

   b. Sample statistics are more reliable than population parameters.

   c. Members of the Electoral College determine the outcome of a presidential election rather than the popular vote.

   d. It would not be practical for the media to determine every person's opinion prior to the election.

3. Which statement best describes the effect of sample size on statistics?

   a. A statistic obtained from a large sample gives a more reliable estimate of a population parameter than a statistic obtained from a small sample.

   b. A statistic obtained from a large sample gives a less reliable estimate of a population parameter than a statistic obtained from a small sample.

   c. A statistic obtained from a large sample has greater variability than the variability in the original population.

   d. A statistic obtained from a large sample has greater variability than a statistic obtained from a small sample.

continued

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Use what you have learned about samples to complete problems 4–7.

4. For his science project, Tyrus tested 40 Suncharged-brand batteries to estimate the mean time that Suncharged batteries last. Identify the population, parameter, sample, and statistic of interest in this situation.

5. Maggie distributed a survey to the students in 5 homerooms to estimate the percent of students at her high school who are in favor of the new dress code. Identify the population, parameter, sample, and statistic of interest in this situation.

6. In a marketing survey, 13 out of 80 participating adults reported that they would like to purchase a new cell phone in the next month. Estimate the number of adults in a community of 7,200 adults who would like to purchase a new cell phone in the next month. Assume that the sample is representative of the population.

7. In a wildlife study, 12 moose in a given region were released with tracking devices. Later, 20 moose were found in the region and 4 of them had tracking devices. Use the results to estimate the number of moose in the region. Assume no moose entered or left the region during the study.

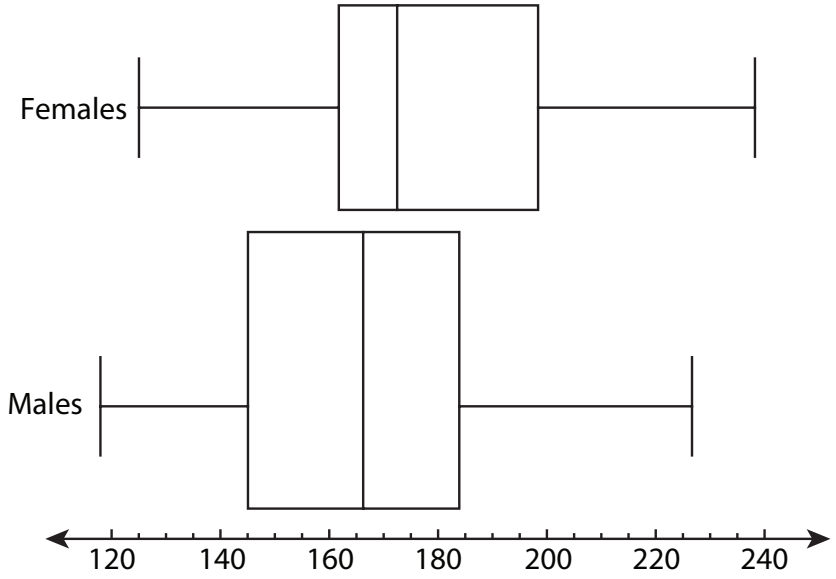
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Use the provided information to complete problems 8–10.

The director of a community health clinic is compiling information on the total blood cholesterol levels of all the patients who regularly visit the clinic. One week, 27 male patients and 23 female patients had their blood cholesterol levels measured at the clinic. The results are shown in the box plots and table of summary statistics below.

Cholesterol levels in mg/dL for males and females



Summary statistics

	Males	Females
Population size	343	298
Sample size	27	23
Sample mean cholesterol (mg/dL)	167.6	179.0
Sample standard deviation (mg/dL)	29.0	28.0
Sample participants with cholesterol greater than 150 mg/dL	20	14

8. Use the results in the table to estimate the number of male patients at the clinic with a cholesterol level greater than 150 mg/dL based on the sample of males.

9. Use the results in the table to estimate the number of female patients at the clinic with a cholesterol level greater than 150 mg/dL based on the sample of females.

10. Estimate the mean cholesterol level of all the clinic's regular patients. Assume that the observed differences between males and females can be attributed to sampling error.

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Lesson 1.2.2: Simple Random Sampling

Warm-Up 1.2.2

Players of a dice game roll five dice and earn points according to the combinations of numbers they roll. After the first roll, a player can pick up as many dice as desired and re-roll them to get an improved score. The following table outlines the points scored for different types of rolls.

Points	Description	Example roll
0	All five dice have different numbers.	1, 2, 3, 5, 6
1	Two dice have the same number.	1, 2, 2, 3, 5
2	Two sets of two dice have the same number.	1, 3, 3, 6, 6
3	Three dice have the same number.	2, 2, 2, 4, 5
4	Three dice have the same number and the other two dice match.	1, 1, 1, 4, 4
5	Four dice have the same number.	3, 3, 3, 3, 6
6	All five dice have the same number.	5, 5, 5, 5, 5

To better understand the game, Chad simulated rolling dice on his graphing calculator. He played 20 games with the following results:

Initial roll	Points	Second roll	Total points
4, 3, 6, 2, 3	1	3, 3, 4, 6, 6	2
6, 1, 3, 6, 2	1	6, 6, 1, 1, 3	2
4, 2, 5, 5, 1	1	5, 5, 4, 1, 5	3
5, 6, 1, 2, 4	0	2, 1, 6, 2, 2	3
1, 2, 1, 2, 5	2	1, 1, 2, 2, 6	2
4, 1, 3, 3, 4	2	3, 3, 4, 4, 6	2
6, 6, 5, 2, 5	2	6, 6, 5, 5, 4	2
1, 2, 4, 3, 1	2	1, 1, 4, 4, 4	4
3, 1, 4, 3, 1	2	3, 3, 1, 1, 5	2
6, 2, 6, 5, 1	1	6, 6, 1, 3, 4	1
4, 5, 3, 1, 1	1	1, 1, 5, 5, 1	2
3, 4, 2, 4, 3	2	3, 3, 4, 4, 1	2
1, 3, 2, 3, 1	2	1, 1, 3, 3, 3	4
2, 2, 1, 1, 1	4	2, 2, 1, 1, 1	4
3, 4, 1, 1, 3	2	3, 3, 1, 1, 5	2
6, 3, 1, 5, 5	1	5, 5, 1, 1, 1	4
6, 6, 6, 3, 5	3	6, 6, 6, 4, 6	5
4, 3, 5, 4, 5	2	4, 4, 5, 5, 5	4
2, 2, 4, 5, 4	2	2, 2, 4, 4, 3	2
4, 2, 6, 4, 3	1	4, 4, 3, 4, 4	5
Total	34	**Total**	57

continued

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1. What percent of the time could you expect to earn 0 points on the first roll based on the results of Chad's simulation?

2. Suppose it is your final turn and you need a minimum of 3 points to win the game. What is the probability that you will win according to the simulation?

3. What is the mean of the values for each of the first rolls in the simulation?

4. What is the mean of the values after each of the second rolls in the simulation?

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Instruction

Lesson 1.2.2: Simple Random Sampling

Common Core State Standard

S–IC.2 Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?*★

Warm-Up 1.2.2 Debrief

1. What percent of the time could you expect to earn 0 points on the first roll based on the results of Chad's simulation?

   To determine the percent of the time you would expect to earn 0 points on the first roll, first find the number of times in the simulation that the first roll yielded 0 points. According to the simulation, only 1 roll earned 0 points.

   Calculate the percent by setting up a ratio of the number of occurrences of 0 points out of the total number of first rolls, 20.

   $$\frac{\text{number of rolls earning 0 points}}{\text{total number of first rolls}} = \frac{1}{20} = 0.05 = 5\%$$

   You could expect to earn 0 points on the first roll 5% of the time, according to the simulation.

2. Suppose it is your final turn and you need a minimum of 3 points to win the game. What is the probability that you will win according to the simulation?

   To determine the probability that you will win, find the number of games in the simulation for which the total points earned for both rolls was at least 3.

   There were 9 games in the simulation with a total score that was greater than or equal to 3 points.

   $$\frac{\text{number of games with a total score} \geq 3}{\text{total number of games}} = \frac{9}{20} = 0.45 = 45\%$$

   According to the simulation, the probability that you will win is 45%.

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3. What is the mean of the values for each of the first rolls in the simulation?

   Recall that the mean of a data set can be found using the formula $\overline{x} = \frac{\sum x_i}{n} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$, where n is the number of data values (x).

   $\overline{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$ Formula for calculating mean

   $\overline{x} = \frac{7(1) + 0 + 10(2) + 4 + 3}{20}$ Substitute the known values. (Repeated values are listed as products for convenience.)

   $\overline{x} = 1.7$ Simplify.

   The mean of the values for the first rolls is 1.7.

4. What is the mean of the values after each of the second rolls in the simulation?

   Again, use the formula $\overline{x} = \frac{\sum x_i}{n} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$ to calculate the mean.

   $\overline{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$ Formula for calculating mean

   $\overline{x} = \frac{10(2) + 2(3) + 5(4) + 1 + 2(5)}{20}$ Substitute the known values. (Repeated values are listed as products for convenience.)

   $\overline{x} = 2.85$ Simplify.

   The mean of the values after each of the second rolls is 2.85.

Connection to the Lesson

- Students will continue to calculate statistics such as mean, standard deviation, and proportion from finite data sets.
- Students will begin to understand that statistical probability is the key connection in inferential statistics.
- Students will connect statistics in finite data sets to probabilities in theoretical distributions.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- distinguishing between a sample and a population (7.SP.1)
- understanding when it is advisable to use a sample instead of an entire population (7.SP.1)
- using proportions to solve for missing values (7.RP.3)
- calculating means, standard deviations, and proportions in data sets (6.SP.5c)
- understanding how to read and construct a box plot (S–ID.1)

Introduction

Suppose that some students from the junior class will be chosen to receive new laptop computers for free as part of a pilot program. You hear that the laptops have powerful processing capabilities and that they make learning more interesting. Suppose you want one of these free laptops, but you understand that some students will not receive them. Here is what many students might think and feel about the selection process:

- I will be happy if I am chosen to receive a free laptop.
- I will be satisfied knowing that I have the same chance as all of my other classmates to receive a laptop.
- I will be upset if I learn that the students who receive the free laptops just happen to be in the right place at the right time and the donors put little time and effort into the selection process.
- I will be furious if I learn that favoritism is involved in the awarding of free laptops.

The possible responses to the laptop selection process vary greatly, and illustrate the importance of representative sampling. It is impractical in most situations to determine parameters by studying all members of a population, but with quality sampling procedures, valuable research can be performed. For research to provide accurate results, the sample that is used must be representative of the population from which it is drawn.

Having a fair laptop selection process also shows the significance of using random samples. Though not every population member can be chosen, it is still possible, in some cases, for every population member to have an equal (or nearly equal) chance of inclusion. This is the goal of random sampling. A **random sample** is a subset or portion of a population or set that has been selected without bias. In a random sample, each member of the population has an equal chance of selection.

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Lesson 2: Populations Versus Random Samples and Random Sampling

Instruction

This lesson will focus on selecting simple random samples using playing cards and graphing calculators. Simple random sampling will be contrasted with biased sampling, and conjectures will be made about how biased samples affect research results. Simulations will be performed with simple random samples to better understand how to classify events that are common, somewhat unusual, or highly improbable. With careful study, these skills will enable you to better conduct quality research as well as evaluate the research of others.

Key Concepts

- **Sampling bias** refers to errors in estimation caused by a flawed, non-representative sample selection.
- A **simple random sample** is a sample in which any combination of a given number of individuals in the population has an equal chance of being selected for the sample.
- Simple random samples do not contain sampling bias since, for any sample size, all combinations of population members have an equal chance of being chosen for the sample.
- By using a simple random sample, a researcher can eliminate intentional and unintentional advantages and disadvantages of any members of the population.
- For example, suppose school administrators decide to survey 100 students about a proposed change in the dress-code policy. The administration assigns each of the 875 students at the school a number and then randomly selects 100 numbers. While there is chance involved regarding who is chosen for the survey, no group of students has a better chance of selection than any other group of students. There is chance, but not intentional or unintentional bias.
- A simple random sample will likely result in sampling error, the difference between a sample result and the corresponding measure in the population, since there will be some variation in sample statistics depending on which members of the population are chosen for the sample.
- For example, suppose school administrators decide to survey two groups of 100 students instead of just one group. It is likely that the percent of students with favorable opinions about the dress-code policy will be slightly higher in one sample than in the other.
- If all other factors are equal, sampling error is greater when there is more variation in a population than when there is less variation. All else being equal, sampling error is less when large samples are used than when small samples are used.
- Researchers analyze data to decide if the results of an experiment can be attributed to chance variation or if it is likely that other factors have an effect. Depending on the researcher and the situation, limits of 1%, 5%, or 10% are normally used to make these decisions.
- To have sufficient evidence that a given factor (such as a personal characteristic, a medical treatment, or a new product) has an effect on the results, a researcher must rule out the possibility that the results can be attributed to chance variation.

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Instruction

Common Errors/Misconceptions

- mistakenly believing that the word *random* in the term *random sampling* means "haphazard, or done quickly without thought"
- not understanding that performing a statistical analysis with biased data can lead to grossly misleading results even if the mathematical analysis is perfect
- not believing that events with low probability are likely to occur some of the time if a population or sample size is large enough

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Guided Practice 1.2.2

Example 1

Mr. DiCenso wants to establish baseline measures for the 21 students in his psychology class on a memory test, but he doesn't have time to test all students. How could Mr. DiCenso use a standard deck of 52 cards to select a simple random sample of 10 students? The students in Mr. DiCenso's class are listed as follows.

Tim	Brion	Victoria	Nick	Quinn	Gigi	Jose
Alex	Andy	Michael	Stella	Claire	Lara	Noemi
Eliza	Morgan	Ian	Dominic	DeSean	Rafiq	Gillian

1. Assign a value to each student.

   Assign a card name (for example, *ace of spades*) to each student, as shown in the following table.

Student	Card	Student	Card	Student	Card
Tim	Ace of spades	Michael	7 of spades	DeSean	King of hearts
Alex	King of spades	Ian	6 of spades	Gigi	Queen of hearts
Eliza	Queen of spades	Nick	5 of spades	Lara	Jack of hearts
Brion	Jack of spades	Stella	4 of spades	Rafiq	10 of hearts
Andy	10 of spades	Dominic	3 of spades	Jose	9 of hearts
Morgan	9 of spades	Quinn	2 of spades	Noemi	8 of hearts
Victoria	8 of spades	Claire	Ace of hearts	Gillian	7 of hearts

2. Randomly select cards.

   Shuffle the 21 cards thoroughly, then select the first 10 cards.

   Identify the students whose names were assigned to the chosen cards.

   Samples may vary; one possibility follows.

   6 of spades: Ian
   9 of spades: Morgan
   10 of spades: Andy
   Ace of hearts: Claire
   2 of spades: Quinn
   King of hearts: DeSean
   Jack of hearts: Lara
   4 of spades: Stella
   Queen of hearts: Gigi
   7 of spades: Michael

   The selected cards indicate which students will be a part of the simple random sample.

SWB p. 61; SRB p. U1-84

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Instruction

Example 2

Mrs. Tilton wants to estimate the number of words per page in a book she plans to have her class read. There are 373 pages in the book, and Mrs. Tilton wants to base her estimation on a sample of 40 pages. Use a graphing calculator to select a simple random sample of 40 page numbers.

1. Determine the starting and ending values for the situation described.

   In order to use a graphing calculator to select a simple random sample, you must identify both the starting and ending values.

   We will assume the book begins on page 1. There are 373 pages in the specified book; therefore, the starting value should be 1 and the ending value should be 373.

2. Determine the number of unique numbers to generate.

   Mrs. Tilton wants to select a simple random sample of 40 page numbers; therefore, we must generate 40 unique numbers.

3. Use a graphing calculator to generate the unique numbers.

   Follow the directions specific to your calculator model.

   On a TI-83/84:

   Step 1: From the home screen, press [MATH]. Arrow over to the PRB menu, then down to 5: randInt(. Press [ENTER].

   Step 2: At randInt(, use the keypad to enter the starting value, 1, and the ending value, 373, separated by a comma and followed by a closing parenthesis. Press [ENTER]. This will generate a random number with a value within the range given.

   Step 3: Press [ENTER] repeatedly until 40 numbers have been generated. Copy each of the random numbers into a table.

(*continued*)

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On a TI-Nspire:

Step 1: From the home screen, arrow down to the calculator icon, the first icon from the left, and press [enter].

Step 2: Press [menu]. Use the arrow key to choose 5: Probability, then 4: Random, then 2: integer. Press [enter]. This will bring up a screen with "randInt()."

Step 3: Inside the parentheses, use the keypad to enter the starting value, 1, and the ending value, 373, separated by a comma. Press [enter]. This will generate a random number with a value within the range given.

Step 4: Press [enter] repeatedly until 40 numbers have been generated. Copy each of the random numbers into a table.

4. Identify the simple random sample of 40 page numbers.

   One potential sample is listed as follows; different samples are also possible.

352	339	55	192	152	274	17	127	372	75
298	356	83	138	3	349	41	158	205	320
365	104	103	46	20	270	5	115	363	11
313	231	77	368	271	113	93	353	346	16

   The randomly generated numbers represent the simple random sample of 40 pages from the 373 total pages of the book.

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Example 3

The following table shows the time it took in 100 trials to recharge a particular brand of cell phone after its battery ran out of charge. Each time is rounded to the nearest minute. Use a random integer generator to select two random samples of size 10 from the population of 100 cell phones. Determine the mean and the standard deviation of each sample. Explain why the mean and standard deviation of the first sample are different from the mean and standard deviation of the second sample.

Trial	Minutes	Trial	Minutes	Trial	Minutes	Trial	Minutes
1	70	26	78	51	73	76	74
2	71	27	75	52	74	77	68
3	72	28	75	53	75	78	65
4	74	29	75	54	69	79	73
5	71	30	70	55	72	80	69
6	70	31	67	56	72	81	73
7	76	32	72	57	73	82	78
8	76	33	79	58	75	83	75
9	75	34	70	59	80	84	68
10	69	35	69	60	73	85	78
11	81	36	75	61	74	86	73
12	73	37	67	62	69	87	70
13	68	38	73	63	77	88	73
14	69	39	81	64	65	89	67
15	65	40	60	65	73	90	77
16	73	41	68	66	71	91	70
17	76	42	72	67	70	92	74
18	72	43	66	68	79	93	67
19	77	44	75	69	76	94	72
20	71	45	71	70	70	95	82
21	75	46	69	71	71	96	69
22	79	47	67	72	70	97	73
23	72	48	69	73	70	98	72
24	72	49	72	74	72	99	65
25	76	50	68	75	66	100	69

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Instruction

1. Use a random integer generator to select two random samples of size 10 from the population of 100 cell phones.

   Follow the directions outlined in Example 1 that are appropriate for your calculator model.

   Let the starting value be 1 and the ending value be 100.

   Generate 10 unique numbers to represent the first sample, Sample A, and record them in a table.

   Generate a second set of 10 unique numbers to represent the second sample, Sample B, and record these numbers in the same table.

Sample A		Sample B	
Trial number	**Minutes**	**Trial number**	**Minutes**
51		50	
81		31	
32		29	
49		13	
80		43	
34		35	
41		93	
9		64	
57		87	
6		37	

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2. Record the minutes corresponding to each random integer in the table.

   Refer to the given table of values to identify the number of minutes associated with each cell phone trial.

Sample A		Sample B	
Trial number	**Minutes**	**Trial number**	**Minutes**
51	73	50	68
81	73	31	67
32	72	29	75
49	72	13	68
80	69	43	66
34	70	35	69
41	68	93	67
9	75	64	65
57	73	87	70
6	70	37	67

3. Use a graphing calculator to determine the mean and standard deviation of each sample.

   Follow the directions specific to your calculator model.

   On a TI-83/84:

   Step 1: Press [STAT] to bring up the statistics menu. The first option, 1: Edit, will already be highlighted. Press [ENTER].

   Step 2: Arrow up to L1 and press [CLEAR], then [ENTER], to clear the list. Repeat this process to clear L2 and L3 if needed.

   Step 3: From L1, press the down arrow to move your cursor into the list. Enter each of the minutes for Sample A from the table into L1, pressing [ENTER] after each number to navigate down to the next blank spot in the list.

   Step 4: Arrow over to L2 and enter the minutes from Sample B as listed in the table.

(*continued*)

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Step 5: To calculate the mean and standard deviation of the minutes for Sample A (L1), press [STAT]. Arrow over to the CALC menu. The first option, 1–Var Stats, will already be highlighted. Press [ENTER]. This brings up the 1–Var Stats menu.

Step 6: In the menu, "L1" should be displayed next to "List." Press [2ND][1] if not. This will enter "L1."

Step 7: Press [ENTER] three times to evaluate the data set. The mean of the sample, 71.5, will be listed to the right of $\bar{x}$ = and the standard deviation of the sample, 2.173067486, will be listed to the right of S_x =.

Step 8: To calculate the mean and standard deviation of the minutes for Sample B, press [STAT], arrow over to the CALC menu, and press 1: 1–Var Stats. When prompted, press [2ND][1] to enter L2 next to "List."

Step 9: Press [ENTER] three times to evaluate the data set. The mean of the sample, 68.2, will be listed to the right of $\bar{x}$ = and the standard deviation of the sample, 2.780887149, will be listed to the right of S_x =.

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the spreadsheet icon, the fourth icon from the left, and press [enter].

Step 3: To clear the lists in your calculator, arrow up to the topmost cell of the table to highlight the entire column, then press [menu]. Use the arrow key to choose 3: Data, then 4: Clear Data, then press [enter]. Repeat for each column as necessary.

Step 4: Arrow to the first cell of the column labeled "A." Enter each of the minutes for Sample A from the table in this column, pressing [enter] after each number to navigate down to the next blank cell.

(*continued*)

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Step 5: Arrow to the first cell of the column labeled "B." Enter the minutes for Sample B from the table in this column, pressing [enter] after each number.

Step 6: To calculate the mean and standard deviation of both data sets, press [menu], arrow down to 4: Statistics, then arrow to 1: Stat Calculations, then 1: One-Variable Statistics. Press [enter]. When prompted, enter 2 for Num of Lists, tab to "OK," and then press [enter].

Step 7: At X1 List, enter A using your keypad to select the data in column A. Tab to the X2 List, then enter B to select the data in column B. Tab to 1st Result Column and enter C. Tab down to "OK" and press [enter] to evaluate the data sets. This will bring you back to the spreadsheet, where column C will be populated with the title of each calculation, and columns D and E will list the values for each data set. Use the arrow key to scroll through the rows of the spreadsheet to find the rows for the sample mean and sample standard deviation. The sample means will be listed to the right of $\overline{x}$, and the sample standard deviations will be listed to the right of "$sx := s_{n-1}x$". For Sample A, the sample mean is 71.5 and the sample standard deviation is 2.173067486. For Sample B, the sample mean is 68.2, and the sample standard deviation is 2.780887149.

Rounded to the nearest tenth, the mean of Sample A is approximately 71.5, and its standard deviation is approximately 2.17.

The mean of Sample B is approximately 68.2, and its standard deviation is approximately 2.78.

4. Explain why the mean and standard deviation of the first sample are different from the mean and standard deviation of the second sample.

   The difference between the means of the two samples and the difference between the standard deviations of the two samples can be attributed to chance variation. These differences are examples of sampling error.

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Instruction

Example 4

The Bennett family believes that they have a special genetic makeup because there are 5 children in the family and all of them are girls. Perform a simulation of 100 families with 5 children. Assume the probability that an individual child is a girl is 50%. Determine the percent of families in which all 5 children are girls. Decide whether having 5 girls in a family of 5 children is probable, somewhat unusual, or highly improbable.

1. Create a simulation using coins.

   Let 5 coins represent each of the 5 children.

   Put all 5 coins into your hands and shake them vigorously.

   Toss the coins into the air and let them land.

   Each coin toss represents 1 family. Let a coin that turns up heads represent a girl and a coin that turns up tails represent a boy.

   In a table, record the number of "girls" for each coin toss. Repeat for a total of 100 coin tosses.

   The sample below depicts the results of 100 coin tosses. Each number indicates the number of girls in that family. This sample is only one possible sample; other samples will be different.

3	2	2	1	2	2	2	2	1	3
2	1	2	1	2	5	3	2	2	3
3	0	1	4	3	4	2	4	2	3
3	3	0	1	2	2	2	2	3	2
4	4	3	4	2	4	1	1	4	3
1	2	1	4	2	2	3	1	3	5
3	4	3	4	1	2	2	3	2	4
5	3	2	2	4	1	1	3	4	2
2	2	1	2	3	3	2	4	3	1
3	3	2	3	3	2	3	3	2	4

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2. Determine the percent of families with all 5 children of the same gender.

   Since the table only records the number of girls, a 0 in the table represents all boys and a 5 represents all girls.

   In the given sample, there are 2 families with all boys and 3 families with all girls; therefore, there are 5 families with all 5 children of the same gender.

   To find the percent, divide the number of families with all 5 children of the same gender by 100, the sample size.

   $$\frac{5}{100} = 0.05 = 5\%$$

   In this sample, 5% of the families have 5 children of the same gender.

3. Determine the percent of families with 5 girls.

   Among the 100 families in the given sample, 3 have all girls.

   To find the percent, divide the number of families with 5 girls by 100, the sample size.

   $$\frac{3}{100} = 0.03 = 3\%$$

   In this sample, 3% of the families have all girls.

4. Interpret your results.

   It is important to note that there is no way to determine with certainty whether the belief that the Bennetts have a special genetic makeup is correct. Based on this sample, we can only determine that in families who have 5 children, there is a 5% chance that all 5 children would be the same gender, and that there is a 3% chance that families with 5 children would have 5 girls.

   The results of the simulation indicate that having 5 girls in a family of 5 children is highly improbable.

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Instruction

Example 5

At the Fowl County Fair, contestants have the opportunity to win prizes for throwing beanbags into the mouth of a large wooden chicken. It costs $2 to play and each contestant gets 3 beanbags to throw. The following table shows the value of each possible prize awarded to a contestant.

Successful beanbag throws	Prize value
0	$0
1	$0
2	$5
3	$25

Assume that there is a 40% chance that a contestant will be successful on any given throw. Use a graphing calculator to simulate 20 games with 3 beanbag tosses in each game. Determine the mean value of the prize won by the sample contestants. According to your simulation, is it worth playing the game?

1. Determine an interval of random numbers that corresponds to a 40% probability of a successful toss.

   Probability can be represented by a decimal greater than or equal to 0 and less than or equal to 1.

   Recall that 40% is equal to 0.40.

   The "rand" (random) function of a calculator generates numbers between 0 and 1.

   Assign a successful outcome (hit) as equivalent to a number less than 0.4 and an unsuccessful outcome (miss) as equivalent to a number greater than 0.4.

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2. Use a graphing calculator to generate 20 random numbers between 0 and 1.

   Follow the directions specific to your model.

   On a TI-83/84:

   Step 1: From the home screen, press [MATH]. Arrow over to the PRB menu, and then press 1:rand.

   Step 2: Press [ENTER] three times to generate three random numbers representing the results of one game (three beanbag tosses).

   Step 3: Repeat this process until 20 games have been simulated. Copy each of the random numbers into a table.

   On a TI-Nspire:

   Step 1: From the home screen, arrow down to the calculator icon, the first icon from the left, and press [enter].

   Step 2: Press [menu]. Use the arrow key to choose 5: Probability, then 4: Random, then 1: Number. Press [enter]. This will bring up a screen with "rand()."

   Step 3: Press [enter] three times to generate three random numbers representing the results of one game (three beanbag tosses).

   Step 4: Repeat this process until 20 games have been simulated. Copy each of the random numbers into a table.

(*continued*)

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The following table lists the possible results of a simulation. Results of other simulations will be different.

Game number	Random number (result)	Random number (result)	Random number (result)	Number of hits	Prize value ($)
1	0.017	0.243	0.486		
2	0.417	0.081	0.254		
3	0.145	0.465	0.695		
4	0.031	0.774	0.084		
5	0.955	0.465	0.398		
6	0.109	0.729	0.539		
7	0.083	0.691	0.935		
8	0.486	0.283	0.624		
9	0.690	0.266	0.593		
10	0.166	0.022	0.999		
11	0.059	0.100	0.227		
12	0.702	0.471	0.331		
13	0.314	0.668	0.598		
14	0.604	0.110	0.102		
15	0.685	0.708	0.503		
16	0.331	0.993	0.325		
17	0.855	0.019	0.385		
18	0.683	0.996	0.435		
19	0.722	0.622	0.997		
20	0.212	0.397	0.523		

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3. Determine the number of hits and enter the value of the prize for each of the 20 games into a list.

   Expand upon the previous table and determine which results are hits and which are misses.

   Recall that a hit is any number less than or equal to 0.4 and a miss is any number greater than 0.4.

Game number	Random number (result)	Random number (result)	Random number (result)	Number of hits	Prize value ($)
1	0.017 (hit)	0.243 (hit)	0.486 (miss)	2	5
2	0.417 (miss)	0.081 (miss)	0.254 (hit)	1	0
3	0.145 (hit)	0.465 (miss)	0.695 (miss)	1	0
4	0.031 (hit)	0.774 (miss)	0.084 (hit)	2	5
5	0.955 (miss)	0.465 (miss)	0.398 (hit)	1	0
6	0.109 (hit)	0.729 (miss)	0.539 (miss)	1	0
7	0.083 (hit)	0.691 (miss)	0.935 (miss)	1	0
8	0.486 (miss)	0.283 (hit)	0.624 (miss)	1	0
9	0.690 (miss)	0.266 (hit)	0.593 (miss)	1	0
10	0.166 (hit)	0.022 (hit)	0.999 (miss)	2	5
11	0.059 (hit)	0.100 (hit)	0.227 (hit)	3	25
12	0.702 (miss)	0.471 (miss)	0.331 (hit)	1	0
13	0.314 (hit)	0.668 (miss)	0.598 (miss)	1	0
14	0.604 (miss)	0.110 (hit)	0.102 (hit)	2	5
15	0.685 (miss)	0.708 (miss)	0.503 (miss)	0	0
16	0.331 (hit)	0.993 (miss)	0.325 (hit)	2	5
17	0.855 (miss)	0.019 (hit)	0.385 (hit)	2	5
18	0.683 (miss)	0.996 (miss)	0.435 (miss)	0	0
19	0.722 (miss)	0.622 (miss)	0.997 (miss)	0	0
20	0.212 (hit)	0.397 (hit)	0.523 (miss)	2	5

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4. Calculate the mean of the prize values.

   Use your graphing calculator to calculate the mean of the prize values. Follow the directions outlined in Example 3 to find the mean for your calculator model. Enter the prize values in the first list (L1) of your calculator.

   The mean prize value of this sample is $3.

5. Compare the mean prize value to the cost of the game to determine if the game is worth playing.

   Mathematically, if the mean prize value is greater than the cost of the game, $2, then the game is worth playing. If the mean prize value is less than the cost of the game, then the game is not worth playing.

   According to this simulation, the game is worth playing because $3 is greater than the cost to play of $2.

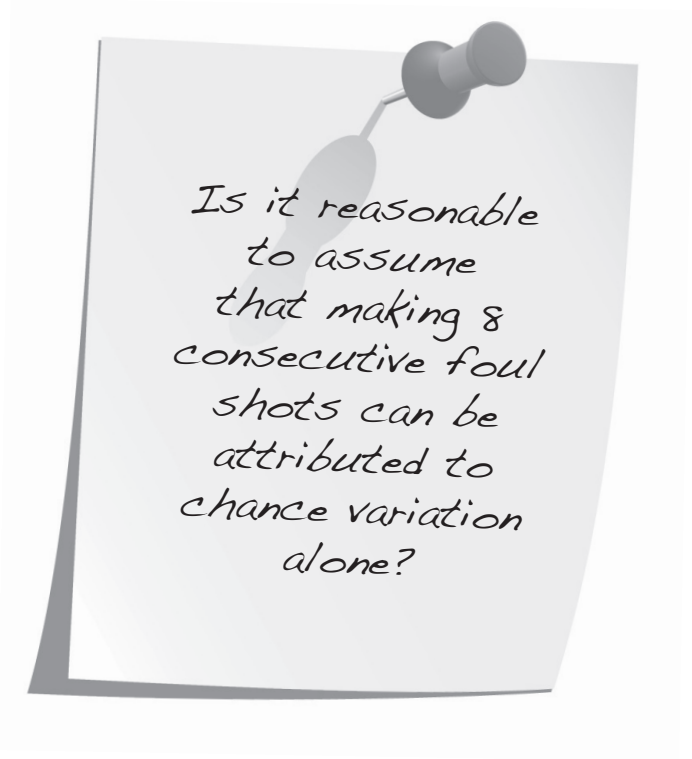
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Problem-Based Task 1.2.2: Chance or Greatness?

SMP	
1	2
3 ✓	4
5 ✓	6 ✓
7	8

During the course of the district basketball championship, Allie sunk 8 consecutive foul shots to lead her team to victory. While leaving the gymnasium, one fan remarked, "Allie has nerves of steel. I don't know if I've ever seen a greater foul-shot performance than that." A second fan had a curious response. "I'm not sure you can call that a *great* performance," he said. "Allie's just a good free-throw shooter. Anyone who makes 80% of their free throws is bound to have a streak of 8 in a row. These just came at the right time."

Is it reasonable to assume that making 8 consecutive foul shots for a player who typically makes 80% of her free throws can be attributed to chance variation alone, or is this performance evidence of other possible factors, such as strength and increased concentration? Run at least 20 simulations of a player shooting 8 foul shots. Assume that each foul shot has an 80% chance of success. Justify your answer based on the results of your simulation.



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Problem-Based Task 1.2.2: Chance or Greatness?

Coaching

a. How can you use a standard deck of 52 cards to simulate a foul shot that has an 80% chance of success?

b. Using this deck of cards, how can you simulate a set of 8 foul shots with the same 80% chance of success?

c. How can you use a graphing calculator to simulate a foul shot that has an 80% chance of success?

d. Using a graphing calculator, how can you simulate a set of 8 foul shots with the same 80% chance of success?

e. Choose either a deck of cards or a graphing calculator to run at least 20 simulations of a player shooting 8 foul shots with an 80% chance of success. Record your results in a table.

f. Determine the number of simulations in which all 8 foul shots are made.

g. Calculate the percent of simulations in which all 8 foul shots are made.

h. Interpret the results using the following guidelines: If 8 foul shots are made in 0% or 5% of the simulations, then it is not reasonable to assume that the streak can be attributed to chance variation alone. If 8 foul shots are made at least 10% of the time, then it is reasonable to assume that the streak could be the result of chance variation alone.

i. What do the results mean in the context of the problem?

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Instruction

Problem-Based Task 1.2.2: Chance or Greatness?

Coaching Sample Responses

a. How can you use a standard deck of 52 cards to simulate a foul shot that has an 80% chance of success?

A standard deck of cards includes 52 cards, 12 of which are face cards (jack, queen, and king for each of the four suits). The remaining 40 cards are considered number cards, ace through 10. To create an appropriate proportion for the simulation, we can remove 2 of the face cards so that 50 total cards remain, leaving 10 face cards in the deck. 40 out of 50 is equal to 80%; therefore, a number card such as ace, 2, 3, etc., could represent a made foul shot, while a jack, queen, or king could represent a missed foul shot.

b. Using this deck of cards, how can you simulate a set of 8 foul shots with the same 80% chance of success?

From the deck of 50 cards, choose 1 card and record the result. Again, a number card (ace through 10) represents a made foul shot while a face card (jack, queen, or king) represents a missed foul shot.

Replace the card and shuffle the deck of 50 cards.

Draw another card and record the result.

Continue this process 6 more times until there are a total of 8 results, each time recording the result.

c. How can you use a graphing calculator to simulate a foul shot that has an 80% chance of success?

Generate a random number to represent a "made" or "missed" foul shot.

An 80% chance of success is equal to $\frac{4}{5}$; therefore, a range of 5 different numbers is enough for this simulation, where four of the numbers each represent a made shot and one number represents a missed shot.

Using the calculator, generate a random integer from 1 to 5.

If the generated number is a 1, 2, 3, or 4, then consider the foul shot made. If the generated number is 5, consider the foul shot missed.

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Instruction

Follow the directions specific to your calculator model.

On a TI-83/84:

Step 1: From the home screen, press [MATH]. Arrow over to the PRB menu, then down to 5:randInt(. Press [ENTER].

Step 2: At randInt(, use the keypad to enter the starting value, 1, and the ending value, 5, separated by a comma and followed by a closing parenthesis. Press [ENTER]. This will generate a random number with a value within the range given.

On a TI-Nspire:

Step 1: From the home screen, arrow down to the calculator icon, the first icon from the left, and press [enter].

Step 2: Press [menu]. Use the arrow key to choose 5: Probability, then 4: Random, then 2: integer. Press [enter]. This will bring up a screen with "randInt()."

Step 3: Inside the parentheses, use the keypad to enter the starting value, 1, and the ending value, 5, separated by a comma. Press [enter]. This will generate a random number with a value within the range given.

d. Using a graphing calculator, how can you simulate a set of 8 foul shots with the same 80% chance of success?

Repeat the calculator directions for generating a random integer between 1 and 5 a total of 8 times to simulate 8 foul shots.

e. Choose either a deck of cards or a graphing calculator to run at least 20 simulations of a player shooting 8 foul shots with an 80% chance of success. Record your results in a table.

Answers will vary. This is a random process, so variation is expected.

A sample simulation follows.

Set	Result	Result	Result	Result	Result	Result	Result	Result
1	missed	missed	made	made	made	made	made	made
2	made	missed	made	made	made	made	made	made
3	made	made	missed	made	made	missed	made	made
4	made	made	made	made	made	made	missed	missed
5	made	made	made	missed	made	made	made	made
6	made	made	made	made	made	made	made	made

(*continued*)

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Set	Result	Result	Result	Result	Result	Result	Result	Result
7	made	made	made	made	made	made	made	made
8	missed	made	made	made	made	made	made	made
9	made	made	made	missed	made	made	missed	missed
10	missed	made	made	made	made	missed	made	made
11	made	missed	made	made	missed	made	made	missed
12	missed	made	made	missed	made	missed	made	missed
13	made	missed	made	made	missed	made	made	made
14	missed	made	made	made	missed	made	missed	missed
15	made	made	made	made	made	made	made	missed
16	made	made	missed	made	made	made	made	made
17	made	made	made	made	made	made	made	made
18	made	made	made	made	made	missed	made	made
19	made	made	missed	made	made	missed	made	made
20	missed	made	made	missed	made	made	made	missed

f. Determine the number of simulations in which all 8 foul shots are made.

Answers will vary. The following table shows sample results for 20 simulations; rows with bold text indicate sets in which all 8 shots were made.

Shots made per set	
Set 1: 6 made shots	Set 11: 5 made shots
Set 2: 7 made shots	Set 12: 4 made shots
Set 3: 6 made shots	Set 13: 6 made shots
Set 4: 6 made shots	Set 14: 4 made shots
Set 5: 7 made shots	Set 15: 7 made shots
Set 6: 8 made shots	Set 16: 7 made shots
Set 7: 8 made shots	**Set 17: 8 made shots**
Set 8: 7 made shots	Set 18: 7 made shots
Set 9: 5 made shots	Set 19: 6 made shots
Set 10: 6 made shots	Set 20: 5 made shots

In this sample, all eight foul shots are made in sets 6, 7, and 17.

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g. Calculate the percent of simulations in which all 8 foul shots are made.

Divide the number of sets in which all 8 foul shots were made (3) by the total number of sets (20). Multiply this result by 100 to find the percent of simulations in which all 8 foul shots were made.

$$\frac{3}{20} \bullet 100$$
$$= 0.15 \bullet 100$$
$$= 15\%$$

Based on our sample data, 15% of the simulations resulted in all 8 foul shots made.

h. Interpret the results using the following guidelines: If 8 foul shots are made in 0% or 5% of the simulations, then it is not reasonable to assume that the streak can be attributed to chance variation alone. If 8 foul shots are made at least 10% of the time, then it is reasonable to assume that the streak could be the result of chance variation alone.

In this particular simulation, all 8 foul shots were made 15% of the time. It is reasonable to assume that the streak could be the result of chance variation alone.

i. What do the results mean in the context of the problem?

If a result can occur 15% of the time by chance variation alone, then it is a reasonable assumption that the result is due to chance variation. This does not mean the assumption is correct, only that it is reasonable. Also, this does not mean that other factors are not involved, only that we don't have strong evidence to conclude whether any other factors are involved.

Based on the results of this simulation, there is a reasonable chance that a player who has an 80% success rate with foul shots would make 8 consecutive free throws at any given time. Thus, while other factors such as strength and increased concentration may be involved in this situation, it is reasonable to assume that Allie would make 8 consecutive free throws regardless.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.2.2: Simple Random Sampling

Jocelyn collected three samples from a standard deck of 52 cards. For each sample, she shuffled the deck thoroughly and then drew the top 20 cards. Jocelyn used the numerical card value system for popular card games as shown below.

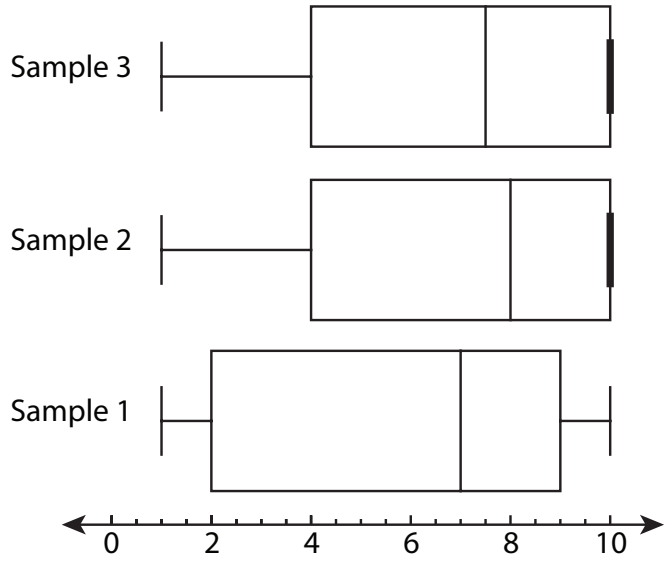
Ace = 1

2 = 2, 3 = 3, etc., through 10 = 10

Jack = 10, queen = 10, king = 10

Jocelyn wants to estimate the mean and standard deviation of the card values in the deck. Box plots and summary statistics for her samples are shown as follows. Use the given information to complete problems 1–4. *Note*: Both the third quartile and the maximum for samples 2 and 3 are equal to 10.

Card values selected from a deck of playing cards



Summary statistics

	Sample 1	Sample 2	Sample 3
Number of cards	20	20	20
Mean	6.2	7.0	6.8
Standard deviation	3.4	3.1	3.1

1. Which of the samples, if any, provide unbiased estimates of the mean card value in a standard deck of 52 cards?

continued

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2. Why are the estimates different if they are all taken from the same deck of cards?

3. Estimate the mean card value in the deck using the information from all three samples.

4. Why is the estimate taken from all three samples more reliable than the estimates taken from the individual samples?

Use the following information to complete problems 5–7.

Ms. Davison is trying to estimate the mean times that the students at Harmony High School spend playing or listening to music every day. Three students in the band also study statistics. Each of the students developed a sampling plan to help Ms. Davison in her research:

- Holly plans to survey all of the 83 students in the band.
- Zach plans to obtain a list of all 857 students at Harmony and randomly select 50 students from the list. He will survey the 50 students.
- Seth randomly selects 6 classes that meet during his third period study hall and plans to survey all the students in the 6 classes.

5. Which of the samples provides the most convenient method of collecting data? Explain your answer.

6. Which of the samples involves the least sampling bias? Explain your answer.

7. How can Seth's plan be improved in order to provide more reliable estimates? Explain your answer.

continued

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Use the following information to complete problems 8–10.

The table below shows the cost of driving 25 miles in several hybrid vehicles built during the 2007 model year.

Car make and model	Cost (in $) per 25 miles driven
Honda Accord	2.60
Honda Civic	1.61
Lexus GS 450h	3.27
Saturn Aura	2.68
Toyota Camry	2.06
Nissan Altima	2.06
Toyota Prius	1.46

Source: U.S. Department of Energy, "Compare Hybrids Side-by-Side."

8. Find the sample of 4 car models with the smallest mean. Find the mean rounded to the nearest hundredth.

9. Find the sample of 4 car models with the greatest standard deviation. Find the standard deviation rounded to the nearest hundredth.

10. Explain how you can select a simple random sample of 4 car models from the 7 models given in the table.

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Lesson 1.2.3: Other Methods of Random Sampling

Warm-Up 1.2.3

A forest manager uses a 10×10 grid of a forest plot to estimate the number of white pine trees and red pine trees on the given plot of land. There are 100 sections in the grid. White pine trees are noted with "w" and red pine trees are noted with "r."

	A	B	C	D	E	F	G	H	I	J
1	wrrr rrrr	www wrrrr	ww rrr	wwww wrrr	wrr	www wrrrr	www wrrrr	www wwrrr	www wwrrr	wrrrrr
2	wwww wrrr	wwww wrrr	www rrrr	wwww wrrr	wwww wrrr	wrrrrrr	www wwrrr	www wwrrr	wrrr rrrr	www wwrrr
3	wrr rrrr	wwww wrrr	wwww wrrr	wwww wrrr	wwwr rrrr	wwww wrrr	wwww wrrr	wrrrrrr	www wrrrr	wwrr rrrr
4	www rrrrr	www rrrrr	wwww wrrr	wwr rrr	wwww rrrr	www wrrrr	wwrr rrrr	www rrrrr	www wwrrr	wwwr rrrr
5	wwww wrrr	www wrrrr	wwwr rrrr	wwrr rrrr	www rrrrr	www wwrrr	www wrrrr	www wwrrr	wrrrr	www wwrrr
6	wwww wrrr	www rrrr	wwww wrrr	wwrr rrrr	www wrrrr	www rrrrr	wwr rrrrr	wrrr rrrr	www wwrrr	www wwrrr
7	www rrrrr	wwrr rrrr	wrrrr	wrrrrr	www wrrrr	www wrrrr	www wwrrr	www rrrrr	wwrr rrrr	wwr rrrr
8	www wrrrr	wrrr rrr	www rrrr	www rrrrr	www wrrrr	www rrrrr	www rrrrr	wrr rrrr	www rrrrr	wrrr rrrr
9	wwrr rrr	wwrr rrrr	wrrrr	wwwr rrrr	wwrrr rrrr	wwww wrrr	www rrrrr	www wrrrr	www wwrrr	wrrrrr
10	ww rrr	wwrr rrrr	wwww wrrr	wwrr rrrr	wwr rrr	wwww wrrr	www wwrrr	www rrrrr	www wwrrr	wwrr rrrr

The forest manager randomly selects the following 10 sections out of the 100 sections of the forest:

B–10 E–3 F–1 C–3 I–9 A–3 H–10 D–9 F–9 B–6

1. Determine the number of white pines and red pines in the sample sections.

2. Estimate the number of white pines and red pines in the forest plot.

3. Explain how a simple random sample could be used in this situation.

4. Why is a simple random sample not practical in this situation?

5. Does the sampling procedure provide a representative sample? Explain your answer.

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Lesson 1.2.3: Other Methods of Random Sampling

Common Core State Standard

S–IC.2 Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?*★

Warm-Up 1.2.3 Debrief

1. Determine the number of white pines and red pines in the sample sections.

   Use the grid to determine the number of white pines and red pines in each sample selection. Count the occurrences of white pine trees, *w*, and red pine trees, *r*, for each section.

   - Section B–10 has 2 white pine trees and 6 red pine trees.
   - Section E–3 has 3 white pine trees and 5 red pine trees.
   - Section F–1 has 4 white pine trees and 4 red pine trees.
   - Section C–3 has 5 white pine trees and 3 red pine trees.
   - Section I–9 has 5 white pine trees and 3 red pine trees.
   - Section A–3 has 1 white pine tree and 6 red pine trees.
   - Section H–10 has 3 white pine trees and 5 red pine trees.
   - Section D–9 has 3 white pine trees and 5 red pine trees.
   - Section F–9 has 5 white pine trees and 3 red pine trees.
   - Section B–6 has 3 white pine trees and 4 red pine trees.

   To find the number of white pine trees and red pine trees in the sample, find the total number of white pine trees and the total number of red pine trees.

   white pine trees = 2 + 3 + 4 + 5 + 5 + 1 + 3 + 3 + 5 + 3 = 34

   The total number of white pine trees is 34.

   red pine trees = 6 + 5 + 4 + 3 + 3 + 6 + 5 + 5 + 3 + 4 = 44

   The total number of red pine trees is 44.

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2. Estimate the number of white pines and red pines in the forest plot.

   Since trees were counted in 10 of the 100 sections, we can use proportions to estimate the number of trees in the entire plot.

   Set up a proportion to estimate the total number of white pines for the entire plot.

   Use the number of sections sampled (10) for the sample size, and the number of sections in the entire forest plot (100) for the population size.

   Create a proportion for white pines, as shown; then, solve it for the unknown variable.

   $$\frac{\text{number of white pines in sample}}{\text{sample size}} = \frac{\text{number of white pines in total population}}{\text{population size}}$$

   $\frac{(34)}{(10)} = \frac{w}{(100)}$ Substitute known values.

   $10w = 3400$ Cross-multiply to solve the proportion.

   $w = 340$ Divide both sides by 10.

   Based on the sample, there are estimated to be 340 white pines in the forest plot.

   Repeat the process to estimate the number of red pines.

   Create a proportion for red pines, as shown; then, solve it for the unknown variable.

   $$\frac{\text{number of red pines in sample}}{\text{sample size}} = \frac{\text{number of red pines in total population}}{\text{population size}}$$

   $\frac{(44)}{(10)} = \frac{r}{(100)}$ Substitute known values.

   $10r = 4400$ Cross-multiply to solve the proportion.

   $r = 440$ Divide both sides by 10.

   Based on the sample, there are estimated to be 440 red pines in the forest plot.

3. Explain how a simple random sample could be used in this situation.

   To use a simple random sample in this situation, every tree would need to be labeled with a number, and then random integers could be generated to correspond to the trees that are selected in the sample.

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4. Why is a simple random sample not practical in this situation?

   It would be extremely time-consuming to label every tree. Also, if the goal is to estimate the number of trees, then the time it takes to label them is more than the time it would take to count them.

5. Does the sampling procedure provide a representative sample? Explain your answer.

   Since each of the 100 sections has an equal chance of selection, the sample is representative. It is not a simple random sample and the estimate of the number of trees will contain sampling error, but the sample is not biased.

Connection to the Lesson

- Students begin to understand why choosing a simple random sample is not always the best method of random sampling.
- Students will expand upon their understanding of random sampling to learn about other methods.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- identifying sources of sampling bias (S–IC.3)
- calculating means, standard deviations, and proportions from samples and populations (S–SP.5c)
- using a standard deck of 52 cards or a graphing calculator to generate random numbers (no standard)

Introduction

Previous lessons focused on the relationship between samples and populations, and on using random sampling to select a representative sample and reduce sampling bias. This lesson introduces the idea that simple random sampling is not the only method for selecting representative samples, and that the sampling method used often depends on the goal of the research being conducted as well as practical considerations.

Different sampling methods can be helpful tools for a wide variety of research situations. Furthermore, familiarity with these methods allows you to understand the methods used by other researchers who often need to mix and match methods in order to meet practical challenges without compromising the representative nature of their samples.

Key Concepts

- Additional sampling methods include cluster sampling, systematic sampling, and stratified sampling.
- All of these methods involve random assignment, although none meet the criteria of simple random sampling.
- With a **cluster sample**, naturally occurring groups of population members are chosen for the sample. This method involves dividing the population into groups by geography or other practical criteria. Some of the groups are randomly selected, while others are not. This method allows each member of the population to have a nearly equal chance of selection. Cluster sampling is usually chosen to eliminate excessive travel or reduce the disruption that a study may cause.
- A **systematic sample** is a sample drawn by selecting people or objects from a list, chart, or grouping at a uniform interval. This method involves using a natural ordering of population members, such as by arrival time, location, or placement on a list. Once the order is established, every *n*th member (e.g., every fifth member) is chosen. If the starting number is randomly selected, then each member of the population has a nearly equal chance of selection. Systematic sampling is usually chosen when relative position in a list may be related to key variables in a study, or when it is useful to a researcher to space out data gathering.

SRB p. U1-102

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Instruction

- For a **stratified sample**, the population is divided into subgroups so that the people or objects within the subgroup share relevant characteristics. This method involves grouping members of the population by characteristics that may be related to parameters of interest. Once the groups are formed, members of each group are randomly selected so that the number of members in the sample with given characteristics is approximately proportional to the number of members in the population with the same characteristics. Stratified sampling has been used for many years to predict the results of state and national elections.
- A **convenience sample** is a sample for which members are chosen in order to minimize time, effort, or expense. Convenience sampling involves gathering data quickly and easily. The advantage of convenience sampling is that, in some cases, preliminary estimates of population parameters can be obtained quickly. The main disadvantage of convenience sampling is that the samples are prone to serious biases. As a result, the estimates obtained are seldom accurate and the statistics are difficult to interpret.
- While simple random samples provide unbiased estimates, there are situations in which the goal of the research is better served by other forms of sampling. These include situations in which the goal is to count all members of a population and situations in which the sample provides a comparison group.
- It is unwise to use a sampling method simply because it is the most convenient. Unless the sample is representative of the population of interest, the statistics that are produced may be misleading.
- A larger sample is not always a better sample. There is less variability in measures taken from a large sample, but if the large sample is biased, the researcher will likely obtain estimates that are inaccurate.

Common Errors/Misconceptions

- mistakenly believing that a larger sample is always a better sample
- ignoring bias when making estimates regarding the entire population

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Guided Practice 1.2.3

Example 1

The following table lists the 30 movies that earned the most money in United States theaters in 2012. Use the table to obtain a systematic sample of 10 movies.

Rank	Title	Total earned in millions ($)	Rank	Title	Total earned in millions ($)
1	*Marvel's The Avengers*	623	16	*Ice Age: Continental Drift*	161
2	*The Dark Knight Rises*	448	17	*Snow White and the Huntsman*	155
3	*The Hunger Games*	408	18	*Les Misérables* (2012 version)	149
4	*Skyfall*	304	19	*Hotel Transylvania*	148
5	*The Hobbit: An Unexpected Journey*	303	20	*Taken 2*	140
6	*The Twilight Saga: Breaking Dawn Part 2*	292	21	*21 Jump Street*	138
7	*The Amazing Spider-Man*	262	22	*Argo*	136
8	*Brave*	237	23	*Silver Linings Playbook*	132
9	*Ted*	219	24	*Prometheus*	126
10	*Madagascar 3: Europe's Most Wanted*	216	25	*Safe House*	126
11	*Dr. Seuss's The Lorax*	214	26	*The Vow*	125
12	*Wreck-It Ralph*	189	27	*Life of Pi*	125
13	*Lincoln*	182	28	*Magic Mike*	114
14	*Men in Black 3*	179	29	*The Bourne Legacy*	113
15	*Django Unchained*	163	30	*Journey 2: The Mysterious Island*	104

Source: Box Office Mojo, "2012 Domestic Grosses."

SWB p. 75; SRB p. U1-104

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Instruction

1. Determine the increment between movies.

   To determine the increment between movies, divide the number of movies in the population by the number of movies required for the sample.

   The number of movies in the population is 30, and we are asked to create a systematic sample of 10 movies.

   $$\frac{30}{10} = 3$$

   The increment between movies is 3.

2. Determine the number of the first sample movie from its position in the list.

   Since we are choosing every third movie, we can start with either the first movie in the list, the second movie, or the third movie. Since these movies are already ranked, we can randomly select one of the top 3 movies as the first sample element.

   We can randomly choose a 1, 2, or 3 by shuffling 3 playing cards (ace, 2, or 3) or by using a random number generator on a graphing calculator.

   Suppose the randomly selected number is 3.

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3. Begin with the first movie selected and choose every third movie after that.

   We randomly determined the starting number to be 3. The third movie on the list is *The Hunger Games*.

   We determined the increment to be 3 as well.

   Referring to the list, we can see that the third movie after *The Hunger Games* is *The Twilight Saga: Breaking Dawn Part 2*.

   Continuing in this manner, we can generate the following systematic sample of 10 movies.

Rank	Title
3	*The Hunger Games*
6	*The Twilight Saga: Breaking Dawn Part 2*
9	*Ted*
12	*Wreck-It Ralph*
15	*Django Unchained*
18	*Les Misérables*
21	*21 Jump Street*
24	*Prometheus*
27	*Life of Pi*
30	*Journey 2: The Mysterious Island*



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Example 2

Pearce wants to conduct a survey of shoppers at the local mall. He obtains a list of the major stores, restaurants, and other establishments and creates the following table that includes each destination's name, location (zone), category, and category rank. The category rank represents where the mall destination falls in a list of all the establishments in the same category; for example, Aéropostale is second in the list of clothing stores, so its category rank is 2.

Use the table and two methods to choose a cluster sample of 5 establishments at which Pearce can interview shoppers.

- Method 1: Give each zone an equal chance of selection.
- Method 2: Give each establishment an equal chance of selection.

Establishment	Zone	Category	Category rank
Abercrombie & Fitch	D	Clothing	1
Aéropostale	D	Clothing	2
Amato's	A	Food	1
American Eagle	B	Clothing	3
Arby's	A	Food	2
AT&T	C	Technology/electronics	1
babyGap	D	Clothing	4
Banana Republic	E	Clothing	5
Barton's Couture	D	Clothing	6
Bath & Body Works	B	Bath/beauty	1
The Body Shop	D	Bath/beauty	2
Build-A-Bear Workshop	B	Toys/hobbies	1
Bureau of Motor Vehicles	D	Services	2
Charley's Subs	A	Food	3
Chico's	D	Clothing	7
The Children's Place	B	Clothing	8
Claire's	A	Accessories	1
Coach	B	Accessories	2
Coldwater Creek	C	Clothing	9
dELiA*s	B	Clothing	10
Dube Travel	A	Services	1
Eddie Bauer	D	Clothing	11
Express	D	Clothing	12

(*continued*)

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Establishment	Zone	Category	Category rank
f.y.e.	A	Technology/electronics	2
Foot Locker	B	Clothing	13
Francesca's	B	Clothing	14
G.M. Pollack & Sons	C	Jewelry	4
GameStop	A	Toys/hobbies	2
Gap	D	Clothing	15
Gloria Jean's Coffee	C	Food	4
Go Games	C	Toys/hobbies	3
Gymboree	E	Clothing	16
Hannoush Jewelers	A	Jewelry	1
Hometown Buffet	C	Food	5
Hot Topic	A	Clothing	17
Icing by Claire's	A	Accessories	3
J.Crew	E	Clothing	18
J.Jill	B	Clothing	19
Johnny Rockets	A	Food	6
Just Puzzles	B	Toys/hobbies	4
Kamasouptra	A	Food	7
Kay Jewelers	D	Jewelry	2
La Biotique	A	Bath/beauty	3
Lane Bryant	D	Clothing	20
LensCrafters	A	Services	3
Lids	A	Accessories	4
LOFT	D	Clothing	21
LUSH	E	Bath/beauty	4
MasterCuts	A	Bath/beauty	5
Mayflower Massage	A	Services	4
Mrs. Field's Cookies	A	Food	8
Olympia Sports	D	Toys/hobbies	5
On Time	A	Accessories	5
Origins	B	Bath/beauty	6
PacSun	A	Clothing	22
Panda Express	A	Food	9
The Picture People	A	Services	5

(*continued*)

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Establishment	Zone	Category	Category rank
Piercing Pagoda	E	Jewelry	3
Pretzel Time/TCBY	C	Food	10
Pro Vision	A	Services	6
Qdoba	A	Food	11
Radio Shack	C	Technology/electronics	3
Red Mango	A	Food	12
Regis Salon	A	Bath/beauty	7
Sarku Japan	A	Food	13
Sbarro	A	Food	14
Sephora	E	Bath/beauty	8
Starbucks	A	Food	15
Sunglass Hut	B	Accessories	6
Super Hearing Aids	A	Services	7
Swarovski	D	Jewelry	5
T & C Nails	A	Bath/beauty	9
T-Mobile	B	Technology/electronics	4
Teavana	D	Food	16
Verizon Wireless	A	Technology/electronics	5

Method 1: Give each zone an equal chance of selection.

1. Number the zones.

   The mall is divided into 5 zones, so assign each zone a number 1 through 5.

   Let A = 1, B = 2, C = 3, D = 4, and E = 5.

2. Select a zone of the mall.

   Randomly select 1 of the 5 zones using 5 cards from a standard deck or a random number generator.

   Suppose that a 4 is chosen. This corresponds to Zone D.

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3. Label the businesses in the chosen zone.

   There are 16 establishments in Zone D, so label each one with a number from 1 to 16.

   1 = Abercrombie & Fitch
   2 = Aéropostale
   3 = babyGap
   4 = Barton's Couture
   5 = The Body Shop
   6 = Bureau of Motor Vehicles
   7 = Chico's
   8 = Eddie Bauer
   9 = Express
   10 = Gap
   11 = Kay Jewelers
   12 = Lane Bryant
   13 = LOFT
   14 = Olympia Sports
   15 = Swarovski
   16 = Teavana

4. Randomly select 5 of the establishments in the selected zone.

   Using 16 cards or a random number generator, randomly select 5 establishments from Zone D. Discard repeats.

   Results will vary, but suppose the numbers 1, 4, 7, 8, and 12 are randomly chosen.

   These numbers correspond to the following establishments:

   1 = Abercrombie & Fitch
   4 = Barton's Couture
   7 = Chico's
   8 = Eddie Bauer
   12 = Lane Bryant

   The corresponding cluster sample of 5 establishments at which Pearce can interview shoppers consists of Abercrombie & Fitch, Barton's Couture, Chico's, Eddie Bauer, and Lane Bryant.

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Method 2: Give each establishment an equal chance of selection.

1. Label each establishment.

   There are 75 establishments, so label each of them with a number from 1 to 75.

2. Randomly select a number from 1 to 75.

   Randomly select one of the 75 establishments using 75 cards or a random number generator.

   Suppose a 10 is chosen. This corresponds to Barton's Couture.

3. Since this is a cluster sample, choose 4 other establishments in the same zone.

   Barton's Couture is in Zone D.

   There are 16 establishments in Zone D, so label each one with a number from 1 to 16.

1 = Abercrombie & Fitch	9 = Express
2 = Aéropostale	10 = Gap
3 = babyGap	11 = Kay Jewelers
4 = Barton's Couture	12 = Lane Bryant
5 = The Body Shop	13 = LOFT
6 = Bureau of Motor Vehicles	14 = Olympia Sports
7 = Chico's	15 = Swarovski
8 = Eddie Bauer	16 = Teavana

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4. Randomly select 4 other establishments in Zone D.

   Using 16 cards or a random number generator, randomly select 4 additional establishments from Zone D. Discard repeats.

   Results will vary, but suppose the numbers 1, 9, 13, and 15 are randomly chosen.

   These numbers correspond to the following stores:

   1 = Abercrombie & Fitch

   9 = Express

   13 = LOFT

   15 = Swarovski

   The corresponding cluster sample of 5 establishments at which Pearce can interview shoppers consists of Barton's Couture, Abercrombie & Fitch, Express, LOFT, and Swarovski.

   Note: Method 1 will probably be more convenient because the smaller zones (Zone C and Zone E) have an equal chance of selection. Since small zones have fewer establishments, the establishments in a small zone will probably be closer together, on average, than the establishments in a large zone, making it easier on Pearce to conduct his survey. Using Method 2 means that the establishments in smaller zones have less chance of being selected.



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Example 3

Kylie wants to estimate the total number of times customers enter different establishments at the same mall described in Example 2. Kylie has 10 electronic devices that can count the number of customers entering a given establishment. Use the tables provided in Example 2 to select a stratified sample (by category) of 10 establishments at which Kylie can install her counting devices.

1. Construct a table that shows the number of establishments in each category.

   Refer to the table in Example 2 to determine the number of establishments in each category. Organize the results in a new table.

Category	Number of establishments
Clothing	22
Food	16
Bath/beauty	9
Services	7
Accessories	6
Jewelry	5
Technology/electronics	5
Toys/hobbies	5
Total	75

2. Determine the number of establishments to select from each category.

   Since Kylie needs to select 10 establishments from only 8 categories, select 2 establishments from the largest 2 categories, and 1 from each remaining category. Two stores each from the Clothing and Food categories will be selected, since these are the largest categories.

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3. Organize the list of establishments by category, then number each item within a category.

   Create tables to organize the 8 categories of establishments.

   Number the stores from 1 to *n*, where *n* is the number ranking of a particular establishment in a list of all the members of the same category. For example, babyGap is fourth in the list of clothing stores, so its value for *n* is 4.

Clothing	
Name	***n***
Abercrombie & Fitch	1
Aéropostale	2
American Eagle	3
babyGap	4
Banana Republic	5
Barton's Couture	6
Chico's	7
The Children's Place	8
Coldwater Creek	9
dELiA*s	10
Eddie Bauer	11
Express	12
Foot Locker	13
Francesca's	14
Gap	15
Gymboree	16
Hot Topic	17
J.Crew	18
J.Jill	19
Lane Bryant	20
LOFT	21
PacSun	22

Food	
Name	***n***
Amato's	1
Arby's	2
Charley's Subs	3
Gloria Jean's Coffee	4
Hometown Buffet	5
Johnny Rockets	6
Kamasouptra	7
Mrs. Field's Cookies	8
Panda Express	9
Pretzel Time/TCBY	10
Qdoba	11
Red Mango	12
Sarku Japan	13
Sbarro	14
Starbucks	15
Teavana	16

(*continued*)

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Bath/beauty	
Name	***n***
Bath & Body Works	1
The Body Shop	2
La Biotique	3
LUSH	4
MasterCuts	5
Origins	6
Regis Salon	7
Sephora	8
T & C Nails	9

Services	
Name	***n***
Dube Travel	1
Bureau of Motor Vehicles	2
LensCrafters	3
Mayflower Massage	4
The Picture People	5
Pro Vision	6
Super Hearing Aids	7

Accessories	
Name	***n***
Claire's	1
Coach	2
Icing by Claire's	3
Lids	4
On Time	5
Sunglass Hut	6

Jewelry	
Name	***n***
Hannoush Jewelers	1
Kay Jewelers	2
Piercing Pagoda	3
G.M. Pollack & Sons	4
Swarovski	5

Technology/electronics	
Name	***n***
AT&T	1
f.y.e.	2
Radio Shack	3
T-Mobile	4
Verizon Wireless	5

Toys/hobbies	
Name	***n***
Build-A-Bear Workshop	1
GameStop	2
Go Games	3
Just Puzzles	4
Olympia Sports	5

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4. Randomly select the appropriate number of stores in each category.

   Using cards or a random number generator, randomly select 2 of the 22 clothing stores, 2 of the 16 food stores, 1 of the 9 bath/beauty stores, 1 of the 7 service stores, 1 of the 6 accessories stores, 1 of the 5 jewelry stores, 1 of the 5 technology/electronics stores, and 1 of the 5 toys/hobbies stores.

   Results will vary, but suppose the following numbers were selected:

   - Clothing: The random integers 12 and 9 were selected.
   - Food: The random integers 9 and 16 were selected.
   - Bath/beauty: The random integer 5 was selected.
   - Services: The random integer 1 was selected.
   - Accessories: The random integer 5 was selected.
   - Jewelry: The random integer 5 was selected.
   - Technology/electronics: The random integer 5 was selected.
   - Toys/hobbies: The random integer 5 was selected.

5. Match each random number with the establishment that falls in that position in the category list.

   From our tables, we can use the randomly generated numbers to select a stratified sample.

   The following stores represent the stratified sample.

   - Clothing: 9 = Coldwater Creek and 12 = Express
   - Food: 9 = Panda Express and 16 = Teavana
   - Bath/beauty: 5 = MasterCuts
   - Services: 1 = Dube Travel
   - Accessories: 5 = On Time
   - Jewelry: 5 = Swarovski
   - Technology/electronics: 5 = Verizon Wireless
   - Toys/hobbies: 5 = Olympia Sports

   Note: It is possible with a simple random sample that one or more of the categories will be left out if 10 stores are selected using simple random sampling. By using stratified sampling, each category is represented.

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Problem-Based Task 1.2.3: Breakfast and Grades

SMP	
1 ✓	2
3	4 ✓
5	6
7	8 ✓

School officials are evaluating a new program that provides a free nutritious breakfast to high school students. Researchers randomly selected 60 students to receive a free breakfast from the 280 students who applied for the program. Now, the researchers want to select 60 students from the 220 applicants who were *not* chosen to receive free breakfast to use as a comparison group. At the end of the program, they will compare the academic performance of students in the two groups.

Does receiving a free nutritious breakfast help a student learn? Use the following tables to guide your response. Table 1 shows the average academic grades and genders of students receiving free breakfast. Table 2 shows the average academic grades and genders of students not receiving free breakfast. Table 3 shows the students not receiving free breakfast, numbered and organized by gender and academic grade.

Table 1: Students Receiving Free Breakfast

Academic average	Female	Male	Total
A	3	0	3
B	19	8	27
C	17	8	25
D	1	4	5
Total	40	20	60

Table 2: Students *Not* Receiving Free Breakfast

Academic average	Female	Male	Total
A	13	7	20
B	61	32	93
C	37	49	86
D	2	19	21
Total	113	107	220

continued

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Table 3: Number, Gender, and Academic Average for Students *Not* Receiving Free Breakfast

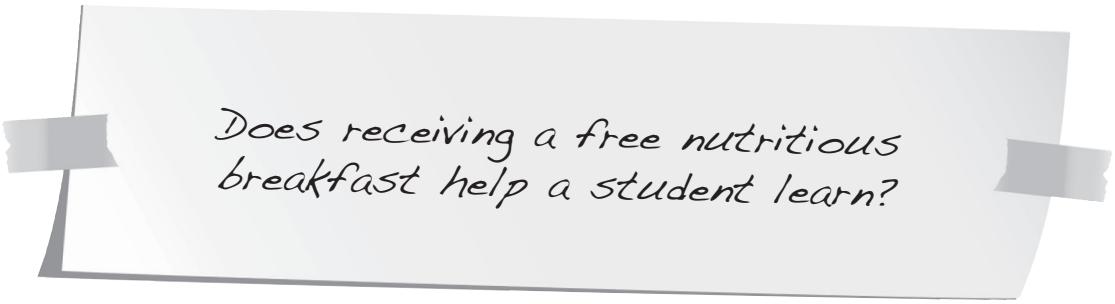
#	M/F	Grade	#	M/F	Grade	#	M/F	Grade	#	M/F	Grade
1	F	A	33	F	B	65	F	B	97	F	C
2	F	A	34	F	B	66	F	B	98	F	C
3	F	A	35	F	B	67	F	B	99	F	C
4	F	A	36	F	B	68	F	B	100	F	C
5	F	A	37	F	B	69	F	B	101	F	C
6	F	A	38	F	B	70	F	B	102	F	C
7	F	A	39	F	B	71	F	B	103	F	C
8	F	A	40	F	B	72	F	B	104	F	C
9	F	A	41	F	B	73	F	B	105	F	C
10	F	A	42	F	B	74	F	B	106	F	C
11	F	A	43	F	B	75	F	C	107	F	C
12	F	A	44	F	B	76	F	C	108	F	C
13	F	A	45	F	B	77	F	C	109	F	C
14	F	B	46	F	B	78	F	C	110	F	C
15	F	B	47	F	B	79	F	C	111	F	C
16	F	B	48	F	B	80	F	C	112	F	D
17	F	B	49	F	B	81	F	C	113	F	D
18	F	B	50	F	B	82	F	C	114	M	A
19	F	B	51	F	B	83	F	C	115	M	A
20	F	B	52	F	B	84	F	C	116	M	A
21	F	B	53	F	B	85	F	C	117	M	A
22	F	B	54	F	B	86	F	C	118	M	A
23	F	B	55	F	B	87	F	C	119	M	A
24	F	B	56	F	B	88	F	C	120	M	A
25	F	B	57	F	B	89	F	C	121	M	A
26	F	B	58	F	B	90	F	C	122	M	A
27	F	B	59	F	B	91	F	C	123	M	B
28	F	B	60	F	B	92	F	C	124	M	B
29	F	B	61	F	B	93	F	C	125	M	B
30	F	B	62	F	B	94	F	C	126	M	B
31	F	B	63	F	B	95	F	C	127	M	B
32	F	B	64	F	B	96	F	C	128	M	B

(*continued*)

continued

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#	M/F	Grade	#	M/F	Grade	#	M/F	Grade	#	M/F	Grade
129	M	B	152	M	B	175	M	C	198	M	C
130	M	B	153	M	B	176	M	C	199	M	C
131	M	B	154	M	B	177	M	C	200	M	C
132	M	B	155	M	C	178	M	C	201	M	C
133	M	B	156	M	C	179	M	C	202	M	D
134	M	B	157	M	C	180	M	C	203	M	D
135	M	B	158	M	C	181	M	C	204	M	D
136	M	B	159	M	C	182	M	C	205	M	D
137	M	B	160	M	C	183	M	C	206	M	D
138	M	B	161	M	C	184	M	C	207	M	D
139	M	B	162	M	C	185	M	C	208	M	D
140	M	B	163	M	C	186	M	C	209	M	D
141	M	B	164	M	C	187	M	C	210	M	D
142	M	B	165	M	C	188	M	C	211	M	D
143	M	B	166	M	C	189	M	C	212	M	D
144	M	B	167	M	C	190	M	C	213	M	D
145	M	B	168	M	C	191	M	C	214	M	D
146	M	B	169	M	C	192	M	C	215	M	D
147	M	B	170	M	C	193	M	C	216	M	D
148	M	B	171	M	C	194	M	C	217	M	D
149	M	B	172	M	C	195	M	C	218	M	D
150	M	B	173	M	C	196	M	C	219	M	D
151	M	B	174	M	C	197	M	C	220	M	D



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Problem-Based Task 1.2.3: Breakfast and Grades

Coaching

a. How many female students with an A average should be chosen for the comparison group?

b. How can these students be selected so that each of the female students with an A average has an equal chance of being chosen for the comparison group?

c. How many female students with a B average should be chosen for the comparison group?

d. How can these students be selected so that each of the female students with a B average has an equal chance of being chosen for the comparison group?

e. Is the chance of a girl with an A average being chosen for the comparison group the same as the chance of a boy with an A average being chosen?

f. Is it important that each of the 220 members of the group that doesn't receive free breakfast has an equal chance of selection?

g. How could you ensure that the proportion of students with each combination of gender and grade is the same for both groups?

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Problem-Based Task 1.2.3: Breakfast and Grades

Coaching Sample Responses

a. How many female students with an A average should be chosen for the comparison group?

Since there are 3 female students with an A average in the study group, there should also be 3 female students with an A average in the comparison group.

b. How can these students be selected so that each of the female students with an A average has an equal chance of being chosen for the comparison group?

Since the students are already numbered, random assignment can be performed by selecting 13 cards, assigning a card value to each of the 13 female students with an A average, shuffling the deck, and drawing 3 cards.

The students could also be selected using a random integer generator to select 3 random integers from 1 to 13, ignoring duplicates.

c. How many female students with a B average should be chosen for the comparison group?

Since there are 19 female students with a B average in the study group, there should also be 19 female students with a B average in the comparison group.

d. How can these students be selected so that each of the female students with a B average has an equal chance of being chosen for the comparison group?

The students could be selected using a random number generator to select 19 random integers from 1 to 61, ignoring duplicates.

e. Is the chance of a girl with an A average being chosen for the comparison group the same as the chance of a boy with an A average being chosen?

No. This sampling technique is not designed to give each member of the population an equal chance of selection. In this case, a female student with an A average has a $\frac{3}{13} \approx 23.1\%$ chance of selection, while a male student with an A average has a $\frac{0}{7} = 0\%$ chance of selection.

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f. Is it important that each of the 220 members of the group that doesn't receive free breakfast has an equal chance of selection?

No. The goal here is to compare the academic achievement of the group that receives free breakfast with the academic achievement of a control group. If the goal were to estimate the academic achievement of 280 members, then a simple random sample would be appropriate. This is a case in which a stratified sample provides better information than a simple random sample.

g. How could you ensure that the proportion of students with each combination of gender and grade is the same for both groups?

Match the numbers for each combination in the group receiving free breakfast when selecting the control group. In other words, continue the procedure outlined in parts b and d with all combinations of gender and grade. As long as there are enough students with each gender and grade combination available, the researcher can match the numbers exactly.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.2.3: Other Methods of Random Sampling

For problems 1–4, identify which type of sampling is used: simple random, cluster, systematic, stratified, or convenience. It is possible that more than one type of sampling is used.

1. George wants to estimate the amount of credit card debt among graduating seniors at his college. George interviews seniors who visit the school store during his lunch break between classes.

2. Ms. L'Heureux wants to collect baseline data for writing before her high school begins a new writing program. Each student provides a timed writing sample. Ms. L'Heureux then randomly selects 20 samples from each grade to score with the school-wide writing rubric.

3. A television station wants to predict the results of a referendum on legalized gambling. The television station randomly selects 8 precincts and conducts exit polling of all voters at each of the selected precincts.

4. Melanie wants to study the changes in stock prices of companies in the S&P 500, a group of 500 stocks chosen because they represent the U.S. economy. She numbers the companies 1 to 500, obtains a random number from 1 to 20 on a graphing calculator (in this case, 18) and then selects every twentieth company starting at 18 (18, 38, 58, ..., 498) to include in her sample.

continued

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The following table contains the number of wins for Major League Baseball teams during the 2012 season. Use the table to select each type of sample requested in problems 5–7. Explain how you selected the teams for each sample.

Team	Wins	Team	Wins
National League East		**American League East**	
Washington Nationals	98	New York Yankees	95
Atlanta Braves	94	Baltimore Orioles	93
Philadelphia Phillies	81	Tampa Bay Rays	90
New York Mets	74	Toronto Blue Jays	73
Miami Marlins	69	Boston Red Sox	69
National League Central		**American League Central**	
Cincinnati Reds	97	Detroit Tigers	88
St. Louis Cardinals	88	Chicago White Sox	85
Milwaukee Brewers	83	Kansas City Royals	72
Pittsburgh Pirates	79	Cleveland Indians	68
Chicago Cubs	61	Minnesota Twins	66
Houston Astros	55		
National League West		**American League West**	
San Francisco Giants	94	Oakland Athletics	94
Los Angeles Dodgers	86	Texas Rangers	93
Arizona Diamondbacks	81	Los Angeles Angels	89
San Diego Padres	76	Seattle Mariners	75
Colorado Rockies	64		

Source: MLB.com, "MLB Standings—2012."

5. a simple random sample with 10 teams

6. a systematic sample with 10 teams

7. a cluster sample with at least 14 teams

continued

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The following table depicts the selling prices of 3-bedroom homes in thousands of dollars for 6 real-estate companies. Use the table to select each type of sample named in problems 8–10. Explain how you chose each sample. *Note*: Some companies sold fewer homes.

Selling Prices for 3-Bedroom Homes in Thousands of Dollars ($)

Listing	Bulldog Realty	Gator Realty	Longhorn Realty	Bruin Realty	Badger Realty	Cornhusker Realty
1	149	130	128	100	190	155
2	150	174	165	159	199	180
3	160	180	210	170	200	183
4	169	195	239	175	219	198
5	180	200	274	175	219	245
6	180	200	399	179	225	270
7	185	210	449	199	350	274
8	190	240	540	235	698	489
9	239	255	—	289	—	—
10	248	260	—	550	—	—
11	259	270	—	598	—	—
12	—	280	—	649	—	—
13	—	375	—	—	—	—

8. a random sample of 20 homes

9. a systematic sample of 20 homes

10. a cluster sample of at least 20 homes

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Progress Assessment

Circle the letter of the best answer.

1. When are samples needed?
   a. when an issue is too important to gather data from an entire population
   b. when gathering data from an entire population is impractical
   c. when an entire population is biased
   d. when a population is unreliable

2. Why are simple random samples desirable?
   a. They are more biased than other types of samples.
   b. They are more reliable than other types of samples.
   c. They are simpler to collect than other types of samples.
   d. For any sample size, every combination of members of the sample has an equal chance of selection.

3. Kenneth chose a simple random sample of 40 students from the population of students from his high school. Kathryn chose a simple random sample of 20 students from the same population. The mean height of the students in Kenneth's sample was 68.2 inches. The mean height of the students in Kathryn's sample was 66.8 inches. Which of the following best explains the difference in sample means?
   a. The differences between sample means can be attributed to chance variation.
   b. One of the samples is probably biased.
   c. Kenneth is a boy and boys are taller on average than girls.
   d. Kenneth has a larger sample, so we would expect the mean height of students in Kenneth's sample to be greater.

continued

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Assessment

4. Shelby chose a simple random sample of 40 students from the population of students taking physical education. Shane chose a simple random sample of 20 students from the same population. Shelby and Shane then determined the mean grade point average of the students in each sample. Which of the following is true about the two sample means?

   a. The mean of Shelby's sample is a more reliable estimate of the population mean.

   b. The mean of Shane's sample is a more reliable estimate of the population mean.

   c. There is less bias in Shelby's estimate.

   d. There is less bias in Shane's estimate.

5. Rachel tossed 5 coins in the air and 4 of the coins landed tails-up. Which statement best describes this result?

   a. This result is not possible to recreate.

   b. This result is somewhat unusual on a given toss.

   c. This result is going to happen on approximately 50% of the tosses.

   d. This result is evidence that the coins are abnormal.

6. Grace rolled 5 six-sided dice and 4 of the dice landed on the number 3. Which statement best describes this result?

   a. This result is highly improbable on a given roll.

   b. This result is somewhat unusual on a given roll.

   c. This result is going to happen on approximately $\frac{1}{6}$ of the rolls.

   d. This result is evidence that the dice are abnormal.

continued

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Assessment

7. Nate wants to estimate the number of times that the main character's name appears in the book he just finished reading. Nate randomly selects 20 of the book's 365 total pages and counts the number of times that the character's name appears on each page. What type of sample does this represent?

   a. a cluster sample
   b. a simple random sample
   c. a stratified sample
   d. a systematic sample

8. Mrs. Waters, a school librarian, would like to estimate the percent of students who are in the school library at any given time of day. Mrs. Waters will count the number of students in the library at 7:45 A.M., 8:15 A.M., 8:45 A.M., and so on until 2:15 P.M. What type of sample does this represent?

   a. a cluster sample
   b. a simple random sample
   c. a stratified sample
   d. a systematic sample

9. The population of 9,300 registered voters in Grapefruit County is 38% Democrat, 36% Republican, and 26% Independent. How many Independent voters should be chosen for a stratified sample of 150 voters?

   a. 26
   b. 39
   c. 65
   d. 75

10. Suppose a simple random sample of 150 voters is chosen from the population of voters from problem 9. Which of the following is most improbable?

    a. Less than 30% of the voters in the sample are Democrat.
    b. Less than 30% of the voters in the sample are Republican.
    c. Less than 30% of the voters in the sample are Independent.
    d. All of the above are equally improbable.

continued

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Assessment

Use what you have learned about populations and random sampling to complete all parts of the following problem.

11. The following table shows the top-selling new cars and trucks in the United States during the first six months of 2013, with the number of vehicles sold for each model listed in thousands. A magazine editor wants to randomly select 8 vehicles from the list to review for the next month's issue. *Note*: The figures in the table are rounded to the nearest thousand.

Rank	Make and model	Vehicles sold in thousands	Rank	Make and model	Vehicles sold in thousands
1	Ford F-Series	367	11	Honda CR-V	146
2	Chevrolet Silverado	253	12	Ford Focus	135
3	Toyota Camry	208	13	Chevrolet Cruze	134
4	Honda Accord	187	14	Chevrolet Equinox	126
5	Dodge Ram 1500	170	15	Hyundai Elantra	126
6	Nissan Altima	168	16	Toyota Prius	120
7	Ford Fusion	161	17	Chevrolet Malibu	111
8	Toyota Corolla/ Matrix	159	18	Hyundai Sonata	103
9	Honda Civic	159	19	Toyota RAV4	101
10	Ford Escape	157	20	Ford Explorer	101

Source: MSN Autos, "Best-Selling Cars of 2013: Halftime Report."

a. Describe a sampling procedure that would give each of the 20 vehicles an equal chance of selection.

b. Describe a sampling procedure that would guarantee that the percent of Fords in a sample of 8 vehicles is the same as the percent of Fords in the top 20 vehicles.

c. Suppose that you found out that a random sample of 5 vehicles was taken, and that only 1 Ford was chosen for the sample. Would you conclude that the sampling process was done unfairly? Explain your answer.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Which of the following best describes an experiment?
   a. Observations and measurements are made in a way that does not change whatever it is that is being measured or studied.
   b. Any process or study that results in the collection of data after a treatment is conducted on a population, the outcome of the study is unknown.
   c. A survey carried out using a sampling method so that only a portion of the population is surveyed rather than the whole population.
   d. An experiment is the study of how to collect, organize, analyze, and interpret numerical information from data.

2. Which of the following would be the most accurate way of finding out how people in a given state feel about gun control laws?
   a. Conduct a survey of all gun owners in the state.
   b. Set up a study in which researchers observe how people react to news of gun violence.
   c. Analyze the results of a survey given to a random sample of the state's population.
   d. Conduct an experiment in which state employees are provided with the ability to be "legislator for a day" and see how they vote.

3. A local homeless shelter would like to gather information about how the children of their clientele are faring in school. The shelter will use the information to apply for a grant to increase services for students who do not have a home. What would be the best way for the shelter staff to gather the information?
   a. one-on-one, in-person interviews
   b. Internet-based feedback forms
   c. telephone interviews
   d. confidential, mailed written surveys

continued

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4. Which of the following course-evaluation questions is neutral?
   a. Agree or disagree: This course assigns too much homework.
   b. Agree or disagree: This course has greatly increased my knowledge of the subject.
   c. Agree or disagree: This course is easy.
   d. Agree or disagree: The instructor provides adequate feedback.

5. Which of the following would be a random sample of the members of a choir?
   a. choosing the person in every fourth seat
   b. choosing 3 people from each section (soprano, alto, tenor, bass)
   c. choosing the first 5% of the choir to arrive at a rehearsal
   d. choosing the youngest 10% of the choir members

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Lesson 3: Surveys, Experiments, and Observational Studies

Instruction

Common Core State Standard

S–IC.3 Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.★

SMP	
1 ✓	2 ✓
3 ✓	4
5 ✓	6 ✓
7	8

Essential Questions

1. In what ways can we collect data?
2. How are studies designed?
3. What are the differences between types of studies?
4. How do studies justify their conclusions?
5. What is the importance of randomization in gathering data?

WORDS TO KNOW

bias	leaning toward one result over another; having a lack of neutrality
confounding variable	an ignored or unknown variable that influences the result of an experiment, survey, or study
control group	the group of participants in a study who are not subjected to the treatment, action, or process being studied in the experiment, in order to form a comparison with participants who are subjected to it
data	numbers in context
double-blind study	a study in which neither the researcher nor the participants know who has been subjected to the treatment, action, or process being studied, and who is in a control group
experiment	a process or action that has observable results
neutral	not biased or skewed toward one side or another; regarding surveys, neutral refers to phrasing questions in a way that does not lead the response toward one particular answer or side of an issue

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observational study	a study in which all data, including observations and measurements, are recorded in a way that does not change the subject that is being measured or studied
outcome	the observable result of an experiment
placebo	a substance that is used as a control in testing new medications; the substance has no medicinal effect on the subject
random	the designation of a group or sample that has been formed without following any kind of pattern and without bias. Each group member has been selected without having more of a chance than any other group member of being chosen.
randomization	the selection of a group, subgroup, or sample without following a pattern, so that the probability of any item in the set being generated is equal; the process used to ensure that a sample best represents the population
sample survey	a survey carried out using a sampling method so that only a portion of the population is surveyed rather than the whole population
skew	to distort or bias, as in data
statistics	a branch of mathematics focusing on how to collect, organize, analyze, and interpret information from data gathered
survey	a study of particular qualities or attributes of items or people of interest to a researcher

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Recommended Resources

- Education.com. "Design of a Study: Sampling, Surveys, and Experiments Free Response Practice Problems for AP Statistics."

  http://www.walch.com/rr/00182

  This collection of challenging problems helps users test their knowledge of how studies are designed.

- Hudler, Eric H. University of Washington. "Data Collection and Analysis."

  http://www.walch.com/rr/00183

  This site offers a concise explanation of sampling and testing authored by Eric Hudler, an associate professor at the University of Washington and publisher of "Neuroscience for Kids."

- Stat Trek. "Bias in Survey Sampling."

  http://www.walch.com/rr/00184

  This site provides tutorials and examples explaining experiment and study design, randomization, and bias. It also includes a random number generator and many interactive statistics calculators and tools.

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Lesson 1.3.1: Identifying Surveys, Experiments, and Observational Studies

Warm-Up 1.3.1

While standing in line at an amusement park with a group of friends, you pose the following question: What is the best way to become healthier? Each friend gives you a different answer.

1. What are some possible responses to your question?

2. List some ways you could gather information to answer this question.

3. How can you scientifically determine which friend has given you the best response?

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Lesson 1.3.1: Identifying Surveys, Experiments, and Observational Studies

Common Core State Standard

S–IC.3 Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.★

Warm-Up 1.3.1 Debrief

1. What are some possible responses to your question?

   Possible responses include: watching less television; playing fewer computer games; eating breakfast; taking vitamins or supplements; exercising; consuming fewer calories; eating more fruits and/or vegetables; and eliminating or reducing consumption of foods with empty calories, such as chips, candy, and soda, from the diet.

2. List some ways you could gather information to answer this question.

   You could gather information by: searching online for reputable information; asking an expert or a family member; reading a trusted source, such as a newspaper, book, or scientific journal; or asking more people.

3. How can you scientifically determine which friend has given you the best response?

   There are several possible methods, including the following: conducting a survey or poll online, in person, or by phone; interviewing experts; researching the question; or creating a feedback form.

Connection to the Lesson

- Students will recognize how varied responses to posed questions can be, as well as how varied follow-up can be depending on the type of question asked. Students will begin to think about how data is collected and surveys are created.

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Prerequisite Skills

This lesson requires the use of the following skills:

- familiarity with surveys (no standard)
- understanding the definition of *random* as it relates to gathering and interpreting data (S–IC.2)

Introduction

Data is vital to every aspect of how we live today. From commerce to industry, the Internet to agriculture, politics to publicity, data is constantly being gathered, analyzed, applied, and reported. **Statistics** is a branch of mathematics that is focused on how to collect, organize, analyze, and interpret information from data gathered. There are many ways to gather **data**, or numbers in context. The most appropriate method for gathering data can vary based on the data that is desired, the situation, or the purpose of the study. In this lesson, we will discuss methods of collecting data and when each method is appropriate.

Key Concepts

Gathering Data Without Influencing It

- Sometimes, we need data about how things in the world exist without outside interference.
- For example, a team of zoologists might want to study the habits of an endangered bird species, but to disturb or interact with the birds may cause the animals to behave differently than they normally would. Therefore, the team may choose to observe the birds from a safe distance using binoculars.
- This sort of study is an **observational study**; that is, a study in which all data, including observations and measurements, are recorded in a way that does not change the subject that is being measured or studied.
- An observational study allows information to be gathered without disturbing or impacting the subject(s) at all.
- Most of the time, observational studies are used when it would be impractical or unethical to perform an experiment.
- For example, researchers trying to establish a link between smoking and lung cancer could pay the study participants to smoke, and then see if the participants develop lung cancer; however, to do so would be highly unethical. An observational study will provide useful data without interfering in people's lives and health.

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Gathering Data on Large Populations

- A **survey** is a study of particular qualities or attributes of items or people of interest to a researcher.
- Many reality shows are competitions, in which winners are determined by gathering votes from every audience member who wishes to enter a vote. Each episode of the show is actually a survey of the audience, using technology to quickly gather and count the votes.
- However, there are instances when surveying an entire audience or population would take too long or cost too much money—for example, conducting a survey of everyone living in New York City to see how many New Yorkers like chocolate ice cream. Since there are millions of people living in New York City, it would be too difficult and too expensive to survey everyone who lives there, let alone record and analyze all that data.
- When data on a large population is needed, it is often gathered through a sample survey. A **sample survey** is carried out using a sampling method so that only a portion of the population is surveyed rather than the whole population.
- In the ice cream example, it would be a better use of time and money to survey only a certain number of New York residents, and then base conclusions on that sample.
- Sample surveys must be carefully designed to produce reliable conclusions:
    - The sample must be representative of the population as a whole, so that the data will lead to conclusions that apply to the entire population.
    - Questions must be **neutral**—that is, asked in a way that does not lead the response toward one particular answer or side of an issue.

Gathering Data to Determine Causes and Effects

- When the purpose of collecting data is to find out how something such as a medical treatment or other outside influence affects a population or subject, often the best method of study involves conducting an experiment.
- An **experiment** is a process or action that has observable results called **outcomes**.
- In an experiment, participants are intentionally subjected to some process, action, or substance. The results of the experiment are observed and recorded.
- Deliberately offering participants an incentive, such as money or free products, often brings about a desired outcome.
- Frequently, researchers conduct experiments to test the effectiveness of new medications. When the new medicine is ready for trials on human subjects, the experiments are carried out on groups of volunteers.

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- A **placebo**, or substance used as a control in testing new medications, is given to one group. The placebo has no medicinal effect on the participants, who may not be told that they are taking a placebo. If, during the experiment, the volunteers taking the medication report dizzy spells, but the placebo group does not, then the researchers can have a better idea that dizziness is a side effect of the new medication.
- The study participants who are taking the placebo make up the control group. A **control group** is a group of study participants who are not subjected to the treatment, action, or process being studied in the experiment. By using a control group, researchers can compare the outcomes of the experiment between this group and the group actually receiving the treatment, and better understand the effects of what is being studied.

Common Errors/Misconceptions

- being unable to differentiate between an experiment and an observational study
- thinking that surveys are generally given to all subjects in a population
- thinking that surveys can only involve human subjects
- not understanding that in order to conduct a experiment, at least a portion of the population studied must be subjected to the process, action, or substance being evaluated

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Guided Practice 1.3.1

Example 1

Spirit Week is approaching, and the student council wants more students to participate in the festivities by dressing up. Student council members plan to collect data on the most popular dress-up themes for the days of Spirit Week by asking other students what their favorite themes are. Since the student council doesn't have much time or funding, members will not be able to talk to every student. What method of data gathering will most closely match what the student council is trying to accomplish?

1. Consider the methods of data collection described in this lesson.

   The lesson described observational studies, experiments, and surveys/sample surveys.

2. Recall the distinguishing characteristics of each method.

   An observational study requires that the researcher observe the subject without interacting with or disturbing the subject.

   In an experiment, participants are intentionally subjected to some process, action, or substance so that the results can be observed and recorded.

   A survey is a study of particular qualities or attributes of items or people of interest to a researcher. A survey involves directly interacting with the subject population, such as by asking questions. A sample survey is conducted using only a portion of the population, rather than the entire population.

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Instruction

3. Evaluate the situation described in the problem scenario to determine the purpose and characteristics of the required data.

   The student council wants to determine the most popular dress-up themes for days during Spirit Week.

   The council wants to use this data to increase the number of students who participate.

   Council members plan to gather data by asking students about their favorite dress-up themes.

   The council knows it doesn't have the time or money to ask every student at the school.

4. Determine which method of data collection best matches the situation.

   Compare each method of data collection with the particulars of the situation to rule out methods that aren't suited to the situation.

   Student council members cannot avoid interacting with the study population (their fellow students); therefore, an observational study isn't appropriate for the situation.

   Council members do not need to subject the student body to any particular treatment, process, or action, so an experiment is not an appropriate method for this situation either.

   The remaining method to collect the needed data is a survey.

   The problem scenario states that council members have the resources to ask some students their preferences for Spirit Week dress-up themes, but not all students.

   Therefore, the method that best matches this situation is a sample survey, in which the council members will survey a portion (sample) of the student population rather than the entire population.



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Example 2

The student council successfully gathered data and used it to choose the themes for each day of Spirit Week. Now that Spirit Week is finally here, council members need to know how each theme affects student participation. They plan to sit in the front of the cafeteria during lunch each day of Spirit Week to count the number of students dressed up for the day's theme. What method of data gathering most closely matches this plan?

1. Recall the distinguishing characteristics of each method of data collection described in this lesson.

   An observational study requires that the researcher observe the subject without interacting with or disturbing the subject.

   In an experiment, participants are intentionally subjected to some process, action, or substance so that the results can be observed and recorded.

   A survey is a study of particular qualities or attributes of items or people of interest to a researcher. A survey involves directly interacting with the subject population, such as by asking questions. A sample survey is conducted using only a portion of the population, rather than the entire population.

2. Evaluate the situation described in the problem scenario to determine the purpose and characteristics of the required data.

   The student council wanted to increase student participation in Spirit Week. They need to determine how the chosen themes are affecting participation.

   Council members plan to gather data by counting the number of students dressed up for each day's theme.

   Council members are going to count dressed-up students from the front of the cafeteria, without directly interacting with them.

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3. Determine which method of data collection best matches the situation.

   The student council members are not giving any particular treatment to the population or subjecting it to any actions or processes, so this is not an experiment.

   Additionally, council members are not going to interact with the population by asking questions to gather their data; therefore, this is not a survey or sample survey.

   Finally, since the council members will be observing (counting) the number of students who dress up, but not interacting with them or experimenting on them, they will be conducting an observational study.



Example 3

To encourage as many students as possible to dress up for the final day of Spirit Week, the student council is giving away raffle prizes donated by local businesses. Every student who dresses up will get a free raffle ticket. Council members will gather data on how many students participate on the last day of Spirit Week, and compare that information with the data they have gathered from their observational study on dress-up participation for the other days of Spirit Week. What method of data gathering will most closely match what the student council is trying to accomplish with the raffle prizes?

1. Evaluate the situation described in the problem scenario to determine the purpose and characteristics of the required data.

   The student council wants as many people as possible to dress up on the last day of Spirit Week.

   The council plans to give away raffle tickets for prizes to students who dress up.

   The council will compare the number of students who dress up on the last day of Spirit Week with data on how many students dressed up on the other days of Spirit Week.



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2. Determine which method of data collection best matches the situation.

   The student council members have to interact with participating students in order to give them raffle tickets. Therefore, this will not be an observational study.

   Additionally, council members are not going to conduct a survey to gather their data; therefore, this is not a survey or sample survey.

   The council members are giving away raffle tickets for prizes as an incentive to dress up. An incentive will directly affect how many students participate, and the desired outcome is increased participation. Since the student council is deliberately subjecting students to an incentive to bring about a desired outcome, the student council is conducting an experiment. ✓

Example 4

Mrs. Webber, the school nurse, keeps a log of all symptoms reported by students. Lately there has been a marked increase in the number of students coming to the office complaining of back pain. After researching factors that lead to back pain in adolescents, Mrs. Webber found heavy backpacks have led to injuries in other schools. The American Academy of Pediatrics recommends that students' backpacks weigh no more than 10 to 20 percent of the student's weight. Mrs. Webber would like to find out the average weight of a backpack in her school.

Which method of data collection will provide Mrs. Webber with the best information for answering her research question: an experiment, an observational study, or a survey?

1. Recall the distinguishing characteristics of each method given as an option in the problem scenario.

   An observational study requires that the researcher observe the subject without interacting with or disturbing the subject.

   In an experiment, participants are intentionally subjected to some process, action, or substance so that the results can be observed and recorded.

   A survey is a study of particular qualities or attributes of items or people of interest to a researcher. A survey involves directly interacting with the subject population, such as by asking questions. A sample survey is conducted using only a portion of the population, rather than the entire population.

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2. Evaluate the situation described in the problem scenario to determine the purpose and characteristics of the required data.

   Mrs. Webber has seen an increase in the number of students complaining of back pain. Her research indicates that heavy backpacks are the cause.

   Mrs. Webber wants to determine the average weight of both the students in her school and the backpacks they carry.

3. Determine which method of data collection best matches the situation.

   At this point, Mrs. Webber is not yet attempting to affect or change what is happening, so she does not need to subject the student population to any treatments, processes, or actions in order to answer her question. Therefore, an experiment is not an appropriate method of study for this situation.

   Mrs. Webber interacts with the student population as a function of her job, so an observational study is also not appropriate.

   The remaining option for collecting the needed data is by conducting a survey.

   Since it may be highly unlikely that Mrs. Webber will be able to survey the entire student population, the most practical option for this situation would be a sample survey.



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Problem-Based Task 1.3.1: Does Soda Cause Cancer?

SMP	
1	2 ✓
3 ✓	4
5	6 ✓
7	8

Your classmate Jimmy presented a project to your class about carcinogens, substances that can cause cancer in living cells. When Jimmy said during his presentation that some soda ingredients may be carcinogens, you nearly spit out your root beer. Now you can't rest until you know whether soda consumption is linked to developing cancer. How would you go about investigating whether soda and cancer are linked?



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Problem-Based Task 1.3.1: Does Soda Cause Cancer?

Coaching

a. What three methods of data collection are described in this lesson?

b. Choose one of the methods to evaluate. Describe how this method could be used to gather information about the situation.

c. What are the benefits and drawbacks of this method?

d. Choose another method to evaluate. Describe how this method could be used to gather information about the situation.

e. What are the benefits and drawbacks of this method?

f. Evaluate the remaining method. Describe how this method could be used to gather information about the situation.

g. What are the benefits and drawbacks of this method?

h. Compare the benefits and drawbacks of each method. Which method offers the strongest benefits? Which methods have drawbacks that would make them ineffective for this investigation?

i. Choose your preferred method for conducting an investigation into soda consumption and cancer. Justify your choice.

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Instruction

Problem-Based Task 1.3.1: Does Soda Cause Cancer?

Coaching Sample Responses

a. What three methods of data collection are described in this lesson?

The lesson describes sample surveys, experiments, and observational studies.

b. Choose one of the methods to evaluate. Describe how this method could be used to gather information about the situation.

Responses will vary according to the method chosen. Sample response: I could conduct a sample survey, asking participants about their habits in drinking soda and their health, including cancer diagnosis.

c. What are the benefits and drawbacks of this method?

One benefit of this method would be that a large number of people drink soda, providing for a large population from which to draw a sample.

Drawbacks include the concern that people may not wish to share their habits in drinking soda, or people may not be truthful in giving their answers. Some people may not know or realize how much soda they consume. Respondents may also not wish to talk to a stranger about their health and cancer status, or may not know whether they have cancer. Furthermore, there are many other carcinogens that people may encounter, knowingly or unknowingly; I would have to construct my survey questions to try and anticipate these encounters. Since cancer can take time to develop, it could prove difficult to sample populations to track their habits in drinking soda over years of consumption in an effort to determine a link to cancer.

d. Choose another method to evaluate. Describe how this method could be used to gather information about the situation.

Responses will vary according to the method chosen. Sample response: I could also conduct an experiment. In this case, I would need participants who would be willing to let me monitor their soda consumption and study their cells over time, and who would be willing to possibly increase their soda consumption, if required by the experiment.

e. What are the benefits and drawbacks of this method?

Benefits include having a large population of soda drinkers from which to recruit participants.

Drawbacks of conducting an experiment include ethical issues. For example, it is possible participants could be at a higher risk of developing cancer than non-participants if there really is a link between soda and cancer, given that the experiment does not discourage drinking soda and may actually encourage drinking more soda. Also, since the development of cancer cannot

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be predicted, it may take some subjects years to develop cancer, and finding subjects willing to be tracked for so long could prove difficult. I may not have the required time and/or resources necessary for such an experiment. Furthermore, there are many known causes of cancer, so I would have to design my experiment to rule out numerous other variables. On the other hand, an experiment that is designed well and controlled for other variables could provide powerful evidence of a link between drinking soda and the development of cancer.

f. Evaluate the remaining method. Describe how this method could be used to gather information about the situation.

Responses will vary according to the method chosen. Sample response: I could conduct an observational study.

g. What are the benefits and drawbacks of this method?

The primary benefit of an observational study is that I don't have to consider the issue of asking subjects to change their soda consumption. Furthermore, as with the other methods, there is a large pool of potential subjects. The drawback is the difficulty of studying soda consumption habits without intruding in subjects' lives. Also, I may not have the time and/or resources to conduct an observational study.

h. Compare the benefits and drawbacks of each method. Which method offers the strongest benefits? Which methods have drawbacks that would make them ineffective for this investigation?

Answers may vary. Justifications include the following: All three methods share the benefit of having a large population of soda drinkers from which to draw subjects. An observational study has the additional benefit of not interfering with the subjects' habits.

It would be difficult to use responses from a sample survey to link the development of cancer to soda because of the many other possible carcinogens that people encounter, and the possibility of people (either intentionally or unintentionally) providing imprecise responses.

An observational study would be difficult to conduct, as direct observation of the subjects' soda consumption in an uncontrolled environment would require a high level of intrusiveness, and interaction would be nearly impossible to prevent.

The drawbacks of conducting an experiment are highly detrimental to the investigation. An experiment would take considerable time and resources, would be difficult to design given other possible variables, and involve ethical problems related to encouraging consumption of potential carcinogens.

A survey would be the least effective method for providing evidence of a link between cancer and soda consumption.

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i. Choose your preferred method for conducting an investigation into soda consumption and cancer. Justify your choice.

While all three methods have serious drawbacks, the best choice for this situation given the time constraints of the student conducting the investigation is a sample survey. In a sample survey, the random selection of the subjects from a large and varied population could mitigate the effects of many other variables. The next best choice would be an observational study; if you could observe a large and varied enough population, the investigation could yield valuable information to prove or disprove any link between soda consumption and the onset of cancer.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.3.1: Identifying Surveys, Experiments, and Observational Studies

For problems 1–3, identify whether the method of study described is a sample survey, experiment, or observational study. Explain your reasoning.

1. A weight-loss program is purchased by 25,000 people. The company registers all 25,000 people in a database, recording each person's starting weight. After 8 weeks, the company checks in with 5,000 of the customers selected at random to record the new weights of these customers to determine their weight-loss progress.

2. A company is conducting market research on a new cleaning product by providing free samples to two groups of people. The samples given to one group are at full strength, and the samples given to the other group are diluted with water. The company then gathers data from each group on product satisfaction and effectiveness.

3. A study of 200 college-age cigarette smokers found that the participants were able to walk on a treadmill set with a steep incline for an average of 0.6 mile before the participants became short of breath.

For problems 4–9, identify which method of study could be used to best accomplish the results sought in each scenario. Explain your reasoning.

4. Membership at the local library continues to decrease. What kind of study should the library conduct in order to increase library membership?

5. The birth rate in first-world countries is decreasing. The government of one country in particular is anticipating negative effects on the economy if the population is reduced. This country's government needs a better understanding of why people are having fewer children. What kind of study would help the government understand this trend?

continued

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6. What kind of study should a teacher conduct in order to improve student grades?

7. The owner of a coffee shop is considering installing a drive-through window, but wants to know the possible effect on parking for current customers. What kind of study might this shop owner conduct to understand the parking patterns of current customers?

8. The owner of the coffee shop would also like to better compete against popular energy and alertness drinks on the market. He would like to create an ad campaign that includes the length of time a small cup of his shop's regular coffee will help customers stay awake and alert. What kind of study might he conduct to find out how long customers, on average, can count on staying awake after consuming a small cup of his shop's coffee?

9. A group of biology students would like to know how the type of light that sunflowers are exposed to impacts the growth of the flowers over time. The students want to explore the effects of natural light, ultraviolet light, and fluorescent light. What kind of study might the students conduct to find out how the type of light impacts the growth of the sunflowers?

Use your understanding of surveys, experiments, and observational studies to complete problem 10.

10. A farmer would like to compare two brands of seeds that both claim to yield more crops. Design a study that she might conduct to test the claims of both brands.

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Lesson 1.3.2: Designing Surveys, Experiments, and Observational Studies

Warm-Up 1.3.2

While watching TV, you heard the following claims about various products.

a. "4 out of 5 dentists recommend Brand A Toothpaste."

b. "Our paper towel absorbs twice as much liquid as Brand B's paper towel."

c. "I lost 4 inches off my waist after wearing Brand X's Weight-Loss Wrap."

1. Identify the type of study that likely led to each claim.

2. After thinking about each statement, write a question you could ask the maker of each product to determine how valid each claim might be.

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Instruction

Lesson 1.3.2: Designing Surveys, Experiments, and Observational Studies

Common Core State Standard

S–IC.3 Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.★

Warm-Up 1.3.2 Debrief

1. Identify the type of study that likely led to each claim.

   a. "4 out of 5 dentists recommend Brand A Toothpaste."

   It appears as though only a portion of the population was asked their toothpaste preference (dentists); therefore, this is an example of the result of a sample survey.

   b. "Our paper towel absorbs twice as much liquid as Brand B's paper towel."

   It appears as though the paper-towel manufacturer compared the amount of liquid absorbed by their paper towel to the amount of liquid absorbed by Brand B paper towels; therefore, this is an example of an experiment.

   c. "I lost 4 inches off my waist after wearing Brand X's Weight-Loss Wrap."

   This seems to be an experiment, in which the consumer measured his or her waist prior to using the advertised wrap and then measured his or her waist again after using the wrap.

2. After thinking about each statement, write a question you could ask the maker of each product to determine how valid each claim might be.

   a. "4 out of 5 dentists recommend Brand A Toothpaste."

   Possible questions include: How many dentists were surveyed? How were the populations and/or samples of dentists selected? Were the dentists provided free samples or compensation for recommending Brand A?

   b. "Our paper towel absorbs twice as much liquid as Brand B's paper towel."

   Possible questions include: Are both brands listed as the same ply? In other words, is one brand single-ply (made of one layer of paper) and the other two-ply (made of two layers)? How many times was the experiment conducted? Was each paper towel tested in an identical manner; i.e., exposed to the same amount of the same liquid, using the same number of sheets of each brand of paper towel to absorb the liquid?

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c. "I lost 4 inches off my waist after wearing Brand X's Weight-Loss Wrap."

Some possible questions include: How many people participated in this experiment? Are these results typical? Did the person quoted take additional measures to lose inches, such as eating less, exercising, getting the recommended amount of sleep regularly, or taking medications or supplements? Was this person an actor who was paid to endorse this product?

Connection to the Lesson

- Students will extend their understanding of surveys, experiments, and observational studies by further examining how these studies are created and then evaluating the design of each.
- Students will analyze how studies are formed and look for possible flaws in their design.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- identifying a survey, an experiment, and an observational study (no standard)
- understanding the definition of *random* as it relates to assembling a sample of study subjects (S–IC.2)

Introduction

Studies are important for gathering information. In this lesson, you will learn how to effectively design a study so that it yields reliable results. A well-designed study, whether it is a survey, experiment, or observational study, has a number of qualities, including:

- a statement describing the study's purpose
- neutral questions
- procedures designed to control for as many confounding variables as possible
- random assignment of subjects
- implementation of a sufficient number of trials in order for the results to be considered representative of the population being studied or surveyed

Key Concepts

- Studies are designed through a careful process meant to ensure that the study outcomes are reliable and relevant to the topic being studied.
- When designing a study, steps must be taken to avoid or eliminate bias. Studies can show bias, leaning toward one result over another, when preferred study subjects are selected from a population, or when survey questions are not neutral.
- A biased study lacks neutrality, and can generate results that are misleading.
- Data or results that have been influenced by bias are referred to as **skewed**.
- When designing an experiment, it is also important to limit confounding variables. **Confounding variables** are ignored or unknown variables that influence the results of an experiment, survey, or study.

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- For instance, researchers conducting human trials for medications often limit confounding variables by giving some volunteers placebos instead of the real medication. If, during the experiment, the volunteers taking the medication report dizzy spells, but the placebo group does not, then the researchers can have a better idea that dizziness is a side effect of the new medication. Without a placebo group, it can't be known for certain whether the dizziness could be attributed to the new medicine or to some other unknown variable(s).
- Careful design of a study helps to avoid bias and skewed results.
- The steps to design an effective study are listed and described as follows.

Steps to Design an Effective Study
1. Create a purpose statement.
2. Determine the population to be studied.
3. Generate neutral questions.
4. Assign subjects or participants randomly in order to avoid bias and to control for confounding variables.
5. Choose a large enough number of subjects depending on the purpose and the situation.

Step 1: Create a purpose statement.

- One of the very first steps in creating a study is to explicitly state the study's purpose. This is very important for both participants and researchers so that both parties have a clear idea of what the study is about. Additionally, a purpose statement keeps the design of the study focused, without additional topics, ideas, or extraneous information.

Step 2: Determine the population to be studied.

- The purpose statement will help determine the characteristics of the population to be studied. For example, a study of the effectiveness of a dandruff shampoo requires a population of participants who have dandruff.

Step 3: Generate neutral questions.

- The wording of interview questions or survey questions has an effect on the results of the survey. Questions need to be phrased so that they are neutral—that is, so the questions don't lead the respondent to answer in one way or another.

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Step 4: Assign subjects or participants randomly in order to avoid bias and to control for confounding variables.

- Once the population to be studied has been determined, a sample of that population must be selected to take part in the study. Selecting members at random helps ensure that the results of the study will be free from bias.
- A group or sample that has been formed without following any kind of pattern and without bias is a **random** group. Each group member has been selected with the same chance of selection as any other group member; no member is more or less likely than another to be chosen.
- **Randomization** is the selection of a group, subgroup, or sample without following a pattern. The probability of any item in the set being generated is equal. This process ensures that a sample best represents the population.
- A sample is either random or not. Samples cannot be "somewhat random," "almost random," or "partially random."
- Applying the treatment, process, or action being studied to every other item or member on a list of subjects is not ever considered random. Choosing members at set intervals, such as every other person, every third person, or every fourth person, is a pattern, and randomization cannot follow a pattern.
- One of the most popular methods of ensuring randomization is to conduct a **double-blind study**, in which neither the researchers nor the participants know who has been subjected to the treatment, action, or process being studied in the experiment, as opposed to who is in a control group (participants who are not subjected to what is being studied).The subjects of an observational study can be randomly selected from a population of interested volunteers. These subjects are often asked to complete surveys during the course of the study. However, participants are not randomly assigned to various treatments. That is why the results of observational studies can only be used to indicate *possible* links between variables, as opposed to *definite* links.

Step 5: Choose a large enough number of subjects depending on the purpose and the situation.

- The sample size must be large enough to make sure the results of the study apply to the population as a whole.
- A study with too few participants may give results that conflict with results gathered from a larger sample.

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Common Errors/Misconceptions

- not understanding that a sample is either random or not random
- mistakenly believing that samples can be "somewhat random," "almost random," or "partially random"
- not realizing that the wording of interview questions or survey questions has an impact on the results of the survey
- not understanding that applying the studied treatment or process to every other member of a sample (or any other set interval) is not considered random
- not considering confounding variables

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Guided Practice 1.3.2

Example 1

The following survey question was sent to managers and business owners who have registered with a local Chamber of Commerce:

> "Don't you agree that people spend too much time on social networking websites, both at home and at work, and that there should be a limit placed on the amount of time people can spend on these sites so that they are more productive and spend more time with family and friends?"

Determine whether bias exists in the question and/or in the population being surveyed. If bias does exist in the question, explain how the question may be rewritten to avoid bias. If bias exists in the population being surveyed, explain how you could create a sample of people to survey to avoid bias.

1. Determine whether bias exists in the question.

   The survey question is not neutral. It includes phrases that indicate what the survey writer believes is the acceptable answer: "Yes, people spend too much time on social networks and are neglecting family and work." The opening phrase, "Don't you agree," exerts pressure on the participant to agree that people spend too much time on social networking websites. The question includes the phrase "both at home and at work," implying that too much time is spent on social networks at both locations. The question also implies that people spending time on social networks are neglecting their work and relationships—hinting at what the survey writer thinks people should be doing with their time instead of visiting social networking sites. Also, invoking the idea of "family and friends" could trigger emotions in the respondents that would affect their answers.

2. Determine whether bias exists in the population being surveyed.

   The population surveyed includes managers and business owners. These participants are in supervisory positions, and may have opinions and expectations about productivity that would skew the results of this survey.

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3. How can this survey be rewritten to eliminate bias?

   Any emotionally charged statements, phrases, or presuppositions need to be removed from the question.

   Furthermore, the core goal of the survey needs to be more focused. Does the survey writer wish to evaluate opinions on the amount of time spent on social networks, or opinions as to whether there should be a time limit on social networking?

   A survey should be comprised of individual questions rather than a single question with many parts in order to yield clear responses.

   Let's focus on determining the respondents' opinions on the amount of time people spend on social networks.

   One possibility for rewriting the question is, "Do people spend an appropriate amount of time on social networking websites?"

   This question doesn't include any emotionally charged elements that might influence the respondent to give what the original question implied as the acceptable answer. The new question also focuses on a single element, so there is less risk of confusing the respondent, or of the respondent only answering part of the question.

   Another option to avoid bias would be to rephrase the survey question as a statement with defined answer choices, as shown:

   Choose the response that reflects your opinion of the following statement:

   People spend an appropriate amount of time on social networking websites.

   Strongly Agree Agree Neutral Disagree Strongly Disagree

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4. How could you create a sample of people to survey to avoid bias?

   The original survey was sent to managers and business owners who have registered with a local Chamber of Commerce. This particular population is more likely to value productivity and less likely to be in favor of the use of social networking sites by employees during work hours.

   The participants in the survey should include representatives from all levels of each company—such as owners and managers, middle-level management, supervisors and coordinators, and administrative assistants—to ensure an adequate representation of the company. An example of a random sample of this population could be to randomly assign each employee a number and then use a table of random numbers or a random number generator to select the desired number of subjects.

Example 2

A chain of department stores has updated its return policy in one store on a trial basis. The chain is gathering customer feedback by hiring researchers to interview customers on the last Sunday of June about their feelings regarding the new policy. Identify any flaws that exist in this sample survey, and suggest a way to eliminate these flaws.

1. Determine how the timing of the study could impact the results.

   A portion of the store's customer base might be missing if the interviews are conducted on a particular day of the week. For example, it is possible that members of clergy and the parishioners of particular denominations that have their services on Sunday would not be present. Other events that draw large numbers of residents who fit a certain demographic may be scheduled on the day of the survey, resulting in that particular group not being represented well or at all in the survey population; for example, a circus parade could draw children and their guardians, skewing the survey population toward people without children.

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2. Determine any limitations of interviewing customers.

   There are many possible limitations to interviewing customers in this way. For example, customers willing to be interviewed could be those who are more likely to have had a poor experience and are seeking a way to voice their discontent. Customers who have returned items in the past could be more likely to participate due to their familiarity with the return policy. Customers who have time to stop and answer interview questions may be those with more lenient schedules; for example, people without young children. These people may have more disposable income, with which they might have made a greater number of purchases in the store, increasing the likelihood that they have made returns.

3. Suggest a way to limit the identified flaws.

   Rather than conducting the survey on one particular day of the week, the store should conduct several surveys at various times of the day on various days of the week throughout the month.

   Surveys could also be mailed or e-mailed to customers to complete at their leisure.



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Example 3

A potentially fatal virus is spreading among birds. The director of a bird sanctuary found an herbal supplement that claims to reduce susceptibility to the virus. The director decided to test the supplement by having his staff put it in the water of every other birdbath in the sanctuary. Can this be considered a randomized experiment?

1. Identify any flaws in this experiment.

   Since the supplement was systematically put in every other birdbath, this selection process follows a pattern. In addition, we do not know if the birds use different baths in this sanctuary. If birdbaths treated with the supplement are in the same enclosure as baths without the supplement, we may not know which birds have used each bath.

   Also, there is no indication that the herbal supplement will be effective when diluted in a birdbath. Birds will drink at different rates and will therefore ingest differing amounts of the supplement.

2. Determine if this experiment is considered a randomized experiment.

   Providing treatment to every other birdbath is not considered random. In any trial, giving treatment to every other participant, or to participants at any other set interval, is never considered random because such intervals follow a pattern. In order for the experiment to be random, the birdbaths that get the supplement need to be selected at random.

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Example 4

Researchers for a treatment facility at a local university are seeking volunteers who have been diagnosed with severe Obsessive-Compulsive Disorder (OCD). The researchers are asking volunteers to spend three months living at the facility and working with faculty and doctoral students to lessen the impact of OCD on their ability to function in society. Determine factors that may skew the sample population. Based on these factors, how might the sample be affected?

1. Determine any factors that may skew the sample population.

   The study requires that participants live in the facility for three months. Since only people who have the ability to spend three months living on-site at the university will be able to participate, the study will include a reduced number of patients from many constituent groups.

2. State how the sample might be impacted.

   Because of the study's three-month, on-site commitment, parents with children at home may be unlikely to participate in the study.

   Anyone who must earn an income and keep a home may also be less likely to participate.

   Consultants or salespeople who travel extensively for work would be less likely to volunteer.

   By not including these people, the sample population could be skewed toward older, retired, unemployed, or childless participants whose requirements and daily experiences would not be representative of all OCD sufferers.



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Example 5

It's the day before your beach vacation, and you're trying to decide which sunscreen to buy. You're most concerned about providing maximum protection for your face. Someone told you that any sunscreen with a sun protection factor (SPF) greater than 50 is no more effective than one with an SPF of exactly 50.

Design a study to determine how well different sunscreens protect your face. Then, describe how to create a random sample if the population is large. Finally, indicate whether you chose to conduct a survey, experiment, or observational study and explain why you chose this type of study.

1. Design a study.

   One possible study design involves purchasing sample bottles of sunscreen, some with an SPF greater than 50 and others that have an SPF of exactly 50. Then apply one sunscreen with an SPF greater than 50 to half of your face, and apply a different sunscreen with an SPF of exactly 50 to the other half. Compare the results after your day in the sun.

2. Describe how to create a random sample if the population is large.

   Since this experiment may prove to be too costly to try all possible sunscreens being sold, you may put one sample of each brand with an SPF greater than 50 into a basket, and then put a sample of each brand with an SPF of exactly 50 into another basket. Close your eyes, and choose a bottle from each basket.

3. Indicate whether you chose to conduct a survey, experiment, or observational study, and explain why you chose this type of study.

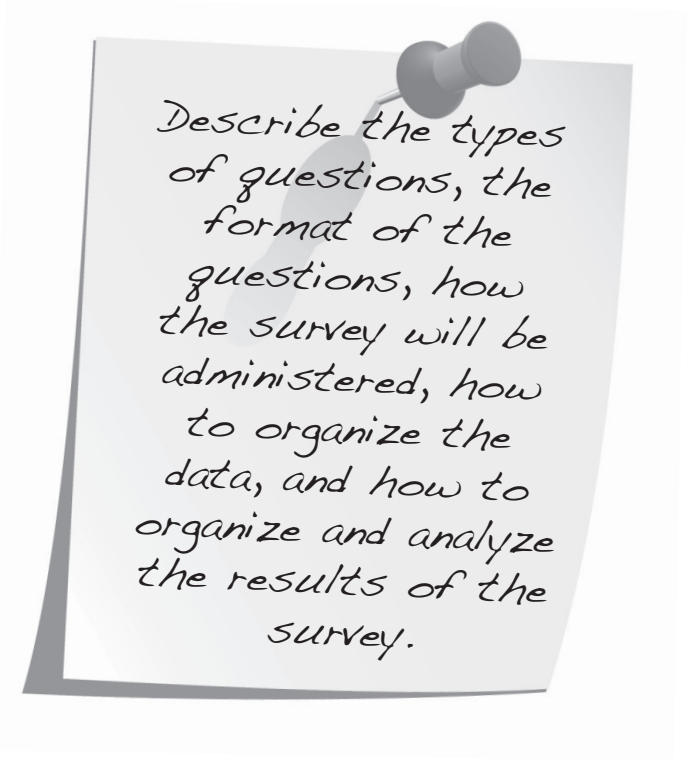
   This is an experiment, since it involves treating different sections of one's face with different sunscreen SPFs and comparing the results. One reason to choose an experiment is that the results of this type of study are easy to observe and record.

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Problem-Based Task 1.3.2: Creating a Survey

SMP	
1 ✓	2
3	4
5 ✓	6
7	8

You are the lead designer of a recently released smartphone. The target demographic of this phone is young adults, aged 18 to 24. You would like to get feedback from some members of this age group before designing an upgraded version of the phone that addresses any flaws in the current version. Create a survey. Describe the types of questions, the format of the questions, how the survey will be administered, how to organize the data, and how to organize and analyze the results of the survey.



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Problem-Based Task 1.3.2: Creating a Survey

Coaching

a. What is the purpose of the survey?

b. Who will be surveyed?

c. What is the best way to reach this population?

d. What questions should be asked?

e. How long after customers receive the phone should the survey be administered?

f. Will you survey the entire population or a sample? If you survey a sample, how will you choose this sample?

g. How will you follow up with surveys that remain unanswered?

h. How will you organize the survey data?

i. How might the results of your survey be used?

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Instruction

Problem-Based Task 1.3.2: Creating a Survey

Coaching Sample Responses

a. What is the purpose of the survey?

The purpose of this survey is to gain insight into what the target population thinks about the new smartphone.

b. Who will be surveyed?

The smartphone's target demographic is young adults aged 18 to 24, so the survey will be administered to members of this age group who have used the phone.

c. What is the best way to reach this population?

One way to reach this population would be to create a survey that users could access on their phones or through feedback via an app store. Most smartphone owners in this age range are also highly engaged in social media, and are frequently exposed to television and Internet advertising. We can consider reaching this population through any of these avenues.

d. What questions should be asked?

To aid in designing upgrades and fixing flaws for the next version, the questions should ask about the populations' favorite and least-favorite features of the phone, as well as about any problems users have experienced. A general request for "suggestions for improvement" could yield fresh ideas and responses that might not be readily provided by asking a specific question.

e. How long after customers receive the phone should the survey be administered?

This survey should be administered after the sample population has been able to use the phone for a trial period.

f. Will you survey the entire population or a sample? If you survey a sample, how will you choose this sample?

Since this population is so large, choose a sample of the population. One option is to use product registration records to randomly select 200 people from a list of current users within the target age range. *Note*: This method would be skewed toward customers who have completed the registration process.

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Instruction

g. How will you follow up with surveys that remain unanswered?

One option would be to glean customers' contact information from product registrations and then call, text, or e-mail customers to encourage them to respond. Offering an incentive, such as a discount, rebate, or prize, could encourage participation.

h. How will you organize the survey data?

One option is to categorize the feedback and provide direct quotes within each category. Some category examples for a smartphone might be ease of customization, keyboard response, battery life, organization of functions, usability, product accessories, speed, and app availability.

i. How might the results of your survey be used?

The data could be shared within the company for review and implementation uses. Positive feedback may be used in advertisements and product information guides. Negative feedback could be used to guide designers in making improvements and solving problems.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.3.2: Designing Surveys, Experiments, and Observational Studies

For each of the following situations, design an appropriate study to find the desired information. State whether your study is a survey, experiment, or observational study.

1. The owner of a tourist attraction on a tropical island wants to know the average daily temperature for the island so she can use it in her advertising.

2. A teacher would like to add student evaluation data to her portfolio.

3. A nursing home administrator would like to include patient satisfaction rates in a new brochure, with comparisons to satisfaction rates at 5 local, competing nursing homes.

4. The dean of students at a local college must report on how a new freshman orientation course has impacted student grade point averages.

5. A dietitian has 100 clients and would like to compare weight-loss results for two different diet plans.

6. A school guidance counselor wants to know if teenagers' music preferences have an effect on their self-esteem.

7. A consultant for a major metropolitan hospital wants to determine the impact on patients, finances, and medical staff of delaying the transfer of patients out of the intensive care unit.

8. A town manager wants to know: How likely are town residents to vote in favor of a proposal to build a new performing arts theater?

9. A group of students wants to know the average number of hours students at their school spend on homework during their senior year.

10. A marketing executive for a grocery store chain wants to know which brand of dish detergent the store's customers prefer: the nationally advertised brand or the store brand?

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Progress Assessment

Circle the letter of the best answer.

1. A school subscribes to an online program to help students learn math facts. The school randomly provides access to the program to half of the students and then evaluates their scores on standardized exams. What is a problem with this experiment?

   a. Giving students access to the program does not guarantee that they will use it, and students will also use the program for differing amounts of time.

   b. This experiment does not take into consideration the demographics of the students.

   c. The makers of the online program likely already have data about the program's impact on student achievement.

   d. This is not an experiment. It is an observational study.

2. Which of the following will *not* help to avoid bias in a study?

   a. randomizing the sample population

   b. making the study double-blind

   c. ensuring that the study is reliable

   d. assigning treatments to subjects randomly

3. Which of the following sample populations are valid for a study that measures the average SAT score for a school?

   a. students with perfect attendance

   b. students selected by teachers

   c. student athletes

   d. none of the above

4. Two children were hiking and got lost for several days. A week after the children were rescued, three school boards in adjacent towns proposed to require that students wear tracking devices. What issues might arise in conducting opinion polls concerning this new proposal?

   a. The school boards may be taking advantage of heightened parental concern.

   b. Parents who may not have agreed to tracking devices before (and who might change their minds later) may have a stronger opinion at this time because of the recent incident.

   c. The survey is not randomized and therefore would not be valid.

   d. Both *a* and *b* are correct.

continued

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Assessment

5. A cable channel that focuses on food would like suggestions for new programs and shows that would attract additional television watchers to the channel. The channel purchases banner advertisements on five major websites. People who click on the banner can respond to a survey and share their ideas for the chance to win a coffeemaker. Which of the following is *not* true?

   a. The target population matches the sampled population because most people have Internet access.

   b. The sampled population is unlikely to include people who do not drink coffee.

   c. The sample population will be skewed toward people who are willing to participate in the survey.

   d. This method will generate ideas for new programs.

6. Which of the following would *not* result in skewed survey data?

   a. conducting a survey of spending habits at a store the night of a midnight sale

   b. asking students to complete course evaluations of professors after students have seen their end-of-course grades

   c. distributing a survey about music preferences during the winter season

   d. asking season ticket holders for a sports team to provide feedback about increasing prices at the stadium

7. A neuroscientist selects 750 patients whose prescriptions are meant to help them relax and ease anxiety. The neuroscientist scans each patient's brain for one hour after taking the medication to look for patterns of reaction from different areas of the brain. What kind of study is this?

   a. a randomized trial

   b. a sample survey

   c. an observational study

   d. an experiment

8. Why is it important to state the purpose of a study before conducting it?

   a. The study and design can be focused on the purpose without straying.

   b. The participants and researchers are able to understand what the study is about.

   c. Stating the purpose will help to avoid adding additional topics or ideas as the study unfolds.

   d. all of the above

continued

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Assessment

9. A company has been providing bottles of vitamin-enhanced water to a local high-school basketball team this season. Every member of the team drank one bottle before each game. At the end of this particular season, the team was undefeated and won the championship for the first time in 16 years. Which of the following describes this scenario?

   a. an experiment
   b. an observational study
   c. a sample survey
   d. none of the above

10. Which of the following are important in the design of studies?

    I. having a stated purpose

    II. asking neutral questions

    III. making every effort to control as many confounding variables as possible

    IV. assigning subjects or participants randomly to avoid bias

    a. II and III
    b. I, III, and IV
    c. II, III, and IV
    d. I, II, III, and IV

Use what you have learned about surveys, experiments, and observational studies to complete all parts of the following problem.

11. A local school board has proposed that all students wear a school uniform and wants to determine the community's response to this new proposal.

    a. What type of studies should the school board review?
    b. How might the board use data from a study to encourage community members to vote in favor of a school uniform policy?
    c. What is the potential for bias in the study you described?

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Pre-Assessment

Circle the letter of the best answer.

1. In a recent poll, 25% of 200 people polled preferred the store brand of bread to the leading brand of bread. What is the standard error of this sample population?

   a. 0.0094
   b. 0.0306
   c. 0.25
   d. 50

2. The probability that Jim will score above a grade of 85 on his final exams is $\frac{4}{7}$. Which of the following expressions can we use to determine the probability that he will score above 85 on exactly 5 out of 9 finals?

   a. $\binom{85}{100}85^4100^7$
   b. $\binom{85}{100}\left(\frac{4}{7}\right)^5\left(\frac{3}{7}\right)^4$
   c. $\binom{9}{5}\left(\frac{4}{7}\right)^5\left(\frac{3}{7}\right)^4$
   d. $\binom{85}{100}\left(\frac{4}{7}\right)^5\left(\frac{5}{9}\right)^4$

3. A sample of 4,500 people was asked to track their text messaging for one week. Based on the following table of responses, what proportion of the sample population sends more than 100 texts each week?

Number of texts per week	0–20	21–50	51–100	101+
Number of people	125	494	1,001	2,880

   a. $\frac{1}{36}$
   b. $\frac{11}{100}$
   c. $\frac{2}{9}$
   d. $\frac{16}{25}$

continued

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4. A television reporter announced the latest poll numbers for a local election. She said, "With 95% confidence, between 60% and 64% of voters chose Candidate X." What does she mean by "95% confidence"?

   a. 95% of voters chose Candidate X.

   b. There is a 95% chance that between 60% and 64% of the voters chose Candidate X.

   c. Viewers can be 95% confident in this news report.

   d. The poll conductors are confident that if the same poll were conducted 100 times, using the same method each time, 95 of the polls would find that between 60% and 64% of the voting population voted for Candidate X.

5. A standardized exam is administered at half of the schools in a district. The exam returns a score that ranges from 1 to 5, with 1 being the lowest score and 5 being the highest. The scores were distributed normally with a mean of $\frac{3}{2}$ and a standard deviation of $\frac{1}{4}$. What is a 95% confidence interval for this test?

   a. (0.2375, 1.425)

   b. (1.25, 1.75)

   c. (1.54, 1.80)

   d. (1.8, 4.6)

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Lesson 4: Estimating Sample Proportions and Sample Means

Instruction

Common Core State Standard

S–IC.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.★

SMP	
1 ✓	2 ✓
3 ✓	4
5	6 ✓
7	8 ✓

Essential Questions

1. How do we estimate measures for populations that are very large?
2. How does *margin of error* explain statistical results?
3. How sure of our findings can we be when using data from statistics?

WORDS TO KNOW

addition rule for mutually exclusive events If events A and B are mutually exclusive, then the probability that A or B will occur is the sum of the probability of each event; $P(A \text{ or } B) = P(A) + P(B)$.

binomial experiment an experiment in which there are a fixed number of trials, each trial is independent of the others, there are only two possible outcomes (success or failure), and the probability of each outcome is constant from trial to trial

binomial probability distribution formula the distribution of the probability, P, of exactly x successes out of n trials, if the probability of success is p and the probability of failure is q; given by the formula

$$P = \binom{n}{x} p^x q^{n-x}$$

confidence interval an interval of numbers within which it can be claimed that repeated samples will result in the calculated parameter; generally calculated using the estimate plus or minus the margin of error

confidence level the probability that a parameter's value can be found in a specified interval; also called *level of confidence*

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critical value	a measure of the number of standards of error to be added to or subtracted from the mean in order to achieve the desired confidence level; also known as z_c-*value*
desirable outcome	the data sought or hoped for, represented by p; also known as *favorable outcome* or *success*
factorial	the product of an integer and all preceding positive integers, represented using a ! symbol; $n! = n \bullet (n-1) \bullet (n-2) \bullet \cdots \bullet 1$. For example, $5! = 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1$. By definition, $0! = 1$.
failure	the occurrence of an event that was not sought out or wanted, represented by q; also known as *undesirable outcome* or *unfavorable outcome*
favorable outcome	the data sought or hoped for, represented by p; also known as *desirable outcome* or *success*
level of confidence	the probability that a parameter's value can be found in a specified interval; also called *confidence level*
margin of error	the quantity that represents the level of confidence in a calculated parameter, abbreviated MOE. The margin of error can be calculated by multiplying the critical value by the standard deviation, if known, or by the SEM.
mutually exclusive events	events that have no outcomes in common. If A and B are mutually exclusive events, then they cannot both occur.
parameter	numerical value(s) representing the data in a set, including proportion, mean, and variance
population	all of the people, objects, or phenomena of interest in an investigation; the entire data set
population average	the sum of all quantities in a population, divided by the total number of quantities in the population; typically represented by μ; also known as *population mean*
population mean	the sum of all quantities in a population, divided by the total number of quantities in the population; typically represented by μ; also known as *population average*

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random sample — a subset or portion of a population or set that has been selected without bias, with each item in the population or set having the same chance of being found in the sample

sample average — the sum of all quantities in a sample divided by the total number of quantities in the sample, typically represented by $\overline{x}$; also known as *sample mean*

sample mean — the sum of all quantities in a sample divided by the total number of quantities in the sample, typically represented by $\overline{x}$; also known as *sample average*

sample population — a portion of the population; the number of elements or observations in a sample population is represented by n

sample proportion — the fraction of favorable results p from a sample population n; conventionally represented by $\hat{p}$, which is pronounced "p hat." The formula for the sample proportion is $\hat{p} = \frac{p}{n}$, where p is the number of favorable outcomes and n is the number of elements or observations in the sample population.

spread — refers to how data is spread out with respect to the mean; sometimes called *variability*

standard deviation — how much the data in a given set is spread out, represented by s or σ. The standard deviation of a sample can be found using the following formula: $s = \sqrt{\frac{\sum \left(x_i - \overline{x}\right)^2}{n-1}}$.

standard error of the mean — the variability of the mean of a sample; given by $\text{SEM} = \frac{s}{\sqrt{n}}$, where s represents the standard deviation and n is the number of elements or observations in the sample population

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Instruction

standard error of the proportion	the variability of the measure of the proportion of a sample, abbreviated SEP. The standard error (SEP) of a sample proportion $\hat{p}$ is given by the formula $\text{SEP}=\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, where $\hat{p}$ is the sample proportion determined by the sample and n is the number of elements or observations in the sample population.
success	the data sought or hoped for, represented by p; also known as *desirable outcome* or *favorable outcome*
trial	each individual event or selection in an experiment or treatment
undesirable outcome	the data not sought or hoped for, represented by q; also known as *unfavorable outcome* or *failure*
unfavorable outcome	the data not sought or hoped for, represented by q; also known as *undesirable outcome* or *failure*
variability	refers to how data is spread out with respect to the mean; sometimes called *spread*
z_c-value	a measure of the number of standards of error to be added or subtracted from the mean in order to achieve the desired confidence level; also known as *critical value*

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Recommended Resources

- Encyclopedia Britannica. "Estimation of a Population Mean."

  http://www.walch.com/rr/00185

  This encyclopedic entry provides a detailed explanation for the concept and formulation of the population mean. It includes context, connecting the population mean to the remainder of the concepts in this lesson.

- Khan Academy. "Confidence Interval 1."

  http://www.walch.com/rr/00186

  This video explains the concept of confidence intervals, and how they are used to estimate the probability that a true population mean can be found within a particular range of values.

- Oswego City School District Regents Exam Prep Center. "Binomial Probability."

  http://www.walch.com/rr/00187

  This exam-prep review website explains the binomial probability formula, offering worked example problems and complete answers.

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Lesson 1.4.1: Estimating Sample Proportions

Warm-Up 1.4.1

A local high school records data about each graduating class. The following table shows graduation and enrollment numbers for members of three given graduation years. The table includes the number of students who graduated in each given class, the number of students in each class who did not graduate but were still enrolled at the end of that year, and the number of students who did not graduate and were no longer enrolled at the end of that year.

Class	Graduated	Still enrolled	No longer enrolled
2012	1,042	345	221
2011	1,120	306	198
2010	1,025	450	222

1. What is the graduation rate for each class: 2012, 2011, and 2010? Round your answer to the nearest percent.

2. What is the high school's graduation rate over the three given years?

3. Which class best represents the overall graduation rate? Explain your reasoning.

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Lesson 1.4.1: Estimating Sample Proportions

Common Core State Standard

S–IC.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.★

Warm-Up 1.4.1 Debrief

1. What is the graduation rate for each class: 2012, 2011, and 2010? Round your answer to the nearest percent.

   To find the graduation rate for each class, first calculate the total number of students in the class.

   The table indicates three possible categories for students in each graduating class: the number of students who graduated that year, the number who did not graduate but were still enrolled at the end of that year, and the number who did not graduate and were no longer enrolled at the end of that year.

   To find the graduation rate for a given class, divide the number of students who graduated by the total number of students in the class.

   Class of 2012

   To find the total number of students in the class, find the sum of the three categories:

   total = graduated + still enrolled + no longer enrolled

graduated + still enrolled + no longer enrolled	Total number of students
(1042) + (345) + (221)	Substitute values from the table.
1608	Simplify.

   The class of 2012 had 1,608 students.

   Calculate the graduation rate for the class of 2012 by dividing the number of graduates (1,042) by the total number of students in the class.

$\text{graduation rate} = \frac{\text{number of students who graduated}}{\text{number of students in the class of 2012}}$	Formula
$\text{graduation rate} = \frac{(1042)}{(1608)}$	Substitute known values.
graduation rate ≈ 0.64801	Simplify.

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To convert this decimal to a percent, multiply by 100.

$0.64801 \cdot 100 = 64.801$

The graduation rate for the class of 2012 is approximately 65%.

Class of 2011

Find the total number of students in the class:

total = graduated + still enrolled + no longer enrolled

graduated + still enrolled + no longer enrolled	Total number of students
(1120) + (306) + (198)	Substitute values from the table.
1624	Simplify.

The class of 2011 had 1,624 students.

Calculate the graduation rate by dividing the number of graduates (1,120) by the total number of students in the class.

$\text{graduation rate} = \frac{\text{number of students who graduated}}{\text{number of students in the class of 2011}}$	Formula
$\text{graduation rate} = \frac{(1120)}{(1624)}$	Substitute known values.
graduation rate ≈ 0.68966	Simplify.

Convert this decimal to a percent.

$0.68966 \cdot 100 = 68.966$

The graduation rate for the class of 2011 is approximately 69%.

Class of 2010

Find the total number of students in the class:

total = graduated + still enrolled + no longer enrolled

graduated + still enrolled + no longer enrolled	Total number of students
(1025) + (450) + (222)	Substitute values from the table.
1697	Simplify.

The class of 2010 had 1,697 students.

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Calculate the graduation rate by dividing the number of graduates (1,025) by the number of students in the class.

$$\text{graduation rate} = \frac{\text{number of students who graduated}}{\text{number of students in the class of 2010}} \qquad \text{Formula}$$

$$\text{graduation rate} = \frac{(1025)}{(1697)} \qquad \text{Substitute known values.}$$

$$\text{graduation rate} \approx 0.60401 \qquad \text{Simplify.}$$

Convert this decimal to a percent.

$0.60401 \bullet 100 = 60.401$

The graduation rate for the class of 2010 is approximately 60%.

2. What is the high school's graduation rate over the three given years?

To find the graduation rate for the entire three years, start by finding the total number of graduates for the three years:

total number of graduates = 2012 graduates + 2011 graduates + 2010 graduates

2012 graduates + 2011 graduates + 2010 graduates	Total number of graduates
(1042) + (1120) + (1025)	Substitute values from the table.
3187	Simplify.

The total number of graduates for all three given years is 3,187.

Find the total number of students in all three classes by summing the totals found for each class in problem 1:

total number of students = number of students in class of 2012 + number of students in class of 2011 + number of students in class of 2010

number of students in class of 2012 + number of students in class of 2011 + number of students in class of 2010	Total number of students
(1608) + (1624) + (1697)	Substitute values found in problem 1.
4929	Simplify.

The total number of students for all three given years is 4,929 students.

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As in problem 1, divide the number of graduates by the total number of students.

$$\text{graduation rate} = \frac{\text{total number of students who graduated}}{\text{total number of students}}$$ Formula

$$\text{graduation rate} = \frac{(3187)}{(4929)}$$ Substitute known values.

$\text{graduation rate} \approx 0.64658$ Simplify.

Convert this decimal to a percent.

$0.64658 \bullet 100 = 64.658$

The high school's graduation rate over the three given years is approximately 65%.

3. Which class best represents the overall graduation rate? Explain your reasoning.

   Compare the graduation rates for each class with the overall graduation rate for the three given years.

   - The class of 2012 had a graduation rate of 65%.
   - The class of 2011 had a graduation rate of 69%.
   - The class of 2010 had a graduation rate of 60%.
   - The overall graduation rate for the three given years is 65%.

   The class whose graduation rate most closely matches the overall rate of 65% is the class of 2012; therefore, the class of 2012 best represents the overall graduation rate.

Connection to the Lesson

- Students will continue to work with data to investigate sample populations.
- Students will also relate samples to populations.

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Prerequisite Skills

This lesson requires the use of the following skills:

- calculating standard deviation (no standard)
- understanding random sampling (S–IC.2)

Introduction

For many survey situations, polling the entire population is impractical or impossible, necessitating the use of random samples as discussed in a previous lesson. It follows that any data collected or averaged from a random sample is not completely descriptive, since data wasn't collected from the entire population. In this lesson, we will explore the process for explaining how close we can say that we have come to estimating conclusions that represent an entire population using data collected from a random sample.

Key Concepts

- Sometimes data sets are too large to measure. When we cannot measure the entire data set, called a population, we take a sample or a portion of the population to measure.
- A **sample population** is a portion of the population. The number of elements or observations in the sample population is denoted by *n*.
- The **sample proportion** is the name we give for the estimate of the population, based on the sample data that we have. This is often represented by $\hat{p}$, which is pronounced "p hat."
- The sample proportion is calculated using the formula $\hat{p} = \frac{p}{n}$, where *p* is the number of favorable outcomes and *n* is the sample population.
- When expressing a sample proportion, we can use a fraction, a percentage, or a decimal.
- **Favorable outcomes**, also known as **desirable outcomes** or **successes**, are those data sought or hoped for in a survey, but are not limited to these data; favorable outcomes also include the percentage of people who respond to a survey.
- The **standard error of the proportion** (SEP) is the variability of the measure of the proportion of a sample. The formula used to calculate the standard error of the proportion is $\text{SEP} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, where $\hat{p}$ is the sample proportion determined by the sample and *n* is the number of elements or observations in the sample population.

SRB p. U1-152

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- This formula is valid when the population is at least 10 times as large as the sample. Such a size ensures that the population is large enough to estimate valid conclusions based on a random sample.

Common Errors/Misconceptions

- forgetting to take the square root of both the numerator and denominator when calculating the standard error of a proportion
- interpreting favorable outcomes as positive experiences rather than as desirable outcomes

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Guided Practice 1.4.1

Example 1

A sample of 480 townspeople were surveyed about their opinions of an elected official's decisions. If 336 responded in support of the official's decisions, what is the sample proportion, $\hat{p}$, for the official's approval rating amongst this sample population?

1. Identify the given information.

   In order to calculate the sample proportion, first identify the number of favorable outcomes, p, and the number of elements in the sample population, n.

   The number of favorable outcomes, p, is 336.

   The number of elements in the sample population, n, is 480.

2. Calculate the sample proportion.

   The formula used to calculate the sample proportion is $\hat{p} = \frac{p}{n}$, where p is the number of favorable outcomes and n is the number of elements in the sample population.

   Substitute the known values into the formula.

   $\hat{p} = \frac{p}{n}$ Sample proportion formula

   $\hat{p} = \frac{(336)}{(480)}$ Substitute 336 for p and 480 for n.

   $\hat{p} = 0.7$ Simplify.

   To convert the decimal to a percentage, multiply by 100.

   $(0.7)(100) = 70$

   The sample proportion for the official's approval rating amongst this sample population is 70%.

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Example 2

Estimate the standard error of the proportion from Example 1 to the nearest hundredth.

1. Identify the known information.

   In order to calculate the standard error of the proportion, we must identify the number of elements in the sample population, n, and the sample proportion, $\hat{p}$.

   The number of elements in the sample population given in Example 1, n, is 480.

   The sample proportion calculated in Example 1, $\hat{p}$, is 70% or 0.70.

2. Calculate the standard error of the proportion to the nearest hundredth.

   The formula used to calculate the standard error of the proportion (SEP) is $\text{SEP} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, where n is the number of elements in the sample population and $\hat{p}$ is the sample proportion.

   Substitute the known quantities.

   $\text{SEP} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ Formula for the standard error of the proportion

   $\text{SEP} = \sqrt{\frac{(0.70)[1-(0.70)]}{(480)}}$ Substitute 0.70 for $\hat{p}$ and 480 for n.

   $\text{SEP} = \sqrt{\frac{0.70(0.30)}{480}}$ Simplify.

   $\text{SEP} = \sqrt{\frac{0.21}{480}}$

   $\text{SEP} = \sqrt{0.000438}$

   $\text{SEP} \approx 0.020917$

   $\text{SEP} \approx 0.02$ Round to the nearest hundredth.

   The standard error of the proportion is approximately 0.02 and represents the amount by which the sample proportion will deviate from the actual measure of the elected official's approval rating for the entire population.

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Example 3

If 540 out of 3,600 high school graduates who answer a post-graduation survey indicate that they intend to enter the military, what is the standard error of the proportion for this sample population to the nearest hundredth?

1. Identify the given information.

   The number of favorable outcomes, p, is 540.

   The number of elements in the sample population, n, is 3,600.

2. Calculate the sample proportion.

   Use the formula for calculating the sample proportion: $\hat{p}=\frac{p}{n}$, where p is the number of favorable outcomes and n is the number of elements in the sample population.

   Substitute the known quantities.

$\hat{p}=\frac{p}{n}$	Sample proportion formula
$\hat{p}=\frac{(540)}{(3600)}$	Substitute 540 for p and 3,600 for n.
$\hat{p}=0.15$	Simplify.

   The sample proportion is 0.15.

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3. Calculate the standard error of the proportion.

   Use the formula for calculating the standard error of the proportion: $SEP = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, where n is the number of elements in the sample population and $\hat{p}$ is the sample proportion.

$SEP = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$	Formula for the standard error of the proportion
$SEP = \sqrt{\frac{(0.15)[1-(0.15)]}{(3600)}}$	Substitute 0.15 for $\hat{p}$ and 3,600 for n.
$SEP = \sqrt{\frac{0.15(0.85)}{3600}}$	Simplify.
$SEP = \sqrt{\frac{0.1275}{3600}}$	
$SEP = \sqrt{0.0000354167}$	
$SEP \approx 0.00595119$	
$SEP \approx 0.01$	Round to the nearest hundredth.

   The standard error of the proportion is approximately 0.01.

Example 4

Shae owns a carnival and is testing a new game. She would like the game to have a 50% win rate, with 0.05 for the standard error of the proportion. How many times should Shae test the game to ensure these numbers?

1. Identify the given information.

   Shae would like the game to have a 50% win rate; therefore, the sample proportion, $\hat{p}$, is 50% or 0.5.

   The standard error of the proportion is given as 0.05.

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Instruction

2. Determine the sample population.

Use the formula for calculating the standard error of the proportion: $\text{SEP} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, where n is the number of elements in the sample population and $\hat{p}$ is the sample proportion.

$\text{SEP} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$	Formula for the standard error of the proportion
$(0.05) = \sqrt{\frac{(0.5)[1-(0.5)]}{n}}$	Substitute 0.05 for SEP and 0.5 for $\hat{p}$.
$0.05 = \sqrt{\frac{0.5(0.5)}{n}}$	Simplify.
$0.05 = \sqrt{\frac{0.25}{n}}$	

Solve the equation for n, the number of elements in the sample population.

$(0.05)^2 = \left(\sqrt{\frac{0.25}{n}}\right)^2$	Square both sides of the equation.
$0.0025 = \frac{0.25}{n}$	Simplify.
$0.0025n = 0.25$	Multiply both sides by n.
$n = 100$	Divide both sides by 0.0025.

The number of elements in the sample population, n, is 100; therefore, Shae should test the game 100 times to ensure a 50% win rate and a standard error of 0.05.



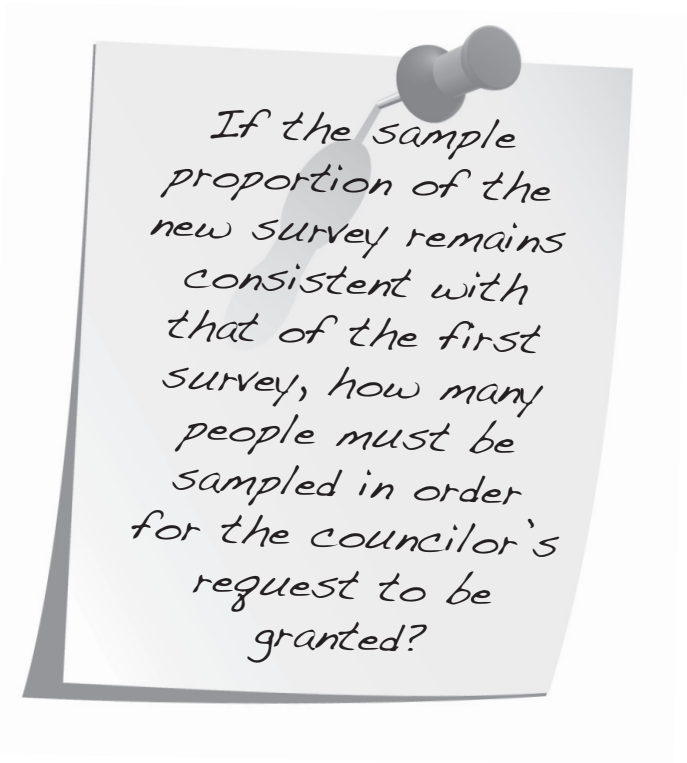
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Problem-Based Task 1.4.1: Traffic-Light Camera Survey

SMP	
1 ✓	2 ✓
3	4
5	6 ✓
7	8

The police chief of a small town wants to add surveillance cameras at all the traffic lights in the town to cut down on accidents. He surveyed some community members, and found that 16 out of 24 people favored the cameras. When the chief shared this data at a town council meeting, a councilor who works as a statistician objected to the small sample size. She said she would not vote in favor of surveillance cameras until the standard error of the proportion for the sample population is reduced to less than 0.03.

The police chief plans to conduct a new survey to fulfill the councilor's request. If the sample proportion of the new survey remains consistent with that of the first survey, how many people must be sampled in order for the councilor's request to be granted?



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Problem-Based Task 1.4.1: Traffic-Light Camera Survey

Coaching

a. What is the sample proportion of the police chief's original survey?

b. What is the standard error of the proportion for the original survey, rounded to the nearest thousandth?

c. Which variable in the formula for the standard error of the proportion must be altered in value in order for the standard error to decrease?

d. What changes can we make to the value of this variable?

e. What is the most logical way to change the value of this variable in order to decrease the SEP? Explain your reasoning.

f. If the sample proportion for the new survey remains consistent with that of the first survey, how many people must be sampled in order for the councilor's request to be granted?

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Instruction

Problem-Based Task 1.4.1: Traffic-Light Camera Survey

Coaching Sample Responses

a. What is the sample proportion of the police chief's original survey?

The formula used to calculate sample proportion is $\hat{p}=\frac{p}{n}$, where p is the number of favorable outcomes and n is the number of elements in the sample population.

The number of favorable outcomes is 16 and the number of elements in the sample population is 24.

$$\hat{p}=\frac{p}{n}$$

$$\hat{p}=\frac{16}{24}$$

$$\hat{p}=\frac{2}{3}=0.\overline{6}$$

$$\hat{p}\approx 66.7\%$$

The sample proportion of the original survey is approximately 66.7%.

b. What is the standard error of the proportion for the original survey, rounded to the nearest thousandth?

The formula used to calculate the standard error of the proportion for the survey is $\text{SEP}=\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}$, where n is the number of elements in the sample population and $\hat{p}$ is the sample proportion.

The number of elements in the sample population, n, is 24 and $\hat{p}$ is $\frac{2}{3}$.

$$\text{SEP}=\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}$$

$$\text{SEP}=\sqrt{\frac{\left(\frac{2}{3}\right)\left[1-\left(\frac{2}{3}\right)\right]}{(24)}}$$

$$\text{SEP}=\sqrt{\frac{\frac{2}{3}\left(\frac{1}{3}\right)}{24}}$$

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$$SEP = \sqrt{\frac{\frac{2}{9}}{24}}$$

$$SEP = \sqrt{0.009259259}$$

$$SEP \approx 0.096225045$$

$$SEP \approx 0.096$$

The standard error of the proportion for the original survey is approximately 0.096.

c. Which variable in the formula for the standard error of the proportion must be altered in value in order for the standard error to decrease?

The formula is $SEP = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.

In order to decrease the standard error to less than 0.03, we must alter the value for the size of the sample population, *n*.

d. What changes can we make to the value of this variable?

The size of a sample population, *n*, can either be increased or decreased.

e. What is the most logical way to change the value of this variable in order to decrease the SEP? Explain your reasoning.

Since the survey has already been administered once, we cannot decrease the population size at this point in the process. Therefore, it makes sense to increase the size of the sample population in order to decrease the standard error.

One possibility is to try doubling the size of the sample and then recalculating the standard error of the proportion.

If the original size of the sample population was 24, then doubling this number would result in a sample population size of 48.

Recalculate the standard error of the proportion using a value of 48 for *n*. As in the original survey, $\hat{p}$ is $\frac{2}{3}$, since the problem scenario assumes the sample proportion will remain unchanged between the two surveys.

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$$\text{SEP} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\text{SEP} = \sqrt{\frac{\left(\frac{2}{3}\right)\left[1-\left(\frac{2}{3}\right)\right]}{(48)}}$$

$$\text{SEP} = \sqrt{\frac{\frac{2}{3}\left(\frac{1}{3}\right)}{48}}$$

$$\text{SEP} = \sqrt{\frac{\frac{2}{9}}{48}}$$

$$\text{SEP} = \sqrt{0.00462963}$$

$$\text{SEP} \approx 0.068041382$$

$$\text{SEP} \approx 0.068$$

The goal is to achieve an SEP of less than 0.03, so we need a larger sample population size to decrease the standard error of the proportion even more. Try multiplying the size of the original sample population by 3 and then calculating the standard error with that number.

$24 \bullet 3 = 72$

Recalculate the SEP, using a value of 72 for n.

$$\text{SEP} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\text{SEP} = \sqrt{\frac{\left(\frac{2}{3}\right)\left[1-\left(\frac{2}{3}\right)\right]}{(72)}}$$

$$\text{SEP} = \sqrt{\frac{\frac{2}{3}\left(\frac{1}{3}\right)}{72}}$$

$$\text{SEP} = \sqrt{\frac{\frac{2}{9}}{72}}$$

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$SEP = \sqrt{0.00308642}$

$SEP \approx 0.055555556$

$SEP \approx 0.056$

To determine the minimum number of people the police chief must sample, increase the value of n until the SEP is less than 0.03.

Continue this process, or one similar, until the desired SEP of less than 0.03 is reached.

The table below lists the results of applying various multipliers to the sample population.

Multiplier	n	SEP
1	24	0.096225
2	48	0.068041
3	72	0.055556
4	96	0.048113
5	120	0.043033
6	144	0.039284
7	168	0.036370
8	192	0.034021
9	216	0.032075
10	240	0.030429
11	264	0.029013

Notice that it is not until the size of the sample population reaches 264 that the standard error of the proportion falls below 0.03.

It is also possible to solve the SEP formula for the value of n. Using this method reveals that when $n = 246$, the SEP is greater than 0.03, but when $n = 247$, the SEP is less than 0.03.

f. If the sample proportion for the new survey remains consistent with that of the first survey, how many people must be sampled in order for the councilor's request to be granted?

The police chief must sample at least 247 people in order to reduce the standard error of the proportion to less than 0.03.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.4.1: Estimating Sample Proportions

For problems 1–5, use the given information to calculate the sample proportion, $\hat{p}$, and the standard error of the proportion, SEP, for each of the described sample populations. Round $\hat{p}$ to the nearest whole percent and round the SEP to the nearest hundredth.

1. A recent opinion poll found that 245 out of 250 people are opposed to a new tax.

2. Marine biologists catching tuna for research found that 16 out of 28 tuna had elevated mercury levels.

3. A new window screen was found to block 1,400 out of 1,540 types of insects from getting through the window.

4. The local meteorologist has been correct in predicting temperatures on 11 of the past 14 days.

5. A gymnast landed without stumbling during 7 out of 13 routine practices.

continued

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Use what you have learned about the sample proportion, $\hat{p}$, and the standard error of the proportion, SEP, to solve problems 6–10. Round $\hat{p}$ to the nearest whole percent and round the SEP to the nearest hundredth.

6. A poll found that 30% of 300 residents polled were opposed to having a state-sponsored lottery. What is the SEP?

7. A survey asked people if they would like to live to the age of 120 if doing so required undergoing special medical treatments. 56% of the 2,012 respondents said they would. About how many people were in favor of undergoing special treatments if it meant living to 120? What is the SEP?

8. An experiment was found to have an SEP of 10% and a sample proportion of 80%. What was the size of the sample, n?

9. If 10,000 students enrolled at a for-profit college in the same year, and 900 of the students graduated within 6 years, what is $\hat{p}$?

10. To celebrate 24 years in business, a clothing store's marketing executive is ordering scratch-off discount coupons to give to customers. She would like 40% of customers in the population to receive the highest possible discount, with an SEP of 0.01 for this population. How many coupons should she order?

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Lesson 1.4.2: The Binomial Distribution

Warm-Up 1.4.2

Carly bought 6 posters of different bands and hung them on the same wall of her bedroom. One poster in particular is her favorite.

1. If Carly accidently knocks 1 random poster off the wall, what is the probability that the poster knocked down was her favorite?

2. How many ways can Carly arrange all 6 posters in a straight line on the same wall?

3. If Carly decides to move 2 of the posters to the opposite wall, how many different combinations of 2 posters can she possibly select?

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Instruction

Lesson 1.4.2: The Binomial Distribution

Common Core State Standard

S–IC.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.★

Warm-Up 1.4.2 Debrief

1. If Carly accidently knocks 1 random poster off the wall, what is the probability that the poster knocked down was her favorite?

   In order to determine a probability, create a fraction in which the number of favorable outcomes is the numerator and the total possible outcomes is the denominator.

   In this case, there is 1 favorite poster indicated out of 6 total posters.

   $$\frac{\text{favorable outcomes}}{\text{total possible outcomes}} = \frac{1}{6}$$

   Therefore, the probability that Carly knocked down her favorite poster is $\frac{1}{6}$.

2. How many ways can Carly arrange all 6 posters in a straight line on the same wall?

   The number of ways to arrange all items in a group of n items is equal to $n!$.

   In this case, there are 6 items.

   Calculate 6!.

   $6! = 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1 = 720$

   There are 720 ways for Carly to arrange the 6 posters in a straight line on the same wall.

3. If Carly decides to move 2 of the posters to the opposite wall, how many different combinations of 2 posters can she possibly select?

   When asked to determine the number ways you can select 2 posters from the total of 6 posters, you are being asked to find a combination. For combinations, the order in which items are chosen does not impact the result.

   The general formula for calculating a combination is ${}_nC_r = \frac{n!}{(n-r)!r!}$, where n is the total possible number of items to choose from and r is the number of items actually chosen.

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In this scenario, $n = 6$ and $r = 2$.

$_nC_r = \frac{n!}{(n-r)!r!}$	Formula for calculating a combination
$_{(6)}C_{(2)} = \frac{(6)!}{[(6)-(2)]!(2)!}$	Substitute 6 for n and 2 for r.
$_6C_2 = \frac{6!}{4!2!}$	Simplify.
$_6C_2 = \frac{6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}{(4 \bullet 3 \bullet 2 \bullet 1)(2 \bullet 1)}$	Apply the factorial.
$_6C_2 = \frac{720}{(24)(2)}$	Simplify.
$_6C_2 = \frac{720}{48}$	
$_6C_2 = 15$	

There are 15 ways Carly can choose 2 out of the 6 posters to put on the opposite wall.

Connection to the Lesson

- Students will expand upon their understanding of probabilities and apply this concept to more sophisticated probability questions.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- calculating the probability of failure given the probability of success (and vice versa) (S–CP.1)
- calculating factorials (no standard)
- calculating combinations (S–CP.9)

Introduction

Previously, we have worked with experiments and probabilities that have resulted in two outcomes: success and failure. Success is used to describe the outcomes that we are interested in and **failure** (sometimes called **undesirable outcomes** or **unfavorable outcomes**) is used to describe any other outcomes. For example, if calculating how many times an even number is rolled on a fair six-sided die, we would describe "success" as rolling a 2, 4, or 6, and "failure" as rolling a 1, 3, or 5. In this lesson, we will answer questions about the probability of x successes given the probability of success, p, and a number of trials, n.

Key Concepts

- A **trial** is each individual event or selection in an experiment or treatment.
- A **binomial experiment** is an experiment that satisfies the following conditions:
  - The experiment has a fixed number of trials.
  - Each trial is independent of the others.
  - There are only two outcomes: success and failure.
  - The probability of each outcome is constant from trial to trial.
- It is possible to predict the number of outcomes of binomial experiments.
- The **binomial probability distribution formula** allows us to determine the probability of success in a binomial experiment.
- The formula, $P = \binom{n}{x} p^x q^{n-x}$, is used to find the probability, P, of exactly x number of successes out of n trials, if the probability of success is p and the probability of failure is q.

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- This formula includes the following notation: $\binom{n}{x}$. You may be familiar with an alternate notation for combinations, such as ${}_nC_r$. The notations $\binom{n}{x}$ and ${}_nC_r$ are equivalent, and both are found using the formula for combinations: ${}_nC_r = \frac{n!}{(n-r)!r!}$, where n is the total number of items available to choose from and r is the number of items actually chosen.
- Recall that the probability of success, p, will always be at least 0 but no more than 1. In other words, the probability of success, p, cannot be negative and cannot be more than 1.
- The probabilities p and q should always sum to 1. This allows you to find the value of p or q given one or the other.
- For example, given p but not q, q can be calculated by subtracting p from 1 $(1 - p)$ or by solving the equation $p + q = 1$ for q.
- Sometimes it is necessary to calculate the probability of "at least" or "at most" of a certain event. In this case, apply the addition rule for mutually exclusive events. With this rule, it is possible to calculate the probability of more than one event occurring.
- **Mutually exclusive events** are events that cannot occur at the same time. For example, when tossing a coin, the coin can land heads up or tails up, but not both. "Heads" and "tails" are mutually exclusive events.
- The **addition rule for mutually exclusive events** states that when two events, A and B, are mutually exclusive, the probability that A or B will occur is the sum of the probability of each event. Symbolically, $P(A \text{ or } B) = P(A) + P(B)$.
- For example, the probability of rolling any number on a six-sided number cube is $\frac{1}{6}$. If you want to roll a 1 or a 2 and you can only roll once, the probability of getting *either* 1 or 2 on that roll is the sum of the probabilities for each individual number (or event):

$$P(1) + P(2) = \left(\frac{1}{6}\right) + \left(\frac{1}{6}\right) = \frac{2}{6} = \frac{1}{3}$$

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- You can use a graphing calculator to determine the probability of mutually exclusive events.

On a TI-83/84:

Step 1: Press [2ND][VARS] to bring up the distribution menu.

Step 2: Scroll down to A: binompdf(and press [ENTER].

Step 3: Enter values for *n, p,* and *x,* where *n* is the total number of trials, *p* is the probability of success entered in decimal form, and *x* is the number of successes.

Step 4: Press [)] to close the parentheses, then press [ENTER].

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow down to the calculator page icon (the first icon on the left) and press [enter].

Step 3: Press [menu]. Arrow down to 5: Probability, then arrow right to bring up the sub-menu. Arrow down to 5: Distributions, then arrow right and choose D: Binomial Pdf by pressing [enter].

Step 4: Enter values for *n, p,* and *x,* where *n* is the total number of trials, *p* is the probability of success entered in decimal form, and *x* is the number of successes. Arrow right after each entry to move between fields.

Step 5: Press [enter] to select OK.

- Either calculator will return the probability in decimal form.

Common Errors/Misconceptions

- mistakenly applying the binomial formula to experiments with more than two possible outcomes
- mistakenly believing that successes include only a positive outcome rather than the desirable outcome
- ignoring key words such as "at most," "no more than," or "exactly" when calculating the binomial distribution

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Guided Practice 1.4.2

Example 1

When tossing a fair coin 10 times, what is the probability the coin will land heads-up exactly 6 times?

1. Identify the needed information.

   To determine the likelihood of the coin landing heads-up on 6 out of 10 tosses, use the binomial probability distribution formula: $P = \binom{n}{x} p^x q^{n-x}$, where p is the probability of success, q is the probability of failure, n is the total number of trials, and x is the number of successes.

   To use this formula, we must determine values for p, q, n, and x.

2. Determine the probability of success, p.

   The probability of success, p, can be found by creating a fraction in which the number of favorable outcomes is the numerator and the total possible outcomes is the denominator.

   $$\frac{\text{favorable outcomes}}{\text{total possible outcomes}} = \frac{\text{tossing heads}}{\text{tossing heads or tails}} = \frac{1}{2}$$

   When tossing a fair coin, the probability of success, p, is $\frac{1}{2}$ or 0.5.

3. Determine the probability of failure, q.

   Since the value of p is known, calculate q by subtracting p from 1 ($q = 1 - p$) or by solving the equation $p + q = 1$ for q.

   Subtract p from 1 to find q.

$q = 1 - p$	Equation to find q given p
$q = 1 - (0.5)$	Substitute 0.5 for p.
$q = 0.5$	Simplify.

   The probability of failure, q, is 0.5.

SWB p. 123; SRB p. U1-163

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4. Determine the number of trials, n.

   The problem scenario specifies that the coin will be tossed 10 times.

   Each coin toss is a trial; therefore, $n = 10$.

5. Determine the number of successes, x.

   We are asked to find the probability of the coin landing heads-up 6 times.

   Tossing a coin that lands heads-up is the success in this problem; therefore, $x = 6$.

6. Calculate the probability of the coin landing heads-up 6 times.

   Use the binomial probability distribution formula to calculate the probability.

$$P = \binom{n}{x} p^x q^{n-x}$$ Binomial probability distribution formula

$$P(6) = \binom{(10)}{(6)} (0.5)^{(6)} (0.5)^{(10-6)}$$ Substitute 10 for n, 6 for x, 0.5 for p, and 0.5 for q.

$$P(6) = \binom{10}{6} 0.5^6 0.5^4$$ Simplify any exponents.

To calculate $\binom{10}{6}$, use the formula for calculating a combination.

$${}_nC_r = \frac{n!}{(n-r)!r!}$$ Formula for calculating a combination

$${}_{(10)}C_{(6)} = \frac{(10)!}{[(10)-(6)]!(6)!}$$ Substitute 10 for n and 6 for r.

$${}_{10}C_6 = \frac{10!}{4!6!}$$ Simplify.

$${}_{10}C_6 = 210$$

(*continued*)

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Substitute 210 for $\binom{10}{6}$ in the binomial probability distribution formula and solve.

$P(6) = \binom{10}{6} 0.5^6 0.5^4$ Previously determined equation

$P(6) = (210)0.5^6 0.5^4$ Substitute 210 for $\binom{10}{6}$.

$P(6) = (210)(0.015625)(0.0625)$ Simplify.

$P(6) \approx 0.205078125$

Written as a percentage rounded to the nearest whole number, $P(6) \approx 21\%$.

To calculate the probability on your graphing calculator, follow the steps appropriate to your model.

On a TI-83/84:

Step 1: Press [2ND][VARS] to bring up the distribution menu.

Step 2: Scroll down to A: binompdf(and press [ENTER].

Step 3: Enter values for *n, p,* and *x,* where *n* is the total number of trials, *p* is the probability of success entered in decimal form, and *x* is the number of desirable successes.

Step 4: Press [)] to close the parentheses, then press [ENTER].

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow down to the calculator page icon (the first icon on the left) and press [enter].

Step 3: Press [menu]. Arrow down to 5: Probability, then arrow right to bring up the sub-menu. Arrow down to 5: Distributions, then arrow right and choose D: Binomial Pdf by pressing [enter].

Step 4: Enter values for *n, p,* and *x,* where *n* is the total number of trials, *p* is the probability of success entered in decimal form, and *x* is the number of desirable successes. Arrow right after each entry to move between fields.

Step 5: Press [enter] to select OK.

Either calculator will return the probability in decimal form.

Converted to a fraction, 0.205078125 is equal to $\frac{105}{512}$.

The probability of tossing a fair coin heads-up 6 times out of 10 is $\frac{105}{512}$.



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Example 2

Of all the students who have signed up for physical education classes at a particular school, 65% are male and 35% are female. What is the likelihood, or probability, that a class of 15 students will include exactly 8 male students? Round your answer to the nearest percent.

1. Identify the needed information.

   To determine the likelihood of a physical education class of 15 students having exactly 8 male students, use the binomial probability distribution formula: $P = \binom{n}{x} p^x q^{n-x}$.

   To use this formula, we need to determine values for p (the probability of success), q (the probability of failure), n (the total number of trials), and x (the number of successes).

2. Identify the given information.

   In this example, the "trial" is choosing a student from the class.

   Since we are choosing from a class of 15 students, the number of trials, n, is equal to 15.

   A "success" would be choosing a male student. Therefore, the value of x is 8, the desired number of male students.

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3. Determine the unknown information.

The remaining variables in the formula for which we need values are p and q.

The problem statement asks for the probability of having 8 males in a class of 15 students, so p = the probability of choosing a male student. Therefore, q must represent the probability of choosing a female student.

We know that 65% of the students taking physical education classes are male.

The value of p, the probability of choosing a male student, can be found by converting 65% to a decimal.

$$65\% = \frac{65}{100} = 0.65$$

The value of p, the probability of choosing a male student, is 0.65.

The value of q, the probability of choosing a female student, can be found by calculating $1 - p$.

$q = 1 - p$	Equation for finding q given p
$q = 1 - (0.65)$	Substitute 0.65 for p.
$q = 0.35$	Simplify.

The value of q, the probability of choosing a female student, is 0.35.

4. Calculate the probability that a physical education class of 15 students will include exactly 8 male students.

Use the binomial probability distribution formula to calculate the probability.

$P = \binom{n}{x} p^x q^{n-x}$	Binomial probability distribution formula
$P(8) = \binom{(15)}{(8)} (0.65)^{(8)} (0.35)^{(15-8)}$	Substitute 15 for n, 8 for x, 0.65 for p, and 0.35 for q.
$P(8) = \binom{15}{8} 0.65^8 0.35^7$	Simplify any exponents.

(*continued*)

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To calculate $\binom{15}{8}$, use the formula for calculating a combination.

$_nC_r = \frac{n!}{(n-r)!r!}$	Formula for calculating a combination
$_{(15)}C_{(8)} = \frac{(15)!}{[(15)-(8)]!(8)!}$	Substitute 15 for n and 8 for r.
$_{15}C_8 = \frac{15!}{7!8!}$	Simplify.
$_{15}C_8 = 6435$	

Substitute 6,435 for $\binom{15}{8}$ in the binomial probability distribution formula and solve.

$P(8) = \binom{15}{8} 0.65^8 0.35^7$	Previously determined equation
$P(8) = (6435)(0.65)^8(0.35)^7$	Substitute 6,435 for $\binom{15}{8}$.
$P(8) = (6435)(0.03186448)(0.000643393)$	Simplify.
$P(8) \approx 0.131851745$	Continue to simplify.
$P(8) \approx 13\%$	Round to the nearest percent.

To calculate the probability on your graphing calculator, follow the steps outlined in Example 1.

The probability of having exactly 8 male students in a physical education class of 15 students is approximately 13%.



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Example 3

A new restaurant's menu claims that every entrée on the menu has less than 350 calories. A consumer advocacy group hired nutritionists to analyze the restaurant's claim, and found that 1 out of 25 entrées served contained more than 350 calories. If you go to the restaurant as part of a party of 4 people, determine the probability, to the nearest tenth of a percent, that half of your party's entrées actually contain more than 350 calories.

1. Identify the needed information.

   To determine the probability that exactly half of the 4 people in your party will be served an entrée that has more than 350 calories, use the binomial probability distribution formula: $P = \binom{n}{x} p^x q^{n-x}$.

   To use this formula, we need to determine values for p (the probability of success), q (the probability of failure), n (the total number of trials), and x (the number of successes).

2. Identify the given information.

   We need to determine the probability of exactly half of the entrées having more than 350 calories.

   The value of n, the number of people in the party, is 4.

   The value of x, half the people in the party, is 2.

   It is stated in the problem that the probability of this event happening is 1 in 25 entrées served; therefore, the value of p, the probability of an entrée being more than 350 calories, is $\frac{1}{25}$.

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3. Determine the unknown information.

The value of q, the probability of an entrée being less than 350 calories, can be found by calculating $1 - p$.

$q = 1 - p$	Equation for q given p
$q = 1 - \left(\frac{1}{25}\right)$	Substitute $\frac{1}{25}$ for p.
$q = \frac{24}{25}$	Simplify.

The value of q, the probability of an entrée being less than 350 calories, is $\frac{24}{25}$.

4. Calculate the probability that half of the meals served to your party contain more than 350 calories.

Use the binomial probability distribution formula to calculate the probability.

$P = \binom{n}{x} p^x q^{n-x}$	Binomial probability distribution formula
$P(2) = \binom{(4)}{(2)}\left(\frac{1}{25}\right)^{(2)}\left(\frac{24}{25}\right)^{(4-2)}$	Substitute 4 for n, 2 for x, $\frac{1}{25}$ for p, and $\frac{24}{25}$ for q.
$P(2) = \binom{4}{2}\left(\frac{1}{25}\right)^2\left(\frac{24}{25}\right)^2$	Simplify any exponents.

(*continued*)

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To calculate $\binom{4}{2}$, use the formula for calculating a combination.

$${}_nC_r = \frac{n!}{(n-r)!r!}$$ Formula for calculating a combination

$${}_{(4)}C_{(2)} = \frac{(4)!}{[(4)-(2)]!(2)!}$$ Substitute 4 for n and 2 for r.

$${}_4C_2 = \frac{4!}{2!2!}$$ Simplify.

$${}_4C_2 = 6$$

Substitute 6 for $\binom{4}{2}$ in the binomial probability distribution formula and solve.

$$P(2) = \binom{4}{2}\left(\frac{1}{25}\right)^2\left(\frac{24}{25}\right)^2$$ Previously determined equation

$$P(2) = (6)\left(\frac{1}{25}\right)^2\left(\frac{24}{25}\right)^2$$ Substitute 6 for $\binom{4}{2}$.

$$P(2) = 6\left(\frac{1}{625}\right)\left(\frac{576}{625}\right)$$ Simplify.

$$P(2) \approx 0.00884736$$ Continue to simplify.

$$P(2) \approx 0.88\%$$ Round to the nearest hundredth of a percent.

To calculate the probability on your graphing calculator, follow the steps outlined in Example 1.

If there are 4 people in your party, there is about a 0.88% chance that half of your party will be served entrées that have more than 350 calories.

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Example 4

Ten members of an extended family have set aside one day per month to get together for game night. If the probability of all 10 family members being present is $\frac{9}{10}$, what is the likelihood of all of them being present at least 10 times in one year?

1. Identify the needed information.

   To determine the likelihood of all 10 of the family members being present at least 10 times in one year, use the binomial probability distribution formula, $P = \binom{n}{x} p^x q^{n-x}$.

   To use this formula, we need to determine values for p (the probability of success), q (the probability of failure), n (the total number of trials), and x (the number of successes).

2. Identify the given information.

   We are being asked about a certain number of events happening out of a given number of events.

   There are two possible outcomes: all family members present or not all family members present.

   The value of n, the number of times the family gets together for one day each month in one year, is 12.

   The problem asks for the likelihood of all 10 family members being present *at least* 10 times; therefore, the value of x, the number of desirable occurrences, is 10, or 11, or 12.

   The value of p, the probability that all 10 family members are present, is $\frac{9}{10}$ or 0.9.

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3. Determine the unknown information.

The value of q, the probability of a family member missing a game night, can be found by calculating $1 - p$.

$q = 1 - p$	Equation for q given p.
$q = 1 - \left(\frac{9}{10}\right)$	Substitute $\frac{9}{10}$ for p.
$q = \frac{1}{10}$	Simplify.

The value of q, the probability of a family member missing a game night, is $\frac{1}{10}$ or 0.1.

4. Calculate the probability that all 10 family members will be present at least 10 times in one year.

In order to determine this probability, calculate the probability that all 10 family members are present 10 times, 11 times, and 12 times.

Use the binomial probability distribution formula to calculate the probability for when x = 10, 11, and 12.

Let x = 10.

$P = \binom{n}{x} p^x q^{n-x}$	Binomial probability distribution formula
$P(10) = \binom{(12)}{(10)} (0.9)^{(10)} (0.1)^{(12-10)}$	Substitute 12 for n, 10 for x, 0.9 for p, and 0.1 for q.
$P(10) = \binom{12}{10} 0.9^{10} 0.1^2$	Simplify any exponents.

(*continued*)

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To calculate $\binom{12}{10}$, use the formula for calculating a combination.

$${}_nC_r = \frac{n!}{(n-r)!r!}$$ Formula for calculating a combination

$${}_{(12)}C_{(10)} = \frac{(12)!}{[(12)-(10)]!(10)!}$$ Substitute 12 for n and 10 for r.

$${}_{12}C_{10} = \frac{12!}{2!10!}$$ Simplify.

$${}_{12}C_{10} = 66$$

Substitute 66 for $\binom{12}{10}$ in the binomial probability distribution formula and solve.

$$P(10) = \binom{12}{10}0.9^{10}0.1^{2}$$ Previously determined equation

$$P(10) = (66)0.9^{10}0.1^{2}$$ Substitute 66 for $\binom{12}{10}$.

$P(10) = (66)(0.3486784401)(0.01)$ Simplify.

$P(10) \approx 0.23013$

Let $x = 11$.

$$P = \binom{n}{x}p^{x}q^{n-x}$$ Binomial probability distribution formula

$$P(11) = \binom{(12)}{(11)}(0.9)^{(11)}(0.1)^{(12-11)}$$ Substitute 12 for n, 11 for x, 0.9 for p, and 0.1 for q.

$$P(11) = \binom{12}{11}0.9^{11}0.1^{1}$$ Simplify any exponents.

(*continued*)

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To calculate $\binom{12}{11}$, use the formula for calculating a combination.

$${}_nC_r = \frac{n!}{(n-r)!r!}$$ Formula for calculating a combination

$${}_{(12)}C_{(11)} = \frac{(12)!}{[(12)-(11)]!(11)!}$$ Substitute 12 for n and 11 for r.

$${}_{12}C_{11} = \frac{12!}{1!11!}$$ Simplify.

$${}_{12}C_{11} = 12$$

Substitute 12 for $\binom{12}{11}$ in the binomial probability distribution formula and solve.

$$P(11) = \binom{12}{11}0.9^{11}0.1^{1}$$ Previously determined equation

$$P(11) = (12)0.9^{11}0.1^{1}$$ Substitute 12 for $\binom{12}{11}$.

$P(11) = (12)(0.3138105961)(0.1)$ Simplify.

$P(11) \approx 0.37657$

Let $x = 12$.

$$P = \binom{n}{x} p^x q^{n-x}$$ Binomial probability distribution formula

$$P(12) = \binom{(12)}{(12)}(0.9)^{(12)}(0.1)^{(12-12)}$$ Substitute 12 for n, 12 for x, 0.9 for p, and 0.1 for q.

$$P(12) = \binom{12}{12}0.9^{12}0.1^{0}$$ Simplify any exponents.

(*continued*)

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To calculate $\binom{12}{12}$, use the formula for calculating a combination.

$${}_nC_r = \frac{n!}{(n-r)!r!}$$ Formula for calculating a combination

$${}_{(12)}C_{(12)} = \frac{(12)!}{[(12)-(12)]!(12)!}$$ Substitute 12 for n and 12 for r.

$${}_{12}C_{12} = \frac{12!}{0!12!}$$ Simplify. (Recall that 0! = 1.)

$${}_{12}C_{12} = 1$$

Substitute 1 for $\binom{12}{12}$ in the binomial probability distribution formula and solve.

$$P(12) = \binom{12}{12}0.9^{12}0.1^{0}$$ Previously determined equation

$$P(12) = (1)0.9^{12}0.1^{0}$$ Substitute 1 for $\binom{12}{12}$.

$P(12) = (1)(0.2824295365)(1)$ Simplify. (Remember that any number raised to a power of 0 is equal to 1.)

$P(12) \approx 0.28243$

When determining the probability of the family being present *at least* 10 times, the total probability is comprised of the sum of the three probabilities.

$P(\text{at least 10 times}) = P(10) + P(11) + P(12)$

$P(\text{at least 10 times}) \approx 0.23013 + 0.37657 + 0.28243$

$P(\text{at least 10 times}) \approx 0.88913$

$P(\text{at least 10 times}) \approx 89\%$

There is about an 89% chance that all 10 family members will be present at least 10 times in a given year.

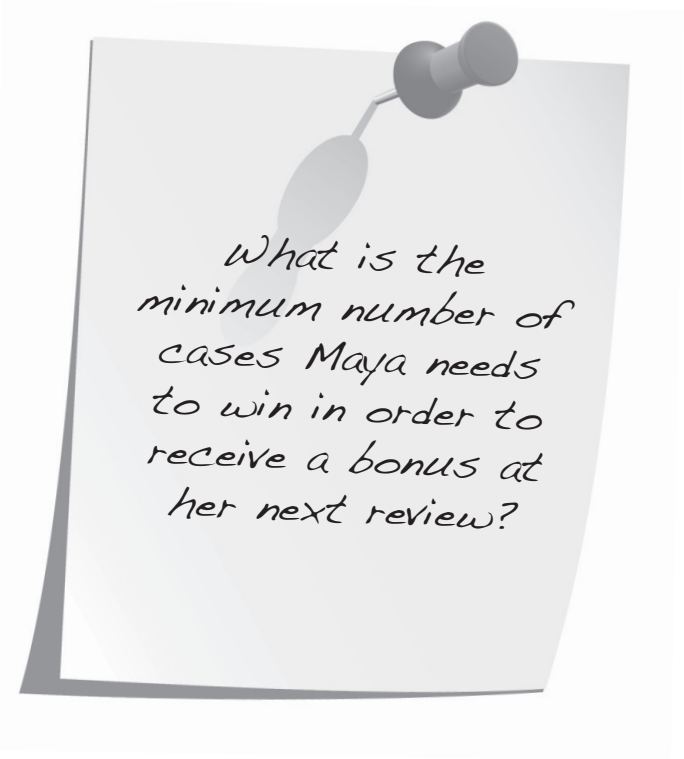


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Problem-Based Task 1.4.2: When Will She Win a Bonus?

SMP	
1 ✓	2 ✓
3	4
5	6
7	8

A law firm awards bonuses to its lead attorneys based on how many cases the attorneys win. Bonuses are determined at each lawyer's performance review, which takes place after every 35 completed cases. Maya is one of the firm's top lawyers; she has a record of winning 78% of her cases. If Maya's statistics-savvy superiors would like her to have a minimum 60% chance of earning her bonus based on her past performance, what is the minimum number of cases Maya needs to win in order to receive a bonus at her next review?



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Problem-Based Task 1.4.2: When Will She Win a Bonus?

Coaching

a. How can Maya's superiors determine that the likelihood of Maya winning her cases will be 60%?

b. What is the probability that Maya will win all 35 cases?

c. What is the probability that Maya will win 34 cases?

d. What is the probability that Maya will win 34 or 35 cases?

e. What is the probability that Maya will win 33 or more cases? 32 or more cases?

f. What is the minimum number of cases Maya will need to win in order to receive a bonus at her next review?

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Problem-Based Task 1.4.2: When Will She Win a Bonus?

Coaching Sample Responses

a. How can Maya's superiors determine that the likelihood of Maya winning her cases will be 60%?

Maya's superiors can use the binomial probability distribution formula, $P = \binom{n}{x} p^x q^{n-x}$, and her record of winning cases to determine the likelihood of her winning each of the 35 cases she must complete before her next review.

b. What is the probability that Maya will win all 35 cases?

In order to use the binomial probability distribution formula, identify n, x, p, and q, where n is equal to the total number of completed cases, p is equal to the probability of success (winning a case), q is equal to the probability of failure, and x is equal to the total number of successes (cases won) we are looking for.

Identify the given information.

The value of n, the total number of cases Maya needs to complete, is 35.

The value of p, the probability of winning a case, is 0.78.

The value of q, the probability of failure, is 1 – 0.78, or 0.12.

The value of x, the total number of cases won, is 35. Substitute these values into the formula to determine $P(35)$, the probability of winning all 35 cases.

$$P = \binom{n}{x} p^x q^{n-x}$$

$$P = \binom{(35)}{(35)} (0.78)^{(35)} (0.12)^{(35-35)}$$

$$P \approx 0.000167$$

$$P \approx 0.0167\%$$

The probability that Maya will win all 35 cases is approximately 0.000167 or 0.0167%.

c. What is the probability that Maya will win 34 cases?

This time, the value of x, the number of cases we are looking to win, is 34.

The values of n, p, and q remain the same.

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Substitute these values into the formula to determine $P(34)$, the probability of winning 34 cases.

$$P = \binom{n}{x} p^x q^{n-x}$$

$$P(34) = \binom{(35)}{(34)} (0.78)^{(34)} (0.12)^{(35-34)}$$

$P(34) \approx 0.0016508$

$P(34) \approx 0.17\%$

The probability of Maya winning 34 cases is 0.0017 or 0.17%.

d. What is the probability that Maya will win 34 or 35 cases?

To determine the likelihood of two events, apply the addition rule for mutually exclusive events.

Use the previously determined values to find $P(34 \text{ or } 35)$, the probability of winning 34 or 35 cases.

$P(34 \text{ or } 35) = P(34) + P(35)$

$P(34 \text{ or } 35) \approx 0.0016508 + 0.000167$

$P(34 \text{ or } 35) \approx 0.001818$

$P(34 \text{ or } 35) \approx 0.18\%$ (rounded to the nearest hundredth)

The probability of winning 34 or 35 cases is 0.0018 or 0.18%.

e. What is the probability that Maya will win 33 or more cases? 32 or more cases?

Apply the binomial probability distribution formula to find values for $P(33)$ and $P(32)$ cases won, then use the addition rule for mutually exclusive events to determine the cumulative probability for $P(33 \text{ or more})$ and $P(32 \text{ or more})$.

$P(33) \approx 0.0079156$

$P(33 \text{ or more}) \approx 0.009733$

$P(32) \approx 0.0245587$

$P(32 \text{ or more}) \approx 0.034292$

The probability that Maya will win 33 or more cases is approximately 0.009733 or 0.97%. The probability that Maya will win 32 or more cases is approximately 0.034292 or 3.43%.

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f. What is the minimum number of cases Maya will need to win in order to receive a bonus at her next review?

The probability of winning 32 or more cases does not approach the 60% win rate required to earn a bonus.

By trial and error, we can continue to apply the binomial probability distribution formula to different numbers of cases won, until we find that the likelihood of winning a certain number of cases is greater than 60%.

Continuing to apply the formula and the addition rule will produce a set of results as follows:

Probability of cases won	Cumulative probability
$P(35) \approx 0.0001670$	
$P(34) \approx 0.0016508$	$P(34 \text{ or more}) \approx 0.001818$
$P(33) \approx 0.0079156$	$P(33 \text{ or more}) \approx 0.009733$
$P(32) \approx 0.0245587$	$P(32 \text{ or more}) \approx 0.034292$
$P(31) \approx 0.0554145$	$P(31 \text{ or more}) \approx 0.089707$
$P(30) \approx 0.0969042$	$P(30 \text{ or more}) \approx 0.186611$
$P(29) \approx 0.1366598$	$P(29 \text{ or more}) \approx 0.323271$
$P(28) \approx 0.1596868$	$P(28 \text{ or more}) \approx 0.482957$
$P(27) \approx 0.1576395$	$P(27 \text{ or more}) \approx 0.640597$

Based on the information in the table, Maya needs to win 27 or more cases in order to earn a bonus, since her probability of winning 27 or more cases is 64%.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.4.2: The Binomial Distribution

For each problem, calculate the probability, *P*, using the given information. Round answers to the nearest hundredth. Use the formulas for binomial probability distribution and for calculating combinations.

1. When rolling a fair six-sided die 12 times, what is the probability of rolling a 5 exactly 2 times?

2. What is the probability of heads coming up 7 times out of 10 when tossing a fair coin?

3. A new product reportedly has a $\frac{1}{150}$ defect rate. What is the probability of having no defective products in a shipment of 100 items?

4. A moving company's website advertises that its movers arrive on time for 90% of appointments. What is the likelihood that the movers are on time once if the movers have 3 appointments in one week?

5. A commercial for eye cream claims that "85% of women saw a reduction in wrinkles" after using the product. What is the likelihood that a focus group of 10 women chosen to try the product contains 2 women who did not see a reduction in wrinkles?

continued

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6. What is the probability of a fair coin landing heads-up 3 times in 6 tosses?

7. What is the likelihood of a fair six-sided die coming up with a number greater than 2 on 9 out of 10 throws?

8. In Las Vegas, it generally rains only once every 51 days. If you have booked a 7-day vacation, what are the chances that all 7 days will be sunny?

9. While playing a board game, you throw 2 dice to determine how many spaces you move per turn. If your roll results in 2 matching numbers, or doubles, you win an extra turn. What is the probability that you roll doubles 3 times in 10 turns?

10. The spinner in a children's game includes 7 equally sized sections: blue, green, purple, green, yellow, red, and orange. What is the probability that the spinner will land on green 4 times in 14 turns?

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Lesson 1.4.3: Estimating Sample Means

Warm-Up 1.4.3

A coach is analyzing his players' weekly off-season exercise routines. He has gathered the following data, including each player's age, the number of miles they run, and the number of hours spent weight training.

Player	Age (years)	Running (miles)	Weight training (hours)
Angel	15	40	2
Blake	17	25	5
Jaime	16	10	7
Pat	15	0	12
Rene	17	30	3

1. What is the mean age of this sample?

2. What is the mean of the miles spent running for this sample?

3. What is the mean of the hours spent weight training for this sample?

4. What is the standard deviation of the miles that were run for this data set to the nearest tenth?

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Instruction

Lesson 1.4.3: Estimating Sample Means

Common Core State Standard

S–IC.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.★

Warm-Up 1.4.3 Debrief

1. What is the mean age of this sample?

   In order to calculate the mean age of the players, first find the sum of the ages.

   15 + 17 + 16 + 15 + 17 = 80

   Then divide the sum by the number of quantities in the data set, 5.

   80 ÷ 5 = 16

   The mean of the sample ages is 16 years.

2. What is the mean of the miles spent running for this sample?

   In order to calculate the mean of the miles run, first find the sum of the miles run.

   40 + 25 + 10 + 0 + 30 = 105

   Divide the sum by the number of quantities in our data set, 5.

   105 ÷ 5 = 21

   The mean of the miles run for this sample is 21 miles.

3. What is the mean of the hours spent weight training for this sample?

   In order to calculate the mean of the hours spent weight training, first find the sum of the hours.

   2 + 5 + 7 + 12 + 3 = 29

   Divide the sum by the number of quantities in our data set, 5.

   29 ÷ 5 = 5.8

   The mean of the hours spent weight training for this sample is 5.8 hours.

4. What is the standard deviation of the miles that were run for this data set to the nearest tenth?

   The standard deviation can be found using the formula $s=\sqrt{\frac{1}{n}\sum\left(x_i-\overline{x}\right)^2}$, where n is the number of values in the data set, x is each value of the data set, and $\overline{x}$ is the mean.

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The number of values in the data set, n, is 5.

The values of x include 40, 25, 10, 0, and 30. We can assign these values as follows: $x_1 = 40$, $x_2 = 25$, $x_3 = 10$, $x_4 = 0$, and $x_5 = 30$.

The value of the mean, $\bar{x}$, was determined to be 21.

$$s = \sqrt{\frac{1}{n}\sum\left(x_i - \bar{x}\right)^2}$$ Formula for standard deviation

$$s = \sqrt{\frac{1}{(5)}\sum\left[x_i - (21)\right]^2}$$ Substitute for 5 for n and 21 for $\bar{x}$.

Expand the summation as shown below.

$$s = \sqrt{\frac{1}{5}\left[\left(x_1 - 21\right)^2 + \left(x_2 - 21\right)^2 + \left(x_3 - 21\right)^2 + \left(x_4 - 21\right)^2 + \left(x_5 - 21\right)^2\right]}$$

Substitute 40 for x_1, 25 for x_2, 10 for x_3, 0 for x_4, and 30 for x_5, as shown.

$$s = \sqrt{\frac{1}{5}\left[(40-21)^2 + (25-21)^2 + (10-21)^2 + (0-21)^2 + (30-21)^2\right]}$$

$$s = \sqrt{\frac{1}{5}\left[(19)^2 + (4)^2 + (-11)^2 + (-21)^2 + (9)^2\right]}$$ Simplify.

$$s = \sqrt{\frac{1}{5}(361 + 16 + 121 + 441 + 81)}$$

$$s = \sqrt{\frac{1}{5}(1020)}$$

$$s = \sqrt{204} = 14.28285686$$

$$s \approx 14.3$$

The standard deviation of miles that were run for this data set is approximately 14.3 miles.

Connection to the Lesson

- Students will continue to work with mean and standard deviation and apply these calculations to sample populations.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- calculating mean (6.SP.5c)
- calculating standard deviation (no standard)

Introduction

The previous lesson discussed how to calculate a sample proportion and how to calculate the standard error of the population proportion. This lesson explores sample means and their relationship to population means. Since this lesson involves surveys with populations that are too large to feasibly calculate, it is necessary to calculate estimates and standard errors based on samples.

Key Concepts

- The **population mean**, or **population average**, is calculated by first finding the sum of all quantities in the population, and then dividing the sum by the total number of quantities in the population. This value is represented by μ.
- The population mean can be estimated when the mean of a sample of the population, $\bar{x}$, is known.
- The **sample mean**, $\bar{x}$, is the sum of all the quantities in a sample divided by the total number of quantities in the sample. It is also called the **sample average**.
- The **standard error of the mean**, SEM, is a measure of the variability of the mean of a sample.
- **Variability,** or **spread**, refers to how the data is spread out with respect to the mean.
- The SEM can be calculated by dividing the standard deviation, *s*, by the square root of the number of elements in the sample, *n*; that is, $\text{SEM} = \frac{s}{\sqrt{n}}$.
- When the standard error of the mean is small, or close to 0, then the sample mean is likely to be a good estimate of the population mean.
- It is also important to note that the standard error of the mean will decrease when the standard deviation decreases and the sample size increases.

Common Errors/Misconceptions

- confusing the formula for standard error of the proportion (SEP) with the formula for the standard error of the mean (SEM)

SRB p. U1-179

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Instruction

Guided Practice 1.4.3

Example 1

The manager of a car dealership would like to determine the average years of ownership for a new vehicle. He found that a sample of 25 customers who bought new vehicles owned that vehicle for 7.8 years, with a standard deviation of 2.5 years. What is the standard error for this sample mean?

1. Identify the given information.

   As stated in the problem, the sample is made up of 25 customers, so $n = 25$.

   We are also given that the standard deviation of the average years of ownership is 2.5 years, so $s = 2.5$.

2. Determine the standard error of the mean.

   The formula for the standard error of the mean is $\text{SEM} = \frac{s}{\sqrt{n}}$, where s represents the standard deviation and n is the sample size.

   $\text{SEM} = \frac{s}{\sqrt{n}}$ Formula for the standard error of the mean

   $\text{SEM} = \frac{(2.5)}{\sqrt{(25)}}$ Substitute 2.5 for s and 25 for n.

   $\text{SEM} = \frac{2.5}{5}$ Simplify.

   $\text{SEM} = 0.5$

   The standard error of the mean for a sample of 25 customers who owned their vehicle for 7.8 years with a standard deviation of 2.5 years is equal to 0.5 year. This mean that although the average ownership is for 7.8 years, the standard error of 0.5 year tells us that the ownership actually varies between 7.8 – 0.5 and 7.8 + 0.5. Therefore, the ownership period for this sample varies from 7.3 years to 8.3 years.

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Example 2

In 2011, the average salary for a sample of NCAA Division 1A head football coaches was $1.5 million per year, with a standard deviation of $1.07 million. If there are 100 coaches in this sample, what is the standard error of the mean? What can you predict about the population mean based on the sample mean and its standard error?

1. Identify the given information.

   As stated in the problem, the sample is 100 coaches, so $n = 100$.

   Also stated is the standard deviation for the sample mean (average salary): $s = 1.07$.

2. Determine the standard error of the mean.

   The formula for the standard error of the mean is $\text{SEM} = \frac{s}{\sqrt{n}}$, where s represents the standard deviation of the sample and n is the sample size.

   $\text{SEM} = \frac{s}{\sqrt{n}}$ Formula for the standard error of the mean

   $\text{SEM} = \frac{(1.07)}{\sqrt{(100)}}$ Substitute 1.07 for s and 100 for n.

   $\text{SEM} = \frac{1.07}{10}$ Simplify.

   $\text{SEM} = 0.107$

   The standard error of the mean is 0.107.

   In this situation, we are calculating salaries in millions of dollars.

   If we multiply 0.107 by $1,000,000, we find that the SEM is about $107,000 for this sample of 100 coaches.

   This means that, based on the sample mean salary of $1.5 million, this amount actually varies from $1.5 million + $107,000 ($1,607,000) to $1.5 million – $107,000 ($1,393,000). The population mean is likely to be within these two values.

   The SEM allows us to determine the range within which the population mean is likely to be. As the sample gets larger, n will get larger, and since n is in the denominator, the SEM will get smaller. As we increase the sample, the mean of the sample becomes a better estimate of the mean of the population.

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Example 3

In a study of 64 patients participating in a test of a new iron supplement, the standard error of the mean for the sample was found to be 1.625. What was the standard deviation for this sample population mean?

1. Identify the given information.

   As stated in the problem, the sample is made up of 64 participants, so $n = 64$.

   Also stated is the standard error of the mean, so SEM = 1.625.

2. Determine the standard deviation for this population mean.

   The formula for the standard error of the mean is $\text{SEM} = \frac{s}{\sqrt{n}}$, where s represents the standard deviation and n is the sample size.

$\text{SEM} = \frac{s}{\sqrt{n}}$	Formula for the standard error of the mean
$(1.625) = \frac{s}{\sqrt{(64)}}$	Substitute 1.625 for the SEM and 64 for n.
$1.625 = \frac{s}{8}$	Simplify.
$13 = s$	Solve for s.

   The standard deviation for this sample population is 13.

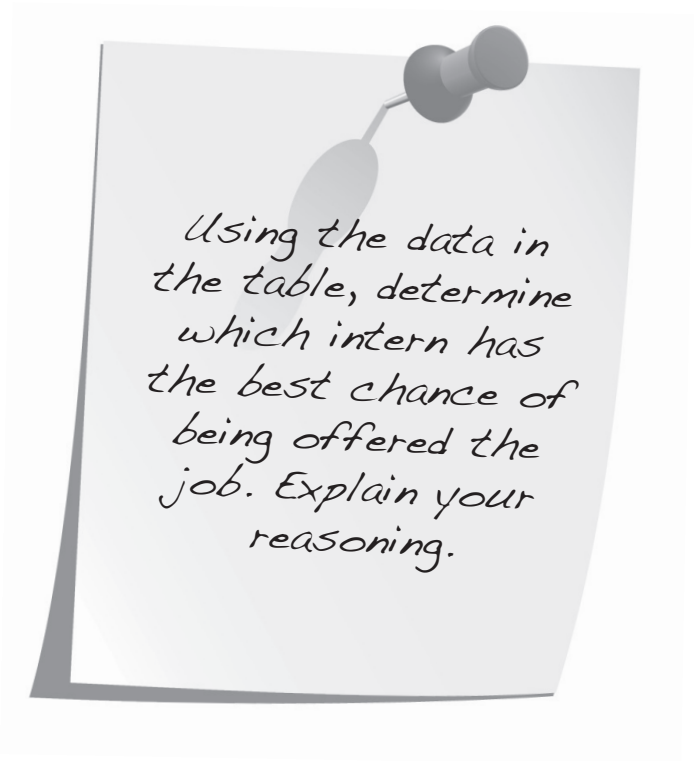
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Problem-Based Task 1.4.3: Job Competition

SMP	
1 ✓	2
3 ✓	4
5	6
7	8 ✓

Some recent graduates working internships for a financial company are comparing their stock picks. Their chances of being offered a full-time job with the company depend on the performance of the stocks in which they've invested the company's money. The following table details each intern's average profit per share purchased, the standard deviation of the profit per share, and the total number of shares each intern purchased on the company's behalf. Each intern has to make a presentation to a supervisor on how much the investments have earned, using statistical data for justification. Using the data in the table, determine which intern has the best chance of being offered the job. Explain your reasoning.

Intern	Average profit per share purchased	Standard deviation	Number of shares purchased
Leonard	$4.25	$0.45	350
Mae	$4.50	$0.58	185
Patrick	$2.75	$2.00	125
Sajeena	$1.75	$1.75	336
William	$2.50	$0.15	512



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Problem-Based Task 1.4.3: Job Competition

Coaching

a. What is the standard error of the mean for each intern? Round your answer to the nearest thousandth.

b. Which intern had the highest average profit per share? How will this benefit the company?

c. Which intern's portfolio had the lowest standard deviation? How will this benefit the company?

d. Which intern had the highest number of shares in his or her portfolio? How will this benefit the company?

e. Which intern's SEM stands out and why?

f. What does the SEM indicate about the performance of the intern identified in part e?

g. Which intern has the best chance of being offered the job? Explain your reasoning.

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Instruction

Problem-Based Task 1.4.3: Job Competition

Coaching Sample Responses

a. What is the standard error of the mean for each intern? Round your answer to the nearest thousandth.

The formula for the standard error of the mean is $\text{SEM} = \frac{s}{\sqrt{n}}$, where s represents the standard deviation and n is the sample size.

To find the SEM for each intern, substitute the values as given in the table.

Leonard's SEM can be found by substituting 0.45 for s and 350 for n.

$$\text{SEM} = \frac{(0.45)}{\sqrt{(350)}}$$

$$\text{SEM} = 0.024054$$

$$\text{SEM} \approx 0.024$$

Mae's SEM can be found by substituting 0.58 for s and 185 for n.

$$\text{SEM} = \frac{(0.58)}{\sqrt{(185)}}$$

$$\text{SEM} = 0.042642$$

$$\text{SEM} \approx 0.043$$

Patrick's SEM can be found by substituting 2.00 for s and 125 for n.

$$\text{SEM} = \frac{(2.00)}{\sqrt{(125)}}$$

$$\text{SEM} = 0.178885$$

$$\text{SEM} \approx 0.1789$$

Sajeena's SEM can be found by substituting 1.75 for s and 336 for n.

$$\text{SEM} = \frac{(1.75)}{\sqrt{(336)}}$$

$$\text{SEM} = 0.09547$$

$$\text{SEM} \approx 0.095$$

William's SEM can be found by substituting 0.15 for s and 512 for n.

$$\text{SEM} = \frac{(0.15)}{\sqrt{(512)}}$$

$$\text{SEM} = 0.006629$$

$$\text{SEM} \approx 0.007$$

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b. Which intern had the highest average profit per share? How will this benefit the company?

Mae had the highest average profit, with $4.50 per share, exceeding second-place Leonard's average profit by $0.25.

Having the highest average profit is a benefit to the company because it shows that Mae's stock picks are earning, on average, more money for the company.

c. Which intern's portfolio had the lowest standard deviation? How will this benefit the company?

The intern with the lowest standard deviation is William. His standard deviation was only $0.15.

Having the lowest standard deviation indicates that William's stock choices are more consistent. His stock choices, on average, earned about the same amount of money and with less fluctuation in profits than the stocks chosen by the other interns.

d. Which intern had the highest number of shares in his or her portfolio? How will this benefit the company?

The intern with the highest number of shares in his portfolio is William, with 512 shares.

Investing in a high number of shares benefits the company by maximizing potential profits while minimizing the risk of investment—concentrating company funds in too few stocks would magnify the damage to profits if the stocks don't perform well.

e. Which intern's SEM stands out and why?

William's SEM (0.007) stands out because it is so much lower than that of the other interns.

f. What does the SEM indicate about the performance of the intern identified in part e?

Standard error of the mean takes into account both standard deviation and the size of the population, so the performance of an intern with a low SEM would indicate, in this situation, a higher number of shares and lower standard deviation; i.e., William has chosen a large number of shares that have generated profits, with relatively little fluctuation in those profits.

g. Which intern has the best chance of being offered the job? Explain your reasoning.

Answers may vary. Mae has a good chance because she had the highest average profit.

William also appears to be in the running for the job offer because he had the most shares, the lowest standard deviation, and the lowest SEM.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.4.3: Estimating Sample Means

Determine the standard error of the mean for each of the following situations. Use the formula $\text{SEM} = \frac{s}{\sqrt{n}}$, where s represents the standard deviation and n is the sample size. Round answers to the nearest hundredth.

1. A survey of 18 students found that they spend $300 per month for car-related expenses, with a standard deviation of $99.

2. A clinical trial found that blood pressure dropped an average of 12 points with a standard deviation of 7 points for 49 participants who regularly meditated for 15 minutes per day.

3. A group of 5 students who did poorly on a college entrance test took a test-preparation course offered on Saturdays. After finishing the course and retaking the test, their scores increased by an average of 100 points, with a standard deviation of 16 points.

4. A randomly selected sample of 100 people was asked to count the number of contacts in their phone. The average number of contacts was 250, with a standard deviation of 100 contacts.

5. Arena workers polled the first 90 people in line for a concert and asked each person how much they had paid for their ticket. The average was $125, with a standard deviation of $57.

continued

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6. A sample of 3,000 middle-aged men found that their average weight was 250, with the standard deviation being 12 pounds.

7. A school district's transportation director reviewed the average distance from a sample of students' homes to their schools. She found that, in the 125-student sample, the average distance was 5.6 miles, with a standard deviation of 1.85 miles.

8. A baseball team with 25 players has an overall batting average of 0.240, with a standard deviation of 0.025.

9. An analysis of 41 items on a café's menu found that the menu items had an average of 450 calories, with a standard deviation of 223 calories.

10. A music reporter studied the average length of CDs issued by a particular record label. On 500 CDs, the average length was 33 minutes, with a standard deviation of 4 minutes.

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Lesson 1.4.4: Estimating with Confidence

Warm-Up 1.4.4

When trying to predict the winner of a major election, it is common for polling companies to ask voters whom they voted for as the voters exit the polling station. These exit poll results are then reported, and predictions about the winner are made well before all of the actual votes have been counted. Commonly, the results are reported along with a margin of error. The results of a particular exit poll are listed in the following table. Notice the footnote showing that the margin of error is ±3%. This indicates that the actual election results could be 3% higher (+3%) or 3% lower (–3%) than the reported exit poll results.

Exit Poll Results*	
Candidate	**Percentage of votes won**
Archer	30%
Benton	34%
Castellano	32%

*Margin of error: ±3%

1. Based on the exit poll results and a margin of error of ±3%, determine the range that would represent candidate Archer's results.

2. Based on the exit poll results and a margin of error of ±3%, determine the range that would represent candidate Benton's results.

3. Based on the exit poll results and a margin of error of ±3%, determine the range that would represent candidate Castellano's results.

4. Based on your answers for problems 1–3, which candidate do you predict won the election? Explain your answer.

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Lesson 1.4.4: Estimating with Confidence

Common Core State Standard

S–IC.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.★

Warm-Up 1.4.4 Debrief

1. Based on the exit poll results and a margin of error of ±3%, determine the range that would represent candidate Archer's results.

   According to the table of tallied exit poll responses, Archer won 30% of the vote. The margin of error tells us that this result may be 3% too high, or 3% too low.

   To determine the range that would represent Archer's results, you must add 3% to the poll result, and then subtract 3% from the poll result.

   To determine the lower bound of the range, subtract 3% from 30%.

   $30 - 3 = 27$

   To determine the upper bound of the range, add 3% to 30%.

   $30 + 3 = 33$

   The range that would represent Archer's results is 27–33%.

   Archer could have between 27% and 33% of the vote.

2. Based on the exit poll results and a margin of error of ±3%, determine the range that would represent candidate Benton's results.

   According to the table, Benton won 34% of the vote. The margin of error tells us that this result may be 3% too high, or 3% too low.

   To determine the range that would represent Benton's results, again add 3% to the poll result, and then subtract 3% from the poll result.

   To determine the lower bound of the range, subtract 3% from 34%.

   $34 - 3 = 31$

   To determine the upper bound of the range, add 3% to 34%.

   $34 + 3 = 37$

   The range that would represent Benton's results is 31–37%.

   Benton could have between 31% and 37% of the vote.

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3. Based on the exit poll results and a margin of error of ±3%, determine the range that would represent candidate Castellano's results.

   According to the table, Castellano won 32% of the vote. The margin of error tells us that this result may be 3% too high, or 3% too low.

   To determine the range that would represent Castellano's results, add 3% to the poll result, and then subtract 3% from the poll result.

   To determine the lower bound of the range, subtract 3% from 32%.

   $32 - 3 = 29$

   To determine the upper bound of the range, add 3% to 32%.

   $32 + 3 = 35$

   The range that would represent Castellano's results is 29–35%.

   Castellano could have between 29% and 35% of the vote.

4. Based on your answers for problems 1–3, which candidate do you predict won the election? Explain your answer.

   At first glance, it appears as though Benton won the election. Benton has the highest percentage in the table and the highest upper-bound percentage; however, there are several instances of overlap in the ranges identified. The ranges of all three candidates include the possibility of earning 31%, 32%, and 33% of the vote. While it is likely that Benton won the election, the overlaps in the margins of error make the election too close to call based on these polling results.

Connection to the Lesson

- News media often report preliminary election results and predictions using a margin of error. Students will explore the meaning of this information.
- Students will also learn how to calculate the margin of error and how it applies to confidence intervals.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- calculating sample proportions (S–IC.4)
- calculating sample means (S–IC.4)
- calculating the standard error of the proportion (S–IC.4)
- calculating the standard error of the mean (S–IC.4)
- calculating standard deviation (no standard)

Introduction

Studying the normal curve in previous lessons has revealed that normal data sets hover around the average, and that most data fits within intervals. Knowing this, it is possible to calculate the range within which most of the population's data stays, to a chosen degree. Calculations can reveal the interval within which 95% of the data will likely be found, or 80% of it, or some other appropriate percentage depending on the information desired. Making these calculations helps with understanding the level of assurance we can have in our estimates.

Key Concepts

- Since we are estimating based on sample populations, our calculations aren't always going to be 100% true to the entire population we are studying.
- Often, a **confidence level** is determined. Otherwise known as the **level of confidence**, the confidence level is the probability that a parameter's value can be found in a specified interval.
- The confidence level is often reported as a percentage and represents how often the true percentage of the entire population is represented.
- A 95% confidence level means that you are 95% certain of your results. Conversely, a 95% confidence level means you are 5% *uncertain* of your results, since 100 – 95 = 5. Recall that you cannot be more than 100% certain of your results. A 95% confidence level also means that if you were to repeat the study several times, you would achieve the same results 95% of those times.
- Once the confidence level is determined, we can expect the data of repeated samples to follow the same general parameters. Parameters are the numerical values representing the data and include proportion, mean, and variance.

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Instruction

- To help us report how accurate we believe our sample to be, we can calculate the **margin of error**.
- The margin of error is a quantity that represents how confident we are with our calculations; it is often abbreviated as MOE.
- It is important to note that the margin of error can be decreased by increasing the sample size or by decreasing the level of confidence.
- **Critical values**, also known as z_c**-values**, measure the number of standards of error to be added to or subtracted from the mean in order to achieve the desired confidence level.
- The following table shows common confidence levels and their corresponding z_c-values.

Common Critical Values

Confidence level	99%	98%	96%	95%	90%	80%	50%
Critical value (z_c)	2.58	2.33	2.05	1.96	1.645	1.28	0.6745

- Use the following formulas when calculating the margin of error.

Margin of error	Formula
Margin of error for a sample mean	$\text{MOE} = \pm z_c \frac{s}{\sqrt{n}}$, where s = standard deviation and n = sample size
Margin of error for a sample proportion	$\text{MOE} = \pm z_c \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, where $\hat{p}$ = sample proportion and n = sample size

- If we apply the margin of error to a parameter, such as a proportion or mean, we are able to calculate a range called a **confidence interval**, abbreviated as CI. This interval represents the true value of the parameter in repeated samples.

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Instruction

- Use the following formulas when calculating confidence intervals.

Confidence interval	Formula
Confidence interval for a sample population with proportion $\hat{p}$	$CI = \hat{p} \pm z_c \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, where $\hat{p}$ = sample proportion, z_c = critical value, and n = sample size
Confidence interval for a sample population with mean $\overline{x}$	$CI = \overline{x} \pm z_c \frac{s}{\sqrt{n}}$, where s = standard deviation, $\overline{x}$ = sample population mean, and n = sample size

- Confidence intervals are often reported as a decimal and are frequently written using interval notation. For example, the notation (4, 5) indicates a confidence interval of 4 to 5.
- A wider confidence interval indicates a less accurate estimate of the data, whereas a narrower confidence interval indicates a more accurate estimate.

Common Errors/Misconceptions

- using the incorrect critical value for a specified confidence level
- using the incorrect formula for calculating the margin of error or for calculating a confidence interval

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Guided Practice 1.4.4

Example 1

In a sample of 300 day care providers, 90% of the providers are female. What is the margin of error for this population if a 96% level of confidence is applied?

1. Determine the given information.

   In order to calculate the margin of error, first identify the information provided in the problem.

   It is stated that the sample included 300 day care providers; therefore, $n = 300$.

   It is also given that 90% of the providers are female. This value does not represent a mean, so it must represent a sample proportion; therefore, $\hat{p} = 90\%$ or 0.9.

   To apply a 96% level of confidence, determine the critical value for this confidence level by referring to the table of Common Critical Values (as provided in the Key Concepts and repeated for reference as follows):

Common Critical Values

Confidence level	99%	98%	96%	95%	90%	80%	50%
Critical value (z_c)	2.58	2.33	2.05	1.96	1.645	1.28	0.6745

   The table of critical values indicates that the critical value for a 96% confidence level is 2.05; therefore, $z_c = 2.05$.

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Instruction

2. Calculate the margin of error.

The formula used to calculate the margin of error of a sample proportion is $\text{MOE} = \pm z_c \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, where $\hat{p}$ is the sample proportion and n is the sample size.

$\text{MOE} = \pm z_c \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ Formula for the margin of error of a sample proportion

$\text{MOE} = \pm(2.05)\sqrt{\frac{(0.9)[1-(0.9)]}{(300)}}$ Substitute 2.05 for z_c, 0.9 for $\hat{p}$, and 300 for n.

$\text{MOE} = \pm 2.05\sqrt{\frac{(0.9)(0.1)}{300}}$ Simplify.

$\text{MOE} = \pm 2.05\sqrt{\frac{0.09}{300}}$

$\text{MOE} = \pm 2.05\sqrt{0.0003}$

$\text{MOE} \approx \pm 2.05(0.0173)$

$\text{MOE} \approx \pm 0.0355$

The margin of error for this population is approximately ±0.0355 or ±3.55%.



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Instruction

Example 2

A group of marine biologists placed tracking tags on 100 fish in Lake Erie one summer. The weight of each fish was recorded at the beginning and end of the summer. The average weight gain for all of the tagged fish was 1.2 pounds, with a standard deviation of 0.4 pound. What is the margin of error with 90% confidence for this study?

1. Determine the given information.

   In order to calculate the margin of error, first identify the information provided in the problem.

   It is stated that the sample included 100 tagged fish; therefore, $n = 100$.

   It is also given that the average weight gain for a fish is 1.2 pounds. This value represents a mean; therefore, $\bar{x} = 1.2$.

   It is stated that the standard deviation is 0.4 pound; therefore, $s = 0.4$.

   We are asked to use a 90% confidence level for this study. The table of Common Critical Values indicates that the critical value for a 90% confidence level is 1.645; therefore, $z_c = 1.645$.

2. Calculate the margin of error.

   The formula used to calculate the margin of error of a sample mean is $\text{MOE} = \pm z_c \frac{s}{\sqrt{n}}$, where s is the standard deviation and n is the sample size.

$\text{MOE} = \pm z_c \frac{s}{\sqrt{n}}$	Formula for the margin of error of a sample mean
$\text{MOE} = \pm (1.645) \frac{(0.4)}{\sqrt{(100)}}$	Substitute 1.645 for z_c, 0.4 for s, and 100 for n.
$\text{MOE} = \pm 1.645 \left(\frac{0.4}{10} \right)$	Simplify.
$\text{MOE} = \pm 1.645(0.04)$	
$\text{MOE} = \pm 0.0658$	

   The margin of error for this population is ±0.0658 or ±6.58%.

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Instruction

Example 3

A random sample of 1,000 retirees found that 28% participate in activities at their local senior center. Find a 95% confidence interval for the proportion of seniors who participate in activities at their local senior center.

1. Determine the given information.

   In order to determine a confidence interval, first identify the information provided in the problem.

   It is stated that the sample included 1,000 retirees; therefore, $n = 1000$.

   It is also given that 28% of the retirees participated in activities at their local senior center. This value does not represent a mean, so it must represent the sample proportion; therefore, $\hat{p} = 28\%$ or 0.28.

   We are asked to find a 95% confidence interval. The table of Common Critical Values indicates that the critical value for a 95% confidence level is 1.96; therefore, $z_c = 1.96$.

2. Determine the confidence interval.

   The formula used to calculate the confidence interval for a sample population with a proportion is $\text{CI} = \hat{p} \pm z_c \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, where $\hat{p}$ is the sample proportion and n is the sample size.

   $\text{CI} = \hat{p} \pm z_c \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ Formula for the confidence interval for a sample population

   $\text{CI} = (0.28) \pm (1.96) \sqrt{\frac{(0.28)[1-(0.28)]}{(1000)}}$ Substitute 1.96 for z_c, 0.28 for $\hat{p}$, and 1,000 for n.

   $\text{CI} = 0.28 \pm 1.96 \sqrt{\frac{(0.28)(0.72)}{1000}}$ Simplify.

   $\text{CI} = 0.28 \pm 1.96 \sqrt{\frac{0.2016}{1000}}$

   $\text{CI} = 0.28 \pm 1.96 \sqrt{0.0002016}$

   $\text{CI} \approx 0.28 \pm 1.96(0.0142)$

   $\text{CI} \approx 0.28 \pm 0.0278$

(*continued*)

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Calculate each value for the confidence interval separately.

$0.28 + 0.0278 \approx 0.3078$

$0.28 - 0.0278 \approx 0.2522$

The confidence interval can be written as (0.2522, 0.3078), meaning the requested confidence interval would fall between approximately 0.2522 and 0.3078. In terms of the study, this means that approximately 25.2% to 30.8% of seniors participate in activities at their local senior center.

These calculations can also be performed on a graphing calculator:

On a TI-83/84:

Step 1: Press [STAT].

Step 2: Arrow over to the TESTS menu.

Step 3: Scroll down to A: 1–PropZInt, and press [ENTER].

Step 4: Enter the following known values, pressing [ENTER] after each entry:

x: 280 (favorable results)

n: 1000 (number in the sample population)

C-Level: 0.95 (confidence level in decimal form)

Step 5: Highlight "Calculate" and press [ENTER].

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow down to the calculator page icon (the first icon on the left) and press [enter].

Step 3: Press [menu]. Arrow down to 6: Statistics. Arrow right to choose 6: Confidence Intervals, and then arrow down to 5: 1–Prop z Interval.

Step 4: Enter the following known values. Arrow right after each entry to move between fields.

Successes, *x*: 280 (favorable results)

n: 1000 (number in the sample population)

C Level: 0.95 (confidence level in decimal form)

Step 5: Press [enter] to select OK.

Your calculator will return approximately the same values as calculated by hand.



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Instruction

Example 4

A sample of 49 randomly selected fifth graders who took the same math test found that the students scored an average of 89 points, with a standard deviation of 11.9 points. Determine a 99% confidence interval for this sample.

1. Determine the given information.

   In order to determine a confidence interval, first identify the information provided in the problem.

   It is stated that the sample included 49 fifth graders; therefore, $n = 49$.

   It is also given that the average test score was 89 points. This value represents a mean; therefore, $\bar{x} = 89$.

   It is stated that the standard deviation is 11.9 points; therefore, $s = 11.9$.

   We are asked to find a 99% confidence level. The table of Common Critical Values indicates that the critical value for a 99% confidence level is 2.58; therefore, $z_c = 2.58$.

2. Determine the confidence interval.

   The formula used to calculate the confidence interval for a sample population with a given mean is $\text{CI} = \bar{x} \pm z_c \frac{s}{\sqrt{n}}$, where s = standard deviation, $\bar{x}$ = mean, and n = sample size.

   $\text{CI} = \bar{x} \pm z_c \frac{s}{\sqrt{n}}$ — Formula for the confidence interval for a sample population

   $\text{CI} = (89) \pm (2.58) \frac{(11.9)}{\sqrt{(49)}}$ — Substitute 89 for $\bar{x}$, 2.58 for z_c, 11.9 for s, and 49 for n.

   $\text{CI} = 89 \pm 2.58\left(\frac{11.9}{7}\right)$ — Simplify.

   $\text{CI} = 89 \pm (2.58)(1.7)$

   $\text{CI} = 89 \pm 4.386$

   Calculate each value for the confidence interval separately.

   $89 + 4.386 = 93.386$

   $89 - 4.386 = 84.614$

(*continued*)

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The confidence interval can be written as (84.614, 93.386), meaning the confidence interval would fall between approximately 84.614 and 93.386. In terms of this study, a 99% confidence level can be found between 84.614 and 93.386 points.

The confidence interval can also be found using a graphing calculator:

On a TI-83/84:

Step 1: Press [STAT].

Step 2: Arrow over to the TESTS menu.

Step 3: Scroll down to 7: ZInterval and press [ENTER].

Step 4: Arrow over to the right to highlight Stats and press [ENTER].

Step 5: Enter the following known values, pressing [ENTER] after each entry:

σ: 11.9 (standard deviation)

$\bar{x}$: 89 (sample population mean)

n: 49 (number in the sample population)

C-Level: 0.99 (confidence level in decimal form)

Step 6: Highlight "Calculate" and press [ENTER].

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow down to the calculator page icon (the first icon on the left) and press [enter].

Step 3: Press [menu]. Arrow down to 6: Statistics. Arrow right to choose 6: Confidence Intervals, and then choose 1: z Interval.

Step 4: Select Stats from the Data Input Method drop-down menu, arrow right to highlight OK, then press [enter].

Step 5: Enter the following known values. Arrow right after each entry to move between fields.

σ: 11.9 (standard deviation)

$\bar{x}$: 89 (sample population mean)

n: 49 (number in the sample population)

C Level: 0.99 (confidence level in decimal form)

Step 6: Press [enter] to select OK.

Your calculator will return approximately the same values as calculated by hand.



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Problem-Based Task 1.4.4: Fitness Analysis

SMP	
1	2 ✓
3 ✓	4
5	6
7	8

Jolie is an instructor of two fitness classes and wants to analyze the weight-loss results of both classes. After receiving the raw data for each class, Jolie groups the sample of people into 8 different categories. For example, participants in the first category are athletes training before the sports season, and participants in the second category have part-time jobs. Each category contains 10 people.

Jolie has determined the standard deviation of Class 1 to be 5.9 pounds and the standard deviation of Class 2 to be 2.3 pounds. Based on this information and the following table, which class shows better weight-loss results? Explain your reasoning.

Weight-Loss Results

Category	Average weight loss (in pounds)	
	Class 1	Class 2
1	12.7	6.5
2	10.4	9.1
3	3	3.9
4	0.75	4.1
5	5	8.9
6	15	10
7	12.9	7.6
8	0.4	6.7

Which class shows better weight-loss results? Explain your reasoning.

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Problem-Based Task 1.4.4: Fitness Analysis

Coaching

a. What is the sample size of this data set?

b. What is the mean of the data representing Class 1?

c. What is the mean of the data representing Class 2?

d. What is the standard deviation of the data representing Class 1?

e. What is the standard deviation of the data representing Class 2?

f. Determine a 99% confidence interval for Class 1.

g. Determine a 99% confidence interval for Class 2.

h. Which class shows better weight-loss results? Explain your reasoning.

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Instruction

Problem-Based Task 1.4.4: Fitness Analysis

Coaching Sample Responses

a. What is the sample size of this data set?

Each of the 8 categories contains 10 people. The sample size of this data set is the product of 10 and 8, or 80.

b. What is the mean of the data representing Class 1?

To determine the mean of the data, add the number of pounds lost for each category and divide by the number of categories.

$$\overline{x} = \frac{12.7+10.4+3+0.75+5+15+12.9+0.4}{8}$$

$$\overline{x} \approx 7.5$$

The average weight loss for Class 1 is approximately 7.5 pounds.

c. What is the mean of the data representing Class 2?

Again, add the number of pounds lost for each category and divide by the sample size.

$$\overline{x} = \frac{6.5+9.1+3.9+4.1+8.9+10+7.6+6.7}{8}$$

$$\overline{x} = 7.1$$

The average weight loss for Class 2 is approximately 7.1 pounds.

d. What is the standard deviation of the data representing Class 1?

As stated in the problem, the standard deviation of Class 1 is 5.9 pounds.

e. What is the standard deviation of the data representing Class 2?

As stated in the problem, the standard deviation of Class 2 is 2.3 pounds.

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f. Determine a 99% confidence interval for Class 1.

The formula used to calculate the confidence interval for a sample population with a given mean is $CI = \overline{x} \pm z_c \frac{s}{\sqrt{n}}$, where s = standard deviation, $\overline{x}$ = mean, and n = sample size.

It is given that $s = 5.9$, and we have determined that $\overline{x} = 7.5$ and $n = 80$.

Based on the table of Common Critical Values, a 99% confidence interval has a critical value of 2.58, so $z_c = 2.58$.

Now substitute the known values into the formula and solve.

$$CI = \overline{x} \pm z_c \frac{s}{\sqrt{n}}$$

$$CI = (7.5) \pm (2.58) \frac{(5.9)}{\sqrt{(80)}}$$

$$CI \approx 7.5 \pm (2.58)(0.6596)$$

$$CI \approx 7.5 \pm 1.702$$

Calculate each value for the confidence interval separately.

$$7.5 + 1.702 \approx 9.202$$

$$7.5 - 1.702 \approx 5.798$$

The requested confidence interval would fall between approximately 5.798 and 9.202 pounds.

g. Determine a 99% confidence interval for Class 2.

It is given that $s = 2.3$, and we have determined that $\overline{x} = 7.1$ and $n = 80$.

The confidence interval is 99% for this program as well, so the critical value has not changed: $z_c = 2.58$.

Now substitute the known values into the same formula and solve.

$$CI = \overline{x} \pm z_c \frac{s}{\sqrt{n}}$$

$$CI = (7.1) \pm (2.58) \frac{(2.3)}{\sqrt{(80)}}$$

$$CI \approx 7.1 \pm (2.58)(0.2571)$$

$$CI \approx 7.1 \pm 0.6633$$

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Calculate each value for the confidence interval separately.

$7.1 + 0.6633 \approx 7.763$

$7.1 - 0.6633 \approx 6.437$

The requested confidence interval would fall between approximately 6.437 and 7.763 pounds.

h. Which class shows better weight-loss results? Explain your reasoning.

Based on the data chosen, Class 1 could appear to have better weight-loss results because the participants' average weight loss is higher. However, it is important to note that the confidence interval of Class 2 is much narrower for a 99% confidence level. This indicates that the weight loss of Class 2 varies less and is more consistent. For this reason, Class 2 shows better weight-loss results.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.4.4: Estimating with Confidence

For problems 1–4, calculate the margin of error for each scenario described. Round answers to the nearest hundredth of a percent.

1. After taking a sample of 70 customers, an online retailer found that 65% of customers make a purchase. The survey has an 80% confidence level.

2. A survey of 125 parents found that they began teaching their children to drive at an average age of 15 years old. The survey found a standard deviation of 0.75 year. The survey has a 90% confidence level.

3. A survey of 6,000 households who contribute to charity found that the average contribution was 5% of the average household income, with a standard deviation of 3%. The survey has a 99% confidence level.

4. A commercial claims, "4 out of 5 dentists recommend our product." The sample included 15 dentists. The survey has a 95% confidence level.

For problems 5–8, determine the confidence interval for each scenario described. Round answers to the nearest tenth.

5. A sample of 78 cars found the average gas mileage to be 22.3 miles per gallon, with a standard deviation of 2.7 miles per gallon. Estimate a 96% confidence interval.

continued

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6. A professor in Canada published a study of how watching television affected 1,024 children over time. He recorded the number of hours per week each child watched TV at age 2. Then, he revisited the same children when they were in fourth grade, and recorded their standardized math test scores and body mass index. The study demonstrated that for every 1-hour increase in TV time for each child at age 2, there was an average 6% reduction in math achievement and a 5% increase in body mass index by the fourth grade. If the standard deviation for both the math and weight data was 0.75%, determine a 95% confidence interval for each.

7. A study of 587 Swedish men who developed dementia before age 54 found nine risk factors associated with the diagnosis. The highest risk factor was adolescent alcohol use, with a mean "hazard ratio" of 4.82 and a standard deviation of 2.01. Determine an 80% confidence interval for this data.

8. A recent study found the rate of glaucoma among patients diagnosed with motion sickness was 11.26 per 1,000 people. Determine a 95% confidence interval if the standard deviation is 0.98.

For problems 9 and 10, use what you have learned about confidence intervals to solve each problem. Round answers to the nearest hundredth of a percent.

9. A new restaurant prides itself on having a short wait time for service and has stopwatches at each table for customers to use. The restaurant will give you your meal for free if you are not served within an 80% level of confidence of their average wait time of 7.2 minutes. The standard deviation is 2.0 minutes. Let the sample size represent the number of tables the restaurant has, 100. How many seconds after 7 minutes would you have to wait to get your meal for free?

10. An animal shelter records the age and weight of rescued cats. If the mean of a 100-cat study is 7.9 pounds with a standard deviation of 1.1 pounds, would a cat weighing 6 pounds fall within an 80% confidence interval?

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Progress Assessment

Circle the letter of the best answer.

1. The following table reflects the contents of a package of jelly beans:

Flavor	Cherry	Lemon	Apple	Grape	Orange	Licorice	Coconut
Number of beans	13	8	17	22	9	3	15

What is the estimated population proportion of the most common flavor?

a. 0.15
c. 0.17
b. 0.25
d. 0.34

2. What is the appropriate critical value in order to calculate a 98% confidence interval for a population mean from a sample of 1,000 moviegoers?

a. 1.25
c. 1.96
b. 1.645
d. 2.33

3. In a survey of 1,500 high school students, 79% do not get an average of 8 hours of sleep. What is the approximate standard error for the population who do not get an average of 8 hours of sleep?

a. 0.0105
c. 0.0118
b. 0.0003
d. 0.8885

4. What is the probability of a fair coin landing heads-up exactly 8 times out of 10 tosses?

a. $\frac{45}{1024}$
c. $\frac{90}{1024}$
b. $\frac{64}{1024}$
d. $\frac{180}{1024}$

5. The average number of calories in a family-size package of chocolate chip cookies is believed to be approximately normally distributed. The mean number of calories in a random sample of 25 of the cookies is 69.72, and the standard deviation is 4.15. What is the standard error of the mean?

a. 0.80
c. 1.85
b. 0.83
d. 2.04

continued

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Assessment

6. A 90% confidence level means:
   a. You are 90% confident that your interval will include the population parameter.
   b. You are 10% confident that your interval will not include the population parameter.
   c. Repeating the sample will capture your parameter 90% of the time.
   d. all of the above

7. What is the margin of error for a sample of 34 students whose average height, with 90% confidence, can be found between 47.5″ and 50.5″?
   a. 0.25″
   b. 3″
   c. 0.4″
   d. ±1.5″

8. The standard error of the mean will decrease when the:
   a. population size decreases
   b. sample size decreases
   c. standard deviation increases and n is constant
   d. standard deviation decreases and n increases

9. A pollster asked a random sample of 269 students about their savings. The average amount of money saved was $258, with a standard deviation of $145. What is a 90% confidence interval for this data?
   a. $235.23 to $280.77
   b. $240.67 to $275.33
   c. $243.46 to $272.54
   d. $245.27 to $270.73

10. A reporter for a city radio station called 40 people living within 3 blocks of the station and conducted a survey about their salaries. The results were used to estimate the average salary in the city. How did the reporter misuse the data?
    a. The sample size is not large enough.
    b. The reporter only called people who listen to that radio station.
    c. People living within 3 blocks do not represent a citywide sample.
    d. The reporter assumed people are telling the truth about how much money they earn.

continued

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Assessment

Apply your knowledge of statistics to answer all parts of problem 11.

11. You are a real estate agent who has been assigned by your supervisor to analyze home sales in your town for the last 6 months. You use the following table, which shows the sale price for several homes in hundreds of thousands of dollars. *Note*: Some months have fewer values.

June	July	August	September	October	November
$155	$125	$198	$204	$310	$149
$213	$47	$303	$197	$219	$178
$210	$410	$276	$69	—	$201
$175	$244	$199	$345	—	—
$350	$255	$299	—	—	—

a. Calculate the mean, standard deviation, and SEM for your sample. If we assume that housing prices are normally distributed, calculate a 95% confidence interval for the mean.

b. Eliminate the two lowest priced home sales and re-evaluate the mean, standard deviation, SEM, and confidence interval using a 95% confidence level.

c. Which set of statistics created for parts a and b will you present to your supervisor? Explain your reasoning.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. What does a *t*-test provide?
   a. a theoretical mean value of a population
   b. a way to compare means of two sets
   c. a table of probability values
   d. a correlation between the mean and the standard deviation of a set of data

2. In order to obtain the *p*-value from a *t*-distribution table, what must you first determine?
   a. the *t*-value and the degrees of freedom
   b. the mean and the standard deviation of the population
   c. the mean and the standard deviation of only one sample
   d. whether the size (n) of the samples is the same

3. What would a *p*-value of less than 0.01 indicate?
   a. a high statistical significance of the sample
   b. a low statistical significance of the sample
   c. a statistical insignificance of the sample
   d. none of the above, since *p*-value is not related to statistical significance

continued

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Assessment

4. A simulation of an outcome uses a well-balanced, fair roulette. After 10 series of 10 rolls each, which of the following is true of the outcomes?

   a. The outcomes will have a normal distribution, and the sums of subsets of rolls will also be normally distributed.

   b. The outcomes will have a uniform distribution, but the sums of subsets of rolls will be normally distributed.

   c. The outcomes will have a uniform distribution, and the sums of subsets of rolls will also be uniformly distributed.

   d. The outcomes will have a normal distribution, but the sums of subsets of rolls will be uniformly distributed.

5. A treatment report shows a *p*-value of more than 0.15 for a "before and after" set of data from a group of experimental patients. What could you state about the treatment?

   a. The report does not show statistically significant improvement.

   b. The treatment may have had counter-effective results.

   c. The treatment shows a highly statistically significant improvement.

   d. The test was not properly designed.

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Lesson 5: Comparing Treatments and Reading Reports

Instruction

Common Core State Standards

SMP
1 ✓ 2 ✓
3 ✓ 4 ✓
5 ✓ 6 ✓
7 ✓ 8

S–IC.5 Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.★

S–IC.6 Evaluate reports based on data.★

Essential Questions

1. How do researchers determine whether their results are significant?
2. What general assumptions do you need to make before the statistical work has validity?
3. Given a data set, what is a *t*-test used for?
4. What are simulations and how can they help us understand data that we are curious about?
5. How do we know we can trust the results of a study or experiment?
6. How would you evaluate a report that uses statistical evidence in order to support a claim?

WORDS TO KNOW

alternative hypothesis	any hypothesis that differs from the null hypothesis; that is, a statement that indicates there is a difference in the data from two treatments; represented by H_a
bias	leaning toward one result over another; having a lack of neutrality
confidence level	the probability that a parameter's value can be found in a specified interval; also called *level of confidence*
confounding variable	an ignored or unknown variable that influences the result of an experiment, survey, or study
correlation	a measure of the power of the association between exactly two quantifiable variables
degrees of freedom (*df*)	the number of data values that are free to vary in the final calculation of a statistic; that is, values that can change or move without violating the constraints on the data
hypothesis	a statement that you are trying to prove or disprove

SRB p. U1-197

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Instruction

hypothesis testing	assessing data in order to determine whether the data supports (or fails to support) the hypothesis as it relates to a parameter of the population
level of confidence	the probability that a parameter's value can be found in a specified interval; also called *confidence level*
measurement bias	bias that occurs when the tool used to measure the data is not accurate, current, or consistent
nonresponse bias	bias that occurs when the respondents to a survey have different characteristics than nonrespondents, causing the population that does not respond to be underrepresented in the survey's results
null hypothesis	the statement or idea that will be tested, represented by H_0; generally characterized by the concept that there is no relationship between the data sets, or that the treatment has no effect on the data
one-tailed test	a *t*-test performed on a set of data to determine if the data could belong in one of the tails of the bell-shaped distribution curve; with this test, the area under only one tail of the distribution is considered
***p*-value**	a number between 0 and 1 that determines whether to accept or reject the null hypothesis
parameter	numerical value(s) representing the data in a set, including proportion, mean, and variance
response bias	bias that occurs when responses by those surveyed have been influenced in some manner
simulation	a set of data that models an event that could happen in real life
statistical significance	a measure used to determine whether the outcome of an experiment is a result of the treatment being applied, as opposed to random chance
***t*-test**	a procedure to establish the statistical significance of a set of data using the mean, standard deviation, and degrees of freedom for the sample or population

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***t*-value**	the result of a *t*-test
treatment	the process or intervention provided to the population being observed
trial	each individual event or selection in an experiment or treatment
two-tailed test	a *t*-test performed on a set of data to determine if the data could belong in either of the tails of the bell-shaped distribution curve; with this test, the area under both tails of the distribution is considered
voluntary response bias	bias that occurs when the sample is not representative of the population due to the sample having the option of responding to the survey

Recommended Resources

- Jackson, Sean. "Bias in Surveys."

  http://www.walch.com/rr/00188

  This video lecture addresses bias in surveys and sampling, and the impact that bias has on the results of a survey.

- Redmon, Angela. "Probability Simulator."

  http://www.walch.com/rr/00189

  This video demonstrates simulating an experiment step-by-step on the TI-84 Plus calculator. Operations demonstrated include graphing the frequency and storing values to a table.

- Stat Trek. "Bias in Survey Sampling."

  http://www.walch.com/rr/00190

  This site defines and addresses types of bias, including sampling bias, nonresponse bias, measurement bias, and response bias. The site also features a link to a video explaining bias in surveys.

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Lesson 1.5.1: Evaluating Treatments

Warm-Up 1.5.1

Alicia is the manager of the school basketball team, and is determining some statistics on the team's recent performance. She has collected the team's scores for the 8 regular-season games the team has played so far.

55 65 45 43 62 48 60 54

1. What is the mean of the team's scores?

2. What is the standard deviation of the team's scores?

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Instruction

Lesson 1.5.1: Evaluating Treatments

Common Core State Standard

S–IC.5 Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.★

Warm-Up 1.5.1 Debrief

1. What is the mean of the team's scores?

   Recall that the mean can be found using the formula $\overline{x}=\frac{x_1+x_2+x_3+\cdots+x_n}{n}$, where n is the number of data values.

   $\overline{x}=\frac{x_1+x_2+x_3+\cdots+x_n}{n}$ Formula for calculating mean

   $\overline{x}=\frac{(55)+(65)+(45)+(43)+(62)+(48)+(60)+(54)}{(8)}$ Substitute the known values.

   $\overline{x}=54$ Simplify.

   The mean of the scores is 54 points.

2. What is the standard deviation of the team's scores?

   Recall that the standard deviation can be found using the formula $s=\sqrt{\frac{\left(x_1-\overline{x}\right)^2+\left(x_2-\overline{x}\right)^2+\left(x_3-\overline{x}\right)^2+\cdots+\left(x_n-\overline{x}\right)^2}{n}}$, with s representing the standard deviation and $\overline{x}$ representing the mean.

   Substitute values for the scores and the mean into this formula, then solve for s, as shown.

   $$s=\sqrt{\frac{[(55)-(54)]^2+[(65)-(54)]^2+[(45)-(54)]^2+[(43)-(54)]^2+[(62)-(54)]^2+[(48)-(54)]^2+[(60)-(54)]^2+[(54)-(54)]^2}{(8)}}$$

   $s=\sqrt{\frac{(1)^2+(11)^2+(-9)^2+(-11)^2+(8)^2+(-6)^2+(6)^2+(0)^2}{8}}$ Simplify.

   $s=\sqrt{\frac{1+121+81+121+64+36+36+0}{8}}$

   $s=\sqrt{\frac{460}{8}}$

   $s\approx 7.58$

   The standard deviation is approximately 7.58.

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Instruction

Connection to the Lesson

- Students will continue to use mean and standard deviation calculations to evaluate the *p*-values for data.
- Students will also use the mean to decide whether a hypothesis requires a one-tailed test or a two-tailed test.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- calculating the mean and standard deviation of a set of data (S–ID.2)
- distinguishing intuitively a normally distributed population from a uniformly distributed one (S–ID.4)
- reading values from a table (no standard)

Introduction

Scientists, mathematicians, and other professionals sometimes spend years conducting research and gathering data in order to determine whether a certain hypothesis is true. A **hypothesis** is a statement that you are trying to prove or disprove. A hypothesis is proved or disproved by observing the effects of a treatment on a population. A **treatment** is a process or intervention provided to the population being observed.

Once the hypothesis has been crafted and the treatment or experiment carefully conducted, the researchers can test their hypothesis. **Hypothesis testing** is the process of assessing data in order to determine whether the data supports (or fails to support) the hypothesis as it relates to a parameter of the population. By testing a hypothesis, it is possible to determine whether the result of an experiment is actually related to the treatment being applied to the population, or if the result is due to random chance. This lesson explores one method of hypothesis testing, called the *t*-test.

Key Concepts

- **Statistical significance** is a measure used to determine whether the outcome of an experiment is a result of the treatment being applied, as opposed to random chance.
- There is a relationship between statistical significance and level of confidence, the probability that a parameter's value can be found in a specified interval. Recall that a parameter is a numerical value representing the data in a set.
- Generally, the results of an experiment are considered to be statistically significant if the chance of a given outcome occurring randomly is less than 5%; that is, if the overall data has a 95% confidence level.
- For example, if 100 trials of the same experiment are conducted, and fewer than 5 of those trials result in data values that fall outside of a 95% confidence level, then the chance that these data values occurred randomly (rather than as a result of the treatment), is only $\frac{5}{100} = 0.05 = 5\%$.

SRB p. U1-200

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Instruction

- A high confidence level corresponds to a low level of significance; therefore, a lower level of significance indicates more precise results.
- A ***t*-test** is used to establish the statistical significance of a set of data. It uses the means and standard deviations of samples and populations, as well as another parameter called degrees of freedom.
- In a data set, the **degrees of freedom (*df*)** are the number of data values that are free to vary in the final calculation of a statistic; that is, values that can change or move without violating the constraints on the data.
- For example, if a student wants to earn an average of 80 points on 4 given tests, there are 3 degrees of freedom: the first 3 test grades. Once the first 3 test grades are determined, the student is not "free," or able, to set the fourth grade to any value other than the value needed to maintain an average of 80 points.
- Therefore, the number of degrees of freedom is a function of the sample size for the situation under study. The specific formula to find the degrees of freedom depends on the type of problem.
- Before a *t*-test can be applied, the population must have a normal (bell-shaped) distribution. Recall that a normal distribution tapers off on either side of the median, forming "tails."
- There are two types of *t*-tests: a one-tailed test and a two-tailed test.
- A **one-tailed test** is used if you are comparing the mean of a sample to values on only one side of the population mean. Values are chosen from either the right-hand side (tail) of the distribution or from the left-hand side of the distribution, but not from both sides.
- When comparing the mean of the sample to values that are *greater* than the mean, focus on the tail of the distribution to the *right* of the mean.
- When comparing the mean of the sample to values that are *less* than the mean, focus on the tail of the distribution to the *left* of the mean.
- A **two-tailed test** is used when comparing the mean of a sample to values on both sides of the population mean—that is, to values that are greater than the mean (on the right side of the distribution) *and* to values that are less than the mean (on the left side of the distribution).
- The result of a *t*-test is called a ***t*-value**.
- When the *t*-value and the degrees of freedom are entered into a *t*-distribution table, a *p*-value can be determined. The sign of the value of *t* does not matter; a value of $t = -1.2345$ has exactly the same location in the *t*-distribution table as a value of $t = 1.2345$.

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Instruction

- A ***p*-value** is a number between 0 and 1, determined from the *t*-distribution table. The *p*-value is used to accept or reject the null hypothesis.
- A **null hypothesis**, or H_0, is a statement or idea that will be tested. It is generally characterized by the concept that the treatment does not result in a change, or that, for a set of data under observation and its associated results, the results could have been selected from the same population 95% of the time by sheer chance. In other words, there is no relationship between the data sets.
- An **alternative hypothesis** is any hypothesis that differs from the null hypothesis; that is, a statement that indicates there is a difference in the data from two treatments. The alternative hypothesis is represented by H_a.
- If the *p*-value is less than a given confidence level (usually 0.05, or 5%), the null hypothesis is rejected.
- To run a *t*-test for two sets of data, first obtain the mean and standard deviation of each set.
- To calculate the *t*-value, use the formula $t = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$, described as follows.
  - $\overline{x}_1$ is the mean of the first set of data.
  - $\overline{x}_2$ is the mean of the second set of data.
  - s_1^2 and s_2^2 are the squares of the standard deviations of the first set and second set, respectively.
  - n_1 and n_2 are the respective sample sizes.
- With the obtained value of *t*, refer to the *t*-distribution table to find the *p*-value on the line corresponding to the degrees of freedom for the sets.
- Degrees of freedom are calculated using the formula $df = \frac{n_1 - 1 + n_2 - 1}{2}$, where n_1 is the sample size of the first set and n_2 is the sample size of the second set.
- Round the calculated degrees of freedom down to a whole number.

Running a *t*-test Between One Set of Sample Data and a Population

- If you run a *t*-test between one sample set and a population whose standard deviation is unknown, first obtain the mean and standard deviation for the sample set.

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Instruction

- To calculate the *t*-value, use the formula $t = \frac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}}$, where $\overline{x}$ is the sample mean, μ_0 is the population mean, *s* is the standard deviation of the sample, and *n* is the sample size.
- To find the *p*-value, refer to the *t*-distribution table. Find the line that corresponds to the degrees of freedom (*df*) for the set.
- For only one set of data, *df* is equal to $n - 1$, where *n* is the sample size.
- A graphing calculator can be used to perform *t*-tests.

On a TI-83/84:

Step 1: Press [STAT] and arrow over to TESTS.

Step 2: Select 2: T-Test... and press [ENTER].

Step 3: Arrow over to Stats and press [ENTER].

Step 4: Enter values for the hypothesized mean, sample mean, standard deviation, and sample size.

Step 5: Select the appropriate alternative hypothesis. For a two-tailed test, select $\neq \mu_0$. For a one-tailed test, select $< \mu_0$ to compare the mean of the set to the left side of the bell-shaped distribution, or select $> \mu_0$ to compare the mean of the set to the right side of the bell-shaped distribution.

Step 6: Select Calculate and press [ENTER]. The *t*-value and *p*-value will be displayed.

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Instruction

On a TI-Nspire:

Step 1: Arrow down to the calculator icon, the first icon on the left, and press [enter].

Step 2: Press [menu], then use the arrow key to select 6: Statistics, then 7: Stat Tests and 2: t Test.... Press [enter].

Step 3: Select the data input method. Choose "Data" if you have the data, or "Stats" if you already know the hypothesized mean, sample mean, standard deviation, and sample size. Select "OK."

Step 4: Enter values for either the data and the population mean, μ_0, or the hypothesized mean, sample mean, standard deviation, and sample size, depending on your selection from the previous step. Beside "Alternate Hyp," select the appropriate alternative hypothesis. For a two-tailed test, select $\mu \neq \mu_0$. For a one-tailed test, select $\mu < \mu_0$ to compare the mean of the set to the left side of the bell-shaped distribution, or select $\mu > \mu_0$ to compare the mean of the set to the right side of the bell-shaped distribution.

Step 5: Select "OK." The *t*-value and *p*-value will be displayed.

Common Errors/Misconceptions

- expecting statistics to provide exact answers to problems rather than ways of looking at and interpreting data
- deciding to run a one-tailed *t*-test when trying to compare a sample set to both sides of the distribution
- conversely, running a two-tailed test when trying to compare the sample set to one side of the distribution
- thinking that the result of a statistics problem is just a number, rather than a report, written in plain language, that draws conclusions after observing data
- forgetting that the sign of the value of *t* is irrelevant

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Instruction

Guided Practice 1.5.1

Example 1

The students of Ms. Stomper's class earned the following scores on a state test:

71 70 69 75 67 73 71 72 68 75 68 70

The population mean of the state scores is 69 points. Based on the test results, did Ms. Stomper's class achieve higher than the state mean, with a statistical significance of 0.05? In other words, if the test were carried out 100 times, would a result like the one represented by the set above occur 5 or more times?

1. Determine the sample size of the data.

   The data values include the values 71, 70, 69, 75, 67, 73, 71, 72, 68, 75, 68, and 70.

   To determine the sample size, count the number of data values.

   There are a total of 12 data values; therefore, $n = 12$.

2. Calculate the sample mean of the data.

   To calculate the sample mean of the data, use the formula for sample mean, $\overline{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$, where n is the sample size.

   Substitute values from the data set for x and 12 for n, as shown below.

   $\overline{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$ Formula for calculating sample mean

   $\overline{x} = \frac{(71)+(70)+(69)+(75)+(67)+(73)+(71)+(72)+(68)+(75)+(68)+(70)}{(12)}$

   $\overline{x} = \frac{849}{12}$ Simplify.

   $\overline{x} = 70.75$

   The sample mean of the data is 70.75.

SWB p. 153; SRB p. U1-205

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Instruction

3. Calculate the standard deviation of the sample data.

To calculate the standard deviation of the sample data, use the formula $s=\sqrt{\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+(x_3-\bar{x})^2+\cdots+(x_n-\bar{x})^2}{n-1}}$, where $\bar{x}$ is the mean, each x is a data value, and n is the sample size.

Substitute values for the scores, the mean, and the sample size into the formula, as shown below.

$$s=\sqrt{\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+(x_3-\bar{x})^2+\cdots+(x_n-\bar{x})^2}{n-1}}$$

Formula for standard deviation of a sample

$$s=\sqrt{\frac{\begin{aligned}&[(71)-(70.75)]^2+[(70)-(70.75)]^2+[(69)-(70.75)]^2+\\&[(75)-(70.75)]^2+[(67)-(70.75)]^2+[(73)-(70.75)]^2+\\&[(71)-(70.75)]^2+[(72)-(70.75)]^2+[(68)-(70.75)]^2+\\&[(75)-(70.75)]^2+[(68)-(70.75)]^2+[(70)-(70.75)]^2\end{aligned}}{(12)-1}}$$

$s\approx 2.633$

The sample standard deviation of the data is approximately 2.633.

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Instruction

4. Determine the *t*-value.

The mean of the population, 69, is known, but the standard deviation of the population is not known.

To determine the *t*-values, use the formula $t = \dfrac{\overline{x} - \mu_0}{\dfrac{s}{\sqrt{n}}}$, where $\overline{x}$ is the sample mean, μ_0 is the population mean, s is the standard deviation of the sample, and n is the sample size.

$t = \dfrac{\overline{x} - \mu_0}{\dfrac{s}{\sqrt{n}}}$	Formula for calculating the *t*-value
$t = \dfrac{(70.75) - (69)}{\dfrac{(2.633)}{\sqrt{(12)}}}$	Substitute values for the sample mean, population mean, standard deviation, and sample size.
$t \approx 2.302$	Simplify.

The *t*-value is approximately 2.302.

5. Determine the degrees of freedom.

Since there is only one set of sample data, the degrees of freedom can be found using the formula $df = n - 1$.

$df = n - 1$	Formula for degrees of freedom
$df = (12) - 1$	Substitute 12 for n.
$df = 11$	Simplify.

The number of degrees of freedom is 11.

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6. Determine the *p*-value.

   Once the *t*-value and degrees of freedom are known, the *p*-value can be found using a *t*-distribution table.

   In a *t*-distribution table, look down the column of degrees of freedom to locate $df = 11$. Then look across this row to determine the two values that a *t*-value of 2.302 falls in between.

   A *t*-value of 2.302 falls between the values of 2.201 and 2.718.

   Look up to the top of these columns to obtain the values within which the *p*-value falls.

   Since we are looking for scores greater than the mean (that is, scores located on only one side of the distribution), refer to the values for a one-tailed *t*-distribution table.

   The entry for $df = 11$ corresponds to $0.025 > p > 0.01$.

7. Summarize your results.

   The problem scenario stated the value for statistical significance in this situation is 0.05, or 5%.

   If the *p*-value obtained from the table is less than 0.05, it can be said that if the same exam were given 100 times, a result such as the one Ms. Stomper's students achieved would only be obtained 5 times or less.

   In the previous step, it was determined that $0.025 > p > 0.01$.

   Since the range of the *p*-values is less than 0.05, we can reject the hypothesis that this result was obtained by sheer chance. In this context, we can conclude that Ms. Stomper's teachings produce statistically significant results.

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Instruction

Example 2

Exequiel and Sigmund are fishermen constantly trying to outdo each other. At the Willow Pond fishing contest, Exequiel caught fish that weighed 2.5, 3.0, and 3.6 pounds. Sigmund caught fish weighing 4.0 and 4.8 pounds. The average weight of fish caught during the contest (that is, the mean of the population, μ_0) is 3.0 pounds.

At award time, Sigmund claims that he should receive a "rare catch" award. His total catch weight is only 0.3 pound less than Exequiel's, but his mean weight is higher. Though Sigmund caught 1 less fish, he insists that if Exequiel fished at Willow Pond 100 times, Exequiel would get a catch like Sigmund's fewer than 10 times.

If you were the judge and had to assess Sigmund's claim to a rare catch, how would you evaluate this claim? Run a *t*-test to determine the statistical significance of each sample compared to the population mean of $\mu_0 = 3.0$.

1. Calculate the mean of each sample.

   For Exequiel's total catch, the sample size is 3.

   To determine the mean of this sample, use the formula $\overline{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$, where n is the sample size.

   $\overline{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$ Formula for calculating mean

   $\overline{x} = \frac{(2.5) + (3.0) + (3.6)}{(3)}$ Substitute known values.

   $\overline{x} \approx 3.0333$ Simplify.

   The mean of Exequiel's total catch is approximately 3.0333 pounds.

   Use the same formula to determine the mean of Sigmund's catch.

   For Sigmund's total catch, the sample size is 2, since he caught one less fish.

   $\overline{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$ Formula for calculating mean

   $\overline{x} = \frac{(4.0) + (4.8)}{(2)}$ Substitute known values.

   $\overline{x} = 4.4$ Simplify.

   The mean of Sigmund's total catch is 4.4 pounds.

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2. Calculate the standard deviation of each sample.

To determine the standard deviation of Exequiel's catch, use the formula for calculating the standard deviation of a sample, $s=\sqrt{\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+(x_3-\bar{x})^2+\cdots+(x_n-\bar{x})^2}{n-1}}$, where $\bar{x}$ is the mean, x is each data value, and n is the sample size.

Substitute known values into the formula, as shown.

$$s=\sqrt{\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+(x_3-\bar{x})^2+\cdots+(x_n-\bar{x})^2}{n-1}}$$

Formula for calculating standard deviation of a sample

$$s=\sqrt{\frac{[(2.5)-(3.0333)]^2+[(3.0)-(3.0333)]^2+[(3.6)-(3.0333)]^2}{(3)-1}}$$

$s \approx 0.55076$

The standard deviation of Exequiel's catch is approximately 0.55076.

Use the same formula to determine the standard deviation of Sigmund's catch.

$$s=\sqrt{\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+(x_3-\bar{x})^2+\cdots+(x_n-\bar{x})^2}{n-1}}$$

Formula for calculating standard deviation of a sample

$$s=\sqrt{\frac{[(4.0)-(4.4)]^2+[(4.8)-(4.4)]^2}{(2)-1}}$$

Substitute known values.

$s \approx 0.56569$

The standard deviation of Sigmund's catch is approximately 0.56569.

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Instruction

3. Determine the *t*-value for each catch.

To determine the *t*-values for each catch, use the formula $t = \frac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}}$, where $\overline{x}$ is the sample mean, μ_0 is the population mean, s is the standard deviation of the sample, and n is the sample size.

Find the *t*-value for Exequiel's catch.

$t = \frac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}}$ Formula for calculating a *t*-value

$t = \frac{(3.0333) - (3)}{\frac{(0.55076)}{\sqrt{(3)}}}$ Substitute known values.

$t \approx 0.10483$ Simplify.

The *t*-value of Exequiel's catch is approximately 0.10483.

Find the *t*-value for Sigmund's catch.

$t = \frac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}}$ Formula for calculating a *t*-value

$t = \frac{(4.4) - (3.0)}{\frac{(0.56569)}{\sqrt{(2)}}}$ Substitute known values.

$t \approx 3.5$ Simplify.

The *t*-value of Sigmund's catch is approximately 3.5.

While you, the judge, are doing your calculations, Exequiel is looking over your shoulder and he begins to dislike what he sees. He knows quite a bit of statistics, and knows that his low *t*-value of 0.10483 will lead to a *p*-value that shows his catch was actually easy to get. Sigmund's *t*-value of 3.5, on the other hand, will lead to a *p*-value denoting a seldom-obtained catch, supporting his claim to the "rare catch" award.

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Instruction

4. Determine the degrees of freedom for each catch.

The degrees of freedom can be found using the formula $df = n - 1$.

Find the degrees of freedom for Exequiel's catch.

$df = n - 1$	Formula for degrees of freedom
$df = (3) - 1$	Substitute 3 for n.
$df = 2$	Simplify.

The degrees of freedom for Exequiel's catch is 2.

Find the degrees of freedom for Sigmund's catch.

$df = n - 1$	Formula for degrees of freedom
$df = (2) - 1$	Substitute 2 for n.
$df = 1$	Simplify.

The degrees of freedom for Sigmund's catch is 1.

5. Determine the p-value for each sample.

Use a one-tailed test to see values greater than the mean.

To find the p-value for each fisherman's t-value, evaluate the t-distribution table at the row for 2 degrees of freedom for Exequiel's catch and then at the row for 1 degree of freedom for Sigmund's catch. These row numbers are each 1 less than the sample size number for each catch.

Exequiel's t-value of 0.10483 at 2 degrees of freedom has the following range of p-values: $0.50 > p > 0.25$.

Convert these values to percents to see how often a catch like Exequiel's would occur.

$0.50(100) = 50\%$

$0.25(100) = 25\%$

It can be expected that a catch like Exequiel's would occur from 25% to 50% of the time—that is, between 25 and 50 times out of 100 fishing expeditions.

(*continued*)

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Sigmund's *t*-value of 3.49805 at 1 degree of freedom has the following range of *p*-values: $0.10 > p > 0.05$.

Convert these values to percents to see how often a catch like Sigmund's would occur.

$0.10(100) = 10\%$

$0.05(100) = 5\%$

It can be expected that a catch like Sigmund's would occur from 5% to 10% of the time—that is, between 5 and 10 times out of 100 fishing contests.

6. Summarize your results.

The *t*-values for each catch led to high *p*-values for Exequiel and very low *p*-values for Sigmund. The one-tailed values of *p* imply that we are looking for significance among values greater than the mean.

The two-tailed value of *p* is always double that of the one-tailed, because the distribution is symmetric about the mean.

Therefore, in a two-tailed test, a catch like Exequiel's would occur between 50 and 100 times out of 100, and a catch like Sigmund's would occur between 10 and 20 times out of 100.

When Exequiel sees these conclusions, he demands a two-sample *t*-test be carried out on the data.

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Instruction

Example 3

Looking at the data from Example 2, could these samples come from the same fish population? If so, with what statistical significance? In other words, is Sigmund fishing out of the same known population as Exequiel, or has he found a spot where the potential mean for a catch is higher than in the rest of the pond? Could Sigmund have been manipulating data? Perform a two-sample *t*-test to determine the probability that the catches of both fishermen came from the same population.

1. Determine the standard deviation and mean of each set of data.

   Recall that Exequiel caught 3 fish weighing 2.5, 3.0, and 3.6 pounds, with a sample mean of approximately 3.0333 and a standard deviation of approximately 0.55076.

   Sigmund caught 2 fish weighing 4.0 and 4.8 pounds, with a sample mean of approximately 4.40 and a standard deviation of approximately 0.56569.

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Instruction

2. Determine the *t*-value for the two catches.

Since we are comparing two samples with known means and standard deviations, use the *t*-value formula $t=\dfrac{\overline{x}_1-\overline{x}_2}{\sqrt{\dfrac{s_1^2}{n_1}+\dfrac{s_2^2}{n_2}}}$, described as follows.

- $\overline{x}_1$ is the mean of the first set.
- $\overline{x}_2$ is the mean of the second set.
- s_1^2 and s_2^2 are the squares of the standard deviations of each respective set.
- n_1 and n_2 are the respective sample sizes.

$$t=\frac{\overline{x}_1-\overline{x}_2}{\sqrt{\dfrac{s_1^2}{n_1}+\dfrac{s_2^2}{n_2}}}$$ Formula for calculating a *t*-value

$$t=\frac{(3.0333)-(4.4)}{\sqrt{\dfrac{(0.55076)^2}{(3)}+\dfrac{(0.56569)^2}{(2)}}}$$ Substitute known values for the means, standard deviations, and sample sizes of each set.

$t\approx -2.6745$

The *t*-value for the two sets is approximately –2.6745.

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Instruction

3. Determine the degrees of freedom.

With two sets of data, the degrees of freedom is the whole number part of the average of each sample size minus 1. Symbolically, $df = \frac{n_1 - 1 + n_2 - 1}{2}$.

$df = \frac{n_1 - 1 + n_2 - 1}{2}$	Formula for degrees of freedom
$df = \frac{(3) - 1 + (2) - 1}{2}$	Substitute 3 for Exequiel's sample size and 2 for Sigmund's sample size.
$df = 1.5$	Simplify.

Notice that the degrees of freedom is a decimal: 1.5. The whole part of this average is 1; therefore, the degree of freedom is 1.

4. Determine the *p*-value.

To determine the value of *p*, evaluate the *t*-distribution table at the row for 1 degree of freedom. Look along this row until you find the two values within which –2.6745 is located.

A *t*-value of –2.6745 at 1 degree of freedom has the following range of *p*-values: $0.15 > p > 0.10$.

Convert these values to percents to see how often two catches like these would occur.

0.15(100) = 15%

0.10(100) = 10%

It can be expected that these two catches would come from the same population between 10% and 15% of the time—that is, from 10 to 15 times out of 100 fishing contests.

5. Summarize your results.

Recall that, in a one-tailed test, it can be expected that a catch like Sigmund's would occur 5% to 10% of the time and a catch like Exequiel's would occur 25% to 50% of the time. Since these two catches would come from the same population only 10 to 15 times out of 100, Exequiel's catch is fairly common. Uniqueness can only be attributed to Sigmund.

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Problem-Based Task 1.5.1: State Scores Compared

SMP	
1 ✓	2 ✓
3	4
5	6 ✓
7	8

The students of Mr. Franklin's class have obtained the following scores on a state test.

71 70 69 76 68 73 76 72 68 76 68 70

The population mean of the state scores is 69 points. Does this sample have statistical significance at a confidence level of 99%?



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Problem-Based Task 1.5.1: State Scores Compared

Coaching

a. What is the sample mean?

b. What is the sample standard deviation?

c. Does this problem involve one sample and a population or two samples?

d. Which formula for t should be used?

e. What is the t-value?

f. How many degrees of freedom are there?

g. Use a t-distribution table to determine the p-value.

h. Does this sample have statistical significance at a 99% confidence level?

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Instruction

Problem-Based Task 1.5.1: State Scores Compared

Coaching Sample Responses

a. What is the sample mean?

To determine the mean of this sample, use the formula $\overline{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$, where n is the sample size, 12.

$$\overline{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

$$\overline{x} = \frac{(71)+(70)+(69)+(76)+(68)+(73)+(76)+(72)+(68)+(76)+(68)+(70)}{(12)}$$

$$\overline{x} \approx 71.417$$

The mean of the sample is approximately 71.417 points.

b. What is the sample standard deviation?

To calculate the standard deviation of the sample data, use the formula $s = \sqrt{\frac{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + (x_3 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2}{n-1}}$, where $\overline{x}$ is the mean and n is the sample size.

For this scenario, the mean is 71.417 and n is 12.

$$s = \sqrt{\frac{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + (x_3 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2}{n-1}}$$

$$s = \sqrt{\frac{\begin{array}{l}[(71)-(71.417)]^2 + [(70)-(71.417)]^2 + [(69)-(71.417)]^2 + \\ [(76)-(71.417)]^2 + [(68)-(71.417)]^2 + [(73)-(71.417)]^2 + \\ [(76)-(71.417)]^2 + [(72)-(71.417)]^2 + [(68)-(71.417)]^2 + \\ [(76)-(71.417)]^2 + [(68)-(71.417)]^2 + [(70)-(71.417)]^2\end{array}}{(12)-1}}$$

$$s = \sqrt{\frac{110.91668}{11}}$$

$$s \approx 3.175$$

The standard deviation of the sample is approximately 3.175.

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Instruction

c. Does this problem involve one sample and a population or two samples?

This problem involves one sample and a population that has a mean of 69.

d. Which formula for t should be used?

Use the formula for t that uses one sample and a population mean, $t = \frac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}}$, where $\overline{x}$ is the sample mean, μ_0 is the population mean, s is the standard deviation of the sample, and n is the sample size.

e. What is the t-value?

Substitute the known values into the formula for t determined in part d: $t = \frac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}}$.

As determined in the previous parts, the sample mean is 71.417, the population mean is 69, s is approximately 3.175, and n is 12.

$$t = \frac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}}$$

$$t = \frac{(71.417) - (69)}{\frac{(3.175)}{\sqrt{(12)}}}$$

$$t \approx 2.63708$$

The value of t is approximately 2.63708.

f. How many degrees of freedom are there?

To determine the degrees of freedom, use the formula $df = n - 1$.

$df = n - 1$

$df = [(12) - 1]$

$df = 11$

There are 11 degrees of freedom.

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g. Use a *t*-distribution table to determine the *p*-value.

Look down the first column of the table to find 11 degrees of freedom.

Read across the row to determine the two values between which 2.63708 is located.

The *t*-value falls between 0.025 and 0.01; therefore, $0.025 > p > 0.01$.

h. Does this sample have statistical significance at a 99% confidence level?

No, the sample does not have statistical significance at a 99% confidence level because *p* is greater than 0.01. For a 99% confidence level, *p* must be less than 1% or 0.01.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.5.1: Evaluating Treatments

Use the information and table that follow to complete problems 1–10.

Roulette is a casino game in which a wheel with sections numbered 0–36 is spun in one direction, and a small ball is spun onto the wheel in the opposite direction. In order to win, players must guess which number on the wheel the ball will land on. A well-balanced roulette wheel has a mean of 18. The following table shows the results of 5 sample sets from 5 different roulette wheels labeled A–E, obtained by spinning the ball 12 times on each roulette wheel.

Wheel	Spin 1	Spin 2	Spin 3	Spin 4	Spin 5	Spin 6	Spin 7	Spin 8	Spin 9	Spin 10	Spin 11	Spin 12
A	1	35	3	27	14	11	16	29	0	19	18	35
B	17	28	4	29	19	25	10	26	27	23	28	25
C	4	2	30	9	16	0	25	34	31	14	18	32
D	32	20	2	10	17	35	7	17	18	26	3	18
E	24	23	2	28	11	32	24	16	6	36	23	15

1. What is the mean for each spin number? Round answers to the nearest tenth.

2. Which spin number has a notable mean? Why?

3. What is the standard deviation for each spin number? Round answers to the nearest hundredth.

continued

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4. Calculate the *t*-value for each of the spin numbers. Round answers to the nearest thousandth.

5. Which spin number has the highest *t*-value?

6. Which spin number has the lowest *t*-value?

7. How can you explain the difference between low and high *t*-values?

8. Use a *t*-distribution table to find the *p*-value for the first spin.

9. Use a *t*-distribution table to find the *p*-value for the highest *t*-value.

10. Use a *t*-distribution table to find the *p*-value for the lowest *t*-value.

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Lesson 1.5.2: Designing and Simulating Treatments

Warm-Up 1.5.2

In sports games, teams playing on their home field are often considered to have a "home-field advantage," meaning the teams play better and win more often when playing on their home field. The following table lists the number of games won and lost by the home team for NFL playoff games from 1970 to 1999. Use the table to answer the questions that follow.

Years	Home team wins	Home team losses
1970–79	43	21
1980–89	59	27
1990–99	73	27

Source: ColdHardFootballFacts.com, "SuperStudy: Home-field advantage in playoffs."

1. What is the percentage of wins for the teams that had a home-field advantage in the playoffs for the 1970s, rounded to the nearest tenth?

2. What is the percentage of wins for the teams that had a home-field advantage in the playoffs for the 1980s, rounded to the nearest tenth?

3. What is the percentage of wins for the teams that had a home-field advantage in the playoffs for the 1990s, rounded to the nearest tenth?

4. Would you say that having a "home-field advantage" really does improve the chances of winning for the home team? Why or why not?

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Instruction

Lesson 1.5.2: Designing and Simulating Treatments

Common Core State Standard

S–IC.5 Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.★

Warm-Up 1.5.2 Debrief

1. What is the percentage of wins for the teams that had a home-field advantage in the playoffs for the 1970s, rounded to the nearest tenth?

   Calculate the percentage of wins by dividing the number of wins by the number of total games played.

   There were 43 wins and 21 losses.

   $43 + 21 = 64$

   A total of 64 playoff games were played in the 1970s.

   Therefore, the percentage of wins was $\frac{43}{64} \approx 0.6719$, or approximately 67.2%.

2. What is the percentage of wins for the teams that had a home-field advantage in the playoffs for the 1980s, rounded to the nearest tenth?

   Again, calculate the percentage of wins by dividing the number of wins by the number of total games played.

   There were 59 wins and 27 losses.

   $59 + 27 = 86$

   A total of 86 playoff games were played in the 1980s.

   The percentage of wins was $\frac{59}{86} \approx 0.6860$, or approximately 68.6%.

3. What is the percentage of wins for the teams that had a home-field advantage in the playoffs for the 1990s, rounded to the nearest tenth?

   Again, calculate the percentage of wins by dividing the number of wins by the total games played.

   There were 73 wins and 27 losses.

   $73 + 27 = 100$

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A total of 100 playoff games were played in the 1990s.

The percentage of wins was $\frac{73}{100} = 0.73$, or 73%.

4. Would you say that having a "home-field advantage" really does improve the chances of winning for the home team? Why or why not?

   Answers may vary, however, it can be assumed that the most skilled and winning teams play in the playoffs. Therefore, these teams can be thought of as having the best records for the season, thus eliminating the idea that the wins were based on the teams playing against weaker or less-skilled teams.

   It could be said that on any given game day, a team has a 50% chance of winning a game. Since the three decades of data show winning percentages (67.2%, 68.6%, and 73%) that all exceed 50%, it can be concluded that the "home-field advantage" does improve the chances of winning for the home teams.

Connection to the Lesson

- Students will be analyzing data to determine if conclusions can be drawn or assumptions can be made. Simulations can be used to draw conclusions when archived data, such as that presented in the warm-up, is not available.

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Prerequisite Skills

This lesson requires the use of the following skills:

- understanding the terms *trial* and *treatment* (S–IC.5)
- understanding the types of data resulting from a trial (S–IC.5)
- understanding the correct application of a *t*-test (S–IC.5)

Introduction

Imagine the process for testing a new design for a propulsion system on the International Space Station. The project engineers wouldn't perform their initial tests on the actual space station—to do so would be impractical because of the expense and time involved in making a trip into space. Instead, the engineers would start by using small models of the propulsion system to simulate how it would perform in real life.

A **simulation** is a set of data that models an event that could happen in real life. It parallels a similar, larger-scale process that would be more difficult, cumbersome, or expensive to carry out. Simulations are often designed for treatments in order to test a hypothesis. What is a well-designed simulation for a treatment? An accurate simulation is made up of smaller sample sets that mimic the larger sample sets that would be extracted from the entire population subjected to the treatment. In this section, we will evaluate simulations by comparing their results to expected or real-world results.

Key Concepts

- Recall that a treatment is the process or intervention provided to the population being observed.
- A trial is each individual event or selection in an experiment or treatment. A single treatment or experiment can have multiple trials.
- In order to understand the effects of treatments and experiments, simulations can be conducted.
- Simulations allow us to generate a set of data that models an event that might happen in real life. For example, you could simulate spinning a roulette wheel 20 times (that is, running 20 trials) in a spreadsheet program, and get data that would replicate the lucky numbers coming from the 20 spins in a casino. The simulation would allow you to collect data that would reflect the conditions a player will be subjected to at the casino.

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Instruction

- However, a simulation must be carefully designed in order to ensure that its results are representative of the larger population.

Steps for Designing a Simulation
1. Identify the simulation you will use.
2. Explain how you will model the simulation trials.
3. Run multiple trials.
4. Analyze the data from the simulation against the theoretically established or known parameter(s).
5. State your conclusion about whether the simulation was effective, or answer the question from the problem.

- There are a number of ways to model a trial:
  - If you have two items in your data set, you could flip a coin—a heads-up toss would represent the occurrence of one item in the data set, and a tails-up toss would represent the occurrence of the other item in the data set.
  - If you have four items in your data set and you have access to a four-section spinner, you could spin to determine an outcome.
  - If you have six items in your data set, you might roll a six-sided die.
  - If you have a larger number of items in your data set, you might make index cards to represent outcomes or numbers.
  - Many graphing calculators have a probability simulator that will flip a coin, roll dice, or choose a card from a deck multiple times to help you simulate large numbers of trials. These calculators also feature a random number generator that can be used to generate sets of random numbers based on your defined parameters.
- After running a simulation, analyze the results to determine if the simulation data seems to be at, above, or below the expected results.

Common Errors/Misconceptions

- mistakenly believing that simulations provide real-life data rather than anticipated results under ideal conditions
- not conducting enough trials of a simulation to gather data that's representative of the population

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Guided Practice 1.5.2

Example 1

Your favorite sour candy comes in a package consisting of three flavors: cherry, grape, and apple. However, the flavors are not equally distributed in each bag. You have found out that 30% of the candy in a bag is cherry, half of the candy is grape, and the rest is apple. How many candies will you have to pull from the bag before you get one of each flavor? Create and implement a simulation for this situation.

1. Identify the simulation.

   There are many possibilities for conducting a simulation of this situation. In this case, let's run a simulation that consists of drawing cards. Since we are dealing with percents, use a 10-card deck.

   Rather than running a simulation of a similar, larger-scale process, you are actually conducting a simulation that closely resembles reality. You are just using cards instead of candy, with the cards representing the percentages of candy flavors selected.

2. Explain how to model the trial.

   It is known that 30% of the candy is cherry and half (or 50%) of the candy is grape.

   Subtract these amounts from 100 to determine the remaining percentage of apple-flavored candy.

   $100 - 30 - 50 = 20$

   The remaining 20% of the candy is apple.

   Model the trial by assigning the 10 number cards to match the proportion of each candy flavor. Following this method, 3 out of 10 cards represents 30%, 5 out of 10 cards represents 50%, and 2 out of 10 cards represents 20%.

   Let numbers 1, 2, and 3 represent the cherry candies.

   Let 4, 5, 6, 7, and 8 represent the grape candies.

   Let 9 and 10 represent the apple candies.

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3. Run multiple trials.

Choose a card from the shuffled deck of 10 cards and record the number. Replace the card, shuffle the deck, choose another number, and record the number. Repeat this process until each candy flavor is represented.

The result of one simulation follows.

Trial	Outcomes	Number of candies chosen before all flavors were represented
1	2, 1, 10, 2, 1, 6	6
2	1, 9, 10, 8, 10, 7	6
3	2, 10, 6	3
4	7, 8, 8, 10, 9, 10, 1	7
5	5, 9, 6, 4, 3	5

4. Analyze the data.

For this example, 5 trials were conducted.

The values for the number of candies chosen before all flavors were represented were 6, 6, 3, 7, and 5.

The average number of candies can be calculated by finding the sum of the candies chosen in each trial and then dividing the sum by the total number of trials, 5.

$\overline{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$ Formula for calculating mean

$\overline{x} = \frac{(6)+(6)+(3)+(7)+(5)}{(5)}$ Substitute known values.

$\overline{x} = 5.4$ Simplify.

The average number of candies chosen per trial is 5.4 candies.

5. State the conclusion or answer the question from the problem.

Based on a simulation of 5 trials, the estimated number of candies that must be chosen before all three flavors will appear is an average of 5.4 candies.



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Instruction

Example 2

Your favorite uncle plays the Pick 3 lottery. The lottery numbers available in this game begin with 1 and end at 65. Since it is a Pick 3 lottery, 3 numbers are chosen. Your uncle believes that even numbers are the luckiest, and would like to know how often all 3 numbers in a drawing are even. Create and implement a simulation of at least 15 trials for this situation.

1. Identify the simulation.

   The simulation will be the selection of 3 numbers from 1 to 65.

2. Explain how to model the trial.

   Since creating 65 number cards is time-consuming and impractical, use the random number generator on a graphing calculator or computer.

3. Run multiple trials.

   A graphing calculator or computer can be used to generate numbers. To access the random number generator on your calculator, follow the directions specific to your model.

   On a TI-83/84:

   Step 1: Press [MATH].

   Step 2: Arrow over to the PRB menu, select 5: randInt(, and press [ENTER].

   Step 3: At the cursor, enter values for the lowest number possible, the highest number possible, and the number of values to be generated, separated by commas. Press [ENTER].

   Step 4: Continue to press [ENTER] to generate additional random numbers using the same range.

(*continued*)

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On a TI-Nspire:

Step 1: At the home screen, arrow down to the calculator icon, the first icon on the left, and press [enter.]

Step 2: Press [menu]. Use the arrow keys to select 5: Probability, then 4: Random, then 2: Integer. Press [enter].

Step 3: At the cursor, use the keypad values for the lowest number possible, the highest number possible, and the number of values to be generated, separated by commas. Press [enter].

Step 4: Continue to press [enter] to generate additional random numbers using the same range.

The following table shows the results of one simulation consisting of 15 trials.

Trial	Outcome	All three numbers even?
1	60, 34, 53	No
2	6, 59, 2	No
3	58, 63, 12	No
4	3, 42, 28	No
5	17, 16, 44	No
6	13, 28, 65	No
7	11, 15, 45	No
8	4, 24, 5	No
9	57, 47, 18	No
10	27, 51, 14	No
11	3, 37, 1	No
12	22, 44, 59	No
13	43, 4, 25	No
14	30, 17, 11	No
15	59, 58, 39	No

4. Analyze the data.

   Of the 15 trials conducted, none resulted in all even numbers.

5. State the conclusion or answer the question from the problem.

   Based on a simulation of 15 trials, 3 even numbers did not occur at all. So, it would probably not be wise for your uncle to pick 3 even numbers.

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Instruction

Example 3

Aspiring lawyers must pass a test called a bar exam before they can be licensed to practice law in a certain location. A local law school claims that, on average, its graduates only take the bar exam twice before passing. The national average pass rate for first-time takers of the bar exam is 52%. The national average pass rate for all other takers (those taking the test 2 or more times) is 36%. What is the average number of tests that aspiring lawyers nationally must take before passing the bar? Is the local law school's program superior to other schools in preparing students for the bar exam? Conduct a simulation with at least 20 trials.

1. Identify the simulation.

   We are asked to compare the local law school's average pass rate for bar exam test takers to the nation's average pass rate.

   This simulation has two parts. Since the national average pass rate for first-time test takers is 52%, the simulation will consist of selecting digits from 1 to 52 to represent a passing score. The second part of the simulation will consist of selecting digits from 1 to 36 to represent a passing score for any additional tests, since that national average pass rate is 36%.

2. Explain how to model the trial.

   Use two random digits for each attempt.

   For the first attempt, let the random numbers 1–52 represent obtaining a passing score and 53–100 represent a failure.

   If the person failed the first test, generate a new random number to simulate the person's second attempt to pass the bar exam. Let the random numbers 1–36 represent a passing score and 37–100 represent a failure.

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3. Run multiple trials.

   A graphing calculator or computer can be used to generate two random digits from 1 to 100. To access the random number generator on your calculator, follow the directions specific to your model, as described in Example 2.

   The problem statement specified that at least 20 trials should be conducted.

   The following table shows the result of one possible simulation consisting of 20 trials.

Trial	Outcome	Number of tests taken before passing
1	19	1
2	85, 7	2
3	73, 83, 88, 69, 61, 94, 14	7
4	14	1
5	66, 33	2
6	32	1
7	51	1
8	26	1
9	44	1
10	55, 35	2
11	88, 24	2
12	62, 66, 23	3
13	38	1
14	16	1
15	92, 14	2
16	70, 20	2
17	38	1
18	48	1
19	73, 61, 23	3
20	66, 56, 18	3

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Instruction

4. Analyze the data.

For this example, 20 trials were conducted.

The average number of tests taken can be calculated by finding the sum of the tests taken in each trial and then dividing the sum by the total number of trials.

$$\overline{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$ Formula for calculating mean

$$\overline{x} = \frac{(10)1 + (6)2 + (3)3 + 7}{(20)}$$ Substitute known values. (Repeated values are listed as products.)

$$\overline{x} = \frac{38}{20}$$ Simplify.

$$\overline{x} = 1.9$$

The average number of tests taken nationally in order to pass the bar exam is 1.9.

5. State the conclusion or answer the question from the problem.

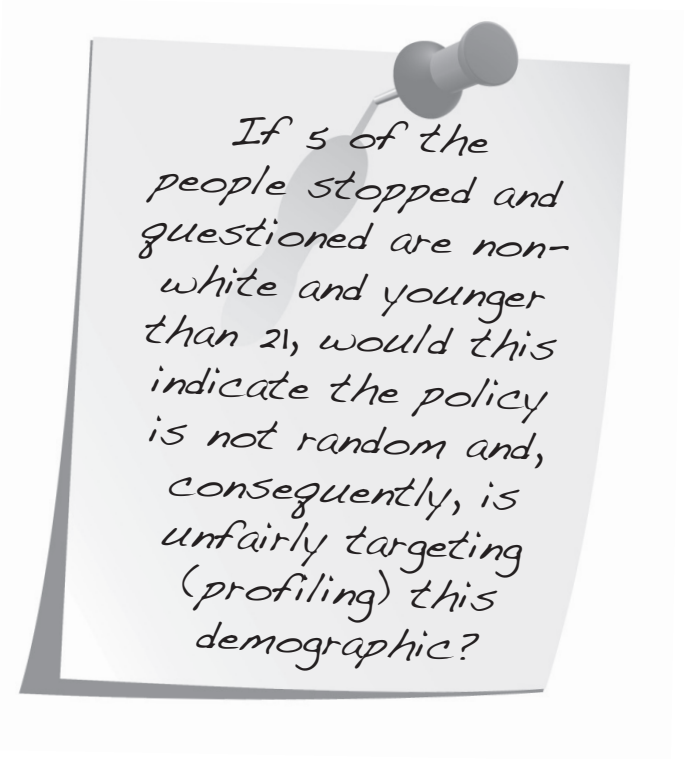
Based on a simulation of 20 trials, on average, test takers across the nation take the exam 1.9 times before passing. The local law school claims that, on average, their students only take the exam twice. Using this data, the local law school is not better at preparing students for the bar than other schools across the nation. There is very little difference between the national bar exam average (1.9) and the local law school's average (2).

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Problem-Based Task 1.5.2: Unfair Profiling?

SMP	
1	2
3 ✓	4 ✓
5	6
7 ✓	8

A controversial policy used by police in a small city is under review. The policy dictates that 1 in 10 people should be stopped and questioned to determine if they may be involved in criminal activity. One day, 2 officers are sent to a particular street to question people. Of 140 people walking down that street while the officers are on duty, 20 people are non-white and under the age of 21. If 5 of the people stopped and questioned are non-white and younger than 21, would this indicate the policy is not random and, consequently, is unfairly targeting (profiling) this demographic? Design and implement a simulation to justify your claim.



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Problem-Based Task 1.5.2: Unfair Profiling?

Coaching

a. How many people on the given street should be stopped and questioned in accordance with this policy?

b. How many people from the under-21, non-white demographic would be stopped and questioned if the number of people in this demographic who are stopped is proportionate to the number calculated in part a?

c. Design a simulation for this data to determine whether the policy unfairly targets those who are non-white and younger than 21.

d. Based on your simulation, what is the average number of simulated stops of under-21, non-white members of the population?

e. Can you justify a claim of profiling using your data?

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Instruction

Problem-Based Task 1.5.2: Unfair Profiling?

Coaching Sample Responses

a. How many people on the given street should be stopped and questioned in accordance with this policy?

The population totals 140 people, and the policy indicates that 1 in 10, or 10%, should be stopped.

$140(0.10) = 14$

Under this policy, 14 people should be stopped on the street and questioned.

b. How many people from the under-21, non-white demographic would be stopped and questioned if the number of people in this demographic who are stopped is proportionate to the number calculated in part a?

Of the 140 people who walked down the street, 20 were both non-white and younger than 21.

Set up and solve a proportion to determine how many from the under-21, non-white demographic would be stopped if the distribution paralleled the population.

$$\frac{20}{140} = \frac{x}{14}$$

$(20)(14) = 140x$

$280 = 140x$

$x = 2$

If the number of people actually stopped is proportionate to the number of non-white, under-21 people stopped, then 2 out of the 14 people stopped would be non-white and younger than 21.

c. Design a simulation for this data to determine whether the policy unfairly targets those who are non-white and younger than 21.

Begin by identifying the treatment.

We are seeking to find out if non-white people who are younger than 21 are disproportionately stopped for questioning.

Next, explain how to model the trial.

Since there are 140 people in this population, identify them by assigning each person a number from 1 to 140. The numbers 1–20 will represent the under-21, non-white population, and the numbers 21–140 will represent the remaining people walking down the street.

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Instruction

For a single trial, select 14 numbers to represent the population being stopped and questioned, then tally the number of values from 1–20 that are generated. This will indicate the number of people who are non-white and under the age of 21 who would be stopped and questioned.

Run multiple trials using a graphing calculator or computer.

The following table shows the result of one possible simulation consisting of 20 trials.

Trial	Assigned values of people selected	Number of values between 1 and 20
1	46, 137, 76, 121, 115, 99, 74, 126, 53, 97, 56, 99, 66, 64	0
2	32, 8, 117, 26, 26, 134, 49, 97, 105, 120, 23, 64, 109, 94	1
3	69, 96, 65, 126, 121, 24, 123, 49, 89, 82, 71, 121, 117, 68	0
4	129, 84, 19, 51, 74, 93, 44, 33, 40, 78, 29, 91, 20, 129	2
5	21, 139, 61, 96, 12, 34, 83, 106, 13, 32, 23, 43, 99, 81	2
6	121, 130, 43, 28, 118, 125, 35, 74, 132, 74, 97, 68, 113, 15	1
7	57, 99, 111, 108, 117, 17, 77, 62, 121, 61, 34, 24, 134, 16	2
8	106, 114, 105, 96, 85, 113, 2, 47, 42, 34, 92, 39, 118, 43	1
9	58, 18, 43, 90, 94, 14, 10, 127, 133, 96, 16, 35, 87, 92	4
10	15, 86, 123, 49, 90, 46, 90, 51, 51, 75, 86, 126, 140, 74	1
11	104, 23, 59, 97, 12, 97, 46, 19, 16, 78, 114, 2, 139, 96	4
12	80, 19, 102, 14, 68, 4, 100, 59, 75, 2, 21, 67, 136, 125	4
13	50, 84, 123, 36, 79, 121, 88, 101, 137, 60, 22, 18, 59, 68	1
14	117, 34, 115, 91, 117, 89, 64, 138, 54, 43, 92, 74, 95, 100	0
15	1, 60, 55, 25, 86, 119, 87, 87, 87, 13, 43, 22, 85, 50	2
16	20, 121, 31, 23, 120, 28, 42, 38, 90, 111, 138, 9, 73, 99	2
17	8, 49, 125, 71, 19, 27, 77, 25, 86, 115, 110, 83, 121, 140	2
18	41, 80, 4, 44, 121, 56, 90, 87, 122, 140, 137, 120, 63, 29	1
19	86, 88, 41, 26, 108, 139, 121, 47, 113, 4, 34, 23, 95, 132	1
20	15, 124, 17, 130, 137, 7, 133, 111, 101, 126, 74, 20, 57	4

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d. Based on your simulation, what is the average number of simulated stops of under-21, non-white members of the population?

For this simulation, 20 trials were conducted.

The average number of simulated stops for this demographic can be calculated by finding the sum of the values between 1 and 20 for each trial and then dividing the sum by the number of trials, 20.

$$\overline{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

$$\overline{x} = \frac{(3)0 + (7)1 + (6)2 + (4)4}{(20)}$$

$$\overline{x} = \frac{35}{20}$$

$$\overline{x} = 1.75$$

The average number of simulated stops for under-21, non-white members of the population is 1.75.

e. Can you justify a claim of profiling using your data?

Of the 14 people stopped, 5 of them are under-21 and non-white. However, as determined in part b, the proportion of the population that is non-white and younger than 21 that should be stopped is just 2. The simulation average was 1.75 people in this demographic group. Therefore, the number of people actually stopped from this subgroup was more than double the proportion and the simulation average. Additionally, in the simulation, none of the 20 trials resulted in 5 members of this subgroup being stopped. Therefore, you could use simulated data to justify the claim that the 5 people were stopped due to unfair profiling and not because of pure probability.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.5.2: Designing and Simulating Treatments

For problems 1–3, explain the flaw in each simulation.

1. NBA legend Wilt Chamberlain missed 5,805 of the 11,862 free throw shots he attempted over the course of his career. You would like to simulate this using a coin flip in which heads represents making the shot and tails represents a missed shot.

2. After simulating lucky numbers for his dad, Johnny predicted, "My dad is going to win the lottery 4% of the time!"

3. Kim invited 5 neighbors to a party. She has a 5-section spinner and will use it to predict who will arrive next.

For problems 4–6, describe a possible method for simulating each situation.

4. Given 5 playing cards from a standard deck of 52 cards, how can you simulate a process to determine which is more likely, drawing 2 pairs or drawing 3 of a kind?

5. There are 85 students who would like to take a statistics course, and three math professors. One professor will teach a class of 25 students, another will teach 2 classes of 25 students, and the third will teach a class of 10 students. What is the likelihood that 3 friends will be in the same class?

6. A manager is reviewing his company's quality-control process. He found that 5% of the company's products are returned defective. After repair, 50% of the repaired items are returned again. How can you simulate the process?

continued

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For problems 7–10, design a simulation for each situation and describe how to implement it.

7. In Maine, hunters are not allowed to hunt female deer without a special permit. In one community, 20 members of the local gun club entered a lottery to obtain the deer-hunting permit along with 37 other townspeople. If only 3 permits are issued, what is the likelihood that all 3 permits will be awarded to members of the gun club?

8. Four pairs of siblings have signed up for a darts tournament. Teams of 2 will be chosen randomly. What is the likelihood that no siblings will be on a team together?

9. In a dice game, players take turns rolling a six-sided die and adding up the value rolled. After rolling the die once, each player continues to roll the die and sum the values of the rolls until achieving a sum greater than or equal to 10. Then the next player gets a turn. If the player achieves a sum of exactly 10, that player wins the game. Suggest an appropriate simulation for this game.

10. The average age at which men marry is now 32 years old, with a standard deviation of 2.5 years. What are the chances that 4 males aren't married by 30 years of age?

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Lesson 1.5.3: Reading Reports

Warm-Up 1.5.3

A new self-serve frozen yogurt shop has been asking customers to fill out a survey after they finish consuming their purchase, but before exiting. The following table shows the results of the exit survey. Use the data from the table to answer the questions that follow.

Statement	Strongly disagree	Disagree	Agree	Strongly agree
I enjoyed my frozen yogurt.	15	6	97	211
I feel the price of my purchase was reasonable.	20	5	88	205
I will recommend this establishment to others.	17	3	90	207

1. Determine the percentage of respondents who agreed or strongly agreed with the first statement, "I enjoyed my frozen yogurt."

2. What percentage of the respondents will not recommend the frozen yogurt shop to others?

3. How many people responded to each statement? Did every person respond to all 3 survey statements? Explain how you know.

4. Is this data taken from a random sample? Explain.

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Instruction

Lesson 1.5.3: Reading Reports

Common Core State Standard

S–IC.6 Evaluate reports based on data.★

Warm-Up 1.5.3 Debrief

1. Determine the percentage of respondents who agreed or strongly agreed with the first statement, "I enjoyed my frozen yogurt."

   The table shows that the first statement relates to whether or not the customers enjoyed their frozen yogurt.

   To find a percentage, calculate the part divided by the whole, then multiply by 100.

   The part of the data we are interested in is the total number of respondents who indicated that they "agree" or "strongly agree" with the statement.

   The table shows that 97 respondents agreed with the statement and 211 respondents strongly agreed.

   $97 + 211 = 308$

   308 people responded that they "agree" or "strongly agree."

   Next, calculate the total number of responses.

   This row in the table shows the following numbers of responses: 15, 6, 97, and 211. Sum the total across the row to determine how many respondents there were.

   $15 + 6 + 97 + 211 = 329$

   There were 329 people who responded to this survey statement.

   To find the percentage, divide the part by the whole, then multiply by 100.

   $$\text{percentage who "agree" or "strongly agree"} = \frac{\text{"agree" + "strongly agree"}}{\text{total responses}} \bullet 100$$

   $$\text{percentage who "agree" or "strongly agree"} = \frac{(308)}{(329)} \bullet 100$$

   $$\text{percentage who "agree" or "strongly agree"} \approx 93.6\%$$

   Approximately 93.6% of respondents agreed or strongly agreed with the statement, "I enjoyed my frozen yogurt."

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2. What percentage of the respondents will not recommend the frozen yogurt shop to others?

   The table shows that the third statement relates to whether customers will recommend the establishment to others.

   To find a percentage, calculate the part divided by the whole, then multiply by 100.

   Since we want to find out the percentage of respondents who will *not* recommend the establishment, determine the total number of respondents who selected "strongly disagree" or "disagree" for this statement.

   The table shows that 17 respondents strongly disagreed and 3 respondents disagreed.

   $17 + 3 = 20$

   20 people responded that they "disagree" or "strongly disagree."

   Next, calculate the total number of responses.

   This row in the table shows the following numbers of responses: 17, 3, 90, and 207.

   $17 + 3 + 90 + 207 = 317$

   There were 317 people who responded to this survey statement.

   To find the percentage, divide the part by the whole, then multiply by 100.

   $$\text{percentage who "disagree" or "strongly disagree"} = \frac{\text{"disagree" + "strongly disagree"}}{\text{total responses}} \bullet 100$$

   $$\text{percentage who "disagree" or "strongly disagree"} = \frac{(20)}{(317)} \bullet 100$$

   $$\text{percentage who "disagree" or "strongly disagree"} \approx 6.3\%$$

   Approximately 6.3% of respondents disagree or strongly disagree with the statement, "I will recommend this establishment to others."

3. How many people responded to each statement? Did every person respond to all 3 survey statements? Explain how you know.

   Calculate how many people responded to each statement.

   - The first statement had response totals of 15, 6, 97, and 211.

     $15 + 6 + 97 + 211 = 329$

     The first statement had 329 responses.

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- The second statement had response totals of 20, 5, 88, and 205.

  $20 + 5 + 88 + 205 = 318$

  The second statement had 318 responses.

- The third statement had response totals of 17, 3, 90, and 207.

  $17 + 3 + 90 + 207 = 317$

  The third statement had 317 responses.

Comparing these three numbers (329, 318, and 317) shows that none of these numbers are equal. Therefore, it can be concluded that some respondents did not respond to all 3 statements.

4. Is this data taken from a random sample? Explain.

   From the information presented, the sample does not appear to be random. For example, customers who took their orders to go were not surveyed. Customers who were in a rush to leave or who have small children requiring their attention may have been less likely to respond.

Connection to the Lesson

- Students will be looking carefully at data and studies to analyze what the data reveals, such as whether the data shows a random sample.
- Students will be seeking to draw conclusions about the outcomes that have been reached from the data presented.

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Prerequisite Skills

This lesson requires the use of the following skills:

- recognizing bias (S–IC.3)
- understanding randomization (S–IC.2)
- determining statistical significance (S–IC.5)

Introduction

Data may be presented in a way that seems flawless, but upon further review, we might question conclusions that are drawn and assumptions that are made. In this lesson, we will seek to analyze underlying critical factors in studies and statistics.

Key Concepts

There are a number of steps to take when analyzing and evaluating reported data.

Investigate Charts and Graphs

- Check to see if the data sums correctly. For example, do the totals match up? Do percentages sum to 100%? What scale is used?
- How many data points does each percentage, picture, or bar represent?
- Charts and graphs can be skewed to produce a particular effect or present a particular view. Are the units compatible? Are the scales compatible? For example, you might have one set of data reported in feet and another reported in miles, or one set reported in seconds compared with a set reported in minutes. Comparing such disparate units would give a different look to the data.

Check for Possible Bias

- Recall that bias refers to surveys that lean toward one result over another or lack neutrality. There are many types of bias.
- **Voluntary response bias** occurs when the sample is not representative of the population due to the sample having the option of deciding whether to respond to the survey. This type of bias invalidates a survey due to overrepresentation of people who have strong opinions or strong motivations for responding.
- **Response bias** occurs when responses by those surveyed have been influenced in some manner. For example, if the survey questions are "leading" the respondent to give certain answers, the survey is biased.

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Instruction

- **Measurement bias** occurs when the tool used to measure the data is not accurate, current, or consistent.
- **Nonresponse bias** occurs when the respondents to a survey have different characteristics than nonrespondents, causing the population that does not respond to be underrepresented in the survey's results. People who do not respond may have a reason not to respond other than just not wanting to; for example, people who are working two jobs might not have time for a survey. The omission of this group will cause the data collected to be inaccurate for the population.
- The following questions can help you determine if there is bias:
  - How was the sample selected?
  - Are some respondents more likely than others to respond based on selection?
  - How was the data collected?
  - Is the wording of the questions unbiased?
  - Are people likely to be honest?
  - Is all of the data included?
  - Who funded the study?
  - Why was the study conducted?

Study the Sample

- While reviewing the sample, use the following questions as a guide:
  - **Is sample size disclosed?** If not, why might the author have left this out? Most statisticians indicate a minimum subgroup of 30 participants in order to generate a conclusion that can be considered reliable.
  - **What was the response rate of the survey?** (How many people responded in relation to how many people were given the survey?) The response rate can be calculated by dividing the number of people who responded by the total number contacted or surveyed. Acceptable response rates differ depending on how the survey is conducted. For example, a 50% response rate to a mailed survey would be considered adequate, while a 30% response rate would be acceptable for an online survey. Researchers conducting in-person interviews would expect a response rate of 70% or more.
  - **Was the sample chosen at random?** This entails randomly assigning subjects to treatments in an experiment to create a fair comparison of the treatment's effectiveness.

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Consider Confounding Variables

- Recall that a confounding variable is an ignored or unknown variable that influences the result of an experiment, survey, or study.
- Consider the following questions:
  - What unaddressed factors might influence a study?
  - Could the results from the data be due to some reason that has not been mentioned?

Check for Correlation

- **Correlation** is the measure of the power of the association between exactly two quantifiable variables (that is, variables that can be counted or quantified). For example, we can investigate the correlation between the length of a person's stride and her foot size, because the dimensions for both can be definitively measured, such as with a tape measure or meterstick. However, correlation cannot be applied to hair color and height because hair color is not quantifiable and is considered qualitative—it cannot be measured.

Mind the Mathematics

- When possible, double-check the arithmetic.
- Also, when data is reported, determine if it is reported as a number or as a rate. For example, one study might find that the number of automobile accidents at a particular intersection has increased. However, if a newly constructed neighborhood or building resulted in an increase in population and traffic, there may be more automobiles crossing this intersection. Additional analysis may reveal that the percentage of automobile accidents has actually not changed, or that it has possibly even decreased.

Review the Results

- While reviewing the results, consider the following questions:
  - What was the null hypothesis? Recall that the null hypothesis is the statement or idea that will be tested, and is based on the concept that there is no relationship between the data sets being studied.
  - How many trials were conducted?
  - Has the result been replicated by others?
  - Is this one person's anecdote or experience?
  - Are the significance levels appropriate for this trial?

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Common Errors/Misconceptions

- not understanding that data can be reported in a variety of ways, and that each reporting method can lead to a different result
- not realizing that much of the data reported is left to the reader to interpret

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Guided Practice 1.5.3

Example 1

A study found that children in homes with vinyl flooring would be twice as likely to be diagnosed with autism. What are some potential factors that could have affected the result of this study?

1. Review the given information for potential issues.

   Since there are no data, charts, or graphs included with this statement, we will not be concerned with the mathematics and assume that the data has been correctly calculated.

   There is also little information that might lead to bias, as this description does not supply us with evidence that this data is from an interview or survey.

2. Evaluate how the results might have been impacted by external factors.

   There may be confounding variables that impacted the results of this study. We might note that vinyl flooring could be considered less expensive, and families with lower incomes might be associated with homes that have vinyl flooring. This may have impacted the study result showing an increase in autism.



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Example 2

The president of a university sends an online survey to all faculty members, requesting feedback about satisfaction levels with university departments, service, and benefits offered. How might the results of this survey be biased?

1. Review the given information for potential issues.

   This survey was performed online. It is possible that the university president could view the identity of the person taking the survey. Respondents may not answer with complete honesty if they believe their responses are not anonymous.

2. Evaluate how the results might have been impacted by external factors.

   Online surveys have a lower expected return rate than other forms of surveys, so there will likely be underrepresentation of multiple populations.

   Faculty members who respond to the survey might have friends in certain departments, and may inadvertently perpetuate response bias by expressing higher satisfaction with the departments in which their friends work.

   Faculty members might fear retaliation for negative comments and be more likely to respond positively when asked for their opinions; i.e., to express higher levels of satisfaction than they really feel.



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Example 3

A group of newly hired campus safety officers boast that there have been 30 fewer reported incidents since the officers were hired. What questions might you have about this result?

1. Review the given information for potential issues.

   The data is reported as a decrease in quantity and has not been reported as a rate or in proportion to any other number.

   Since this data is reported by the officers as a decrease in quantity, we might ask if there has also been a decrease in student enrollment (a population decrease) that could affect the number of reported incidents. Another factor could be the time of year when the safety officers recorded their information—if it's during time periods when students are on break, we might expect that the numbers of incidents would go down.

2. Evaluate how the results might have been impacted by external factors.

   We could also ask if the officers are recording fewer incidents in order to change the results. It's possible that the new officers might leave incidents out of their official records in order to make themselves look better. It's also possible that students are intimidated by the new safety officers and under-report any incidents.



Example 4

Review the following survey questions and determine if the questions are unbiased or if they might create bias:

- Question A: Given America's great tradition of promoting democracy, do you think we should intervene in other countries?
- Question B: Should all high school students be required to apply to college?
- Question C: Since there has been an increase in pedestrian injuries in this intersection, should we have crosswalks painted onto the streets?
- Question D: Should restaurants be required to include ingredients and calorie counts on their menus for food and beverage items, for just the food items, for just the beverages, or for neither?

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Instruction

1. Review each question for words or leading phrases that would inform or encourage bias.

   Question A includes the leading phrase, "Given America's great tradition of promoting democracy," which is designed to put the respondent in a patriotic frame of mind before they are exposed to the actual question. Also, the word "great" is not neutral. Both the leading phrase and the word "great" might create bias.

   Question B does not include leading phrases that would inform or encourage bias.

   Question C includes the leading phrase "Since there has been an increase in pedestrian injuries in this intersection," which might create bias by affecting the respondent's opinion of the dangerousness of the intersection.

   Question D does not include leading phrases that would inform or encourage bias. However, it does ask the respondent to consider more than one option for including caloric and ingredient information on menus, making it difficult for a respondent to address all parts of the question with a simple "yes" or "no" answer. The respondent may agree with including the information for one section of the menu, but not all.

2. Interpret your findings.

   Question A might create bias.

   Question B seems to be an unbiased question and therefore will not likely create bias.

   Question C might create bias.

   While Question D doesn't have any leading phrases, it does include several questions within one. The question has too many components, and the respondent may be confused about which one(s) to answer; therefore, Question D might create bias.

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Problem-Based Task 1.5.3: A Voice for Our Schools

SMP	
1 ✓	2 ✓
3	4
5 ✓	6
7	8

A school district would like to obtain more information about how the district's stakeholders perceive their schools. A stakeholder is a group or member of the community that is interested in helping an organization achieve success. Create an action plan for gathering such data through a survey.



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Problem-Based Task 1.5.3: A Voice for Our Schools

Coaching

a. Who would be considered the stakeholders of the school district?

b. What are possible survey questions you could ask?

c. How will you administer the survey?

d. Identify possible sources of bias.

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Instruction

Problem-Based Task 1.5.3: A Voice for Our Schools

Coaching Sample Responses

a. Who would be considered the stakeholders of the school district?

Stakeholders include any person or group who has an interest in the school district.

Common school stakeholders include parents, teachers, students, school staff, neighbors, administrators, community activists, police, crossing guards, and politicians.

b. What are possible survey questions you could ask?

Possible questions regarding each school in the district include: Does the school offer an adequate variety of classes? Does the school offer a sufficient amount of extracurricular activities? Does the school have adequate parking? In general, can the teachers at this school be considered experts in their area of instruction? Does this school adequately prepare students for the next level of education?

c. How will you administer the survey?

The survey might be distributed in multiple ways depending upon the stakeholder group to be surveyed.

The survey could be administered at parent/teacher meetings or at a town council meeting.

A website could be made available for online access.

Copies of the survey could also be given to students in classes.

d. Identify possible sources of bias.

One possible source of bias concerns the ability of the stakeholders to choose whether to respond, leading to voluntary response bias. Stakeholders who have strong opinions about the school district or who have strong motivations may be more likely to respond.

Another possible source of bias comes from the method for distributing the survey. For example, students might feel pressured to write positive comments about their teachers if their teachers are collecting the surveys, leading to response bias.

Measurement bias is possible if different survey questions are administered to different stakeholder groups, meaning the "measurement" of stakeholders' opinions is applied inconsistently.

Finally, the timing of the survey may result in nonresponse bias. If, for example, the survey is administered during a town council meeting, people who work at night or who are attending their children's extracurricular activities would be underrepresented, altering the data.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.5.3: Reading Reports

Use your knowledge of statistical reporting to answer the following questions.

1. A study recently reported that 6 out of 7 respondents favor lower taxes. A political action committee in favor of lower taxes ran a television ad claiming that the study showed 87% of respondents favored lower taxes. What is the flaw in this ad?

2. The table below shows the number of seventh-graders who achieved a passing score on a standardized test in a particular school district. The author of a report commissioned by the superintendent of the district included the table as evidence that test results are improving. Do you agree? Explain.

Year	2010	2011	2012
Students with passing scores	345	567	656

3. A company promoted a new anti-clotting and blood-thinning drug to cardiologists, who then prescribed the drug to their patients. However, trauma and emergency room surgeons have noticed a marked decrease in their ability to stop bleeding in injured patients taking this medication, since there is no way to reverse its effects. What might be said about the studies that led to the approval of this drug?

4. A report and subsequent publications have claimed that genetically modified corn causes cancer in rats. The researchers divided 200 rats into groups of 10 and each group of 10 rats was provided a different treatment (control, a 100% corn diet, a 75% corn diet, etc.). Are there any issues with the design of this study? Explain.

5. A psychology research paper has indicated a correlation between violent video games and aggression in teenagers. Would you cite these results in a term paper? Explain.

continued

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6. Scores on a 10-question quiz on pop culture were accumulated and calculated. The mean score was 5 with a standard deviation of 4.3. Do you think this large standard deviation is the result of miscalculation? Explain.

7. You have been playing a game where you roll a six-sided die in order to move your playing piece along a game board. You notice that the number 5 has come up on most rolls. You would like to conduct an experiment to test the dice for fairness. What would be the null hypothesis for this experiment?

8. A medical team is conducting research on a new arthritis treatment. A team from a national nonprofit is also conducting similar arthritis research. Which team's results should have a lower level of significance? Why?

9. Some high school students believe that they can improve childhood cancer patients' experiences by reading positive books to the patients. They raise money and collect donations of children's books with positive messages. Each week, the group visits a local children's hospital to read to the children. They find improvement in the children as indicated by hospital staff, parents, and the patients themselves, and decide that the books have made a difference. What is the confounding variable in this situation?

10. You are asking for opinions about how well your last school photo turned out. You ask 30 of your friends and family, and the results of the survey indicate that your photos are wonderful and amazing. What can you conclude about the results of this survey?

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Progress Assessment

Circle the letter of the best answer.

1. In a *t*-test, a larger value of *t* will lead to which of the following?
   a. a larger value for *p*
   b. a smaller value for *p*
   c. a larger or smaller value for *p*, depending on the sample size
   d. The value of *t* and the value of *p* are not connected.

2. Researchers at a state research university are studying the effects of radiation waves from smartphones on the brains of laboratory rats. If the study was funded by a manufacturer of smartphones and found no statistical difference between the control and treatment groups, what might you conclude about the study?
   a. Each aspect of the study should be scrutinized because the funders would benefit from such a conclusion.
   b. State research universities should always be assumed to have correct methodology and assumed to be free from bias.
   c. The study is conclusively biased.
   d. There is no conclusion that can be made about this study because we do not know how the rats were treated.

3. A student gathers data and simulates the average wait time for customer service at a local department store. After properly conducting her simulation, she concludes that there will be a wait time of 4.2 minutes per customer. What can be determined about this conclusion?
   a. The conclusion is correct and the department store can use this figure.
   b. The data should be analyzed further in additional trials.
   c. This conclusion cannot be made because simulations do not predict exact figures.
   d. The simulation was accurate and people should be prepared to wait for the given amount of time.

continued

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4. A smaller value of p obtained as a result of a t-test indicates which of the following?
   a. a smaller probability of the result being obtained by pure chance
   b. a greater probability of the result being obtained by pure chance
   c. a smaller or greater probability of the result being due to chance, depending on the degrees of freedom
   d. The p-value is not related to any probability.

5. A nationally syndicated television show allows audience members to call or text a single vote for their favorite singer in a contest. At the end of the contest, the show's host states that the winner is "America's favorite singer." What can we say about this statement?
   a. There is voluntary response bias and therefore this conclusion is not valid.
   b. The contest is fixed.
   c. We do not know how the show is able to enforce the "one vote per person" rule.
   d. There is response bias in this survey.

6. You have only one set of data from a population whose mean is known, but whose standard deviation is unknown. Could you use a form of the t-test to establish a confidence level for the sample?
   a. No; you need two sets of data to run a t-test.
   b. Yes; just use the t-value formula with the data that you have.
   c. It depends on how large the standard deviation of the sample is.
   d. It depends on the sample size.

7. Data from a survey of 15,000 people showed that the average number of patients diagnosed with depression decreased when the patients regularly took vitamin D supplements. In a properly conducted simulation, this data set was found to have a 1% chance of occurring randomly. What might we conclude about this survey?
   a. Taking vitamin D reduces the average number of patients diagnosed with depression, since the simulation shows such a low chance of random occurrence.
   b. The sample is too large to draw a valid conclusion.
   c. There may have been response bias in this survey, so we cannot make any valid conclusions.
   d. We must know who funded the study before allowing this information to be shared.

continued

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Assessment

8. A restaurant manager has given buttons to the restaurant's servers in order to increase tips. The manager found that the more buttons the servers wear, the higher the amount of tips the servers receive. What might be confounding factors in this study?

   a. Servers on different shifts may get different tips.

   b. There may be more male than female servers in this restaurant.

   c. The servers who are given more buttons may provide better service or feel more positive about their jobs.

   d. The customers may miscalculate tips because they are distracted by the buttons.

9. How could we determine if a machine that generates lottery numbers is truly random?

   a. Create a simulation and compare the results.

   b. Test the machine's outcomes to determine if each is equally likely.

   c. Review a list of lottery winners for the past 40 years to analyze winners of multiple jackpots.

   d. Calculate the probabilities of winning the lottery and compare the results to the actual winners.

10. Which of the following could invalidate the conclusion of a study?

    a. nonresponse bias

    b. applying correlation to qualitative results

    c. a sample size less than 5

    d. all of the above

Use the data from the table to answer the questions about the treatment of this study.

11. The manufacturers of a new cleaning product conducted in-person surveys of their customers at local grocery stores and found the following customer satisfaction rates:

	Satisfied	Dissatisfied	Never used the product
Store X	145	14	78
Store Y	216	2	90
Store Z	310	30	112

    a. What percentage of respondents of the survey are satisfied?

    b. What are possible sources of bias in this survey?

    c. The manufacturers ask a marketing expert if they should use these results in their television ads. How will she advise them?

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. A game that consists of rolling one standard, six-sided die numbered 1–6 pays out $2 for getting a number greater than 5 and $0.50 for getting an even number. What is the expected value of this game, rounded to the nearest cent?

   a. $0.58

   b. $0.92

   c. $2.50

   d. The expected value cannot be calculated because we do not know what number(s) came up on the dice.

2. A mother of 4 children is flipping a coin to decide which of her children gets to read a story first. Is this a fair way for her to make this decision?

   a. Yes, because flipping a coin results in a random outcome.

   b. No, because the child doesn't get to decide "heads" or "tails."

   c. Yes, because the mom does not choose between her children.

   d. No, because there is not an equally likely chance for each child.

3. Which of the following describes a random number generator?

   a. Any amount of numbers can be chosen as needed.

   b. Each number must have an equally likely chance of occurring.

   c. No pattern can evolve from the numbers generated.

   d. all of the above

continued

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4. What is the probability that a family with 5 children has all boys?

   a. 1 in 5

   b. 1 in 10

   c. 1 in 32

   d. Probability cannot be calculated in this scenario.

5. At a party, each guest writes his or her name on a piece of paper to be entered into a raffle for prizes. Then, the guests put their slips of paper into a hat, and the host chooses the prizewinners by selecting slips of paper one at a time from the hat. Is this a fair way to select prizewinners?

   a. No, because guests might write down the name of a friend instead of their own name as directed, leading to some names being entered more than once.

   b. Yes, because putting all of the papers into a hat will make the choice random.

   c. Yes, because the host will not see anyone's names or papers.

   d. No, because the host could look into the hat.

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Lesson 6: Making and Analyzing Decisions

Instruction

Common Core State Standards

SMP
1 ✓ 2
3 ✓ 4
5 ✓ 6
7 ✓ 8 ✓

S–MD.6 (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).★

S–MD.7 (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).★

Essential Questions

1. How can probability be applied in making decisions?
2. How can we determine the fairness of a decision?
3. What are the implications of conditional probabilities in the real world?

WORDS TO KNOW

conditional probability of *B* given *A*
the probability that event B occurs, given that event A has already occurred. If A and B are two events from a sample space with $P(A) \neq 0$, then the conditional probability of B given A, denoted $P(B|A)$, has two equivalent expressions:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\text{number of outcomes of } (A \text{ and } B)}{\text{all outcomes in } A}$$

empirical probability
the number of times an event actually occurs divided by the total number of trials, given by the formula $P(E) = \frac{\text{number of occurrences of the event}}{\text{total number of trials}}$; also called *experimental probability*

expected value
an estimate of value that is determined by finding the product of a total value and a probability of a given event; symbolized by $E(X)$

SRB p. U1-240

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Instruction

experimental probability the number of times an event actually occurs divided by the total number of trials, given by the formula $P(E) = \frac{\text{number of occurrences of the event}}{\text{total number of trials}}$; also called *empirical probability*

fair describes a situation or game in which all of the possible outcomes have an equal chance of occurring

false negative result a determination that an experiment has produced an incorrect negative result

false positive result a determination that an experiment has produced an incorrect positive result

random number generator a tool used to select a number without following a pattern, where the probability of generating any number in the set is equal

theoretical probability the probability that an outcome will occur as determined through reasoning or calculation, given by the formula $P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}$

trial each individual event or selection in an experiment or treatment

true negative result a determination that an experiment has produced a correct negative result

true positive result a determination that an experiment has produced a correct positive result

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Instruction

Recommended Resources

- Interactivate. "Advanced Monty Hall."

  http://www.walch.com/rr/00191

  Users explore conditional probability with this interactive game. Given a choice of doors, users select a door in hopes of winning the grand prize. One of the remaining doors is opened to reveal a pig, and users are given the option to either stick with their original door or choose a new door.

- Mr. Nussbaum.com. "Probability Fair."

  http://www.walch.com/rr/00192

  This interactive game engages users in practicing calculation of probabilities and analyzing the fairness of carnival games such as Plinko, Duck Pluck, and the Shell Game.

- National Collegiate Athletic Association. "Probability of competing in athletics beyond high school."

  http://www.walch.com/rr/00193

  This chart details the probabilities of playing various men's and women's sports professionally and for the NCAA. The data here can be used to generate discussions on probabilities in the real world.

- Stat Trek. "Probability Calculator."

  http://www.walch.com/rr/00194

  This interactive calculator allows users to compute the probability of an event based on the probabilities of other events. The site provides explanations for each calculated answer.

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Lesson 1.6.1: Making Decisions

Warm-Up 1.6.1

Luanne is the events director for a family festival that's held each year at the local park. To encourage more families with children to attend this year, Luanne wants to award prizes at random to attendees. A local company has donated two bikes for girls for the festival to give away as prizes. To better understand the probability that the prizes will be awarded to girls, Luanne wants to answer the following questions.

1. What is the probability of a girl being chosen at random from a group of 4 boys and 8 girls?

2. What is the probability that the children in a family with only 2 children are both girls?

3. Based on your answers to the previous questions, are these prizes appropriate? Explain.

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Instruction

Lesson 1.6.1: Making Decisions

Common Core State Standard

S–MD.6 (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).★

Warm-Up 1.6.1 Debrief

Luanne is the events director for a family festival that's held each year at the local park. To encourage more families with children to attend this year, Luanne wants to award prizes at random to attendees. A local company has donated two bikes for girls for the festival to give away as prizes. To better understand the probability that the prizes will be awarded to girls, Luanne wants to answer the following questions.

1. What is the probability of a girl being chosen at random from a group of 4 boys and 8 girls?

   Probability is calculated by dividing the number of ways an event E can occur by the total number of possible outcomes in the sample space:

   $$P(E)=\frac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}$$

   In this scenario, the event E is choosing a girl from the group. Since there are 8 girls in the group, there are 8 ways this can occur.

   The number of outcomes in the sample space is the total number of children in the group.

   There are 4 boys and 8 girls, so sum these numbers to determine the number of outcomes in the sample space.

   $4 + 8 = 12$

   There are a total of 12 children in the group.

   Substitute these results into the equation for the probability of an event E.

   $$P(E)=\frac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}$$ Formula for probability of an event E

   $$P(E)=\frac{(8)}{(12)}$$ Substitute previously determined values.

   $$P(E)=\frac{2}{3}$$ Reduce to lowest terms.

   The probability of choosing a female child from a group of 4 boys and 8 girls is $\frac{8}{12}$, or $\frac{2}{3}$.

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2. What is the probability that the children in a family with only 2 children are both girls?

   First, determine the probability of having 1 child who is a girl.

   Use the formula for the probability of an event E:

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}$$

   The event E is having a girl. There is 1 way this can occur.

   The sample space is the total possible genders. Since the possible genders are male or female, there are 2 possible genders in the sample space.

   Substitute these values into the formula.

$$P(E) = \frac{(1)}{(2)}$$

   The chance of having 1 child who is a girl is $\frac{1}{2}$.

   The gender of the first child has no impact on the gender of the second child; therefore, these are independent events.

   To determine the probability of having 2 girls, apply the Multiplication Rule.

   The Multiplication Rule states that the probability of the occurrence of two independent events, A and B, is $P(A \text{ and } B) = P(A) \bullet P(B)$.

$P(A \text{ and } B) = P(A) \bullet P(B)$	Multiplication Rule for independent events
$P(2 \text{ girls}) = P(1 \text{ girl}) \bullet P(1 \text{ girl})$	Substitute events for A and B.
$P(2 \text{ girls}) = \left(\frac{1}{2}\right) \bullet \left(\frac{1}{2}\right)$	Substitute previously determined values for the probability of each event.
$P(2 \text{ girls}) = \frac{1}{4}$	Simplify.

   The probability that the children in a family with only 2 children are both girls is $\frac{1}{4}$ or 25%.

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Instruction

3. Based on your answers to the previous questions, are these prizes appropriate? Explain.

   Although there is a $\frac{2}{3}$ probability of choosing a female from a group of 12 children that is made up of 4 boys and 8 girls, there is no guarantee that a group of this makeup will actually attend the festival. Also, the probability of there being a family with 2 children that are both girls is only $\frac{1}{4}$. Based on these probabilities alone, it doesn't appear to be appropriate to offer only prizes for girls. Luanne should either provide prizes for boys as well, or find prizes that are gender neutral. Luanne might also determine different probabilities that will allow her to better understand the likelihood of a girl being chosen at random from the entire group of festival attendees.

Connection to the Lesson

- Students will apply their understanding and knowledge of probability to real-world problems.
- Students will use the results of calculating probabilities to inform decisions.

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Prerequisite Skills

This lesson requires the use of the following skills:

- calculating compound probability (7.SP.8)
- calculating expected value (S–MD.5)
- calculating probability (7.SP.7)

Introduction

At the beginning of each American football game, there is a coin toss. The team that wins the coin toss may then choose whether they want to begin the game on offense or defense (or defer the decision until the beginning of the second half of the game). The toss of a coin with 2 sides is considered to be a fair way to make this decision. This lesson will explore ways of making decisions that are fair based on the sample space or participants.

Key Concepts

- Probabilities are often used to make fair decisions.
- Tools such as a fair coin, a numbered die, or a random number generator are often used when determining probabilities.
- Recall that a random number generator is a tool to select a number without following a pattern, where the probability of any number in the set being generated is equal.
- When discussing probability, there are two types to consider: empirical (or experimental) probability and theoretical probability.
- **Empirical probability** (also known as **experimental probability**) is the number of times an event *actually occurs* divided by the total number of trials. Empirical probability is determined using the formula $P(E)=\frac{\text{number of occurrences of the event}}{\text{total number of trials}}$.
- Recall that a trial is each individual event or selection in an experiment or treatment.
- In order to calculate empirical or experimental probabilities, the results or data must be provided. That is, empirical or experimental probability is based on data that has been gathered already—results that have been observed.

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Instruction

- **Theoretical probability**, on the other hand, is the probability that an outcome *will occur* as determined through reasoning or calculation, and is given by the formula $P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}$.
- Theoretical probability does not require any previously gathered data from trials; it is based on what is possible given the circumstances of the event.
- Values that are determined based on empirical or experimental probability are often different from values obtained based on theoretical probability; in other words, what is observed is frequently different from what is theoretically possible.
- Experimental or empirical probability will grow closer and closer in value to that of theoretical probability as the number of trials increases.
- **Expected value**, $E(X)$, is an estimate of value that is determined by finding the product of a total value and a probability of a given event.
- Expected value is calculated by multiplying the value by the probability.
- In situations where there are multiple values and multiple probabilities, multiply the corresponding value (p_n) by its probability $P(X_n)$, and then find the sum of the results: $E(X) = p_1P(X_1) + p_2P(X_2) + p_3P(X_3)$.
- In expected value problems, the probabilities must sum to 1 or 100%.
- A situation or game is said to be **fair** if all of the possible outcomes have an equal chance of occurring.
- Be certain to consider any assumptions when deciding whether a situation is "fair." You may think that a situation or game is fair at first glance, but upon further review find that it is not. Calculate probabilities as a justification of any intuition or first impressions you may have.
- Sometimes, partial evidence or information is provided. If this happens, an inference or conclusion can be made about the given information. For example, a passage in a news article may state, "The coach went with her instincts." We can infer from the use of "her" that the coach in this situation is female.

Common Errors/Misconceptions

- incorrectly calculating probabilities
- confusing theoretical probability with empirical or experimental probability

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Instruction

Guided Practice 1.6.1

Example 1

Suppose the U.S. State Department distributes 140,000 visas for people to come to work in the United States each year. The visas are distributed among five preference categories (E1–E5), with E1 being the highest-priority category, then E2, and so on. The first three categories each receive 28.6% of the available visas, and the fourth receives 7.1%. Is the process fair for all applicants?

1. Analyze the given information.

   The first piece of information is that 140,000 visas are distributed.

   The total number of possible visas does not impact the fairness of the distribution.

   This is extraneous information and we will not use this number to determine fairness.

   The next piece of given information is that there are five preference categories.

   Since people who fall into the highest preference category, E1, will be given first choice at a visa, we know that all applicants who do not fall into this category have a reduced possibility of being granted a visa.

   The final piece of given information indicates the percentage of visas distributed to each category.

   The percentage of visas given to applicants in each category is as follows:

   - E1: 28.6%
   - E2: 28.6%
   - E3: 28.6%
   - E4: 7.1%
   - E5: Unknown

   The total percentage of visas issued for the first four categories can be determined by summing the known values.

   $28.6 + 28.6 + 28.6 + 7.1 = 92.9$

   To determine the percentage for the final category, subtract this total from 100:

   $100 - 92.9 = 7.1$

   E5 applicants received 7.1% of the visas.

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2. Determine if the process is fair to all applicants.

   Applicants who fall outside of the first three categories have a reduced possibility of being granted a visa, so we can determine that this process is not fair; not all applicants have an equal opportunity to receive a visa.



Example 2

In the Olympics, swimmers are assigned to lanes based on what place they earned in previous swim trials—first place, second place, etc. Swimmers in the inner lanes have an advantage, since the waves created when water hits the sides of the pool create resistance that can slow swimmers down, and the effect of these waves is greater in the lanes that are closer to the sides of the pool. The lane assignments are as follows:

Lane	Swimmers by rank
1	Seventh place
2	Fifth place
3	Third place
4	First place
5	Second place
6	Fourth place
7	Sixth place
8	Eighth place

Is this a fair way to assign a swimmer to each lane?

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Instruction

1. Analyze the given information.

   The first piece of information stated is that swimmers are assigned to lanes based on how well they performed in previous swim trials.

   This statement seems to be a fair distribution because it can be presumed that each Olympian followed the rules and is an exceptional swimmer, and therefore had an equal chance of winning the previous race.

   The next piece of information given is that swimmers in the inner lanes meet less resistance from waves. It is possible that the swimmer's lane assignment in the last race had an effect on his or her result, since particular lanes provide an advantage (or disadvantage).

2. Review the table of lane assignments.

   If there are 8 swimmers—1 for each available lane—then the swimmers do not have an equal chance of being placed in each lane because, once assigned, swimmers do not have an opportunity to be chosen by chance for any other lane.

   We can also see from the table that the innermost lane is awarded to the first-place swimmer of the previous trial. The outermost lanes are assigned to the seventh- and eighth-place swimmers.

3. Determine if this is a fair way to assign swimmers to each lane.

   Since the 8 swimmers do not have an equal chance of being placed in any given lane, this method of assigning lanes cannot be considered fair. ✓

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Example 3

Valerie is playing a dice game at a carnival. She pays $3 to play and roll one regular, 6-sided die. If she rolls a 3, she wins $10. If she does not roll a 3, she gets a second chance. If she rolls a 3 this time, she wins $5. If she doesn't roll a 3 either time, she loses the game. What is the expected amount Valerie could win playing this game? Based on the expected amount and the fee to play, is this game fair?

1. Determine what information the problem is asking you to find.

   Since the question asks, "what is the expected amount," this problem is asking for the expected value of an event.

   There are three possible events with corresponding payout amounts: winning on the first roll, winning on the second roll, or losing.

   We can determine the fairness of this game by calculating the expected value.

2. Determine the given information.

   This game involves rolling a regular, 6-sided die.

   Therefore, the sample space of a regular die includes 6 possibilities: 1, 2, 3, 4, 5, and 6.

   The given information includes values for a number of events:

   - earning $10 for rolling a 3 on the first try
   - earning $5 for rolling a 3 on the second try (if a 3 is not rolled on the first try)
   - paying $3 to play, regardless of the number rolled

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3. Determine the probabilities of each outcome listed in the given information.

   To determine the probability of winning \$10, determine the probability of rolling a 3.

   There are 6 sides to a fair die; this is the sample space.

   There is 1 side with a 3 on it.

   Rolling a 3 has a probability of $\frac{1}{6}$; therefore, the probability of winning \$10 is $\frac{1}{6}$.

   To determine the probability of winning \$5, there are two events that must be determined—not rolling a 3, and then rolling a 3.

   First determine the probability of not rolling a 3.

   Again, there are 6 sides to a fair die, so this is the sample space.

   There are 5 sides that do not have a 3 on them.

   Thus, the probability of not rolling a 3 is $\frac{5}{6}$.

   If a 3 is not rolled on the first try, a second roll is given.

   The probability of rolling a 3 on the second try is $\frac{1}{6}$.

   The probability of winning \$5 can be found using the Multiplication Rule, which states that the probability of two independent events, A and B, occurring is $P(A \text{ and } B) = P(A) \bullet P(B)$. In this case, A is "not rolling a 3" (not 3) and B is "rolling a 3" (3).

$P(A \text{ and } B) = P(A) \bullet P(B)$	Multiplication Rule
$P(\text{not } 3 \text{ and } 3) = P(\text{not } 3) \bullet P(3)$	Substitute events for A and B.
$P(\text{not } 3 \text{ and } 3) = \left(\frac{5}{6}\right) \bullet \left(\frac{1}{6}\right)$	Substitute values for the probability of each event.
$P(\text{not } 3 \text{ and } 3) = \frac{5}{36}$	Simplify.

   The probability of winning \$5 is $\frac{5}{36}$.

(*continued*)

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Instruction

To determine the probability of losing the game, calculate the probability of not rolling a 3 on the first roll and then not rolling a 3 again on the second roll.

As determined earlier, the probability of not rolling a 3 is $\frac{5}{6}$.

$P(A \text{ and } B) = P(A) \bullet P(B)$	Multiplication Rule
$P(\text{not 3 and not 3}) = P(\text{not 3}) \bullet P(\text{not 3})$	Substitute events for A and B.
$P(\text{not 3 and not 3}) = \left(\frac{5}{6}\right) \bullet \left(\frac{5}{6}\right)$	Substitute values for the probability of each event.
$P(\text{not 3 and not 3}) = \frac{25}{36}$	Simplify.

The probability of losing the game is $\frac{25}{36}$.

4. Calculate the expected value.

   Calculate the expected value by taking the sum of each payout, p_n, times each probability, $P(X_n)$.

   Let p_1 equal 7, the payout from winning on the first try ($10) minus the cost to play the game ($3): 10 – 3 = 7.

   Let $P(X_1)$ represent the probability of that event occurring, $\frac{1}{6}$.

   Let p_2 equal 2, the payout from winning on the second try ($5) minus the cost to play the game ($3): 5 – 3 = 2.

   Let $P(X_2)$ represent the probability of that event occurring, $\frac{5}{36}$.

   Let p_3 equal –3, the payout from not winning either time ($0) minus the cost to play the game ($3): 0 – 3 = –3.

(*continued*)

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Instruction

Let $P(X_3)$ represent the probability of that event occurring, $\frac{25}{36}$.

$E(X) = p_1P(X_1) + p_2P(X_2) + p_3P(X_3)$ Formula to calculate expected value

$E(X) = (7)\left(\frac{1}{6}\right) + (2)\left(\frac{5}{36}\right) + (-3)\left(\frac{25}{36}\right)$ Substitute known values for p_n and $P(X_n)$.

$E(X) \approx -0.64$

The expected value, or the amount that Valerie could expect to win over a long period of time playing this game, is approximately –$0.64.

5. Determine if the game is fair.

The expected value is –0.64, meaning Valerie would expect to lose $0.64 over a long period time.

As a result, this game is unfair to the player and favors the carnival.

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Instruction

Example 4

Pulmonary tuberculosis is a serious and sometimes fatal bacterial infection of the lungs. Prior to World War II, the standard treatment for patients diagnosed with pulmonary tuberculosis was bed rest. After the war and the success of treating wounded soldiers with the antibiotic penicillin, doctors sought to determine if a new medicine called streptomycin could help tuberculosis sufferers. Diagnosed patients were first screened for eligibility of this trial. Upon admission, patients were given randomly numbered envelopes that were distributed equally by gender. Inside the envelopes were cards labeled "S" for "streptomycin" or "C" for "control." The patients receiving an "S" would be given streptomycin in addition to bed rest. The control group would be given bed rest only. Did the patients have a fair chance for receiving the experimental treatment?

1. Analyze the given information.

   The envelopes were numbered randomly.

   The envelopes were then divided equally by gender.

   The envelopes were given to patients—the patients did not choose their own envelopes.

2. Make an inference about the given information.

   Since the labeled cards were inside the envelopes, there was no way to determine which patient would be in the control group and which would receive the treatment.

3. Determine if the patients had a fair chance for receiving the experimental treatment.

   Since each patient had an equal chance of receiving the treatment, this experiment can be considered fair.

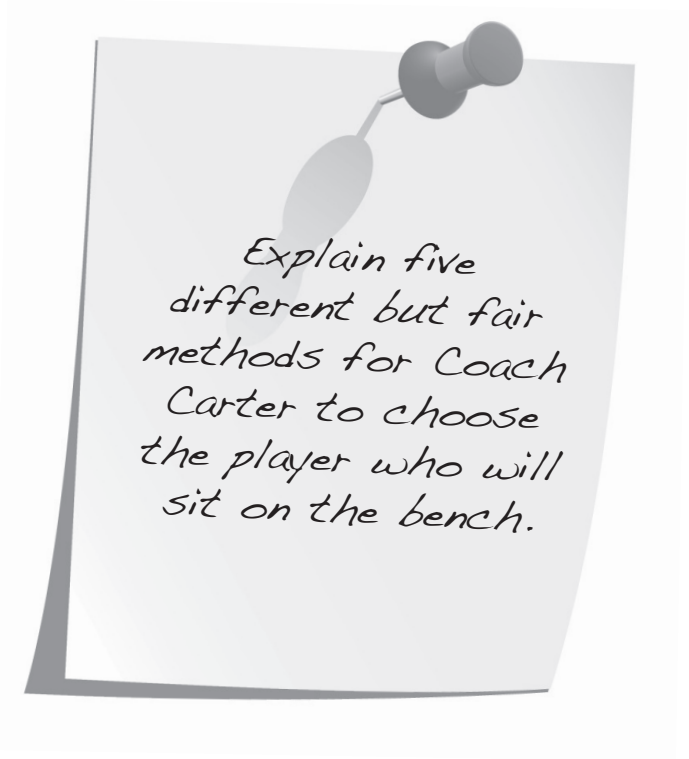


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Problem-Based Task 1.6.1: Put Me In, Coach!

SMP	
1	2
3 ✓	4
5 ✓	6
7	8 ✓

Carter is the coach of a traveling basketball team made up of 6 equally talented players. Because basketball games start with 5 players on the court at one time, 1 player must be chosen to sit out. Explain five different but fair methods for Coach Carter to choose the player who will sit on the bench.



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Problem-Based Task 1.6.1: Put Me In, Coach!

Coaching

a. There are 6 players from which to choose. What is one method to choose the player that comes to mind?

b. What device might Carter create that could help in choosing the 1 player who will sit out?

c. If Carter has only 1 piece of paper and a pencil on hand, how might he use these tools to choose which player sits out?

d. How might Carter use a bag of marbles to choose the player?

e. How might Carter use a deck of cards to choose the player?

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Problem-Based Task 1.6.1: Put Me In, Coach!

Coaching Sample Responses

a. There are 6 players from which to choose. What is one method to choose the player that comes to mind?

Since there are exactly 6 players, rolling a fair six-sided die in order to choose the player who will sit out seems like a logical choice.

Each player can have a designated number, and the player whose number appears will be the one to sit out.

b. What device might Carter create that could help in choosing the 1 player who will sit out?

Carter might create a spinner divided into 6 equally sized sections. The name of each player could be written in each section. The section the spinner lands on will designate the player who will sit out.

c. If Carter has only 1 piece of paper and a pencil on hand, how might he use these tools to choose which player sits out?

In this case, Carter could fold and rip the paper into 6 pieces that are equal in size and write each player's name on a different piece. Then, without looking, he could choose 1 of the pieces of paper. The paper chosen would designate the player who sits out.

d. How might Carter use a bag of marbles to choose the player?

If the bag has at least 6 marbles that are all different colors, Carter could assign a color to each player, put the 6 marbles into the bag, and choose 1 marble. The marble chosen would designate the player who sits out.

e. How might Carter use a deck of cards to choose the player?

He could ask each player to choose 1 card from the deck; this is the card that will represent the player. After shuffling these 6 cards, Carter could choose a card at random. The card chosen would designate the player who sits out.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.6.1: Making Decisions

Determine whether each of the following scenarios is fair. Explain your reasoning.

1. Students in grade 4 at an elementary school are assigned to 1 of 7 teachers. One teacher will work with students with special needs. Another teacher will guide students who are learning English. A third teacher will provide accelerated instruction for students who have exceeded the grade level score on a standardized exam. All the remaining students will be randomly assigned to open spaces in one of the remaining 4 teachers' classes. Does a child have an equal chance of being assigned one of the 4 teachers?

2. Claudia has lived a long and prosperous life. Her 6 children are all grown; most have families of their own, of varying sizes. Now that she's writing her will, Claudia has decided to distribute her wealth to her children according to the size of each child's family. Is this a fair way for Claudia to allocate her wealth?

3. Rico has been pestering his father to let him fly to Florida alone to visit a friend, but Rico's father has been hesitant. Finally, his father comes up with an idea—drawing lots from a hat. His father writes the following on 3 separate pieces of paper: "Yes, you can go;" "Maybe next year;" and "No, never, and stop asking." Since Rico is anxious to fly to Florida as soon as possible, is this fair?

4. In the Greek classic *The Iliad*, there was concern amongst the Greek warriors over who would have to fight Hector, the admired and feared Trojan soldier. Each Greek warrior placed a piece of paper with his own name on it into King Agamemnon's helmet. A knight shook the helmet and Ajax's name fell out. Ajax would be the one to fight Hector. Was this a fair way of choosing who battled Hector?

5. In times of war in the late 18th century in Europe, drafts were a popular method of selecting men to serve in the army. Suppose that in order to prepare for war, the names of men who were between the ages of 17 and 45 were listed and submitted to the government. Each county in the nation was expected to provide 1,500 men to serve in the military. If the 1,500 names were randomly chosen from each county, would this be considered a fair way to create a draft?

continued

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6. You draw a card from a standard deck of 52 cards. If you choose a diamond, you win $50. If you choose a heart, you win $25. If you choose the ace of spades, you win $100. What is the expected value of this game?

7. An automobile insurance company is recalculating its rates for the next year. The company's actuaries have found that 1 in 10,000 customers of a given population had a serious accident costing an average of $50,000. About 30 customers in that same population of 10,000 had minor accidents that cost an average of $15,000 each. What is the expected value of an insurance policy for this population?

8. In order to create 3 teams at a birthday party, a mother put an equal number of black, red, and white marbles in a box. Each child picked a marble, with each color representing a different team. List an additional detail that would make this an unfair way to choose teams.

9. Which of the following dice games can be considered fair, if both involve 2 players and 2 regular, 6-sided dice?

   - Game 1: One player wins when one die is even and the other is odd. The other player wins all other rolls.
   - Game 2: One player wins when the product of the two numbers rolled is prime. The other player wins when the product of the two numbers rolled is not prime.

10. Justin, Hayden, Noriko, and Terrence would each like to be captain of the Math Team. Terrence suggests that they put 4 pieces of paper in a hat and mark one piece with "C." Each person selects and keeps a piece of paper from the hat. The first person to choose the paper with the "C" gets to be captain. Is this a fair way to choose the team captain?

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Lesson 1.6.2: Analyzing Decisions

Warm-Up 1.6.2

Movie tickets are priced according to age group: child tickets (anyone younger than 12); adult tickets (ages 12–59); and senior tickets (anyone older than 59). A movie theater manager recorded the number of male and female customers for one Sunday and listed the number of tickets sold by demographic, as shown in the table. Use this table to answer the questions that follow.

	Child tickets	Adult tickets	Senior tickets
Females	46	190	78
Males	55	265	34

1. What is the percentage of female ticket holders?

2. If a customer is chosen at random, what is the probability that the customer is a senior?

3. If a customer is chosen at random, what is the probability that the customer is a male?

4. If a customer is chosen at random, what is the probability that the customer is a senior and a male?

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Lesson 1.6.2: Analyzing Decisions

Common Core State Standard

S–MD.7 (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).★

Warm-Up 1.6.2 Debrief

Movie tickets are priced according to age group: child tickets (anyone younger than 12); adult tickets (ages 12–59); and senior tickets (anyone older than 59). A movie theater manager recorded the number of male and female customers for one Sunday and listed the number of tickets sold by demographic, as shown in the table. Use this table to answer the questions that follow.

	Child tickets	Adult tickets	Senior tickets
Females	46	190	78
Males	55	265	34

1. What is the percentage of female ticket holders?

   The total number of female ticket holders can be calculated by adding the number of female ticket holders from all three age groups.

   46 female children + 190 adult females + 78 senior females = 314 female ticket holders

   There are 314 female ticket holders.

   The total number of ticket holders can be calculated by adding the number of female ticket holders to each of the male ticket holder subtotals.

   314 females + 55 male children + 265 adult males + 34 senior males = 668 total ticket holders

   There are a total of 668 ticket holders.

   The percentage of female ticket holders can be calculated by dividing the number of female ticket holders, 314, by the total number of ticket holders, 668.

   $$\frac{314}{668} \approx 0.47005 \approx 0.47$$

   Multiply this amount by 100 to determine the percentage.

   0.47 • 100 = 47%

   Approximately 47% of ticket holders are female.

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2. If a customer is chosen at random, what is the probability that the customer is a senior?

   Determine the total number of senior ticket holders.

   78 senior females + 34 senior males = 112 seniors

   There are a total of 112 senior ticket holders.

   The likelihood of choosing a senior ticket holder can be calculated by dividing the number of senior ticket holders, 112, by the total number of ticket holders, 668.

   $$\frac{112}{668} \approx 0.16766 \approx 0.168$$

   Probability can also be expressed as a percent; multiply the result by 100 to convert the probability to a percent.

   0.168 • 100 = 16.8%

   The probability that a randomly selected customer is a senior is approximately 0.168 or 16.8%.

3. If a customer is chosen at random, what is the probability that the customer is a male?

   The total number of male ticket holders can be calculated by adding the number of male ticket holders from all three age groups.

   55 male children + 265 adult males + 34 senior males = 354 total male customers

   There are 354 male ticket holders.

   The percentage of male ticket holders can be calculated by dividing the number of male ticket holders, 354, by the total number of ticket holders, 668.

   $$\frac{354}{668} \approx 0.52994 \approx 0.53$$

   Multiply the result by 100 to express this probability as a percent.

   0.53 • 100 = 53%

   The probability that a randomly selected customer is a male is approximately 0.53 or 53%.

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4. If a customer is chosen at random, what is the probability that the customer is a senior and a male?

   There are 34 senior male ticket holders.

   Divide the number of senior male ticket holders, 34, by the total number of ticket holders, 668.

   $$\frac{34}{668} \approx 0.05089 \approx 0.05$$

   To express this as a percent, multiply by 100.

   $0.05 \bullet 100 = 5\%$

   The probability that a randomly selected customer is a male senior is approximately 0.05, or 5%.

Connection to the Lesson

- Students will apply probability concepts to real-life situations in order to make informed decisions based on these calculations.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- calculating expected value (S–MD.5)
- calculating empirical (experimental) probability (7.SP.6)
- calculating theoretical probability (7.SP.7)
- calculating conditional probability (S–CP.6)

Introduction

Probabilities can be fun and interesting to calculate and consider. But they can also be serious business, especially when used to assist in the following tasks: making financial decisions, understanding the results of medical tests, planning for upcoming events, and developing strategies. Knowledge of probability is used to weigh in on decisions and analyze situations. This section will focus on how to evaluate the accuracy and fairness of situations and outcomes.

Key Concepts

- Recall that fairness is the absence of bias in possible outcomes. Each outcome of a fair situation has an equal chance or probability of occurring.
- Conditional probabilities can be applied when there are multiple factors in a situation or decision.
- The **conditional probability of *B* given *A*** is the probability that event *B* occurs, given that event *A* has already occurred. If *A* and *B* are two events from a sample space with $P(A) \neq 0$, then the conditional probability of *B* given *A*, denoted $P(B|A)$, has two equivalent expressions, as follows.

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\text{number of outcomes of } (A \text{ and } B)}{\text{all outcomes in } A}$$

- When making decisions about the fairness of a situation, expected values are often determined.
- Recall that expected value, $E(X)$, is an estimate of value that is determined by finding the product of a total value and a probability of a given event.
- An expected value can be used to evaluate strategies. For example, calculating expected value can help in assessing the outcomes of an investment, game, or business venture.

SRB p. U1-256

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- A situation or game is fair if all of the possible outcomes have equal chances.
- There are a number of possible results that are important to know when evaluating statements about results of experiments. These include: true positive, true negative, false positive, and false negative.
- A **true positive result** means that the experiment has produced a correct positive result. For example, if your doctor tests your sore throat for strep bacteria, the culture test comes back positive, and you do in fact have strep throat.
- A **true negative result** means that the experiment has produced a correct negative result. For example, a woman has her DNA tested to see if she is a carrier for Huntington's disease, a hereditary disease resulting in damage to brain cells. The test comes back negative; thus, she is not in fact a carrier for the disease.
- A **false positive result** means that the experiment has produced an incorrect positive result. For example, a child tests positive for the chicken pox virus but does not actually have chicken pox.
- A **false negative result** means that the experiment has produced an incorrect negative result. For example, a suspicious mole is tested to see if it could be melanoma (skin cancer). This test comes back negative, however, repeated tests reveal that the mole is indeed cancerous—the first test result was a false negative.
- These terms are most often used in medical studies, but may be found elsewhere as well.
- When analyzing decisions and strategies, be sure to consider personal bias prior to calculating. Double check calculations to ensure that the results do not reflect your own intuition about what the results "should" be.
- When evaluating a problem statement, watch for terms such as *and*, *as well as*, and *or*, because this language will result in different calculations when dealing with probabilities.
- In many cases, there is no "right" or "wrong" answer when analyzing data. Use what is known to make correct calculations, and then apply the calculations to the argument. Different people may not agree on what makes a quantity or percentage desirable; the best strategy is to justify opinions with accurate data.

Common Errors/Misconceptions

- miscalculating expected values or probabilities
- incorrectly applying probabilities
- mistaking compound probability with conditional probability

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Guided Practice 1.6.2

Example 1

In basketball, there is a strategy called intentional fouling, in which the defensive team's coach directs a player to foul someone on the offensive team. The fouled player will have the opportunity to shoot a foul shot for 1 point. If she makes the basket, she can take another shot. If she does not make the first basket, she does not get another chance. The fouled player might score 1 or 2 points, but the defensive team has a chance to regain the ball and attempt a 2- or 3-point field goal.

Coach Baxter tells Kia to foul a certain player on the opposing team who makes 70% of her foul shots. Explain why Coach Baxter decided to use the strategy of intentional fouling against this player.

1. Analyze the given information.

   The fouled player makes 70% of her foul shots.

   Through subtraction, we can determine that this player misses 100% – 70% = 30% of her foul shots.

2. Calculate the likelihood of each event.

   There are three possible outcomes: the fouled player makes both foul shots, the player makes the first shot but misses the second, or the player misses the first foul shot (and, therefore, does not get a chance for a second shot).

   These are independent events, where the probability of one event's occurrence does not affect the probability of another event; therefore, use the Multiplication Rule, where $P(A \text{ and } B) = P(A) \bullet P(B)$.

   To calculate the likelihood of the fouled player making both foul shots, multiply the probability of the player making the first foul shot by the probability of the player making the second foul shot.

   Since the fouled player makes 70% of her foul shots, the probability of making any individual shot is 70% or 0.70; therefore, the probability of making each shot is the same.

   $P(\text{make both shots}) = P(\text{make 1 shot}) \bullet P(\text{make 1 shot})$

   $P(\text{make both shots}) = (0.70) \bullet (0.70)$

   $P(\text{make both shots}) = 0.49$

   The likelihood of the player making both foul shots is 0.49 or 49%.

(*continued*)

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For the next probability, the player needs to make the first shot in order to have the opportunity for a second shot. Thus, we must calculate the likelihood of the player making the first foul shot but missing the second. This is different from calculating the likelihood of making at least 1 shot out of 2 possible shots. Therefore, multiply the probability of the player making 1 shot by the probability of the player missing 1 shot.

Since the fouled player misses 30% of her foul shots, the probability of missing any individual shot is 30% or 0.30.

$P(\text{make 1 shot and miss 1 shot}) = P(\text{make 1 shot}) \bullet P(\text{miss 1 shot})$

$P(\text{make 1 shot and miss 1 shot}) = (0.70) \bullet (0.30)$

$P(\text{make 1 shot and miss 1 shot}) = 0.21$

The likelihood of the player making 1 foul shot and missing 1 shot is 0.21 or 21%.

Now determine the likelihood of the player missing the first shot, since a missed shot means the player does not get to try for a second shot.

Since the probability of missing 1 shot is already known to be 0.30, the likelihood of the player missing the first shot (and, thus, missing both shots) is 0.30 or 30%.

3. Explain why Coach Baxter might have called for an intentional foul on this player.

   Since the chance of the player making both baskets is less than 50%, Coach Baxter might think that this strategy is worth the risk in order to get possession of the ball, so that her team can attempt a 2- or 3-point field goal.



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Example 2

There is a certification exam required to complete a Certified Nursing Assistant program. Each candidate must take the exam and pass in order to be granted a diploma. The pass rate for candidates taking the exam the first time is 59%, and the pass rate for those taking the exam the second time is 89%. The program supervisor must open a summer examination preparation course for any student who fails the exam twice. Determine a way for the supervisor to plan a number of seats for the summer preparation course.

1. Analyze the given information.

   Since the passing rate the first time around is 59%, these students need to take only one exam.

   $P(\text{pass in 1 attempt}) = 0.59$

   By subtracting this pass rate from 1, it can be determined that $1 - 0.59 = 0.41$ or 41% of students fail in the first attempt; that is, they need to take the exam a second time.

   The given information also tells us that 89% of these students pass on their next attempt.

2. Calculate the percentage of students who pass the exam on the second try.

   First, calculate the probability of a student passing the exam on the second try.

   To calculate the probability of a student passing the exam on the second try, multiply the probability of the student failing the exam on the first try by the probability of the student passing on the second try.

   $P(\text{pass in 2 attempts}) = P(\text{fail in 1 attempt}) \bullet P(\text{pass in next attempt})$

   $P(\text{pass in 2 attempts}) = (0.41)(0.89)$

   $P(\text{pass in 2 attempts}) = 0.3649$

   The probability of a student passing the exam on the second try is 0.3649 or 36.49%.

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3. Determine the number of students who pass the test on either their first or second attempt.

   Add the probability of passing the first time to the probability of passing the second time to find the rate of students who pass the test on the first or second try.

   P(pass in 1 or 2 attempts) = P(pass in 1 attempt) + P(pass in 2 attempts)

   P(pass in 1 or 2 attempts) = (0.59) + (0.3649)

   P(pass in 1 or 2 attempts) = 0.9549

   95.49% of students in a given year pass the exam on the first or second try.

4. Determine a way for the supervisor to plan a number of seats for the summer preparation course.

   If 95.49% of students pass the exam on the first or second attempt, then we can subtract this amount from 100% to determine that 100% – 95.49% = 4.51% of students do not pass on either the first or second attempt.

   Therefore, we can advise the program supervisor to plan for enough seats in the summer examination preparation course to accommodate 4.51% of each class.

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Example 3

An outbreak of a virus has infected 10% of the population. There is a test, though not perfect, that can help determine if an individual needs treatment for the virus. There is a 15% chance that the test will be negative even if you have the virus; that is, that the test will show a false negative result. Additionally, there is a 30% chance of the test showing a false positive result—that an uninfected person tests positive for the virus. If a person is chosen and tested at random, what are the chances of each possible result?

1. Identify the possible results.

   In this situation, there are four possible results.

   The first is a true positive result, which means that an infected person has a correct positive result.

   The second is a true negative result, which means that an uninfected person has a correct negative result.

   The third is a false positive result, which means that an uninfected person has an incorrect positive result.

   The last is a false negative result, which means that an infected person has an incorrect negative result.

2. Interpret the given information.

   The given information states that 10% of the population is infected.

   It can be calculated, by subtracting this percent from 100%, that the remaining 90% of the population is uninfected.

   The given information also states that 15% of the tests are false negatives, leaving 100% – 15% = 85% of tests that show true positive results. That is, if 100% of those who are infected are tested, 15% will have false negative test results (indicating, falsely, that they are not infected), and the remaining 85% will have true positive results showing they are, in fact, infected.

(*continued*)

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The final piece of given information states that 30% of positives are false positives. By subtraction, we can determine that 100% – 30% = 70% of test results are true negatives. In other words, of the healthy people tested, 30% will get test results incorrectly indicating that they have the virus, and 70% will get test results correctly showing that they do not have the virus.

It can be helpful to summarize this information in a table.

	Positive test result (%)	Negative test result (%)	Total (%)
Infected	85	15	100
Uninfected	30	70	100

3. Calculate the likelihood of each of the possible results for the given population.

   To calculate the likelihood of a true positive result for the given population, multiply the probability of encountering those infected by the probability of getting a true positive result.

   $P(\text{true positive for the population}) = P(\text{infected}) \bullet P(\text{true positive})$

   $P(\text{true positive for the population}) = (0.10) \bullet (0.85)$

   $P(\text{true positive for the population}) = 0.085$

   The probability of getting a true positive for the given population is 0.085.

   To calculate the likelihood of a true negative result for the given population, multiply the probability of encountering those uninfected by the probability of getting a true negative result.

   $P(\text{true negative for the population}) = P(\text{uninfected}) \bullet P(\text{true negative})$

   $P(\text{true negative for the population}) = (0.90) \bullet (0.70)$

   $P(\text{true negative for the population}) = 0.63$

   The probability of getting a true negative for the given population is 0.63.

(*continued*)

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To calculate the likelihood of a false positive result for the given population, multiply the probability of encountering those uninfected by the probability of getting a false positive result.

$P(\text{false positive for the population}) = P(\text{uninfected}) \bullet P(\text{false positive})$

$P(\text{false positive for the population}) = (0.90) \bullet (0.30)$

$P(\text{false positive for the population}) = 0.27$

The probability of getting a false positive for the given population is 0.27.

To calculate the likelihood of a false negative result for the given population, multiply the probability of encountering those infected by the probability of getting a false negative result.

$P(\text{false negative for the population}) = P(\text{infected}) \bullet P(\text{false negative})$

$P(\text{false negative for the population}) = (0.10) \bullet (0.15)$

$P(\text{false negative for the population}) = 0.015$

The probability of getting a false negative result for the given population is 0.015.

Check your results by totaling all of the probabilities and ensuring they sum to 1.

$0.085 + 0.63 + 0.27 + 0.015 = 1$

4. Summarize your findings.

The probability of a true positive result is 0.085, meaning that the likelihood of randomly selecting an infected person who tests positive for the virus is 8.5%.

The probability of a true negative result is 0.63, meaning that the likelihood of randomly selecting an uninfected person who tests negative for the virus is 63%.

The probability of a false positive result is 0.27, meaning that the likelihood of randomly selecting a person who is not infected but tests positive for the virus is 27%.

The probability of a false negative result is 0.015, meaning that the likelihood of randomly selecting a person who is actually infected but tests negative for the virus is 1.5%.



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Example 4

A musician has decided to create an album on her own label at a cost of $40,000. The chances of producing a successful, or profitable, album are 10%. She predicts that the profit—the money earned after accounting for the start-up cost—from a successful album would be $159,000. She is asking for your advice: do you think she should go forward with this venture?

1. Identify the given information.

   It will cost $40,000 to create the album.

   The album has a 10% chance of being successful.

   Subtracting 10% from 100%, it can also be said that the album has a 90% chance of being unsuccessful.

   If the album is successful, the musician would see a profit of $159,000.

2. Determine how to best apply the given information.

   There is a 10% chance of making a $159,000 profit.

   On the other hand, there is a 90% chance of being unsuccessful and therefore losing the $40,000 invested in creating the album.

   There is a value associated with each of the two probabilities.

   Calculating the expected value of each outcome is necessary to determine how to advise the musician.

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Instruction

3. Calculate the expected value of each outcome.

The expected value can be found using the formula $E(X) = p_1P(X_1) + p_2P(X_2)$, where p_1 is the amount of money that would be lost, $P(X_1)$ is the probability of losing money (being unsuccessful), p_2 is the profit, and $P(X_2)$ is the probability of being successful.

Since p_1 is the amount of money that would be lost, it is a negative value: –40,000.

Recall that p_2 is the amount of profit, which means the cost of spending \$40,000 has already been accounted for; therefore, $p_2 = 159{,}000$.

$E(X) = p_1P(X_1) + p_2P(X_2)$	Formula to calculate expected value
$E(X) = (-40{,}000)(0.90) + (159{,}000)(0.10)$	Substitute values for each variable.
$E(X) = -36{,}000 + 15{,}900$	Simplify.
$E(X) = -20{,}100$	

The expected value, or the expected amount of profit, is –\$20,100.

4. Determine if the musician should move forward with this venture.

Since the expected value is –\$20,100 (a loss), and there is a 90% chance of failure, the musician should not produce her own album.



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Instruction

Example 5

Riley had the unfortunate luck of being struck by lightning—twice. Her friend told her, "With luck like that, you should play the lottery." The chance of being struck by lightning once in a lifetime is approximately 1 in 6,250. How do Riley's chances of a dual lightning strike compare to winning a six-number lottery jackpot, if the numbers 1 through 59 are the possible numbers chosen (without replacement and order does not matter)? Given this information, would you agree with the friend's assessment of Riley's rare luck?

1. Identify the given information.

   There are two scenarios in this example.

   The first scenario discusses the "luck" of being struck by lightning twice.

   The second scenario discusses the likelihood of winning the lottery.

   The question is asking you to compare the probabilities of each scenario.

   Therefore, it is necessary to calculate each probability in order to compare them.

2. Calculate Riley's chance of being struck by lightning twice.

   Since these are independent events, the chance of being struck by lightning twice is found by multiplying the probability of being struck once (1 in 6,250) by the probability of being struck once.

   $P(\text{struck by lightning twice}) = P(\text{struck by lightning once}) \bullet P(\text{struck by lightning once})$

   $$P(\text{struck by lightning twice}) = \left(\frac{1}{6250}\right) \bullet \left(\frac{1}{6250}\right)$$

   $$P(\text{struck by lightning twice}) = \frac{1}{39{,}062{,}500}$$

   The chance of being struck by lightning twice is 1 in 39,062,500.

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Instruction

3. Calculate the chance of winning a six-number lottery with 59 number choices.

The lottery game described in the problem scenario requires that players choose 6 numbers from 1 to 59 without replacement, and states that order does not matter.

The general formula for calculating a combination is ${}_nC_r = \frac{n!}{(n-r)!r!}$, where n is the total possible number of items we are choosing from and r is equal to the number of items actually chosen. In this example, $n = 59$ and $r = 6$.

${}_nC_r = \frac{n!}{(n-r)!r!}$	Formula for calculating a combination
${}_{(59)}C_{(6)} = \frac{(59)!}{[(59)-(6)]!(6)!}$	Substitute 59 for n and 6 for r.
${}_{59}C_6 = \frac{59!}{53!6!}$	Simplify.
${}_{59}C_6 = 45{,}057{,}474$	

The chance of winning the lottery described is 1 in 45,057,474.

4. Compare the probabilities to determine whether Riley's friend is correct about her luck.

Riley's friend compared her chance of being struck by lightning twice, or 1 in 39,062,500, to her chance of winning the lottery, which is 1 in 45,057,474.

Both incidents are very unlikely.

However, Riley is more likely to be struck by lightning twice than to win the lottery.

Therefore, Riley's friend is mistaken in making a comparison of the odds of each of these situations.

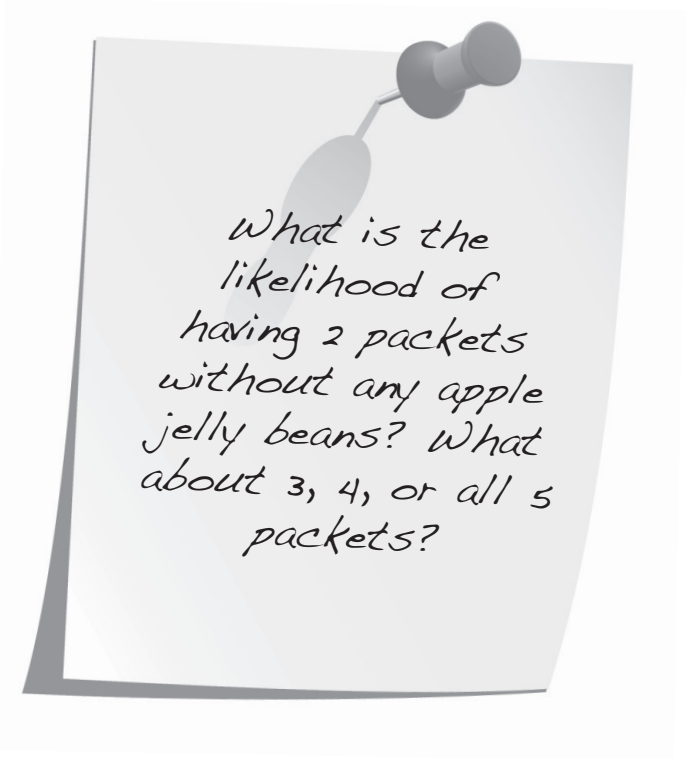
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Problem-Based Task 1.6.2: Niko's Jelly Beans

SMP	
1 ✓	2
3	4
5	6
7 ✓	8

You're holding a birthday party for your aunt, and bought a bag of jelly bean packets for the 5 children who will attend. There are 15 jelly beans per individual packet, and the bag contains 5 packets. The jelly beans come in 8 flavors, including your brother Niko's favorite, apple. The manufacturer claims that each flavor is produced equally and is equally likely to be in each packet.

At the party, you randomly give out all 5 packets of jelly beans. Niko tears his packet open, and finds that there are no apple jelly beans. Soon the other children start to find that their packets don't have apple jelly beans either. What is the likelihood of having 2 packets without any apple jelly beans? What about 3, 4, or all 5 packets? Would you question the manufacturer's claim after this incident? What would you think if there were no apple jelly beans in any of the packets from the bag of jelly beans you bought?



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Problem-Based Task 1.6.2: Niko's Jelly Beans

Coaching

a. What proportion of the population of jelly beans should be apple flavored?

b. On average, how many apple jelly beans should Niko expect to find in a packet of 15 jelly beans?

c. What is the proportion of the population that is not apple?

d. What is the probability that a packet of 15 jelly beans would have no apple jelly beans?

e. What is the probability that 2 packets would have no apple jelly beans?

f. What is the probability that 3 packets would have no apple jelly beans?

g. What is the probability that 4 packets would have no apple jelly beans?

h. What is the probability that none of the 5 packets would have a single apple jelly bean in them?

i. Would you question the manufacturer's claim based solely on the contents of Niko's packet?

j. What would you think if there were no apple jelly beans in any of the packets in this bag?

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Problem-Based Task 1.6.2: Niko's Jelly Beans

Coaching Sample Responses

a. What proportion of the population of jelly beans should be apple flavored?

If each of the 8 flavors occurs equally in the population, then 1 in every 8 jelly beans should be apple. The proportion of the total population would be $\frac{1}{8}$, which is equal to 0.125 or 12.5%.

b. On average, how many apple jelly beans should Niko expect to find in a packet of 15 jelly beans?

It was calculated that 12.5% of the population of jelly beans should be apple.

So, in a sample of 15 jelly beans, we can use the following equation:

average number of apple jelly beans = (percentage of apple jelly beans)(number in sample population)

average number of apple jelly beans = (12.5%)(15)

average number of apple jelly beans = 1.875

Rounded to the nearest whole number, this means that there will be about 2 apple jelly beans in each package of 15.

c. What is the proportion of the population that is not apple?

The percentage of the population of jelly beans that are apple is 12.5%; subtract this figure from 100% to calculate the percentage of jelly beans that are not apple.

100% – 12.5% = 87.5%

87.5% of the jelly beans are not apple.

d. What is the probability that a packet of 15 jelly beans would have no apple jelly beans?

If none of the 15 randomly selected jelly beans are apple, then all 15 jelly beans are non-apple.

The percentage of the population that consists of non-apple jelly beans is 87.5%.

To calculate the probability that all 15 jelly beans are not apple, find the likelihood of choosing non-apple jelly beans 15 times; in other words, multiply the probability of selecting non-apple jelly beans by itself 15 times.

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$P(15 \text{ non-apple}) = (\text{non-apple})^{15}$

$P(15 \text{ non-apple}) = (0.875)^{15}$

$P(15 \text{ non-apple}) = 0.1349$

The probability that a single packet of 15 jelly beans contains no apple jelly beans is 0.1349 or 13.49%.

e. What is the probability that 2 packets would have no apple jelly beans?

Use the calculation from part d to determine the probability of 2 packets having no apple jelly beans.

$P(2 \text{ non-apple packets}) = (\text{non-apple packet})^{2}$

$P(2 \text{ non-apple packets}) = (0.1349)^{2}$

$P(2 \text{ non-apple packets}) = 0.018$

The probability of having 2 packets that do not contain any apple jelly beans is 0.018 or 1.8%.

f. What is the probability that 3 packets would have no apple jelly beans?

Use the calculation from part d to determine the probability of 3 packets having no apple jelly beans.

$P(3 \text{ non-apple packets}) = (\text{non-apple packet})^{3}$

$P(3 \text{ non-apple packets}) = (0.1349)^{3}$

$P(3 \text{ non-apple packets}) = 0.0025$

The likelihood of having 3 packets that do not contain any apple jelly beans is 0.0025 or 0.25%.

g. What is the probability that 4 packets would have no apple jelly beans?

Use the calculation from part d to determine the probability of 4 packets having no apple jelly beans.

$P(4 \text{ non-apple packets}) = (\text{non-apple packet})^{4}$

$P(4 \text{ non-apple packets}) = (0.1349)^{4}$

$P(4 \text{ non-apple packets}) = 0.0003$

The likelihood of having 4 packets that do not contain any apple jelly beans is 0.0003 or 0.03%.

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h. What is the probability that none of the 5 packets would have a single apple jelly bean in them?

Use the calculation from part d to determine the probability of 5 packets having no apple flavored jelly beans.

$P(\text{5 non-apple packets}) = (\text{non-apple packet})^5$

$P(\text{5 non-apple packets}) = (0.1349)^5$

$P(\text{5 non-apple packets}) = 0.000045$

The likelihood of having 5 packets that do not contain any apple jelly beans is 0.000045 or 0.0045%.

i. Would you question the manufacturer's claim based solely on the contents of Niko's packet?

The manufacturer claims that each flavor is produced equally and is equally likely to be in each packet.

Since 13.49% of the packages chosen at random would contain no apple jelly beans, there is a possibility that Niko ended up with one of these packages. This occurs a little more than 13 times in every 100 packages. We would not question the equal distribution of apple jelly beans based on Niko's packet.

j. What would you think if there were no apple jelly beans in any of the packets in this bag?

If there were no child at the party who had apple jelly beans, we would have a more difficult situation to fathom. If this event should occur only 0.0045% of the time, as we calculated, then the likelihood of this happening would be about 45 times in 1,000,000. These odds seem too low to happen at random. We would, at this time, question whether something happened to the apple jelly beans when this bag was made.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.6.2: Analyzing Decisions

Use the following information to complete problems 1–3.

> You are a partner in a law firm. There are 2 junior lawyers on your team: Cathy, who wins 75% of all of her cases, and Carlos, who wins 55% of his morning cases but 95% of his afternoon cases. Each lawyer handles one case in the morning, and a different case in the afternoon. One day, both Cathy and Carlos lost both their morning and afternoon cases.

1. What is the likelihood of Cathy losing both cases?

2. What is the likelihood of Carlos losing both cases?

3. Would you call either of the lawyers in to reprimand them on their losses? Why or why not?

Use the following information to complete problems 4–6.

> In a 3-round table tennis tournament, all teams have an equal chance of winning the first round, where the probability of winning is $\frac{1}{2}$. In subsequent rounds, however, the probability of winning has changed for each team based on its previous win or loss. Teams who won previously have a positive momentum and outlook, and therefore have a winning probability of $\frac{2}{3}$. Losing teams, then, have a winning probability of $\frac{1}{3}$.

4. What is the likelihood of a team losing the first round and then winning the next 2 rounds?

5. What is the likelihood of a team losing the first 2 rounds and then winning the third round?

6. Would you bet on a team that loses in the first round? Explain your answer.

continued

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Use the following information to complete problems 7–10.

> When a hockey team pulls its goalie and puts all players on the offense, a change in scoring rates occurs. The team that pulls its goalie scores at an estimated rate of 0.16 goals per minute, while the opposing team scores at a rate of 0.47 goals per minute.

7. What is the expected number of goals scored for each team if there are 2 minutes left in the game?

8. What is the expected number of goals scored for each team if there are 4 minutes left in the game?

9. What is the expected number of goals scored for each team if there are 5 minutes left in the game?

10. Would you pull your goalie if you were coaching? Explain why or why not.

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Progress Assessment

Circle the letter of the best answer.

1. A game that consists of rolling one standard, six-sided die numbered 1–6 pays $40 for getting an even number and $500 for getting a 5. What is the most you should pay to play this game?

   a. $40
   b. $83
   c. $103
   d. $183

2. There are 6 qualified candidates who wish to be chosen for 4 student office assistant positions. Which would not be a fair way to choose which students are selected?

   a. rolling a fair 6-sided die
   b. spinning a spinner with 6 equal-sized portions
   c. flipping a coin 6 times
   d. All of the options are fair.

3. If a person has been vaccinated against tetanus, his or her likelihood of being exposed to tetanus and contracting it is 5%. Without the vaccination, a person has a 35% chance of being exposed to tetanus and contracting it. About 1 in 3 people have been vaccinated against tetanus. If a person contracts tetanus, what is the probability that he or she was vaccinated?

   a. 5%
   b. 6.7%
   c. 23.3%
   d. 24.97%

4. A middle school has had an outbreak of a bacterial infection. The nurse has tested all students for the infection. Johnny's mother was informed that he tested positive for the infection, so she took him to his pediatrician. After a number of subsequent tests, it was discovered that Johnny did not have the bacterial infection. What does this indicate about the test the nurse gave Johnny?

   a. The test produced a false positive result.
   b. The test produced a false negative result.
   c. The test produced a true positive result.
   d. The test produced a true negative result.

continued

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5. Why are random number generators used?

   a. to select a specific number in a set

   b. to select a number in a set, where the probability of generating a specific number in the set is higher for certain numbers

   c. to select a number in a set without following a pattern, so that each number in the set has an equal probability of being generated

   d. to select a number that follows a pattern

6. A friend pitches an investment opportunity to you. He says that the up-front investment cost is $400, but you have a 10% chance of making $10,000. What is true about this opportunity?

   a. It is a bad idea because the chance of making money is so low.

   b. The expected value of the venture is $640.

   c. The expected value of this venture is $1,400.

   d. If we invest enough times, we will make $10,000.

7. A widely used test for a type of virus is 99% accurate, and 0.1% of the population is actually infected with the virus. What is the probability that someone tests positive but is not infected?

   a. 0.99%

   b. 3%

   c. 59%

   d. 91%

8. Which of the following scenarios would make you question the fairness of the result?

   a. tossing a coin 70 times and getting the result of "heads" every time

   b. rolling a die 100 times and getting a 3 on 80 of the rolls

   c. drawing an ace from a deck of cards on 35 out of 50 draws

   d. all of the above

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9. Which of the following strategies could be used to determine fairness?

   a. Calculate the expected value of a game, then compare it to the cost to play the game.

   b. Calculate the probability of each possible outcome in a situation, then compare the probabilities.

   c. Calculate the conditional probability to analyze the likelihood of a result.

   d. all of the above

10. A small bag of candy contains 5 different colors: red, blue, orange, yellow, and purple. The colors are equally distributed. If the bag contains 16 candies, what is the likelihood that none of the candy in the bag is purple?

    a. 2.8%

    b. 31%

    c. 35%

    d. 80%

Apply your knowledge of probability to determine a strategy for the following game.

11. You have been selected to come to the stage on a game show to play "Green or Goat?" You are met with 3 doors. The winning door opens to a large stack of cash—the losing doors open to goats. You pick a door, hoping to win the cash. The game show host opens one of the doors that you *did not* pick, revealing a goat! You now have the choice to stay with the door you originally chose or change to the remaining closed door. What is your strategy? Justify your reasoning with probability.

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Unit 1 Assessment

Circle the letter of the best answer.

1. Which of the following describes a data set that is not normally distributed?

   a. The histogram representing the data is skewed.

   b. The mean and median are not equal.

   c. The data set includes extreme outliers.

   d. all of the above

2. A candy store owner is running a contest for her customers. Each purchase allows a customer to guess the percentage of blue candies in a jar of 1,500 candies. The customer whose guess is closest to the actual percentage wins a free hula hoop. The store owner allows a few children to take a handful of candies to help them guess. The table below details each child's age and the makeup of his or her handful.

Child	Child's age	Total candies	Blue candies
Blake	11	15	5
Hailey	6	7	2
Jackson	3	5	1
Andre	7	9	3
Malika	10	12	3
Nevy	5	7	5

   Which of the following conclusions would *not* be supported by this data?

   a. Since Blake and Andre had the same percentage of blue candies in their handfuls, they are most likely to have the most accurate result.

   b. As the children's ages increased, the number of candies in their handfuls increased.

   c. Nevy's handful appears to be an outlier.

   d. When we combine the children's samples, the result is that 35% of the candies are blue in the sample of 55 candies.

continued

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3. Which of the following indicates a population and a random sample from that population?

   a. population: music artists; sample: country music stars

   b. population: bicycles; sample: any 5 bicycles from each available manufacturer

   c. population: makeup; sample: 2 lipsticks from each available brand name

   d. population: high school students in the United States; sample: students who attend charter schools

4. A researcher designed a study to determine whether intellectual exertion has an impact on the effectiveness of physical workouts. He plans to monitor the number of calories burned for each participant's workouts over a period of time. Participants will be directed to watch calm, serene movies prior to some workouts. Other times, participants will play an intense computer game reliably known to cause mental fatigue before working out. Which of the following has the researcher designed?

   a. a survey

   b. an observational study

   c. an experiment

   d. none of the above

5. The average amount of garbage waste generated in the homes of a particular town is approximately normally distributed about the mean, μ. The average number of gallons of waste in a sample of 25 homes is $\overline{x}$ = 39.25 gallons, with a standard deviation, s, of 14.15. What is the standard error of $\overline{x}$?

   a. 0.566

   b. 1.57

   c. 2.83

   d. 7.85

6. You own a store that sells new and used computers. Your best customer ordered 2 new laptops. After the laptops were shipped, you discovered that a new stockroom employee had been putting used laptops with the new ones instead of keeping the two types separate. You realized this customer might have been sent used laptops instead of the new ones she ordered. Out of the total inventory of 20 laptops, 4 were used. If your customer was sent 2 brand-new laptops, there will be no problem. If 1 of the laptops turns out to be used, she will return it, which will cost you $75 in shipping. On the other hand, if both laptops were used, the customer will cancel her order, costing you $2,500. What is the expected value of this mistake over time?

   a. $32.60

   b. $104.01

   c. $124.60

   d. $824.60

continued

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7. A research firm calculated a 90% confidence interval for the average savings accounts of all adults in a certain town. Which of the following statements accurately interprets this data?

   a. 90% of all adults in this town have between $1,090 and $11,654 in their savings accounts.

   b. The firm is 90% confident that a randomly selected adult in this town will have between $1,090 and $11,654 in savings.

   c. 90% of all samples of adults in this town will have an average savings account balance of between $1,090 and $11,654.

   d. none of the above

8. A study found that students who attend schools that implement drivers' education programs are more likely to be victims of car accidents. What can we conclude from this study?

   a. Drivers' education programs cause student drivers to have car accidents.

   b. Drivers' education programs cause students to become victims of car accidents.

   c. Drivers' education programs are ineffective.

   d. none of the above

9. The makers of a household cleaner claim their product kills 99% of viruses and bacteria on surfaces. What is the likelihood that a surface that has been sprayed with the product will test positive for viruses and bacteria even if none are present?

   a. 0.01

   b. 0.005

   c. 0.0495

   d. 0.055

10. If 3,304 out of 5,600 students who answer a post-graduation survey indicate that they intend to enter a four-year college, what is the standard error for this sample population to the nearest hundredth?

    a. 0.00

    b. 0.24

    c. 0.01

    d. 0.59

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11. You bought a high-end "battery life extender" for your cell phone battery. You are recording the amount of time you use your phone each day until it shuts down from a drained battery. Assume that the battery is fully charged at the beginning of each day. Use a 95% confidence interval to estimate the average battery life given the information in the following table.

Battery life in hours per day							
Day 1	**Day 2**	**Day 3**	**Day 4**	**Day 5**	**Day 6**	**Day 7**	**Day 8**
19.5	16.8	9.2	11.1	22.7	15.5	14.0	15.7

a. (11.52, 19.61)

b. (11.24, 19.89)

c. (12.67, 18.37)

d. (12.24, 18.89)

12. A survey questioned 3,000 community members about whether a certain parcel of land should be used for a large shopping center or for a town park. If 1,599 of those surveyed responded in support of a park, what is the sample proportion, $\hat{p}$, for this population, rounded to the nearest hundredth?

a. 0.47

b. 0.53

c. 1.89

d. 2.14

Use what you have learned about making inferences and drawing conclusions from data to complete all parts of the following problems.

13. Jenna's doctor's office is providing patients a new online service. With this service, patients can view their blood test results on a website as soon as the results are available. Jenna recently logged on to the website to review the results of her white blood cell count. Healthy adults have normally distributed white blood cell counts, with a mean count of 6.29×10^3 cells per microliter (μL) and a standard deviation of 0.93×10^3 μL.

a. Approximately what percentage of the white blood cell counts are found between 4.43×10^3 cells per μL and 8.15×10^3 cells per μL?

b. Jenna's white blood cell count is listed as 14.1×10^3 cells per μL with an alert icon next to it. What is a mathematical explanation for the alert?

c. Jenna is pregnant and in her third trimester. Her doctor forgot to adjust the computer program to indicate Jenna's current status. During pregnancy, a woman's white blood cell count increases. In the third trimester, the mean white blood cell count is 11.24×10^3 cells per μL, with a standard deviation of 1.88×10^3 μL. Should Jenna be worried? How would you explain her result of 14.1×10^3 cells per μL given this new context?

continued

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14. You are taking a 20-question final examination. All of the questions are multiple-choice questions with 4 possible answer options. Find the probabilities for each situation described, rounded to the nearest thousandth.

    a. If you did not have a chance to study and guess for all of the questions, what are the chances that you will answer exactly half of the questions correctly?

    b. If you are able to read through all the questions and eliminate 1 incorrect answer option from each, what are the chances that you will answer exactly half of the questions correctly?

    c. If you are able to read through all the questions and eliminate 2 incorrect answer options from each, what are the chances that you will answer exactly half of the questions correctly?

15. A foundation that awards scholarships based on a national college entrance examination just announced the top 1% of scorers for this year's test. The foundation intentionally distributes the scholarships geographically. A perfect score would total 250 points, and the scores result in a normal distribution. Use the data from the table below to complete the parts that follow.

State	Average score	Standard deviation	Scholarship qualifying score
Massachusetts	146.9	34.4	224
New Jersey	143.7	34.8	203

    a. Jesse's test score was 214. Explain where his score would fall on a normal curve depending on which of the two states he lives in.

    b. Is it more difficult to qualify for a scholarship depending on the state in which you live? Explain in terms of the respective means and standard deviations for each provided state.

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Answer Key

Lesson 1: Using the Normal Curve

Pre-Assessment, p. U1-1

1. b
2. d
3. c
4. c
5. c

Warm-Up 1.1.1, p. U1-6

1. 108 ft^2
2. 50%
3. Both columns are 6 feet tall.
4. 55.6%

Practice 1.1.1: Normal Distributions and the 68–95–99.7 Rule, p. U1-31

1. 68%
2. 97.5%
3. 1.25 minutes or 1 minute and 15 seconds
4. about 2.5%
5. 11.1%
6. 27.8%
7. 95%
8. 81.5%
9. The distribution is not normal.
10. Answers may vary. Sample answer: It can't be normal if there are no houses with values less than 1.6 standard deviations below the mean.

Warm-Up 1.1.2, p. U1-33

1. 14
2. There are more combinations of 10 than 5.
3. 0.027
4. 0.5671
5. Answers may vary. Sample answer: The overall shapes of the distributions are similar (symmetry, clustering, etc.). However, the sums of dice are discrete, while the normal distribution is continuous.

Practice 1.1.2: Standard Normal Calculations, p. U1-58

1. $z = 1.08$
2. 86%
3. 83%
4. 90%
5. 0.25 mm
6. 7.4 minutes
7. 91%
8. –1.62
9. 94.7%
10. 10%

Warm-Up 1.1.3, p. U1-60

1. 5%
2. 20%. There are several possible reasons for why there are so many 3-letter words. For example, the sample includes words such as *the* and *was* that are commonly used in the English language. There are also several 3-letter pronouns and prepositions.
3. $\mu \approx 4.7$ letters per word, median = 4.0 letters per word
4. $\sigma \approx 2.7$ letters per word
5. No, the word lengths are not symmetric because 50% of the data does not lie above the mean, while 50% of the data lies below the mean.

Practice 1.1.3: Assessing Normality, p. U1-88

1. Histogram A
2. Histogram D
3. Histogram B
4. The value $1,500 is an outlier. "Company president" is a different type of job than the others listed; the other positions are more likely to work outdoors on landscaping projects, while the company president would most likely work in an office. If the outlier is removed, the analysis will be based on "outdoor workers" rather than all employees. It is acceptable to remove the outlier as long as the interpretation takes the removal into consideration. The mean and standard deviation are inappropriate if the outlier is kept.
5. 25%
6. No. The distribution is highly skewed to the right.
7. 70%
8. 93.3%
9. 100%
10. Yes. The histogram and normal probability plots show approximately normal distribution.

Progress Assessment, p. U1-91

1. b
2. a
3. c
4. b
5. c
6. a
7. d
8. a
9. d
10. c
11. a. smallest z-score: –1.34; largest z-score: 2.71
    b. about 70%
    c. No. The distribution is skewed to the right. The mean is greater than the median.

Lesson 2: Populations Versus Random Samples and Random Sampling

Pre-Assessment, p. U1-94

1. c
2. d
3. b
4. d
5. b

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Warm-Up 1.2.1, p. U1-99

1. 15
2. 70
3. 97.5
4. 78.75
5. 92.5

Practice 1.2.1: Differences Between Populations and Samples, p. U1-122

1. b
2. d
3. a
4. population: all Suncharged-brand batteries
   parameters: the mean time that Suncharged batteries last
   sample: the 40 batteries that Tyrus tested
   statistic of interest: the mean time that the batteries in the sample last
5. population: all of the students at the high school
   parameter: the percent of students in the population in favor of the dress code
   sample: students in home rooms who took the survey
   statistic of interest: the percent of students in the sample who are in favor of the dress code
6. 1,170 adults
7. 60 moose
8. 254 males
9. 181 females
10. 173.3 mg/dL

Warm-Up 1.2.2, p. U1-125

1. 5%
2. 45%
3. 1.7
4. 2.85

Practice 1.2.2: Simple Random Sampling, p. U1-153

1. All samples are unbiased.
2. sampling error or chance variation
3. 6.7
4. The combined samples provide a larger sample size than any individual sample.
5. Holly's sample; this is a convenience sample, since she only surveys one class that she is in.
6. Zach's sample; he is conducting a simple unbiased random sample.
7. Seth's plan can be improved by selecting more classes, because a larger sample size increases reliability.
8. sample with the smallest mean: Toyota Prius, Honda Civic, Toyota Camry, and Nissan Altima; mean = $1.80
9. sample with the greatest standard deviation: Honda Civic, Toyota Prius, Saturn Aura, and Lexus GS 450h; standard deviation = $0.87
10. Use 7 cards or a random number generator to generate 4 random integers from 1 to 7.

Warm-Up 1.2.3, p. U1-156

1. white pines: 34; red pines: 44
2. white pines: 340; red pines: 440
3. Number every tree and use a random number generator to select the desired number of trees.
4. It would be too time-consuming, and since the point is to count the trees, there is no point in selecting a sample once all the trees are numbered.
5. Since each of the 100 sections has an equal chance of selection, the sample is representative.

Practice 1.2.3: Other Methods of Random Sampling, p. U1-181

1. convenience
2. stratified
3. cluster
4. systematic
5. Number the teams from 1 to 30. Use 30 cards or a random number generator to select 10 teams.
6. Randomly select a starting number from 1 to 3, then select every third team.
7. Number the divisions from 1 to 6, then use cards or a random number generator to select 3 divisions.
8. Number the houses from 1 to 60 and use a random number generator to select 20 houses.
9. Number the houses from 1 to 60, then use a random number generator to select a starting number from 1 to 3. Select every third house beginning with the starting number.
10. Number the real estate companies from 1 to 6, then use cards or a random number generator to select 2 companies. Randomly select a third company if there are fewer than 20 homes in those sold by the first 2 companies selected.

Progress Assessment, p. U1-184

1. b
2. d
3. a
4. a
5. b
6. a
7. b
8. d
9. b
10. a
11. a. Create a simple random sample by numbering each vehicle and randomly selecting a desired number.
    b. Use a stratified sample that includes two Fords. Randomly select 2 Fords from the 5 in the list.
    c. Not necessarily; while this result is somewhat unusual, it is not highly improbable. Therefore, the process could still be fair.

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Lesson 3: Surveys, Experiments, and Observational Studies

Pre-Assessment, p. U1-188

1. b
2. c
3. a
4. d
5. b

Warm-Up 1.3.1, p. U1-193

1. Possible responses include: watching less television; playing fewer computer games; eating breakfast; taking vitamins or supplements; exercising; consuming fewer calories; eating more fruits and/or vegetables; and eliminating or reducing consumption of foods with empty calories, such as chips, candy, and soda, from the diet.
2. You could gather information by: searching online for reputable information; asking an expert or a family member; reading a trusted source, such as a newspaper, book, or scientific journal; or asking more people.
3. There are several possible methods, including the following: conducting a survey or poll online, in person, or by phone; interviewing experts; researching the question; or creating a feedback form.

Practice 1.3.1: Identifying Surveys, Experiments, and Observational Studies, p. U1-209

1. sample survey. Checking in with the customers to record their progress describes a survey. Since the company is only asking 5,000 out of the 25,000 customers, they have taken a sample.
2. experiment. Since the product is provided to different groups at different strengths, this constitutes an experiment.
3. observational study. This study does not indicate treatment, nor does it ask questions of the participants. It observes and records data without intervention.
4. sample survey or experiment. Observing the library's membership through an observational study would not likely provide the librarians with data about how to increase membership. A survey would give insightful information about what people like about the library and how the library might be improved to attract additional members. A library's membership would likely be too large to expect to survey the entire population, so a sample would be necessary. The librarians also might try an experiment with a membership drive or incentive program.
5. sample survey. Asking potential parents will provide the most beneficial information. In this case, the population would be too large and would need to be sampled in order to be reasonable.
6. Answers may vary. A teacher might conduct an experiment to see if methods of teaching increase student grades. The teacher might also conduct an observational study of his or her successful students or of successful teachers in order to replicate the success. Asking students what they think and how they learn best in a survey or sample survey would also provide insight into increasing student grades.
7. observational study. The shop owner might review footage from security cameras set up in the parking lot to note patterns in how customers park and what impact a drive-through lane would have on how and when customers usually park.
8. sample survey. The coffee shop's population would likely be too large to survey, and it's unlikely the owner would be able to conduct an observational study. An experiment would provide treatment or alter normal behavior, so it seems that a sample survey would allow customers to respond and share how long they stay awake and alert from drinking a small coffee.
9. experiment. Students could plant groups of sunflowers and expose each group to only one type of light: natural, ultraviolet, or fluorescent. Students could then record the amount of flower growth over time and compare the results for each group.
10. The farmer might conduct an experiment. She might randomly divide her land into parcels and then randomly assign one of the two brands of seed to each parcel. She will need to track the production of the crop in each parcel and compare the results at the end of the growing season.

Warm-Up 1.3.2, p. U1-211

1. a. sample survey; b. experiment; c. experiment
2. Answers may vary. Possible questions include:
   a. How many dentists were surveyed? How were the populations and/or samples of dentists selected? Were the dentists provided free samples or compensation for recommending Brand A?
   b. Are both brands listed as the same ply? In other words, is one brand single-ply (made of one layer of paper) and the other two-ply (made of two layers)? How many times was the experiment conducted? Was each paper towel tested in an identical manner; i.e., exposed to the same amount of the same liquid, using the same number of sheets of each brand of paper towel to absorb the liquid?
   c. How many people participated in this experiment? Are these results typical? Did the person quoted take additional measures to lose inches, such as eating less, exercising, getting the recommended amount of sleep regularly, or taking medications or supplements? Was this person an actor who was paid to endorse this product?

Practice 1.3.2: Designing Surveys, Experiments, and Observational Studies, p. U1-229

1. The owner of the tourist attraction might conduct an observational study, recording daily temperatures for a period of time and averaging them to share on the attraction's website.

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2. The teacher could conduct a survey of her classes and average the results. The teacher might also include anonymous quotes from open-ended questions.
3. The nursing home administrator could conduct a sample survey to gather data. He might randomly choose room numbers to obtain a random sample.
4. The dean could create an experiment. The dean could split the incoming group of students in half and randomly assign each student to take the orientation course, or not. Then the dean could review the GPA data at the end of the semester.
5. The dietitian might conduct an experiment by randomly assigning the 100 clients to one of three groups (the first diet, the second diet, or no special diet), and then track the clients' weight-loss results after following each program.
6. A sample survey could provide data to answer this question. The guidance counselor could give the sample survey to randomly chosen teenagers at school asking about their music choices and self-esteem.
7. An observational study could determine how patients, finances, and medical staff are impacted by keeping patients in intensive care for additional time. The consultant could review medical data, financial records, and staffing needs before, during, and after transfer.
8. This may be concluded by a sample survey. Participants could be chosen at random from a list of registered voters.
9. A survey of the average number of hours seniors spend on homework each week during the school year could be given to seniors at a mandatory event. An average of the results could be calculated.
10. An observational study of the inventory records of randomly selected grocery stores throughout the country could measure consumers' preferences for the national brand versus the store brand.

Progress Assessment, p. U1-230

1. a
2. c
3. d
4. d
5. a
6. c
7. c
8. d
9. d
10. d
11. a. The school board could review observational studies from other schools or districts that have implemented a school uniform policy. They could seek data on any of the following: student achievement; incidents and student behavior reports; and the impact of uniforms on student self-esteem.
    b. If the data from these studies indicates that students wearing uniforms have higher rates of achievement, less behavioral incidents and issues, and increased self-esteem, the board would present this data to strengthen its argument for the new policy.
    c. There is a potential for bias since schools that have uniforms may not represent a typical school. For example, data may be collected from private or religious schools. Students who attend these schools may not have the same socioeconomic background as students at any given public school. On the other hand, some schools with significant difficulties and very low student achievement may have adopted school uniform policies in recent years, which would lead to skewed data as well.

Lesson 4: Estimating Sample Proportions and Sample Means

Pre-Assessment, p. U1-233

1. b
2. c
3. d
4. d
5. c

Warm-Up 1.4.1, p. U1-240

1. 2012: 65%; 2011: 69%; 2010: 60%
2. 65%
3. The class of 2012 best represents the overall graduation rate. The rates, in decimal form, differ by less than or about 0.05 when compared to the mean. The class of 2011 had a 4% higher graduation rate and the class of 2010 had a 5% lower graduation rate than the overall graduation rate of 65%.

Practice 1.4.1: Estimating Sample Proportions, p. U1-258

1. $\hat{p} \approx 98\%$; SEP ≈ 0.01
2. $\hat{p} \approx 57\%$; SEP ≈ 0.09
3. $\hat{p} \approx 91\%$; SEP ≈ 0.01
4. $\hat{p} \approx 79\%$; SEP ≈ 0.11
5. $\hat{p} = 54\%$; SEP ≈ 0.14
6. SEP ≈ 0.03
7. 1,127 people; SEP ≈ 0.01
8. $n = 16$
9. $\hat{p} \approx 9\%$
10. She should order 2,400 coupons.

Warm-Up 1.4.2, p. U1-260

1. $\frac{1}{6}$
2. There are 720 ways to arrange the posters.
3. There are 15 ways to choose 2 out of 6 posters.

Practice 1.4.2: The Binomial Distribution, p. U1-285

1. 0.03
2. 0.12
3. 0.51
4. 0.03
5. 0.28
6. 0.31
7. 0.09
8. 0.88
9. 0.16
10. 0.23

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Warm-Up 1.4.3, p. U1-287

1. 16 years
2. 21 miles
3. 5.8 hours
4. $s \approx 14.3$

Practice 1.4.3: Estimating Sample Means, p. U1-298

1. 23.33
2. 1.00
3. 7.16
4. 10.00
5. 6.01
6. 0.22
7. 0.17
8. 0.01
9. 34.83
10. 0.18

Warm-Up 1.4.4, p. U1-300

1. 27–33%
2. 31–37%
3. 29–35%
4. At first glance, it appears as though Benton won the election. Benton has the highest percentage in the table and the highest upper-bound percentage; however, there are several instances of overlap in the ranges identified. The ranges of all three candidates include the possibility of earning 31%, 32%, and 33% of the vote. While it is likely that Benton won the election, the overlaps in the margins of error make the election too close to call based on these results.

Practice 1.4.4: Estimating with Confidence, p. U1-318

1. ±7.30%
2. ±11.03%
3. ±0.10%
4. ±20.24%
5. (21.66, 22.94)
6. math achievement: (5.95, 6.05); BMI increase: (4.95, 5.05)
7. (4.7, 4.9)
8. (11.199, 11.321)
9. The upper level of the confidence interval is 7.4563; $0.2563(60) \approx 15.4$, so after 7 minutes had passed, you would have to wait for an additional 15.4 seconds to get a free meal.
10. The confidence interval is 7.759 to 8.041 pounds. A cat weighing 6 pounds would not be found within an 80% confidence interval.

Progress Assessment, p. U1-320

1. b
2. d
3. a
4. a
5. b
6. d
7. b
8. d
9. c
10. c
11. a. mean: \$222; standard deviation: \$86.40; standard error of the mean: 17.63; confidence interval: (187.43, 256.57)
    b. mean: \$237; standard deviation: \$73.25; standard error of the mean: 14.95; confidence interval: (207.69, 266.31)
    c. Answers may vary. Sample answer: The second data set has a lower standard of error; therefore, the data is less spread out and more closely represents the population mean. However, by excluding the two lowest values, you are also decreasing the sample size, and perhaps skewing the results.

Lesson 5: Comparing Treatments and Reading Reports

Pre-Assessment, p. U1-323

1. b
2. a
3. a
4. b
5. a

Warm-Up 1.5.1, p. U1-328

1. 54 points
2. 7.58

Practice 1.5.1: Evaluating Treatments, p. U1-353

1. Spin 1: 15.6; Spin 2: 21.6; Spin 3: 8.2; Spin 4: 20.6; Spin 5: 15.4; Spin 6: 20.6; Spin 7: 16.4; Spin 8: 24.4; Spin 9: 16.4; Spin 10: 23.6; Spin 11: 18.0; Spin 12: 25.0
2. Spin 3 has a particularly low mean of 8.2 when compared to the means of the other spin numbers.
3. Spin 1: 11.74; Spin 2: 11.04; Spin 3: 10.93; Spin 4: 9.09; Spin 5: 2.73; Spin 6: 13.22; Spin 7: 7.23; Spin 8: 6.95; Spin 9: 11.88; Spin 10: 7.39; Spin 11: 8.37; Spin 12: 7.72
4. Spin 1: –0.457; Spin 2: 0.729; Spin 3: –2.005; Spin 4: 0.640; Spin 5: –2.130; Spin 6: 0.440; Spin 7: –0.495; Spin 8: 2.059; Spin 9: –0.301; Spin 10: 1.694; Spin 11: 0.000; Spin 12: 2.028
5. The spin number with the highest *t*-value is Spin 5. A low standard combined with a low mean will cause a low *t*-value.
6. The spin number with the lowest *t*-value is Spin 11. The mean of this spin number is the same as the population mean of 18, which creates a *t*-value of 0.
7. When calculating the *t*-value, a large difference in the means in the numerator of the formula or a small standard deviation in the denominator will cause a high *t*-value.
8. $1.0 > p > 0.50$
9. $0.20 > p > 0.10$
10. $1.0 > p > 0.50$

Warm-Up 1.5.2, p. U1-355

1. 67.2%
2. 68.6%
3. 73%
4. Since the three decades showed winning percentages (67.2%, 68.6%, and 73%) that all exceed 50%, it could be concluded that the "home-field advantage" does improve the chances of winning for the home teams.

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Practice 1.5.2: Designing and Simulating Treatments, p. U1-372

1. If we calculate 5805 ÷ 11,862, he missed 49% of his shots. His free-throw average would be about 51%. A coin flip (though close) would not exactly simulate this actual average.
2. Notice that since the problem does not describe the simulation, we cannot decide if it is a well-designed simulation or a flawed simulation. He should work out the theoretical probability to make the prediction.
3. A spinner has 5 sections that are constant—the spinner can land on each section more than once. After a guest arrives, he is no longer eligible to arrive again, so the simulation does not mimic reality well.
4. We can use the card simulator in a graphing calculator. This would allow us to track the cards chosen and how often each hand is dealt.
5. We can use a random number generator to place the students in the classes, assigning certain numbers (such as 1–3) to the 3 friends to calculate how many times they would end up in the same class.
6. Assign numbers 1–5 to the defective products. Assign numbers 5–100 to the non-defective products. If a defective product is chosen, then number the products 1–50 and designate these as returns, with the numbers 51–100 assigned to non-returned items.
7. Put 20 blue algebra tiles in a bag, then add 37 red tiles. Draw 3 tiles and record the colors. Perform multiple trials. Drawing 3 blue tiles indicates that gun club members received all 3 permits.
8. Build a spinner with 8 sections and paint each section 1 of 4 colors. Spin for each team member. Record how many times a series of 2 spins results in different colors. A series of 2 spins of the same color represents siblings as teammates.
9. An appropriate simulation would be to use spinners with 6 sections instead of dice.
10. Find, experimentally, a distribution of ages with an average of 32 and standard deviation of 2.5 (the standard deviation will depend on the ages you determine to be at the extremes of the population). Label squares of paper with the age of each person in the data set and put the squares in a bag. Draw 4 squares of paper several times to represent multiple trials.

Warm-Up 1.5.3, p. U1-374

1. ≈ 93.6%
2. ≈ 6.3%
3. The numbers of respondents to each statement are not equal, so it can be concluded that some respondents did not respond to all 3 statements.
4. From the information presented, the sample does not appear to be random. For example, customers who took their orders to go were not surveyed. Customers who were in a rush to leave or who have small children requiring their attention may have been less likely to respond.

Practice 1.5.3: Reading Reports, p. U1-389

1. The ad has an arithmetic error: 6 ÷ 7 is 0.85714..., which, rounded to the nearest whole percent, is 86%, not the 87% cited in the ad.
2. While the chart clearly indicates that more students are passing the standardized test, we must ask an important question. Namely, "how many students took the test?" The increase in the number of students passing may be a result of increased numbers of students taking the test. If the number of students taking the test *did not* increase, then the passing rate would be more indicative of improved test results and provide more significant data to review.
3. There are many possible failings in this research. The researchers may not have conducted enough trials. They may have only studied the positive effects of the drug and not the cause of the injury. The study may have been funded by the company that would be producing and profiting from the medicine.
4. While the total number of rats studied is sizable (200), the subgroups of 10 rats are too small for the results to be significant. The sample size of each group is so small that the results might be caused by chance alone.
5. In general, this is a vague statement, but the fact that there are no numbers assigned to these factors (violent video games and aggression) tells us that we cannot attribute correlation. We know that numerical or quantitative studies are the only ones that can be considered to have statistical correlation, so we should not cite these results in a term paper.
6. This standard deviation might be attributed to the nature of the questions and those who were asked the questions. For a pop culture quiz, you might find participants who are particularly knowledgeable on the topics of celebrities, music, and movies, and score very high, while other participants who don't have much knowledge of pop culture would have low scores. This standard deviation shows a large disparity in the quiz participants, indicating that some know all or most of the answers and some know few or no answers, which is very possible in such a case. There could be a mathematical error, but it is also possible that the data is correct.
7. The null hypothesis would be "the probability of rolling a 5 on this dice is 1/6 or H_0: $P = 1/6$."
8. The medical research team's results should have a lower level of significance. The lower the level of significance, the more likely that the results are accurate—that is, a significance level of 0.01 is more accurate than a significance level of 0.05. The medical research team is

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seeking to help people's health and reduce danger, so they should have more precise results than a team doing research for a nonprofit organization.

9. The visits from the high school students are a confounding variable in this study. We can't be sure whether it was the positive message of the books that impacted the patients or if it was the visits from the students who read the books that had an effect.
10. The participants in this survey are your friends and family and therefore would likely be biased. These results would not be statistically valid due to the bias of those polled.

Progress Assessment, p. U1-391

1. b
2. a
3. c
4. a
5. a
6. b
7. a
8. c
9. b
10. d
11. a. 67.3%
    b. Voluntary response bias: the sample is not representative of all customers since the sample members have the option to respond.
    Response bias: the survey required in-person interviews, and respondents may have been reluctant to express dissatisfaction about the product or to disagree with the interviewer.
    Measurement bias: we would have to check to make sure all of the customers were asked the exact same question in the same way.
    Nonresponse bias: the people who didn't respond might have been in a rush to leave; the omission of this group would alter the data.
    c. The marketing expert will not advise the company to use these results. To begin with, 67.3% is not a high enough percentage of satisfied customers to indicate that the product is outstanding. In addition, the possible biases in the sample and survey do not give the survey a great deal of validity.

Lesson 6: Making and Analyzing Decisions

Pre-Assessment, p. U1-394

1. a
2. d
3. d
4. c
5. a

Warm-Up 1.6.1, p. U1-399

1. $\frac{2}{3}$
2. $\frac{1}{2}$
3. Although there is a $\frac{2}{3}$ probability of choosing a female from a group of 12 children that is made up of 4 boys and 8 girls, there is no guarantee that a group of this makeup will actually attend the festival. Also, the probability of there being a family with 2 children who are both girls is only $\frac{1}{4}$. Based on these probabilities alone, it doesn't appear to be appropriate to offer only prizes for girls. Luanne should either provide prizes for boys as well, or find prizes that are gender neutral. Luanne might also determine different probabilities that will allow her to better understand the likelihood of a girl being chosen at random from the entire group of festival attendees.

Practice 1.6.1: Making Decisions, p. U1-416

1. No. Students are assigned to teachers based on their learning needs.
2. Claudia's adult children who have larger families will inherit more of her wealth. This can be considered fair, since larger families require more money. However, you might also argue that the method is not fair since each adult child will not inherit an equal portion of Claudia's wealth; those who have started families will inherit more than those who haven't, and whoever has the largest family will inherit the most money.
3. This is not fair to Rico, since he does not have an equal chance of being allowed to fly to Florida right away. Two of the three papers would either postpone the trip or do away with it entirely.
4. If all of the papers that the warriors wrote on were the same size, and each paper was equally likely to fall out when the helmet was shaken, then this can be considered fair.
5. No, unless each county were exactly the same size and had the same number of men who were eligible for the draft.
6. The expected value of this game is $20.67. This price would be fair in order to play this game.
7. $50
8. The marbles might be different sizes. The box might be made of a material that can be seen through.
9. Game 1 is fair because both players are equally likely to win. Game 2 is not fair because most of the products will be considered "not prime" (other than rolling 1 and 2, 1 and 3, and 1 and 5).
10. No. The likelihood of choosing the "C" paper changes based on the order in which the students choose from the hat.

Warm-Up 1.6.2, p. U1-418

1. 47%
2. 16.8%
3. 53%
4. 5%

Practice 1.6.2: Analyzing Decisions, p. U1-440

1. 6.25%
2. 2.25%

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3. Answers will vary, but should include justification. Sample answer: Cathy had a 6.25% chance of losing both cases. With her record, this will happen 6 days out of every 100. Carlos had an even lower chance of losing both cases, since with his record he is only likely to lose both cases on 2 days out of every 100. Both lawyers have very good win rates. Based on this information, I would call the lawyers in to congratulate them on their good work and let them know that the two losses are rare and shouldn't bring them down.
4. $\frac{2}{18}$ or $\frac{1}{9}$
5. $\frac{1}{18}$
6. No, because the team that loses in the first round is not likely to win the tournament.
7. The expected number of goals is 0.32 for the team that pulls the goalie and 0.94 for their opponent.
8. The expected number of goals is 0.64 for the team that pulls the goalie and 1.88 for their opponent.
9. The expected number of goals is 0.80 for the team that pulls the goalie and 2.35 for their opponent.
10. Answers may vary, but should include justification. Students may note that their decision would depend on the score (for example, whether the game is tied). Sample answer: With 5 minutes left, the average goals scored for the team that pulls their goalie is 0.8, while the average goals scored for the opponent is 2.35. This may be enough justification to leave the goalie in and attempt to win the game.

Progress Assessment, p. U1-442

1. c
2. c
3. b
4. a
5. c
6. b
7. a
8. d
9. d
10. a
11. Change to the closed door that you did not originally select. When you first made your choice, there was a 1 in 3 chance that there would be a goat behind the door. However, since the host has knowledge of which doors hide a goat, the host has knowingly eliminated 1 goat. Staying with your first choice does not change your original 1 in 3 chance of winning the money. Switching, however, increases your chances of winning to 2 out of 3.

Unit Assessment p. U1-445

1. d
2. a
3. b
4. c
5. c
6. b
7. d
8. d
9. a
10. c
11. c
12. b
13. a. 95%
    b. A white blood cell count of about 14.1×10^3 cells per μL would be out of a normal range and would likely cause an alert. The count of 14.1×10^3 cells per μL would be more than 3 standard deviations from the mean, or higher than 99.7% of normal counts.
    c. No, she should not be worried. Her white blood cell count is within normal limits for a pregnant woman in her third trimester. She falls within 2 standard deviations, or 95% of the population.
14. a. 0.010
    b. 0.054
    c. 0.176
15. a. Jesse's score would fall in the top 2.6% in Massachusetts and in the top 2.2% in New Jersey. This is due to the combination of lower mean and greater standard deviation in New Jersey compared to Massachusetts.
    b. It is more difficult for a student in Massachusetts to score within the qualifying range. In Massachusetts, students' scores must be higher than 2.24 standard deviations above the mean, whereas students in New Jersey would only have to score 1.7 standard deviations above the mean to qualify for the scholarship.

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Student Book Answer Key

Lesson 1: Using the Normal Curve

Practice 1.1.1: Normal Distributions and the 68–95–99.7 Rule, p. U1-21

1. 95%
2. 84%
3. 0.6°F
4. 99.7%
5. 8.3%
6. 41.7%
7. 95%
8. 15.7%
9. The number of college applications is not normally distributed.
10. Answers may vary. Sample answer: There are no students with a number of applications less than 1.67 standard deviations below the mean. This does not match or nearly match the percentages in a normal distribution.

Practice 1.1.2: Standard Normal Calculations, p. U1-38

1. $z = 1.42$
2. 92nd percentile
3. 63%
4. 91%
5. $\sigma = 0.20$ ounce
6. 7.1 inches
7. 45%
8. –1.16
9. 12.3%
10. 16.4%

Practice 1.1.3: Assessing Normality, p. U1-59

1. Histogram B
2. Histogram B and Histogram C
3. Histogram C
4. There are no outliers.
5. 70%
6. Yes. The histogram and normal probability plots show approximately normal distribution.
7. 59%
8. 100%
9. smallest z-score: –1.59; largest z-score: 1.63
10. No. The minimum possible value is only 1.66 standard deviations below the mean. The maximum possible is only 1.66 standard deviations above the mean. These do not match normal distribution. The birthdays are spread out fairly evenly, rather than concentrated in the middle of the year. A normal distribution would have birthdays centered about the middle of the year.

Lesson 2: Populations Versus Random Samples and Random Sampling

Practice 1.2.1: Differences Between Populations and Samples, p. U1-79

1. c
2. a
3. c
4. population: all quarters in circulation
   parameter: the mean mass of all quarters
   sample: the 40 quarters that Talia weighed
   statistic of interest: the mean mass of the 40 quarters
5. population: all students at the high school
   parameter: the percent of students at the school in favor of the late arrival policy
   sample: students in the 5 homerooms that Amber chose
   statistic of interest: the percent of students in the 5 homerooms who favor the late arrival policy
6. about 115 students
7. about 57 turtles
8. About 90 incoming ninth graders plan to ride the bus.
9. About 93 incoming ninth graders don't plan to ride the bus.
10. 3.4 miles

Practice 1.2.2: Simple Random Sampling, p. U1-99

1. All samples are unbiased.
2. sampling error or chance variation
3. 3.6
4. The combined samples provide a larger sample size than any individual sample.
5. Clarissa's sample; this is a convenience sample, since she surveys just the people in the soccer program.
6. Kerry's sample; he is conducting a simple unbiased random sample.
7. Selecting more homerooms would give more data, reducing the sampling error.
8. sample with the greatest mean: ultra-premium blend, ground beef and rice, and puppy formula; mean = 466.7 calories
9. sample with the smallest standard deviation from the population mean: fish and potato, organic formula, chicken and potato; standard deviation = 13.2 calories
10. Use 8 cards or a random number generator to generate 3 random integers from 1 to 8.

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Practice 1.2.3: Other Methods of Random Sampling, p. U1-117

1. systematic
2. cluster
3. convenience
4. stratified
5. Number the teams from 1 to 30, then use cards or a random number generator to select 10 teams.
6. Use cards or a random number generator to select a starting number from 1 to 3, then select every third team.
7. Number the divisions from 1 to 6, then use cards or a random number generator to select two divisions.
8. Number the NASDAQ-100 companies from 1 to 98. Use a random number generator to select 20 companies from the list.
9. Randomly select a starting number from 1 to 5, then select every fifth company starting with the first number. If the first number is a 4 or 5, then select 1 company from the remaining companies for the twentieth member of the sample.
10. Number the industries from 1 to 11, then randomly select 1 company. Select as many companies as needed to reach a sample size of 20 or more.

Lesson 3: Surveys, Experiments, and Observational Studies

Practice 1.3.1: Identifying Surveys, Experiments, and Observational Studies, p. U1-134

1. sample survey. Asking consumers about the features of their phones describes a survey. Since the company is randomly selecting consumers, they have taken a sample.
2. observational study. This study does not indicate that a treatment or action is being studied, nor does it ask questions of participants. It observes and records data without intervention.
3. experiment. Since the technologies were applied to different groups, this constitutes an experiment.
4. observational study. Researchers might look through the telescope at various times and record their results.
5. sample survey. The dermatologist might conduct a survey of 400 local college students asking about their tanning habits.
6. observational study. This group of scientists might observe the surface of Mars and compare it to the surface on Earth.
7. observational study. Researchers might record income and other indicators of poverty as well as obtain the health records of each mother, then obtain their children's growth and educational records.
8. observational study. Scientists might observe how dolphins respond to a recording of their own sound (the sound used by other dolphins to communicate with them) to determine if the dolphins use a specific "name."
9. experiment. The educators could present hands-on projects to several classes of students prior to instruction, then compare student achievement to several classes that were not exposed to hand-on projects prior to instruction.
10. An experiment would take more effort to design. While an observational study does not assign groups to treatments, in an experiment this process must be included. An observational study would choose the time and place and then take data from an existing situation.

Practice 1.3.2: Designing Surveys, Experiments, and Observational Studies, p. U1-146

1. The designer might review average electricity-use rates for families, and then conduct an experiment that would measure how much electricity the new bulb uses when compared to traditional bulbs.
2. The yoga instructor might conduct an experiment in which he randomly assigns participants in two groups (one that participates in yoga and one that doesn't), and then measure the participants' blood pressure daily for four weeks and compare the results.
3. The teacher might conduct an experiment in which he randomly assigns classes of the same course to the treatment of allowing calculators, while the remaining classes do not use calculators. He would compare the results of final averages at the end of the term or year. He would need to conduct this experiment over multiple years in order to draw conclusions about this data.
4. The maker could conduct an experiment by recruiting volunteers from a population of patients receiving chemotherapy, and requiring randomly assigned patients to use the new head wrap while comparing their hair-loss results to patients who did not use the wrap.
5. The salon owner might conduct a sample survey, asking random clients why they have not made appointments for their next styling session as they exit the salon.
6. This can be determined by conducting an observational study. The physical education teacher might observe the use of various spaces in a given area, evaluating the quality of the space and its use by children.
7. A sample survey would be useful in this situation. The national teaching commission might randomly choose former teachers throughout the country to take part in the survey.
8. This can be determined by conducting an observational study. The doctor could use patient records to determine a population of obese adult patients and a population of adult patients who aren't obese, then choose a random sample from each population for his study. The patients' records would include addresses; the doctor could then use maps to determine what percentage of the patients in each sample have accessible green space within a certain radius of their homes. Finally, the doctor would compare the results for each sample.

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9. An experiment may help in this case. An experiment could be conducted in which women with similar talents, skills, education, and job experience are separated into two groups. One group of women wears make-up to job interviews and the other does not. The number of jobs offered to both groups would be compared. This experiment would need to be conducted at various businesses in several professions.
10. A sample survey could answer this question. The life coach could survey clients who met their weight-loss goals and kept the weight off for a certain amount of time, asking questions to determine the factors that helped the clients persist.

Lesson 4: Estimating Sample Proportions and Sample Means

Practice 1.4.1: Estimating Sample Proportions, p. U1-158

1. $\hat{p} \approx 40\%$; SEP ≈ 0.06
2. $\hat{p} \approx 62\%$; SEP ≈ 0.03
3. $\hat{p} \approx 75\%$; SEP ≈ 0.04
4. $\hat{p} \approx 80\%$; SEP ≈ 0.02
5. $\hat{p} \approx 0\%$; SEP ≈ 0.00
6. SEP ≈ 0.04
7. About 679 people responded favorably; SEP ≈ 0.01.
8. $n = 5$
9. $\hat{p} \approx 46\%$
10. The manufacturer must survey 225 people.

Practice 1.4.2: The Binomial Distribution, p. U1-177

1. 0.11
2. 0.01
3. 0.41
4. 0.11
5. 0.06
6. 0.41
7. 0.15
8. 0.04
9. 0.10
10. 0.20

Practice 1.4.3: Estimating Sample Means, p. U1-183

1. 8.60
2. 0.21
3. 16,855
4. 1.52
5. 10.84
6. 0.13
7. 0.22
8. 1.35
9. 866.03
10. 0.16

Practice 1.4.4: Estimating with Confidence, p. U1-195

1. ±10.00%
2. ±93.2%
3. ±1.76%
4. ±3.28%
5. (434.4, 465.6)
6. 95% confidence interval for a sample of 40: (276.8, 289.2); 95% confidence interval for a sample of 80: (278.6, 287.4); There is a difference of 1.8 grams.
7. 98% confidence interval for Country A: (0.23845, 0.36155); 98% confidence interval for Country B: (0.00709, 0.05291); The confidence interval for Country B is more narrow than that of Country A.
8. (696.63, 703.37)
9. The 90% confidence interval is 59.195 to 70.815 minutes. Your 1-hour wait is within the 90% confidence interval.
10. The developers must charge no more than $76.99 per square foot.

Lesson 5: Comparing Treatments and Reading Reports

Practice 1.5.1: Evaluating Treatments, p. U1-217

1. A: 8.5; B: 10.5; C: 8.6; D: 10.6
2. A: 3.5; B: 2.4; C: 2.8; D: 2.6
3. $t = -0.077$
4. $t = -0.098$
5. $t = -1.633$
6. $t = -1.668$
7. A and C seem to belong to one subpopulation and B and D to another subpopulation. This is because the *t*-values of both sets are close to 0.
8. A and C: 8.5; B and D: 10.5
9. Minimum: 4; maximum: 14; this information represents the minimum and maximum weight of the rabbits in the entire data set.
10. The data could have been gathered from subpopulations of different ages, where the mean of 8.5 pounds corresponds to the younger rabbits and the mean of 10.5 pounds corresponds to the older rabbits.

Practice 1.5.2: Designing and Simulating Treatments, p. U1-228

1. Including the 0 would give you an extra number. Since there are 9 different colors, you would need to assign the digits 1–9 to different colors. The correct calculator command would be RandInt(1, 9).
2. Since the teams have been chosen to play in the playoffs, it is not likely that they all have a 50% likelihood of winning. By now, they have played games and have records of wins and losses. A coin will simply choose randomly with a probability of 0.5.
3. Using a random number generator with the values 1–13 would omit the fact that there are 4 suits for each card, and once a card has been chosen in a hand, it is not returned.
4. We can use a random number generator, with 1–15 indicating a dunked basketball that breaks the backboard and 16–100 indicating a dunk that does not break the backboard.

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5. Since there are 4 groups with exactly 13 members each, we might simulate this using a deck of 52 cards, with each suit representing a different family. If the teams are formed at random, we would draw cards at random from the deck.
6. We could use the random number generator of a graphing calculator, assigning numbers 1–6 to the respective dice rolls. Also, we could use the probability simulator in the calculator, which has a dice simulator built in.
7. Use 20 cards: 15 labeled "correct" and 5 labeled "incorrect." Flip 10 of the 20 cards. If the 10 cards are labeled "correct," you will score 100% on the test.
8. If the probability is 0.50 for both *P*(girl) and *P*(boy), a coin is a good simulator.
9. You can use a spinner instead of the cards.
10. Put 20 algebra tiles in a bag: 13 white tiles and 7 black tiles. If you draw 6 white tiles in a row, you have emulated the lawyer's success through the simulation.

Practice 1.5.3: Reading Reports, p. U1-237

1. There are many questions that might arise from such a general statement: What was the sample size? How was the sample chosen? What was the procedure to ensure randomization? Who funded the research? How was the question worded? How can we ensure that people were honest in their response?
2. $p = 0.05$ is the level of statistical significance, or the probability of the test being replicated and obtaining the same result. $p = 0.05$ means that the data used a 95% level of confidence, which can be calculated by subtraction: $1 - 0.05 = 0.95$.
3. This is an error of omitting data. If all of the data were included and the calculation were incorrect, we could call this a mathematical or arithmetic error.
4. Non-vaccinated children may have parents who choose to treat their children holistically as opposed to bringing them to the hospital. Non-vaccinated children may not be representative of the population as a whole in that they may disproportionately live in rural areas, be homeschooled, live in a religious community, or be part of some other demographic that would lead them to be less likely to be taken to the hospital.
5. The total sales for the year were $1,260,000. The quarterly sales can be calculated as percentages: first quarter: 44.4%; second quarter: 35.7%; third quarter: 9.5%; fourth quarter: 10.3%. The circle graph does not accurately represent the data provided. For example, the circle graph shows first quarter sales to be more than half of the circle, but we know that 44.4% is less than half or 50% of the sales for the year.
6. This study revealed that only about 12% of the trials conducted were pediatric, while almost 60% of the diseases studied were pediatric. We see that there is an imbalance in the pediatric trials versus adult trials.
7. If the "happiness" of the state could be measured in some quantity, the study may show a correlation.
8. We might find bias based on the sponsors of each study. This does not indicate the population or samples that have been chosen, so we might find bias in each population or sample taken.
9. This data is in numerical form. We should ask what the rate of cases for each state is compared to the overall population. There is a large difference in the populations of California and Maine, for example.
10. The null hypothesis can be stated as "Grades have an effect on student motivation."

Lesson 6: Making and Analyzing Decisions

Practice 1.6.1: Making Decisions, p. U1-253

1. No. Students who are in the honors program and those who are on a sports team get first choice of rooms.
2. Yes. Three straws that appeared to be of equal length were held by a neutral party. Each sister could choose one straw and there were not any straws left over. Each sister had an equal chance to select the short straw and, therefore, to use the car.
3. No. The chance of drawing a paper that delays or prevents a proposal is greater than the chance of drawing the paper allowing the proposal. Furthermore, it seems that the prospective bride and groom do not have equal say in the decision, because only the man is allowed to decide whether the drawing takes place.
4. Game 1 is fair; there are 18 even outcomes and 18 odd outcomes. Game 2 is not fair. There are 15 prime results and 21 results that are not prime.
5. Yes. Each potential successor was represented by an identical ball of barley meal, so each successor had an equal chance of being selected.
6. Since there are only 5 friends, rolling a die will not be fair because there are 6 possible outcomes. Flipping a coin will also not be fair because there are only 2 possible outcomes. In order to be fair, their names should be written on equally sized pieces of paper and drawn from a hat, as Oliver suggested.
7. No, because patients are not equally likely to receive the shots.
8. No, because some parents may not have Internet access and therefore couldn't submit an application.
9. No, because the outcomes are based on the personal opinions of each voter. All contestants do not have an equal chance of being voted off the show.
10. Yes. The expected value is $0, meaning money will not be lost or won over a long period of time—that is, the amount of money lost and won balances out over time.

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Practice 1.6.2: Analyzing Decisions, p. U1-269

1. 0.0005
2. 0.009995
3. 0.047 or 4.7%
4. 0.5212
5. 0.0020
6. Even though the polygraph seems to have a high level of accuracy, the likelihood of the polygraph reporting a lie when the person is telling the truth is more than 52%. In matters of the law, this is not accurate enough to be used to prosecute people.
7. 0.33%
8. 20%
9. 135
10. 35%

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Instruction

Goal: To guide students to an understanding of *z*-scores in relationship to standard deviation, probability, and intervals

Common Core State Standard

S–ID.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.★

Student Activities Overview and Answer Key

Station 1

Working with groups, students calculate the *z*-score from a given mean and standard deviation.

Answers

1. $z = \dfrac{13 - 14.54545}{1.50756} = -1.0251$
2. $z = \dfrac{6 - 5.875}{1.24642} = 0.1003$
3. $z = \dfrac{70 - 56.81818}{21.82576} = 0.6040$
4. $z = \dfrac{20 - 21.5}{1.28602} = -1.1664$
5. $z = \dfrac{32 - 30.9}{1.44914} = 0.7591$

Station 2

Working with groups, students calculate the *z*-score from a given set of data and standard deviation. Students should begin to develop a sense of the relationship between *z*-scores and real-world data.

Answers

1.
   a. 14.625
   b. 3.1795
   c. $z = \dfrac{16 - 14.625}{3.1795} = 0.4325$

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2.
   a. 75.95238
   b. 12.7521
   c. $z = \frac{90 - 75.95238}{12.7521} = 1.1016$
   d. $z = \frac{65 - 75.95238}{12.7521} = -0.8589$

3.
   a. 94.8
   b. 5.1381
   c. $z = \frac{100 - 94.8}{5.1381} = 1.0121$
   d. $z = \frac{89 - 94.8}{5.1381} = -1.1288$

4.
   a. 49.8
   b. 4.1425
   c. $z = \frac{60 - 49.8}{4.1425} = 2.4623$

Station 3

Working with groups, students use *z*-scores to determine probability.

Answers

1.
   a. 19.84615
   b. 2.24794
   c. 1.403
   d. 0.9192
   e. 8.08%

2.
   a. 37.5
   b. 2.6118
   c. –2.1058
   d. 0.0174
   e. 1.74%

3.
   a. 13.6
   b. 4.7371
   c. –0.75996
   d. 0.2236
   e. 22.36%

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Instruction

Station 4

Working with groups, students use *z*-scores and probability to determine the approximate number of items in a particular interval.

Answers

1. 24.8%
2. 32.29%
3. 8.19%
4. a. 0.39%

   b. 12.4%

   c. The shift between midnight and 8 A.M. should get first priority.

   d. Answers will vary.

Materials List/Setup

Station 1 calculator

Station 2 calculator

Station 3 calculator; *z*-scores table

Station 4 calculator; *z*-scores table

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Station Activities Set 1: *z*-scores

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

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z	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00
–3.4	0.0002	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
–3.3	0.0003	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0005	0.0005	0.0005
–3.2	0.0005	0.0005	0.0005	0.0006	0.0006	0.0006	0.0006	0.0006	0.0007	0.0007
–3.1	0.0007	0.0007	0.0008	0.0008	0.0008	0.0008	0.0009	0.0009	0.0009	0.0010
–3.0	0.0010	0.0010	0.0011	0.0011	0.0011	0.0012	0.0012	0.0013	0.0013	0.0013
–2.9	0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019
–2.8	0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026
–2.7	0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035
–2.6	0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047
–2.5	0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062
–2.4	0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082
–2.3	0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107
–2.2	0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139
–2.1	0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179
–2.0	0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228
–1.9	0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287
–1.8	0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359
–1.7	0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446
–1.6	0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548
–1.5	0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668
–1.4	0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793	0.0808
–1.3	0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951	0.0968
–1.2	0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131	0.1151
–1.1	0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335	0.1357
–1.0	0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562	0.1587
–0.9	0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814	0.1841
–0.8	0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090	0.2119
–0.7	0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389	0.2420
–0.6	0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709	0.2743
–0.5	0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050	0.3085
–0.4	0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409	0.3446
–0.3	0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783	0.3821
–0.2	0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168	0.4207
–0.1	0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562	0.4602
–0.0	0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960	0.5000

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. What is probability?
2. What do *z*-scores represent?
3. What is the formula for converting a raw score to a *z*-score?
4. What is the meaning of a negative *z*-score? Positive? 0?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in the same station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. Probability is the mathematical likelihood of an occurrence.
2. *z*-scores represent a standard distribution of values about a mean of 0.
3. $z = \dfrac{\text{score} - \text{mean}}{\text{standard deviation}}$
4. A negative *z*-score means the original score was below the mean. A positive *z*-score means the original score was above the mean. A *z*-score of 0 means the original score was equal to the mean.

Possible Misunderstandings/Mistakes

- Incorrectly setting up the *z*-score calculation or incorrectly calculating the *z*-score
- Misinterpreting the data
- Misreading the *z*-scores table
- Incorrectly using a calculator's statistics function
- Subtracting *z*-scores rather than probabilities to find the probability
- Rounding too soon in the series of calculations

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Station Activities Set 1: *z*-scores

Station 1

Work with your group to find each *z*-score. M is the mean of a data set, and s is the standard deviation for the data set. You may use a calculator to find the *z*-score. Show how you used the formula.

1. $M = 14.54545$

   $s = 1.50756$

   What is the *z*-score for a sample value of 13?

2. $M = 5.875$

   $s = 1.24642$

   What is the *z*-score for a sample value of 6?

3. $M = 56.81818$

   $s = 21.82576$

   What is the *z*-score for a sample value of 70?

4. $M = 21.5$

   $s = 1.28602$

   What is the *z*-score for a sample value of 20?

5. $M = 30.9$

   $s = 1.44914$

   What is the *z*-score for a sample value of 32?

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Station 2

Work with your group to answer each question. Use a calculator to find the standard deviation. You may use a calculator to find the *z*-score. Show how you used the formula.

1. At a track meet, students land long jumps of the following distances, in feet:

   10, 12, 13, 15, 20, 19, 18, 15, 15, 16, 17, 12, 11, 9, 14, 18

   a. What is the mean?

   b. What is the standard deviation?

   c. Find the *z*-score that represents the likelihood of a student jumping at least 16 feet.

2. On a recent test, students received the following grades:

   90, 75, 76, 55, 40, 92, 88, 80, 80, 81, 78, 65, 67, 92, 89, 72, 78, 83, 84, 62, 68

   a. What is the mean?

   b. What is the standard deviation?

   c. Find the *z*-score that represents the likelihood of a student scoring at least a 90.

   d. If a failing grade is anything below 65, find the *z*-score that represents the likelihood of failing.

continued

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3. A food company is trying to create a new shipping crate for their most popular apple. Apples that weigh 89 grams or less will not be desirable to consumers. The new shipping crate will not accommodate apples of 100 grams or more. Apples outside the weight range are used for apple sauce, juice, and frozen pies. The apples in a sample have the following weights, in grams:

   100, 90, 85, 92, 93, 95, 98, 99, 94, 102

   a. What is the mean?

   b. What is the standard deviation?

   c. Find the *z*-score that represents the likelihood of an apple weighing 100 grams or more.

   d. Find the *z*-score that represents the likelihood of an apple weighing 89 grams or less.

4. Job applicants receive the following scores on a skills aptitude test:

   50, 45, 49, 47, 48, 52, 53, 60, 48, 46

   a. What is the mean?

   b. What is the standard deviation?

   c. Find the *z*-score that represents the likelihood of an individual's scoring higher than 60.

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Station 3

Work with your group to solve the following problems about *z*-scores and probability. Use a calculator and *z*-scores table.

1. At a track meet, shot-put competitors record the following distances, in feet:

   20, 19, 18, 20, 21, 23, 22, 15, 16, 20, 21, 22, 21

   a. What is the mean?
   b. What is the standard deviation?
   c. Find the *z*-score that represents the likelihood of a shot-put distance over 23 feet.
   d. What is the corresponding number on the *z*-scores table?
   e. What is the probability that someone will record a distance over 23 feet?

2. To determine whether he can safely plant a crop, a farmer records the following overnight low temperatures, in degrees Fahrenheit:

   40, 43, 39, 35, 34, 37, 39, 36, 41, 38, 38, 35, 36, 34

   a. What is the mean?
   b. What is the standard deviation?
   c. Find the *z*-score that represents the likelihood of the temperature falling below freezing (32°F).
   d. What is the corresponding number on the *z*-scores table?
   e. What is the probability that the weather will turn freezing overnight?

3. The same farmer has a record of April rainfalls for the past 10 years, in inches:

   20, 15, 3, 17, 11, 10, 19, 16, 13, 12

   If April rainfall is less than 10 inches, the farmer must plan extra watering.

   a. What is the mean?
   b. What is the standard deviation?
   c. Find the *z*-score that represents the likelihood of rainfall less than 10 inches.
   d. What is the corresponding number on the *z*-scores table?
   e. What is the probability that the farmer will have to plan extra watering?

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Station 4

Work with your group to solve the following problems about *z*-scores, probability, and intervals. Use a calculator and *z*-scores table.

1. Amy learns that on this morning's history exam, the mean of all the scores for her class was 75.66667, with a standard deviation of 11.19216. There are 30 students in the class. What is the probability that Amy's score was between 80 and 90?

2. Using the same information from problem 1, what is the probability that any given student earned a score between 65 and 75?

3. A medical association conducts a study on cholesterol levels in men in a certain population. If the mean cholesterol is 200, with a standard deviation of 39.8, what is the probability that a man in this population will have cholesterol between 170 and 180?

4. A city needs to allocate its traffic officers to the times of day when traffic violations are most likely to occur. It assigns each reported violation a number corresponding to the time of day the violation occurred, between 1 and 24, with 1 representing 1 P.M., 2 representing 2 P.M., and so on, all the way up to 24, or noon. The city nicknames these numbers "time stamps." After several months a pattern becomes clear. The mean time stamp of traffic violations is 13.5, with a standard deviation of 3.
   a. What is the probability that a violation will occur between the hours of 5 P.M. and 6 P.M.?
   b. What is the probability that a violation will occur between the hours of 2 A.M. and 3 A.M.?
   c. Given the standard employment shifts of 8 A.M. to 4 P.M., 4 P.M. to midnight, and midnight to 8 A.M., when should the city plan to have the most traffic officers on call?
   d. Is the city's method of assessing its data on traffic violations a valid approach? Explain.

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Instruction

Goal: To provide opportunities for students to analyze distributions of simulated data and confidence intervals and to calculate point estimates

Common Core State Standards

S–ID.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.★

S–IC.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.★

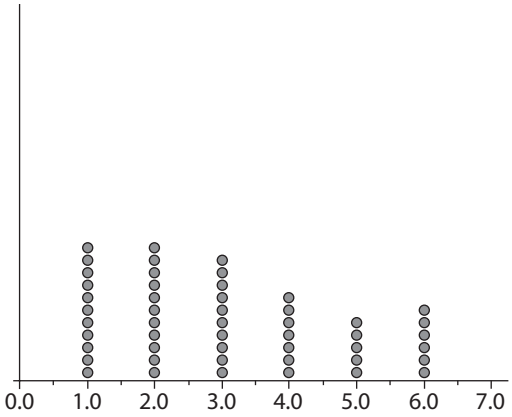
Student Activities Overview and Answer Key

Station 1

Students will work in small groups. Each student will simulate collecting 50 numbers between 1 and 6, and will plot these values on a line graph. The distribution of the plots will vary by student. Students will be asked to describe the line plot's shape and also to determine the mean and standard deviation. Students will then collaborate with their group members. Initially, they will discuss their individual plots, and will then compile their data to create a larger data set. Theoretically, the data set will appear more normal. They will be given the definition of a normal distribution, and will be asked to analyze the percent of the data that falls within one, two, and three standard deviations. After this analysis they are asked to decide if the compiled data is more normal.

Answers

1. a. Answers will vary, but should include 50 integers between 1 and 6. Sample line plot:



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b. Answers will vary based on simulated data. For the sample line plot, the data appears skewed to the right.

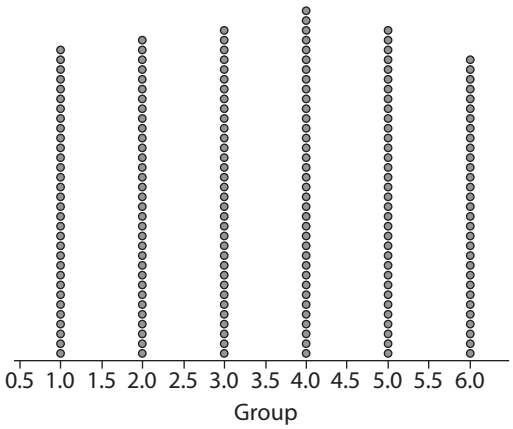
c. Answers will vary based on simulated data. For the sample data, the mean is 3.04 and the sample standard deviation is 1.666.

2. a. Distributions will vary by student. The range of all data will be the same, but the line plots will differ.

b. Means will likely be around 3, but student data will vary. Standard deviations will also likely be close to the sample data in part a of problem 1.

3. a. Answers will vary. 28 of the data values, or 56% of the data, lie within one standard deviation of the sample data from part a of problem 1. 44 of the data values, or 88% of the data, lie within two standard deviations, and 100% of the data lies within three standard deviations. In this sample, there is not enough data within one standard deviation of the mean for it to be considered a normal distribution.

b. The compiled data will be fairly evenly distributed among the 6 integers. Distribution will vary based on individual simulations. Sample plot for four group members:

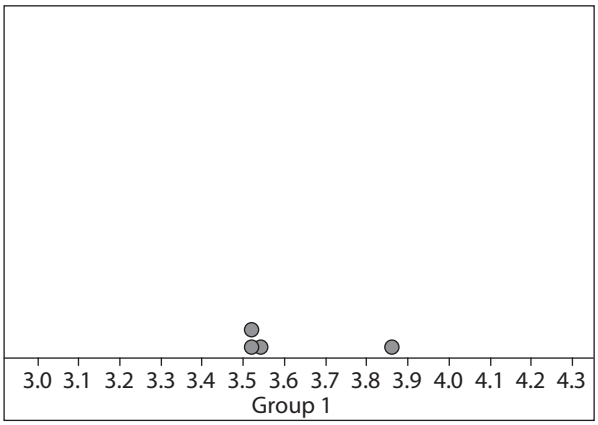


Station 2

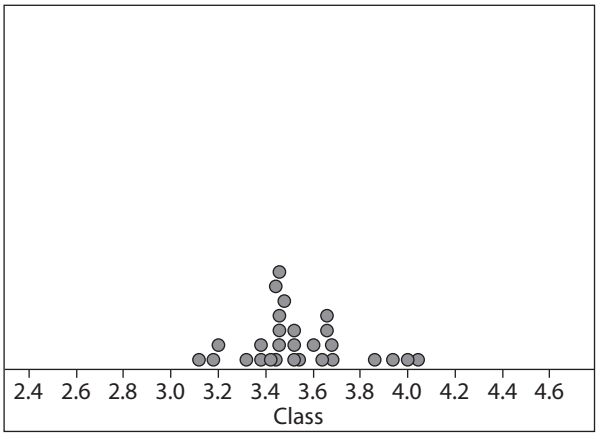
Students will work with simulated data. Students will describe the shape, mean, and standard deviation of the sample means. They will then determine if the sample means fulfill the requirements of a normal distribution.

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Answers

1. a. 

   b. The data set is too small to meaningfully analyze on a dot plot. Based on the sample data from part a, the mean is 3.61 and the sample standard deviation is 0.17.

2. a. 

   b. The shape of the sample means appears more normal. The mean is 3.54 and the sample standard deviation is 0.227.

   c. 21 of the 28 data points, or 75%, lie within one standard deviation. 26 of the 28 data points, or 93%, lie within two standard deviations. All of the data, or 100%, lie within three standard deviations. This closely resembles the percentages of a normal distribution, and the data can be considered normally distributed.

Station 3

Students will analyze confidence intervals. As the sample size increases, and the confidence level remains constant, the margin of error will decrease. The confidence interval will also, respectively, decrease. Students can then compare the mean of a class's sample means—which should closely resemble the mean of the population—to the confidence interval for the mean of a group's sample means. They should see that the mean of the sample means falls within this interval, reinforcing that the confidence interval is a good indicator of the population mean.

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Answers

1. margin of error: 0.088
2. confidence interval: $3.45 < \mu < 3.63$
3. margin of error: 0.266
4. confidence interval: $3.34 < \mu < 3.88$
5. The margin of error of the population mean, with 95% confidence, is much smaller when working with a sample of 200 data items versus 50.
6. The mean of the sample means is 3.54. The mean of the sample means is exactly in the center of the confidence interval, with 95% confidence, of our population mean.
7. Based on the confidence interval and the mean of the sample means, the population mean of all rolls of a six-sided die is approximately 3.5.

Station 4

Students will examine the differences among estimates of a population on different confidence intervals. As the confidence level decreases, the confidence interval decreases. Being less confident about the interval of a population parameter implies that fewer values are within that interval.

Answers

1. confidence interval: $27.01 < \mu < 30.07$
2. Yes; the estimate is within the range, so it would be acceptable for the car manufacturer to state the average mpg is 30.
3. The confidence interval, for 90% confidence, is $27.59 < \mu < 29.48$. The stated mpg is not within this range, so it would not be a good estimate with 90% confidence.
4. For an identical data set, a lower confidence level has a smaller confidence interval. This is because it is with less certainty that the mean is within a smaller range. A larger range of values allows for more error in the parameter, and allows us to be more certain.

Materials List/Setup

All Stations graphing calculators

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. How did the number of data points affect the distribution of the data?
2. Does the distribution of a collected set of data affect the distribution of the sample means?
3. How is the confidence interval of a population mean affected by an increase in sample size?
4. How does the confidence level affect the confidence interval?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in the same station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. The more data points, the more the data will represent data from an entire population. When trying to analyze sample means, it was difficult to discuss any distribution when only comparing 4 pieces of data. However, once all the class data was compiled, it was easy to determine if the data was normally distributed.
2. It shouldn't. The distribution of the data in Station 1 is fairly evenly distributed, but the sample means were normal. The data does not need to be normal for the distribution of the sample means to be normal.
3. The larger sample size greatly reduced the margin of error. The confidence interval was therefore much smaller, and narrowed down the possibilities for the population mean.
4. A higher confidence level means the interval is larger.

Possible Misunderstandings/Mistakes

- Making general observations when only working with small numbers of data
- Omitting qualifying language, such as "within my sample" or "in our group"

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Station 1

Work with your group to answer the following questions about distribution.

1. Individually, simulate rolling a six-sided die 50 times.
   a. Plot each data point on a dot plot.

   b. Describe the shape of the distribution.

   c. Determine the mean and standard deviation.

continued

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2. Compile your data within your group.

   a. Compare the distributions on the dot plot.

   b. Compare the means and standard deviations.

3. The formal definition of a normal curve is that 68% of the data falls within one standard deviation of the mean, 95% falls within two standard deviations of the mean, and 99.7% falls within three standard deviations of the mean.

   a. Look at your individual data. Does your data qualify as normal?

   b. Reflect on the compiled data. Does the compilation within your group satisfy the definition of a normal curve?

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Station 2

Work with your group to answer the questions about distribution.

1. The following means are from a simulation of a class of seven groups, each containing 4 students, each rolling a six-sided die 50 times.

Group #	Student #	Mean	Group #	Student #	Mean
1	1	3.54	4	3	3.46
1	2	3.52	4	4	4.00
1	3	3.52	5	1	4.04
1	4	3.86	5	2	3.94
2	1	3.64	5	3	3.46
2	2	3.44	5	4	3.38
2	3	3.18	6	1	3.48
2	4	3.60	6	2	3.44
3	1	3.20	6	3	3.68
3	2	3.38	6	4	3.46
3	3	3.68	7	1	3.66
3	4	3.42	7	2	3.66
4	1	3.32	7	3	3.52
4	2	3.46	7	4	3.12

   a. Plot the means of group 1 on a dot plot.

continued

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b. Describe the shape of the distribution, and determine the mean and standard deviation.

2. Look at the data collected by the entire class.

   a. Plot the sample means of each classmate on a dot plot.

   b. Describe the shape of the distribution, and determine the mean and standard deviation.

   c. The formal definition of a normal curve is that 68% of the data falls within one standard deviation of the mean, 95% falls within two standard deviations of the mean, and 99.7% falls within three standard deviations of the mean. Is the distribution normal?

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Station 3

Work within your group to answer the questions. For all questions, use a 95% confidence interval, and a corresponding *z*-score of 1.96. The following means are from a simulation of a class of 7 groups, each containing 4 students, each rolling a six-sided die 50 times.

Group #	Student #	Mean	Group #	Student #	Mean
1	1	3.54	4	3	3.46
1	2	3.52	4	4	4.00
1	3	3.52	5	1	4.04
1	4	3.86	5	2	3.94
2	1	3.64	5	3	3.46
2	2	3.44	5	4	3.38
2	3	3.18	6	1	3.48
2	4	3.60	6	2	3.44
3	1	3.20	6	3	3.68
3	2	3.38	6	4	3.46
3	3	3.68	7	1	3.66
3	4	3.42	7	2	3.66
4	1	3.32	7	3	3.52
4	2	3.46	7	4	3.12

1. Determine the margin of error of the population mean based on the class data.

2. Determine the confidence interval of the population mean based on the class data.

continued

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3. Determine the margin of error of the population mean based on group 1's data.

4. Determine the confidence interval of the population mean, based on group 1's data.

5. Compare the results from problems 2 and 4.

6. How does this range compare to the mean of the class sample means?

7. What would we expect the sample mean to be of all rolls of a six-sided die?

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Station 4

Work within your group to answer the following questions.

A car manufacturer is determining the average miles per gallon of gas used by an automobile on a highway. The manufacturer records the average miles per gallon, while driving an average of 65 miles per hour, in 35 different vehicles. The data is recorded below.

Car #	Average mpg	Car #	Average mpg	Car #	Average mpg
1	26.7	13	30	25	31.9
2	31.8	14	24.8	26	27.9
3	29.3	15	22.5	27	28.2
4	25.4	16	27.5	28	29
5	24.7	17	24.7	29	23.3
6	27.5	18	32.4	30	30.5
7	28.6	19	30.8	31	24.9
8	26.8	20	28.1	32	27
9	33.1	21	22.4	33	30
10	32.6	22	30.7	34	29.7
11	24.5	23	32.8	35	33.3
12	30.7	24	34.7		

1. Determine the confidence interval of the population mean with 98% confidence.

continued

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2. If the manufacturer wanted to state that the average mpg, or miles per gallon, rounded to the nearest gallon of its vehicle on the highway was 30 mpg, would this be correct with 99% confidence? Explain.

3. Would this be correct with 90% confidence? Why or why not?

4. Describe the difference between a confidence interval of an identical data set for confidence levels of 99% and 90%.

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APPENDIX

Problem-Based Task Implementation Guides

Introduction

This addendum contains a collection of Implementation Guides (IGs) designed to support the use of the Problem-Based Tasks found throughout this unit.

Each Implementation Guide includes an overview and suggestions for introducing and facilitating the task. These contain specific recommendations for selected Standards for Mathematical Practice (SMPs) to highlight, guidance on addressing common errors and misconceptions associated with the task, and ideas for monitoring and coaching students as they work through the task. Finally, there is guidance for debriefing the task, with special attention to the SMPs and relevant Key Concepts from the instruction, as well as information on suitable extensions and alternate strategies.

Throughout the Implementation Guides, suggestions are provided in a question/prompt format to support your efforts to elicit student thinking and facilitate classroom discourse.

We developed these Implementation Guides for our Problem-Based Tasks at the request of math educators, and with advice and feedback from mathematics supervisors and teachers. Please let us know if they are helpful. Visit us at www.walch.com, follow us on Twitter (@WalchEd), or e-mail suggestions to customerservice@walch.com.

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Implementation Guide

Problem-Based Task 1.1.1: Lily's Lemonade Stand

Common Core State Standard

S–ID.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.★

Task Overview

Focus

How can two lemonade machines be adjusted so that customers consistently receive the advertised amount of lemonade? In this task, students will explore the concept of standard deviation to make decisions about automated lemonade machines.

This activity will provide practice with:

- interpreting the meaning of standard deviation in a context
- comparing and analyzing two standard deviations
- calculating the percentage that lies above or below a given standard deviation
- interpreting results in the context of a problem
- using the results of standard deviation analysis to make decisions

Introduction

This task should be used to explore or to apply the concepts of the normal curve and standard deviation.

Begin by reading the problem and clarifying the meaning of *normal distribution* and *standard deviation*, as defined in the lesson.

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  The concept of standard deviation can seem confusing and difficult to students when they are first exposed to it. Encourage students to persevere and work together to make sense of the problem. Also encourage students to draw the curve and mark in the standard deviations, as this visual representation often helps clarify the situation for students when they are trying to calculate the percentages.

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- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students should be discussing their findings with each other. If students disagree with each other's answers, encourage them to explain their reasoning and kindly critique the reasoning of other students with whom they disagree. By the end of the activity, each student should be able to explain, using precise mathematical language, which machine is better and why.

- **SMP 4:** Model with mathematics.

  Students will be constructing visual mathematical models of this scenario through the use of normal curves. If students attempt to solve the problem without making a sketch, encourage them to try drawing the normal curve to help clarify the situation. If students come up with another model besides drawing the normal curve, make sure they can explain their model and their reasoning to others.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- assuming that all normal distributions have a standard deviation of 1

  Have students clarify that the standard deviations that appear in this problem are 0.10 and 0.05.

- not applying symmetry in a normal distribution to calculate percentages

  Remind students that when they calculate the percentages, they must take into account the symmetry of the normal curve, so that they add the 50% that lies on one side of the center to the percentage determined by the standard deviation.

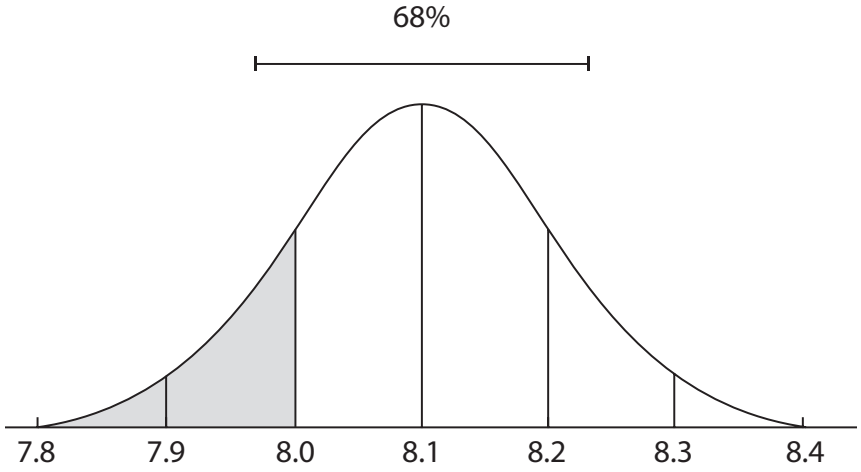
Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students have difficulty starting, suggest that they construct a diagram of a normal curve for each machine, and write in the values that they know.
- Ask what the mean is for each machine. (**Answer:** 8.10 fluid ounces)
- Ask what the standard deviation is for each machine, and what this means. (**Answer:** The standard deviation for Machine A is 0.10 ounces, and the standard deviation for Machine B is 0.05 ounces. This means that 68% of the cups of lemonade dispensed by the machine are within this distance from the mean of 8.10 ounces.)

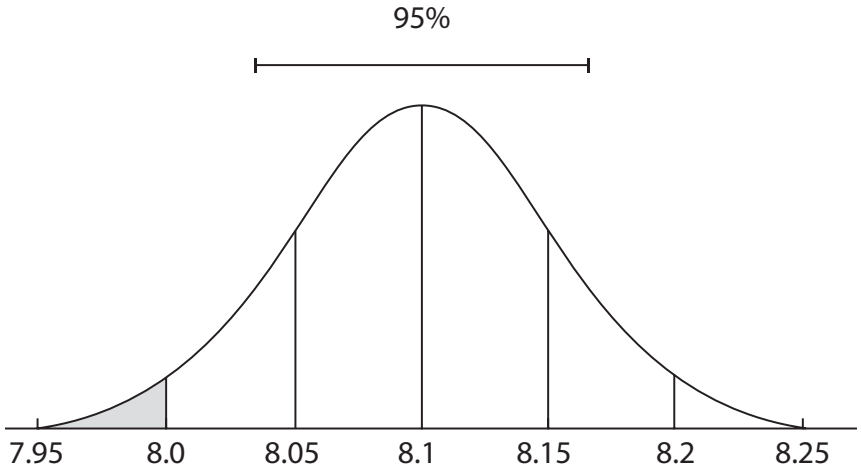
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- If students need help focusing on the requirements of the task, ask them to consider Machine A. What percent of the customers who used this machine will receive less than the advertised 8 ounces of lemonade? (**Answer:** 16%) Ask students to explain how they found this answer. (**Sample answer:** I drew a normal curve, drew in lines for ± 1, 2, and 3 standard deviations, and wrote in the values for the mean and the standard deviations, like this:



I know that 50% of the cups of lemonade are to the left of the mean, and 34% (half of 68%) are between 8 ounces and 8.1 ounces. Therefore, the region to the left of 8 ounces must be 50% – 34% = 16%.)

- Ask students to consider Machine B. What percent of the customers who used this machine will receive less than the advertised 8 ounces of lemonade? (**Answer:** 2.5%) Ask students to explain how they got this answer. (**Sample answer:** I drew a normal curve, drew in lines for ± 1, 2, and 3 standard deviations, and wrote in the values for the mean and the standard deviations, like this:



The percentage of cups of lemonade that have fewer than 8 ounces is represented by the region to the left of 8.0. I know that 50% of the cups of lemonade are to the left of the mean, and 47.5% (half of 95%) are between 8 and 8.1 ounces, so the region to the left of 8.0 is 50% – 47.5% = 2.5%.)

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- Ask students to use this information to calculate what percentage of the cups of lemonade contain 8 ounces or more. (**Answer:** Machine A: 84%; Machine B: 97.5%)
- Ask if this is enough information to determine which machine, at the current settings, is better. (**Answer:** Machine B is better, because 97.5% of the cups it dispenses contain at least 8 ounces of lemonade, while Machine A only meets this criteria 84% of the time.)
- Ask students how they can decide how to change the settings on Machine A so that it dispenses 8 ounces of lemonade 97.5% of the time. (**Answer:** Machine A should be set at 8.2 ounces per serving, so that if it dispenses 2 standard deviations below this (2×0.10), the cup will still contain 8 ounces of lemonade.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

After students have completed the task, have volunteers explain how they arrived at their solutions. Students should present their diagrams and calculations along with their verbal explanation. Allow students to ask questions of each other, and make sure to clear up any confusion or remaining misunderstandings.

Connecting to Key Concepts

- Continuous data is a set of values for which there is at least one value between any two values.

  The scenario in this task represents continuous data because the amount of lemonade in each cup can be any number of ounces that fits in the cup, including fractional amounts.
- When using continuous data, we need to assign probabilities to an interval or range of values.

  In this task, for Machine A for example, we determined that 68% of the cups of lemonade contain between 8.0 and 8.2 ounces.
- Normal distributions are symmetric to the mean. This means that 50% of the data is to the right of the mean and 50% of the data is to the left of the mean.

  We used this fact when calculating the percentage of cups of lemonade that contain more than 8 ounces.

Extending the Task

To extend the task, change a parameter in the problem. For example:

- Suppose that the advertised amount of lemonade is 16 ounces. What should each machine be set to so that 97.5% of her customers receive at least 16 ounces? (**Answer:** Machine A: 16.2 ounces; Machine B: 16.2 ounces)

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- Suppose Lily would be content for 84% of her customers to receive 8 ounces or more. What should she set each machine to in order to maximize her profits? (**Answer:** To maximize her profits, she would want to set the machines so that 8 ounces falls one standard deviation below the mean. Thus, Machine A is already set to meet this criterion, and Machine B should be set to 8.05 ounces.)

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Sample answer:** Even though the problem was confusing when I first read it, I re-read it a couple of times and drew a diagram to help me. I also asked my friends questions when I got stuck.)
- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Sample answer:** Yes, I was able to explain how to solve the problem to another student, and when someone made the error of forgetting to add the 50% to the percentage he found, I was able to explain his mistake to him.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Sample answer:** I drew diagrams of the normal curves for each machine, and modeled the situation with these diagrams.)

Alternate Strategies or Solutions

- Students may reach the correct conclusion without drawing the normal curves, by reasoning that Machine B is better than Machine A since they both have the same mean, but the standard deviation for Machine B is less, meaning that most cups of lemonade are closer to the mean than for Machine A. Suggest to these students that they justify their reasoning with diagrams or find a way to prove their answer to someone who doesn't understand the problem.
- Students may calculate the percentages either by adding regions below the curve (such as adding 50% plus 34%), or they may find the region by calculating the smaller region below the curve and subtracting it from 1.

Technology

Students do not need technology for this task.

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Problem-Based Task 1.1.3: White Pines

Common Core State Standard

S–ID.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.★

Task Overview

Focus

How is a data set analyzed to determine if it follows a normal distribution model or not? What are the key components in the proper description of a data set? In this task, students will create a histogram of a data set representing the length of needles of white pine trees, and they will analyze the histogram and determine if the data follows a normal distribution model or not. They will also create a normal probability plot for further analysis of the data distribution.

This activity will provide practice with:

- creating a histogram and a normal probability plot using graphing technology
- analyzing a normal probability plot
- identifying key components of the description of a data set
- determining the mean and standard deviation of a data set
- drawing conclusions based on a histogram and normal probability plot
- classifying a data distribution

Introduction

This task should be used to explore or to apply the skill of using the mean and standard deviation of a data set to fit it to a normal distribution, and analyze a data set to assess if it follows a normal distribution model or not.

Begin by reading the problem and reviewing and clarifying the meaning of *histogram*, *mean*, *normal distribution*, *normal probability plot*, *outlier*, *sample*, and *standard deviation*.

histogram a graphical display where the data is grouped into ranges and then plotted as rectangular bars

mean a measure of center in a set of numerical data, computed by adding the values in a data set and then dividing the sum by the number of values in the data set; denoted asthe Greek lowercase letter *mu*, μ

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normal distribution	a set of values that are continuous, are symmetric to a mean, and have higher frequencies in intervals close to the mean than equal-sized intervals away from the mean
normal probability plot	a graphical technique for assessing whether or not a data set is approximately normally distributed. The data are plotted against a theoretical normal distribution in such a way that the points should form an approximate straight line.
outlier	a value far above or below other values of a distribution
sample	a subset of the population
standard deviation	how much the data in a given set is spread out, represented by s or σ

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for classifying the distribution of the data set as reasonably close to a normal distribution. They will conclude that the a normal distribution is an adequate model for the data set of the lengths of pine needles of white pine trees, based on their analysis of the symmetry of the histogram, as well as the distribution of the data points in the normal probability plot. When presenting their arguments to others, they will explain their analysis of the results, specifically the different methods in which the data was displayed and interpreted, as well as how the mean and standard deviation was used to further support their conclusions.

- **SMP 5:** Use appropriate tools strategically.

  Students will use a calculator or graphing software throughout this task to display and analyze the data points given in the sample of the lengths of needles. They will first display the data with a histogram. Then they will create a normal probability plot to display and analyze the distribution of the data. Finally, they will use the tools to determine the mean and standard deviation of the data, which will help them justify their classification of the data set as an adequate normal distribution model.

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- **SMP 6:** Attend to precision.

  Students will first carefully enter all 30 pine needle lengths into their calculators, or graphing software, to create a histogram and a normal probability plot of the data. They will interpret and analyze the results of the histogram and normal probability plot, specifically noting the symmetry of the histogram, and the proximity of the data points to a straight line in the normal probability plot. Remind students to round the value for the mean to the nearest tenth, to be consistent with the place value of the sample data points. Encourage them to use the correct terminology throughout the task, specifically when referring to the symmetry of the histogram, the mean and standard deviation, and the components involved in classifying a data set as following a normal distribution model.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly inputting data points into a calculator or graphing software

  Remind students to take their time and carefully enter the data points. Encourage them to use a system, such as entering the data points by columns, or by rows. Have them check their calculator screens to make sure they have 30 data points entered.

- adhering too strictly to the rules of thumb for assessing normality

  Remind students to consider and analyze all methods of assessing the distribution of the data set. Also remind them that the data does not have to follow a uniform distribution model, but it can be close to a normal distribution model. Encourage them to use the word "approximation" when applicable.

- incorrectly classifying the two smallest data points, or the two largest data points, as outliers

  Review the definition of outliers with students. Remind them that outliers are values that are either far above or far below the other data points. Have them carefully examine the histogram in order to recognize that the data points on both ends of the histogram are not far above or below the other data values.

- incorrectly classifying the distribution as a perfect normal distribution

  Review the definition of a normal distribution with students. Remind them that a perfect normal distribution is perfectly symmetric, and the mean, median, and mode are all identical. Discuss how the distribution of data values representing pine needle lengths is not perfect, but it is close enough to the model of a normal distribution.

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Implementation Guide

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, discuss the importance of sample size. Ask them, "Why do you think it is helpful to have a large number of data points when trying to assess a distribution? What if Lisa only recorded the lengths of 10 white pine needles instead of 30?" (**Answer:** It is helpful to have a large number of data points, because a larger sample size will allow for a more accurate assessment of a distribution. If Lisa only recorded the lengths of 10 white pine needles, it is possible that her assessment would not be as valid, or accurate, as it would be if she recorded the lengths of 30 needles.)
- Before students begin the task, review measures of center and spread with them. Ask them, "What are the most common measures of center and spread used when describing a data distribution?" (**Answer:** The most common measures of center are mean, median, and mode. The most common measures of spread are range, standard deviation, and variance. Some students may list the interquartile range as a common measure of spread.)
- Before students begin the task, ask them, "What are the advantages of using a calculator or graphing software to create a histogram and a normal probability plot? Could these displays be created without the use of technology?" (**Answer:** Yes, it is possible to create a histogram and normal probability plot of 30 data points without technology. However, the use of a calculator of graphing software is more efficient, and the results will be more accurate and precise, assuming there is not any error in inputting the values.)
- Before students begin the task, discuss the concept of outliers. Ask them, "What if Lisa found a pine needle with a length of 2.7 centimeters. What would you classify this data point as?" (**Answer:** Since a needle with a length of 2.7 centimeters is far below the other values in the sample, this value would be classified as an outlier in the data set.)
- As an extension of the previous question, ask students, "Should a pine needle with a length of 2.7 centimeters should be included in the data set? Why or why not?" (**Answer:** Since there are 30 other data points which range from 6.5 centimeters to 8.5 centimeters, the needle measuring 2.7 centimeters could be eliminated from the set. One outlier in a data set that is this far away from the range of the values is not representative of the entire data set.)
- As students are working on the task and have completed the histogram, ask them, "What can you conclude about the distribution based solely on the histogram? Is it a perfect normal distribution? Why or why not?" (**Answer:** The histogram shows that the distribution is approximately symmetrical. It is not a perfect normal distribution, because the bars on the sides of the tallest bar are not even with each other.)
- As students are using their calculators or graphing software to create a normal probability plot, ask them, "What is the purpose of a normal probability plot in terms of assessing a data distribution?" (**Answer:** A normal probability plot displays the data against a theoretical normal distribution in such a way that the points plotted should form an approximate straight line. The closer the points are to this straight line, or line of best fit, the closer the distribution is to a perfect normal distribution.)

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- Ask students, "Why is more practical to use a calculator to determine the mean and standard deviation, as opposed to calculating these values by hand?" (**Answer:** Since there are 30 data values, it would take a greater amount of time to perform the calculations necessary for determining the mean and standard deviation without the use of a calculator. There is also the possibility of calculation errors when determining these values by hand.)
- Once students have calculated the mean, ask them to interpret the value in terms of the histogram and normal probability plot. Ask them, "The mean is calculated as 7.6 centimeters. Does this make sense with the two displays of the data distribution – the histogram and the normal probability plot? Explain." (**Answer:** On the histogram, the mean value of 7.6 centimeters is represented graphically in the highest bar on the histogram. It would not have made sense for the mean to be represented in one of the bars lower bars on the lower or upper end of the range of values. On the normal probability plot, the mean value is located approximately in the center of the 30 dots, which represent the data values in the sample. Both the histogram and the normal probability plot display the mean in the approximate center of the displays.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial analysis of the data values in the sample set of pine needle lengths. Ask them if they noticed any outliers in the data set. Then ask them to discuss the results of the histogram they created to display the data. Specifically discuss how the shape of the distribution relates to a normal distribution. Discuss why a second display of data can strengthen their conclusion about the data distribution. Ask them to explain how the normal probability plot supports the idea that Lisa's sample set of pine needle length is represented by a normal distribution model. Also discuss the relevance and meaning of the values of the mean and standard deviation. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process for supporting the claim that the lengths of white pine needles in Lisa's sample study plot are close enough to a normal distribution model. Ask them to explain the validity of the histogram and normal probability plot that were created, as well as other evidence used to support their conclusions. Focus on the use of precise mathematical language and clarity, specifically when discussing the description of the data displays in terms of symmetry of the histogram and clustering of the data points near the mean in the normal probability plot. Also focus on the use of correct terminology when discussing the measures of center and spread, specifically the mean and standard deviation.

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Implementation Guide

Connecting to Key Concepts

- Assessing a distribution for normality requires evaluating the distribution's four key components: a sample or population size, a sketch of the overall shape of the distribution, a measure of average (or central tendency), and a measure of variation

  In this task, students will assess a data set with a sample size of 30 by creating a histogram used to display the overall shape of the distribution, they will create a normal probability plot, and they will find the mean and standard deviation of the data set.

- Histograms and other graphs provide more efficient methods to assess the normality of a distribution.

  Students will use a calculator or graphing software to create a histogram as a way to display the data in order to assess the distribution. They will also use their tools to create a normal probability plot, which is another method of displaying the data and assessing the normality of the distribution.

- If a histogram is approximately symmetric with a concentration of values near the mean, then using a normal distribution is reasonable (assuming there are no outliers).

  Students will observe that the histogram is roughly symmetric and the data values are clustered around the mean. They will also note that there are not any outliers in the data set.

- Graphing calculators and computers can be used to construct normal probability plots, which are a more advanced system for assessing normality.

  Students will use a calculator or other graphing software to construct a normal probability plot as another method of assessing the distribution of the data.

- If the points in the normal plot are approximately linear with no systematic pattern of values above and below the line of best fit, then it is reasonable to assume that the data set is normally distributed.

  Students will observe that the data points in the probability plot deviate only slightly from the line of best fit, but overall the plot is roughly linear.

Extending the Task

- To extend the task, provide the students with 5 additional data points to add to Lisa's sample set of white pine needle lengths. Give them the values of 2.8 centimeters, 3.4 centimeters, 4.3 centimeters, 8.7 centimeters, and 9.3 centimeters. Ask them to create a new histogram and a new probability plot with the additional 5 data points. Ask them to work with a partner and determine if the data follows a normal distribution model, based on the new data points. Ask for volunteers to discuss their results, specifically which values are considered outliers and how these outliers affect the mean and standard deviation of the sample. Also ask students to discuss their conclusion about whether the data is represented by a normal distribution model.

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- Another option for extending the task it to ask students to work through the task again, this time using only the first two columns of data points in the sample, for a total of 10 data points. Ask them to create a histogram and normal probability plot with these 10 data points, and then ask them to compare the data displays with the displays from the task when 30 data points were used. Discuss the difficulties and disadvantages of working with a small sample size. Encourage students to use proper terminology when discussing their analysis of the distribution.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining my choices for classifying the distribution of the data set as an adequate normal distribution model. I concluded this for the data set of the lengths of pine needles of white pine trees, based on my analysis of the symmetry of the histogram, as well as the distribution of the data points in the normal probability plot. I also explained my analysis of the results, specifically the different methods in which the data was displayed and interpreted, as well as how the mean and standard deviation was used to further support my conclusions.)
- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** I used a calculator and graphing software strategically to display and analyze the data points given in the sample of the lengths of needles from white pine trees. I first used these tools to display the data with a histogram. Then I created a normal probability plot as another method to display and analyze the distribution of the data. I also used the tools to determine the mean and standard deviation of the data, which provided further evidence for justifying my classification of the data set as a normal distribution model.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision by carefully entering all 30 pine needle lengths into my calculator, or graphing software, in order to create a histogram and a normal probability plot of the data. I interpreted the results of the histogram and normal probability plot, specifically noting the symmetry of the histogram, and the proximity of the data points to a straight line in the normal probability plot. I rounded the value for the mean to the nearest tenth, in order to be consistent with the place value of the data points given from the sample. I also made sure to use correct terminology throughout the task, specifically when I referred to the symmetry of the histogram, the mean and standard deviation, and the components involved in classifying a data set as following a normal distribution model.)

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Alternate Strategies or Solutions

- Some students may choose to display the sample data in another format, such as a frequency table, or a stem and leaf plot. Encourage them to think about the advantages and disadvantages of each display, in terms of the abilities of their calculators and/or graphing software to create displays other than histograms, as well as the limitations of the analysis that can be made with each display.
- Some students may also choose to determine all measures of center and spread to better analyze the data set. They may find the mean, median, mode, range, standard deviation, interquartile range, and variance. Encourage them to assess which of these measures are more appropriate for this data set, and which measures provide the most information in terms of analyzing the distribution.

Technology

Students can use a graphing calculator or other graphing software in this task.

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Problem-Based Task 1.2.1: Song Requests

Common Core State Standards

S–IC.1 Understand statistics as a process for making inferences about population parameters based on a random sample from that population.★

Task Overview

Focus

How can a valid sample be obtained from a population? How can the mean and standard deviation of a sample means be compared to the mean and standard deviation of a population? In this task, students will assess a set of data representing the frequency of the most requested songs over a six-year time period. From this data, they will choose a sample of all combinations of 3 songs from the 6 years, and they will compare the mean and standard deviation of the sample means to the mean and standard deviation of the population.

This activity will provide practice with:

- using graphing technology to determine the mean and standard deviation of a data set
- calculating the possible number of combinations of a sample from a population
- identifying all possibilities of a sample set
- creating a table to display data
- interpreting the absolute value of the difference in means
- comparing values of mean and standard deviation for the sample means and the population

Introduction

This task should be used to explore or to apply the skill of understanding statistics as a process for making inferences about population parameters based on a random sample from that population.

Begin by reading the problem and reviewing and clarifying the meaning of *combination*, *mean*, *population*, *sample*, and *standard deviation*.

combination	a subset of a group of objects taken from a larger group of objects; the order of the objects does not matter, and objects may be repeated
mean	a measure of center in a set of numerical data, computed by adding the values in a data set and then dividing the sum by the number of values in the data set
population	all the people, objects, or phenomena of interest in an investigation
sample	a subset of the population
standard deviation	how much the data in a given set is spread out

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to compare the mean and standard deviation of the sample means of a population to the mean and standard deviation of the population itself. They will accomplish this goal by first determining the number of possible 3-song combinations and using these combinations as the sample from the population of the 6 most requested songs over a six-year period. Students will use a graphing calculator to calculate the mean and standard deviation of the population and the sample means. They will also create a table in order to organize their results and determine which of the sample means represents the best and worst estimates of the population mean.

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they make sense of the goal of the task by recognizing that they will need to first determine an appropriate sample from the population. After they have determined that there are 20 possible combinations of 3 songs from the group of 6 songs, they will organize their results in a table. Students will also reason abstractly as they decide which of the sample means represents the best and worst estimate of the population mean by calculating the absolute value of the differences between each sample mean and the population mean. They will reason quantitatively as they perform the calculations for the means and standard deviations of the sample means and the population, as well as when they determine the absolute value of the difference of each sample mean and the population mean.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure and procedures involved in calculating the mean and standard deviation for both the population and the sample means. They will also apply structure as they create a table of the data in order to organize the results of finding the sample mean for each combination, as well as the absolute value of the difference between the population mean and each sample mean. Students will recognize that the same procedures are applied when calculating the values in the table, as they repeat these same procedures 20 times to represent each of the possible combinations of 3 songs from the list of 6 songs.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly inputting data points into a calculator or graphing software

  Remind students to take their time and carefully enter the data points given in the chart. Have them check their calculator screens to make sure they have all data points entered. Encourage them to perform each calculation twice in order to verify the accuracy of their results.

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- miscalculating the number of possible 3-song combinations from the population of 6 songs

  Have students write the formula for combinations on the top of their paper: ${}_nC_r = \frac{n!}{(n-r)!r!}$. Remind them that n is the total number of items from which to choose, and that r is the number of items chosen. In this problem, the value of n is 6 and the value of r is 3.

- mistakenly concluding that the samples with the largest absolute value of the difference between the population mean and sample mean is the best estimate of the population mean

  Remind students that the closer the sample mean is to the population mean, the better an estimate it is to the actual population mean. The smaller absolute value will represent a closer value to the mean. Therefore, the samples with the smallest absolute value will represent the best estimate of the population mean.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, discuss the terms *population* and *sample*. Ask them, "What is the difference between population and sample?" (**Answer:** The population is all of the subjects, or items, in an investigation or study. A sample is a subset, or part, of the population.)
- Discuss with students the necessity of collecting samples from a population. Ask them, "Why is it often necessary to collect a sample from a population, as opposed to using all members of a population for a study?" (**Answer:** If the population is too large, then it is often too difficult to use the entire population in a study. Sometimes not all members of a population are accessible, and sometimes it is impractical in terms of the amount of time it would take to find all members of a population.)
- Before students begin the task, clarify which parts of the scenario represent the population and which parts represent the sample. Ask them, "What is the population represented in the scenario, and what is the sample that is taken from the population?" (**Answer:** The population is the number of song requests for the most popular song each year for six years at a radio station. The sample is all possible combinations of 3 songs from the list of 6 songs.)
- As students are working through the task and calculating the means and standard deviations, ask them, "What do you notice about the place value of the final results of the means and standard deviations? Why do you think these values are not rounded to the nearest whole number?" (**Answer:** Both the values for the mean and standard deviation are rounded to the nearest hundredth. If the values were rounded to the nearest whole number, there would only be a single digit with which to compare the different statistics. By rounding to the nearest hundredth, there are more digits to compare, which will allow for a better analysis of the sample.)

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- As students are determining which of the sample means is the best estimate of the population mean, ask them, "Explain what it means that the song combination with the least absolute value of the difference between the sample mean and the population mean is the best estimate of the population mean." (**Answer:** The least absolute value indicates that the mean of the sample means is closest to the population mean. Therefore, the sample song combination with the smallest absolute value is most representative of the entire population.)
- As students are completing the task and comparing the mean of the sample means to the mean of the population, ask them, "How can you interpret the fact that the mean of the sample means is the same as the mean of the population? (**Answer:** Since the two means are identical, this indicates that the sample is representative of the entire population and is a valid and reliable sample.)
- Ask students, "How can you interpret the fact that the standard deviation of the sample means is less than the standard deviation of the population? Explain this in the context of the sample size and the population size." (**Answer:** The smaller standard deviation of the sample means indicates that the data values are closer to the mean than the data values of the population. Since the sample size is much smaller than the size of the population, it makes sense for the data values of the population to be spread out more and farther away from the mean, because there is a larger amount of data values in the population.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts about how to accomplish the goal of the task. Ask them to explain how they applied the formula for calculating a combination in order to determine the possible number of 3-song combinations from the group of 6 songs. Discuss the importance of creating a table in order to display the results of 20 combinations, and how the table was used to analyze the results when comparing the mean of each sample to the population mean. Ask students to explain their results when finding the absolute value of the difference between the population mean and each sample mean, and how these results helped them conclude which samples were the best estimates of the population mean. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process for determining whether the sample of 20 combinations of songs represented the entire population. Ask them to explain their thoughts by interpreting the values of the means and standard deviations for both the sample means and the population. Focus on the use of precise mathematical language and clarity, specifically when discussing the analysis of the sample means and the population, in terms of the validity of the sample. Also focus on the use of correct terminology when discussing the formula used for combinations, as well as when explaining the values for the mean and standard deviation.

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Connecting to Key Concepts

Make explicit connections to key concepts:

- A measure of the population that we are interested in is a **parameter**, a numerical value that represents the data in a set.

  In this task, students will determine the values of the parameters of means and standard deviations of an entire population and also a sample of that population. They will compare these values to draw conclusions about how representative the sample is of the population.

- When using a graphing calculator to find standard deviations of data sets, it is important to recognize whether the data set is a population or a sample so that the proper measure of standard deviation is selected.

  Students will use their graphing calculators for the calculations of the means and standard deviations of both the population and the sample means. They will use the proper functions on their calculators in order to differentiate between the two data sets.

- In general, estimates of a population based on data from large samples are more reliable than estimates from small samples.

  Students will be comparing the means and standard deviations of the sample means to the population means. The sample chosen represents combinations of 3 songs chosen from the 6 songs on the list. Each song represents thousands of requests, so the sample size is large enough in order to provide a reliable representation of the population.

Extending the Task

To extend the task, ask students to work through the task again, but this time let all possible samples of size 4 be the sample. Ask them to first determine the number of combinations possible with 4 songs from the group of 6 songs. Once students have calculated that there are 15 possible combinations, ask them to predict whether they think a sample size which is smaller by 5 combinations will result in a significant difference when comparing the means and standard deviations of the sample means and the population, in comparison to the results when 20 possible combinations were used as the sample size. Once students have performed the calculations, ask for volunteers to share their results and their analysis regarding whether a smaller sample size made a significant difference in representing the population or not. As a further extension, ask students to repeat the task with all possible samples of size 5 as the sample. Ask them to discuss how they think a sample of only 5 combinations will represent the entire population.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the task is to compare the mean and standard deviation of the sample means of a population to the mean and standard deviation of the population itself. I accomplished this goal by first determining the number of possible 3-song combinations and using these combinations as the sample from the population of the 6 most requested songs over a six-year period. I used a graphing calculator to calculate the mean and standard deviation of the population and the sample means. I also created a table in order to organize my results and determine which of the sample means represents the best and worst estimates of the population mean.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I made sense of the goal of the task by recognizing that the first step was to determine an appropriate sample from the population. After I determined that there were 20 possible combinations of 3 songs from the group of 6 songs, I organized my results in a table. I also reasoned abstractly as I decided which of the sample means represented the best and worst estimate of the population mean by calculating the absolute value of the differences between each sample mean and the population mean.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning as I performed the calculations for the means and standard deviations of the sample means and the population, and I also determined the absolute value of the difference of each sample mean and the population mean.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** When solving this problem, I made use of structure by applying the structure and procedures involved in calculating the mean and standard deviation for both the population and the sample means. I also applied structure as I created a table of the data in order to organize the results of finding the sample mean for each combination, as well as the absolute value of the difference between the population mean and each sample mean. I recognized that the same procedures were applied when calculating the values in the table, as I repeated these same procedures 20 times to represent each of the possible combinations of 3 songs from the list of 6 songs.)

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Alternate Strategies or Solutions

- Some students may choose to calculate the mean and standard deviation of the population by hand, as opposed to using a calculator. Encourage them to use a calculator also, in order to check their paper-and-pencil calculations.
- Some students may choose to create a bar graph in order to display the data given in the stem of the problem. Remind students that a bar graph may help with visualizing the frequency of the song requests for each of the six years.

Technology

Students can use a graphing calculator or other graphing software in this task containing statistical functions.

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Problem-Based Task 1.2.3: Breakfast and Grades

Common Core State Standard

S–IC.2 Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?*★

Task Overview

Focus

How can random sampling be used effectively to provide analysis on a study involving an entire population? How can specific sampling techniques be performed to ensure fairness to all members of the sample groups? In this task, students will analyze two groups of high school students to draw conclusions about whether or not providing a free, nutritious breakfast helps students learn.

This activity will provide practice with:

- interpreting data given in a table
- calculating selection chances for groups of people
- determining appropriate sampling techniques
- comparing sampling techniques
- analyzing results of a sample
- drawing conclusions about a study

Introduction

This task should be used to explore or to apply the skill of deciding whether a specified model is consistent with results from a given data-generating process, e.g., using simulation, as well as analyzing sampling techniques. Students should already be familiar with the concept of random sampling as well as using a standard deck of 52 cards or a graphing calculator to generate random numbers (no standard).

Begin by reading the problem and clarifying the meanings of the terms *random number generator*, *simple random sample*, and *stratified sample*.

random number generator	a tool used to select a number without following a pattern, where the probability of generating any number in the set is equal
simple random sample	a sample in which any combination of a given number of individuals in the population has an equal chance of selection
stratified sample	a sample chosen by first dividing a population into subgroups of people or objects that share relevant characteristics, then randomly selecting members of each subgroup for the sample

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to analyze a sample of a group of high school students who receive a free, nutritious breakfast each day and compare this to a group of high school students who do not receive a free, nutritious breakfast each day. Students will examine and interpret tables that display data of gender and grades in the two groups. They will also identify the specific sampling technique used and determine whether the samples chosen are reliable and valid. Finally, students will assess the overall results to draw conclusions about whether or not a free, nutritious breakfast influences academic performance in school.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled with statistical sampling that can be interpreted and examined to draw conclusions about an entire population. They will use tables displaying the data obtained from the sample groups of students chosen to justify their conclusions and to explain their analysis of the sampling technique used, as well as the reliability of the samples.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  Students will express regularity in repeated reasoning as they explain and justify their conclusions about whether students learn better when provided with a free and nutritious breakfast each day. They will cite evidence from the tables displaying the grades and genders of the students chosen from the groups that did and did not receive the free breakfasts.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- mistakenly believing that the word *random* in the term *random sampling* means "haphazard, or done quickly without thought"

  Review the definition of random sampling with students. Remind them of the different methods of performing a random sample on a population.

- incorrectly choosing the number of female students with a B average in the comparison group compared with the number chosen for the study group

  Remind students that it is necessary to have the same number of subjects when comparing two groups in a population. Therefore, because 19 female students with a B average were chosen in the study group, then to fairly compare the two groups, 19 female students with a B average should also be chosen in the comparison group.

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- comparing genders in each group as opposed to comparing average academic grades for the two groups

Have students reread the task. Remind them that the goal of the task is to draw conclusions about whether a free, nutritious breakfast helps students' academic performance in general, not specifically whether or not males or females have a better academic performance.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "Without examining the results of the study given in the tables, what are your initial predictions and/or thoughts about whether providing students with a free, nutritious breakfast each day will cause them to have better grades?" (**Answer:** Answers will vary. Ask for volunteers to explain their predictions for either case.)
- As an extension to the previous question, ask students, "If you think students who have a free, nutritious breakfast each day will make better grades, why do you think the breakfast will help them perform better?" (**Answer:** Although student responses may vary, most will note that a nutritious breakfast can provide students with energy, and this energy can lead to better focus and concentration throughout the day. In contrast, students who are hungry and do not receive proper nutrition may be tired and distracted during their school day.)
- Before students begin the task, ask them, "How could the study have been performed such that a larger sample size was taken, in terms of the entire school? What would the two comparison groups be?" (**Answer:** The study could be performed such that the entire population of high school students was used in the study. Then, the 280 students who applied for the free breakfast each day could all be given the free breakfast, as this would be one of the comparison groups. The other group would be the remaining high school students who did not receive a free, nutritious breakfast. For simplification we would assume the school is large enough!)
- As an extension to the previous question, ask students, "Why would it not be practical to use the entire population of high school students in the study?" (**Answer:** Answers will vary. One possible reason it would not be practical is that it might be time-consuming to monitor the academic performance of each high school student. Another reason is because it might not be practical to determine and accurately assess the breakfast habits of all the high school students.)
- As students are working through the task, ask, "Although the study is not comparing the academic performance of males and females, what would be one way to ensure the fairness of the study in terms of males and females?" (**Answer:** One way to ensure fairness would be to sample the exact same number of males and females in both the study group and the comparison group.)

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- Discuss the methods of random number generation. Ask students, "Besides a random number generator, what are some other methods of choosing random items in a sample group for comparison?" (**Answer:** Some other methods are using a deck of playing cards and assigning the value of the cards to a sample item, then shuffling the cards and choosing a card. Other methods include choosing every third item in the list, or every fourth item, etc.)
- Ensure students' understanding of a stratified sample as a sampling technique. Ask them, "How is the sampling technique used in this study classified as stratified sampling?" (**Answer:** The sample chosen from the population of high school students, which is 280 students, is first divided into two subgroups, which are the students out of the 280 sample size who receive a free, nutritious breakfast each day, and the students who do not receive a free, nutritious breakfast. Then, members of each group are randomly selected for the study. This type of sampling technique is known as stratified sampling.)
- Ask students to consider alternate ways to examine and analyze the data presented in the tables. Ask them, "If *all* 280 people who are represented in the tables were considered in the study, what other type of parameter could be used to compare the academic performance of the group of 60 students who received a free breakfast to the group of 220 students who did not receive a free breakfast?" (**Answer:** The percentages, or relative frequencies, of students who received each letter grade could be given to analyze and compare the results of the students who received a free breakfast and those who did not. For example, the percentage of students in the group of students who received free breakfasts and achieved an academic average of an A is $\frac{3}{60} = \frac{1}{20} = 0.05 = 5\%$. The percentage of students in the group of students who did not receive free breakfasts and achieved an academic average of an A is $\frac{20}{220} = \frac{1}{11} \approx 0.09 \approx 9\%$. These percentages could also be calculated for the remaining B, C, and D academic averages, then conclusions could be drawn based on these results.)
- Ask students, "Given that the sample size of students chosen that receive the free, nutritious breakfast is 60, and the sample size of students chosen that do not receive the free, nutritious breakfast is 220, what could be modified so that random number generation was not necessary?" (**Answer:** If the sample sizes were equal for both groups, then random number generation would not be necessary. So if only 60 total students were chosen out of the 220 who do not receive the free, nutritious breakfast, then the sample sizes for the two groups would be equal.)

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- As students are completing the task, ask, "Although this study only focused on the concept of a nutritious breakfast as a factor for better academic performance, what other factors do you think lead to a student having better grades?" (**Answer:** Answers will vary. Some possible factors include the amount of sleep a student receives each night, the amount of extra-curricular activities the student is involved in, and the amount of study time a student spends each day.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial predictions about whether or not high school students who are given a free, nutritious breakfast each day will have better academic performance than students who do not have a free, nutritious breakfast each day. Discuss the reliability and validity of using a stratified sampling technique for this study. Ask students to specifically discuss their assessment of students having an equal chance of being selected for the sample groups. Also discuss ways to ensure a random sample, such as the use of a random number generator. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and analysis for concluding whether providing a free breakfast to students impacts their academic performance. Ask them to explain how they interpreted the data displayed in the tables and used this data to draw their conclusions. Focus on the use of precise mathematical language and clarity, specifically when discussing the analysis of the sampling techniques applied in the study, and the reliability and validity of the study. Also focus on the use of correct terminology when discussing random sampling and stratified sampling, as well as the use of a random number generator.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Simple random samples do not contain sampling bias since, for any sample size, all combinations of population members have an equal chance of being chosen for the sample

  In this task, students will recognize and compare the differences between a simple random sample and a stratified sample, based on the context of the scenario of the task.

- Additional sampling methods include cluster sampling, systematic sampling, and stratified sampling. All of these methods involve random assignment, although none meet the criteria of simple random sampling.

  Students will see how the sampling technique applied to the scenario in the task is an example of a stratified sample.

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- For a **stratified sample**, the population is divided into subgroups so that the people or objects within the subgroup share relevant characteristics. This method involves grouping members of the population by characteristics that may be related to parameters of interest.

  Students will recognize how the overall population in this task was divided into subgroups of students who did and did not receive a free, nutritious breakfast. They will also recognize that the students chosen in the sample shared relevant characteristics.

- The use of a **random number generator** can be helpful in selecting samples. A random number generator is a tool used to select a number without following a pattern, where the probability of generating any number in the set is equal.

  Students will observe that one way to ensure the students with certain characteristics have an equal chance of being selected for the comparison group is to use a random number generator for the results given in the tables.

Extending the Task

- To extend the task, ask students to work together to analyze the results of the sample groups to draw conclusions about whether a free, nutritious breakfast helps males or females have better academic performance. Ask them to write a paragraph justifying their conclusions about which gender benefits more from having a free, nutritious breakfast. Ask for volunteers to share their results and explain their conclusions. Discuss the type of sampling techniques that may provide better results, as well as how sample size may also affect the results. Also discuss methods of ensuring reliable results with this type of study.
- Another option for extending the task is to ask students to work together to determine specific probabilities, based on the results of the study that are displayed in the tables. Ask students to create a two-way table displaying the results. Ask them to create their own sample probabilities based on the gender and grade (i.e., academic average grade) of the students in the two sample groups. Also discuss how relative frequencies can be determined. Ask for volunteers to share their two-way tables, as well as their calculations for their sample probabilities. Discuss any errors that occur in their calculations.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by recognizing that the goal of the task is to analyze a sample of a group of high school students who receive a free, nutritious breakfast each day to a group of high school students who do not receive a free, nutritious breakfast each day. I examined and interpreted tables that displayed data of gender and grades in the two groups. I also identified the specific sampling technique used and determined whether the samples chosen were reliable and valid. I used the results from the tables to draw conclusions about whether a free, nutritious breakfast influences academic performance in school.)

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- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this scenario by recognizing the scenario can be modeled with statistical sampling that could be interpreted and examined to draw conclusions about an entire population. I used tables displaying the data obtained from the sample groups of students chosen to justify my conclusions and explain my analysis of the sampling technique used, as well as the reliability of the samples.)
- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** I looked for and expressed regularity in repeated reasoning by explaining and justifying my conclusions about whether students learn better when they are provided with a free and nutritious breakfast each day. I cited evidence from the tables displaying the grades and genders of the students chosen from the groups that did and did not receive the free breakfasts.)

Alternate Strategies or Solutions

Some students may choose to apply their own method of random number generation to choose the samples from the subgroups. Also, some students may choose to calculate the relative frequencies for each academic average grade in each table for the two groups so that they may have a numerical means in which to compare the two groups. Encourage students to either add another column onto the one used in the task, or else create their own table with additional columns.

Technology

Students can use scientific calculators for this task.

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Problem-Based Task 1.3.1: Does Soda Cause Cancer?

Common Core State Standard

S–IC.3 Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.★

Task Overview

Focus

What types of methods can be used to study the link between soda consumption and cancer? What are the advantages and disadvantages of taking a sample survey, performing an observational study, and conducting an experiment? In this task, students will examine these three methods of data collection, and determine the benefits and drawbacks to each method when applied to an investigation about if the carcinogens in soda can lead to the development of cancer.

This activity will provide practice with:

- identifying methods of data collection
- describing the processes involved in methods of data collection
- comparing methods of data collection
- evaluating benefits and drawbacks of data collection methods
- justifying the choice of preferring one data collection method over other methods

Introduction

This task should be used to explore or apply the skill of recognizing and describing the purposes of and differences among sample surveys, experiments, and observational studies. Students should already be familiar with surveys and the definition of random as it relates to gathering and interpreting data.

Begin by reading the problem and clarifying the meaning of *experiment*, *observational study*, and *sample survey*.

experiment	a process or action that has observable results
observational study	a study in which all data, including observations and measurements, are recorded in a way that does not change the subject that is being measured or studied
sample survey	a survey carried out using a sampling method so that only a portion of the population is surveyed rather than the whole population

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they make sense of the goal of the task by recognizing that they need to evaluate three methods of data collection to draw conclusions about which method is most appropriate for investigating the possible link between soda consumption and cancer. They will examine and analyze the three methods and compare the benefits and drawbacks of each. They will reason abstractly as they draw conclusions about which method is the most practical, most ethical, and promises the best results.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for determining which method of data collection is the most preferred method for investigating the link between soda consumption and cancer. They will examine the benefits and drawbacks of each method, and they will compare these factors when choosing their preferred method of data collection. When presenting their arguments to others, they will explain their analysis of the results of comparing the three methods, specifically which method allows for the most practical and ethical collection of data.

- **SMP 6:** Attend to precision.

  Students will define and describe the processes involved in conducting a sample survey, an experiment, and an observational study. They will evaluate these three methods by analyzing the benefits and drawbacks of each method in terms of the context of the scenario. Encourage them to use the correct terminology throughout the task, specifically when referring to the specifics involved in collecting data when conducting a sample survey, an experiment, and an observational study.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- being unable to differentiate between an experiment and an observational study

  Have students create a table or chart which contains the three methods of data collection discussed in this task. Ask them to briefly describe each method in the chart so that they can differentiate between the methods.

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- thinking that surveys are generally given to all subjects in a population

  Remind students of the limitations of conducting a survey of an entire population, if the population is large. Discuss how a sample survey is often used to obtain data from a portion of a population that can represent the entire population.

- not understanding that to conduct an experiment, at least a portion of the population studied must be subjected to the process, action, or substance being evaluated

  Remind students that conducting an experiment in the context of the scenario of the task would require that people intentionally drink soda. Discuss the ethical issues involved with this, as people could be drinking sodas with carcinogens that could lead to the development of cancer.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them to discuss their initial thoughts and predictions about the goal of the task. Ask them, "If you were given the task of researching the possible link between soda consumption and cancer, how would go about it?" (**Answer:** Answers will vary. Some students may suggest researching types of cancer and possible causes of cancer. Some students may suggest performing experiments with lab rats, as they are commonly used in medical studies.)

- After initially reading through the task, ask students, "If you were preparing to investigate the possible link between soda consumption and cancer, what might you want to know first about carcinogens?" (**Answer:** It might be helpful to research carcinogens to understand what they are, as well as what other types of beverages, or even food, in which they are found. It also might be helpful to research the history of cancer deaths that can be specifically related to carcinogens.)

- As students begin working on the task and examining the data collection methods, ask them, "In all three data collection methods, how can the size of the sample, or number of participants, contribute to a successful investigation? Explain." (**Answer:** The larger the sample size, the more reliable the results. A small sample may not produce results that are represent the entire population of the study.)

- As students are working on the task, ask them, "What age group of people do you think should be studied to obtain the most accurate data? Explain." (**Answer:** Since the development of cancer can be lengthy, it would not be practical to study children. It would be most practical to study either middle-age or older adults, since they would have had years of soda consumption.)

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- As students are analyzing the benefits and drawbacks of the method of conducting an experiment, ensure their understanding of ethical concerns. Ask them, "Why is it unethical to conduct an experiment where participants are encouraged to drink a large amount of soda so they can be compared to a group of participants who did not drink a large amount of soda?" (**Answer:** To conduct an experiment about the possible link of soda consumption to cancer, one group of participants would have to drink large amounts of soda. Since sodas may contain carcinogens, which are harmful, it would be unethical to ask people to knowingly drink a beverage which could possibly harm them.)
- As students are analyzing the benefits and drawbacks of the method of conducting an observational study, ask them, "What do you think are the largest drawbacks of performing an observational study in the context of this scenario? Explain." (**Answer:** Since an observational study would require observing people who drink soda and determining if these people have had cancer, it would be difficult to find this information about without observing people for a long period of time, in terms of their daily lives. It would also be difficult to find out information on their medical history and/or present medical issues without intruding.)
- As an extension to the previous question, ask students, "What type of subjects do you think an observational study might work best for, and why?" (**Answer:** An observational study might work best when the subjects of an investigation are animals. Since animals cannot talk, they cannot provide verbal data, so sometimes observing them is the only method of collecting data. Also, animals are typically not subject to bias, as they can be observed in their natural environments without human interference. Therefore, the data collected from observing animals would be free of bias.)
- Ask students, "What are some specific factors of creating a sample survey that need to be considered to ensure a successful survey?" (**Answer:** A large sample size would be necessary for a survey to be representative of the entire population of interest. Also, the questions created for the survey would need to be neutral questions so that participants are not skewed towards one particular outcome.)
- As students are considering the drawbacks to the three methods, ask them, "In the context of this scenario, how could the issue of time constraints be considered a drawback to all three methods? (**Answer:** Since the topic of this study is the possible link between soda consumption and cancer, time is an issue for all methods because the development of cancer can be a lengthy process and it may take people years to show symptoms of cancer, or to be diagnosed with cancer. Therefore, collecting data on the possibility that soda consumption is linked to cancer could involve years.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask for volunteers to discuss their initial thoughts and predictions about which of the data collection methods would be the best overall method to investigate the possible link between soda consumption and cancer. Ask them to explain how they compared the benefits and drawbacks of each method, and how this analysis led to their conclusion about the preferred method of data collection for this scenario. Discuss the ethical concerns that could arise if an experiment was conducted. Also discuss the practical and realistic limitations of conducting an observational study. Ask them to summarize the reasons why a sample survey is the preferred method of data collection. If students did not agree that a sample survey was the preferred method, ask them to present their arguments to the class about their preferred method. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and methods of comparison of the three data collection methods. As they are explaining their analysis of the results of comparing the three methods, focus on the use of precise mathematical language and clarity, specifically when discussing the description of the processes involved in conducting an experiment, taking a sample survey, and performing an observational study. Also focus on the correct use of terms such as "ethical concerns" and "uncontrolled environment" when discussing the benefits and drawbacks of each method.

Connecting to Key Concepts

- An observational study allows information to be gathered without disturbing or impacting the subject(s) at all.

  In this task, students will discuss how an observational study could be used to collect data on people who drink soda.
- Most of the time, observational studies are used when it would be impractical or unethical to perform an experiment.

  Students will recognize and discuss the ethical issues in performing an experiment on people by asking them to consume more soda, as opposed to the more ethical method of conducting an observational study.
- When data on a large population is needed, it is often gathered through a sample survey. A **sample survey** is carried out using a sampling method so that only a portion of the population is surveyed rather than the whole population.

  Students will examine and discuss how a sample survey could be used to collect data on people who drink soda and the possible link to the development of cancer. They will recognize that a sample survey is possibly the most practical method for this type of study.

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- In an experiment, participants are intentionally subjected to some process, action, or substance. The results of the experiment are observed and recorded.

  Students will discuss the advantages and disadvantages of conducting an experiment to investigate whether drinking soda can lead to the development of cancer. They will examine the practical and ethical limitations to this particular data collection method.

Extending the Task

To extend the task, ask students to work with a partner to design a sample survey that they could use to investigate the possible link between soda consumption and cancer. Ask them to write a plan about where and how the sample survey could be taken, as well as the questions they would ask participants of the survey. Encourage students to consider the practical and ethical concerns of the survey, as well as the wording of their questions to ensure neutrality and eliminate bias. Also encourage them to consider the size of their sample population. Once they have completed their sample survey design, ask for volunteers to share their survey plans. Discuss if their surveys are designed so that the results of the data collected would be representative of the entire population. Encourage other students to offer ideas on how to strengthen the sample surveys.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I reasoned abstractly by first recognizing that I needed to evaluate three different methods of data collection to draw conclusions about which method is most appropriate for investigating the possible link between soda consumption and cancer. I examined and analyzed the three methods and compared the benefits and drawbacks of each. I reasoned abstractly as I drew conclusions about which method is the most practical, most ethical, and would allow for the best possible results.)
- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining my process for determining which method of data collection I prefer most. I explained my results by presenting my comparison of the benefits and drawbacks of each method, specifically how I reached the conclusion that a sample survey was the most practical and most ethical method of data collection for this scenario.)

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- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision by first defining and describing the processes involved in conducting a sample survey, an experiment, and an observational study. Then I analyzed the benefits and drawbacks of each method in terms of the scenario. I used the correct terminology when explaining my analysis of the methods, as well as when I explained my conclusion about the preferred method of data collection.)

Alternate Strategies or Solutions

Students may choose a different order in which to explain and analyze the three methods. They may also discuss factors involved in the implementation of each method that are not presented in the task. Encourage students to explain their thought processes when presenting their arguments about the benefits and drawbacks to each method. Some students may choose to create some type of organizational table or chart which displays the three methods. They may choose to put columns in their charts which describe each method, as well as listing the benefits and drawbacks of each. Ask students to discuss how their method of analysis of the three methods helped them draw conclusions about the best method for investigating the link between soda consumption and cancer.

Technology

No technology is needed for this task.

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Implementation Guide

Problem-Based Task 1.4.1: Traffic-Light Camera Survey

Common Core State Standard

S–IC.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.★

Task Overview

Focus

How can data from a sample survey be used to estimate a sample proportion and a standard error for the proportion of the sample population? How large does the sample population need to be to achieve a specific standard error? In this task, students will calculate the minimum number of people who need to be surveyed about whether or not they favor the use of surveillance cameras at traffic lights in a small town, in order to obtain a standard error of the proportion of less than 0.03.

This activity will provide practice with:

- identifying variables in a sample proportion
- calculating sample proportions and standard errors of the proportion
- interpreting standard errors in the context of the scenario
- analyzing a formula to determine methods necessary to achieve a desired value
- creating a table to display and organize data
- determining solution methods for obtaining a specific SEP
- interpreting results of values of the SEP

Introduction

This task should be used to explore or to apply the skill of using data from a sample survey to estimate a population mean or proportion. Students should already be familiar with understanding random sampling and data collection methods such as surveys and random samples.

Begin by reading the problem and clarifying the meaning of *favorable outcome*, *sample population*, *sample proportion*, and *standard error of the proportion*.

favorable outcome	the data sought or hoped for, represented by p; also known as *desirable outcome* or *success*
sample population	a portion of the population; the number of elements or observations in a sample population is represented by n
sample proportion	the fraction of favorable results p from a sample population n; conventionally represented by $\hat{p}$, which is pronounced "p hat"
standard error of the proportion	the variability of the measure of the proportion of a sample, abbreviated SEP

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine the necessary size of a sample population of residents of a small town in order for the standard error of a proportion (SEP) to be less than 0.03. Students will recognize that it will be necessary to calculate the SEP for different trial sample sizes and determine which sample size allows for the SEP to reach the desired value. They will perform the calculations to find the value of the SEP, and they will create a table to organize and display their results. They will interpret and analyze the results to identify the least number of town residents that must be surveyed so that the SEP is less than 0.03.

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they recognize that they will need to perform calculations for the SEP in order to achieve the goal of the task, which is to determine the size of the sample population of the town residents that will result in a desired SEP of the sample population of less than 0.03. They will reason that the size of the sample population must be increased, and not decreased, to decrease the value of the SEP found from the first survey conducted, to a desired value of 0.03. They will reason quantitatively as they perform the necessary calculations of the sample proportion first, and then the SEP. Students will identify and substitute the appropriate values for the variables in the formulas, and they will follow the steps necessary to simplify expressions under the radical in the formula for the SEP. They will also record the results of their calculations in a table of values, and they will use these results to determine the necessary size of the sample population.

- **SMP 6:** Attend to precision.

  Students will apply the formulas for the sample proportion and the standard error of the proportion to accomplish the goal of this task. They will use these formulas to first determine the sample proportion of the original survey, then to determine the SEP for the first survey. They will carefully substitute values for the appropriate variables, such as n, p, and $\hat{p}$, into these formulas. Remind students to apply order of operations as they simplify the expressions under the radical in the formula for the SEP. Encourage them to use the correct terminology throughout the task, specifically when referring to the values of the sample population, the sample proportion, and the SEP.

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Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- forgetting to take the square root of both the numerator and denominator when calculating the standard error of a proportion

  Have students write the formula for the SEP at the top of their papers. Remind them of order of operations, as well as the sequence of steps necessary for simplifying expressions under a radical.

- interpreting favorable outcomes as positive experiences rather than as desirable outcomes

  Remind students that in the context of the scenario, the favorable outcomes are the town residents who are in favor of having traffic surveillance cameras installed at the stoplights in town. In the context of the scenario, remind them that this does not necessarily mean that this outcome is positive, but rather the favorable outcomes are the outcomes that the police chief desired when he conducted the survey.

- incorrectly interpreting the size of the sample population in terms of reaching the desired SEP value

  Remind students that the goal of the task is for the value of the SEP to be less than 0.03; that when the sample size is 246, the SEP is greater than 0.03; and that when the sample size is 247, the SEP is less than 0.03. Have students carefully review the table of values to make this distinction.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, and after they have read through the stem, discuss the context of the scenario. Ask them, "What do you think are some advantages and disadvantages of having traffic surveillance cameras installed at stoplights?" (**Answer:** Answers will vary. One of the advantages of the traffic cameras is that they might cause town residents to obey the stoplights more often and not try to drive through yellow lights. This may reduce traffic accidents. Another advantage is that if people are caught by the cameras for running red lights, the revenue made from ticket fees may benefit the town. Disadvantages of the traffic cameras might be the cost of installation, as well as residents feeling disgruntled by the presence of cameras at the stoplights.)

- Before students begin the task, discuss their initial thoughts and predictions about the sample size of 24. Ask them, "Do you think a sample size of 24 residents of a town is large enough to allow for reliable and valid results of a study? Explain." (**Answer:** Answers may vary. Some students may state that the town's population was not given, so it is possible that there may not be many residents in the town. Some may note that in general, a town will most likely have a population greater than 24, so in the context of the scenario, a larger sample size should be taken.)

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- As students begin working on the task, ask them, "What do the numbers 16 and 24 represent in the context of the scenario?" (**Answer:** The number 24 represents the total number of people surveyed, which is the number of elements in the sample population. The number 16 represents the number of favorable outcomes of the survey.)
- After the students have found the sample proportion as a fraction, ask them, "What other representations of the value of the sample proportion can you use besides a fraction? Explain how $\frac{2}{3}$ represents the sample proportion in the context of the scenario." (**Answer:** Instead of using the fraction $\frac{2}{3}$, we could use the decimal $0.\overline{6}$ or the percentage $\approx 66.7\%$ to represent the sample proportion. In the context of the scenario, the fractional value represents the result of the survey showing that 2 out of every 3 people like the idea of having traffic surveillance cameras installed at stoplights.)
- As students are working through the task, ensure their understanding of the connection between a larger sample size and a smaller standard error. Ask them, "Why does it make sense to increase the size of the sample population to decrease the standard error of the proportion, as opposed to decreasing the size of the sample in order to decrease the standard error?" (**Answer:** The larger the size of the sample taken from a population, the more valid and reliable the results of an experiment will be. Therefore, there will be less error in the validity of the results if a larger sample size is taken.)
- As students are working through the task, ask them, "Why do you think the SEP values in the table are represented with several decimal place values, instead of only decimal place values in the hundredths, since the desired value for the SEP is represented in the hundredths place value?" (**Answer:** If the SEP values were only represented to the hundredths place value, then 4 of the entries in the table would round to 0.03, which would not allow for a precise determination of the necessary size of the sample population.)
- As students are interpreting the results of the SEP values in the chart, ask them, "Without solving for the exact value of n, how could you choose a value for n that would safely meet the requirements of having the SEP value less than 0.03?" (**Answer:** Since the SEP value when the multiplier of the sample population is 10 equals 0.030429, and the value when the multiplier is 11 equals 0.029013, it would be safe to choose the size of the sample population that represents the SEP value that is less than 0.03, which is 264 when the multiplier is 11.)
- Ask students, "Why is it better to overestimate the size of the sample population than to underestimate it? Explain your answer in the context of the scenario." (**Answer:** It is better to have a larger sample population size because then the standard error will be lower. If the police chief surveys too few people, then the desired SEP value will be more than 0.03. If the police chief interviews a few more people than are necessary, then the value of the SEP will be lower than 0.03.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask for volunteers to discuss their initial thoughts about how to accomplish the goal of the task. Ask them to explain why the first calculation was the sample proportion and how this value was applied in the formula for the standard error of the proportion. Discuss how the values for the variables were substituted in the formula, as well as the steps necessary to simplify the expressions under the radical. Ask students to explain their methods for determining the size of the sample population that caused the SEP value to be less than 0.03. Also ask them to explain how creating a table of values for the SEP when there are various multipliers to the sample population helped them achieve the goal of determining the value for the sample size. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process for determining whether the size of the sample population should be increased or decreased to obtain an SEP value of less than 0.03. Also ask them to explain their thoughts and methods of determining the test values of the sample population. Discuss how they interpreted and analyzed the values given in the table, as well as the rationale for solving for the exact value of n, so they could conclude that the size of the sample population needed to obtain an SEP value of less than 0.03 was 247 or more people. Focus on the use of precise mathematical language and clarity when discussing the terms relating to the survey, specifically the sample population and favorable outcomes of the survey. Also focus on the use of correct terminology when discussing the standard error of the proportion formula and what the variables in the formula represent.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Sometimes data sets are too large to measure. When we cannot measure the entire data set, called a population, we take a sample or a portion of the population.

  In this task, students will recognize that in the context of the scenario, it would have been impractical to survey all members of the town to get their opinion on the use of traffic surveillance cameras at stoplights. This is why a sample, or a portion of the town population, was surveyed.

- A **sample population** is a portion of the population. The number of elements or observations in the sample population is denoted by n.

  Students will identify the sample population of the task as the value 24, which represents the number of people who were surveyed in the portion of the town population.

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- The **sample proportion** is the name we give for the estimate of the population, based on the sample data we have. This is often represented by $\hat{p}$, which is pronounced "p hat."

  Students will calculate the sample proportion, based on the number of people in the sample population. They will also calculate the number of people surveyed in the sample population who favored traffic surveillance cameras.

- When expressing a sample proportion, we can use a fraction, a percentage, or a decimal.

  Students will represent the results of their calculations of the sample proportion as a fraction, percentage, and decimal, specifically $\frac{2}{3}$, $0.\overline{6}$, and $\approx 66.7\%$.

- The **standard error of the proportion** (SEP) is the variability of the measure of the proportion of a sample. The formula used to calculate the standard error of the proportion is $\text{SEP} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, where $\hat{p}$ is the sample proportion determined by the sample and n is the number of elements or observations in the sample population.

  Students will apply this formula for the SEP in order to determine the minimum number of people who must be surveyed to obtain an SEP value of less than 0.03.

Extending the Task

To extend the task, ask students to work together to create a new task, modeled after the original task, with new values for the sample population and a new minimum value for the SEP. Once the tasks are created, ask students to trade with a partner so the partner can work to solve the problem. Encourage students to discuss any errors they might encounter in calculations or analysis. Ask for volunteers to share their tasks, and ask them to discuss any alternate solution methods they used.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by recognizing that the goal of the task is to determine the necessary size of a sample population of residents of a small town for the standard error of a proportion to be less than 0.03. I recognized that it was necessary to calculate the SEP for different trial sample sizes and determine which sample size allowed for the SEP to reach the desired value. I performed the calculations necessary to find the value of the SEP, then I created a table to organize and display my results. I interpreted and analyzed the results to identify the least number of town residents that must be surveyed so that the SEP is less than 0.03.)

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- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first made sense of the goal of the task by recognizing that I needed to determine the size of the sample population of the town residents which resulted in a desired SEP of less than 0.03. I reasoned that the size of the sample population must be increased, not decreased, to decrease the value of the SEP I found from the first survey to a desired value of 0.03.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning as I performed the calculations of the sample proportion first, then the SEP. I identified and substituted the appropriate values for the variables in the formulas, and I followed the steps necessary to simplify expressions under the radical in the formula for the SEP. I recorded the results of my calculations in a table of values, and I used these results to determine the necessary size of the sample population.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision by applying the formulas for the sample proportion and the SEP to accomplish the goal of this task. I used these formulas to first determine the sample proportion of the original survey, then to determine the SEP for the first survey. I carefully substituted values for the appropriate variables, such as n, p, and $\hat{p}$, into these formulas. I applied order of operations as I simplified the expressions under the radical in the formula for the SEP. I used correct terminology throughout the task, specifically when referring to the values of the sample population, the sample proportion, and the SEP.)

Alternate Strategies or Solutions

- Some students may choose to represent the value of the sample proportion as a decimal when applying it in the standard error of the proportion formula, instead of the fractional representation of the value used in the task. Encourage them to be careful when converting fractions to decimals, as well as to round their values in a consistent manner when they simplify the expressions under the radical.
- Some students may choose to rearrange the standard error of the proportion formula such that the variable for n is isolated. Remind them that although this may eliminate the trial and error calculations involved in the task, isolating n will require several steps. Encourage them to perform these steps twice to verify their results. Also encourage them to be careful when interpreting the result of 246 when they solve for n, in terms of the context of the problem.

Technology

Students can use scientific calculators.

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Problem-Based Task 1.4.3: Job Competition

Common Core State Standard

S–IC.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.★

Task Overview

Focus

How can statistical data such as standard deviation and standard error of the mean be used to draw conclusions on a population? Which parameters of statistics provide better information in a study or survey? In this task, students will analyze a table of statistical data representing the number of shares and profits of stocks selected by five interns working at a financial company. Students will determine which intern has the best chance of being offered a full-time job at the company based on how much their stock investments have earned.

This activity will provide practice with:

- interpreting data given in a table
- calculating standard error of the mean
- comparing statistical data
- analyzing statistical data in the context of the given real-world scenario
- drawing conclusions based on data analysis

Introduction

This task should be used to explore or to apply the skill of using data from a sample survey to estimate a population mean or proportion, specifically the standard error of the mean. Students should already be familiar with calculating the mean and standard deviation of a sample population.

Begin by reading the problem and clarifying the meaning of *sample population*, *standard deviation*, *standard error of the mean*, and *stock portfolio*.

sample population	a portion of the population; the number of elements or observations in a sample population is represented by n
standard deviation	how much the data in a given set is spread out, represented by s or σ
standard error of the mean	the variability of the mean of a sample; given by SEM = $\frac{s}{\sqrt{n}}$, where s represents the standard deviation and n is the number of elements or observations in the sample population
stock portfolio	a grouping, or compilation, of individual stocks owned by an investor

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine which of five interns working at a financial company will have the best chance of being offered a full-time job, based on how much they have earned with their stock investments. To accomplish this goal, students will analyze the data given in a table representing the investment of stocks for the five interns, specifically the average profit per share purchased, the standard deviation, and the number of shares purchased. Students will also calculate the standard error of the mean for each intern's data, and they will use these values to compare this parameter, as well as the other parameters given in the table, to draw conclusions about which intern is most likely to receive a full-time job from the company.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for determining which intern has the best chance of being offered a full-time job at the financial company where they are currently working. Students will examine and analyze the data given in the table, as well as calculate the standard error of the mean for each intern, using the standard deviation and number of shares purchased. They will compare parameters such as the standard error of the mean, the average profit per share, the standard deviation, and the highest number of shares purchased to draw their conclusions. When presenting their arguments to others, students will explain their analysis of the results of comparing these parameters, as well as how these parameters relate to each other in the context of the scenario.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  Students will express regularity in repeated reasoning as they calculate the standard error of the mean for each intern. They will also demonstrate repeated reasoning as they analyze the parameters of each intern's data for the stock investments, focusing on which parameters are stronger for each intern. Students will draw conclusions on which intern has the strongest stock portfolio based on their reasoning and comparison of the various parameters.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- confusing the formula for standard error of the proportion (SEP) with the formula for the standard error of the mean (SEM)

  Have students write the formula for the standard error of the mean (SEM) at the top of their papers. Remind them that the SEM uses the standard deviation and the number of elements in the sample population in the formula.

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- incorrectly calculating the values for the SEM

  Remind students which variables are represented in the formula for the SEM. Also remind them of order of operations, and that the value of the standard deviation is divided by the value of the square root of the sample size of the population.

- incorrectly concluding that the higher standard deviation represents a stronger parameter

  Review the meaning of standard deviation with students. Remind them that a smaller standard deviation value indicates that the elements in the sample are closer to the mean value, and therefore there is less fluctuation in values.

- drawing conclusions based on only one parameter of data

  Discuss how in the context of the scenario, the analysis of many different parameters is needed to draw valid conclusions about which intern has the strongest stock portfolio. Remind them to compare the average profit per share, the standard deviation, the standard error of the mean, and the number of shares owned when drawing their conclusions.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask, "What factors do you think contribute to an intern's success in earning money for stock investments?" (**Answer:** The factors that contribute to having success with earning money on stock investments include the price per share of the stock, the total number of shares purchased, and how much average profit is made on each share that is purchased.)

- Before students begin the task, review the concept of standard deviation. Ask them, "In your own words, what does the standard deviation of a sample population mean?" (**Answer:** The standard deviation is a value that represents the amount of spread, or variability, in the data, in respect to the mean. In regards to the sample mean, the standard deviation is the average of how far away the data values are from the mean of the sample population.)

- Before students begin the task, ask them to make initial predictions about the data presented in the table. Ask them, "Based on the average profit per share purchased, the standard deviation, and the number of shares purchased for each intern, who do you predict has the best chance of being offered a full-time job at the financial company?" (**Answer:** Answers will vary. Some students may predict that Mae has the best chance at the job because she has the highest average profit per share. Some may predict that William has the best chance because he has the smallest standard deviation and the highest number of shares purchased. Some students may pick Leonard because he made the most money. Some students may conclude that more information is needed to make a valid choice.)

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- As an extension of the previous question, ask students, "Why do you think knowing the value of the standard error of the mean will help provide better information about each intern's stock investments, and why?" (**Answer:** Because the standard error of the mean is the variability, or standard deviation, of the distribution of sample means of a population, this parameter will provide information about how much each intern's profits fluctuate from the norm.)
- As students are working through the task and analyzing the different parameters represented in the table of data, ask them, "Why do you think it is generally more beneficial to the company for an intern to have a larger number of shares invested?" (**Answer:** Investing a large amount of shares into the stock market allows for the potential of greater profits, as opposed to lesser profits if only a few shares of stock are invested.)
- As students are analyzing each intern's data, ask them, "What do you think is the most important factor for the company in terms of deciding which intern should be offered the full-time job? Explain." (**Answer:** Answers may vary. Some students may choose the overall average profit per share purchased, because it means that the intern who had the highest overall average profit has invested stocks that are performing well and making money. Some students may choose the number of shares purchased as the most important factor because, in general, the more shares that are purchased leads to more profit. Some students may simply state that the most important factor is the total amount of money made.)
- Ask students, "Who do you think has the lowest chance of being offered the full-time job, and why?" (**Answer:** Answers may vary. Some students may think Sajeena has the lowest chance of being offered the full-time job because she has the lowest average profit per share purchased, the second-highest standard deviation, and the second-highest SEM. Some students may think Patrick has the lowest chance because he has the least number of shares purchased, and he also has the highest SEM.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts and predictions about which of the interns they think would be the most likely to be offered the full-time job at the financial company, based only on the data presented in the table. Discuss why more information might be needed to determine which intern has the best chance of receiving the full-time job offer. Ask them to explain their calculations of the SEM and how this value was used to further analyze the data. Discuss how each parameter affected the interns' overall strength of their stock investments and money earned from these investments. Also discuss how stock price, number of shares, and average profit per share purchased all work together to create each intern's stock portfolio. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.

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- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and methods of choosing which intern they think has the best chance of being offered a full-time job. Encourage them to explain their methods based on their analysis of the statistical data given in the table, as well as their calculations for the standard error of the mean for each intern. Ask them to explain what each parameter means in regards to the strength or weakness of each intern's investments. Focus on the use of precise mathematical language and clarity, specifically when discussing the variables involved in the formula for the SEM, as well as when discussing the standard deviation.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The **standard error of the mean**, SEM, is a measure of the variability of the mean of a sample.

  In this task, students will calculate the SEM in order to compare the variability of the mean for each intern's sample. They will use this parameter, as well as others, when determining which intern should be offered a full-time job at the financial company.

- **Variability,** or **spread**, refers to how the data is spread out with respect to the mean.

  Students will calculate the SEM as one parameter that examines the variability of the data given in the table representing information about the stock investments of five interns at a financial company.

- The SEM can be calculated by dividing the standard deviation, s, by the square root of the number of elements in the sample, n; that is, SEM = $\frac{s}{\sqrt{n}}$.

  Students will use the given table of data to find the values of the standard deviation and sample size (which is the number of stock shares purchased) for each intern, and substitute these values into the formula to find the value of the SEM.

- When the standard error of the mean is small, or close to 0, then the sample mean is likely to be a good estimate of the population mean.

  Students will compare the values of each intern's SEM and determine which intern has the lowest SEM. They will recognize that William has the lowest SEM, which indicates that he has chosen a large number of shares that have generated profits, with relatively little fluctuation in those profits.

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Extending the Task

- To extend the task, ask students to work together to create a new table of data with the goal of trying to determine values for the standard deviation and the sample size (number of shares purchased) for Leonard, Mae, Patrick, and Sajeena, which will result in a SEM value of equal to or below William's SEM value of 0.007. Remind students to use different values for the standard deviation and sample size for each of the four interns. Ask them to keep the values for the average profit per share purchased the same as in the original task for each intern. Encourage them to examine the relationship between the standard deviation and sample size, in regards to which value needs to increase or decrease in order to reach the desired value for the SEM of 0.007 or lower. Ask for volunteers to share their results and discuss their processes for determining the new values for each intern. Also discuss how the SEM decreases when the standard deviation decreases and the sample size increases.
- Another option for extending the task is to ask students to create a full analysis of the strength of each intern's stock portfolio in the form of a table showing the strengths and weaknesses of each, based on the analysis of the data provided in the task. Ask students to create a rating system, where a "1" indicates the intern most likely to be offered a full-time job and a "5" represents the intern least likely to be offered a full-time job. After students have rated the interns on this scale of 1 to 5, ask for volunteers to share their results. Have them discuss their conclusions and analysis of the data, and allow other students to present questions and counterpoints based on the results.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by recognizing that the goal of the task is to determine which of five interns working at a financial company will have the best chance of being offered a full-time job, based on how much they have earned with their stock investments. To accomplish this goal, I analyzed the data given in a table representing the investment of stocks for the five interns, specifically the average profit per share purchased, the standard deviation, and the number of shares purchased. I also calculated the standard error of the mean for each intern's data, and I used these values to compare this parameter, as well as the other parameters given in the table, to draw my conclusions about the intern most likely to receive a full-time job from the company.)

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- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining my choice for determining which intern had the best chance of being offered a full-time job at the financial company. I explained how I examined and analyzed the data given in the table, and how I calculated the standard error of the mean for each intern, using the standard deviation and number of shares purchased. I also explained how I compared parameters such as the standard error of the mean, the average profit per share, the standard deviation, and the highest number of shares purchased to construct my argument about which intern had the best portfolio and should receive the full-time job offer. When presenting my arguments to others, I explained my analysis of the results I got by comparing these parameters, and how they related to each other in the context of the scenario of the task.)
- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** I looked for and expressed regularity in repeated reasoning by calculating the standard error of the mean for each intern. I also demonstrated repeated reasoning as I analyzed the parameters of each intern's data for stock investments, focusing on which parameters were stronger for each intern. I drew conclusions on which intern had the strongest stock portfolio based on my reasoning and comparison of the various parameters.)

Alternate Strategies or Solutions

Some students may choose to recreate the table of data and extend it by adding an extra column for the SEM. They may also choose to create other columns in order to analyze the "rankings" of the interns in terms of which has the best chance of being offered the full-time job. Some students may choose to calculate other parameters for the data given in the table, such as the sample means of the average profit per share purchased and the sample means of the number of shares purchased. Discuss how these parameters do or do not provide additional information about each intern's chances of being offered the full-time job.

Technology

Students can use scientific calculators for this task.

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Implementation Guide

Problem-Based Task 1.5.1: State Scores Compared

Common Core State Standard

S–IC.5 Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.★

Task Overview

Focus

What statistical parameters can researchers use to determine whether their results of an experiment, or a study, are valid? How can the statistical significance of an experiment be determined? In this task, students will calculate the sample mean and standard deviation of the test scores in a class of 12 students. They will use these values to calculate the *t*-value and analyze the results to determine whether the sample of test scores has statistical significance at a 99% confidence level.

This activity will provide practice with:

- calculating a sample mean and sample standard deviation
- identifying the correct formula to use for the *t*-value
- substituting values into the formula for the *t*-value
- determining the number of degrees of freedom (*df*) of a sample
- interpreting the values given in a *t*-distribution table
- drawing conclusions about statistical significance at a specified confidence level

Introduction

This task should be used to explore or apply the skill of using data from an experiment to compare two treatments, and determining the statistical significance of a sample. Students should already be familiar with calculating a sample mean and standard deviation, as well as reading values from a table.

Begin by reading the problem and clarifying the meaning of *confidence level*, *degrees of freedom*, *p-value*, *statistical significance*, and *t-test*.

confidence level	the probability that a parameter's value can be found in a specified interval; also called *level of confidence*
degrees of freedom (*df*)	the number of data values that are free to vary in the final calculation of a statistic; that is, values that can change or move without violating the constraints on the data
***p*-value**	a number between 0 and 1 that determines whether to accept or reject the null hypothesis

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statistical significance	a measure used to determine whether the outcome of an experiment is likely a result of the treatment being applied, as opposed to random chance
***t*-test**	a procedure to establish the statistical significance of a set of data using the mean, standard deviation, and degrees of freedom for the sample or population

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine whether the sample of class test scores has statistical significance at a confidence level of 99%. Students will first calculate the mean and standard deviation of the sample. Then they will use these results to find the *t*-value, and then the degrees of freedom for the sample. They will use these parameters and find the *p*-value in a *t*-distribution table. Once they have the appropriate *p*-value, they will interpret this value and determine if the value supports the sample having statistical significance at a 99% confidence level.

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they make sense of the goal of the task by interpreting the results of the calculations for the *t*-value, and subsequently the corresponding *p*-value. They will apply their knowledge about statistical significance in order to reach a conclusion about whether or not the sample of 12 test scores has statistical significance at a 99% confidence level. They will reason quantitatively as they perform the calculations for the sample mean, sample standard deviation, and the *t*-value. They will also demonstrate quantitative reasoning as they locate the correct *p*-value in a *t*-distribution table and apply this value to support their conclusion about if the sample is statistically significant at a 99% confidence level or not.

- **SMP 6:** Attend to precision.

  Students will apply the formulas for the sample mean, sample standard deviation, and the *t*-value to accomplish this task. They will use these formulas to determine the mean and standard deviation of the 12 sample test scores, and they will use these values and substitute them in the formula for the *t*-value, as well as substituting in the given value for the population mean. Remind students to take care in entering the data values in their calculators, as well as when rounding the results. Encourage them to use the correct terminology throughout the task, specifically when referring to the values necessary for the *t*-value formula, specifically the sample mean, the sample standard deviation, and the population mean.

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Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- neglecting to input all 12 values when calculating the sample mean or the sample standard deviation

  Encourage students to double check their calculator screens as they are inputting the 12 data values to ensure that all 12 values have been entered. Also encourage them to perform the calculations twice in order to verify their results.

- incorrectly substituting the values into the formula for the *t*-value

  Have students write the formula for the *t*-value at the top of their papers. Ask them to write out what each variable represents; i.e., $\bar{x}$ is the sample mean, μ_0 is the population mean, *s* is the standard deviation of the sample, and *n* is the sample size. Encourage them to double check the value for each variable before substituting them into the formula.

- incorrectly identifying the appropriate *p*-value from the *t*-distribution table

  Have students use a ruler in order to help them read the correct value from the *t*-distribution table. Encourage them to find the value twice to verify their results.

- misinterpreting the *p*-value in terms of determining statistical significance of the sample at a 99% confidence level

  Remind students that in order for the sample to have statistical significance at a 99% confidence level, the *p*-value would have to be less than 1%, or 0.01. Since the *p*-value is greater than 0.01, then the sample is not statistically significant at a 99% confidence level.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them to discuss their initial predictions on whether or not they think the 12 sample test scores will have a statistical significance at a 99% confidence level, based on the information given in the stem of the problem. Ask them, "What are your predictions about the statistical significance of the sample of 12 test scores at a 99% confidence level, based on what you are given in the stem of the problem?" (**Answer:** Answers will vary. Some students may think that the sample size is too small to be statistically significant at such a high level. Some students may think that because there are no outliers in the 12 test scores, then the sample should be statistically significant at a 99% confidence level.)

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- As students begin working on the task and find the mean and standard deviation of the sample, ask them, "Can you draw any conclusions about the mean and standard deviation of the sample? What do you notice about the sample mean in comparison to the population mean?" (**Answer:** The standard deviation of the sample is not a very large number, so the values do not fluctuate a lot from the mean. The sample mean is very close to the population mean, so from a general point of view, the sample of 12 test scores seems to be a valid sample.)
- After students have found the sample mean and standard deviation, ask them, "Why do you think the values for the mean and standard deviation were not rounded to either the nearest tenth, or nearest whole number, but instead both values are given to the nearest thousandths?" (**Answer:** Since both values are used in the formula for the *t*-value, the result of the *t*-value calculation will be more precise and accurate with more decimal places represented for the values in the formula.)
- As an extension to the previous question, ask students to calculate the *t*-value with rounded values of the sample mean and sample standard deviation. Ask them, "What would the *t*-value be if the value for the sample mean was 71, and the sample standard deviation was 3? Do you think the difference between the *t*-values is significant when 71 and 3 are used, as opposed to the values of 71.417 and 3.175 that are used in the task? Explain." (**Answer:** Using values of 71 and 3, the *t*-value is 2.309401, and when the values of 71.417 and 3.175 are used in the task, the *t*-value is 2.63708. Although these values differ by approximately three-tenths, when both values are used in the *t*-distribution table with 11 degrees of freedom, they both still fall between the same values of 0.025 and 0.01.)
- Ensure students' understanding of the term "degrees of freedom". Ask them, "What does it mean in the context of the scenario of the 12 test scores that there are 11 degrees of freedom?" (**Answer:** The degrees of freedom are the number of data values that are allowed to vary before a final statistical parameter is calculated. If a specific mean of test scores was desired, the first 11 test scores can vary. The twelfth score, however, must be a specific value in order to achieve the desired mean value.)
- As students are working through the task, discuss the term "99% confidence level" with them. Ask them, "What does it mean if the sample of 12 test scores has a statistical significance at a 99% confidence level?" (**Answer:** The sample of 12 test scores would be statistically significant at a 99% confidence level if the chance of a given test score occurring randomly is less than 1%. In other words, the probability that the test scores fall within a given interval is 99%.)
- Once students have determined that the sample of 12 test scores does not have statistical significance at a 99% confidence level, ask them, "Do you think that the sample of 12 test scores has statistical significance at a 95% confidence level? Explain." (**Answer:** The sample does have statistical significance at a 95% confidence level, because the *t*-value falls between an interval which has endpoints that are less than 0.05, which is the necessary range for a sample to be statistically significant at a 95% confidence level.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask for volunteers to discuss their initial thoughts about how to accomplish the goal of the task. Ask them to explain why more information is needed to compare a sample of 12 class test scores to the population of state test scores than just the means of each. Discuss what parameters were needed for the calculation of the *t*-value, and discuss what each variable in the formula for the *t*-value represented in the context of the scenario. Ask students to explain how they used the *t*-distribution chart to find the *p*-value, and then discuss how they used the *p*-value in order to draw conclusions about if the sample of test scores was statistically significant at a 99% confidence level or not. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process for determining if the sample of test scores had statistical significance at the 99% level or not. Ask them to use evidence from their calculations of the *t*-value, as well how the *t*-value and the degrees of freedom were used to locate the correct *p*-value in the *t*-distribution chart, in order to justify their conclusion about the statistical significance. Focus on the use of precise mathematical language and clarity, specifically when discussing the variables involved in the formula for the *t*-value, as well as the terminology involved in discussing statistical significance.

Connecting to Key Concepts

- **Statistical significance** is a measure used to determine whether the outcome of an experiment is a result of the treatment being applied, as opposed to random chance.

  In this task, students will perform the calculations necessary in order to obtain the necessary parameters for determining whether or not the sample of test scores is statistically significant at a specific confidence level.
- A ***t*-test** is used to establish the statistical significance of a set of data. It uses the means and standard deviations of samples and populations, as well as another parameter called degrees of freedom.

  Students will calculate the mean and standard deviation of the sample, and the will use these values, as well as the given value of the population mean, in the formula in order to calculate the *t*-value.
- In a data set, the **degrees of freedom (*df*)** are the number of data values that are free to vary in the final calculation of a statistic; that is, values that can change or move without violating the constraints on the data.

  Students will identify the degrees of freedom of the sample of test scores as 11, since the sample size is 12, and the degrees of freedom are found by subtracting 1 from the sample size.

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- With the obtained value of *t*, refer to the *t*-distribution table to find the *p*-value on the line corresponding to the degrees of freedom for the sets.

  Students will use the obtained value of *t* (2.36708), and the degrees of freedom (11), and they will determine the *p*-value by identifying the correct line in the table corresponding to the degrees of freedom.

- A ***p*-value** is a number between 0 and 1, determined from the *t*-distribution table. The *p*-value is used to accept or reject the null hypothesis.

  Students will use the *t*-distribution table to determine the *p*-value. They will then interpret this value in order to determine the statistical significance of the sample at a 99% confidence level.

Extending the Task

To extend the task, ask students to work with a partner in order to accomplish the goal of determining which parameters of the experiment could be changed in order for a sample of test scores to be statistically significant at a 99% confidence level. Encourage them to consider how the sample size and sample standard deviation could change, as well as how the sample size would affect the degrees of freedom. Ask them to focus on the variables involved in the formula for the *t*-value, and ask them to determine how the values for these variables could be altered to obtain a *t*-value that, in combination with the degrees of freedom, would cause the resulting *p*-value to fall within the acceptable range for a 99% confidence level. Encourage them to create a chart of trial and error calculations. Ask for volunteers to share their results and explain their procedures and methods.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by recognizing that the goal of the task is to determine whether or not the sample of class test scores has a statistical significance at a confidence level of 99%. To accomplish this goal, first I calculated the mean and standard deviation of the sample. Then I applied these results and found the *t*-value and the degrees of freedom for the sample. I used these parameters to find the *p*-value in a *t*-distribution table. Once I identified the appropriate *p*-value, I interpreted this value and determined if the value supported the sample having statistical significance at a 99% confidence level.)

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- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first made sense of the goal of the task by interpreting the results of the calculations for the *t*-value, and then the corresponding *p*-value. I applied my knowledge about statistical significance in order to reach a conclusion about whether or not the sample of 12 test scores had statistical significance at a 99% confidence level.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning as I performed the calculations for the sample mean, sample standard deviation, and the *t*-value. I also demonstrated quantitative reasoning when I located the correct *p*-value in a *t*-distribution table and applied this value to support my conclusion about if the sample was statistically significant at a 99% confidence level or not.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision by applying the formulas for the sample mean, sample standard deviation, and the *t*-value in order to accomplish the goal of this task. I used these formulas to first determine the mean and standard deviation of the 12 sample test scores, and then I used these values and substituted them in the formula for the *t*-value, as well as substituting in the given value for the population mean. I used correct terminology throughout the task, specifically when referring to the values necessary for the *t*-value formula, specifically the sample mean, the sample standard deviation, and the population mean.)

Alternate Strategies or Solutions

Some students may choose to represent the data and calculations in a more organized format, such as a table or chart. Some students may choose to calculate the sample mean without their calculators. Encourage students to verify any pencil and paper calculations with their calculators. Some students may choose to round the values of the sample mean and standard deviation. Remind students that values containing several decimal places will allow for more accurate and precise results.

Technology

Students can use graphing calculators for this task.

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Problem-Based Task 1.5.3: A Voice for Our Schools

Common Core State Standard

S–IC.6 Evaluate reports based on data.★

Task Overview

Focus

What are some methods of gathering data through a survey? How can a survey be administered such that the results are as reliable and valid as possible? In this task, students will create a plan for administering a survey to obtain more information about how a district's stakeholders perceive their schools.

This activity will provide practice with:

- defining types of participants in a survey
- creating survey questions
- determining methods of administering a survey
- identifying possible sources of bias of a survey

Introduction

This task will apply the skills of evaluating surveys and identifying types of bias. Students should be familiar with recognizing bias and understanding randomization.

Begin by reading the problem and clarifying the meaning of *bias*, *measurement bias*, *nonresponse bias*, *response bias*, and *voluntary response bias*.

bias	leaning toward one result over another; having a lack of neutrality
measurement bias	bias that occurs when the tool used to measure the data is not accurate, current, or consistent
nonresponse bias	bias that occurs when the respondents to a survey have different characteristics than nonrespondents, causing the population that does not respond to be underrepresented in the survey's results
response bias	bias that occurs when responses by those surveyed have been influenced in some manner
voluntary response bias	bias that occurs when the sample is not representative of the population due to the sample having the option of responding to the survey

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to create a plan for gathering data through a survey about how the stakeholders in a district perceive their schools. To accomplish this goal, students will first create survey questions. Then they will determine possible methods to administer the survey in order for the results to be reliable and valid. Finally, they will identify and explain types of possible bias that could occur in their survey.

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they make sense of the goal by creating a plan in order to gather data through the use of a survey. They will recognize and consider the participants in the survey, which are the stakeholders in a school district, and they will create survey questions and determine methods of administering the survey with the purpose of eliminating as much bias as possible. They will reason through the different types of bias, and they will adjust their questions and survey methods accordingly.

- **SMP 5:** Use appropriate tools strategically.

  Students will use pencil and paper as they organize and display their action plan for obtaining information from stakeholders in a school district regarding what they think of the schools in their district. They will also discuss tools which might be used to conduct the survey, including paper and pencil and websites, and the tradeoffs.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- not considering members of the community who could be considered stakeholders

  Remind students that stakeholders are any persons or group who have an interest in the school district.

- limiting their survey questions to the ones presented in the task

  Encourage students to think about the questions they might ask about their schools. Also encourage them to think about questions that their parents might ask from their point of view. Remind students that different groups of people will generally have their own viewpoints on topics, depending on their roles and relationships with the schools in their district.

- forgetting all types of possible bias when creating their action plan

  Have students write the four types of bias discussed in the lesson on the tops of their papers. Ask them to think of examples of each type of bias for a scenario different from the one given in the task.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin, review the terms "reliability" and "validity". Ask them, "What do "reliability" and "validity" mean in regards to the results of a survey?" (**Answer:** If the results of a survey are reliable, then the survey should produce similar results if performed over and over again. If the results of a survey are valid, then the survey measures what it intended to measure.)
- As an extension to the previous question, ask students, "In the context of the task, how do the terms "reliability" and "validity" apply to the survey of stakeholders?" (**Answer:** If the survey given to the stakeholders is reliable, then if the same survey were to be administered again, then the results should be similar to the results of the original survey. The survey would be considered valid if the information obtained was what the school district was looking for, in terms of the answers to their specific questions. The school district would be satisfied that they had more information about what the stakeholders perceived about their schools.)
- Ask students to consider other members of the community who could be stakeholders. Ask them, "What types of community members might be considered stakeholders, and could be included in the survey?" (**Answer:** Answers will vary. Some possible community members might be public librarians, business owners, or homeowners. These people might encounter students in the school district on occasions, or their taxes might help pay for the schools, for example.)
- After students have thought about the task, ask them, "What are some survey questions about schools that could be asked?" (**Answer:** Answers will vary. Some questions might include, "What is the graduation rate for the high schools in the district?"; "How do the district's test scores compare to other local districts' test scores?"; "How large are the class sizes in the elementary, middle, and high schools?"; "Do the schools provide support staff for counseling students?")
- Ask students to think about the different methods of distributing the survey. Ask them, "Do you think distributing the survey in written form, or providing a website so that participants can take it online will get a higher response"? Explain some different scenarios for each method." (**Answer:** Answers will vary. Depending on where the written survey is distributed, the response might be good. For example, if a written survey was given to community members as they walked into a town hall meeting, people might be inclined to complete the survey right there. If the written survey was left outside of a grocery store, people might not be inclined to pick it up and fill it out. Providing a website might lend to a better response, as people would be able to complete the survey on their own time. It might also be more efficient and less troublesome, as opposed to having to turn a survey back in somewhere.)

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- As an extension to the previous question, ask students, "Why do you think some participants, such as students and teachers in the schools, might be hesitant to complete a survey about their schools?" (**Answer:** Depending on the format of the survey, if participants are asked to write their responses out in sentences, some students may not provide honest answers because of fear their handwriting could be recognized. Some teachers may also hesitate to share honest responses, since they work for the school district.)
- As students are discussing the types of possible bias, ask them, "What are some ways to prevent nonresponse bias with this survey?" (**Answer:** One way to minimize nonresponse bias is to make sure the survey is distributed in multiple venues. For example, it could be distributed at a town council meeting, to each class in each school, to the offices of politicians and other community members, as well as possibly in the mailboxes of people in the district. Another way to prevent nonresponse bias is to possibly offer an incentive for completing the survey, such as a free ice cream cone at the local ice cream shop, though this might encourage more response from ice cream lovers!.)
- Ask students, "Do you think it is possible to eliminate all types of bias in the context of this scenario? Why or why not?" (**Answer:** Answers will vary. Some students may reason that as long as the survey questions are worded carefully enough such that bias is not present, then they should be without any bias. Some students may reason that because the stakeholders represent members of the community with many different jobs/positions, there will always be some type of bias present with a survey, depending on each stakeholder's relationship within the district.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts about how to accomplish the goal of the task. Ask them to discuss how they decided on the additional survey questions and distribution methods they created. Discuss the advantages and disadvantages of the various distribution methods. Ask students to explain the practical limitations of any of the distribution methods. Also discuss the four types of bias and ask students how they were able to modify their survey questions to eliminate as much bias as possible. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to defend their choices for their survey questions, specifically focusing on how the wording of their questions was designed to eliminate as much bias as possible. Also ask students to explain why they think some distributions are more advantageous than others, and why certain methods have a better chance of garnering a higher response rate than others. Ask them to share any additional thoughts they may have on how to create a better quality survey. Focus on the use of precise mathematical language and clarity, specifically when discussing the four types of bias that could occur in the survey.

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Connecting to Key Concepts

- **Voluntary response bias** occurs when the sample is not representative of the population due to the sample having the option of deciding whether to respond to the survey. This type of bias invalidates a survey due to overrepresentation of people who have strong opinions or strong motivations for responding.

  In this task, students will recognize that voluntary response bias could be present in the survey because stakeholders who have strong opinions, or have strong motivations about the school district, may be more likely to respond than those stakeholders who do not have strong opinions or motivations.

- **Response bias** occurs when responses by those surveyed have been influenced in some manner. For example, if the survey questions are "leading" the respondent to give certain answers, the survey is biased.

  Students will recognize and discuss how response bias could occur in the survey if students taking the survey give only positive comments about their teachers if they know the teachers are collecting the surveys.

- **Measurement bias** occurs when the tool used to measure the data is not accurate, current, or consistent.

  Students will recognize that measurement bias could be present if different survey questions are given to different groups of stakeholders.

- **Nonresponse bias** occurs when the respondents to a survey have different characteristics than nonrespondents, causing the population that does not respond to be underrepresented in the survey's results. People who do not respond may have a reason not to respond other than just not wanting to; for example, people who are working two jobs might not have time for a survey. The omission of this group will cause the data collected to be inaccurate for the population.

  Students will recognize that the timing of the distribution of a survey may introduce nonresponse bias. They will discuss how a survey needs to be distributed at multiple venues and times in order to avoid a large part of the population being underrepresented.

Extending the Task

- To extend the task, ask students to work together to create an organized and thorough action plan for creating and distributing a survey for this task. Tell them that although they may include the survey questions and distribution methods presented in the task, their plan must include three additional survey questions and three additional methods of distribution. Also ask them to list the four types of bias discussed in the task, and then explain how they will attempt to eliminate each in their survey. Ask for volunteers to share their plans, and create a running list of survey questions and distribution methods on the board, so students can see how many different types of questions could be asked in one survey.

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- Another option for extending the task is to ask students to work with a partner to create a new scenario that will need an action plan for obtaining information from a specific population. Provide them with ideas such as wanting to obtain information about students' favorite vacation places, students' favorite places to shop, or even broader topics such as how members of the community feel about community recycling programs. Encourage students to consider topics of interest to them, so that they will create a survey with questions that will help them obtain more information about their topic. Remind them to consider types of bias when creating their survey questions and methods of distributing their surveys. Ask for volunteers to share their scenarios and action plans.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by recognizing that the goal of the task is to create an action plan for gathering data through a survey about how the stakeholders in a district perceive their schools. To accomplish this goal, first I created possible survey questions that could be asked of the stakeholders. Then I determined possible methods that could be used to administer the survey so that the results of the survey would be reliable and valid. Finally, I identified and explained types of possible bias that could occur in the survey which would prevent the results from being reliable and valid.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first made sense of the goal of the task by creating a plan to gather data through the use of a survey. I recognized and considered the participants in the survey, which are the stakeholders in a school district, and I created survey questions and determined methods of administering the survey with the purpose of eliminating as much bias as possible. I reasoned through the different types of bias, and I adjusted my questions and survey methods accordingly.)
- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** I used appropriate tools strategically as I used pencil and paper to organize and display my action plan for obtaining information from stockholders in a school district regarding what they think of their schools. I wrote survey questions which were appropriate and would result in learning how the stakeholders perceive the district's schools. I also wrote down types of possible bias that could occur in this survey, and I reasoned through examples of each type of bias.)

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Alternate Strategies or Solutions

Some students may create different survey questions than the ones presented in the task. Encourage students to think beyond what is discussed in the task. They might also create different survey methods than the ones presented in the task. Discuss the reliability and validity of their survey questions and methods, and remind them to create their questions and survey methods so that potential bias is minimized.

Technology

No technology is needed for this task.

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Implementation Guide

Problem-Based Task 1.6.1: Put Me In, Coach!

Common Core State Standard

S–MD.6 (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).⋆

Task Overview

Focus

How can probabilities be applied to help make fair decisions? What are some methods that can be used to ensure fairness in making a decision? In this task, students will identify and explain five methods that a basketball coach can use to decide which of 6 equally talented basketball players sits out of a basketball game.

This activity will provide practice with:

- applying probabilities to make fair decisions
- identifying different methods of making fair decisions
- explaining different methods of fairness
- analyzing different methods of fairness

Introduction

This task should be used to explore or to apply the skill of using probabilities to make fair decisions. Students should be familiar with calculating probabilities.

Begin by reading the problem and clarifying the meaning of *fair* and *random number generator*.

fair	describes a situation or game in which all possible outcomes have an equal chance of occurring
random number generator	a tool used to select a number without following a pattern, where the probability of generating any number in the set is equal

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for determining five different but fair methods of choosing one basketball player on a team of six to sit out for a game. They will first recognize that because all six players are equally talented, each player should have an equal chance of sitting out for a game. As students present their arguments, they will describe each method used and they will explain why each method gives each player the same chance of sitting out.

- **SMP 5:** Use appropriate tools strategically.

  Students will use pencil and paper as they describe and explain their five different methods. They will display their descriptions of each method in an organized manner. They will write down and explain any other methods they identify as being fair for choosing which player sits out.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  Students will express regularity in repeated reasoning as they explain each method they choose to determine which basketball player out of six sits out of a game. They will apply the repeated reasoning that each method must ensure fairness, such that each player is equally likely to be designated to sit out. They will reason that each method must provide an equal chance of a player being chosen each time the method is used.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly calculating the probability of a player being chosen to sit out of the game

  Remind students that the information in the task states that each of the six players is equally talented. Therefore, each player has a one out of six chance, or a $\frac{1}{6}$ chance, of being selected to sit out of the game.

- forgetting to include the words "equally sized" or "fair die" when describing the different methods

  Discuss the concept of fairness with students in regards to the context of the scenario. Remind them that for each method chosen, they must state that the method contains "equal parts," such as equally-sized spaces on a spinner or a fair die, where the numbers 1–6 are each represented on the die once.

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- assuming that players choosing a card out of a deck of 52 cards, without replacement, and then designating the player with the highest value card to sit out of the game, is a fair method

  Remind students that each method chosen must be fair. If a player chooses a card out of a deck of 52 cards, and does not replace it, then the deck is no longer a fair deck, because a card is missing. Therefore, the probabilities have changed and are no longer the same with each successive player choosing a card out of the deck without replacement.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, discuss the reality of the scenario as presented: that "each player is equally talented." Ask them, "In a real-world scenario, what are some facts or statistics a basketball coach might look at when trying to choose which player should sit out for a game?" (**Answer:** A coach might look at the overall performance of each player in games past. Some useful statistics might be free throw percentages, average number of rebounds per game, or average points scored per game.)
- Before students begin the task, ask them, "Do you think it would be fair for the coach to choose the player to sit out based on the idea of putting the last names of the players in alphabetical order, then choosing the player whose last name is closest to the beginning of the alphabet? Why or why not?" (**Answer:** Answers may vary. Some students may think this is a fair method for making a decision, because a player's last name does not have anything to do with talent. Some students may think this is not a fair method because the player with the last name closest to the beginning of the alphabet may feel as though they cannot control the choice of their last name.)
- As students are working through the task, ask them to consider the efficiency of the methods. Ask them, "Which of the methods do you think might be the most efficient and practical to use, in the context of the scenario? Explain your answer." (**Answer:** Answers will vary. Rolling a fair die might be the most efficient and practical method to use, as the coach could keep a die in his pocket. It takes very little time to roll a die. The other methods might take longer and involve preparation, such as creating a spinner with six equal sections, or writing names down on a piece of paper.)
- After students have examined the five methods described in the task, ask them, "How could each method described in the task be altered such that it would no longer be considered a fair method? Explain." (**Answer:** A die would not be considered fair if it contained the same number twice. A spinner would not be considered fair if the sections were not all the same size. The method of drawing a marble would not be considered fair if there were two or more of the same color of marbles in the bag. The method of writing players' names on paper would not be considered fair if either the pieces of paper were different sizes, or a player's name was written on two pieces of paper. Finally, the method of drawing a card out of a deck of cards would not be considered fair if two players drew the same card from the deck.)

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- Once students have completed the task, ask them to consider other fair methods that Coach Carter could use to choose a player to sit out of a game. Ask them, "What are some other possible fair methods Coach Carter could use to designate which player sits out?" (**Answer:** Coach Carter could use a random number generator to assign each player a number, specifying beforehand that the highest number will sit out. He could also write each player's jersey number on equally-sized pieces of paper and choose one number out of a hat.)
- Ask students to consider how Coach Carter could use these methods for future games. Ask them, "Assuming Coach Carter wants to be fair to all the players when choosing a player to sit out of each game, what should he consider?" (**Answer:** Coach Carter should consider making sure that the same player is not chosen to sit out again until each player has sat out for a game. To do this, he should keep a record of which player sits out each game. After a player has sat out, then that player should not be in the next round of choosing a player to sit out until six games have been played.)
- As an extension to the previous question, ask students, "After a player has been chosen to sit out for one game, then there are five players remaining to choose from to sit out the next game. Which of the methods presented in the task is not really a "good" method to use now?" (**Answer:** Because there are only five players remaining, rolling a fair die would no longer be a good method because it has six possible outcomes. Using the other methods would still be practical, because a new spinner with 5 equally-sized sections could be created, only 5 marbles would be placed in the bag, etc.)
- Although in the context of the task there is only one trial, ask students to think about what would happen if multiple trials of a method were performed. Ask them, "If you were to roll a fair die 30 times, how many times should you expect the number 3 to appear? Explain." (**Answer:** Because the probability of rolling a 3 on a fair die is $\frac{1}{6}$, then theoretically the outcome of a 3 should occur one out of every six rolls. If the die were rolled 30 times, then the outcome of a 3 should occur 5 times, because $\frac{1}{6} = \frac{5}{30}$.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask for volunteers to discuss their initial thoughts about how to accomplish the goal of the task. Ask them to describe and explain each method chosen in detail, in the context of the scenario. Discuss with them why one method might be preferred over the other. Encourage them to discuss their thoughts on the advantages and disadvantages of each method, as well as the practicality and efficiency of using each method in the context of the scenario. Also discuss any different methods they thought of that Coach Carter could use to designate which player will sit out of the game. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process for determining whether each of the five methods described are fair methods. Discuss how each method allows for each student to have an equal chance of sitting out of a game. Encourage students to discuss the details of each method, focusing on terms such as "equally-sized" and "fair." Also focus on the use of precise mathematical language and clarity, specifically when discussing the concept of fairness with each method, as well as ensuring that students apply the word "equal" in the correct context of each method.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Probabilities are often used to make fair decisions.

  In this task, students will recall their prior knowledge about probabilities and use the probability that each player is likely to be selected to sit out of a game in order to justify five different methods for choosing a player.

- Tools such as a fair coin, a numbered die, or a random number generator are often used when determining probabilities.

  Students will recognize and apply the method of choosing a fair die in order to choose which player out of six will sit out of a basketball game. They will recognize that a fair die has numbers 1–6 represented, and that each number has the same chance of being rolled as the other numbers.

- A situation or game is said to be **fair** if all of the possible outcomes have an equal chance of occurring.

  Students will recognize that, because all players are equally talented, a fair method of choosing one player to sit out of a game is required, so that each player has an equal chance of sitting out.

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Extending the Task

- To extend the task, divide students into groups of six. Create as many six-person groups as possible, and let the remaining students create a smaller group. Ask each group to choose a method described in the task, such as rolling a fair die, creating a spinner with six equal sections, drawing six different-colored marbles out of a bag, writing the names of students down on equal-sized papers and drawing one from a hat (or some other object), or drawing a card from a deck of six cards and designating each student with a card.
- Ask the groups to perform an experiment to determine the experimental probability that one particular student is chosen to sit out. Before each group begins its experiment, remind the groups that the theoretical probability of a particular student being chosen is $\frac{1}{6}$. Have them perform their chosen method 12 times, or 12 trials (i.e., roll a die 12 times, spin the spinner 12 times, draw a marble out of the bag 12 times, draw a name out of a hat 12 times, or choose a card out of the six cards 12 times.) Discuss that theoretically, a designated student should be chosen 2 out of 12 times. Ask groups to record the results of their 12 trials. Ask for each group to share its results, and discuss with each group how close its experimental probability was to the theoretical probability. Discuss how increasing the number of trials significantly could lead to having experimental probabilities closer to $\frac{1}{6}$.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by justifying my choices for determining five different but fair methods of choosing one basketball player on a team of six players to sit out for a game. I first recognized that because all six players are equally talented, each player has an equal chance of sitting out for a game. I also described each method used to designate which player sits out, and I explained why each method gives each player the same chance as the other players of sitting out of the game.)
- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** I used appropriate tools strategically by using pencil and paper to describe and explain my five different methods. I displayed my descriptions of each method in an organized manner, such that the explanation showing that each method is a fair method was described. I also wrote down and explained other methods I identified as being fair methods of choosing which player sits out of the game.)

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- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** I looked for and expressed regularity in repeated reasoning when I explained each of the methods I chose to determine which basketball player will sit out. I applied the repeated reasoning that each method must ensure fairness, such that each player is equally likely to be designated to sit out. I also reasoned that each method must provide an equal chance of a player being chosen each time the method is used.)

Alternate Strategies or Solutions

Students may create their own methods, other than the ones presented in the task. Ask them to write their methods down, along with a description and explanation of their methods. Remind students that in order for their method to be fair, each player must have an equally likely chance of being chosen to sit out of the game.

Technology

No technology is needed for this task.

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Common Core State Standards

Mathematics III

Integrated Pathway

Teacher Resource

Unit 2A

WALCH

INTEGRATED MATH

This program was developed and reviewed by experienced math educators who have both academic and professional backgrounds in mathematics. This ensures: freedom from mathematical errors, grade level appropriateness, freedom from bias, and freedom from unnecessary language complexity.

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Standards Correlations

Each lesson in this Integrated Pathway: Mathematics III program was written specifically to address the Common Core State Standards. Each lesson lists the standards covered in all the sub-lessons, and each sub-lesson lists the standards addressed in that particular section. In this section, you'll find a comprehensive list mapping the sub-lessons to the CCSS.

Guide to Common Core State Standards Annotation

As you use this program, you will come across a symbol included with the Common Core standards for some of the lessons and activities. These symbols are explained below.

Symbol: ⋆

Denotes: Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (⋆).

From http://www.walch.com/CCSS/00003

Symbol: (+)

Denotes: College and Career Readiness Standards

Advanced mathematics standards that are required in higher-level courses such as advanced statistics may also be included in lower-level courses. These additional standards are denoted by (+). According to the Common Core State Standards Initiative, "the evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6–8."

From http://www.walch.com/CCSS/00004

Connections to Future Courses

This section provides a map between topics introduced in each unit of this course and subsequent courses where each topic is revisited and built upon.

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CCSS INTEGRATED PATHWAY: MATHEMATICS III STANDARDS CORRELATIONS

Unit 2A: Polynomial Relationships				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Polynomial Structures and Operating with Polynomials**			
	2A.1.1	Structures of Expressions	A–SSE.1a★	U2A-4
	2A.1.2	Adding and Subtracting Polynomials	A–APR.1	U2A-16
	2A.1.3	Multiplying Polynomials	A–APR.1	U2A-26
Lesson 2	**Proving Identities**			
	2A.2.1	Polynomial Identities	A–SSE.1b★ A–SSE.2 A–APR.4	U2A-43
	2A.2.2	Complex Polynomial Identities	N–CN.8 (+) A–SSE.1b★ A–SSE.2 A–APR.4	U2A-59
	2A.2.3	The Binomial Theorem	A–SSE.1a★ A–SSE.1b★ A–SSE.2 A–APR.4 A–APR.5 (+)	U2A-72
Lesson 3	**Graphing Polynomial Functions**			
	2A.3.1	Describing End Behavior and Turns	F–IF.7c★	U2A-96
	2A.3.2	The Remainder Theorem	A–APR.2	U2A-115
	2A.3.3	Finding Zeros	A–APR.3 N–CN.9 (+) F–IF.7c★	U2A-135
	2A.3.4	The Rational Root Theorem	A–APR.3	U2A-153

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Lesson	Sub-lesson number	Title	Standard(s)	Pages
Lesson 4	**Solving Systems of Equations with Polynomials**			
	2A.4.1	Solving Systems of Equations Graphically	A–REI.11★	U2A-178
Lesson 5	**Geometric Series**			
	2A.5.1	Geometric Sequences	A–SSE.4★	U2A-217
	2A.5.2	Sum of a Finite Geometric Series	A–SSE.4★	U2A-239
	2A.5.3	Sum of an Infinite Geometric Series	A–SSE.4★	U2A-267

TOPICS FROM FUTURE COURSES: CCSS INTEGRATED PATHWAY MATHEMATICS III

Math III	Topic introduced	Course/unit where addressed	How addressed
Unit 2A	Interpreting structures of higher-order polynomial expressions; systems of equations, including linear-polynomial, polynomial-polynomial, and absolute value-polynomial systems; sequences as infinite and finite series	Precalculus, Calculus, or Math IV	extended to systems of equations, including larger, more complex systems (e.g., 3×3 and larger systems) using matrices

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Conceptual Activities

Use these interactive open education and/or Desmos resources to build conceptual understanding of mathematical ideas. (*Note*: Activity links will be monitored and repaired or replaced as necessary.)

- Desmos. "Constructing Polynomials."

  http://www.walch.com/ca/01054

  In this activity, students will consider properties of polynomial functions such as end behavior, leading terms, and properties of roots. They will explore connections between those properties and the factored forms of the equations of the polynomials.

- Desmos. "Polygraph: Polynomial Functions."

  http://www.walch.com/ca/01055

  This activity is designed to spark vocabulary-rich conversations about polynomial functions. Key vocabulary terms that may appear in student questions include *degree*, *roots*, *end behavior*, *limit*, *quadrant*, *axis*, *increasing*, *decreasing*, *maximum*, *minimum*, *extrema*, *concave up*, and *concave down*.

- Desmos. "Polynomial Equation Challenges."

  http://www.walch.com/ca/01056

  In this activity, students will create polynomial equations (of degree 2, 3, and 4) to match given zeros and points. Students will explore how the factored form of the equations relates to the zeros and the order of those zeros.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. What is the degree of the polynomial $5x^6 - 2x^4 + x^3 + 10x$?

   a. 1
   b. 5
   c. 6
   d. 10

2. What is the result of $\left(4x^5 - 2x + 8\right) + \left(-7x^2 + 9x + 3\right)$?

   a. $11x^2 + 11x + 11$
   b. $3x^2 + 7x + 11$
   c. $-3x^5 + 7x + 11$
   d. $4x^5 - 7x^2 + 7x + 11$

3. What is the result of $\left(-x^6 + 10x^3 + 4x^2 + 22\right) - \left(5x^6 - 2x^2 + 16\right)$?

   a. $-6x^6 + 10x^3 + 6x^2 + 6$
   b. $4x^6 + 10x^3 + 2x^2 + 6$
   c. $5x^6 + 12x^3 + 4x^2 + 6$
   d. $4x^6 + 10x^3 + 2x^6 + 38$

4. What is the result of $\left(x^3 + 6x\right)\left(2x^2 + 1\right)$?

   a. $2x^5 + x^3$
   b. $2x^5 + 13x^3 + 6x$
   c. $12x^3 + 6x$
   d. $x^3 + 2x^2 + 6x + 1$

5. What is the result of $\left(-3x^4 + x^2 - 1\right)\left(5x^2 + 4\right)$?

   a. $-15x^6 - 12x^4$
   b. $-15x^6 - 7x^4 - x^2 - 4$
   c. $-15x^6 + 17x^4 + 9x^2 + 4$
   d. $2x^4 - 5x^2 - 4$

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Lesson 1: Polynomial Structures and Operating with Polynomials

Instruction

Common Core State Standards

SMP	
1 ✓	2
3 ✓	4 ✓
5	6 ✓
7 ✓	8 ✓

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

A–APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Essential Questions

1. How are terms arranged in a polynomial expression?
2. How can you determine which terms can be combined when simplifying polynomial expressions?
3. What does it mean for an operation to be closed in a system?

WORDS TO KNOW

closure	a system is closed, or shows closure, for a specific operation if the result of the operation is within the system
coefficient	a number that is multiplied by a variable in an algebraic expression
constant term	a term whose value does not change
degree of a one-variable polynomial	the greatest exponent of the variable in a polynomial
descending order	polynomials ordered by the power of the variables, with the greatest power listed first and the constant last
exponential expression	an expression that contains a base raised to a power
factor	one of two or more numbers or expressions that are being multiplied
leading coefficient	the coefficient of the term with the highest power
like terms	terms that contain the same variables raised to the same powers

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Instruction

polynomial	an expression of the form $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, where the coefficients are real or complex numbers, and the variables are raised to integer powers greater than or equal to 0
polynomial function	a function of the general form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, where the coefficients are real or complex numbers, and the variables are raised to integer powers greater than or equal to 0
term	a number, a variable, or the product of a number and variable(s)

Recommended Resources

- Khan Academy. "Multiply Binomials by Polynomials."

  http://www.walch.com/rr/00158

  Users can practice multiplying polynomials at this site. Four possible answers are given as well as options to show hints. If a question is answered incorrectly, users can click "Show solution" to be guided through the steps to find the answer before moving on to the next problem.

- Math Warehouse. "Polynomial Equations."

  http://www.walch.com/rr/00159

  This site defines polynomials and their components, with examples and non-examples.

- Quia. "Battleship for Polynomials—Adding and Subtracting."

  http://www.walch.com/rr/00160

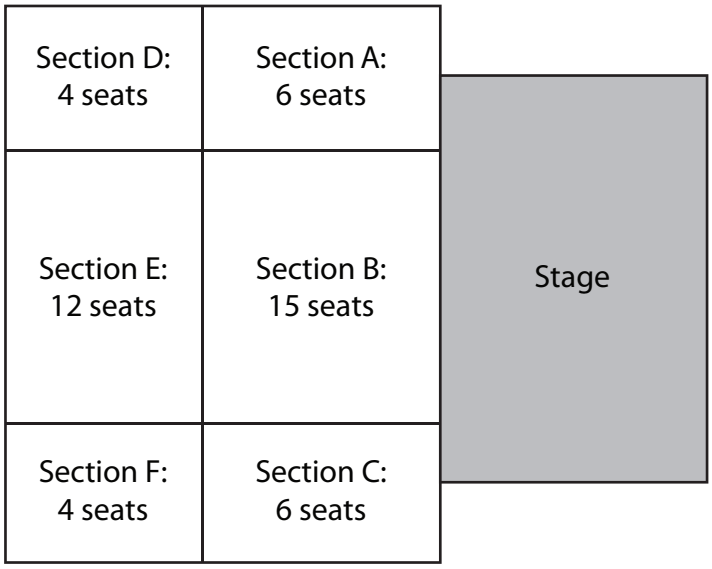
  This site uses the classic game of Battleship to help players practice adding and subtracting polynomials. Competing against the computer, players must answer the problems correctly in order to "hit" a battleship and ultimately sink it. Users may set the computer's skill level to easy, medium, or hard.

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Lesson 2A.1.1: Structures of Expressions

Warm-Up 2A.1.1

A band is holding a small concert. The seating is being sold in sections, and different sections have different numbers of seats. A diagram showing the sections and the number of seats in each section is shown as follows.



1. Mario bought all the seats in sections A and D. How many total seats did he buy?

2. Evan bought all the seats in sections B and C. How many total seats did he buy?

3. Tara is trying to buy enough seats for a total of 18 people. She would like her group to sit in sections E and F, which have all seats available. Will she be able to buy enough seats for her group?

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Instruction

Lesson 2A.1.1: Structures of Expressions

Common Core State Standard

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

Warm-Up 2A.1.1 Debrief

1. Mario bought all the seats in sections A and D. How many total seats did he buy?

   Use the quantities shown in the diagram to find the total seats in sections A and D.

   Section A has 6 seats and Section D has 4 seats.

   The total number of seats bought is the sum of the seats in the two sections: 6 + 4 = 10.

   Mario bought 10 total seats.

2. Evan bought all the seats in sections B and C. How many total seats did he buy?

   Use the quantities shown in the diagram to find the total seats in sections B and C.

   Section B has 15 seats and Section C has 6 seats.

   The total number of seats bought is the sum of the seats in the two sections: 15 + 6 = 21.

   Evan bought 21 total seats.

3. Tara is trying to buy enough seats for a total of 18 people. She would like her group to sit in sections E and F, which have all seats available. Will she be able to buy enough seats for her group?

   Use the quantities shown in the diagram to find the total seats in sections E and F.

   Section E has 12 seats and Section F has 4 seats.

   The total number of seats in sections E and F is 12 + 4 = 16.

   Therefore, there are 16 seats available in these two sections.

   Compare the number of seats available and the number of seats Tara needs.

   Tara's group needs 18 seats.

   $16 < 18$

   16 is less than 18; because there are only 16 seats available, and Tara needs 18 seats, Tara will not be able to buy enough seats in sections E and F for her group.

Connection to the Lesson

- Students will write expressions by writing a sum of terms, including constants.
- Students will write expressions using quantities represented in a diagram.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- writing sums using given quantities (5.OA.2)
- representing unknown quantities with variables (6.EE.2a)

Introduction

Expressions can be used to represent quantities when those quantities are a sum of unknown values. Variables are used to represent the unknown values. Expressions with variables raised to powers are often written in a standard order. This allows expressions to be easily compared.

Key Concepts

- A **factor** is one of two or more numbers or expressions that are being multiplied. In the expression $2a$, 2 and a are factors.
- An **exponential expression** is an expression that contains a base raised to a power. For example, a^2 is an exponential expression: a is the base, and 2 is the exponent. The entire expression, a^2, is the second power of a, or a raised to the second power.
- A **polynomial** is an expression of the form $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, where the coefficients are real or complex numbers, and the variables are raised to integer powers greater than or equal to 0. For example, $b^6 + 2a - 4$ is a polynomial.
- A **term** is a number, a variable, or the product of a number and variable(s). The terms in the polynomial $b^6 + 2a - 4$ are b^6, $2a$, and -4.
- The **degree of a one-variable polynomial** is the greatest exponent of the variable.
- A **polynomial function** is a function written in the following form:

  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, where the coefficients are real or complex numbers, and the variables are raised to integer powers greater than or equal to 0.
- For example, the function $f(x) = 8x^5 + 7x^3 + x^2 + x$ is a polynomial function.
- When a term is the product of a number and a variable, the numeric portion is called the **coefficient** of the term. For example, in the term $8x^5$, the coefficient is 8.
- When the term is just a variable, the coefficient is understood to be 1.

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Instruction

- The terms in a polynomial are ordered by the power of the variables, with the term containing the greatest exponent listed first. This is known as **descending order**. A term whose value does not change, called a **constant term**, is listed after any terms that contain variables. Constant terms can be rewritten as the product of a numeric value and a variable raised to the power of 0. For example, in the polynomial expression $4x^5 + 9x^4 + 3x + 12$, 12 is the constant term, and it can be thought of as $12x^0$.
- The **leading coefficient** of a polynomial is the coefficient of the term with the highest power. For example, in the polynomial expression $4x^5 + 9x^4 + 3x + 12$, the term with the greatest exponent is $4x^5$; therefore, the leading coefficient is 4.

Common Errors/Misconceptions

- incorrectly ordering terms with different powers
- combining terms in an expression that have variables raised to different powers

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Guided Practice 2A.1.1

Example 1

Identify the terms in the expression $5a^2 - a + 7$. What is the highest power of the variable a?

1. Rewrite any subtraction using addition.

   Subtraction can be rewritten as adding a negative quantity.

   $5a^2 - a + 7 = 5a^2 + (-a) + 7$

2. List the terms being added.

   There are three terms in the expression: $5a^2$, $-a$, and 7.

3. Identify the power of the variable a in each term.

   In the first term, $5a^2$, a is raised to the power of 2. In the second term, $-a$, no power is shown. When no power is shown, the power is understood to be 1. Therefore, the power of $-a$ is 1. The third term, 7, is a constant and contains only a numeric value.

4. Determine the highest power of a.

   The highest power of a is 2.

Example 2

Identify the terms in the expression $-2x^8 + 3x^2 - x + 11$, and note the coefficient, variable, and power of each term.

1. Rewrite any subtraction using addition.

   Subtraction can be rewritten as adding a negative quantity.

   $-2x^8 + 3x^2 - x + 11 = -2x^8 + 3x^2 + (-x) + 11$

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2. List the terms being added.

   There are four terms in the expression: $-2x^8$, $3x^2$, $-x$, and 11.

3. Identify the coefficient, variable, and power of each term.

   The coefficient is the number being multiplied by the variable. The variable is the quantity represented by a letter. The power is determined by the exponent of the variable.

   The term $-2x^8$ has a coefficient of -2, a variable of x, and a power of 8.

   The term $3x^2$ has a coefficient of 3, a variable of x, and a power of 2.

   The term $-x$ has a coefficient of -1, a variable of x, and a power of 1.

   The term 11 is a constant; it contains only a numeric value.

Example 3

Write a polynomial function in descending order that contains the terms $-x$, $10x^5$, $4x^3$, and x^2. Determine the degree of the polynomial function.

1. Identify the power of the variable of each term.

   The power of the variable is the value of the exponent of the variable.

   When no exponent is shown, the exponent is understood to be 1. Therefore, the power of the term $-x$ is 1.

   The power of the term $10x^5$ is 5.

   The power of the term $4x^3$ is 3.

   The power of the term x^2 is 2.

2. Order the terms in descending order using the powers of each term.

   List the term with the highest power first, followed by the term with the next highest power, and so on. If there were a constant, it would be listed last.

   $10x^5$, $4x^3$, x^2, $-x$

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Instruction

3. Sum the terms to write the polynomial function of the given variable.

   The variable in each term is x, so this will be a function of x, written $f(x)$. Write $f(x)$ as the sum of the terms in descending order, with the term that has the greatest exponent listed first.

   $f(x) = 10x^5 + 4x^3 + x^2 + (-x)$

   $f(x) = 10x^5 + 4x^3 + x^2 - x$

4. Determine the degree of the polynomial.

   The degree of a polynomial is the greatest exponent.

   The term with the greatest exponent is $10x^5$; the exponent is 5.

   Therefore, the degree of the polynomial function $f(x)$ is 5.

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Problem-Based Task 2A.1.1: Laying Tile

SMP	
1	2
3	4 ✓
5	6 ✓
7 ✓	8

A contractor is creating a design using different-sized rectangular tiles. The area of each tile is shown in the diagram. What is the total area of the shown strip of tile? (Diagram not shown to scale.)

36 in^2	$3x$ in^2	x^2 in^2



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Problem-Based Task 2A.1.1: Laying Tile

Coaching

a. How do you find the total area of the strip of tile?

b. What expression can be written to show the total area of the strip of tile?

c. Typically, in what order are the terms in a polynomial expression written?

d. What is the polynomial expression that shows the total area of the strip of tile?

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Instruction

Problem-Based Task 2A.1.1: Laying Tile

Coaching Sample Responses

a. How do you find the total area of the strip of tile?

To find the total area, add together the areas of each component.

b. What expression can be written to show the total area of the strip of tile?

The areas of each component of the strip of tile are 36 in^2, $3x$ in^2, and x^2 in^2. The total area is the sum of these three smaller areas, or $(36 + 3x + x^2)$ in^2.

c. Typically, in what order are the terms in a polynomial expression written?

In a polynomial expression, the terms are typically listed in descending order based on the powers of the variable; the term with the greatest exponent is listed first, followed by the term with the next greatest exponent, and so on. The constant term is listed last.

d. What is the polynomial expression that shows the total area of the strip of tile?

The term with the highest power is x^2, so it is listed first. This is followed by $3x$, and the constant term, 36, is last.

Therefore, the total area can be represented by the expression $(x^2 + 3x + 36)$ in^2.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2A.1.1: Structures of Expressions

Identify the terms in each expression, and note the coefficient, variable, and power of each term.

1. $y^4 + 13$

2. $8c^3 - c^2 + 8c$

3. $12z^5 + 9z^2 - z - 7$

4. $-5m^{10} + m^8 + 5m^6 - m^4$

Write a polynomial function using the given terms. Determine the degree of each polynomial function.

5. $30x$, x^8, $-3x^3$

6. $14x^2$, $-6x^3$, 10, $-x^6$

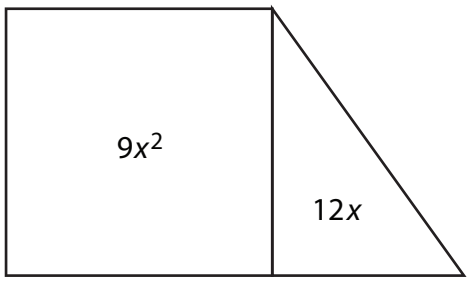
7. $-x^5$, $2x$, 22, x^7, $7x^3$

continued

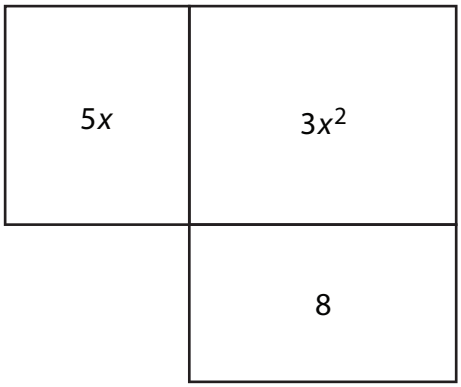
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Each of the figures below is divided into separate parts with each area written within that part. Find the total area of each figure. All units are in square inches.

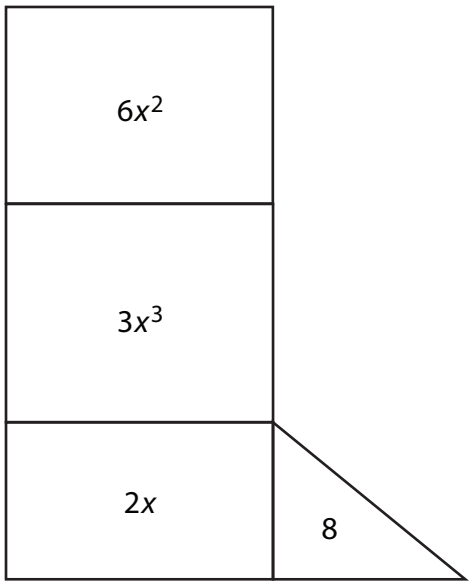
8.



9.



10.

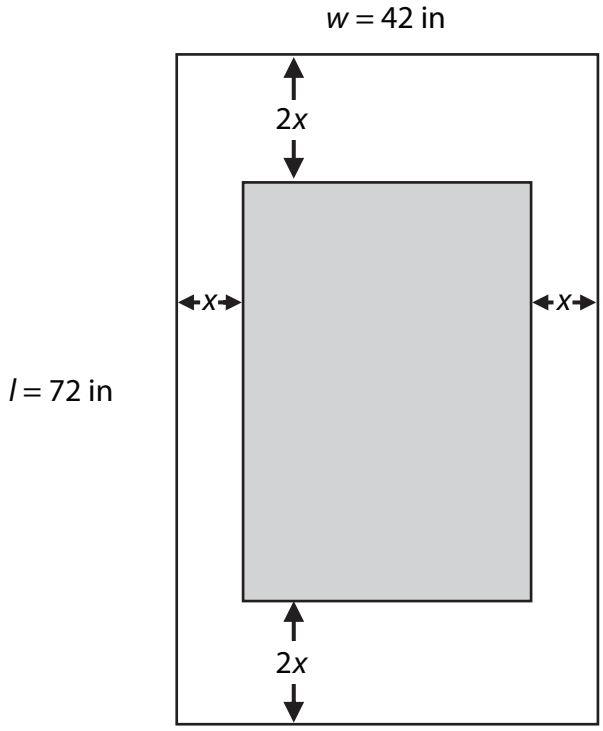


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Lesson 2A.1.2: Adding and Subtracting Polynomials

Warm-Up 2A.1.2

Alyssa is creating a blanket using two types of fabric. The dimensions of the blanket, and the two different fabrics, are shown in the diagram that follows. Alyssa is still deciding on some of the dimensions, and the variable x represents the undecided lengths.



1. The perimeter of a rectangle is the sum of its sides. What will be the perimeter of the completed blanket?

2. Write an expression to show the width of the inner piece of fabric.

3. Write an expression to show the length of the inner piece of fabric.

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Lesson 2A.1.2: Adding and Subtracting Polynomials

Common Core State Standard

A–APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Warm-Up 2A.1.2 Debrief

1. The perimeter of a rectangle is the sum of its sides. What will be the perimeter of the completed blanket?

   The perimeter of the blanket is the perimeter of the larger rectangle. In a rectangle, opposite sides have the same length. Find the sum of the side lengths.

   $42 + 72 + 42 + 72 = 228$

   The perimeter of the completed blanket will be 228 inches.

2. Write an expression to show the width of the inner piece of fabric.

   The width of the blanket, shown by the larger rectangle, is 42 inches. The width of the inner piece of fabric is 42 minus the distance between the inner piece and the left and right edges of the blanket. This distance is shown as x inches to the right and left of the inner piece of fabric. Therefore, the width of the inner piece of fabric is $(42 - x - x)$ inches, or $(42 - 2x)$ inches.

3. Write an expression to show the length of the inner piece of fabric.

   The length of the blanket, shown by the larger rectangle, is 72 inches. The length of the inner piece of fabric is 72 minus the distance between the inner piece and the top and bottom edges of the blanket. This distance is shown as $2x$ inches above and below the inner piece of fabric. Therefore, the length of the inner piece of fabric is $(72 - 2x - 2x)$ inches, or $(72 - 4x)$ inches.

Connection to the Lesson

- Students will write expressions with unknown quantities.
- Students will calculate the perimeter of rectangles.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- using variables to represent unknown quantities (6.EE.2a)
- adding and subtracting real numbers (8.NS.2)
- calculating perimeter (3.MD.8)

Introduction

Polynomials, or expressions that contain variables, numbers, or combinations of variables and numbers, can be added and subtracted. Adding and subtracting polynomials is a way to combine and simplify expressions in order to find a different, but equivalent, representation for a sum or difference.

Key Concepts

- **Like terms** are terms that contain the same variables raised to the same powers. For example, the terms $2a^3$ and $-9a^3$ are like terms because both contain the variable a raised to the third power.
- To add two polynomials, combine like terms by adding the coefficients of the terms. The variable and exponent do not change when adding like terms.
- When two polynomials are added, the result is another polynomial. Because the sum of two polynomials is also a polynomial, the system of polynomials is closed under the operation of addition. A system is closed, or shows **closure**, for a given operation if the result of the operation is within the system.
- For example, the set of integers is closed under the operation of multiplication because the product of two integers is always an integer. However, the set of integers is *not* closed under division, because in some cases the quotient of two integers is not an integer—for example, $3 \div 2 = 1.5$, and 1.5 is not an integer.
- To subtract two polynomials, first rewrite the subtraction using addition by distributing the subtraction to each term in the second polynomial. For example, $a + 1 - (a - 10) = a + 1 + (-a + 10)$. After rewriting using addition, combine like terms.
- Because subtraction of polynomials can be rewritten as addition, and polynomials are closed under the operation of addition, polynomials are also closed under the operation of subtraction.

Common Errors/Misconceptions

- combining two terms with the same variable raised to different powers
- incorrectly changing a subtraction problem to an addition problem by inaccurately distributing the negative sign

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Guided Practice 2A.1.2

Example 1

Simplify $(2x^2 + x + 10) + (7x^2 + 14)$.

1. Rewrite the sum so that like terms are together.

   The first polynomial, $2x^2 + x + 10$, has a term with an exponent of 2, a term with an exponent of 1, and a constant term. The second polynomial, $7x^2 + 14$, has a term with an exponent of 2 and a constant term. The terms with the same exponents and variables are like terms, and the constants are also like terms.

   Rearrange the terms of the given expression in descending order.

   $(2x^2 + x + 10) + (7x^2 + 14)$

   $= 2x^2 + 7x^2 + x + 10 + 14$

2. Find the sum of the constant terms.

   The previous expression contains two constant terms: 10 and 14.

   $2x^2 + 7x^2 + x + 10 + 14$

   $= 2x^2 + 7x^2 + x + 24$

3. Find the sum of any terms with the same variable raised to the same power by adding the coefficients of the terms.

$2x^2 + 7x^2 + x + 24$	Expression from the previous step
$= (2 + 7)x^2 + x + 24$	Combine like terms by adding coefficients.
$= 9x^2 + x + 24$	Simplify.

   $(2x^2 + x + 10) + (7x^2 + 14)$ is equivalent to $9x^2 + x + 24$.

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Instruction

Example 2

Simplify $(6x^4 - x^3 - 3x^2 + 20) + (10x^3 - 4x^2 + 9)$.

1. Rewrite subtraction using addition.

   Subtraction can be rewritten as adding a negative.

   $(6x^4 - x^3 - 3x^2 + 20) + (10x^3 - 4x^2 + 9)$

   $= [6x^4 + (-x^3) + (-3x^2) + 20] + [10x^3 + (-4x^2) + 9]$

2. Rewrite the sum so that like terms are together.

   Be sure to keep any negative signs with the coefficients.

   $[6x^4 + (-x^3) + (-3x^2) + 20] + [10x^3 + (-4x^2) + 9]$

   $= 6x^4 + (-x^3) + 10x^3 + (-3x^2) + (-4x^2) + 20 + 9$

3. Find the sum of the constant terms.

   The previous expression contains two constant terms: 20 and 9.

   $6x^4 + (-x^3) + 10x^3 + (-3x^2) + (-4x^2) + 20 + 9$

   $= 6x^4 + (-x^3) + 10x^3 + (-3x^2) + (-4x^2) + 29$

4. Find the sum of any terms with the same variable raised to the same power.

   The previous expression contains the following like terms: $(-x^3)$ and $10x^3$; $(-3x^2)$ and $(-4x^2)$.

   Add the coefficients of like terms, being sure to keep any negative signs with the coefficients.

$6x^4 + (-x^3) + 10x^3 + (-3x^2) + (-4x^2) + 29$	Expression from the previous step
$= 6x^4 + (-1 + 10)x^3 + [-3 + (-4)]x^2 + 29$	Combine like terms by adding coefficients.
$= 6x^4 + 9x^3 - 7x^2 + 29$	Simplify.

$(6x^4 - x^3 - 3x^2 + 20) + (10x^3 - 4x^2 + 9)$ is equivalent to $6x^4 + 9x^3 - 7x^2 + 29$.

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Instruction

Example 3

Simplify $(-x^6 + 7x^2 + 11) - (12x^6 + 4x^5 - 2x + 1)$.

1. Rewrite the difference as a sum.

   A difference can be written as adding a negative quantity.

   Distribute the negative sign in the second polynomial.

   $(-x^6 + 7x^2 + 11) - (12x^6 + 4x^5 - 2x + 1)$

   $= (-x^6 + 7x^2 + 11) + [(-12x^6) + (-4x^5) + 2x + (-1)]$

2. Rewrite the sum so that like terms are together.

   Be sure to keep any negative signs with the coefficients. Write the terms in descending order.

   $(-x^6 + 7x^2 + 11) + [(-12x^6) + (-4x^5) + 2x + (-1)]$

   $= -x^6 + (-12x^6) + (-4x^5) + 7x^2 + 2x + 11 + (-1)$

3. Find the sum of the constant terms.

   The previous expression contains two constant terms: 11 and –1.

   $-x^6 + (-12x^6) + (-4x^5) + 7x^2 + 2x + 11 + (-1)$

   $= -x^6 + (-12x^6) + (-4x^5) + 7x^2 + 2x + 10$

4. Find the sum of any terms with the same variable raised to the same power.

   The previous expression contains the following like terms: $-x^6$ and $(-12x^6)$.

$-x^6 + (-12x^6) + (-4x^5) + 7x^2 + 2x + 10$	Expression from the previous step
$= [(-1) + (-12)]x^6 + (-4x^5) + 7x^2 + 2x + 10$	Combine like terms by adding coefficients.
$= -13x^6 + (-4x^5) + 7x^2 + 2x + 10$	Simplify.
$= -13x^6 - 4x^5 + 7x^2 + 2x + 10$	Rewrite "$+ (-4x^5)$" as subtraction.

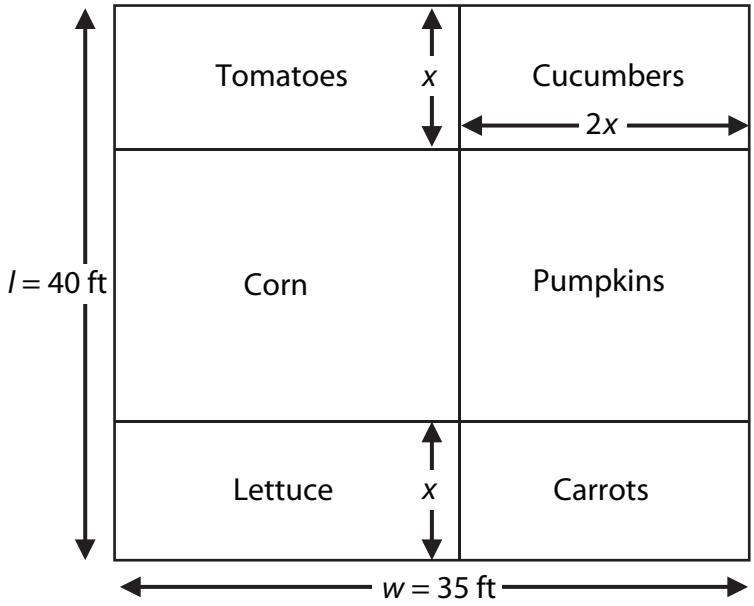
$(-x^6 + 7x^2 + 11) - (12x^6 + 4x^5 - 2x + 1)$ is equivalent to $-13x^6 - 4x^5 + 7x^2 + 2x + 10$.

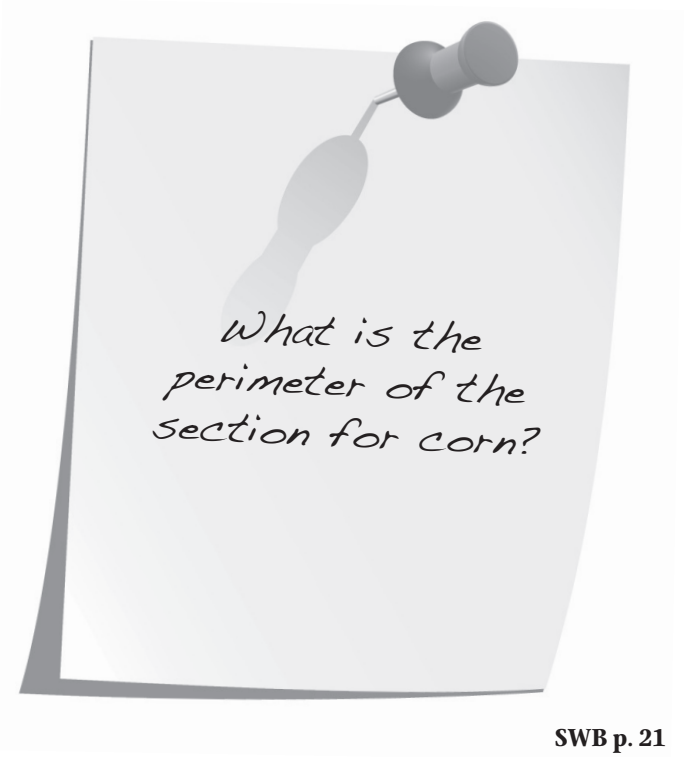
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Problem-Based Task 2A.1.2: Garden Perimeter

SMP	
1 ✓	2
3	4 ✓
5	6
7	8 ✓

Marcus is planting a vegetable garden, and he has already determined what some of the sections of the garden will be. Below is a diagram of the garden, with all units in feet. (Diagram not shown to scale.) Marcus wants to place a fence around the corn. What is the perimeter of the section for corn?





SWB p. 21

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Problem-Based Task 2A.1.2: Garden Perimeter

Coaching

a. What is the difference between the length of the garden and the length of the corn section?

b. What is the length of the corn section, with x representing the unknown dimension?

c. What is the difference between the width of the garden and the width of the corn section?

d. What is the width of the corn section, with x representing the unknown dimensions?

e. What is the perimeter of the corn section?

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Instruction

Problem-Based Task 2A.1.2: Garden Perimeter

Coaching Sample Responses

a. What is the difference between the length of the garden and the length of the corn section?

The total length of the garden is 40 feet. The length of the tomato section, shown above the corn section in the diagram, is x feet. The length of the lettuce section shown below the corn section is also x feet. The corn section is $x + x = 2x$ feet shorter than the length of the garden.

b. What is the length of the corn section, with x representing the unknown dimension?

The length of the corn section is the total garden length minus the lengths of the tomato and lettuce sections: $(40 - 2x)$ feet.

c. What is the difference between the width of the garden and the width of the corn section?

The total width of the garden is 35 feet. The width of the pumpkin section, shown to the right of the corn section, is $2x$ feet. The corn section is $2x$ feet shorter than the width of the garden.

d. What is the width of the corn section, with x representing the unknown dimensions?

The width of the corn section is the total garden width minus the width of the pumpkin section: $(35 - 2x)$ feet.

e. What is the perimeter of the corn section?

The perimeter of the corn section is the sum of the sides. In a rectangle, opposite sides are equal in length, so two of the sides will be equal to the length of the corn section, $(40 - 2x)$ feet, and two of the sides will be equal to the width of the corn section, $(35 - 2x)$ feet.

Sum the four sides.

$(40 - 2x) + (40 - 2x) + (35 - 2x) + (35 - 2x)$

Rearrange the expression so like terms are next to one another.

$= 40 + 40 + 35 + 35 - 2x - 2x - 2x - 2x$

Combine like terms to simplify the expression for the perimeter.

$= 150 - 8x$

Rewrite the polynomial expression with the term containing the variable first and the constant last.

$-8x + 150$

The perimeter of the corn section is $(-8x + 150)$ feet.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2A.1.2: Adding and Subtracting Polynomials

Simplify each expression.

1. $(-x^6 + 10x^3 + 5x + 6) + (12x^3 - 8x + 7)$

2. $(14y^3 + 8y^2 - 8y - 19) + (18y^2 + 5y - 14)$

3. $(20x^4 + x^3 + 18) - (3x^4 + 14x^3 + 11x^2 + 2)$

4. $(11x^5 - 4x^4 + 19x) - (15x^4 + 13x^3 - 6x + 10)$

5. $(-10z^4 + 2z^3 + 14z^2 + 15) + (5z^3 - 17z^2 - 13z + 8)$

6. $(-4x^6 + 3x^4 + 20) - (-17x^6 + 12x^5 - 6x^4)$

7. $(12h^2 - 9h - 15) - (3h^3 + 7h^2 + 8h + 10)$

The perimeter of a rectangle is the sum of its sides. Find the perimeter of a rectangle with the given length and width. All measurements are given in centimeters.

8. length: $x - 5$; width: $x + 10$

9. length: $x^2 + 1$; width: $4x$

10. length: $-8x + 24$; width: $2x + 3$

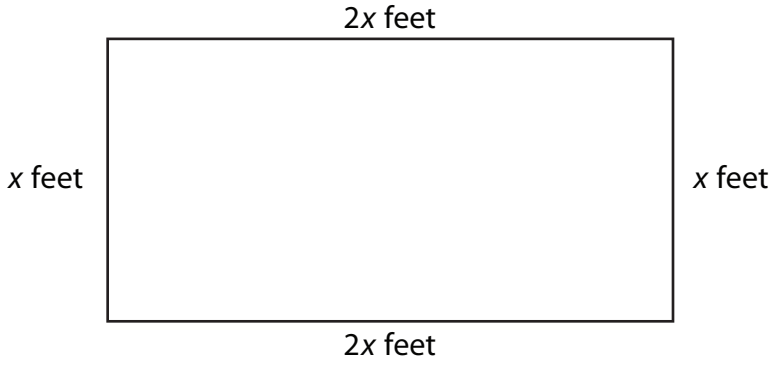
SWB p. 23; SRB Practice 2A.1.2 p. U2A-14

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Lesson 2A.1.3: Multiplying Polynomials

Warm-Up 2A.1.3

Charlie is fencing in a rectangular area of his backyard for a garden, but he hasn't yet decided on the specific dimensions. The diagram below shows the general dimensions of the area Charlie is planning to fence. He knows that the hardware store where he will buy the fencing charges by the foot.



1. Write an expression in terms of x to show the perimeter of the fence.

2. The style of fencing Charlie wants costs \$5 per foot. Use the expression you wrote for the perimeter to find an expression for the total cost of the fencing.

3. If Charlie decides to use 15 feet for the length of x, how much will the entire fence cost?

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Lesson 2A.1.3: Multiplying Polynomials

Common Core State Standard

A–APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Warm-Up 2A.1.3 Debrief

1. Write an expression in terms of x to show the perimeter of the fence.

   The perimeter of a rectangle is the sum of the lengths of its sides.

   $x + 2x + x + 2x = 6x$

   The perimeter of the fence is $6x$ feet.

2. The style of fencing Charlie wants costs \$5 per foot. Use the expression you wrote for the perimeter to find an expression for the total cost of the fencing.

   The perimeter is $6x$ feet. If the fencing Charlie wants costs \$5 per foot, the total cost will be the cost per foot times the perimeter.

   $5 \cdot 6x = 30x$

   The total cost of the fencing is $30x$.

3. If Charlie decides to use 15 feet for the length of x, how much will the entire fence cost?

   Replace x with 15 in the expression for the total cost of the fencing, $30x$.

   $30(15) = 450$

   The entire fence will cost \$450.

Connection to the Lesson

- Students will combine like terms to simplify expressions.
- Students will extend their understanding of multiplication to that of polynomials.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- using variables to represent unknown quantities (6.EE.2a)
- using properties of exponents to simplify expressions (8.EE.1)
- combining like terms to simplify expressions (8.EE.7b)
- calculating areas of rectangles (4.MD.3)

Introduction

To simplify an expression that contains a product of polynomials, such as $(2x + 3)(4x + 5)$, polynomials can be multiplied. Unlike addition and subtraction of polynomials, any two terms can be multiplied, even if the variables or exponents are different. Products of polynomials can be simplified by using the properties of exponents and combining like terms.

Key Concepts

- To multiply two polynomials, multiply each term in the first polynomial by each term in the second polynomial.
- The Distributive Property can be used to simplify the product of two polynomials: $(a + b)(c + d) = (a + b) \bullet c + (a + b) \bullet d = ac + bc + ad + bd$.
- To find the product of two variables with exponents, use the following property of exponents: $x^n \bullet x^m = x^{n+m}$.
- To find the product of a term that contains a variable with a coefficient and a numeric quantity, multiply the coefficient by the numeric quantity; for real numbers a and b, $ax \bullet b = abx$.
- After multiplying all terms, simplify the expression by combining like terms.
- The product of two polynomials is a polynomial, so the system of polynomials is closed under multiplication.

Common Errors/Misconceptions

- forgetting to multiply all terms in each polynomial
- incorrectly simplifying the resulting product
- incorrectly applying the properties of exponents

SRB p. U2A-15

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Guided Practice 2A.1.3

Example 1

Simplify the expression $(x^2 + 3)(x + 6)$.

1. Rewrite the product using the Distributive Property.

   Multiply each term in the first polynomial by each term in the second polynomial.

   $(x^2 + 3)(x + 6) = (x^2 \bullet x) + (3 \bullet x) + (x^2 \bullet 6) + (3 \bullet 6)$

2. Use properties of exponents to simplify the expression.

   The term x is raised to the first power, or x^1.

$(x^2 \bullet x) + (3 \bullet x) + (x^2 \bullet 6) + (3 \bullet 6)$	Expression from the previous step
$= (x^2 \bullet x^1) + (3 \bullet x) + (x^2 \bullet 6) + (3 \bullet 6)$	Substitute x^1 for x when multiplying terms that both have variables.
$= (x^{2+1}) + (3 \bullet x) + (x^2 \bullet 6) + (3 \bullet 6)$	Since $x^n \bullet x^m = x^{n+m}$, add the exponents.
$= x^3 + (3 \bullet x) + (x^2 \bullet 6) + (3 \bullet 6)$	Simplify.

3. Simplify any remaining products.

   The product of a number and a variable is written with the number first, as the coefficient of the variable.

$x^3 + (3 \bullet x) + (x^2 \bullet 6) + (3 \bullet 6)$	Expression from the previous step
$= x^3 + 3x + 6x^2 + 18$	Simplify.
$= x^3 + 6x^2 + 3x + 18$	Rewrite in descending order.

   The expression $(x^2 + 3)(x + 6)$ is equivalent to $x^3 + 6x^2 + 3x + 18$.

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Example 2

Simplify the expression $(-5x + 2)(3x^2 - x + 4)$.

1. Rewrite the product using the Distributive Property.

   Multiply each term in the first polynomial by each term in the second polynomial.

   $(-5x + 2)(3x^2 - x + 4)$

   $= (-5x \bullet 3x^2) + (-5x \bullet -x) + (-5x \bullet 4) + (2 \bullet 3x^2) + (2 \bullet -x) + (2 \bullet 4)$

2. Use properties of exponents and multiplication to simplify the expression.

   Start with the expression from the previous step.

   $(-5x \bullet 3x^2) + (-5x \bullet -x) + (-5x \bullet 4) + (2 \bullet 3x^2) + (2 \bullet -x) + (2 \bullet 4)$

   Substitute x^1 for x when multiplying terms that both have variables.

   $= (-5x^1 \bullet 3x^2) + (-5x^1 \bullet -x^1) + (-5x \bullet 4) + (2 \bullet 3x^2) + (2 \bullet -x) + (2 \bullet 4)$

   Since $x^n \bullet x^m = x^{n+m}$, add the exponents of any two terms that are being multiplied that contain variables. Remember that the product of a number and a variable is written with the number first, as the coefficient of the variable.

   $= [(-5)(3)x^{1+2}] + [(-5)(-1)x^{1+1}] + (-5x \bullet 4) + (2 \bullet 3x^2) + (2 \bullet -x) + (2 \bullet 4)$

   Simplify.

   $= [(-5)(3)x^3] + [(-5)(-1)x^2] + (-5x \bullet 4) + (2 \bullet 3x^2) + (2 \bullet -x) + (2 \bullet 4)$

   $= -15x^3 + 5x^2 + (-20x) + 6x^2 + (-2x) + 8$

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3. Combine like terms.

   Only terms with the same variable raised to the same power can be combined. Write the terms in descending order according to the power of each term, from highest power to lowest.

$-15x^3 + 5x^2 + (-20x) + 6x^2 + (-2x) + 8$	Expression from the previous step
$= -15x^3 + 5x^2 + 6x^2 + (-20x) + (-2x) + 8$	Reorder like terms in descending order.
$= -15x^3 + 11x^2 + (-22x) + 8$	Combine like terms.
$= -15x^3 + 11x^2 - 22x + 8$	Simplify.

The expression $(-5x + 2)(3x^2 - x + 4)$ is equivalent to $-15x^3 + 11x^2 - 22x + 8$.

Example 3

Simplify the expression $(3x^4 + 10x^2 - 4x)(x^3 - 8x^2 + x)$.

1. Rewrite the product using the Distributive Property.

   Multiply each term in the first polynomial by each term in the second polynomial.

   $(3x^4 + 10x^2 - 4x)(x^3 - 8x^2 + x)$

   $= (3x^4 \bullet x^3) + (3x^4 \bullet -8x^2) + (3x^4 \bullet x) + (10x^2 \bullet x^3) + (10x^2 \bullet -8x^2) + (10x^2 \bullet x) + (-4x \bullet x^3) + (-4x \bullet -8x^2) + (-4x \bullet x)$

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Instruction

2. Use properties of exponents and multiplication to simplify any expressions.

   Start with the expression from the previous step.

   $(3x^4 \bullet x^3) + (3x^4 \bullet -8x^2) + (3x^4 \bullet x) + (10x^2 \bullet x^3) + (10x^2 \bullet -8x^2) + (10x^2 \bullet x) + (-4x \bullet x^3) + (-4x \bullet -8x^2) + (-4x \bullet x)$

   Substitute x^1 for x when multiplying terms that both have variables.

   $= (3x^4 \bullet x^3) + (3x^4 \bullet -8x^2) + [3x^4 \bullet (x^1)] + (10x^2 \bullet x^3) + (10x^2 \bullet -8x^2) + [10x^2 \bullet (x^1)] + [-4(x^1) \bullet x^3] + [-4(x^1) \bullet -8x^2] + [-4(x^1) \bullet (x^1)]$

   Since $x^n \bullet x^m = x^{n+m}$, add the exponents of any two terms that are being multiplied that contain variables. Remember that the product of a number and a variable is written with the number first, as the coefficient of the variable.

   $= [(3)(1)x^{4+3}] + [(3)(-8)x^{4+2}] + [(3)(1)x^{4+1}] + [(10)(1)x^{2+3}] + [(10)(-8)x^{2+2}] + [(10)(1)x^{2+1}] + [(-4)(1)x^{1+3}] + [(-4)(-8)x^{1+2}] + [(-4)(1)x^{1+1}]$

   Simplify.

   $= 3x^7 + (-24x^6) + 3x^5 + 10x^5 + (-80x^4) + 10x^3 + (-4x^4) + 32x^3 + (-4x^2)$

3. Combine like terms.

   Only terms with the same variable raised to the same power can be combined.

   Write the polynomial in descending order according to the power of each term, from highest power to lowest.

$3x^7 + (-24x^6) + 3x^5 + 10x^5 + (-80x^4) + 10x^3 + (-4x^4) + 32x^3 + (-4x^2)$	Expression from the previous step
$= 3x^7 + (-24x^6) + 3x^5 + 10x^5 + (-80x^4) + (-4x^4) + 10x^3 + 32x^3 + (-4x^2)$	Reorder like terms in descending order.
$= 3x^7 + (-24x^6) + 13x^5 + (-84x^4) + 42x^3 + (-4x^2)$	Combine like terms.
$= 3x^7 - 24x^6 + 13x^5 - 84x^4 + 42x^3 - 4x^2$	Simplify.

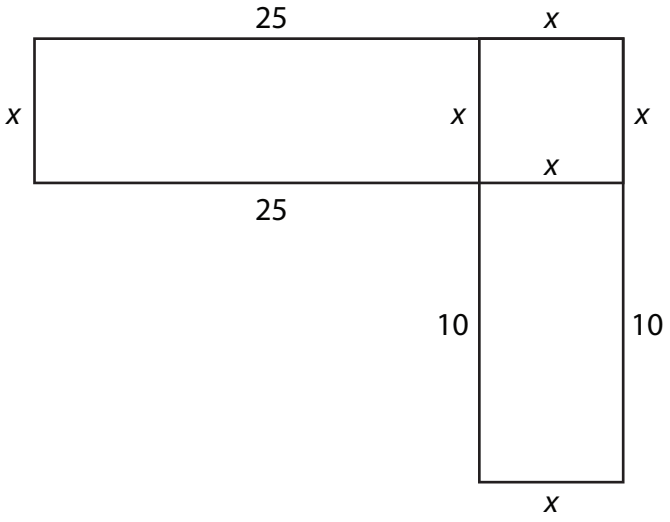
   The expression $(3x^4 + 10x^2 - 4x)(x^3 - 8x^2 + x)$ is equivalent to $3x^7 - 24x^6 + 13x^5 - 84x^4 + 42x^3 - 4x^2$.

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Problem-Based Task 2A.1.3: Not Quite Set in Stone

SMP	
1	2
3 ✓	4 ✓
5	6 ✓
7	8

A landscaper is drafting a plan for a stone walkway. Some of the dimensions of the walkway have not yet been determined. The design is below, with all dimensions shown in feet. (Diagram not shown to scale.) What expression can be used to represent the area of the walkway?





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Problem-Based Task 2A.1.3: Not Quite Set in Stone

Coaching

a. What is the length of the top portion of the walkway?

b. What expression can be written to represent the area of the top portion of the walkway?

c. What is the length of the remaining portion of the walkway?

d. What expression can be written to represent the area of the remaining portion of the walkway?

e. What is the total area of the walkway?

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Instruction

Problem-Based Task 2A.1.3: Not Quite Set in Stone

Coaching Sample Responses

a. What is the length of the top portion of the walkway?

The top portion of the walkway is made up of two lengths, 25 and x.

The total length of the top portion of the walkway is $25 + x$ feet.

b. What expression can be written to represent the area of the top portion of the walkway?

The area of a rectangle is found by multiplying the length by the width.

The length is $(25 + x)$ and the width is x.

The expression representing the area of the top portion of the walkway is $(25 + x)(x)$ or $x^2 + 25x$.

c. What is the length of the remaining portion of the walkway?

To find the length of the remaining portion of the walkway, exclude x, the portion that was included in the area of the top portion of the walkway.

The length of the remaining portion of the walkway is 10 feet.

d. What expression can be written to represent the area of the remaining portion of the walkway?

The area of the remaining portion is the length, 10, times the width, x.

The expression representing the area of the remaining portion of the walkway is $(10)(x)$ or $10x$.

e. What is the total area of the walkway?

The total area of the walkway is the sum of the expressions representing the two smaller areas.

$x^2 + 25x + 10x = x^2 + 35x$

The total area of the walkway is $(x^2 + 35x)$ ft^2.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2A.1.3: Multiplying Polynomials

Simplify each expression.

1. $(11x + 3)(-x^2 + 7)$

2. $(6x^3 + 5x^2 - 1)(2x^3 + 4)$

3. $(-y^2 + 10)(8y^3 + 2y)$

4. $(10x^5 + 4x^2)(2x^2 - 6x + 3)$

5. $(-3x^4 + 5x^3 - x^2)(x^3 - 6)$

6. $(7y^6 - 9y^4 + 2)(4y^3 + y^2 - 1)$

7. $(5x^2 + 4x + 3)(7x^3 - 5x - 2)$

The area of a rectangle is found using the formula *length* • *width*. Find the area of a rectangle with the given length and width. All measurements are given in meters.

8. length: $x + 8$; width: $3x - 2$

9. length: $x^2 + 1$; width: $4x + 10$

10. length: $6x + 5$; width: $2x^2 - 3$

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Progress Assessment

Circle the letter of the best answer.

1. What is the coefficient of the second term of $9x^8 - 15x^3 + 3x^2 - 20$?
   a. –15
   b. –1
   c. 8
   d. 15

2. What is the result of $\left(-10x^4 + 3x^3 - 4\right) + \left(8x^4 + 2x^3 + x\right)$?
   a. $-2x^4 + 5x^3 - 4x$
   b. $18x^4 + 5x^3 + 3x$
   c. $-2x^4 + 5x^3 + x - 4$
   d. $-10x^4 + 6x^3 + x - 4$

3. What is the result of $\left(12x^3 + 7x^2 - 1\right) - \left(-x^3 + 8x^2 + x\right)$?
   a. $11x^3 - x^2 - x - 1$
   b. $13x^3 - x^2 - x - 1$
   c. $11x^3 + 15x^2 + x - 1$
   d. $13x^3 - x^2 - x$

4. What is the result of $\left(-x^4 + 2x^3 + 10\right)\left(x^2 + 4\right)$?
   a. $-x^6 + 2x^5 - 4x^4 + 8x^3 + 10x^2 + 40$
   b. $-x^6 + 2x^5 - 4x^4 + 40$
   c. $-x^8 + 2x^6 + 10x^2 + 40$
   d. $-x^8 + 2x^6 - 4x^4 + 8x^3 + 10x^2 + 40$

5. What is the result of $\left(5x^2 - x - 6\right)\left(3x^2 + x + 3\right)$?
   a. $15x^4 - 2x^2 - 9x - 18$
   b. $15x^4 + 8x^3 + 34x^2 + 9x + 18$
   c. $15x^4 + 2x^3 + 14x^2 - 3x$
   d. $15x^4 + 2x^3 - 4x^2 - 9x - 18$

6. What is the degree of the polynomial $10x^9 + 12x^7 - 7x^3 + 13$?
   a. 7
   b. 9
   c. 10
   d. 12

7. What is the result of $\left(x^5 - x^2 + 14\right) + \left(7x^5 + x^2 - 3\right)$?
   a. $8x^5 + 11$
   b. $8x^5 + 2x^2 + 17$
   c. $8x^4 - x^3 + x^2 + 11$
   d. $x^5 + 7x^4 - x^3 + x^2 - 11$

continued

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8. What is the result of $\left(-13x^6+9x\right)-\left(x^5-6x+22\right)$?

   a. $-12x^6+3x-22$
   b. $-13x^6-x^5+15x-22$
   c. $-13x^6-x^5+3x-22$
   d. $13x^6-x^5+15x-22$

9. What is the result of $\left(-3x^4-1\right)\left(3x^3+5x^2-4\right)$?

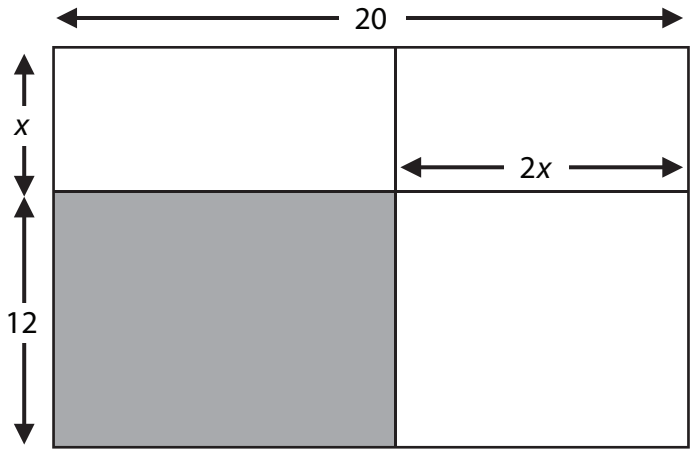
   a. $9x^7+15x^6+12x^4+3x^3+5x^2-4$
   b. $3x^7+5x^6-12x^4-3x^3-5x^2+4$
   c. $-9x^7-15x^6+12x^4-3x^3-5x^2+4$
   d. $-3x^7-5x^6+12x^4+3x^3+5x^2-4$

10. What is the result of $\left(2x^3+x^2-x\right)\left(5x^2-8x+3\right)$?

    a. $10x^5-11x^4-7x^3+11x^2-3x$
    b. $2x^3+6x^2-9x+3$
    c. $10x^6+21x^4+14x^3+3x^2$
    d. $5x^5-3x^4-9x^3-5x^2-3x$

Use what you have learned about operating with polynomials to solve problem 11.

11. Gina is designing a mailer for a political campaign. The figure below represents the layout she's using for her design. Use the figure to answer the questions that follow. All units are in centimeters.



The perimeter of a rectangle is the sum of its sides. The area of a rectangle is *length* • *width*.

   a. What is the perimeter of the shaded rectangle?
   b. What is the area of the shaded rectangle?

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Which expression represents the factored form of $9x^2 - 36$?
   a. $9(x + 2)(x - 2)$
   b. $9(x + 2)^2$
   c. $3(x - 3)^2$
   d. $(9x + 6)(x - 6)$

2. Which expression is equivalent to $(4x + 2i)(4x - 2i)$?
   a. $16x^2 - 4$
   b. $16x^2 - 4i$
   c. $16x^2 + 4$
   d. $4x + 2$

3. Which expression represents the expanded form of $(x^2 + 3x + 1)^2$?
   a. $x^4 + 9x^2 + 1$
   b. $x^4 + 6x^3 + 11x^2 + 6x + 1$
   c. $x^4 + 3x^3 + 10x^2 + 3x + 1$
   d. $2x^2 + 6x + 2$

4. Which expression represents the expanded form of $(-x + 2y)^4$?
   a. $x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4$
   b. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
   c. $-4x + 8y$
   d. $x^2 - 4xy + 4y^2$

5. What is term 8 of the expanded form of $(5x - y)^9$?
   a. $-y^9$
   b. $45xy^8$
   c. $-900x^2y^7$
   d. $-5xy$

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Lesson 2: Proving Identities

Instruction

SMP
1 ✓ 2 ✓
3 4 ✓
5 6 ✓
7 ✓ 8 ✓

Common Core State Standards

N–CN.8 (+) Extend polynomial identities to the complex numbers. *For example, rewrite $x^2 + 4$ as $(x + 2i)(x – 2i)$.*

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.*

A–APR.4 Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 – y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.*

A–APR.5 (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[1]

Essential Questions

1. How do polynomial identities help when expanding or factoring polynomials?
2. How can multiplication of binomials be used to find products of complex numbers?
3. What patterns are used to find powers of binomials?

WORDS TO KNOW

Binomial Theorem a theorem stating that a binomial $(a + b)^n$ can be expanded using the formula

$$\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \bullet a^{n-k}b^k = 1a^n b^0 + \frac{n}{1} \bullet a^{n-1}b^1 + \frac{n(n-1)}{1 \bullet 2} \bullet a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \bullet 2 \bullet 3} \bullet a^{n-3}b^3 + \cdots + 1a^0 b^n$$

[1]The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.

SRB p. U2A-20

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complex conjugate — the complex number that when multiplied by another complex number produces a value that is wholly real; the complex conjugate of $a + bi$ is $a - bi$

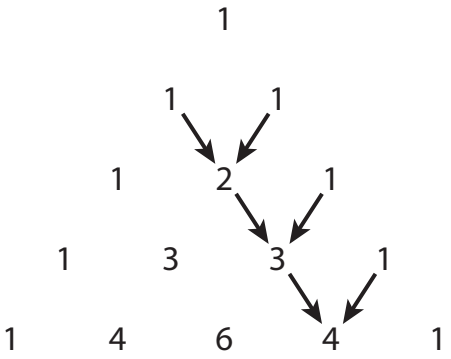
complex number — a number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit

factorial — the product of an integer and all preceding positive integers, represented using a ! symbol; $n! = n \bullet (n - 1) \bullet (n - 2) \bullet \cdots \bullet 1$. For example, $5! = 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1$. By definition, $0! = 1$.

imaginary number — any number of the form bi, where b is a real number, $i = \sqrt{-1}$, and $b \neq 0$

imaginary unit, *i* — the letter i, used to represent the non-real value $i = \sqrt{-1}$

Pascal's Triangle — a triangle displaying a pattern of numbers in which the terms in additional rows are found by adding pairs of terms in previous rows, so that any given term is the sum of the two terms directly above it. The number 1 is the top number of the triangle, and is also the first and last number of each row.



polynomial identity — a true equation that is often generalized so it can apply to more than one example

***sigma* (uppercase), Σ** — a Greek letter used to represent the summation of values

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Recommended Resources

- Interactive Mathematics. "The Binomial Theorem."

  http://www.walch.com/rr/00161

  This site includes a review of binomials, several examples showing binomial expansion, and a summary of Pascal's Triangle. Users can enter values into the interactive applets to find factorials of positive integers up to 30, and expand binomials up to powers of 6.

- MathIsFun.com. "Binomial Theorem."

  http://www.walch.com/rr/00162

  This site offers a simple yet effective review of binomials that leads into an explanation of the Binomial Theorem for $(a + b)^n$. The review covers exponents, polynomials, and Pascal's Triangle. Links are provided to sample questions that generate immediate feedback, along with explanations of answers.

- MathIsFun.com. "Factoring in Algebra."

  http://www.walch.com/rr/00163

  This resource offers instruction on factoring polynomials, including how to apply polynomial identities to factors. The site also includes links to 10 multiple-choice practice questions, with immediate feedback and explanations.

- Virtual Nerd. "What's the Formula for the Square of a Sum?"

  http://www.walch.com/rr/00164

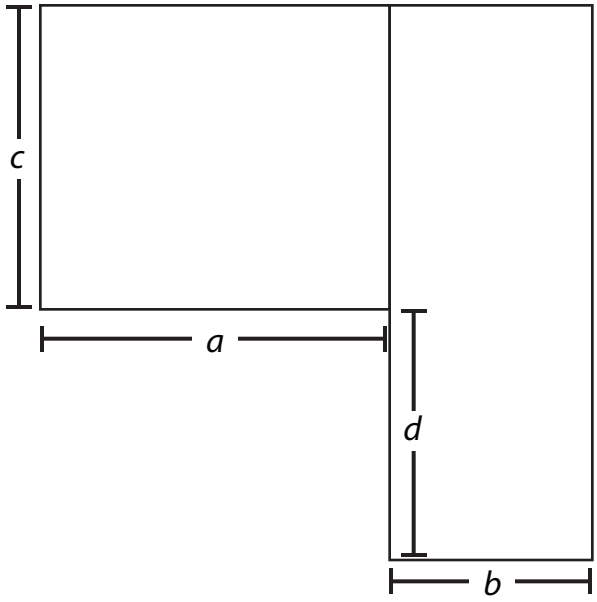
  This brief video tutorial illuminates each step in deriving the formula for the square of a sum.

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Lesson 2A.2.1: Polynomial Identities

Warm-Up 2A.2.1

A new playground is being built at a school. The rubberized ground under the playground will have the design shown below.



A planning committee is considering a few different sizes for the playground. The committee would like to calculate the area of the rubberized ground for the different sizes. What is the area of the ground for a playground with each of the following dimensions for a, b, c, and d? The area of a rectangle is the product of the length and width, or $A = lw$.

1. a = 20 feet, b = 10 feet, c = 15 feet, d = 10 feet

2. a = 30 feet, b = 15 feet, c = 25 feet, d = 20 feet

3. a = 40 feet, b = 20 feet, c = 30 feet, d = 35 feet

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Instruction

Lesson 2A.2.1: Polynomial Identities

Common Core State Standards

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.*

A–APR.4 Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 – y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.*

Warm-Up 2A.2.1 Debrief

1. a = 20 feet, b = 10 feet, c = 15 feet, d = 10 feet

   The rubberized ground will be made up of two rectangles. The total area of the ground is the sum of the areas of the two rectangles. The rectangle on the left has a length of a feet and a width of c feet. Replace a and c with the given values in the formula for the area, $A = lw$.

   Rectangle 1

   Area = $lw = ac = 20 \bullet 15 = 300$ ft^2

   The rectangle on the right has a length of b feet and a width of $c + d$ feet.

   Replace the value of b with the given value of 10 feet.

   The width is the sum of the given values of c and d: 15 + 10 = 25 feet.

   The area is the product of the length and width.

   Rectangle 2

   Area = $lw = 10 \bullet 25 = 250$ ft^2

   The total area is the sum of the two areas: 300 + 250 = 550 ft^2.

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2. a = 30 feet, b = 15 feet, c = 25 feet, d = 20 feet

Rectangle 1

Area = $lw = ac$ = 30 • 25 = 750 ft^2

Rectangle 2

Length = b = 15 feet

Width = $c + d$ = 25 + 20 = 45 feet

Area = lw = 15 feet • 45 feet = 675 ft^2

The total area is the sum of the two areas: 750 + 675 = 1425 ft^2.

3. a = 40 feet, b = 20 feet, c = 30 feet, d = 35 feet

Rectangle 1

Area = $lw = ac$ = 40 • 30 = 1200 ft^2

Rectangle 2

Length = b = 20 feet

Width = $c + d$ = 30 + 35 = 65 feet

Area = lw = 20 feet • 65 feet = 1300 ft^2

The total area is the sum of the two areas: 1200 + 1300 = 2500 ft^2.

Connection to the Lesson

- Students will simplify expressions involving addition and multiplication.
- Students will evaluate expressions by replacing values with numeric quantities.

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Prerequisite Skills

This lesson requires the use of the following skills:

- using variables to represent unknown quantities (6.EE.2a)
- finding sums, products, and differences of polynomial terms (A–APR.1)
- simplifying expressions by combining like terms (8.EE.7b)
- applying properties of exponents to simplify expressions (8.EE.1)
- evaluating numbers and variables raised to powers (8.EE.1)
- organizing polynomial expressions according to the power of the variables (A–APR.1)

Introduction

Polynomials are often added, subtracted, and even multiplied. Doing so results in certain sums, differences, or products of polynomials appearing more commonly. Becoming familiar with certain polynomial identities can make the processes of expanding and factoring polynomials easier.

Key Concepts

- Identities can be used to expand or factor polynomial expressions.
- A **polynomial identity** is a true equation that is often generalized so it can apply to more than one example.
- The table that follows shows the most common polynomial identities, including the steps of how to work them out.

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Instruction

Common Polynomial Identities

Square of Sums Identity	
Formula	**Steps**
With two variables: $(a + b)^2 = a^2 + 2ab + b^2$	$(a + b)^2$ $= (a + b)(a + b)$ $= a^2 + ab + ab + b^2$ $= a^2 + 2ab + b^2$
With three variables: $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$	$(a + b + c)^2$ $= (a + b + c)(a + b + c)$ $= a^2 + ab + ac + ab + b^2 + bc + ac + bc + c^2$ $= a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$
Square of Differences Identity	
Formula	**Steps**
$(a - b)^2 = a^2 - 2ab + b^2$	$(a - b)^2$ $= (a - b)(a - b)$ $= a^2 - ab - ab + b^2$ $= a^2 - 2ab + b^2$
Difference of Two Squares Identity	
Formula	**Steps**
$a^2 - b^2 = (a + b)(a - b)$	$a^2 - b^2$ $= a^2 + ab - ab + b^2$ $= (a + b)(a - b)$
Sum of Two Cubes Identity	
Formula	**Steps**
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	$a^3 + b^3$ $= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$ $= (a + b)(a^2 - ab + b^2)$
Difference of Two Cubes Identity	
Formula	**Steps**
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	$a^3 - b^3$ $= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3$ $= (a - b)(a^2 + ab + b^2)$

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- Note that the square of sums, $(a + b)^2$, is different from the sum of squares, $a^2 + b^2$.
- Polynomial identities are true for any values of the given variables.
- When dealing with large numbers, the use of identities often results in simpler calculations.

Common Errors/Misconceptions

- confusing identities, such as applying the Square of Differences Identity instead of the Difference of Two Squares Identity
- incorrectly applying an identity to find an equivalent expression
- misplacing a negative sign in an identity, resulting in an incorrect equivalent expression

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Guided Practice 2A.2.1

Example 1

Use a polynomial identity to expand the expression $(x - 14)^2$.

1. Determine which identity is written in the same form as the given expression.

   The expression $(x - 14)^2$ is written in the same form as the left side of the Square of Differences Identity: $(a - b)^2 = a^2 - 2ab + b^2$.

   Therefore, we can substitute the values from the expression $(x - 14)^2$ into the Square of Differences Identity.

2. Replace a and b in the identity with the terms in the given expression.

   For the given expression $(x - 14)^2$, let $x = a$ and $14 = b$ in the Square of Differences Identity.

$(x - 14)^2$	Original expression
$(a - b)^2 = a^2 - 2ab + b^2$	Square of Differences Identity
$[(x) - (14)]^2 = (x)^2 - 2(x)(14) + (14)^2$	Substitute x for a and 14 for b in the Square of Differences Identity.

   The rewritten identity is $(x - 14)^2 = x^2 - 2(x)(14) + 14^2$.

3. Simplify the expression by finding any products and evaluating any exponents.

   The first term, x^2, cannot be simplified further, but the other terms can.

$(x - 14)^2 = x^2 - 2(x)(14) + 14^2$	Equation from the previous step
$= x^2 - 28x + 14^2$	Simplify $2(x)(14)$.
$= x^2 - 28x + 196$	Square 14.

   When expanded, the expression $(x - 14)^2$ can be written as $x^2 - 28x + 196$.

SWB p. 39; SRB p. U2A-26

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Example 2

Use a polynomial identity to factor the expression $x^3 + 125$.

1. Determine which identity is written in the same form as the given expression.

   The first term in the expression, x^3, is a cube. Determine whether the second term in the expression, 125, is also a cube.

   $125 = 5 \bullet 25 = 5 \bullet 5 \bullet 5 = 5^3$

   Since 125 can be rewritten as a cube, the expression $x^3 + 125$ can be rewritten as $x^3 + 5^3$.

   The identity $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ is in the same form as $x^3 + 5^3$.

   Therefore, the expression $x^3 + 125$, rewritten as $x^3 + 5^3$, is in the same form as the left side of the Sum of Two Cubes Identity: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

2. Replace a and b in the identity with the terms in the given expression.

   For the rewritten expression $x^3 + 5^3$, let $x = a$ and $5 = b$ in the Sum of Two Cubes Identity.

$x^3 + 5^3$	Rewritten expression
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	Sum of Two Cubes Identity
$(x)^3 + (5)^3 = [(x) + (5)][(x)^2 - (x)(5) + (5)^2]$	Substitute x for a and 5 for b in the Sum of Two Cubes Identity.

   The rewritten identity is $x^3 + 5^3 = (x + 5)[x^2 - (x)(5) + 5^2]$.

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3. Simplify the expression by finding any products and evaluating any exponents.

   The first factor, $x + 5$, cannot be simplified further.

   The second factor, $x^2 - (x)(5) + 5^2$, has three terms: the first term, x^2, cannot be simplified; the other terms, $(x)(5)$ and 5^2, can be simplified.

$x^3 + 5^3 = (x + 5)[x^2 - (x)(5) + 5^2]$	Equation from the previous step
$= (x + 5)(x^2 - 5x + 5^2)$	Simplify $(x)(5)$.
$= (x + 5)(x^2 - 5x + 25)$	Square 5.

   When factored, the expression $x^3 + 125$, which is equivalent to $x^3 + 5^3$, can be rewritten as $(x + 5)(x^2 - 5x + 25)$.

Example 3

Use a polynomial identity to expand the expression $(3x^2 - 2x + 8)^2$.

1. Determine which identity is written in the same form as the given expression.

   The expression that is being squared can be rewritten as a sum.

$(3x^2 - 2x + 8)^2$	Original expression
$= [3x^2 + (-2x) + 8]^2$	Rewrite the expression as a sum.

   The rewritten expression $[3x^2 + (-2x) + 8]^2$ is in the same form as the left side of the Square of Sums Identity: $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$.

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2. Replace a, b, and c in the identity with the terms in the given expression.

   For the expression $[3x^2 + (-2x) + 8]^2$, let $3x^2 = a$, $-2x = b$, and $8 = c$ in the Square of Sums Identity.

$[3x^2 + (-2x) + 8]^2$	Rewritten expression
$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$	Square of Sums Identity
$[(3x^2) + (-2x) + (8)]^2 = (3x^2)^2 + (-2x)^2 + (8)^2 + 2(3x^2)(-2x) + 2(-2x)(8) + 2(3x^2)(8)$	Substitute $3x^2$ for a, $-2x$ for b, and 8 for c in the Square of Sums Identity.

   The rewritten identity is $[3x^2 + (-2x) + 8]^2 = (3x^2)^2 + (-2x)^2 + 8^2 + 2(3x^2)(-2x) + 2(-2x)(8) + 2(3x^2)(8)$.

3. Simplify each term in the expression by finding any products and evaluating any exponents.

$[3x^2 + (-2x) + 8]^2 = (3x^2)^2 + (-2x)^2 + 8^2 + 2(3x^2)(-2x) + 2(-2x)(8) + 2(3x^2)(8)$	Equation from the previous step
$= 9x^4 + (-2x)^2 + 8^2 + 2(3x^2)(-2x) + 2(-2x)(8) + 2(3x^2)(8)$	Simplify $(3x^2)^2$.
$= 9x^4 + 4x^2 + 8^2 + 2(3x^2)(-2x) + 2(-2x)(8) + 2(3x^2)(8)$	Simplify $(-2x)^2$.
$= 9x^4 + 4x^2 + 64 + 2(3x^2)(-2x) + 2(-2x)(8) + 2(3x^2)(8)$	Square 8.
$= 9x^4 + 4x^2 + 64 - 12x^3 + 2(-2x)(8) + 2(3x^2)(8)$	Simplify $2(3x^2)(-2x)$.
$= 9x^4 + 4x^2 + 64 - 12x^3 - 32x + 2(3x^2)(8)$	Multiply $2(-2x)(8)$.
$= 9x^4 + 4x^2 + 64 - 12x^3 - 32x + 48x^2$	Multiply $2(3x^2)(8)$.

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4. Rearrange the polynomial in descending order and combine like terms to simplify the expression.

$[3x^2 + (-2x) + 8]^2 = 9x^4 + 4x^2 + 64 - 12x^3 - 32x + 48x^2$	Equation from the previous step
$= 9x^4 - 12x^3 + 4x^2 + 48x^2 - 32x + 64$	Rearrange the terms in descending order of their powers.
$= 9x^4 - 12x^3 + 52x^2 - 32x + 64$	Combine like terms.

When expanded, the expression $(3x^2 - 2x + 8)^2$ can be rewritten as $9x^4 - 12x^3 + 52x^2 - 32x + 64$.

Example 4

Show how 36^2 can be evaluated using polynomial identities.

1. Determine which polynomial identities can be used to find the square of a value.

   There are three identities that can be used to find the square of a value:

   - Square of Sums Identity (for two variables): $(a + b)^2 = a^2 + 2ab + b^2$
   - Square of Differences Identity: $(a - b)^2 = a^2 - 2ab + b^2$
   - Square of Sums Identity (for three variables): $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$

2. Rewrite the expression as a sum.

   Choose a sum that includes two numbers that can be used to easily find products and squares.

   36^2 can be rewritten as $(30 + 6)^2$.

   The rewritten expression is in the form of the Square of Sums Identity.

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3. Use the sum and a polynomial identity to rewrite the expression.

   For the expression $(30 + 6)^2$, let $30 = a$ and $6 = b$ in the Square of Sums Identity.

$(30 + 6)^2$	Rewritten expression
$(a + b)^2 = a^2 + 2ab + b^2$	Square of Sums Identity
$[(30) + (6)]^2 = (30)^2 + 2(30)(6) + (6)^2$	Substitute 30 for a and 6 for b in the Square of Sums Identity.
$(30 + 6)^2 = 900 + 360 + 36$	Simplify each term.
$(30 + 6)^2 = 1296$	Sum the terms.

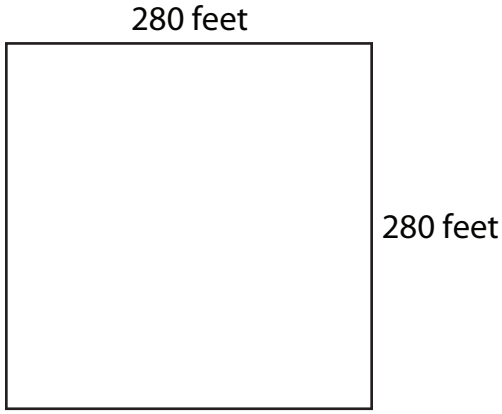
   The expression 36^2 is equal to 1,296.

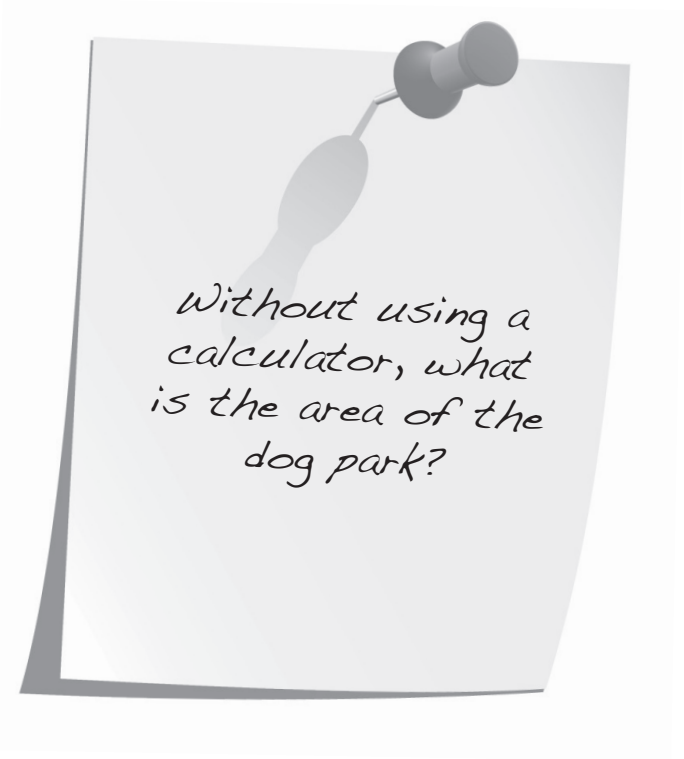
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Problem-Based Task 2A.2.1: How Big Is the Dog Park?

SMP	
1 ✓	2 ✓
3	4
5	6 ✓
7	8

City planners are designing a new dog park. The park will be a square, with the dimensions shown in the diagram below. The area of the park is the square of the side length: area = 280^2 $feet^2$. Without using a calculator, what is the area of the dog park? Use polynomial identities to support your answer.





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Problem-Based Task 2A.2.1: How Big Is the Dog Park?

Coaching

a. Which polynomial identities can be used to find the square of a number?

b. How can the side length of the park be rewritten as a sum or difference?

c. Using the sum or difference of the side length, and a polynomial identity, what is an equivalent expression that represents the area of the dog park?

d. Using the expression and the identity, what is the area of the park?

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Problem-Based Task 2A.2.1: How Big Is the Dog Park?

Coaching Sample Responses

a. Which polynomial identities can be used to find the square of a number?

There are three identities that can be used to find the square of a value:

- Square of Sums Identity (for two variables): $(a + b)^2 = a^2 + 2ab + b^2$
- Square of Differences Identity: $(a – b)^2 = a^2 – 2ab + b^2$
- Square of Sums Identity (for three variables): $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$

b. How can the side length of the park be rewritten as a sum or difference?

Choose a sum or difference that includes two numbers that can be used to easily find products and squares.

One way to rewrite the side length, 280 feet, is to use the difference 300 – 20.

c. Using the sum or difference of the side length, and a polynomial identity, what is an equivalent expression that represents the area of the dog park?

The area of the park is the square of the side length, or 280^2.

If this number is rewritten using the difference 300 – 20, the area is $(300 – 20)^2$.

The Square of Differences Identity, $(a – b)^2 = a^2 – 2ab + b^2$, can be used to rewrite the expression for the area.

$(300 – 20)^2 = 300^2 – 2(300)(20) + 20^2$

d. Using the expression and the identity, what is the area of the park?

Evaluate each term in the expression.

$(300 – 20)^2 = 300^2 – 2(300)(20) + 20^2$

$= 90{,}000 – 2(300)(20) + 20^2$

$= 90{,}000 – 12{,}000 + 20^2$

$= 90{,}000 – 12{,}000 + 400$

$= 78{,}400$

The area of the park is 78,400 feet^2.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2A.2.1: Polynomial Identities

Use polynomial identities to expand or factor each expression.

1. $(x - 16)^2$

2. $(4x + 21)^2$

3. $100x^2 - 4$

4. $x^2 - 81$

5. $27x^3 + 343$

6. $512x^3 - 8$

7. $(x^2 + x + 3)^2$

Find the area of a square with the given side length, without using a calculator. The area of a square is the square of the side length, or $area = side^2$.

8. side length = 530 feet

9. side length = 470 meters

10. side length = 310 centimeters

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Lesson 2A.2.2: Complex Polynomial Identities

Warm-Up 2A.2.2

Isaac is comparing the sizes of three different storage boxes by calculating the volume of each box. The volume of each box is found using the formula *volume* = (*length*)(*width*)(*height*), or $V = lwh$. Use this formula and the given dimensions to calculate the volume of each box.

1. length: 20 inches; width: 20 inches; height: 16 inches

2. length: 24 inches; width: 24 inches; height: 12 inches

3. length: 18 inches; width: 18 inches; height: 18 inches

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Lesson 2A.2.2: Complex Polynomial Identities

Common Core State Standards

N–CN.8 (+) Extend polynomial identities to the complex numbers. *For example, rewrite $x^2 + 4$ as $(x + 2i)(x – 2i)$.*

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.*

A–APR.4 Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 – y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.*

Warm-Up 2A.2.2 Debrief

1. length: 20 inches; width: 20 inches; height: 16 inches

   Replace the dimensions in the formula for volume with the given values.

$V = lwh$	Volume formula
$= (20)(20)(16)$	Substitute values for l, w, and h.
$= (20^2)(16)$	Rewrite the expression using exponents.
$= (400)(16)$	Evaluate the exponent.
$= 6400$	Find the product.

   The volume of the first box is 6,400 in^3.

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2. length: 24 inches; width: 24 inches; height: 12 inches

   Replace the dimensions in the formula for volume with the given values.

$V = lwh$	Volume formula
$= (24)(24)(12)$	Substitute values for l, w, and h.
$= (24^2)(12)$	Rewrite the expression using exponents.
$= (576)(12)$	Evaluate the exponent.
$= 6912$	Find the product.

   The volume of the second box is 6,912 in^3.

3. length: 18 inches; width: 18 inches; height: 18 inches

   Replace the dimensions in the formula for volume with the given values.

$V = lwh$	Volume formula
$= (18)(18)(18)$	Substitute values for l, w, and h.
$= (18^3)$	Rewrite the expression using exponents.
$= 5832$	Evaluate the exponent.

   The volume of the third box is 5,832 in^3.

Connection to the Lesson

- Students will simplify expressions involving exponents.
- Students will apply the order of operations to simplify expressions and they will evaluate numerical expressions with exponents.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- using variables to represent unknown quantities (6.EE.2a)
- finding sums, differences, and products of polynomial terms (A–APR.1)
- simplifying expressions by combining like terms (8.EE.7b)
- applying properties of exponents to simplify expressions (8.EE.1)
- evaluating numbers and variables raised to powers (8.EE.1)
- organizing polynomial expressions according to the power of the variables (A–APR.1)
- finding square roots (8.EE.2)
- adding fractions (5.NF.1)

Introduction

Polynomial identities can be used to find the product of complex numbers. A **complex number** is a number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit. An expression that cannot be written using an identity with real numbers can be factored using the imaginary unit i.

Key Concepts

- An **imaginary number** is any number of the form bi, where b is a real number, $i = \sqrt{-1}$, and $b \neq 0$. The **imaginary unit *i*** is used to represent the non-real value $i = \sqrt{-1}$.
- Recall that $i^2 = -1$.
- Polynomial identities and properties of the imaginary unit i can be used to expand or factor expressions with complex numbers.
- **Complex conjugates** are two complex numbers of the form $a + bi$ and $a - bi$. Both numbers contain an imaginary part, but multiplying them produces a value that is wholly real. Therefore, the complex conjugate of $a + bi$ is $a - bi$, and vice versa.
- The sum of two squares can be rewritten as the product of complex conjugates: $a^2 + b^2 = (a + bi)(a - bi)$, where a and b are real numbers and i is the imaginary unit.
- Rewriting the sum of two squares in this way can allow you to either factor the sum of two squares or to find the product of complex conjugates.

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Instruction

- To prove this, find the product of the conjugates and simplify the expression.

  $(a + bi)(a - bi) = a \bullet a + a \bullet bi + a(-bi) + bi(-bi) = a^2 + abi - abi - b^2i^2 = a^2 - b^2(-1) = a^2 + b^2$

- This factored form of the sum of two squares can also include variables, such as $a^2x^2 + b^2 = (ax + bi)(ax - bi)$, where x is a variable, a and b are real numbers, and i is the imaginary unit.

Common Errors/Misconceptions

- confusing the sum of squares with the Square of Sums Identity
- incorrectly calculating a and b in the expression $a^2 + b^2 = (a + bi)(a - bi)$, when a^2 and b^2 are given
- incorrectly squaring quantities

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Guided Practice 2A.2.2

Example 1

Find the result of $(10 + 7i)(10 - 7i)$.

1. Determine whether an identity can be used to rewrite the expression.

   Since $(10 + 7i)$ and $(10 - 7i)$ are complex conjugates, the expression $(10 + 7i)(10 - 7i)$ can be rewritten as the sum of squares: $(a + bi)(a - bi) = a^2 + b^2$.

2. Identify a and b in the sum of squares.

   Let $10 = a$ and $7 = b$.

$(10 + 7i)(10 - 7i)$	Given expression
$(a + bi)(a - bi) = a^2 + b^2$	The product of two complex conjugates is the sum of squares.
$[(10) + (7)i][(10) - (7)i] = (10)^2 + (7)^2$	Substitute 10 for a and 7 for b.

   The rewritten identity is $(10 + 7i)(10 - 7i) = 10^2 + 7^2$.

3. Simplify the equation as needed.

$(10 + 7i)(10 - 7i) = 10^2 + 7^2$	Equation from the previous step
$= 100 + 49$	Evaluate the exponents.
$= 149$	Sum the terms.

   The result of $(10 + 7i)(10 - 7i)$ is 149.

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Example 2

Factor the expression $9x^2 + 169$.

1. Determine whether an identity can be used to rewrite the expression.

   Both 9 and 169 are perfect squares; therefore, $9x^2 + 169$ is a sum of squares.

   The original expression can be rewritten using exponents.

$9x^2 + 169$	Original expression
$= (3x)^2 + (13)^2$	Rewrite each term using exponents.
$= 3^2x^2 + 13^2$	Rewrite $(3x)^2$ as the product of two squares.

2. Identify a and b in the sum of squares.

   The expression $3^2x^2 + 13^2$ is in the form $a^2x^2 + b^2$, which can be rewritten using the factored form of the sum of squares: $a^2x^2 + b^2 = (ax + bi)(ax - bi)$.

   In the rewritten expression $3^2x^2 + 13^2$, let $3 = a$ and $13 = b$.

3. Factor the sum of squares.

$a^2x^2 + b^2 = (ax + bi)(ax - bi)$	The sum of squares is the product of complex conjugates.
$(3)^2x^2 + (13)^2 = [(3)x + (13)i][(3)x - (13)i]$	Substitute 3 for a and 13 for b.
$9x^2 + 169 = (3x + 13i)(3x - 13i)$	Evaluate the exponents.

   When factored, the expression $9x^2 + 169$ is written as $(3x + 13i)(3x - 13i)$.

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Instruction

Example 3

Find the result of $(8x + 15i)(8x - 15i)$.

1. Determine whether an identity can be used to rewrite the expression.

   Since $(8x + 15i)$ and $(8x - 15i)$ are complex conjugates, the expression can be rewritten as the sum of squares: $(ax + bi)(ax - bi) = a^2x^2 + b^2$.

2. Identify a and b in the sum of squares.

   Let $8 = a$ and $15 = b$ in the expression.

3. Use the factored form to determine the sum of squares.

$(8x + 15i)(8x - 15i)$	Original expression
$(ax + bi)(ax - bi) = a^2x^2 + b^2$	The product of complex conjugates is the sum of squares.
$[(8)x + (15)i][(8)x - (15)i] = (8)^2x^2 + (15)^2$	Substitute 8 for a and 15 for b.
$(8x + 15i)(8x - 15i) = 64x^2 + 225$	Evaluate the exponents.

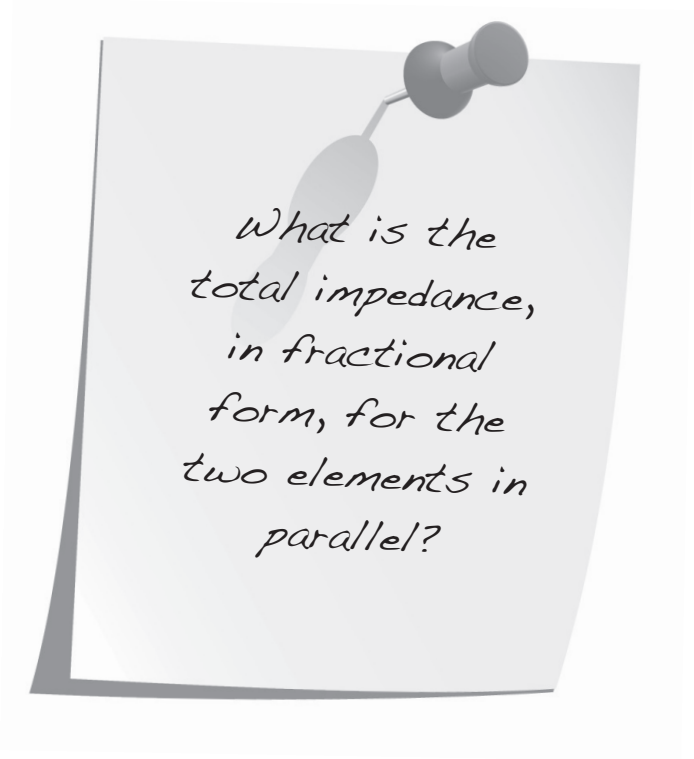
The result of $(8x + 15i)(8x - 15i)$ is $64x^2 + 225$.

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Problem-Based Task 2A.2.2: Measuring Electrical Impedance

SMP	
1	2
3	4 ✓
5	6 ✓
7 ✓	8

Impedance measures the total opposition that a circuit presents to an electric current. The impedance of an element can be represented using a complex number $V + Ii$, where V is the element's voltage and I is the element's current. If the impedance of Element 1 is $Z_1 = 12 + i$, and the impedance of Element 2 is $Z_2 = 14 + i$, the total impedance of the two elements in parallel is $\frac{1}{Z_1} + \frac{1}{Z_2}$. What is the total impedance, in fractional form, for the two elements in parallel?



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Problem-Based Task 2A.2.2: Measuring Electrical Impedance

Coaching

a. What is the reciprocal of each element's impedance?

b. What is the conjugate of each denominator?

c. Rationalize each denominator using a complex conjugate.

d. What is the total impedance, in fractional form, for the two elements in parallel?

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Instruction

Problem-Based Task 2A.2.2: Measuring Electrical Impedance

Coaching Sample Responses

a. What is the reciprocal of each element's impedance?

The impedance of Element 1 is $Z_1 = 12 + i$.

The reciprocal of the impedance of Element 1 is $\frac{1}{Z_1} = \frac{1}{12+i}$.

The impedance of Element 2 is $Z_2 = 14 + i$.

The reciprocal of the impedance of Element 2 is $\frac{1}{Z_2} = \frac{1}{14+i}$.

b. What is the conjugate of each denominator?

The conjugate of a rational number $a + bi$ is $a - bi$.

The conjugate of the denominator $12 + i$ is $12 - i$.

The conjugate of the denominator $14 + i$ is $14 - i$.

c. Rationalize each denominator using a complex conjugate.

To rationalize a denominator, the imaginary unit i must be removed from the denominator.

The product of a complex number and its conjugate is a real number.

Multiply the numerator and denominator of each fraction by the conjugate.

Element 1

$$\frac{1}{Z_1} = \frac{1}{12+i} \bullet \frac{12-i}{12-i}$$

$$= \frac{12-i}{12^2+1}$$

$$= \frac{12-i}{145}$$

Element 2

$$\frac{1}{Z_2} = \frac{1}{14+i} \bullet \frac{14-i}{14-i}$$

$$= \frac{14-i}{14^2+1}$$

$$= \frac{14-i}{197}$$

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d. What is the total impedance, in fractional form, for the two elements in parallel?

To add the two fractions, first find a common denominator using the product of the two denominators.

Element 1

$$\frac{1}{Z_1}=\frac{12-i}{145}\bullet\frac{197}{197}$$

$$=\frac{2364-197i}{28{,}565}$$

Element 2

$$\frac{1}{Z_2}=\frac{14-i}{197}\bullet\frac{145}{145}$$

$$=\frac{2030-145i}{28{,}565}$$

Add the two reciprocals with the rationalized denominators.

$$\frac{1}{Z_1}+\frac{1}{Z_2}=\frac{2364-197i}{28{,}565}+\frac{2030-145i}{28{,}565}$$

$$=\frac{4394-342i}{28{,}565}$$

The total impedance in fractional form for the two elements in parallel is $\frac{4394-342i}{28{,}565}$.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2A.2.2: Complex Polynomial Identities

Use properties of complex numbers to expand or factor each expression.

1. $(20 + i)(20 - i)$

2. $(4 + 3i)(4 - 3i)$

3. $(7x + i)(7x - i)$

4. $x^2 + 64$

5. $25x^2 + 484$

6. $(9x + 15i)(9x - 15i)$

7. $36x^2 + 144$

Use the following information to solve problems 8–10.

> Impedance measures the total opposition that a circuit presents to an electric current. The impedance of an element can be represented using a complex number $V + Ii$, where V is the element's voltage and I is the element's current. If the impedance of Element 1 is Z_1 and the impedance of Element 2 is Z_2, the total impedance of the two elements in parallel is $\frac{1}{Z_1} + \frac{1}{Z_2}$. Find the total impedance for each given set of elements.

8. Element 1: $10 + i$; Element 2: $8 + 2i$

9. Element 1: $13 + 2i$; Element 2: $11 + i$

10. Element 1: $11 + i$; Element 2: $12 + 3i$

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Lesson 2A.2.3: The Binomial Theorem

Warm-Up 2A.2.3

Isabella is on a road trip with her family. She records the time her family has been traveling, in hours, and the total distance they have traveled, in miles, in the table below.

Hours	Miles
2	130
3	195
4	260

1. How many miles is her family traveling each hour?

2. If her family travels at the same rate for the next hour, what will be the total distance traveled after 5 hours?

3. If her family continues to travel at the same rate, what distance will they have traveled after 6.5 hours?

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Instruction

Lesson 2A.2.3: The Binomial Theorem

Common Core State Standards

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1 + r)^n$ *as the product of P and a factor not depending on P.*

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$*, thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.

A–APR.4 Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*

A–APR.5 (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[1]

Warm-Up 2A.2.3 Debrief

1. How many miles is her family traveling each hour?

   The distance traveled is increasing by 65 miles after each hour, so her family is traveling at 65 miles per hour.

2. If her family travels at the same rate for the next hour, what will be the total distance traveled after 5 hours?

   As the time spent traveling increases by 1 hour, the distance traveled increases by 65 miles. Add 65 miles to the distance traveled after 4 hours to find the total distance traveled after 5 hours.

   $260 + 65 = 325$

   The family will have traveled 325 miles after 5 hours.

[1]The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.

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3. If her family continues to travel at the same rate, what distance will they have traveled after 6.5 hours?

   Isabella's family is traveling at 65 miles per hour, so after 6.5 hours the total distance traveled is 65 miles/hour • 6.5 hours.

   $65(6.5) = 422.5$

   The family will have traveled 422.5 miles after 6.5 hours.

Connection to the Lesson

- Students will recognize patterns, and will use observed patterns to find unknown values.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- using terminology related to polynomials, such as *terms* and *powers* (A–SSE.1)
- finding products of polynomials (A–APR.1)
- finding powers of numbers and variables (8.EE.1)

Introduction

Expanding binomial expressions can sometimes be tedious and time-consuming, but the expansion of binomials raised to a power follows patterns. Understanding these patterns not only helps with determining the binomial expansion, but also with finding single terms in a binomial.

Key Concepts

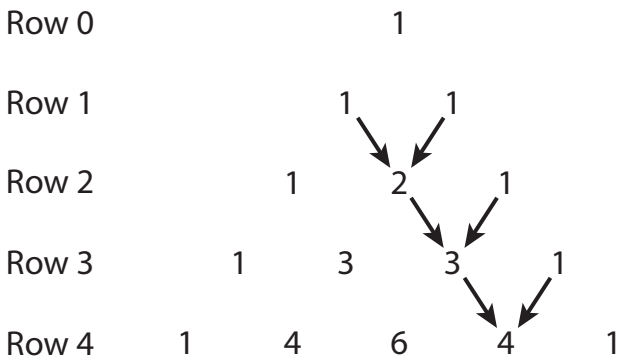
- When a binomial is raised to a power, $(a + b)^n$, there is a pattern in both the powers of the terms and the coefficients of the terms; a and b can be numbers, variables, or a combination of numbers and variables.
- The powers of the terms of the expanded form of $(a + b)^n$ follow the pattern $a^n b^0$, $a^{n-1} b^1$, $a^{n-2} b^2$, ... , $a^2 b^{n-2}$, $a^1 b^{n-1}$, $a^0 b^n$.
- The power of a is $n - k$, where k is the term number. Note that the first term is term number 0, or $k = 0$. The power of b is k.
- For example, if $n = 1$, $(a + b)^1$, the powers of the terms are a^1 and b^1. If $n = 2$, $(a + b)^2$, the powers of the terms are a^2, $a^1 b^1$, and b^2. If $n = 3$, $(a + b)^3$, the powers of the terms are a^3, $a^2 b^1$, $a^1 b^2$, and b^3.

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Instruction

- **Pascal's Triangle** is a triangle displaying a pattern of numbers in which the terms in additional rows are found by adding pairs of terms in the previous rows, so that any given term is the sum of the two terms directly above it.

The sum of two terms is directly below them.



- Notice that the number 1 is the top number of the triangle, and is also the first and last number of each row.
- The top row, containing only the number 1, is called "row 0." The next row is "row 1" and so on.
- The first term in each row is "term 0." The next term is "term 1," and so on.
- The pattern in Pascal's Triangle can be used to find the coefficients of terms in the expanded form of $(a + b)^n$. The coefficients of the terms depend on the power of the binomial, n.
- The coefficients of $(a + b)^n$ can be found in the nth row of Pascal's Triangle.
- For example, if $n = 1$, the coefficients of the terms are the terms in the row 1 of the triangle: 1 and 1. If $n = 2$, the coefficients of the terms are the terms in row 2 of the triangle: 1, 2, and 1. If $n = 3$, the coefficients of the terms are the terms in row 3 of the triangle: 1, 3, 3, and 1.
- The pattern of the powers and coefficients can be used to describe the pattern for finding any power of a binomial.
- The **Binomial Theorem** states that $(a + b)^n$ can be expanded using the following expression:

$$\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \bullet a^{n-k}b^k = 1a^n b^0 + \frac{n}{1} \bullet a^{n-1}b^1 + \frac{n(n-1)}{1 \bullet 2} \bullet a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \bullet 2 \bullet 3} \bullet a^{n-3}b^3 + \cdots + 1a^0 b^n$$

- Recall that the symbol ! represents a **factorial**, which is the product of an integer and all preceding positive integers; $n! = n \bullet (n-1) \bullet (n-2) \bullet \cdots \bullet 1$. For example, $5! = 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1$. By definition, $0! = 1$. Factorials can be performed on most calculators.

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Instruction

- The symbol Σ stands for ***sigma***, a Greek letter used to represent the summation of values.
- The values for $\frac{n!}{(n-k)!k!}$ can be found using the rows of Pascal's Triangle. Each expanded expression contains $n + 1$ terms.
- For example, if $n = 1$, use row 1 of Pascal's Triangle to determine the coefficients, and then follow the pattern of the powers using the Binomial Theorem: $(a + b)^1 = 1a^1 + 1b^1$. If $n = 2$, use row 2 of Pascal's Triangle and the pattern of the powers: $(a + b)^2 = 1a^2 + 2a^1b^1 + 1b^2$. If $n = 3$, use row 3 of Pascal's Triangle and the pattern of the powers: $(a + b)^3 = 1a^3 + 3a^2b^1 + 3a^1b^2 + 1b^3$.
- To expand a binomial expression raised to a power, replace a, b, and n in $(a + b)^n$ with the actual values.
- For example, to expand $(2x + 3)^2$, use the expansion $(a + b)^2 = 1a^2 + 2a^1b^1 + 1b^2$. Replace a with $2x$ and b with 3: $(2x + 3)^2 = (2x)^2 + 2(2x)(3) + (3)^2 = 4x^2 + 12x + 9$.
- Binomial expansion as it relates to Pascal's Triangle is displayed in the following table. (Coefficients and powers of 1 are not shown, but are implied.)

Binomial expansion	Pascal's Triangle	Row
$(a + b)^0 = 1$	1	0
$(a + b)^1 = a + b$	1 1	1
$(a + b)^2 = a^2 + 2ab + b^2$	1 2 1	2
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	1 3 3 1	3
$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$	1 4 6 4 1	4
$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$	1 5 10 10 5 1	5

- Pascal's Triangle can also be used to find the number of combinations, or unique groups of objects, from a larger group.
- The row number of the triangle is the total number of objects in the group, and the term number in the row is the number of objects being selected from the group. The coefficient for that term number is the number of combinations.

Common Errors/Misconceptions

- incorrectly replacing a and b with the actual binomial terms
- incorrectly calculating powers of numbers and variables
- misidentifying the coefficients from Pascal's Triangle

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Instruction

Guided Practice 2A.2.3

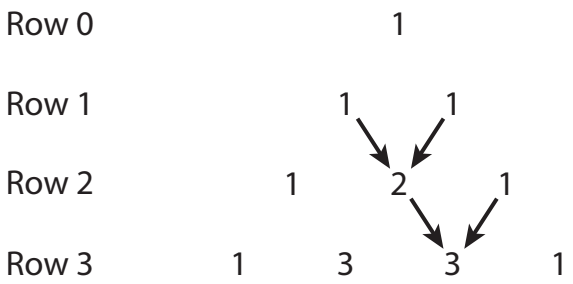
Example 1

Use the Binomial Theorem to expand $(6x + 2y)^3$.

1. Create Pascal's Triangle to the appropriate row.

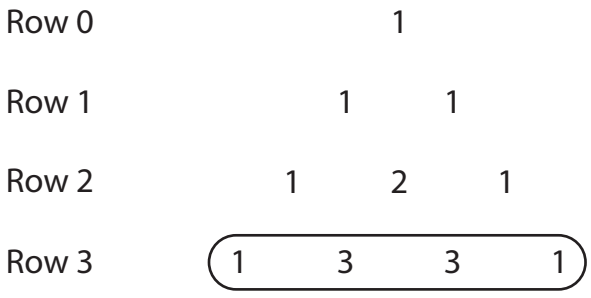
   Note that the first line of the triangle is "row 0." The expression $(6x + 2y)$ is raised to the third power, so the first four rows of the triangle are needed; that is, rows 0–3.

   Find the terms for each new row by adding pairs of terms in the row above.



2. Identify the row of Pascal's Triangle with the coefficients of the expanded expression.

   The power of the binomial is 3, so the coefficients will come from row 3 of Pascal's Triangle.



   The coefficients of the terms in row 3 are 1, 3, 3, and 1.

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Instruction

3. Write the expanded expression with the coefficients and powers of each term.

   The powers of the terms will follow the pattern a^3, a^2b^1, a^1b^2, b^3.

   Replace a with $6x$ and b with $2y$.

   The coefficients of the terms are from row 3 of Pascal's Triangle.

   $(6x)^3 + 3(6x)^2(2y) + 3(6x)(2y)^2 + (2y)^3$

4. Evaluate each term.

   Use the order of operations to evaluate each term, first evaluating any exponents, then finding the products.

$(6x)^3 + 3(6x)^2(2y) + 3(6x)(2y)^2 + (2y)^3$	Expression from the previous step
$= 216x^3 + 3(36x^2)(2y) + 3(6x)(4y^2) + 8y^3$	Evaluate the exponents.
$= 216x^3 + 216x^2y + 72xy^2 + 8y^3$	Find the products.

   The expression $(6x + 2y)^3$, when expanded, is $216x^3 + 216x^2y + 72xy^2 + 8y^3$.

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Instruction

Example 2

Find term 4 from row 6 of Pascal's Triangle.

1. Create Pascal's Triangle to the appropriate row.

   Note that the first line of the triangle is "row 0."

   To include row 6, seven lines of Pascal's Triangle need to be created.

   Find the terms for each new row by adding pairs of terms in the row above.

Row 0			1				
Row 1			1		1		
Row 2		1		2		1	
Row 3		1		3		3	1
Row 4	1		4		6		4	1
Row 5	1		5		10		10	5	1
Row 6	1	6		15		20		15	6	1

2. Find term 4 in the row.

   Recall that the first term in each row is named "term 0," so term 4 is actually the fifth term in the row.

Row 0			1				
Row 1			1		1		
Row 2		1		2		1	
Row 3		1		3		3	1
Row 4	1		4		6		4	1
Row 5	1		5		10		10	5	1
Row 6	1	6		15		20		(15)	6	1

   Term 4 from row 6 of Pascal's Triangle is 15.

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Instruction

Example 3

Find term 4 in the expanded expression of $(4x - 9y)^7$.

1. Identify the row of Pascal's Triangle with the coefficients of the expanded expression.

   The power of the binomial is 7, so the coefficients will come from row 7 of Pascal's Triangle.

Row 0				1				
Row 1			1		1			
Row 2		1		2		1		
Row 3	1		3		3		1	
Row 4		1		4		6		4		1	
Row 5	1		5		10		10		5		1
Row 6		1		6		15		20		15		6		1
Row 7	1		7		21		35		35		21		7		1

   The coefficients of the terms are 1, 7, 21, 35, 35, 21, 7, and 1.

2. Identify the powers of the binomial terms in term 4.

   In the expression $(4x - 9y)^7$, $n = 7$.

   The term number, k, is 4.

   The power of $4x$ is $n - k = 7 - 4 = 3$.

   The power of $-9y$ is $k = 4$.

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Instruction

3. Write the expanded term using the coefficient and the powers of the binomial terms.

   Term 4 from row 7 of Pascal's Triangle is 35.

   Combining this information with the powers found in step 2, we can determine that term 4 of the expanded expression is equal to $35 \bullet (4x)^3 \bullet (-9y)^4$.

   Use the order of operations to simplify the term.

   $35 \bullet (4x)^3 \bullet (-9y)^4$

   $= 35 \bullet 64x^3 \bullet 6561y^4$

   $= 14{,}696{,}640x^3y^4$

   Term 4 in the expanded expression of $(4x - 9y)^7$ is $14{,}696{,}640x^3y^4$.

Example 4

Kamali has 7 unique bracelets, and is trying to decide which 3 to wear on a date. Use Pascal's Triangle to find the number of different combinations of 3 bracelets that Kamali could choose from the 7 she owns.

1. Create Pascal's Triangle to the appropriate row.

   Note that the first line of the triangle is "row 0," so the first eight rows of the triangle are needed.

   Find the terms for each new row by adding pairs of terms in the row above.

Row 0				1				
Row 1			1		1			
Row 2		1		2		1		
Row 3	1		3		3		1	
Row 4		1		4		6		4		1	
Row 5			1		5		10		10		5		1	
Row 6		1		6		15		20		15		6		1
Row 7	1		7		21		35		35		21		7		1

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Instruction

2. Find term 3 in the row.

   The first term in each row is named "term 0," so term 3 is actually the fourth term in the row.

Row 0						1							
Row 1					1		1						
Row 2				1		2		1					
Row 3			1		3		3		1				
Row 4		1		4		6		4		1			
Row 5	1		5		10		10		5		1		
Row 6		1		6		15		20		15		6		1
Row 7	1		7		21		(35)		35		21		7		1

   Term 3 from row 7 of Pascal's Triangle is 35.

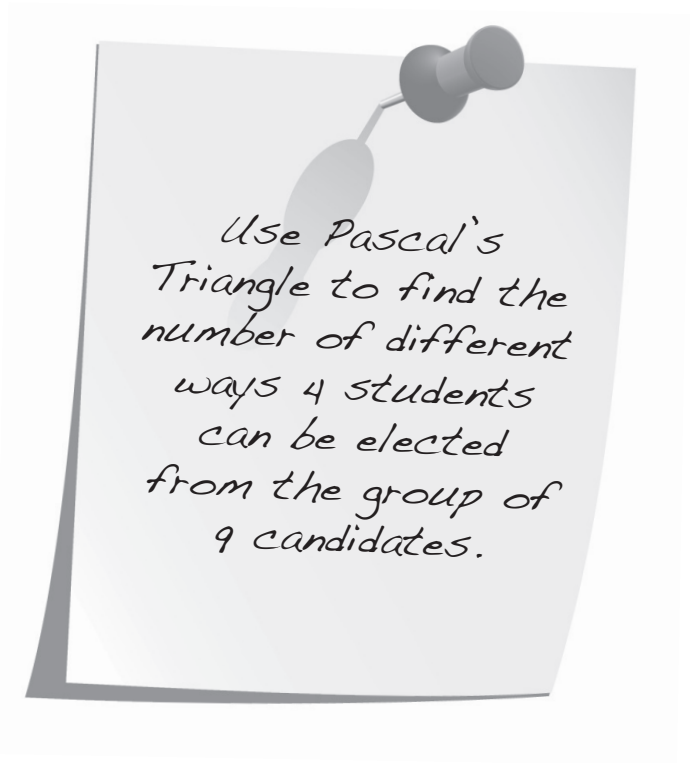
   There are 35 different combinations of 3 bracelets that Kamali could choose to wear out of the 7 bracelets she owns.

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Problem-Based Task 2A.2.3: Combinations of Candidates

SMP	
1	2 ✓
3	4 ✓
5	6
7	8 ✓

There are 9 students running for 4 openings on the student council. Only students who are running for election can be elected; no write-in candidates are allowed. An elected student can only hold one position at a time. Use Pascal's Triangle to find the number of different ways 4 students can be elected from the group of 9 candidates.



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Problem-Based Task 2A.2.3: Combinations of Candidates

Coaching

a. How many students are running for election?

b. Create Pascal's Triangle to the appropriate row.

c. How many students can be elected?

d. How many different combinations of 4 elected students can be created using the 9 candidates?

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Instruction

Problem-Based Task 2A.2.3: Combinations of Candidates

Coaching Sample Responses

a. How many students are running for election?

There are 9 students running for student council.

b. Create Pascal's Triangle to the appropriate row.

Note that the first line of Pascal's Triangle is "row 0," so ten lines of the triangle are needed to find row 9.

Row 0					1					
Row 1				1		1				
Row 2			1		2		1			
Row 3		1		3		3		1		
Row 4	1		4		6		4		1	
Row 5		1		5		10		10		5		1
Row 6			1		6		15		20		15		6		1
Row 7				1		7		21		35		35		21		7		1
Row 8					1		8		28		56		70		56		28		8		1
Row 9				1		9		36		84		126		126		84		36		9		1

c. How many students can be elected?

There are 4 openings on the student council, so 4 students can be elected.

d. How many different combinations of 4 elected students can be created using the 9 candidates?

Find term 4 in row 9 of the triangle.

Remember that the first term is "term 0," so you are actually finding the fifth term.

There are 126 ways 4 students can be elected.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2A.2.3: The Binomial Theorem

Find the coefficient of the given term in the specified row of Pascal's Triangle.

1. row 4, term 1

2. row 10, term 3

Expand each binomial using the Binomial Theorem.

3. $(-8x + 4)^3$

4. $(2x + y)^5$

Find the given term in the expanded form of each binomial.

5. $(5x + 1)^9$, term 7

6. $(-x - 2y)^{12}$, term 3

7. $(4x + y)^{14}$, term 13

Tamara is choosing cities to visit on her train trip across Europe. For problems 8–10, use Pascal's Triangle and the given numbers to determine the number of different combinations of unique cities Tamara can visit, if there are 15 total cities along the route. The train moves in one direction, so she can only stop in each city once.

8. number of cities to visit: 3

9. number of cities to visit: 5

10. number of cities to visit: 8

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Progress Assessment

Circle the letter of the best answer.

1. Which expression represents a factored form of $x^4 - 81$?

   a. $(x - 9)(x + 9)$
   c. $(x + 3)(x - 3)$
   b. $(x^2 - 9)(x^2 + 9)$
   d. $(x^2 + 3)(x^2 - 9)$

2. Which expression represents the factored form of $x^3 + 64$?

   a. $(x + 4)(x^2 - 4x + 16)$
   c. $(x + 4)(x - 4)$
   b. $(x - 4)(x^2 + 4x + 16)$
   d. $(x+4)(x^2+16)$

3. Which expression represents the factored form of $4x^2 + 100$?

   a. $4(x + 5i)^2$
   c. $4(x + 5i)(x - 5i)$
   b. $4(2x + 5i)^2$
   d. $(4x + i)(x - 100i)$

4. What is term 4 in row 12 of Pascal's Triangle?

   a. 48
   c. 495
   b. 220
   d. 792

5. Which expression represents the expanded form of $(-2x + y)^5$?

   a. $-32x^5 + 80x^4y - 80x^3y^2 + 40x^2y^3 - 10xy^4 + y^5$
   b. $-x^5 + 5x^4y - 10x^3y^2 + 10x^2y^3 - 5xy^4 + y^5$
   c. $x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$
   d. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

continued

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6. What is term 2 in the expanded form of $(x-3y)^{11}$?

   a. $-33x^{10}y$
   c. $-4{,}455x^8y^3$
   b. $55x^9y^2$
   d. $495x^9y^2$

7. Which expression represents the factored form of $125x^3-216$?

   a. $(5x-6)^3$
   c. $(5x^2+6)(25x-36)$
   b. $(5x-6)(25x^2+30x+36)$
   d. $(5x+6)(25x^2-30x+36)$

8. What is the result of $(11x + 18i)(11x - 18i)$?

   a. $121x^2+324$
   c. $121x^2+396i+324$
   b. $121x^2-324$
   d. $121x^2-396i+324$

9. What is term 6 in row 12 of Pascal's Triangle?

   a. 792
   c. 1,287
   b. 924
   d. 1,716

10. What is term 8 of the expanded form of $(5x-y)^9$?

    a. $45xy^8$
    c. $45x^8y$
    b. $-45xy^8$
    d. $-45x^8y$

Use what you have learned about Pascal's Triangle and the Binomial Theorem to complete all parts of the following problem.

11. Find the terms requested below for the expanded form of $(-x+3y)^7$.

    a. term 1
    c. term 5
    b. term 2
    d. term 7

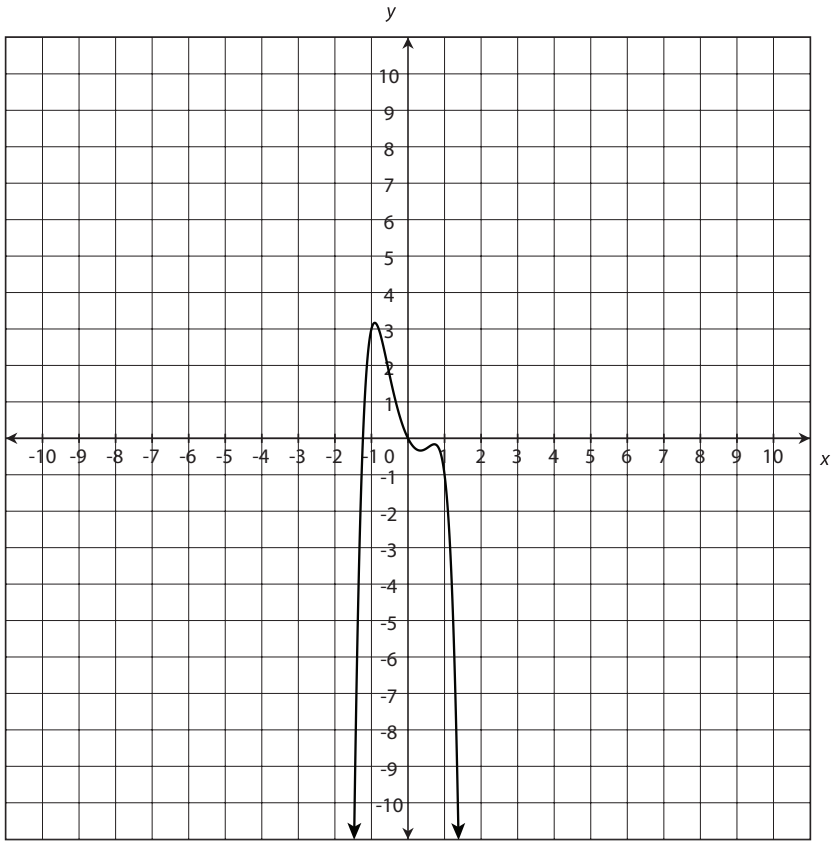
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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. What must be true about the degree and leading coefficient of the graphed polynomial?



a. The polynomial is an odd-degree polynomial and has a positive coefficient.

b. The polynomial is an odd-degree polynomial and has a negative coefficient.

c. The polynomial is an even-degree polynomial and has a positive coefficient.

d. The polynomial is an even-degree polynomial and has a negative coefficient.

continued

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Assessment

2. What is the remainder when $(x^3 - 3x^2 + 6x - 5)$ is divided by $(x - 2)$?

   a. –37

   b. 3

   c. –7

   d. 5

3. Which polynomial function has zeros of –3, 1, and 2?

   a. $f(x) = x^3 - 7x + 6$

   b. $f(x) = x^3 - 7x - 6$

   c. $f(x) = x^3 + 6x^2 + 11x + 6$

   d. $f(x) = x^3 - 6x^2 + 11x - 6$

4. One root of the equation $x^3 - 5x^2 + 11x - 15$ is 3. What are the other two roots?

   a. $2 + i$ and $2 - i$

   b. $-1 + 2i$ and $-1 - 2i$

   c. $1 + 2i$ and $1 - 2i$

   d. $-2 + i$ and $-2 - i$

5. What are all the possible rational roots for the polynomial equation $4x^3 + 2x^3 + 5x + 8 = 0$?

   a. $\pm\frac{1}{4}, \pm\frac{1}{2}, \pm 1, \pm 2, \pm 4, \pm 8$

   b. $\pm\frac{1}{8}, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm 2, \pm 4$

   c. $\pm\frac{1}{4}, \pm 1, \pm 2, \pm 8$

   d. $\pm\frac{1}{8}, \pm\frac{1}{2}, \pm 1, \pm 8$

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Lesson 3: Graphing Polynomial Functions

Instruction

Common Core State Standards

SMP
1 ✓ 2 ✓
3 ✓ 4 ✓
5 ✓ 6
7 ✓ 8 ✓

N–CN.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

A–APR.2 Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x – a$ is $p(a)$, so $p(a) = 0$ if and only if $(x – a)$ is a factor of $p(x)$.

A–APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Essential Questions

1. How can the degree and sign of the leading coefficient of a polynomial function be used to determine the end behavior of that function?
2. How can the degree and sign of the leading coefficient of a polynomial function be used to determine the number of turns of that function?
3. How can you use the number of sign changes in a function to determine the number and type of real zeros of that function?
4. How can synthetic substitution be used to find the value of a function?
5. How are factors, zeros, and x-intercepts of a polynomial function related?

WORDS TO KNOW

complex conjugate	the complex number that when multiplied by another complex number produces a value that is wholly real; the complex conjugate of $a + bi$ is $a – bi$
Complex Conjugate Theorem	Let $p(x)$ be a polynomial with real coefficients. If $a + bi$ is a root of the equation $p(x) = 0$, where a and b are real and $b \neq 0$, then $a – bi$ is also a root of the equation.

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Instruction

depressed polynomial	the result of dividing a polynomial by one of its binomial factors
end behavior	the behavior of the graph as x approaches positive or negative infinity
even-degree polynomial function	a polynomial function in which the highest exponent is an even number. Both ends of the graph of an even-degree polynomial function will extend in the same direction, either upward or downward.
factor of a polynomial	any polynomial that divides evenly into the function $p(x)$
Factor Theorem	The binomial $x - a$ is a factor of the polynomial $p(x)$ if and only if $p(a) = 0$.
Fundamental Theorem of Algebra	If $p(x)$ is a polynomial function of degree $n \geq 1$ with complex coefficients, then the related equation $p(x) = 0$ has at least one complex solution (root).
integer	a number that is not a fraction or decimal
Integral Zero Theorem	If the coefficients of a polynomial function are integers such that $a_n = 1$ and $a_0 \neq 0$, then any rational zeros of the function must be factors of a_0.
Irrational Root Theorem	If a polynomial $p(x)$ has rational coefficients and $a + b\sqrt{c}$ is a root of the polynomial equation $p(x) = 0$, where a and b are rational and $\sqrt{c}$ is irrational, then $a - b\sqrt{c}$ is also a root of $p(x) = 0$.
local maximum	the greatest value of a function for a particular interval of the function; also known as a *relative maximum*
local minimum	the least value of a function for a particular interval of the function; also known as a *relative minimum*
multiplicity (of a zero)	the number of times a zero of a polynomial function occurs
odd-degree polynomial function	a polynomial function in which the highest exponent is an odd number. One end of the graph of an odd-degree polynomial function will extend upward and the other end will extend downward.

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Instruction

polynomial function	a function with a general form of $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x^1 + a_0$, where a_1 is a rational number, $a_n \neq 0$, and n is a nonnegative integer and the highest degree of the polynomial
Rational Root Theorem	If the polynomial $p(x)$ has integer coefficients, then every rational root of the polynomial equation $p(x) = 0$ can be written in the form $\frac{p}{q}$, where p is a factor of the constant term $p(x)$ and q is a factor of the leading coefficient of $p(x)$.
relative maximum	the greatest value of a function for a particular interval of the function; also known as a *local maximum*
relative minimum	the least value of a function for a particular interval of the function; also known as a *local minimum*
Remainder Theorem	For a polynomial $p(x)$ and a number a, dividing $p(x)$ by $x - a$ results in a remainder of $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.
repeated root	a polynomial function with a root that occurs more than once
root	the x-intercept of a function; also known as *zero*
synthetic division	a shorthand way of dividing a polynomial by a linear binomial
synthetic substitution	the process of using synthetic division to evaluate a function by using only the coefficients
turning point	a point where the graph of the function changes direction, from sloping upward to sloping downward or vice versa
zero	the x-intercept of a function; also known as *root*

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Recommended Resources

- Hotmath.com. "Rational Zeros and the Fundamental Theorem of Algebra."

  http://www.walch.com/rr/00165

  This site provides numerous practice problems centering on determining the zeros of a function using the Fundamental Theorem of Algebra. Users may click to show hints and step-by-step guidance toward solutions.

- Illuminations. "Function Matching."

  http://www.walch.com/rr/00166

  Given a graphed function, users manipulate this interactive applet to recreate the graph of the identical function. Users can choose the type of function to match, or match a random function generated by the site.

- MathIsFun.com. "Remainder Theorem and Factor Theorem."

  http://www.walch.com/rr/00167

  This site summarizes both the Remainder Theorem and the Factor Theorem, and features examples as well as practice problems with worked solutions.

- Purplemath.com. "Solving Polynomials."

  http://www.walch.com/rr/00168

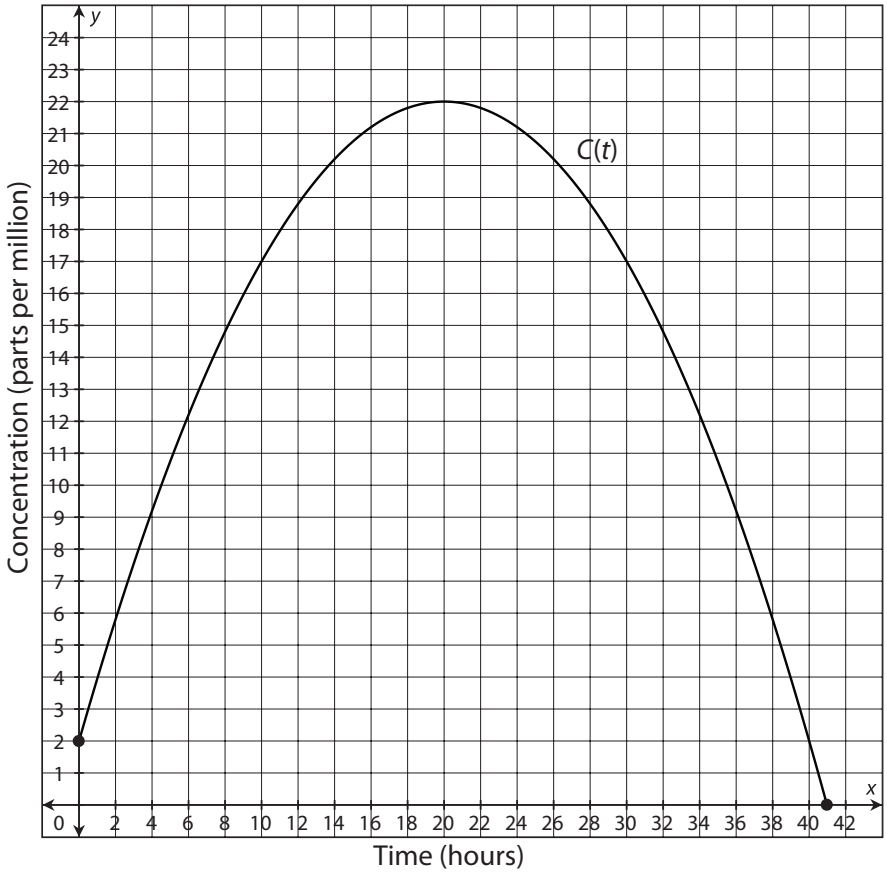
  This site offers a summary of solving large polynomials in addition to information on factoring and graphing polynomial functions.

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Lesson 2A.3.1: Describing End Behavior and Turns

Warm-Up 2A.3.1

The following graphed function represents the concentration, in parts per million, of a particular medication in the bloodstream after t hours. Use the graph to answer the questions that follow.



1. When is the concentration increasing? Decreasing?

2. After how many hours is the concentration of medicine at its highest? What is the concentration at this time?

3. What is the x-intercept and what does it represent?

4. What is the y-intercept and what does it represent?

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Instruction

Lesson 2A.3.1: Describing End Behavior and Turns

Common Core State Standard

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Warm-Up 2A.3.1 Debrief

1. When is the concentration increasing? Decreasing?

   The function starts increasing when the amount of time passed is at 0, and continues increasing until the function reaches the vertex, (20, 22). From that point, the function is decreasing. This means that the concentration of medication is increasing when the amount of time passed is less than 20 hours, or $t < 20$. The concentration is decreasing when the amount of time passed is greater than 20 hours, or $t > 20$.

2. After how many hours is the concentration of medicine at its highest? What is the concentration at this time?

   Use the vertex of the function to determine the maximum concentration.

   The vertex of this function is (20, 22); therefore, 20 hours after a person takes the medication, the medication is at its highest concentration of 22 parts per million.

3. What is the *x*-intercept and what does it represent?

   The *x*-intercept is the point at which the function crosses the *x*-axis. In this graph, the *x*-intercept appears to be (41, 0) and represents when, 41 hours after the dose is first taken, the concentration of medicine in the bloodstream is 0 parts per million.

4. What is the *y*-intercept and what does it represent?

   The *y*-intercept is the point at which the function crosses the *y*-axis. In this graph, the *y*-intercept appears to be (0, 2). It represents the time when the dose is first taken, at $t = 0$, and the concentration in the bloodstream at that time, 2 parts per million.

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Instruction

Connection to the Lesson

- Students will extend their understanding of when functions are increasing and decreasing.
- Students will also extend their understanding of maximums and minimums of quadratic functions to that of identifying local maximums and minimums of polynomial functions.
- Students will interpret maximums and minimums as well as *x*- and *y*-intercepts of polynomial functions.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- determining the degree and leading coefficient of a polynomial (A–SSE.1)
- writing a polynomial in standard form (A–SSE.2)

Introduction

By this point in your mathematics experience, you have worked extensively with functions: determining the slopes of linear functions, identifying the vertices of quadratic functions as maxima or minima, determining the end behavior of exponential functions, and identifying intercepts of many types of functions. You can extend the skills you have used to analyze functions you have previously studied in order to understand the graphs of other polynomial functions.

Key Concepts

- Recall that a polynomial function is a function with a general form of $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x^1 + a_0$, where a_1 is a rational number, $a_n \neq 0$, and n is the highest degree of the polynomial.
- Polynomial functions are defined for any function that contains positive integer exponents.
- Recall that **integers** are numbers that are not fractions or decimals.
- The degree of a polynomial function is the highest exponent to which the dependent variable is raised.
- For example, the equation $y = 3x^7 + 9x^3 - x + 4$ is a seventh-degree polynomial function because its highest exponent is 7 and all other exponents are positive whole numbers.
- $y = 4x^{\frac{3}{5}} + 6x - 3$ is not a polynomial function because the exponent is not a whole number. $y = -3\sqrt{x} + 4$ is also not a polynomial function since the square root of a number can be written as a power of $\frac{1}{2}$, which is not a whole number either. And $y = 5x^3 + 2x^{-4} + 6$ is not a polynomial function because not all exponents are non-negative integers.

End Behavior

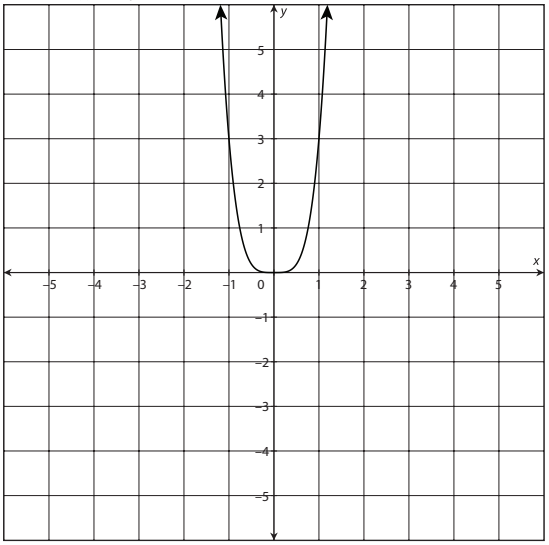
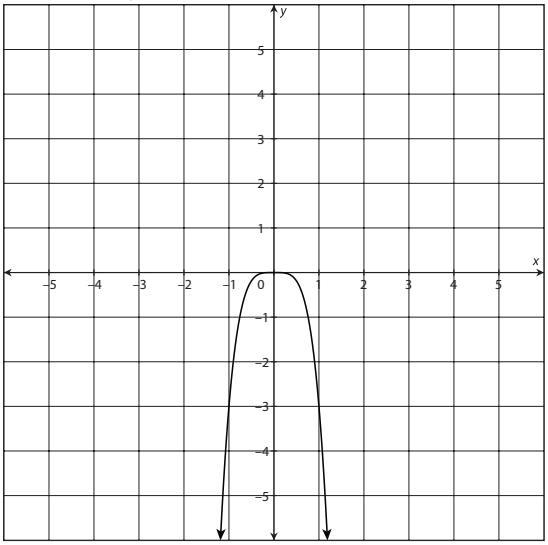
- To determine the **end behavior** of a polynomial function, or the behavior of the graph as x approaches positive or negative infinity, consider the highest degree of the polynomial and its coefficient, ax^n.
- If n is even, the polynomial function is considered an **even-degree polynomial function**.

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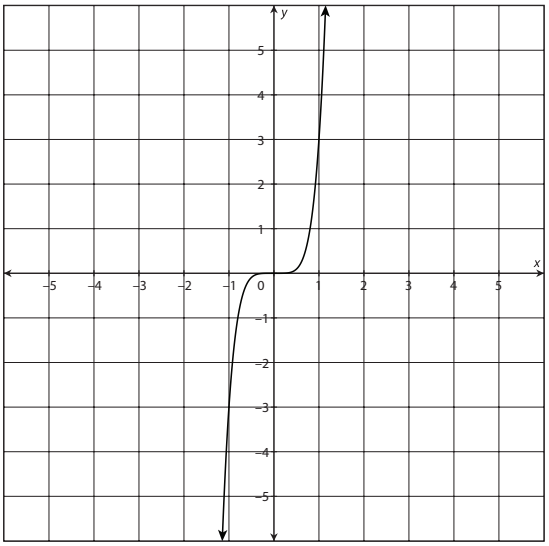
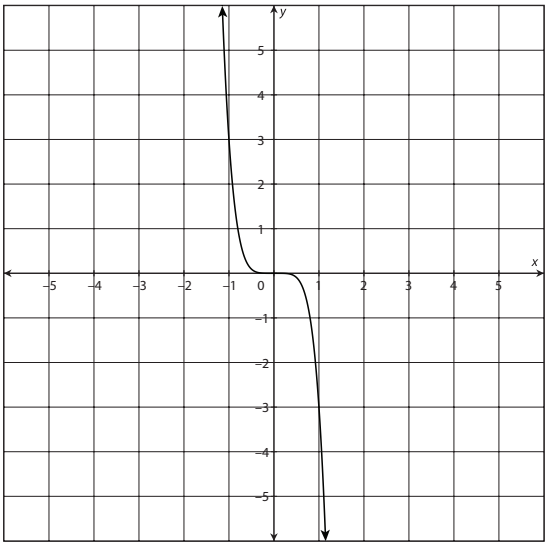
Instruction

- When n is even and a is positive, then both ends of the function will extend upward. That is, the value of $f(x)$ approaches positive infinity as x approaches negative infinity, and also when x approaches positive infinity. Symbolically, this can be written $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.
- When n is even and a is negative, then both ends of the function will extend downward. That is, the value of $f(x)$ approaches negative infinity as x approaches negative infinity, and also when x approaches positive infinity. Symbolically, this can be written $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$.
- If n is odd, the polynomial function is considered an **odd-degree polynomial function**.
- When n is odd and a is positive, then one end of the function will extend down to the left and the other end will extend up to the right. That is, the value of $f(x)$ approaches positive infinity as x approaches positive infinity, and the value of $f(x)$ approaches negative infinity as x approaches negative infinity. Symbolically, this can be written $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.
- When n is odd and a is negative, then one end of the function will extend up to the left and the other end will extend down to the right. That is, the value of $f(x)$ approaches positive infinity as x approaches negative infinity, and the value of $f(x)$ approaches negative infinity as x approaches positive infinity. Symbolically, this can be written $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$.

Even-degree polynomials	
Positive leading coefficient	**Negative leading coefficient**
Example: $y = 3x^4$	Example: $y = -3x^4$
	
$f(x) \to +\infty$ as $x \to -\infty$ $f(x) \to +\infty$ as $x \to +\infty$	$f(x) \to -\infty$ as $x \to -\infty$ $f(x) \to -\infty$ as $x \to +\infty$

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Instruction

Odd-degree polynomials	
Positive leading coefficient Example: $y = 3x^5$	**Negative leading coefficient** Example: $y = -3x^5$
	
$f(x) \to -\infty$ as $x \to -\infty$ $f(x) \to +\infty$ as $x \to +\infty$	$f(x) \to +\infty$ as $x \to -\infty$ $f(x) \to -\infty$ as $x \to +\infty$

Turning Points

- A **turning point** of a function is a point where the graph of the function changes from sloping upward to sloping downward or, alternatively, from sloping downward to sloping upward.
- To determine the maximum number of turning points of a function, subtract 1 from the highest degree of the polynomial. In other words, find $n - 1$.
- For instance, the polynomial function $y = 3x^7 + 9x^3 - x + 4$ can have no more than 7 – 1, or 6, turning points.
- The maximum number of turning points does not necessarily indicate the actual number of turning points of a function, just that it can have no more than that number. Some functions may have fewer turning points than the number calculated.
- A turning point corresponds to a **local maximum**, the greatest value of a function for a particular interval of the function, or a **local minimum**, the least value of a function for a particular interval of the function. A local maximum may also be referred to as a **relative maximum** and a local minimum may also be referred to as a **relative minimum**.

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Instruction

Roots of a Polynomial Function

- The highest degree of the polynomial determines the maximum number of **roots**, or *x*-intercepts of a function.
- A polynomial function with a degree of 10 could have up to 10 roots, but could also have 0 to 9 roots, depending on the specific equation.
- Recall that real numbers include all rational and irrational numbers, but do not include imaginary and complex numbers.

Sketching a Polynomial Function

- Being able to identify the general end behavior, the possible number of turning points, and the maximum number of roots of a polynomial function can be helpful in creating a rough sketch of the function.
- Start by choosing at least six *x*-values that are both positive and negative. It is also useful to choose the value of 0.
- As you've done in previous courses, substitute each chosen *x*-value into the given function and evaluate to determine the corresponding *y*-value. Then, plot the points on a graph.
- Be sure to smoothly connect all chosen points to illustrate the graph of the function.
- Graphing calculators are especially helpful when sketching a complicated function.

Common Errors/Misconceptions

- assuming the maximum number of calculated turning points is the actual number of turning points of a function
- assuming the maximum number of roots is the actual number of roots of a function
- not using the highest degree term to identify the end behavior of the polynomial function
- not using the highest degree of the polynomial function to determine the maximum number of turns
- confusing even-degree and odd-degree functions with even and odd functions

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Guided Practice 2A.3.1

Example 1

Determine the end behavior, maximum number of turning points, and maximum number of real roots of the function $f(x) = 6x^5 - 3x^4 + 2x + 7$.

1. Identify the leading coefficient and degree of the polynomial function.

   The function $f(x) = 6x^5 - 3x^4 + 2x + 7$ is a fifth-degree polynomial function because the highest exponent is 5.

   The coefficient of the term containing the highest exponent is 6; therefore, 6 is the leading coefficient.

2. Determine the end behavior of the function.

   The leading coefficient, 6, is positive and the degree of the function, 5, is odd; therefore, the graph of the function will extend down to the left and up to the right.

   Symbolically, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

3. Determine the maximum number of turning points of the polynomial function.

   To determine the maximum number of turning points, subtract 1 from the degree of the polynomial; that is, find $n - 1$.

   The degree of the polynomial is 5, and $5 - 1 = 4$.

   The maximum number of turning points in the graph is 4.

4. Determine the maximum number of real roots of the polynomial function.

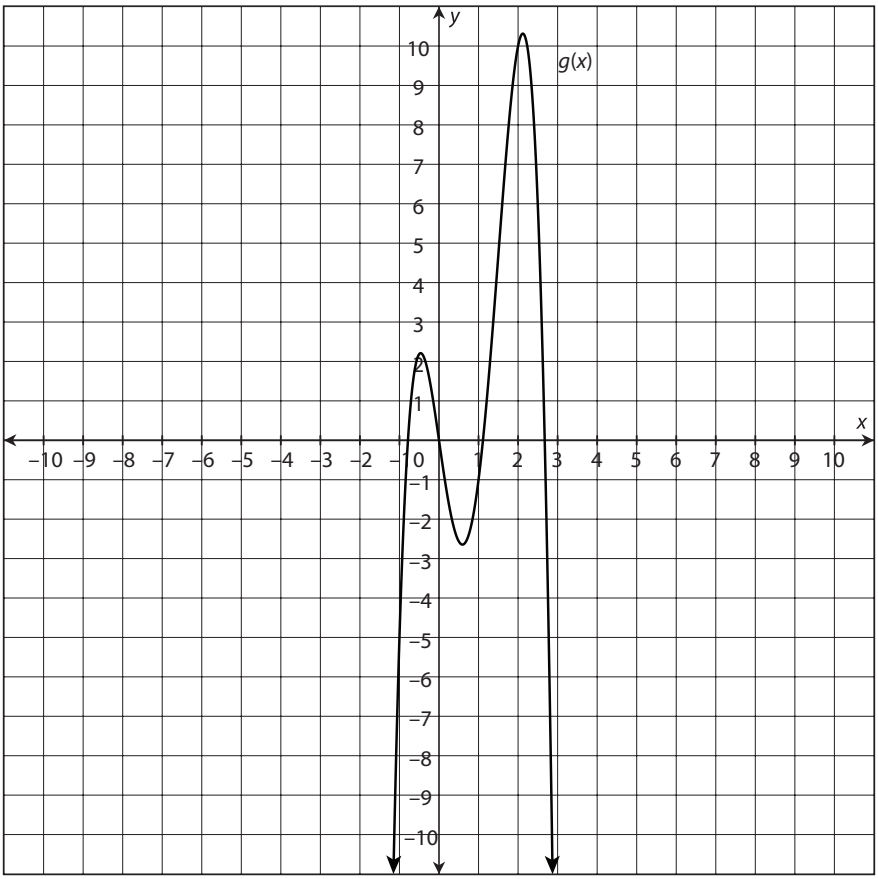
   The maximum number of real roots is equal to the degree of the polynomial; therefore, the maximum number of roots of the function $f(x) = 6x^5 - 3x^4 + 2x + 7$ is 5.

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Instruction

Example 2

Describe the end behavior of the given graph of $g(x)$. Determine whether the graph represents an even-degree or odd-degree function, and determine the number of real roots.



1. Describe the end behavior of the graphed function, $g(x)$.

   Refer to the graph to determine the function's end behavior as x gets larger or smaller. The value of $g(x)$ approaches negative infinity as x approaches negative infinity, and also when x approaches positive infinity. Symbolically, this can be written as $g(x) \to -\infty$ as $x \to -\infty$ and $g(x) \to -\infty$ as $x \to +\infty$.

2. Determine whether the graph of $g(x)$ represents an even-degree or odd-degree function.

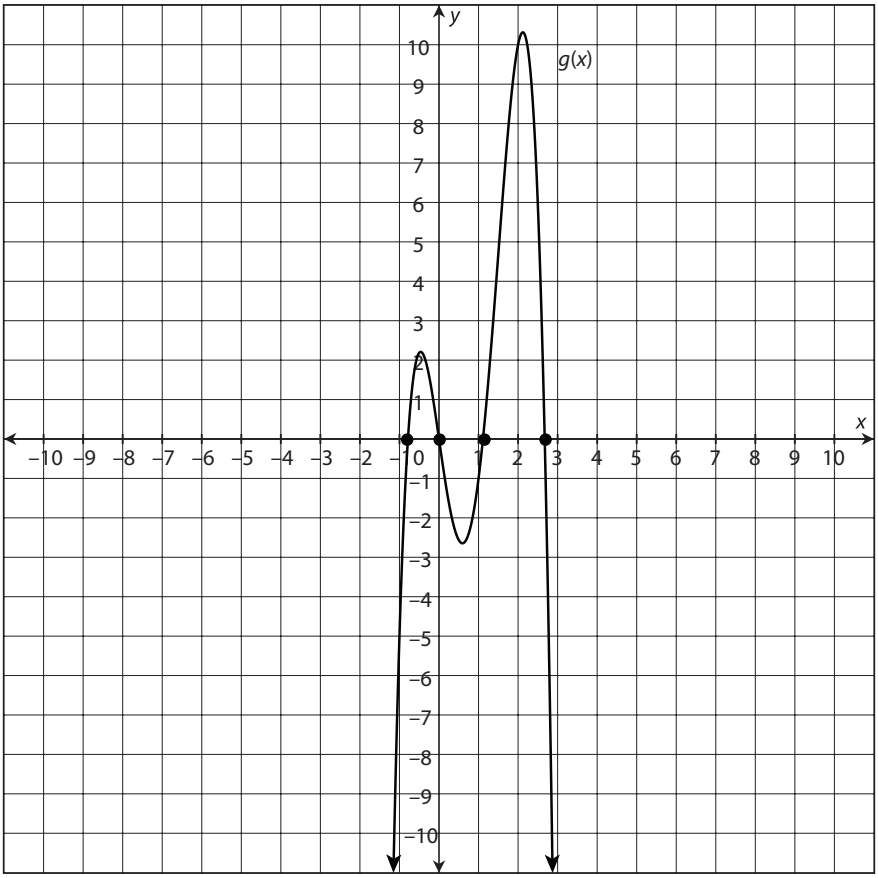
   Both ends of the graph are pointing in the same direction: downward. Therefore, the function must be an even-degree function.

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Instruction

3. Determine the number of real roots.

   Real roots are the points at which the graphed function intersects the *x*-axis. Because the *x*-axis contains only real numbers, only the real roots can be graphed. Determine the number of times the graph intersects the *x*-axis.



The graph intersects the *x*-axis at four points, so there are four real roots.

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Instruction

Example 3

Use a graphing calculator to graph the function $p(x) = -x^4 + 3x^2 + 4$. Summarize the end behavior and turning points of the function.

1. Graph the equation on your calculator.

On a TI-83/84:

Step 1: Press [Y=].

Step 2: Type the function into Y_1. Use the [X, T, θ, n] button for the variable x. To enter exponents, choose [CATALOG] and arrow down to ^.

Step 3: Press [WINDOW] to change the viewing window as needed.

Step 4: Press [GRAPH].

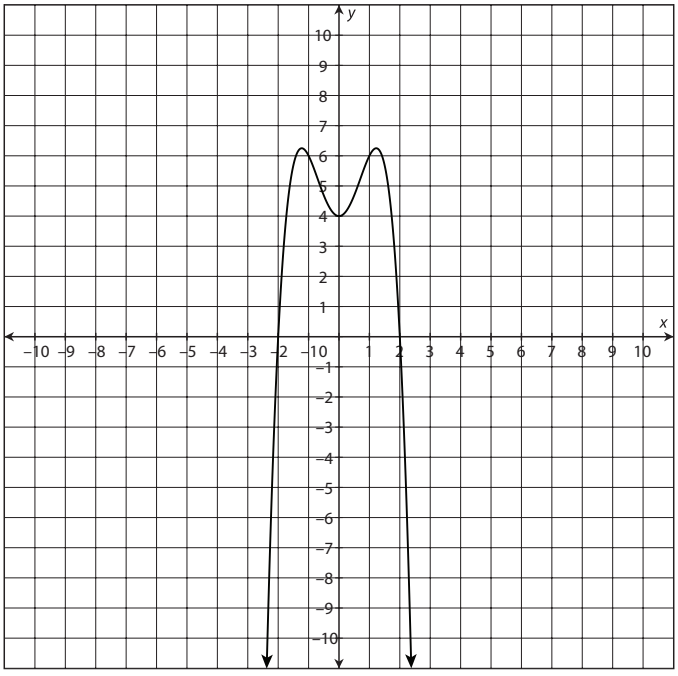
On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Type the function next to *f1*(*x*), or any available equation, and press [enter]. Use the [X] button for the letter x. Use the [^] button for exponents.

Step 4: To change the viewing window, press [menu]. Select 1: Window Settings.



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Instruction

2. Analyze the table of values.

Refer to the table of values for the function.

On a TI-83/84:

Step 1: Choose [TABLE]. Scroll up and down the table to view various points on the graph.

On a TI-Nspire:

Step 1: Choose [menu]. Navigate to 7: Table, 1: Split-Screen Table. Arrow up and down the table to view various points on the graph.

Points include:

x	y
–3	–50
–2	0
–1	6
0	4
1	6
2	0
3	–50

Both ends of the graph extend in the same direction, and the graph opens downward. Therefore, we know that this is an even-degree polynomial with a negative leading coefficient, so $p(x) \to -\infty$ as $x \to -\infty$, and $p(x) \to -\infty$ as $x \to +\infty$.

We can see from the graph that this function has three turning points. It can be seen from the graph and the table of values that one turning point is at (0, 4). $x = 0$ is less than the surrounding points, so it is a local (or relative) minimum. Two other approximate turning points are (–1, 6) and (1, 6). $x = -1$ and 1 are greater than the surrounding points, so they are both local (or relative) maximums.

The graph intersects the x-axis at two points, indicating that there are two real roots of this function.



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Instruction

Example 4

Create a rough sketch of the graph of a sixth-degree polynomial function with a positive leading coefficient.

1. Determine the end behavior of the possible function.

   The function is a sixth-degree polynomial; therefore, it is an even-degree polynomial. Both ends of the graphed function will extend in the same direction. Because the leading coefficient is positive, both ends of the graphed function will extend upward. Symbolically, $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

2. Determine the maximum number of turns.

   The maximum number of turns is defined by the degree of the polynomial minus 1.

   maximum number of turns = $n - 1$

   $= 6 - 1$

   $= 5$

   A sixth-degree polynomial will have no more than 5 turning points.

3. Determine the maximum number of real roots.

   The maximum number of real roots, or x-intercepts, is equal to the degree of the polynomial. This particular function will have no more than 6 real roots.

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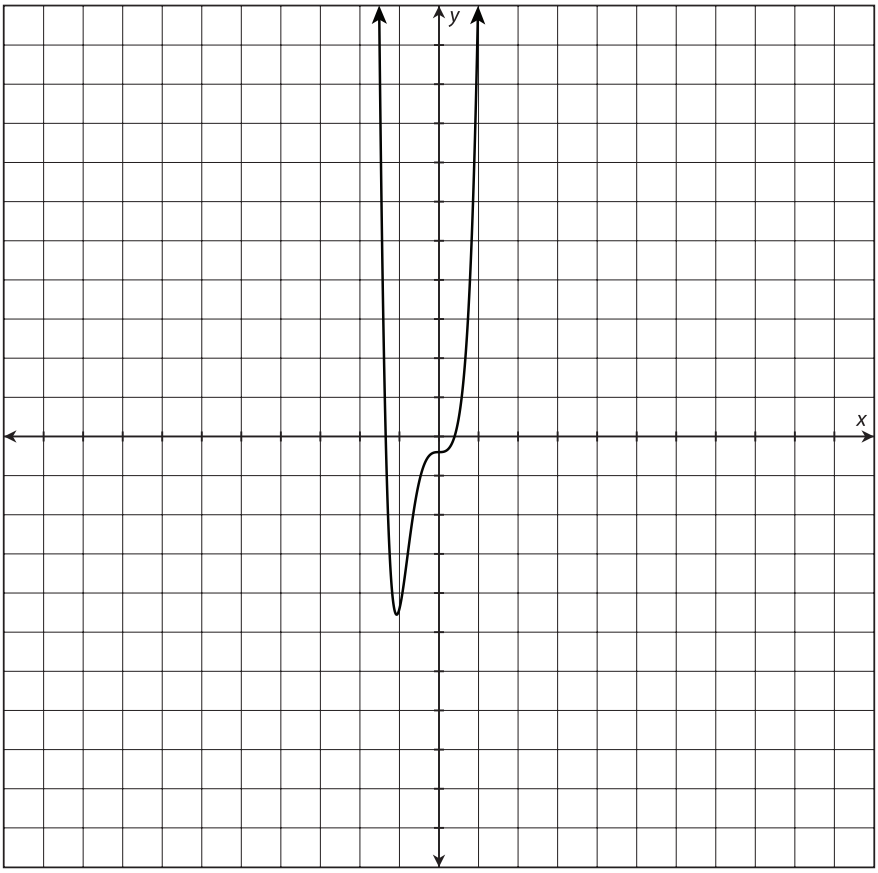
Instruction

4. Use the information from the previous steps to sketch a possible graph of a sixth-degree polynomial with a positive leading coefficient.

   We have determined the following about a sixth-degree polynomial function with a positive leading coefficient:

   - Both ends of the function will extend upward.
   - It can have no more than 5 turning points.
   - It can have no more than 6 real roots.

   The following graph extends upward at both ends, has two turning points, and has two real roots. It satisfies all the requirements of a sixth-degree polynomial function with a positive leading coefficient, so it is a possible graph.

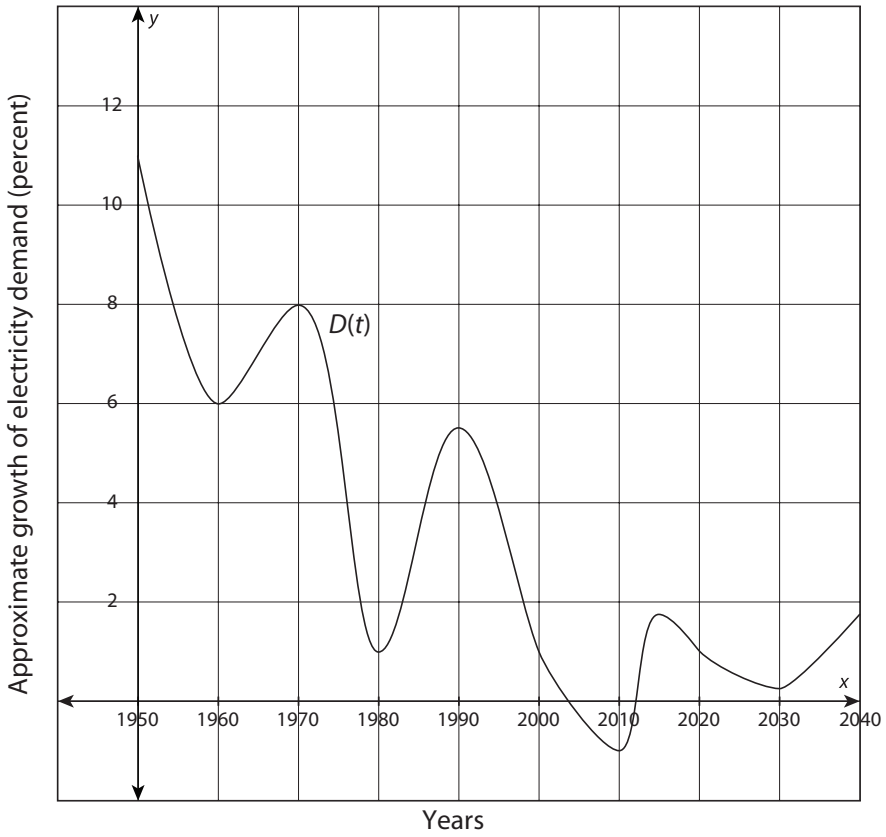


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Problem-Based Task 2A.3.1: It's Electric!

SMP	
1 ✓	2
3 ✓	4 ✓
5	6
7	8

The growth of demand for electricity in the United States changes from year to year. Below is a sketch of a polynomial function that represents that approximate growth since 1950 and includes projections to 2040. Consider a possible polynomial function that could represent this information. Using end behavior, turning points, and roots, do you expect the growth of electricity demand to increase or decrease from 2040 to 2050? Explain your reasoning.



Source: U.S. Energy Information Administration: Annual Energy Outlook 2013

Do you expect the growth of electricity demand to increase or decrease from 2040 to 2050?

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Problem-Based Task 2A.3.1: It's Electric!

Coaching

a. What are the turning points of the polynomial function?

b. What does each turning point represent?

c. What are the real roots of this polynomial?

d. What do the real roots represent?

e. What is the behavior of this function as t approaches positive infinity?

f. Do you expect the growth of demand for electricity to increase or decrease from 2040 to 2050? Explain.

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Instruction

Problem-Based Task 2A.3.1: It's Electric!

Coaching Sample Responses

a. What are the turning points of the polynomial function?

The turning points of the graph are where the graph of the function changes from sloping upward to downward or vice versa. This function appears to have 7 turning points. The turning points are approximately (1960, 6), (1970, 8), (1980, 1), (1990, 5.5), (2010, –1), (2012, 1.75), and (2030, 0.25).

b. What does each turning point represent?

The point (1960, 6) is a local minimum; this indicates the growth of demand for electricity decreased from 1950 to 1960 and then began to increase until the year 1970, when the growth of demand reached a local maximum. After 1970, the growth of demand again decreased until it reached another local minimum in 1980. The point (1990, 5.5) represents another local maximum. The growth of demand decreased from this point until 2010, when the growth of demand was approximately –1%. The growth of demand increased until 2012 and is projected to decrease until the year 2030, when it is projected that the growth of demand will increase until at least 2040.

c. What are the real roots of this polynomial?

The real roots of this function are found at the x-intercepts. This function appears to have two real roots, at the points (2003, 0) and (2011, 0).

d. What do the real roots represent?

The real roots represent the years in which the growth of demand for electricity was 0%.

e. What is the behavior of this function as t approaches positive infinity?

It is difficult to tell from the graph what the end behavior of the function is, but it appears as though $D(t)$ approaches $+\infty$ as t approaches $+\infty$.

f. Do you expect the growth of demand for electricity to increase or decrease from 2040 to 2050? Explain.

Based on the number of turning points and the end behavior of the function represented by the graph, as well as possible advances in technology, it is possible that the growth in demand for electricity will continue to increase from 2040 to 2050. However, it is also possible that because of alternate power sources, the growth in demand for electricity will decline from 2040 to 2050.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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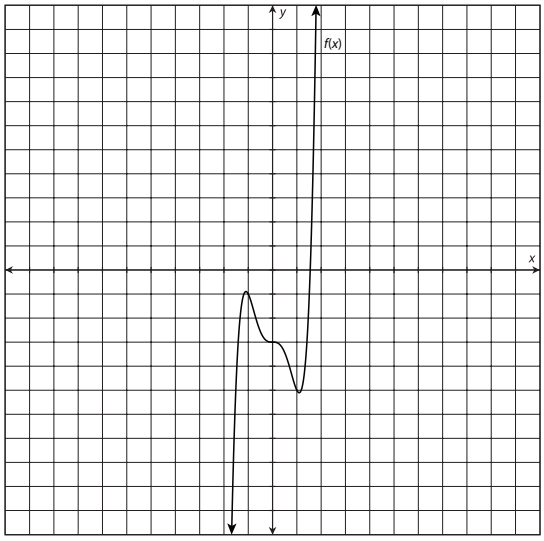
Practice 2A.3.1: Describing End Behavior and Turns

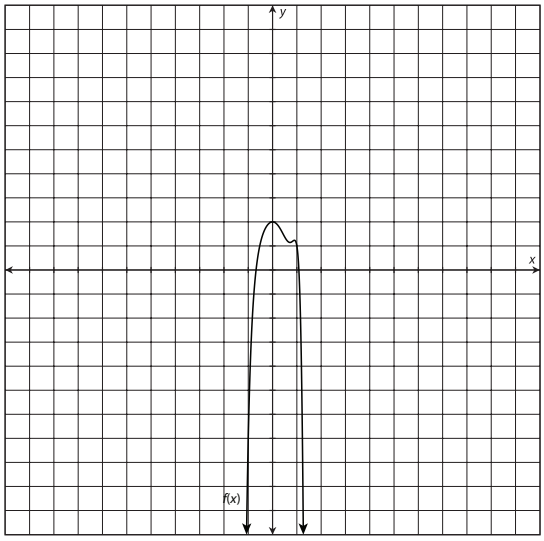
For problems 1 and 2, determine the end behavior, the maximum number of turning points, and the maximum number of real roots of each function.

1. $f(x) = -2x^4 + 6x^3 + 5x$

2. $g(x) = 4x^3 - 7x^2 + 8$

For problems 3 and 4, describe the end behavior of each graph. Determine whether the graph represents an even-degree or odd-degree function, and determine the number of real roots.

3. 

4. 

For problems 5 and 6, create a rough sketch of a possible graph of the function described.

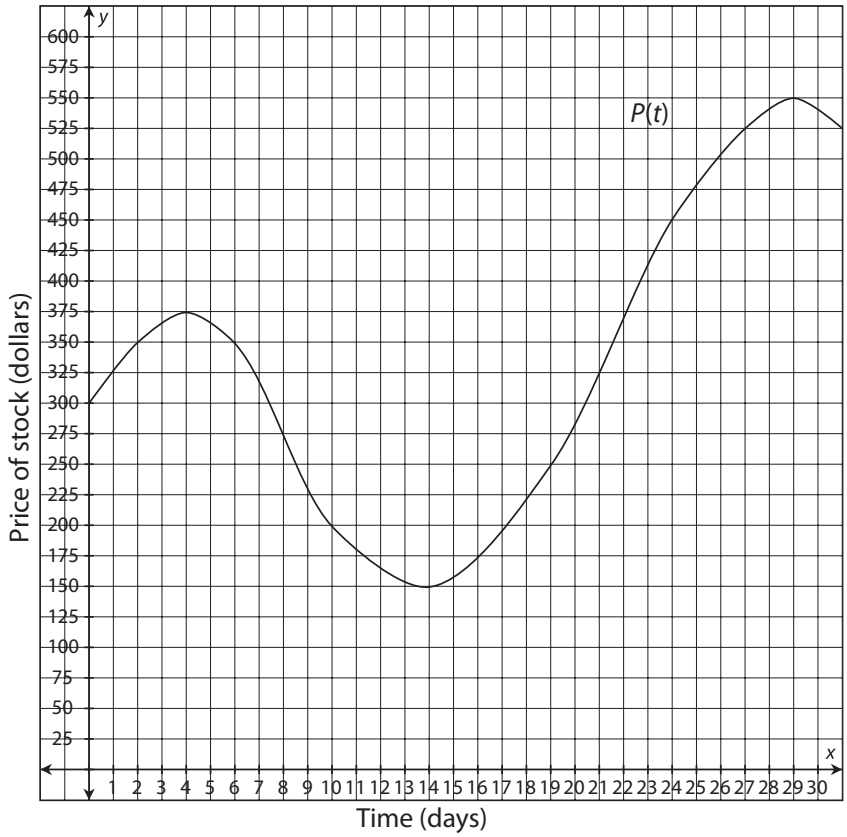
5. a fifth-degree polynomial function with a negative leading coefficient

6. a fourth-degree polynomial function with a positive leading coefficient

continued

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The following graph models the price of a particular stock over a period of time. Use this graph to complete problems 7–10.



7. Estimate the turning points of the graph of this function.

8. What do the turning points mean in terms of the price of the stock?

9. Describe the end behavior of this graph.

10. If this graph were modeled by a polynomial function, what is the least degree the equation could have? Explain your answer.

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Lesson 2A.3.2: The Remainder Theorem

Warm-Up 2A.3.2

A factory produces sweatshirts for a sports team, and packages the sweatshirts in boxes of 20. Every hour, the factory produces 9,284 sweatshirts.

1. How many full boxes of sweatshirts are packaged in 1 hour?

2. How many sweatshirts are produced but not packaged in 1 hour?

3. If the remaining sweatshirts were set aside, what fraction of a box would they make up?

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Instruction

Lesson 2A.3.2: The Remainder Theorem

Common Core State Standard

A–APR.2 Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

Warm-Up 2A.3.2 Debrief

1. How many full boxes of sweatshirts are packaged in 1 hour?

   Divide the number of sweatshirts produced each hour, 9,284, by the number of sweatshirts that are in one box, 20.

$$
\begin{array}{r}
464 \\
20\overline{)9284} \\
\underline{80} \\
128 \\
\underline{120} \\
84 \\
\underline{80} \\
4
\end{array}
$$

   9,284 divided by 20 is approximately 464. It is unnecessary to continue the division past the whole number, because only the number of full boxes concerns us.

   464 full boxes of sweatshirts are packaged in 1 hour.

2. How many sweatshirts are produced but not packaged in 1 hour?

   First, determine the number of sweatshirts that are packaged in 1 hour: 464 • 20 = 9280.

   In 1 hour, 9,280 sweatshirts are packaged in 464 boxes.

   Now subtract the number of packaged sweatshirts from the number of produced sweatshirts: 9284 – 9280 = 4.

   The remaining number of sweatshirts that are produced but not packaged is 4.

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Instruction

3. If the remaining sweatshirts were set aside, what fraction of a box would they make up?

   At the end of 1 hour, 4 sweatshirts remain unpackaged. Each box holds 20 sweatshirts when full. Therefore, the remaining sweatshirts can be represented as the fraction $\frac{4}{20}$ or, in reduced form, $\frac{1}{5}$. The remaining unpackaged sweatshirts make up $\frac{1}{5}$ of a box.

Connection to the Lesson

- Students will extend their understanding of long division of whole numbers to division of polynomials.
- Students will write the remainders of division problems as fractions.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- multiplying and dividing monomials (A–SSE.2)
- evaluating functions for a given value of x (F–IF.2)

Introduction

In mathematics, the word "remainder" is often used in relation to the process of long division. You are probably familiar with dividing whole numbers. For example, the result (or quotient) of 8 divided by 4 is 2; that is, 8 ÷ 4 = 2. In other words, if the dividend of 8 units were divided by a divisor of 4, each group would have 2 units.

The process of division does not always result in a whole number. For instance, the result of 14 divided by 5, or 14 ÷ 5, is 2 with a remainder of 4. Recall that this means that if a dividend of 14 units were divided by a divisor of 5, then each group would have 2 units and there would be 4 units left over. 4 is referred to as the remainder. In earlier grades, you often wrote the quotient and its remainder as 2R4 or $2\frac{4}{5}$. The idea of having remainders also extends to dividing polynomials. Sometimes when polynomials are divided, the result is a polynomial; other times there is a remainder.

Key Concepts

- Long division, used to divide whole numbers, can also be used to divide polynomials. Recall that the dividend is the quantity being divided, and the divisor is the quantity by which it is being divided: dividend ÷ divisor = quotient.
- Long division with polynomials can sometimes be long and tedious; fortunately, there is another way to divide polynomials.
- The process of **synthetic division**, a shorthand way of dividing a polynomial by a linear binomial by using only the coefficients, is often used instead.
- In order to use synthetic division, the dividend must be a polynomial written in standard form, ordered by the power of the variables, with the largest power listed first. If a term is missing, 0 must be used in its place.
- For example, compare $4x^2 + 16$ to the standard form of a polynomial, $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x^1 + a_0$. Notice that $4x^2 + 16$ does not have a value for $a_{n-1}x^{n-1}$, or in this case, ax. To prepare this polynomial for synthetic division, substitute 0 as the coefficient (a) in the ax term: $4x^2 + 0x + 16$. Now you can use the coefficients 4, 0, and 16 to perform the synthetic division.

SRB p. U2A-66

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Instruction

- For synthetic division, the divisor must be of the form $(x - a)$, where a is a real number.
- Use the following steps to divide polynomials using synthetic division. An example has been provided for clarity.

Synthetic Division of Polynomials	
Example: $(3x^2 - 20x + 12) \div (x - 3)$	
1. Write the coefficients of the dividend, $3x^2 - 20x + 12$: 3, –20, and 12.	3 −20 12
2. Identify the value of a in the divisor, $(x - a)$. Write this value, 3, in the upper left corner.	3 | 3 −20 12
3. Create a horizontal line below the coefficients, allowing for space to write above and below the line.	3 | 3 −20 12 _____
4. Write the first coefficient in the dividend below the horizontal line.	3 | 3 −20 12 _____ 3
5. Multiply the number below the horizontal line by the value of a and write the product under the next coefficient and above the horizontal line.	3 | 3 −20 12 9 _____ 3
6. Add the numbers in the new column. Write the result below the horizontal line in that column.	3 | 3 −20 12 9 _____ 3 −11
7. Repeat steps 5 and 6 until addition has been completed for all columns.	3 | 3 −20 12 9 −33 _____ 3 −11 −21
8. Draw a box around the far right sum.	3 | 3 −20 12 9 −33 _____ 3 −11 $\boxed{-21}$
9. The numbers below the horizontal line represent the quotient. These numbers are the coefficients of the polynomial quotient in the order of decreasing degree. The boxed number is the remainder. Place the remainder over the divisor to express the final term of the polynomial.	$= 3x - 11 - \frac{21}{x-3}$

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Instruction

- If you are dividing by a linear function, $(x - a)$, the order of the quotient is 1 less than the dividend. The remainder, if any, is a constant.
- If the remainder is 0, then the divisor is a factor of the polynomial.
- Synthetic division can also be used to find the value of a function. This is known as **synthetic substitution**.
- To evaluate a polynomial using synthetic substitution, follow the same process described for synthetic division. For example, given the function $3x^2 - 20x + 12$, if you must determine the value of the function at $x = 3$, use 3 as the a value in the divisor of the synthetic division. The resulting remainder gives the value of the polynomial when evaluated at $x = 3$.
- If a polynomial $p(x)$ is divided by $(x - a)$, then the remainder, r, is equal to $p(a)$.
- This process leads to the **Remainder Theorem**.

Remainder Theorem
For a polynomial $p(x)$ and a number a, dividing $p(x)$ by $x - a$ results in a remainder of $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

- To take a closer look at the Remainder Theorem, let's work with the same polynomial we used to demonstrate synthetic division of polynomials: $3x^2 - 20x + 12$.
- Let $p(x) = 3x^2 - 20x + 12$. We can use synthetic substitution (by following the process of synthetic division), to evaluate this function for $x = 3$. The result is the remainder, or –21. Because this result is a number other than 0, the Remainder Theorem allows us to conclude that $(x - 3)$ is not a factor of $p(x)$. Only when the remainder is 0 will $(x - a)$ be a factor of the polynomial. A remainder of any number other than 0 indicates that $(x - a)$ is not a factor of the given polynomial.
- If we evaluate the same function, $p(x) = 3x^2 - 20x + 12$, for $x = 6$, the remainder is 0. By the Remainder Theorem, $(x - 6)$ is a factor of the given polynomial.
- Dividing a polynomial by one of its binomial factors results in a **depressed polynomial**. When $3x^2 - 20x + 12$ is divided by 6, the depressed polynomial is $3x - 2$.
- It is sometimes easier to evaluate a function for a given value by using direct substitution. For instance, when evaluating the expression $3x^2 + 2$ when $x = 4$, directly substitute 4 into the polynomial and follow the order of operations: $3(4)^2 + 2 = 3(16) + 2 = 48 + 2 = 50$. Other times, when the polynomials are more complicated, synthetic substitution is more efficient. Both methods are helpful when working with polynomials.
- As you will see in this lesson, synthetic division and substitution can also be applied to real-world problems.

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Instruction

Common Errors/Misconceptions

- not using the standard form of a polynomial before attempting to use synthetic division
- using the incorrect sign when substituting or dividing
- forgetting to use 0 as a placeholder for coefficients that are not present in the standard form of the dividend

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Guided Practice 2A.3.2

Example 1

Find the quotient of $(x^2 - 5x - 20) \div (x - 4)$ using polynomial long division.

1. Set up the division.

   The dividend is $x^2 - 5x - 20$ and the divisor is $x - 4$.

   $$x-4\overline{)x^2-5x-20}$$

2. Divide the leading term of the dividend by the leading term of the divisor.

   The leading term of the dividend is x^2. The leading term of the divisor is x.

   The result of x^2 divided by x is x.

   Record the result in a manner similar to long division of whole numbers.

   $$\begin{array}{r} x \\ x-4\overline{)x^2-5x-20} \end{array}$$

   Multiply the result, x, by the divisor, $x - 4$.

   $x(x - 4) = x^2 - 4x$

   Write the result under the dividend, lining up the terms of equal degree.

   $$\begin{array}{r} x \\ x-4\overline{)x^2-5x-20} \\ x^2-4x \end{array}$$

   Subtract the last line from the line above it.

   $(x^2 - 5x - 20) - (x^2 - 4x) = -x - 20$

   Record the result in the division.

   $$\begin{array}{r} x \\ x-4\overline{)x^2-5x-20} \\ x^2-4x \\ \hline -x-20 \end{array}$$

(*continued*)

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Instruction

Repeat the process. Divide the leading term of the new dividend, $-x-20$, by the leading term of the divisor, $x-4$.

The result of $-x$ divided by x is -1.

Record the result in the division.

$$\begin{array}{r} x\ -1 \\ x-4\overline{)\ x^2-5x-20} \\ \underline{x^2-4x\ \ \ \ } \\ -x-20 \end{array}$$

Multiply the result, -1, by the divisor, $x-4$.

$-1(x-4) = -x+4$

Write the result under the division, again lining up the terms of equal degree.

$$\begin{array}{r} x\ -1 \\ x-4\overline{)\ x^2-5x-20} \\ \underline{x^2-4x\ \ \ \ } \\ -x-20 \\ \underline{\ \ \ x+4\ \ } \end{array}$$

Subtract the last line from the line above it.

$$\begin{array}{r} x\ -1 \\ x-4\overline{)\ x^2-5x-20} \\ \underline{x^2-4x\ \ \ \ } \\ -x-20 \\ \underline{\ \ \ x+4\ \ } \\ -24 \end{array}$$

Notice that x has been divided out from the problem. -24 represents the remainder of $(x^2-5x-20) \div (x-4)$.

Write the remainder over the divisor: $-\dfrac{24}{x-4}$. Remember to include the remainder in the result of the long division.

The result of $(x^2-5x-20) \div (x-4)$ using polynomial long division is $x-1-\dfrac{24}{x-4}$.

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Instruction

Example 2

Find the quotient of $(x^2 - 5x - 20) \div (x - 4)$ using synthetic division.

1. Identify the coefficients of the dividend.

   The dividend is $x^2 - 5x - 20$. The coefficients of this expression are 1, –5, and –20.

2. Identify the value of a.

   The divisor is $x - 4$, which is of the form $x - a$; therefore, the value of a is 4.

3. Record the coefficients of the dividend and the value of a in the divisor in the synthetic division format.

   $$\begin{array}{r|lll} 4 & 1 & -5 & -20 \\ \hline & & & \end{array}$$

4. Carry out the process of synthetic division.

   Bring down the first coefficient of the dividend below the horizontal line.

   $$\begin{array}{r|lll} 4 & 1 & -5 & -20 \\ \hline & 1 & & \end{array}$$

   Multiply the first coefficient by the value of a and write the value under the second coefficient.

   $$\begin{array}{r|lll} 4 & 1 & -5 & -20 \\ & & 4 & \\ \hline & 1 & & \end{array}$$

   Add the second column.

   $$\begin{array}{r|lll} 4 & 1 & -5 & -20 \\ & & 4 & \\ \hline & 1 & -1 & \end{array}$$

(*continued*)

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Instruction

Continue multiplying and adding for the remaining column.

$$\begin{array}{r|rrr} 4 & 1 & -5 & -20 \\ & & 4 & -4 \\ \hline & 1 & -1 & -24 \end{array}$$

Draw a box around the remainder.

$$\begin{array}{r|rrr} 4 & 1 & -5 & -20 \\ & & 4 & -4 \\ \hline & 1 & -1 & \boxed{-24} \end{array}$$

5. Write the quotient.

   Each sum represents the coefficient of the quotient.

   Recall that the order of the quotient is 1 less than the dividend, x^2.

   Write the remainder over the divisor.

   The result of $(x^2 - 5x - 20) \div (x - 4)$ using synthetic division is $x - 1 - \frac{24}{x-4}$.

Example 3

Find the quotient of $(3x^3 + 16x^2 + 18x + 8) \div (x + 4)$ using synthetic division.

1. Identify the coefficients of the dividend.

   The dividend is $3x^3 + 16x^2 + 18x + 8$. The coefficients of this expression are 3, 16, 18, and 8.

2. Identify the value of a.

   Recall that, in general, the divisor is written in the form $(x - a)$. Here, the divisor is written as $(x + 4)$ or $[x - (-4)]$; therefore, the value of a is -4.

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Instruction

3. Record the coefficients of the dividend and the value of a in the divisor in the synthetic division format.

$$\begin{array}{r|llll} -4 & 3 & 16 & 18 & 8 \\ \hline & & & & \end{array}$$

4. Carry out the process of synthetic division.

Bring down the first coefficient of the dividend below the horizontal line.

$$\begin{array}{r|llll} -4 & 3 & 16 & 18 & 8 \\ \hline & 3 & & & \end{array}$$

Multiply the first coefficient by the value of a and write the value under the second coefficient.

$$\begin{array}{r|llll} -4 & 3 & 16 & 18 & 8 \\ & & -12 & & \\ \hline & 3 & & & \end{array}$$

Add the second column.

$$\begin{array}{r|llll} -4 & 3 & 16 & 18 & 8 \\ & & -12 & & \\ \hline & 3 & 4 & & \end{array}$$

Continue multiplying and adding for the remaining columns.

$$\begin{array}{r|llll} -4 & 3 & 16 & 18 & 8 \\ & & -12 & -16 & -8 \\ \hline & 3 & 4 & 2 & 0 \end{array}$$

Draw a box around the remainder.

$$\begin{array}{r|llll} -4 & 3 & 16 & 18 & 8 \\ & & -12 & -16 & -8 \\ \hline & 3 & 4 & 2 & \boxed{0} \end{array}$$

Notice that the remainder is 0. Therefore, according to the Remainder Theorem, $x + 4$ is a factor of the expression $3x^3 + 16x^2 + 18x + 8$.

5. Write the quotient.

Each sum represents a coefficient of the quotient.

$(3x^3 + 16x^2 + 18x + 8) \div (x + 4) = 3x^2 + 4x + 2$

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Instruction

Example 4

Use synthetic substitution to evaluate $p(x) = x^2 - 32$ for $x = -7$.

1. Identify the coefficients of the polynomial function.

   The polynomial function is $p(x) = x^2 - 32$. The coefficients of this function are 1 and –32.

2. Use synthetic division to evaluate the function for $x = -7$.

   Write the divisor in the form $x - a$.

   The value of x is –7; therefore, it is written as $(x - a) = [x - (-7)]$ or $x + 7$.

   Determine the coefficients of the dividend.

   Compare $x^2 - 32$ to the standard form of a polynomial, $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x^1 + a_0$. Notice that $x^2 - 32$ does not have a value for $a_{n-1} x^{n-1}$, or in this case, ax. Therefore, be sure to include 0 in the setup of the synthetic division to represent the missing coefficient.

   The other coefficients are 1 and –32.

   Record the coefficients of the dividend and the value of a in the divisor in the synthetic division format.

   $$\begin{array}{r|rrr} -7 & 1 & 0 & -32 \\ & & & \\ \hline & & & \end{array}$$

   Work through the process of synthetic division.

   $$\begin{array}{r|rrr} -7 & 1 & 0 & -32 \\ & & -7 & 49 \\ \hline & 1 & -7 & 17 \end{array}$$

   Draw a box around the remainder.

   $$\begin{array}{r|rrr} -7 & 1 & 0 & -32 \\ & & -7 & 49 \\ \hline & 1 & -7 & \boxed{17} \end{array}$$

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Instruction

3. Identify the value of $p(x)$ when $x = -7$.

   The remainder of the synthetic division is the value of $p(x)$ for the given value of x.

   The value of $p(-7)$ is equal to 17, the remainder found using synthetic division.

4. Verify your answer by substituting -7 for x in the polynomial function $p(x) = x^2 - 32$.

$p(x) = x^2 - 32$	Original function
$p(-7) = (-7)^2 - 32$	Substitute -7 for x.
$p(-7) = 49 - 32$	Simplify.
$p(-7) = 17$	

   The process of synthetic substitution reveals the same answer as direct substitution.

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Instruction

Example 5

If the remainder of $(x^2 + kx + 34) \div (x - 5)$ is -21, what is the value of k?

1. Create a polynomial function given the dividend, $x^2 + kx + 34$.

   $p(x) = x^2 + kx + 34$

2. Determine the value of k.

   The divisor, $(x - 5)$, is of the form $(x - a)$.

   According to the Remainder Theorem, when the value of a, 5, is substituted into the polynomial function, the result is the remainder, -21. Write the dividend as a function, $p(x) = x^2 + kx + 34$.

$p(5) = -21$	Remainder Theorem
$p(x) = x^2 + kx + 34$	Original function
$p(5) = (5)^2 + k(5) + 34$	Substitute the value of a, 5, for x.
$-21 = (5)^2 + k(5) + 34$	Substitute the remainder, -21, for $p(5)$.
$-21 = 25 + 5k + 34$	Simplify.
$-21 = 59 + 5k$	Solve for k.
$-80 = 5k$	
$k = -16$	

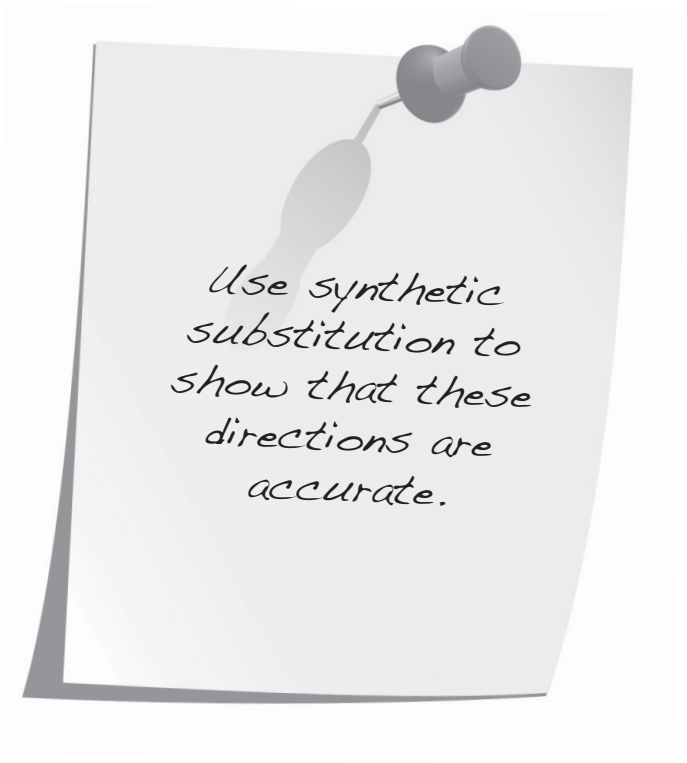
   The value of k is -16 because $(x^2 - 16x + 34) \div (x - 5)$ gives a remainder of -21.

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Problem-Based Task 2A.3.2: When Is the Next Dose Due?

SMP	
1	2 ✓
3	4 ✓
5	6
7	8 ✓

The amount of a certain medication remaining in the bloodstream x hours after taking the medicine is modeled by the equation $M(x) = -x^3 + 5x^2 + 3x + 18$. Package directions recommend taking a second dose 4–6 hours after the initial dose. Use synthetic substitution to show that these directions are accurate.



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Problem-Based Task 2A.3.2: When Is the Next Dose Due?

Coaching

a. Using synthetic substitution, what is $M(3)$?

b. Using synthetic substitution, what is $M(4)$?

c. Using synthetic substitution, what is $M(5)$?

d. Using synthetic substitution, what is $M(6)$?

e. For each value of x given in parts a–d, what does a remainder mean in terms of the context of the problem?

f. What does a remainder of 0 mean in the context of the problem?

g. After how many hours is the medication completely eliminated from the bloodstream?

h. Why would the package directions suggest taking a second dose after 4–6 hours?

i. Are these directions accurate?

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Problem-Based Task 2A.3.2: When Is the Next Dose Due?

Coaching Sample Responses

a. Using synthetic substitution, what is $M(3)$?

By the Remainder Theorem, $M(3)$ should be the remainder when you divide the polynomial, $M(x) = -x^3 + 5x^2 + 3x + 18$, by $x - 3$.

3	−1	5	3	18
		−3	6	27
	−1	2	9	45

The remainder is 45; therefore, using synthetic substitution, $M(3) = 45$.

b. Using synthetic substitution, what is $M(4)$?

By the Remainder Theorem, $M(4)$ should be the remainder when you divide the polynomial, $M(x) = -x^3 + 5x^2 + 3x + 18$, by $x - 4$.

4	−1	5	3	18
		−4	4	28
	−1	1	7	46

The remainder is 46; therefore, using synthetic substitution, $M(4) = 46$.

c. Using synthetic substitution, what is $M(5)$?

By the Remainder Theorem, $M(5)$ should be the remainder when you divide the polynomial, $M(x) = -x^3 + 5x^2 + 3x + 18$, by $x - 5$.

5	−1	5	3	18
		−5	0	15
	−1	0	3	33

The remainder is 33; therefore, using synthetic substitution, $M(5) = 33$.

d. Using synthetic substitution, what is $M(6)$?

By the Remainder Theorem, $M(6)$ should be the remainder when you divide the polynomial, $M(x) = -x^3 + 5x^2 + 3x + 18$, by $x - 6$.

6	−1	5	3	18
		−6	−6	−18
	−1	−1	−3	0

The remainder is 0; therefore, using synthetic substitution, $M(6) = 0$.

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e. For each value of x given in parts a–d, what does a remainder mean in terms of the context of the problem?

A remainder of a value other than 0 indicates that the medication is still in the bloodstream.

f. What does a remainder of 0 mean in the context of the problem?

A remainder of 0 means that the medication is no longer in the bloodstream.

g. After how many hours is the medication completely eliminated from the bloodstream?

The medication is completely eliminated from the bloodstream after 6 hours.

h. Why would the package directions suggest taking a second dose after 4–6 hours?

Comparing the remainders for $M(3)$, $M(4)$, and $M(5)$ shows that the amount of medication in the bloodstream peaks at the 4-hour mark, then starts to decrease, meaning the medication is wearing off. Taking medication before 4 hours have passed could be harmful, since the first dose is still building up in the bloodstream. If the second dose is taken after the first dose starts wearing off but before it is completely eliminated from the bloodstream, then the patient is less likely to feel the effects of the ailment as the first dose wears off, since some medication will still be in the bloodstream.

i. Are these directions accurate?

Yes, the directions are accurate. Taking a second dose after 4 hours have passed but before 6 hours have passed will lessen the effects of the ailment, without resulting in an overdose.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2A.3.2: The Remainder Theorem

For problems 1 and 2, use synthetic division to find each quotient.

1. $(x^3 - 7x^2 + 36) \div (x - 3)$

2. $(2x^4 - 6x^2 + 8x + 2) \div (x + 2)$

For problems 3 and 4, use synthetic substitution to evaluate each function.

3. $p(x) = 3x^2 + 6x + 10$ for $x = -4$

4. $p(x) = x^5 + 4x^3 + 2x - 16$ for $x = 2$

For problems 5 and 6, find the value of k.

5. $(x^2 + kx + 10) \div (x - 1)$ has a remainder of 4.

6. $(x^2 + 5x + k) \div (x + 6)$ has a remainder of 9.

For problems 7–10, use the Remainder Theorem to solve each problem.

7. The area in square feet of a rectangular garden can be expressed as the product of the garden's length and width, or $A(x) = 3x^2 + 13x + 14$. If the width of the garden is $(x + 2)$ feet, what is the length of the garden?

8. The area in square meters of a rectangular patio can be expressed as the product of the patio's length and width, or $A(x) = 7x^2 - 34x + 24$. If the length of the patio is $(x - 4)$ meters, what is the width of the patio?

9. A generator produces voltage using levels of current modeled by $I(t) = t + 4$, where $t > 0$ represents the time in seconds. The power of the generator can be modeled by $P(t) = 0.5t^3 + 8t^2 + 24t$. If voltage is calculated by dividing $P(t)$ by $I(t)$, what expression represents the voltage of the generator?

10. A second generator produces voltage using levels of current modeled by $I(t) = t + 4$, where $t > 0$ represents the time in seconds. The power of the generator can be modeled by $P(t) = 0.2t^3 + 8.8t^2 + 32t$. What expression represents the voltage of this generator?

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Lesson 2A.3.3: Finding Zeros

Warm-Up 2A.3.3

Gregory is the proud new owner of a sailboat, the *Cutty Snark.* It's the most common type of sailboat, a Bermuda sloop, and has two sails: a headsail in front, and a mainsail in back. The mainsail is in the shape of a right triangle. On the *Cutty Snark,* one leg of the triangular mainsail is 31 feet shorter than the other. The longest side of the mainsail is 41 feet. In order to make the proper adjustments, Gregory must know the area of the sail.

1. What is the length of the short leg in terms of x?

2. Using the Pythagorean Theorem, what quadratic equation relates the three sides of the mainsail?

3. What are the factors of this equation?

4. What is the area of the mainsail?

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Instruction

Lesson 2A.3.3: Finding Zeros

Common Core State Standards

A–APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

N–CN.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Warm-Up 2A.3.3 Debrief

1. What is the length of the short leg in terms of x?

   The short leg is 31 feet shorter than the other leg.

   Let x represent the unknown length of the longer leg.

   The shorter leg is $(x - 31)$ feet.

2. Using the Pythagorean Theorem, what quadratic equation relates the three sides of the mainsail?

   Substitute each length of the sail into the Pythagorean Theorem and simplify.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$(x)^2 + (x - 31)^2 = (41)^2$	Substitute values for each side length: x for a, $(x - 31)$ for b, and 41 for c.
$x^2 + x^2 - 62x + 961 = 1681$	Square the binomial.
$2x^2 - 62x - 720 = 0$	Subtract 1,681 from both sides and combine like terms.

   The quadratic equation $2x^2 - 62x - 720 = 0$ relates the three sides of the mainsail.

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Instruction

3. What are the factors of this equation?

   Factor out the greatest common factor, which is 2.

   $2(x^2 - 31x - 360) = 0$

   Find two numbers whose product is –360 and whose sum is –31.

   The numbers are –40 and 9.

   Write these numbers as factors.

   The factors are $(x - 40)$ and $(x + 9)$.

4. What is the area of the mainsail?

   Use the factors to determine the area of the mainsail.

   The factors can be used to determine the actual lengths of each leg of the triangular mainsail.

   Set each factor equal to 0 and solve for x.

   $x + 9 = 0$

   $x = -9$

   One possible length of x is –9.

   $x - 40 = 0$

   $x = 40$

   The other possible length of x is 40.

   The factor $(x + 9)$ results in a negative length (–9) and, therefore, is not a possible leg length of the sail.

   The factor $(x - 40)$ results in a length of 40 feet.

   The other leg is 31 feet shorter than this length, or 9 feet.

   Use the area formula for a triangle, $A = \frac{1}{2}bh$, to calculate the area of the sail.

   $A = \frac{1}{2}(9)(40)$

   $A = 180$

   The area of the sail is 180 ft^2.

Connection to the Lesson

- Students will expand their knowledge of factoring polynomials to assist in finding zeros of polynomial functions as well as sketching graphs of polynomial functions.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- solving quadratic functions using a variety of methods (A–REI.4)
- simplifying polynomial expressions using synthetic division (A–APR.3)

Introduction

Synthetic division, along with your knowledge of end behavior and turning points, can be used to identify the *x*-intercepts of a polynomial function. This information allows for more accurate sketches of functions.

Key Concepts

- Recall that roots are the *x*-intercepts of a function. In other words, these are the *x*-values for which a function equals 0. These **zeros** are another way of referring to the roots of a function.
- When a polynomial equation with a degree greater than 0 is solved, it may have one or more real solutions, or it may have no real solutions (in which case it would have complex solutions).
- Recall that both real and imaginary numbers belong to the set of complex numbers; therefore, all polynomial functions with a degree greater than 0 will have at least one root in the set of complex numbers. This is referred to as the **Fundamental Theorem of Algebra**.

Fundamental Theorem of Algebra
If $p(x)$ is a polynomial function of degree $n \geq 1$ with complex coefficients, then the related equation $p(x) = 0$ has at least one complex solution (root).

- A **repeated root** is a root that occurs more than once in a polynomial function.
- Recall that the solutions to a quadratic equation that contains imaginary numbers come in pairs. These are called complex conjugates, the complex number that when multiplied by another complex number produces a value that is wholly real; the complex conjugate of $a + bi$ is $a - bi$.
- If an imaginary number is a zero of a function, its conjugate is also a zero of that function. This is true for all polynomial functions, and is known as the **Complex Conjugate Theorem**.

Complex Conjugate Theorem
Let $p(x)$ be a polynomial with real coefficients. If $a + bi$ is a root of the equation $p(x) = 0$, where a and b are real and $b \neq 0$, then $a - bi$ is also a root of the equation.

SRB p. U2A-79

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Instruction

- For a polynomial function $p(x)$, the **factor of a polynomial** is any polynomial that divides evenly into that function. Recall that when a polynomial is divided by one of its factors, there is a remainder of 0 and the result is a depressed polynomial. This is an illustration of the **Factor Theorem**.

Factor Theorem
The binomial $x - a$ is a factor of the polynomial $p(x)$ if and only if $p(a) = 0$, where a is a real number.

- The Factor Theorem can help to find all the factors of a polynomial.
- To do this, first show that the binomial in question is a factor of the polynomial. If the remainder is 0, then the binomial is a factor.
- Then, determine if the resulting depressed polynomial can be factored.
- The identified factors indicate where the function crosses the x-axis.
- The zeros of a function are related to the factors of the polynomial. The graph of a polynomial function shows the zeros of the function, which are the x-intercepts of the graph.
- It is often helpful to know which integer values of a to try when determining $p(a) = 0$.
- Use the **Integral Zero Theorem** to determine the zeros of a polynomial function.
- Identify the factors of the constant term of a polynomial function and use substitution to determine if each number results in a zero.
- Use synthetic division to determine the remaining factors.

Integral Zero Theorem
If the coefficients of a polynomial function are integers such that $a_n = 1$ and $a_0 \neq 0$, then any rational zeros of the function must be factors of a_0.

- For example, consider the equation $p(x) = x^2 + 10x + 25$. $a_n = 1$ and $a_0 = 25$. A possible zero of this function is –5 because –5 is a factor of 25.
- The zeros of any polynomial function correspond to the x-intercepts of the graph and to the roots of the corresponding equation.
- If a polynomial has a factor $x - a$ that is repeated n times, then the root is called a repeated root and $x = a$ is a zero of multiplicity. **Multiplicity** refers to the number of times a zero of a polynomial function occurs.
- If the multiplicity is odd, then the graph intersects the x-axis at the point $(x, 0)$. If the multiplicity is even, then the graph just touches the axis at the point $(x, 0)$.

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Instruction

- If $p(x)$ is a polynomial with real coefficients whose terms are arranged in descending powers of the variable, then the number of positive zeros of $y = p(x)$ is the same as the number of sign changes of the coefficients of the terms, or is less than this by an even number. Recall that when solving a quadratic function, we used the quadratic formula, which generated pairs of roots. Often, these roots were complex and the x-intercepts could not be graphed on the coordinate plane. For this reason, we must count down from our maximum number of zeros by twos to determine the number of real zeros. For example, for a polynomial with 3 sign changes, the number of positive zeros may be 3 or 1, since $3 - 2 = 1$.
- Also, the number of negative zeros of $y = p(x)$ is the same as the number of sign changes of the coefficients of the terms of $p(-x)$, or is less than this number by an even number.

Common Errors/Misconceptions

- not finding all the factors of a polynomial function
- making sign errors when performing synthetic division
- misusing the terms "roots" and "factors"

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Guided Practice 2A.3.3

Example 1

Given the equation $x^3 + 4x^2 - 3x - 18 = 0$, state the number and type of roots of the equation if one root is –3.

1. Use synthetic division to find the depressed polynomial.

   The related polynomial is $x^3 + 4x^2 - 3x - 18$, with coefficients 1, 4, –3, and –18.

   One root of the equation is –3; therefore, a factor of the related polynomial is $[x - (-3)]$, which simplifies to $x + 3$.

   Divide the related polynomial by $x + 3$ to find the depressed polynomial. Let the value of a be –3.

$$\begin{array}{r|rrrr} -3 & 1 & 4 & -3 & -18 \\ & & -3 & -3 & 18 \\ \hline & 1 & 1 & -6 & \boxed{0} \end{array}$$

   The depressed polynomial is $x^2 + x - 6$.

2. Factor the depressed polynomial to find the remaining factors.

   Use previously learned strategies to factor the polynomial.

   $x^2 + x - 6$ Depressed polynomial

   $(x - 2)(x + 3)$ Factor the polynomial.

   The remaining factors are $(x - 2)$ and $(x + 3)$.

3. State the number of roots and type of roots of the equation.

   The factors of the related polynomial are $(x + 3)$, $(x - 2)$, and $(x + 3)$. Recall that we divided the depressed polynomial by $(x + 3)$ before finding the remaining factors, so we must include $(x + 3)$ as a factor. Therefore, the roots of the equation are –3, –3, and 2. Since –3 appears twice, it is a repeated root. Because of the repeated root, this equation has only 2 real roots: –3 and 2.

SWB p. 91; SRB p. U2A-82

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Instruction

Example 2

Find all the zeros of the function $f(x) = x^3 + x^2 + 11x + 51$. Verify the zeros by creating a graph.

1. Determine the possible number of zeros.

   The term with the highest power is x^3, so $f(x)$ has a degree of 3; therefore, it has 3 zeros.

2. Determine the type of zeros.

   Identify the signs for each term of the function $f(x) = x^3 + x^2 + 11x + 51$.

   Notice that there are no sign changes between each term; therefore, the function $f(x)$ has no positive real zeros.

   Find $f(-x)$ to determine the number of negative real zeros.

$f(x) = x^3 + x^2 + 11x + 51$	Original function
$f(-x) = (-x)^3 + (-x)^2 + 11(-x) + 51$	Substitute $-x$ for x.
$f(-x) = -x^3 + x^2 - 11x + 51$	Simplify.

   Identify the signs for each term of the function $f(-x)$, and count the number of sign changes.

   $f(-x) = -x^3 + x^2 - 11x + 51$

   There are a total of three sign changes, so the function could have 3 negative real zeros.

   However, recall that the number of negative zeros is either the number of sign changes of $f(-x)$, or the number of sign changes less an even number. In this case, $3 - 2 = 1$, so the function could have 1 negative real zero.

   The function $f(x)$ has either 3 real zeros, or 1 real zero and 2 imaginary zeros.

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Instruction

3. Find the possible zeros of the function.

We have already determined that none of the zeros are positive.

We can determine from the equation that $f(0) = 51$, since $f(0) = (0)^3 + (0)^2 + 11(0) + 51 = 51$.

Evaluate $f(x)$ for negative integral values from –4 to –1 using synthetic substitution. Record each value of x, the coefficients of the depressed polynomial, and the remainder in a table.

x	1	1	11	51
–4	1	–3	23	–41
–3	1	–2	17	**0**
–2	1	–1	13	25
–1	1	0	11	40

Notice that one zero occurs at $x = -3$.

From the coefficients in the table, we can determine that the depressed polynomial of this zero is $x^2 - 2x + 17$.

The depressed polynomial is already written in standard form, $ax^2 + bx + c$.

Use the quadratic formula to find the zeros of the related equation, $x^2 - 2x + 17 = 0$.

Let $a = 1$, $b = -2$, and $c = 17$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic formula

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(17)}}{2(1)}$$ Substitute 1 for a, –2 for b, and 17 for c.

$$x = \frac{2 \pm \sqrt{-64}}{2}$$ Simplify.

$$x = \frac{2 \pm 8i}{2}$$

$$x = 1 \pm 4i$$

The function $f(x) = x^3 + x^2 + 11x + 51$ has one real zero at $x = -3$ and two imaginary zeros at $x = 1 + 4i$ and $x = 1 - 4i$.

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Instruction

4. Graph the function $f(x) = x^3 + x^2 + 11x + 51$.

   Use a table of values to sketch the graph, or use a graphing calculator to create a graph of the function.

   To graph a function using a graphing calculator, follow these general steps for your calculator model.

   On a TI-83/84:

   Step 1: Press [Y=].

   Step 2: Type the function into Y_1, or any available equation. Use the [X, T, θ, n] button for the variable x. Use [^] to enter powers greater than 2; use the [x^2] button for a square.

   Step 3: Press [WINDOW]. Enter values for Xmin, Xmax, Ymin, and Ymax in relation to the change in the viewing window. The Xscl and Yscl are arbitrary. Leave Xres = 1.

   Step 4: Press [GRAPH].

   On a TI-Nspire:

   Step 1: Press the [home] key.

   Step 2: Arrow over to the graphing icon and press [enter].

   Step 3: Type the function next to $f1(x)$, or any available equation, and press [enter]. Use the [X] button for the letter x. Use [^] to enter powers greater than 2; use the [x^2] button for a square.

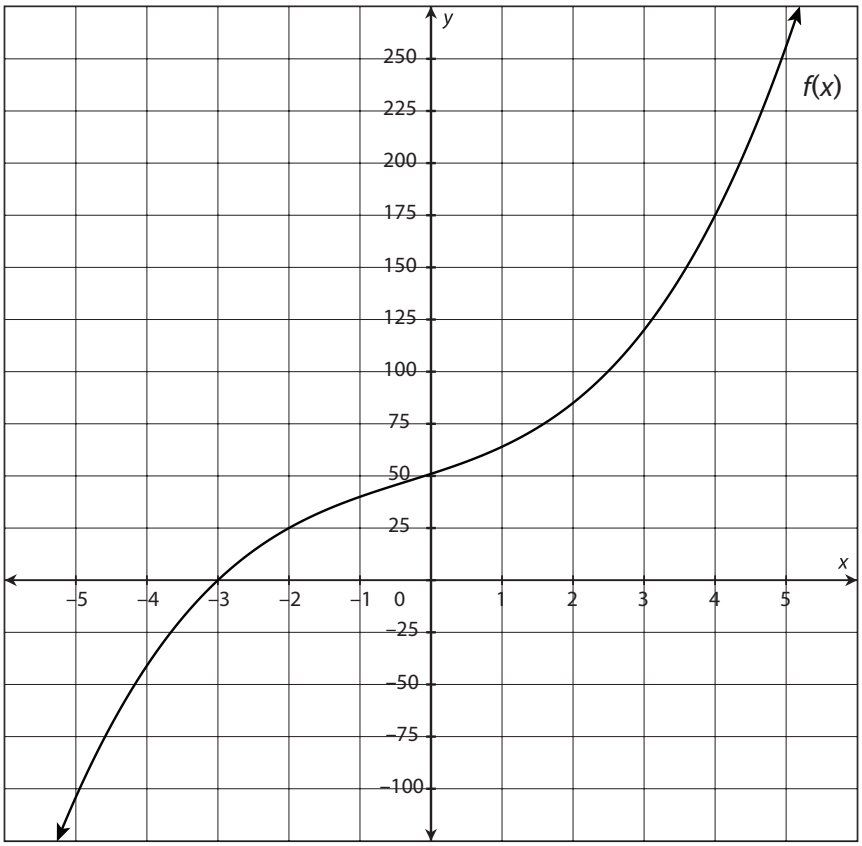
   Step 4: To change the viewing window, press [menu]. Select 1: Window Settings. Enter values for Xmin, Xmax, Ymin, and Ymax in relation to the change in the viewing window. The Xscl and Yscl are arbitrary. Leave Xres = 1.

   Step 5: Press [enter].

(*continued*)

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Instruction



Notice that the graph of the function crosses the x-axis at only one point, (–3, 0). This verifies there is only 1 real root.

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Instruction

Example 3

Write the simplest polynomial function with integer coefficients that has the zeros 5 and $3 - i$.

1. Determine additional zeros of the function.

   Because $3 - i$ is a zero, then according to the Complex Conjugate Theorem, the conjugate $3 + i$ is also a zero.

2. Use the zeros to write the polynomial as a product of the factors.

   The zeros are 5, $3 - i$, and $3 + i$. These can be written as the factors $x - 5$, $x - (3 - i)$, and $x - (3 + i)$.

   The polynomial function written in factored form with zeros 5 and $3 - i$ is $f(x) = (x - 5)[x - (3 - i)][x - (3 + i)]$.

3. Multiply the factors to determine the polynomial function.

$f(x) = (x - 5)[x - (3 - i)][x - (3 + i)]$	Polynomial function written in factored form
$f(x) = (x - 5)[(x - 3) - i][(x - 3) + i]$	Regroup terms.
$f(x) = (x - 5)[(x - 3)^2 - i^2]$	Rewrite as the difference of two squares.
$f(x) = (x - 5)(x^2 - 6x + 9 - i^2)$	Simplify.
$f(x) = (x - 5)[x^2 - 6x + 9 - (-1)]$	Replace i^2 with -1, since $i^2 = -1$.
$f(x) = (x - 5)(x^2 - 6x + 9 + 1)$	Simplify.
$f(x) = x^3 - 11x^2 + 40x - 50$	Distribute.

The polynomial function of least degree with integral coefficients whose zeros are 5 and $3 - i$ is $f(x) = x^3 - 11x^2 + 40x - 50$.



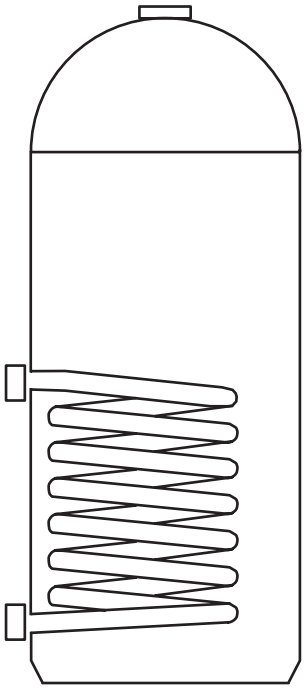
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Problem-Based Task 2A.3.3: In Hot Water

SMP	
1 ✓	2
3	4 ✓
5 ✓	6
7	8

A household hot-water tank is designed to store enough hot water to meet the demands of all the members of the household. If the tank is too small, the household will continuously run out of hot water. If the tank is too large, more than enough hot water will be stored, causing a waste of resources. The design of hot-water tanks varies depending on location, use, and resources.

Luz wants to buy a new hot-water tank that is shaped like a cylinder with a hemisphere on the top. Using the formula $V = \frac{2}{3}\pi r^3 + \pi r^2 h$, where r is the radius of the tank and h is the height of the cylinder, she found that this tank has an approximate volume of $99{,}000\pi$ cm^3. The height of the cylindrical portion of the tank is 90 cm. The spot in Luz's basement for the hot-water tank can only accommodate a tank that is not wider than 55 cm. Does Luz have enough space for this particular hot-water tank? Explain your reasoning.



Does Luz have enough space for this particular hot-water tank?

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Problem-Based Task 2A.3.3: In Hot Water

Coaching

a. What is the height of the cylinder?

b. What is the volume of the hot-water tank?

c. What is the equation that represents the volume of the hot-water tank?

d. What is the equation of the volume of the tank written in standard form?

e. Graph the equation using a graphing calculator. What are the approximate roots of this equation?

f. Are all approximated roots possible values for the radius of the hot-water tank?

g. Using synthetic division, what is the depressed polynomial?

h. What are the dimensions of the hot-water tank?

i. Is there enough space for this hot-water tank? Explain your answer.

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Problem-Based Task 2A.3.3: In Hot Water

Coaching Sample Responses

a. What is the height of the cylinder?

As stated in the problem, the height of the cylinder is 90 cm.

b. What is the volume of the hot-water tank?

As stated in the problem, the volume of the hot-water tank is approximately $99{,}000\pi$ cm^3.

c. What is the function that represents the volume of the hot-water tank?

The general formula for finding the volume of the tank is $V = \frac{2}{3}\pi r^3 + \pi r^2 h$. We know that the cylinder's height is 90 cm, but we do not know the radius of the hot-water tank. Let r represent the unknown radius, and then write the equation in function notation.

$$f(x) = V = \frac{2}{3}\pi r^3 + \pi r^2 h$$

$$f(x) = \frac{2}{3}\pi r^3 + \pi r^2 (90)$$

$$f(x) = \frac{2}{3}\pi r^3 + 90\pi r^2$$

The function $f(x) = \frac{2}{3}\pi r^3 + 90\pi r^2$ represents the volume of the hot-water tank.

d. What is the equation of the volume of the tank written in standard form?

First, substitute the known volume into the equation from part c.

$$99{,}000\pi = \frac{2}{3}\pi r^3 + 90\pi r^2$$

Now write the equation in standard form by setting the equation equal to 0.

$$\frac{2}{3}\pi r^3 + 90\pi r^2 - 99{,}000\pi = 0$$

Divide both sides by π.

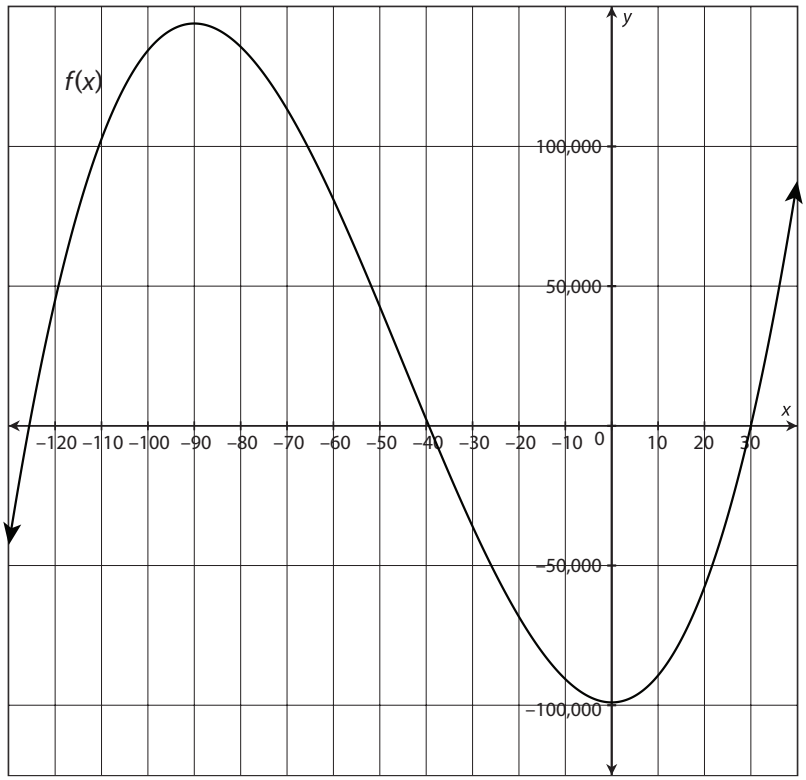
$$\frac{2}{3}r^3 + 90r^2 - 99{,}000 = 0$$

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Instruction

e. Graph the equation using a graphing calculator. What are the approximate roots of this equation?

Use a graphing calculator to determine the graph of the equation.



From the graph, it can be estimated that the roots of the equation are approximately –125, –40, and 30.

f. Are all approximated roots possible values for the radius of the hot-water tank?

It is not possible for the radius to have a negative value; therefore, the only possible root is 30.

g. Using synthetic division, what is the depressed polynomial?

Set up synthetic division using the value of 30 for a and solve.

$$\begin{array}{r|cccc} 30 & \frac{2}{3} & 90 & 0 & -99{,}000 \\ & & 20 & 3300 & 99{,}000 \\ \hline & \frac{2}{3} & 110 & 3300 & \boxed{0} \end{array}$$

The depressed polynomial is $\frac{2}{3}r^2 + 110r + 3300$.

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h. What are the dimensions of the hot-water tank?

The radius of the hot-water tank is 30 cm and the height is 90 cm.

i. Is there enough space for this hot-water tank? Explain your answer.

The location designated for the hot-water tank can accommodate a tank with a diameter no larger than 55 cm. The tank has a radius of 30 cm, or a diameter of 60 cm. The tank will not fit in the space allotted.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2A.3.3: Finding Zeros

For problems 1–3, determine the number and type of roots for each equation using one of the given roots. Then find each root.

1. $x^3 - 7x + 6 = 0$; 1

2. $x^3 - 3x^2 + 25x + 29 = 0$; –1

3. $x^3 - 4x^2 - 3x + 18 = 0$; 3

For problems 4–6, find all the zeros of each function. Then graph each function to verify your answers.

4. $f(x) = x^2 + 4x - 12$

5. $f(x) = x^3 - 3x^2 + x + 5$

6. $f(x) = x^3 - 4x^2 - 7x + 10$

For problems 7 and 8, write the simplest polynomial function with integral coefficients that has the given zeros.

7. –5, –1, 3, 7

8. 4, $2 + 3i$

Use the following information to solve problems 9 and 10.

> A portable fuel tank used to transport gasoline is in the shape of a rectangular prism. The tank is 5 feet longer than it is wide and 29 feet deeper than it is wide. The volume of the tank is 33,488 cubic feet.

9. What equation represents the volume of the portable fuel tank?

10. Use synthetic division to find all the roots of the polynomial equation. What are the dimensions of the fuel tank?

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Lesson 2A.3.4: The Rational Root Theorem

Warm-Up 2A.3.4

The manufacturer of an MP3 player sends its product to various retailers. Each shipping box contains the same number of MP3 players. The table below shows the number of MP3 players each retailer received.

Retailer	MP3 players received
Electronic Envy	210
Sounds Swell	240
Listen-Up	120
Mostly Music	150

1. What is the greatest possible number of MP3 players in each shipping box?

2. How many shipping boxes, in total, were sent?

3. Once the MP3 players were received, Mostly Music requested two additional boxes of MP3 players. How many additional MP3 players did Mostly Music request?

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Lesson 2A.3.4: The Rational Root Theorem

Common Core State Standard

A–APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Warm-Up 2A.3.4 Debrief

1. What is the greatest possible number of MP3 players in each shipping box?

   To determine the greatest number of MP3 players in each shipping box, first find the factors of each number of MP3 players received.

   - Electronic Envy received 210 MP3 players.

     The factors of 210 are 1, 2, 3, 5, 6, 7, 10, 14, 15, 30, 35, 42, 70, 105, and 210.

   - Sounds Swell received 240 MP3 players.

     The factors of 240 are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 40, 48, 60, 80, and 240.

   - Listen-Up received 120 MP3 players.

     The factors of 120 are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, and 120.

   - Mostly Music received 150 MP3 players.

     The factors of 150 are 1, 2, 3, 5, 6, 10, 15, 25, 30, 50, 75, and 150.

   Next, determine the common factors among each number of MP3 players received.

   The common factors are 1, 2, 3, 5, 6, 10, 15, and 30.

   The greatest common factor of each number of MP3 players received is 30; therefore, the greatest possible number of MP3 players in each shipping box is 30.

2. How many shipping boxes, in total, were sent?

   Determine the number of shipping boxes sent to each retailer.

   Electronic Envy received 210 MP3 players. There are 30 MP3 players in each box, so divide 210 by 30: $210 \div 30 = 7$. Electronic Envy received 7 boxes of MP3 players.

   Sounds Swell received 240 MP3 players. There are 30 MP3 players in each box, so divide 240 by 30: $240 \div 30 = 8$. Sounds Swell received 8 boxes of MP3 players.

   Listen-Up received 120 MP3 players. There are 30 MP3 players in each box, so divide 120 by 30: $120 \div 30 = 4$. Listen-Up received 4 boxes of MP3 players.

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Mostly Music received 150 MP3 players. There are 30 MP3 players in each box, so divide 150 by 30: 150 ÷ 30 = 5. Mostly Music received 5 boxes of MP3 players.

The total number of shipping boxes shipped to all the retailers is the sum of the number of boxes shipped to each retailer.

$7 + 8 + 4 + 5 = 24$

In total, 24 shipping boxes were sent to the retailers.

3. Once the MP3 players were received, Mostly Music requested two additional boxes of MP3 players. How many additional MP3 players did Mostly Music request?

   Mostly Music requested two additional boxes of MP3 players. If there are 30 MP3 players in each box, then Mostly Music requested the result of 2 • 30, or 60 MP3 players.

Connection to the Lesson

- Students will determine the factors of the leading coefficient and constant term of a polynomial equation to determine the possible rational factors of the equation.

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Prerequisite Skills

This lesson requires the use of the following skills:

- identifying factors (A–SSE.2)
- understanding the number system, including integers, rational numbers, irrational numbers, and complex numbers (8.NS.1)
- using synthetic division (A–APR.3)
- solving equations (A–REI.3)

Introduction

You have learned several methods for solving polynomial equations by determining the factors, but not all equations are factorable. In this lesson, you will learn how to find the roots of a polynomial equation with integer coefficients that is not factorable.

Key Concepts

- You can find the possible rational roots of a polynomial equation with integer coefficients using the **Rational Root Theorem**. This theorem states that if a polynomial has rational roots, then those roots will be the ratio of the factors of the constant term to the factors of the leading coefficient written in lowest terms.
- Be sure to include all factors of both the constant term and the leading coefficient, including both the positive and negative factors. It is not necessary to repeat duplicate ratios.

Rational Root Theorem
If the polynomial $p(x)$ has integer coefficients, then every rational root of the polynomial equation $p(x) = 0$ can be written in the form $\frac{p}{q}$, where p is a factor of the constant term of $p(x)$ and q is a factor of the leading coefficient of $p(x)$.

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- It is possible for polynomial equations to have no rational roots; polynomial equations could also have irrational roots as stated in the **Irrational Root Theorem**.

Irrational Root Theorem
If a polynomial $p(x)$ has rational coefficients and $a+b\sqrt{c}$ is a root of the polynomial equation $p(x) = 0$, where a and b are rational and $\sqrt{c}$ is irrational, then $a-b\sqrt{c}$ is also a root of $p(x) = 0$.

- Recall that it is also possible for an equation to have imaginary roots in the form $a + bi$ and $a - bi$, referred to as complex conjugates. If the imaginary number $a + bi$ is a root of a polynomial equation with real coefficients, then the conjugate $a - bi$ is also a root.

Common Errors/Misconceptions

- misinterpreting the Rational Root Theorem as stating that all zeros of a polynomial function must be rational
- not identifying both the positive and negative factors of the constant term and leading coefficients
- not finding all factors of a polynomial function

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Instruction

Guided Practice 2A.3.4

Example 1

Use the Rational Root Theorem to list all the possible rational roots of the polynomial equation $2x^4 - 3x^3 + 4x^2 - 9x + 6 = 0$.

1. Identify the constant term and the leading coefficient of the polynomial equation.

   The constant term of the polynomial equation is 6.

   The polynomial equation is written in descending order; therefore, the leading coefficient comes from the term $2x^4$. The leading coefficient is 2.

2. Identify the factors of the constant term and the leading coefficient.

   The constant, 6, has the factors ±1, ±2, ±3, and ±6.

   The leading coefficient, 2, has the factors ±1 and ±2.

3. Identify the possible rational roots of the polynomial equation.

   The possible rational roots of the polynomial equation are made up of the ratios of the factors of the constant term and the factors of the leading coefficient.

   Use the factors determined in step 2 to set up the ratios.

   $\frac{\pm1,\ \pm2,\ \pm3,\ \pm6}{\pm1,\ \pm2}$ Ratio of the factors of the constant term and the factors of the leading coefficient

   $\frac{\pm1}{\pm1}, \frac{\pm1}{\pm2}, \frac{\pm2}{\pm1}, \frac{\pm2}{\pm2}, \frac{\pm3}{\pm1}, \frac{\pm3}{\pm2}, \frac{\pm6}{\pm1}, \frac{\pm6}{\pm2}$ Write a separate ratio for each of the factors.

   $\pm1, \pm\frac{1}{2}, \pm2, \pm1, \pm3, \pm\frac{3}{2}, \pm6, \pm3$ Simplify each ratio.

   $\pm1, \pm\frac{1}{2}, \pm2, \pm3, \pm\frac{3}{2}, \pm6$ Remove any repeated ratios.

   The possible rational roots of the polynomial equation $2x^4 - 3x^3 + 4x^2 - 9x + 6 = 0$ are $\pm1, \pm\frac{1}{2}, \pm2, \pm3, \pm\frac{3}{2}$, and ±6.

SWB p. 101; SRB p. U2A-90

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Instruction

Example 2

Find all of the roots to the polynomial equation $4x^3 - x^2 + 36x - 9 = 0$.

1. List all the possible rational roots.

   The Rational Root Theorem states that the only possible rational roots of the equation must be ratios of the factors of the constant term and the factors of the leading coefficient.

   The constant term is –9 and the leading coefficient is 4.

   The factors of the constant term, –9, are ±1, ±3, and ±9.

   The factors of the leading coefficient, 4, are ±1, ±2, and ±4.

   Use these factors to set up the ratios.

   $\frac{\pm 1,\ \pm 3\ \pm 9}{\pm 1,\ \pm 2,\ \pm 4}$ Ratio of the factors of the constant term and the factors of the leading coefficient

   $\frac{\pm 1}{\pm 1}, \frac{\pm 1}{\pm 2}, \frac{\pm 1}{\pm 4}, \frac{\pm 3}{\pm 1}, \frac{\pm 3}{\pm 2}, \frac{\pm 3}{\pm 4}, \frac{\pm 9}{\pm 1}, \frac{\pm 9}{\pm 2}, \frac{\pm 9}{\pm 4}$ Write a separate ratio for each of the factors.

   $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 3, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm 9, \pm\frac{9}{2}, \pm\frac{9}{4}$ Simplify each ratio.

2. Determine the number of possible roots of the polynomial equation.

   $4x^3 - x^2 + 36x - 9 = 0$ is a third-degree polynomial.

   Therefore, this equation will have either 3 real roots or 1 real root and 2 complex roots.

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3. Test each possible rational root to find one root.

   Begin by selecting either a positive or negative possible rational root.

   Substitute the chosen possibility into the original polynomial. If the result is 0, the number substituted is a rational root. If the result is not 0, then the chosen number is not a rational root.

   First, try the possible root 1.

$4x^3 - x^2 + 36x - 9$	Original polynomial
$4(1)^3 - (1)^2 + 36(1) - 9$	Substitute 1 for x.
$4 - 1 + 36 - 9$	Simplify.
30	

   Substituting 1 for x does not result in the polynomial equaling 0; therefore, 1 is not a rational root of the polynomial equation.

   Next, try the possible root $\frac{1}{2}$.

$4x^3 - x^2 + 36x - 9$	Original polynomial
$4\left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^2 + 36\left(\frac{1}{2}\right) - 9$	Substitute $\frac{1}{2}$ for x.
$\frac{1}{2} - \frac{1}{4} + 18 - 9$	Simplify.
$\frac{37}{4}$	

   Substituting $\frac{1}{2}$ for x does not result in the polynomial equaling 0; therefore, $\frac{1}{2}$ is not a rational root of the polynomial equation.

(*continued*)

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Instruction

Next, try the possible root $\frac{1}{4}$.

$4x^3 - x^2 + 36x - 9$	Original polynomial
$4\left(\frac{1}{4}\right)^3 - \left(\frac{1}{4}\right)^2 + 36\left(\frac{1}{4}\right) - 9$	Substitute $\frac{1}{4}$ for x.
$\frac{1}{16} - \frac{1}{16} + 9 - 9$	Simplify.
0	

Substituting $\frac{1}{4}$ for x does result in the polynomial equaling 0; therefore, $\frac{1}{4}$ is a rational root of the polynomial equation.

4. Use synthetic division to find the roots of the depressed polynomial equation.

The divisor is $\frac{1}{4}$.

$$\begin{array}{c|cccc} \frac{1}{4} & 4 & -1 & 36 & -9 \\ & & 1 & 0 & 9 \\ \hline & 4 & 0 & 36 & \boxed{0} \end{array}$$

The depressed polynomial is $4x^2 + 36$.

Find the roots of $4x^2 + 36 = 0$.

$4x^2 + 36 = 0$	Depressed polynomial
$4(x^2 + 9) = 0$	Factor out the greatest common factor, 4.
$x^2 + 9 = 0$	Divide both sides by 4.
$x^2 = -9$	Subtract 9 from both sides.
$x = \sqrt{-9}$	Take the square root of both sides.
$x = i\sqrt{9}$	Simplify.
$x = \pm 3i$	

The roots of the depressed polynomial equation $4x^2 + 36 = 0$ are $\pm 3i$.

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5. List the roots of the original polynomial equation.

   The roots of the equation $4x^3 - x^2 + 36x - 9 = 0$ are $\frac{1}{4}$, $3i$, and $-3i$.



Example 3

Find a third-degree polynomial with rational coefficients that has the roots 6 and $3 - i$.

1. Use the Imaginary Root Theorem to find the remaining root.

   $3 - i$ is one root. The complex conjugate of $3 - i$ is $3 + i$.

2. Write the polynomial expression.

   The factors of the polynomial are $(x - 6)$, $x - (3 - i)$, and $x - (3 + i)$.

$(x - 6)[x - (3 - i)][x - (3 + i)]$	Write the polynomial in factored form.
$(x - 6)[x^2 - x(3 + i) - x(3 - i) + (3 - i)(3 + i)]$	Distribute.
$(x - 6)(x^2 - 3x - ix - 3x + ix + 9 + 3i - 3i - i^2)$	Simplify.
$(x - 6)[x^2 - 6x + 9 - (-1)]$	Replace i^2 with -1.
$(x - 6)(x^2 - 6x + 10)$	Simplify.
$x^3 - 12x^2 + 46x - 60$	Distribute.

   $x^3 - 12x^2 + 46x - 60$ is a third-degree polynomial with rational coefficients and the roots 6 and $3 - i$.



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Problem-Based Task 2A.3.4: Fan-tastic!

SMP	
1	2 ✓
3 ✓	4
5	6
7 ✓	8

Many residential bathrooms have overhead fans to remove the humidity caused by running a shower or filling a bathtub. If a bathroom has too much moisture, or humidity, the walls, floor, and ceiling can rot. The best size for the overhead fan depends on the volume of the bathroom.

Homeowners who want to install a new fan can determine the appropriate size of fan to purchase by first calculating the volume of the bathroom and then multiplying the product by $\frac{2}{15}$. The resulting value gives the minimum rate at which a fan should circulate the air in that bathroom, in cubic feet per minute. For an overhead fan to be effective, the amount of air it circulates in cubic feet per minute must be greater than the homeowner's calculated air circulation rate based on the volume of the bathroom.

Nolan is remodeling his bathroom. The volume of Nolan's bathroom is represented by the equation $f(x) = x^3 - 30x^2 + 296x - 960$, with the room's dimensions (length, width, and height) measured in feet. Nolan has selected a fan that moves the air at 130 cubic feet per minute. Is this fan appropriate for his bathroom? Explain your answer.



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Problem-Based Task 2A.3.4: Fan-tastic!

Coaching

a. What are the possible rational roots of $f(x) = x^3 - 30x^2 + 296x - 960$?

b. Graph the equation using a graphing calculator. Looking at the graph, which of the possible rational roots are likely to be actual roots of the polynomial equation?

c. Using substitution, verify that one of the estimated roots is an actual root of the polynomial equation.

d. Using synthetic division, what is the depressed polynomial?

e. What are the roots of the related polynomial equation?

f. What are the roots of the polynomial equation that represents the volume of Nolan's bathroom?

g. What are the actual dimensions of the bathroom?

h. What is the volume of the bathroom?

i. If the product of the volume and $\frac{2}{15}$ represents the cubic feet per minute needed to properly ventilate the bathroom, what is the minimum number of cubic feet per minute a fan needs to be able to circulate for a bathroom with the dimensions found in part g?

j. Is a bathroom fan that circulates air at a rate of 130 cubic feet per minute appropriate for Nolan's bathroom? Explain your answer.

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Instruction

Problem-Based Task 2A.3.4: Fan-tastic!

Coaching Sample Responses

a. What are the possible rational roots of $f(x) = x^3 - 30x^2 + 296x - 960$?

The constant term of the polynomial equation is –960. The factors of –960 are ±1, ±2, ±3, ±4, ±5, ±6, ±8, ±10, ±12, ±15, ±16, ±20, ±24, ±30, ±32, ±40, ±48, ±60, ±64, ±80, ±96, ±120, ±192, ±240, ±320, ±480, and ±960.

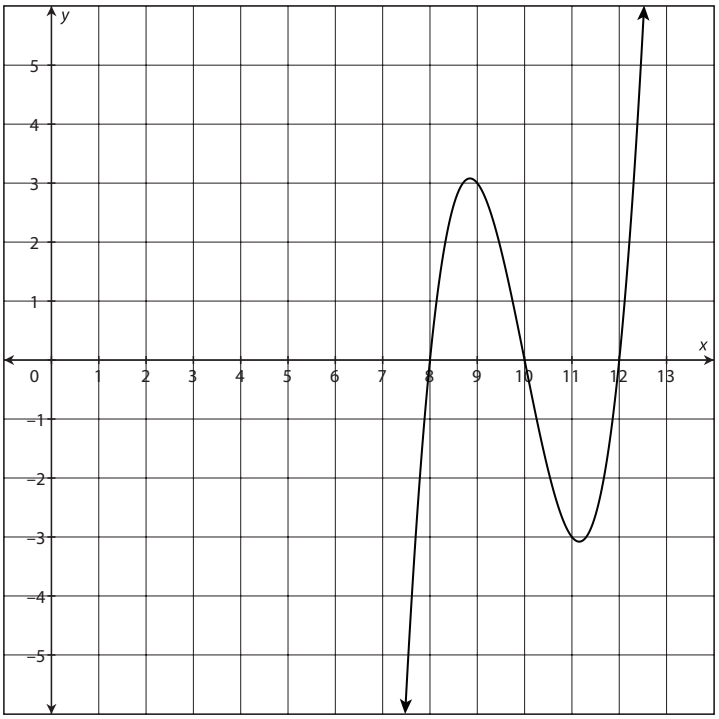
The leading coefficient of the polynomial equation is 1. The factors of 1 are ±1.

The possible rational roots of the polynomial equation are the ratios of the factors of the constant term to the factors of the leading coefficient.

Because the factors of the leading coefficient are ±1, the ratios of the factors of the constant term to the factors of the leading coefficient are whole numbers: ±1, ±2, ±3, ±4, ±5, ±6, ±8, ±10, ±12, ±15, ±16, ±20, ±24, ±30, ±32, ±40, ±48, ±60, ±64, ±80, ±96, ±120, ±192, ±240, ±320, ±480, and ±960.

b. Graph the equation using a graphing calculator. Looking at the graph, which of the possible rational roots are likely to be actual roots of the polynomial equation?

Graph the polynomial equation on a graphing calculator, then use the graph to estimate the roots of the equation.



It appears as though the graph crosses the x-axis at three points: (8, 0), (10, 0), and (12, 0).

The estimated roots of the polynomial equation are 8, 10, and 12.

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c. Using substitution, verify that one of the estimated roots is an actual root of the polynomial equation.

Choose any one of the estimated roots to substitute into the given equation. Let's use $x = 8$.

$x^3 - 30x^2 + 296x - 960 = 0$

$(8)^3 - 30(8)^2 + 296(8) - 960 = 0$

$512 - 1920 + 2368 - 960 = 0$

$0 = 0$

The estimated root of 8 is an actual root of the polynomial equation because when 8 is substituted into the polynomial equation, the result is 0.

d. Using synthetic division, what is the depressed polynomial?

The divisor is 8.

8	1	−30	296	−960
		8	−176	960
	1	−22	120	0

The depressed polynomial is $x^2 - 22x + 120$.

e. What are the roots of the related polynomial equation?

Use the quadratic formula to determine the roots of the polynomial equation $x^2 - 22x + 120 = 0$.

Let $a = 1$, $b = -22$, and $c = 120$. Substitute these values into the quadratic formula and solve for x.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-22) \pm \sqrt{(-22)^2 - 4(1)(120)}}{2(1)}$$

$$x = \frac{22 \pm \sqrt{(4)}}{2}$$

$$x = \frac{22 \pm 2}{2}$$

$x = 11 \pm 1$

$x = 10$ and $x = 12$

The roots of the related polynomial equation are 10 and 12. This is also shown in the graph by the x-intercepts.

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f. What are the roots of the polynomial equation that represents the volume of Nolan's bathroom?

The roots of the polynomial equation that represents the volume of the bathroom are 8, 10, and 12.

g. What are the actual dimensions of the bathroom?

The bathroom is 8 feet × 10 feet × 12 feet.

h. What is the volume of the bathroom?

The volume of the bathroom is found by multiplying the three dimensions of the bathroom. The product of the three dimensions is 8(10)(12), or 960 cubic feet.

i. If the product of the volume and $\frac{2}{15}$ represents the cubic feet per minute needed to properly ventilate the bathroom, what is the minimum number of cubic feet per minute a fan needs to be able to circulate for a bathroom with the dimensions found in part g?

The product of 960 and $\frac{2}{15}$ is 128; therefore, a bathroom with dimensions of 8 feet × 10 feet × 12 feet requires a fan that circulates a minimum of 128 cubic feet per minute in order to properly ventilate the room.

j. Is a bathroom fan that circulates air at a rate of 130 cubic feet per minute appropriate for Nolan's bathroom? Explain your answer.

Yes, a bathroom fan that circulates air at a rate of 130 cubic feet per minute is appropriate, because the fan's rate of air circulation is greater than the minimum air circulation rate of 128 cubic feet per minute that Nolan's bathroom requires.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2A.3.4: The Rational Root Theorem

For problems 1–4, use the Rational Root Theorem to list all of the possible rational roots for each polynomial equation.

1. $x^3 + 3x^2 + 3x + 5 = 0$

2. $x^3 + 2x^2 + 5x - 9 = 0$

3. $4x^3 + x^2 + 2x - 1 = 0$

4. $2x^3 - 6x^2 + 3x + 12 = 0$

For problems 5–8, find all of the solutions of each polynomial equation.

5. $2x^3 + 11x^2 + 8x - 21 = 0$

6. $x^3 + 3x^2 + 3x + 9 = 0$

7. $x^3 - 9x^2 + 22x - 10 = 0$

8. $9x^4 - 24x^3 - 16x^2 + 20x + 3 = 0$

For problems 9 and 10, find a third-degree polynomial with rational coefficients that has the given numbers as roots.

9. -3 and $2i$

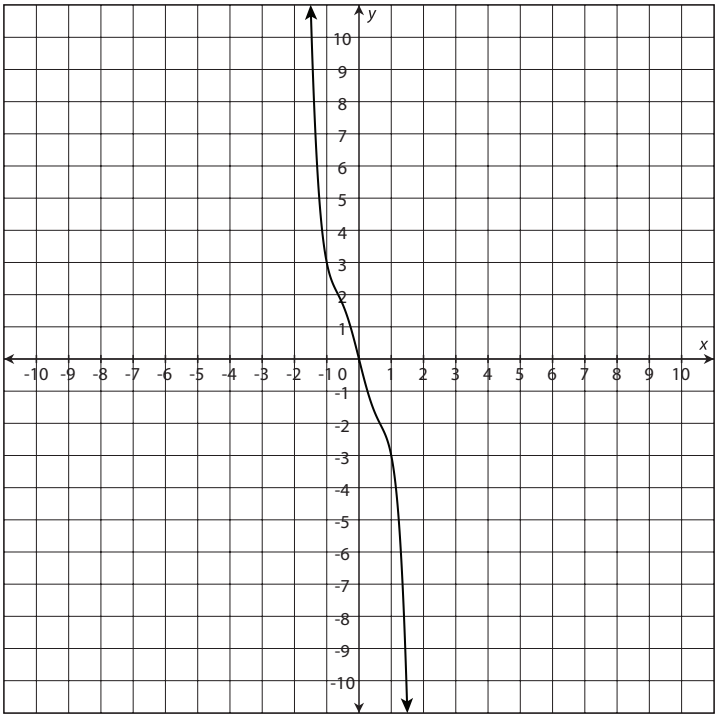
10. 4 and $3-\sqrt{2}$

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Progress Assessment

Circle the letter of the best answer.

1. What must be true about the degree and leading coefficient of the graphed polynomial?



a. The polynomial is an odd-degree function and has a positive coefficient.

b. The polynomial is an odd-degree function and has a negative coefficient.

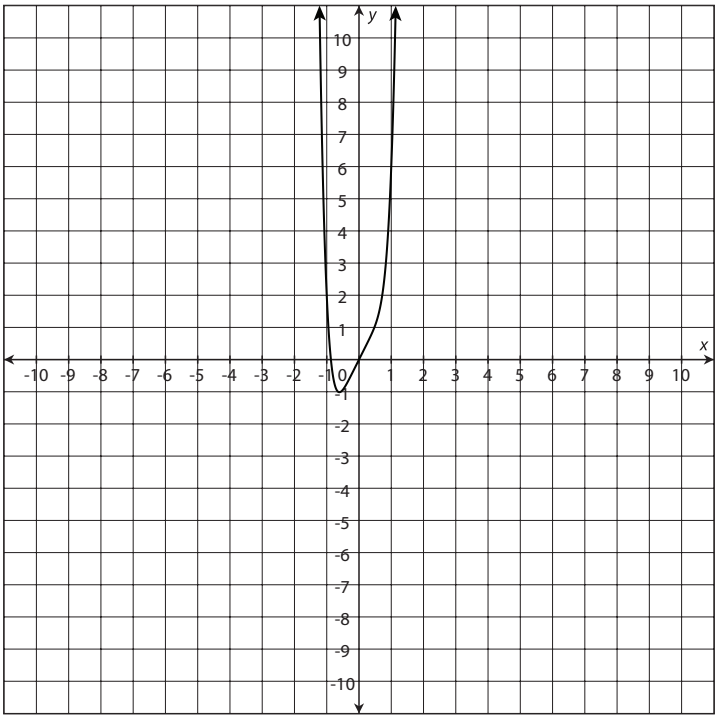
c. The polynomial is an even-degree function and has a positive coefficient.

d. The polynomial is an even-degree function and has a negative coefficient.

continued

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2. Which function could possibly represent the graphed function?



a. $f(x) = -4x^6 - 2x$

b. $f(x) = 4x^5 + 8x$

c. $f(x) = 4x^6 + 2x$

d. $f(x) = -4x^5 + 3x$

3. What is the remainder when $(x^3 - 4x^2 + 10)$ is divided by $(x - 3)$?

a. 1

b. –53

c. 31

d. 7

4. Using synthetic substitution, what is $p(x) = x^3 - 6x^2 - 5x + 12$ when $x = -4$?

a. –40

b. –128

c. –6

d. 72

continued

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Assessment

5. Which of the following is the result of using synthetic division to divide $(x^3 - 4x^2 + 9)$ by $(x - 1)$?

   a. $x^2 - 3x - 3 + \frac{6}{x+1}$

   b. $x - 3 + \frac{6}{x-1}$

   c. $x^2 - 5x + 5 + \frac{4}{x-1}$

   d. $x^2 - 3x - 3 + \frac{6}{x-1}$

6. Which polynomial function has zeros of –5, –3, and 4?

   a. $f(x) = x^3 + 12x^2 + 47x + 60$

   b. $f(x) = x^3 - 12x^2 + 47x - 60$

   c. $f(x) = x^3 - 4x^2 - 17x + 60$

   d. $f(x) = x^3 + 4x^2 - 17x - 60$

7. One solution to the equation $x^3 - 9x^2 + 33x - 25 = 0$ is $x = 1$. What are the other two roots?

   a. $3 + 4i$ and $3 - 4i$

   b. $4 + 3i$ and $4 - 3i$

   c. $-3 + 4i$ and $-3 - 4i$

   d. $-4 + 3i$ and $-4 - 3i$

8. A polynomial has zeros of 3, 5, and $2i$. What is the minimum degree of the polynomial?

   a. 3

   b. 4

   c. 5

   d. 6

9. What are all the possible rational roots for the polynomial equation $3x^3 + 8x^2 - 4x - 12 = 0$?

   a. $\pm\frac{1}{12}, \pm\frac{1}{6}, \pm\frac{1}{4}, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm\frac{3}{4}, \pm 1, \pm\frac{3}{2}, \pm 3$

   b. $\pm\frac{1}{3}, \pm 1, \pm 4, \pm 12$

   c. $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \pm\frac{4}{3}, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

   d. $\pm\frac{1}{12}, \pm\frac{1}{4}, \pm 1, \pm 3$

continued

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Assessment

10. Which of the following is a possible root of the polynomial $12x^3 + 9x^2 - 7x + 8$?

a. 3

b. 6

c. $\frac{3}{2}$

d. $\frac{8}{3}$

Use what you have learned about polynomial functions to answer the following questions.

11. A third-degree polynomial with rational coefficients has the roots 3 and $(2 + i)$.

a. What are the additional roots?

b. What is this polynomial?

c. What is the degree of this polynomial?

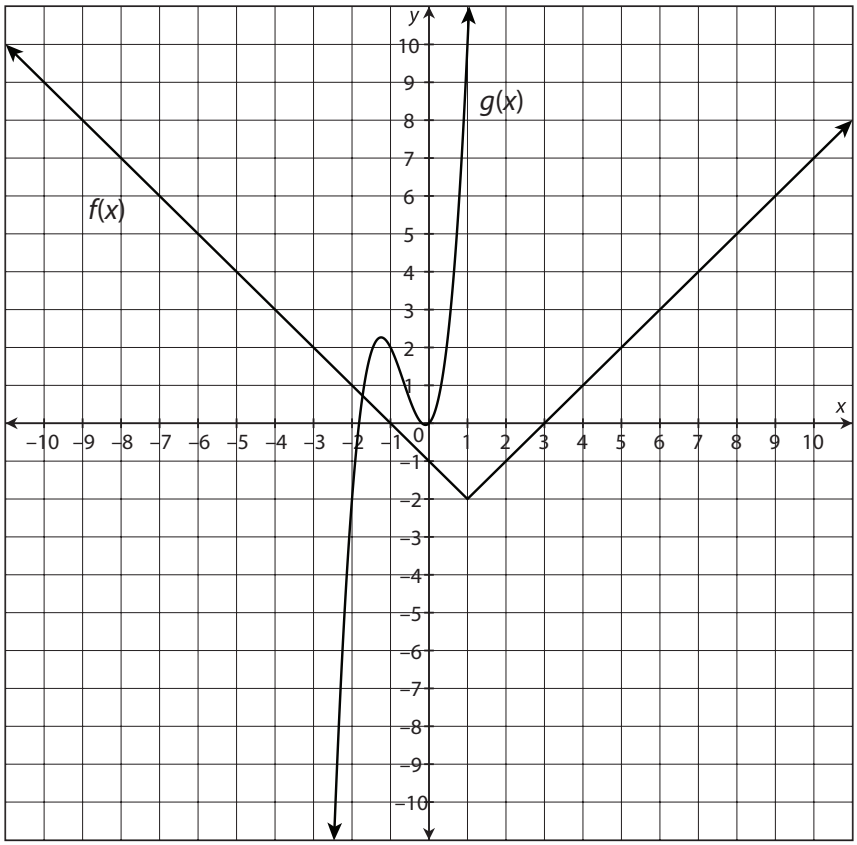
d. Identify the end behavior and maximum number of turning points of the graph of the polynomial.

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Pre-Assessment

Circle the letter of the best answer.

1. How many real solutions does the following graphed system have?



a. no real solutions

b. one real solution

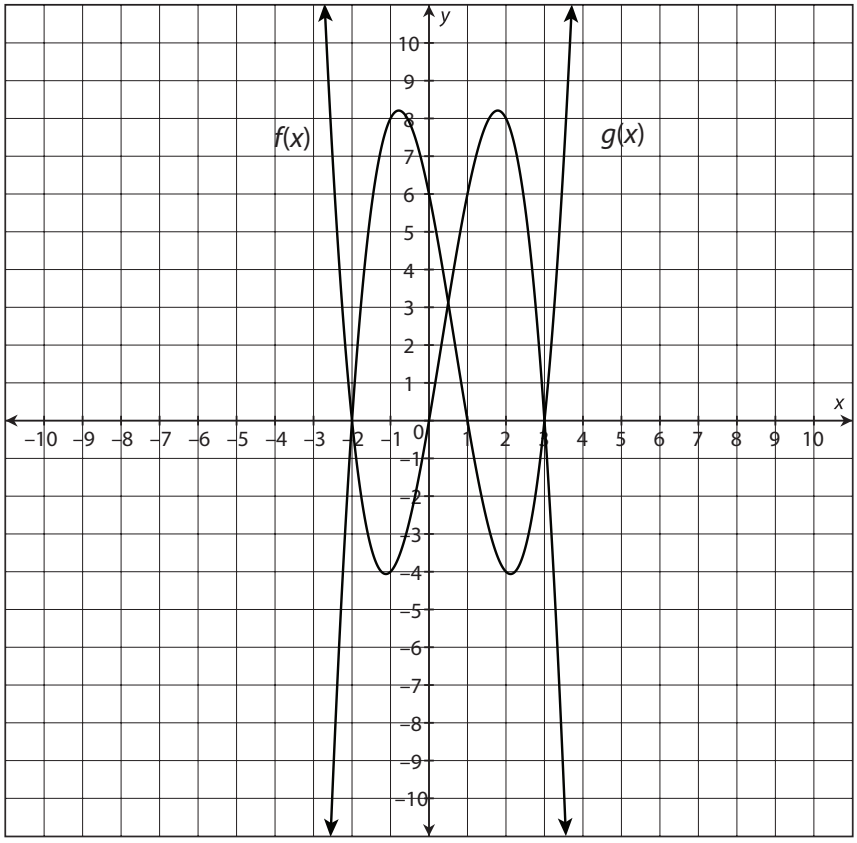
c. two real solutions

d. The number of solutions cannot be determined.

continued

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2. How many real solutions does the following graphed system have?



a. no real solutions

b. two real solutions

c. three real solutions

d. The number of solutions cannot be determined.

3. Graph the following system to estimate the real solution(s), if any.

$$\begin{cases} f(x) = -|x+2| \\ g(x) = x^3 - 2x^2 - 5x + 6 \end{cases}$$

a. {(–2, 0), (2, –4)}

b. {(–2, 2)}

c. {(–2, 0), (2, 0)}

d. There are no real solutions to this system.

continued

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4. Graph the following system to estimate the real solution(s), if any.

$$\begin{cases} f(x)=3x^4+6x^3-3x^2-6x \\ g(x)=-(3)^x-5 \end{cases}$$

   a. {(–2, 0)}

   b. {(1, 0)}

   c. {(–2, 0), (1, 0)}

   d. There are no real solutions to this system.

5. Based on the table, what conclusion can you draw about the number of solutions to the system of equations?

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$	–23	–18	–13	–8	–3	2	7	12	17
$g(x)$	–71	–28	–5	–4	5	4	7	20	49

   a. There is exactly one real solution to this system.

   b. There are at least two real solutions to this system.

   c. There are no solutions to this system.

   d. It is not possible to determine the number of solutions to this system.

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Lesson 4: Solving Systems of Equations with Polynomials

Instruction

Common Core State Standard

SMP	
1 ✓	2 ✓
3	4 ✓
5	6
7	8

A–REI.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★

Essential Questions

1. What is a system of equations?
2. How are tables and graphs used to solve systems of equations?
3. What are the different solution scenarios encountered when solving systems of equations?
4. How can systems of equations be used to model real-world situations?

WORDS TO KNOW

absolute value equation	an equation of the form $y = \lvert ax + b \rvert + c$, where x is the independent variable, y is the dependent variable, and a, b, and c are real numbers
dependent system	a system of equations that intersect at every point
empty set	a set that has no elements, denoted by ∅; the solution to a system of equations with no intersection points, denoted by { }
exponential equation	an equation of the form $y = ab^x$, where x is the independent variable, y is the dependent variable, and a and b are real numbers
independent system	a system of equations with a finite number of points of intersection
linear equation	an equation that can be written in the form $y = mx + b$, in which m is the slope, b is the y-intercept, and the graph is a straight line. The solutions to the linear equation are the infinite set of points on the line.

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Instruction

point(s) of intersection	the ordered pair(s) where graphed functions intersect on a coordinate plane; these are also the solutions to systems of equations
polynomial equation	an equation of the general form $y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, where a_1 is a rational number, $a_n \neq 0$, and n is a nonnegative integer and the highest degree of the polynomial
quadratic equation	an equation that can be written in the form $y = ax^2 + bx + c$, where x is the variable, a, b, and c are constants, and $a \neq 0$
system of equations	a set of equations with the same unknowns

Recommended Resources

- MathIsFun.com. "Function Grapher and Calculator."

  http://www.walch.com/rr/00169

  This site provides an online graphing utility. Users can enter values to create their own system and then zoom in on the functions to determine approximate intersection points.

- Purplemath.com. "Systems of Non-Linear Equations."

  http://www.walch.com/rr/00170

  This site provides a summary of nonlinear systems as well as worked examples.

- Virtual Math Lab. "Solving Systems of Nonlinear Equations in Two Variables."

  http://www.walch.com/rr/00171

  This site reviews possible solutions to systems of nonlinear equations. Methods to solve these systems are not limited to graphing.

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Lesson 2A.4.1: Solving Systems of Equations Graphically

Warm-Up 2A.4.1

The height of a ball thrown into the air can be modeled by the equation $f(x) = -16x^2 + 18x + 6$, where $f(x)$ is the height of the ball in feet x seconds after the ball is thrown. A paintball launched into the air is modeled by the equation $g(x) = \frac{5}{2}x + \frac{11}{2}$, where $g(x)$ is the distance the paintball has traveled. The ball is released at the same time the paintball is launched.

1. Graph each equation on the same coordinate plane.

2. How many points of intersection exist?

3. How long will it take the paintball to cross the path the ball traveled?

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Instruction

Lesson 2A.4.1: Solving Systems of Equations Graphically

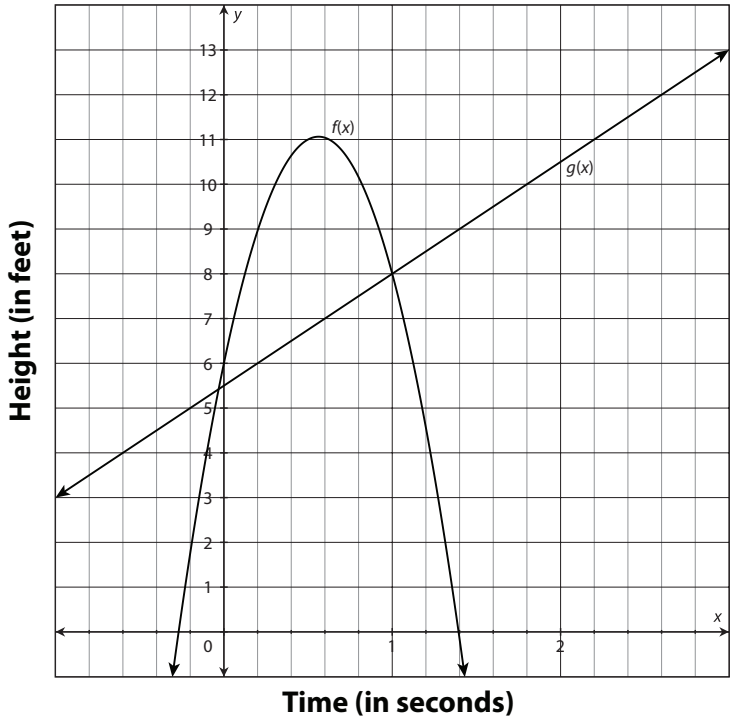
Common Core State Standard

A–REI.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★

Warm-Up 2A.4.1 Debrief

1. Graph each equation on the same coordinate plane.

   The graph of each equation is shown as follows.



   Note: All four quadrants are graphed to show the general shape of each equation. Because the equations represent height and time, only the first quadrant applies to the given problem.

2. How many points of intersection exist?

   This system has two points of intersection. They appear to be at approximately (–0.03, 5.4) and (1, 8).

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3. How long will it take the paintball to cross the path the ball traveled?

   The approximate solution of (–0.03, 5.4) does not apply to this problem because time (represented by x) cannot be negative. Therefore, the only solution that is relevant to the problem is (1, 8). After 1 second, the paintball will cross the path the ball traveled.

Connection to the Lesson

- Students will extend their understanding of solving linear and quadratic-linear systems by graphing to solving systems of equations involving polynomial equations.
- Students will use graphs, as well as tables, to find the intersection(s) of two functions.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- understanding that the graph of an equation is the set of all the solutions of the equation plotted on a coordinate plane (A–REI.10)
- creating a table of values for a given equation (F–IF.2)
- graphing various forms of equations, including polynomial, linear, absolute value, and exponential (F–IF.7)
- graphing multiple equations on a graphing calculator (A–REI.11)

Introduction

A **system of equations** is a set of equations with the same unknowns. It is likely that you have solved a system of linear equations or a quadratic-linear system in the past. In this lesson, we will learn to solve systems that include at least one polynomial by graphing.

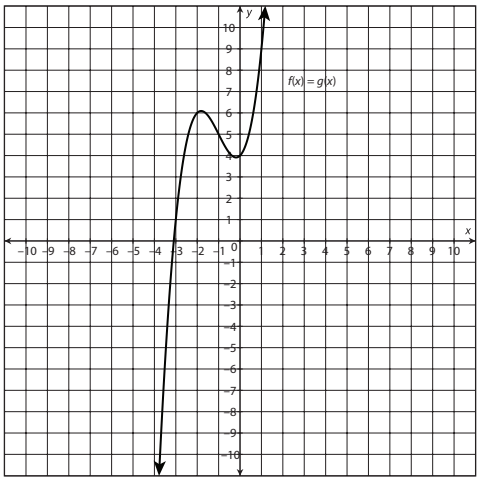
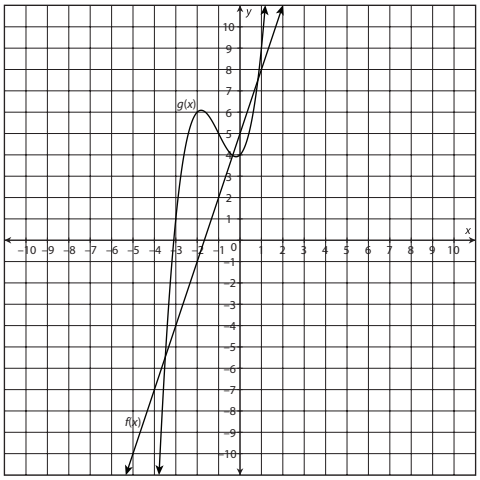
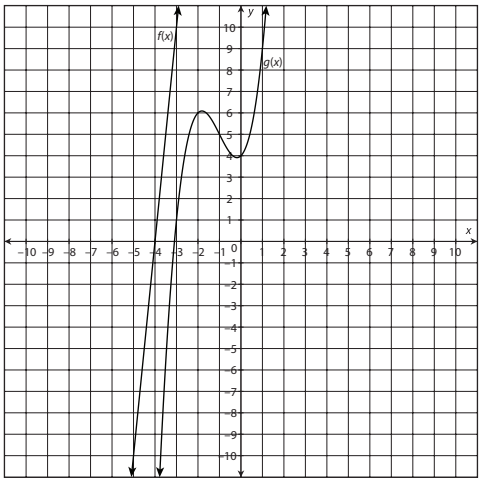
Key Concepts

- The real solutions of a system of equations can often be found by graphing. Recall that the graph of an equation is the set of all the solutions of the equation plotted on a coordinate plane.
- The **points of intersection** of the graphed system of equations are the ordered pairs where graphed functions intersect on a coordinate plane. These are also the solutions to the system. The solution or solutions to a system of equations with $f(x)$ and $g(x)$ occurs where $f(x) = g(x)$ and the difference between these two values is 0. Also recall that the solutions to a system of equations are written using braces: {}.
- A system may have an infinite number of solutions, a finite number of solutions, or no real solutions.
- A system with a finite number of points of intersection is called an **independent system**.
- It is possible that a system of equations does not intersect. The solution to such a system of equations is the **empty set**, denoted by { }.
- A system of equations could also intersect at every point. This type of system is known as a **dependent system**.

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Instruction

Solutions to Systems of Equations with Polynomials

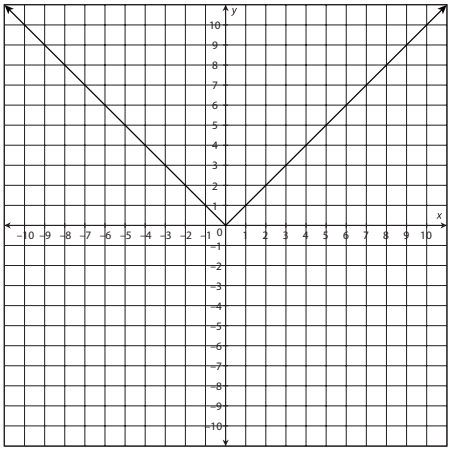
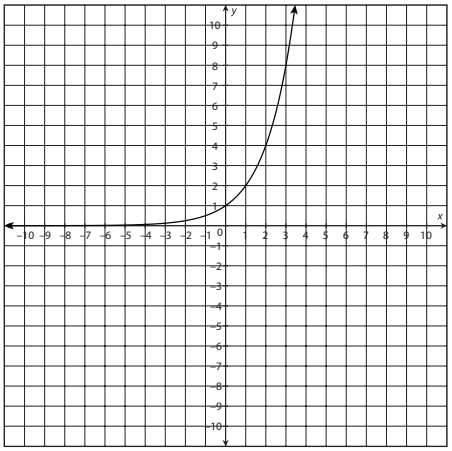
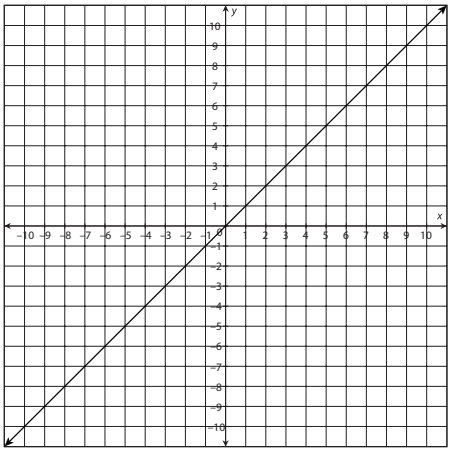
	• If the equations overlap, they have an infinite number of solutions. • This is an example of a dependent system.
	• If the equations intersect, they have a finite number of solutions. • This is an example of an independent system.
	• If the equations do not intersect, they have no solutions. • The solution to this system is the empty set.

- When graphing a system of equations with polynomials, it is helpful to recall various forms of equations and their general shapes.

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Instruction

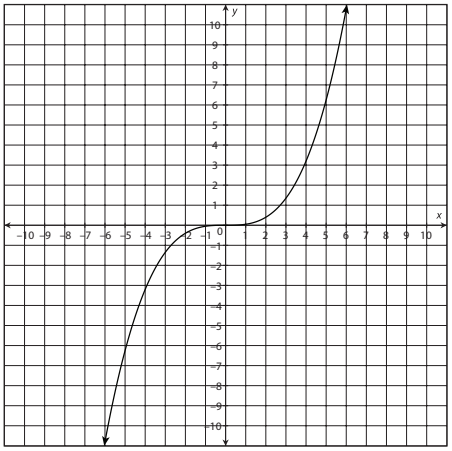
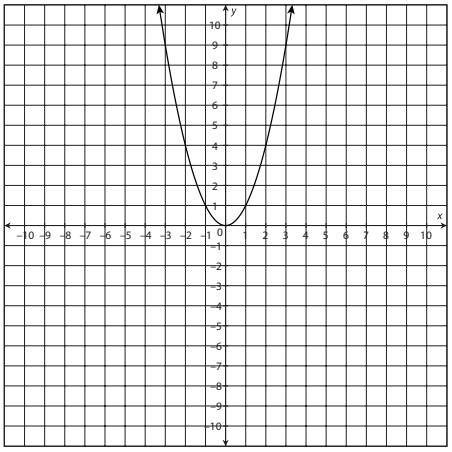
Graphs of General Equations

Type of equation	General equation	Graph
Absolute value equation x is the independent variable, y is the dependent variable, and a, b, and c are real numbers	$y = \lvert ax + b \rvert + c$	
Exponential equation x is the independent variable, y is the dependent variable, and a and b are real numbers	$y = ab^x$	
Linear equation m is the slope and b is the y-intercept	$y = mx + b$	

(*continued*)

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Instruction

Polynomial equation a_1 is a rational number, $a_n \neq 0$, and n is a nonnegative integer and the highest degree of the polynomial	$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$	
Quadratic equation x is the variable, a, b, and c are constants, and $a \neq 0$	$y = ax^2 + bx + c$	

- To graph a system of equations, create a table of values and plot each coordinate on the same coordinate plane. It may be necessary to find additional points to determine where the point(s) of intersection may occur. Each point of intersection represents a solution to the system.
- Solutions to a system of equations can also be identified in a table by finding points such that the y-value of each function is the same for a particular x-value.
- Using a graphing calculator to graph both equations is often helpful. Most calculators have a trace feature to approximate where the equations intersect, or can display a table of values for each equation so you can analyze the values. Some calculators will find intersection points for you.
- To confirm a solution to a system, substitute the coordinates for x and y in both of the original equations. If the intersection point is approximated, the results will be nearly equal.

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Instruction

Common Errors/Misconceptions

- not identifying all points of intersection
- incorrectly graphing each equation
- incorrectly entering the equations into the calculator
- not understanding that solutions must satisfy every function in a system
- believing that every system has at least one solution

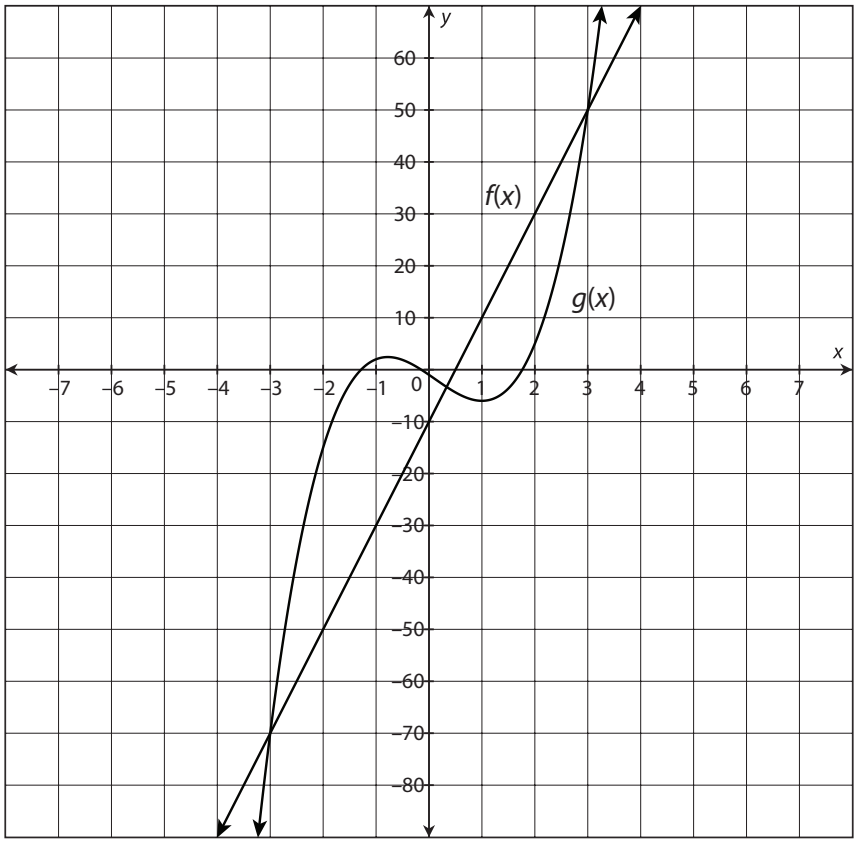
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Instruction

Guided Practice 2A.4.1

Example 1

Use the graph to estimate the solution(s), if any, to the system of equations.



1. Determine the number of times the graphs of $f(x)$ and $g(x)$ intersect.

   The graphs of $f(x)$ and $g(x)$ appear to intersect at three points.

2. Estimate the points of intersection.

   The scale of the graph provided determines how accurately the solution can be approximated. It appears as though the graphs intersect at approximately (–3, –70), (0.4, –4), and (3, 50). It is not possible to more accurately approximate this solution, as the original equations are not provided.

   The approximate solution to this system of equations is {(–3, –70), (0.4, –4), (3, 50)}.

SWB p. 111; SRB p. U2A-102

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Instruction

Example 2

Use a graph to estimate the real solution(s), if any, to the system of equations $\begin{cases} f(x)=4x+1 \\ g(x)=x^3+1 \end{cases}$. Verify that any identified coordinate pairs are solutions.

1. Create a graph of the two functions, $f(x)$ and $g(x)$.

   You can use what you have learned about graphing equations to graph each equation on the same coordinate plane, or you can use a graphing calculator.

   To graph the system on your graphing calculator, follow the directions appropriate to your calculator model.

   On a TI-83/84:

   Step 1: Press the [Y=] button.

   Step 2: Type the first function into Y_1, using the [X, T, θ, n] button for the variable *x:* [4][X, T, θ, n][+][1]. Press [ENTER].

   Step 3: Type the second function into Y_2, using the [^] button for powers: [X, T, θ, n][^][3][+][1]. Press [ENTER].

   Step 4: Press [ZOOM]. Arrow down to 6: ZStandard. Press [ENTER].

   On a TI-Nspire:

   Step 1: Press the [home] key.

   Step 2: Arrow over to the graphing icon and press [enter].

   Step 3: Type the first function next to *f1*(*x*), using the [X] button for the letter *x* or the [x^2] button for a square: [4][X][+][1]. Press [enter].

   Step 4: To graph the second function, press [menu] and arrow down to 3: Graph Type. Arrow right to bring up the sub-menu, then select 1: Function. Press [enter].

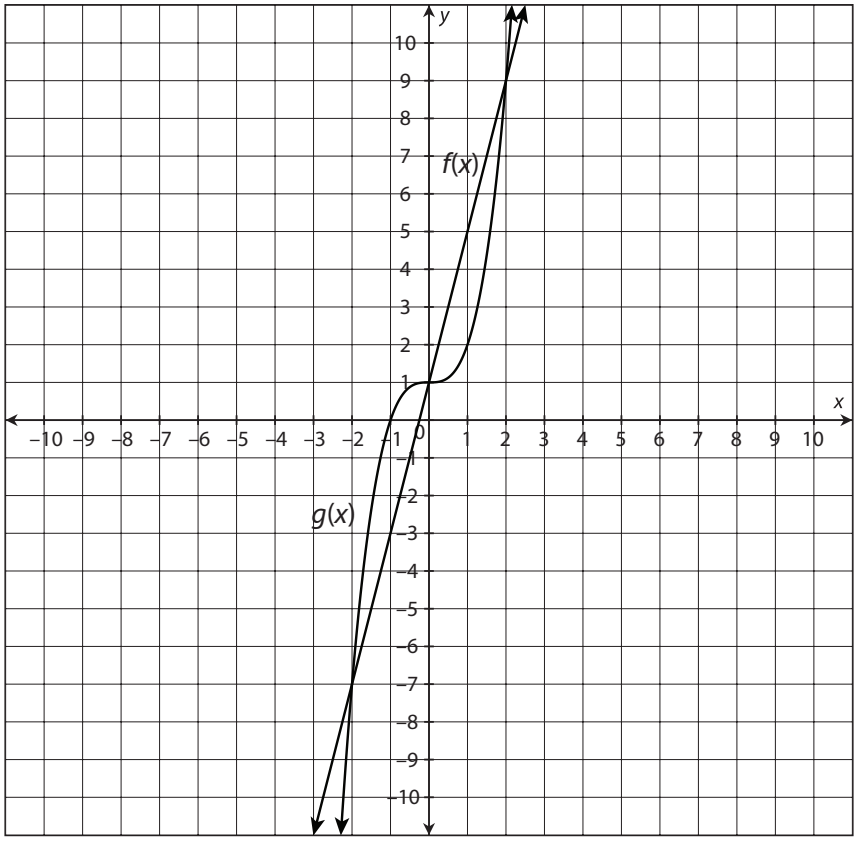
   Step 5: At *f2*(*x*), type [X][^][3][+][1] and press [enter].

   Step 6: To change the viewing window, press [menu]. Select 4: Window/Zoom and select A: Zoom – Fit.

(*continued*)

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The graph appears as follows.



2. Find the coordinates of any apparent intersections.

   On your graph, estimate where the two equations intersect. It may be necessary to plot additional points.

   To determine the approximate point(s) of intersection on your graphing calculator, use the following directions.

   On a TI-83/84:

   Step 1: Press [2ND][TRACE] to call up the CALC screen. Arrow down to 5: intersect. Press [ENTER].

   Step 2: At the prompt, use the up/down arrow keys to select the Y_1 equation, and press [ENTER].

(*continued*)

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Step 3: At the prompt, use the up/down arrow keys to select the Y_2 equation, and press [ENTER].

Step 4: At the prompt, use the arrow keys to move the cursor close to an apparent intersection, and press [ENTER]. The coordinates of the intersection points are displayed.

On a TI-Nspire:

Step 1: Press [menu]. Arrow down to 6: Analyze Graph, then arrow right to 4: Intersection. Press [enter].

Step 2: Use the pointing hand to click on each graphed line. The coordinates of the intersection points are displayed.

Repeat for each intersection as needed.

The points of intersection are (–2, –7), (0, 1), and (2, 9).

3. Verify that the identified coordinate pairs are solutions to the original system of equations.

In order for a coordinate pair to be a solution to a system, the coordinates must satisfy both equations of the system.

Substitute the coordinates into each of the equations, then evaluate the equations to see if the coordinates result in a true statement.

At the point (–2, –7), let $x = -2$ and $f(x)$ and $g(x) = -7$.

$f(x) = 4x + 1$	First equation
$(-7) = 4(-2) + 1$	Substitute –2 for x and –7 for $f(x)$.
$-7 = -7$	

$g(x) = x^3 + 1$	Second equation
$(-7) = (-2)^3 + 1$	Substitute –2 for x and –7 for $g(x)$.
$-7 = -7$	

The identified point (–2, –7) satisfies both $f(x)$ and $g(x)$.

(*continued*)

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Instruction

At the point (0, 1), let $x = 0$ and $f(x)$ and $g(x) = 1$.

$f(x) = 4x + 1$	First equation
$(1) = 4(0) + 1$	Substitute 0 for x and 1 for $f(x)$.
$1 = 1$	

$g(x) = x^3 + 1$	Second equation
$(1) = (0)^3 + 1$	Substitute 0 for x and 1 for $g(x)$.
$1 = 1$	

The identified point (0, 1) satisfies both $f(x)$ and $g(x)$.

At the point (2, 9), let $x = 2$ and $f(x)$ and $g(x) = 9$.

$f(x) = 4x + 1$	First equation
$(9) = 4(2) + 1$	Substitute 2 for x and 9 for $f(x)$.
$9 = 9$	

$g(x) = x^3 + 1$	Second equation
$(9) = (2)^3 + 1$	Substitute 2 for x and 9 for $g(x)$.
$9 = 9$	

The identified point (2, 9) satisfies both $f(x)$ and $g(x)$.

The solution set to the system of equations $\begin{cases} f(x) = 4x + 1 \\ g(x) = x^3 + 1 \end{cases}$ is {(–2, –7), (0, 1), (2, 9)}, as identified on the graph as the intersections of the lines $f(x)$ and $g(x)$.



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Example 3

Create a table to approximate the real solution(s), if any, to the system $\begin{cases} f(x)=|x+4|+1 \\ g(x)=x^4+3x^3+1 \end{cases}$.

1. Create a table of coordinates for the two functions $f(x)$ and $g(x)$.

   You can create a table of values by hand similarly to how you have created them in the past, or you can use your graphing calculator.

   To create a table of values using your graphing calculator, follow the directions appropriate to your calculator model.

   On a TI-83/84:

   Step 1: Press the [Y=] button. Clear any equations from previous calculations.

   Step 2: For the first function, you will need to enter absolute value symbols. There are two ways to enter the absolute value symbols: press [2ND][0] to call up the CATALOG menu, then select abs(for the absolute value symbols; or press [MATH], then arrow over to NUM and select 1: abs(.

   Step 3: Type the first function into Y_1, using the [X, T, θ, n] button for the variable x, and the arrows to move the cursor to outside the absolute value symbols. Press [ENTER].

   Step 4: Type the second function into Y_2. Use the [^] button for powers. After entering a power, press the right arrow button to get out of exponent mode: [X, T, θ, n][^][4][+][3][X, T, θ, n][^][3][+][1]. Press [ENTER].

   Step 5: Press [2ND][WINDOW] to call up the TABLE SETUP screen. Set TblStart to –5 by clearing out any current values and typing [(–)][5][ENTER]. Set ΔTbl to 1, then press [2ND][GRAPH] to call up the TABLE screen. A table of values for both equations will be displayed.

(*continued*)

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On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the calculator page, the first icon over, and press [enter].

Step 3: Define $f(x)$ by entering "define f1(x)=abs(x+4)+1" using the buttons on your keypad. Remember to enter a space after "define". Press [enter].

Step 4: Define $g(x)$ by entering "define g1(x)=x^4+3x^3+1". Remember to press the right arrow key after entering an exponent. Press [enter].

Step 5: Press the [home] key. Arrow over to the spreadsheets page, the fourth icon over, and press [enter].

Step 6: Press [ctrl][T] to switch to a table window. From the list of defined functions that appears, select $f(x)$f1, and press [enter].

Step 7: Press the right arrow key. From the list of defined functions that appears, select $f(x)$g1, and press [enter].

A table of values for both equations is displayed.

The table appears as follows.

x	–5	–4	–3	–2	–1	0	1	2	3
$f(x)$	2	1	2	3	4	5	6	7	8
$g(x)$	251	65	1	–7	–1	1	5	41	163

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2. Estimate the points of intersection.

   Compare the y-values for the same x-value in your completed table.

   Look for places where the difference between the values for $g(x)$ and $f(x)$ is decreasing. If the difference between $g(x)$ and $f(x)$ is increasing, then any intersection occurs in the other direction. The point of intersection occurs when $f(x)$ is equal to $g(x)$ or the difference is 0.

x	–5	–4	–3	–2	–1	0	1	2	3
$f(x)$	2	1	2	3	4	5	6	7	8
$g(x)$	251	65	1	–7	–1	1	5	41	163
$g(x) - f(x)$	+249	+64	–1	–10	–5	–4	–1	+34	+155

   Notice that in the table, there are no values for which $f(x)$ equals $g(x)$. Based on the table, the smallest difference between $g(x)$ and $f(x)$, –1, occurs in two places: at $x = -3$ and at $x = 1$. Therefore, there appear to be intersection points close to these x-values.

   To better approximate the points of intersection, you must find additional values.

   Use the following directions to determine additional values on your graphing calculator.

   On a TI-83/84:

   Step 1: Press [2ND][WINDOW] to call up the TABLE SETUP screen. Set the TblStart to –3.5 and ΔTbl to 0.01, clearing out any previous values. Press [ENTER], then press [2ND] [GRAPH] to call up the TABLE screen. A table of values for both equations will be displayed.

   Step 2: Scroll through the table of values to determine the approximate solutions.

   On a TI-Nspire:

   Step 1: Press [menu]. Use the arrow keys to select 5: Table, then 5: Edit Table Settings. Set the Table Start to –4 and Table Step to 0.01. Arrow down to OK and press [enter]. You may need to readjust the Table Step setting to get a better approximation of when $f(x) = g(x)$.

(*continued*)

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The following tables display information for values of x close to the points we're interested in: $x = -3$ and $x = 1$.

x	–3.08	–3.07	–3.06	–3.05	–3.04	–3.03	–3.02	–3.01
$f(x)$	1.92	1.93	1.94	1.95	1.96	1.97	1.98	1.99
$g(x)$	3.3374	3.0254	2.7192	2.4186	2.1238	1.8345	1.5509	1.2727
$g(x) - f(x)$	+1.4174	+1.0954	+0.7792	+0.4686	+0.1638	–0.1355	–0.4291	–0.7173

x	1.04	1.05	1.06	1.07	1.08	1.09	1.10	1.11
$f(x)$	6.04	6.05	6.06	6.07	6.08	6.09	6.10	6.11
$g(x)$	5.5445	5.6884	5.8355	5.9859	6.1396	6.2967	6.4571	6.6210
$g(x) - f(x)$	–0.4955	–0.3616	–0.2245	–0.0841	+0.0596	+0.2067	+0.3571	+0.5110

Since we know that the difference between $g(x)$ and $f(x)$ is smallest for values of x close to –3 and close to 1, review the tables near those values. Determine where the values of $g(x)$ and $f(x)$ have the smallest difference. According to the generated tables of values, the values of x with the smallest difference between $g(x)$ and $f(x)$ are –3.03 and 1.08. Because the differences in values are so small and because the coordinates are approximate, it is appropriate to use either the $f(x)$ value or the $g(x)$ value when stating the solution.

It can be seen from the tables that –3.03 and 1.08 are the approximate x-coordinates where $f(x)$ and $g(x)$ intersect; therefore, the approximate solutions to the system are (–3.03, 1.97) and (1.08, 6.08).

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Instruction

3. Verify that the identified coordinate pairs are solutions to the original system of equations.

   In order for a coordinate to be a solution to a system, the coordinate must satisfy both equations of the system. Because our coordinates are estimated, it is quite possible that our results will be nearly equal, but not exactly equal.

   Substitute the coordinates into each of the equations, then evaluate the equations to see if the coordinates result in a true statement.

   At the point (–3.03, 1.97), let $x = -3.03$ and $f(x)$ and $g(x) = 1.97$.

   $f(x) = |x + 4| + 1$ First equation

   $(1.97) = |(-3.03) + 4| + 1$ Substitute –3.03 for x and 1.97 for $f(x)$.

   $1.97 = 1.97$

   $g(x) = x^4 + 3x^3 + 1$ Second equation

   $(1.97) = (-3.03)^4 + (-3.03)^3 + 1$ Substitute –3.03 for x and 1.97 for $g(x)$.

   $1.97 \approx 1.83$

   The identified point (–3.03, 1.97) nearly satisfies both $f(x)$ and $g(x)$, and can be considered an approximate solution to the system of equations.

(*continued*)

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Instruction

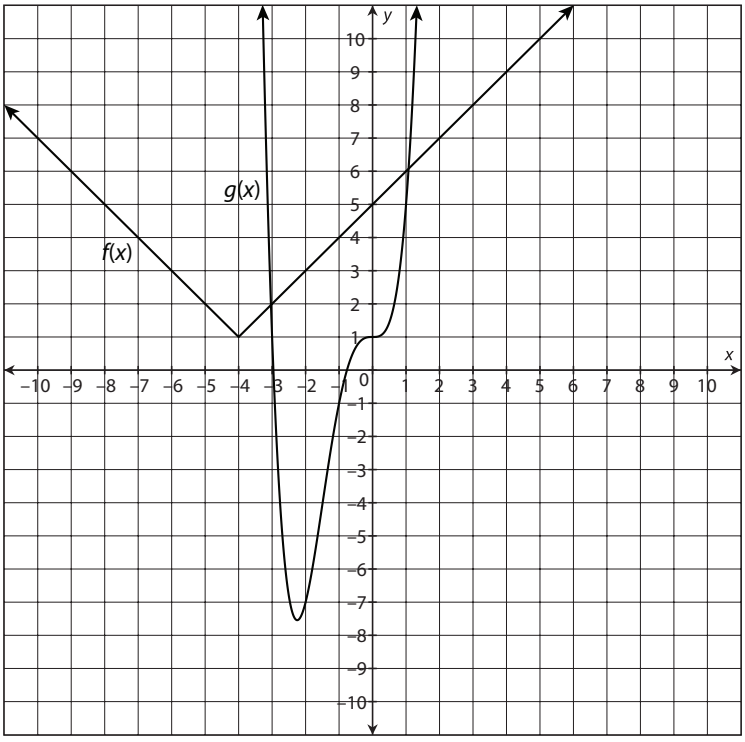
At the point (1.08, 6.08), let $x = 1.08$ and $f(x)$ and $g(x) = 6.08$.

$f(x)=\lvert x+4\rvert+1$	First equation
$(6.08)=\lvert(1.08)+4\rvert+1$	Substitute 1.08 for x and 6.08 for $f(x)$.
$6.08 = 6.08$	
$g(x) = x^4 + 3x^3 + 1$	Second equation
$(6.08) = (1.08)^4 + (1.08)^3 + 1$	Substitute 1.08 for x and 6.08 for $g(x)$.
$6.80 \approx 6.14$	

The identified point (1.08, 6.08) nearly satisfies both $f(x)$ and $g(x)$ and can be considered an approximate solution to the system of equations.

The approximate solution set to the system of equations $\begin{cases} f(x)=\lvert x+4\rvert+1 \\ g(x)=x^4+3x^3+1 \end{cases}$ is {(–3.03, 1.97), (1.08, 6.08)}, as identified in the tables as the intersections of the equations $f(x)$ and $g(x)$.

Graphing the system shows that our approximated solution set is valid.



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Instruction

Example 4

The following table shows values for two functions, $f(x)$ and $g(x)$. Based on the table, what conclusion can you draw about the solutions to the system of equations $f(x)$ and $g(x)$?

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$	–10	–7	–4	–1	2	5	8	11	14
$g(x)$	55	25	11	7	7	5	–5	–29	–73

1. Observe the values of $f(x)$ and $g(x)$ for each x-value.

   Look for places where the difference between the values for $g(x)$ and $f(x)$ is decreasing. If the difference between $g(x)$ and $f(x)$ is increasing, then any intersection occurs in the other direction. The point of intersection occurs when $f(x)$ is equal to $g(x)$ or the difference is 0.

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$	–10	–7	–4	–1	2	5	8	11	14
$g(x)$	55	25	11	7	7	5	–5	–29	–73
$g(x) - f(x)$	+65	+32	+15	+8	+5	0	–13	–40	–87

   The difference between $g(x)$ and $f(x)$ gets smaller as the value of x approaches 1. Notice that when $x = 1$, $f(x)$ and $g(x) = 5$; therefore, (1, 5) is one point of intersection and thus is one solution to the system of equations.

   From this point, the difference between $g(x)$ and $f(x)$ starts increasing.

   There do not appear to be any other points of intersection, or solutions to the system of equations, based on the limited number of values.



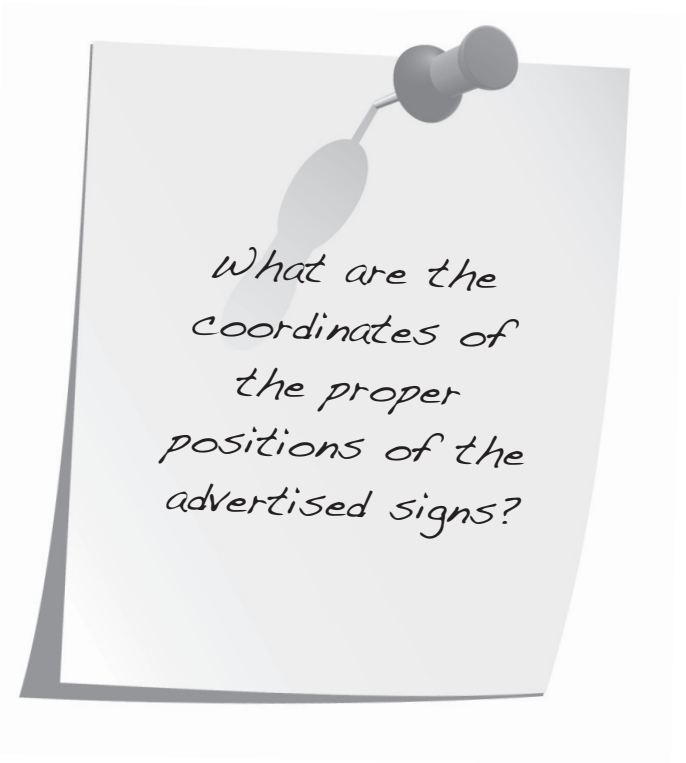
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Problem-Based Task 2A.4.1: Advertise Here!

SMP	
1 ✓	2 ✓
3	4 ✓
5	6
7	8

An advertising agency paid for the right to post signs along a footbridge at a new racetrack. The footbridge passes over the new racetrack's signature S-curve, providing multiple placements for advertising directly above the track. The signs need to be posted in the proper positions so that fans watching the race on TV will be able to read them as the cars race through the S-curve and under the bridge.

The location of the S-curve in the racetrack can be described by the equation $f(x)=\frac{1}{30}\left(x^3-3x^2-270x\right)$. The location of the footbridge can be described by the linear equation $g(x) = 2 - 3x$. What are the coordinates of the proper positions of the advertised signs?



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Problem-Based Task 2A.4.1: Advertise Here!

Coaching

a. Graph the equations that represent the S-curve and the footbridge.

b. How many points of intersection are there?

c. What do the points of intersection represent in terms of the S-curve and the footbridge?

d. In how many places will the footbridge pass directly over the S-curve in the track?

e. Based on your observations of the number of points of intersection, estimate the coordinates of each of the advertised signs.

f. Verify that the estimated coordinates are solutions to the system of equations.

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Instruction

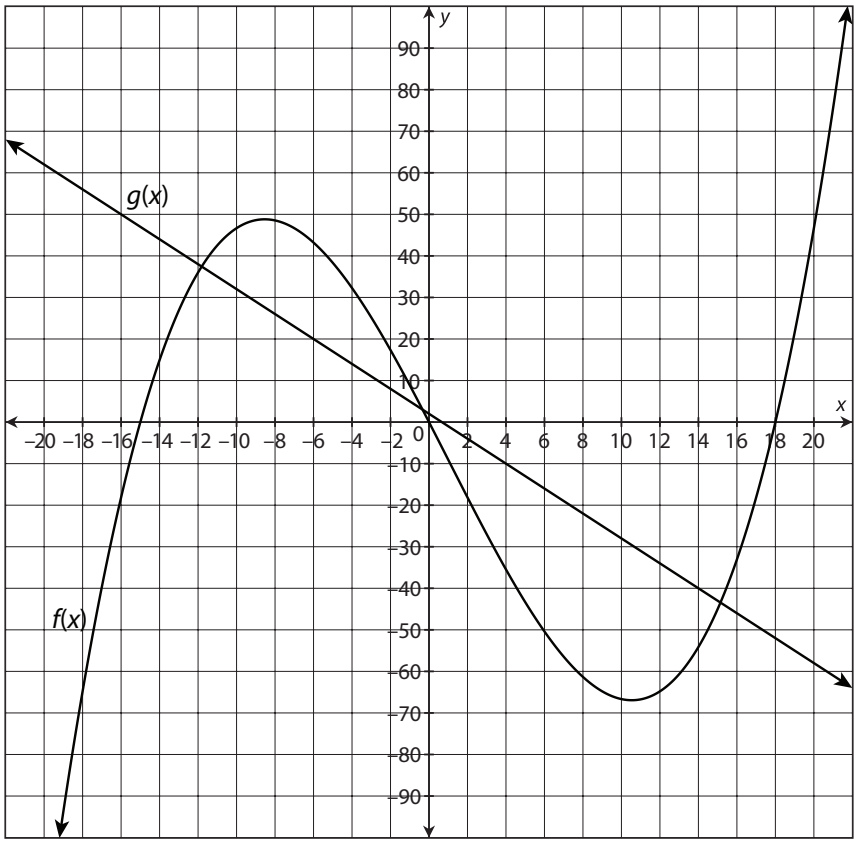
Problem-Based Task 2A.4.1: Advertise Here!

Coaching Sample Responses

a. Graph the equations that represent the S-curve and the footbridge.

Create a table of values or use a graphing calculator to graph the equations.

The equations that represent the S-curve and the footbridge are graphed below.



b. How many points of intersection are there?

There appear to be three points of intersection on the graph.

c. What do the points of intersection represent in terms of the S-curve and the footbridge?

The points of intersection on the graph represent the locations on the racetrack where the advertisements should be placed to best be seen on television.

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d. In how many places will the footbridge pass directly over the S-curve in the track?

There are three places where the footbridge will pass directly over the S-curve in the racetrack. These locations correspond to the intersection points of the equations.

e. Based on your observations of the number of points of intersection, estimate the coordinates of each of the advertised signs.

It can be seen from the graph that the approximate x-values for the points of intersection are between –12 and –10, –1 and 0, and 14 and 16.

Through the use of a graphing calculator, it can be determined that closer approximations for each x-value are –11.81, –0.34, and 15.15.

Consulting the table of values on the graphing calculator shows that the coordinates for these points of intersection are approximately (–11.81, 37.44), (–0.34, 3.05), and (15.15, –43.39).

f. Verify that the estimated coordinates are solutions to the system of equations.

To verify that each coordinate is a solution to the system of equations, substitute the approximated values into both of the original equations, then evaluate. The results will be nearly equal.

Let $x = -11.81$. Recall from the points of intersection that for this value of x, $f(x) = 37.44$ and $g(x) = 37.44$ as well.

$$f(x) = \frac{1}{30}\left(x^3 - 3x^2 - 270x\right)$$

$$f(-11.81) = \frac{1}{30}\left[(-11.81)^3 - 3(-11.81)^2 - 270(-11.81)\right]$$

$$(37.44) = \frac{1}{30}\left[(-11.81)^3 - 3(-11.81)^2 - 270(-11.81)\right]$$

$$37.44 \approx 37.44$$

$g(x) = 2 - 3x$

$g(-11.81) = 2 - 3(-11.81)$

$37.44 = 2 - 3(-11.81)$

$37.44 \approx 37.43$

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Instruction

Let $x = -0.34$. Recall from the points of intersection that for this value of x, $f(x) = 3.05$ and $g(x) = 3.05$ as well.

$$f(x)=\frac{1}{30}\left(x^3-3x^2-270x\right)$$

$$f(-0.34)=\frac{1}{30}\left[(-0.34)^3-3(-0.34)^2-270(-0.34)\right]$$

$$3.05=\frac{1}{30}\left[(-0.34)^3-3(-0.34)^2-270(-0.34)\right]$$

$$3.05 \approx 3.05$$

$$g(x) = 2 - 3x$$

$$g(-0.34) = 2 - 3(-0.34)$$

$$3.05 = 2 - 3(-0.34)$$

$$3.05 \approx 3.02$$

Let $x = 15.15$. Recall from the points of intersection that for this value of x, $f(x) = -43.39$ and $g(x) = -43.39$ as well.

$$f(x)=\frac{1}{30}\left(x^3-3x^2-270x\right)$$

$$f(15.15)=\frac{1}{30}\left[(15.15)^3-3(15.15)^2-270(15.15)\right]$$

$$-43.39=\frac{1}{30}\left[(15.15)^3-3(15.15)^2-270(15.15)\right]$$

$$-43.39 \approx -43.39$$

$$g(x) = 2 - 3x$$

$$g(15.15) = 2 - 3(15.15)$$

$$-43.39 = 2 - 3(15.15)$$

$$-43.39 \approx -43.45$$

The substitution of each coordinate results in nearly equal values for $f(x)$ and $g(x)$; therefore, each coordinate is an approximate solution of the given system.

Recommended Closure Activity

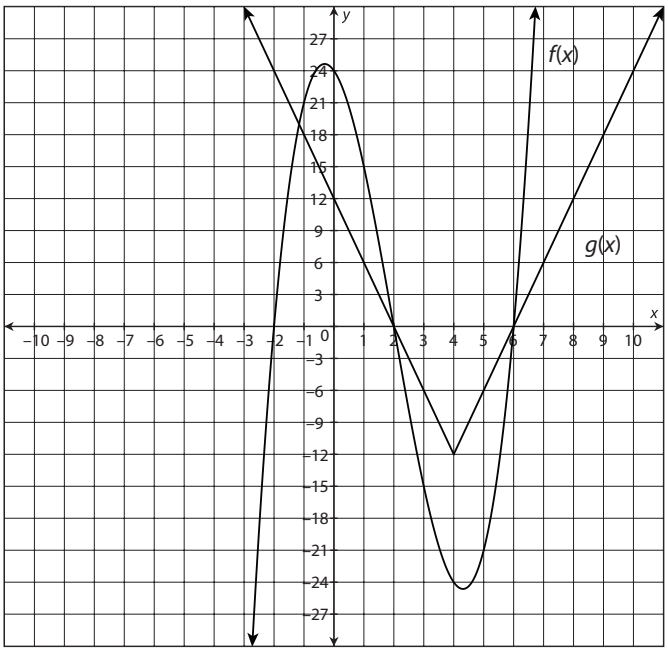
Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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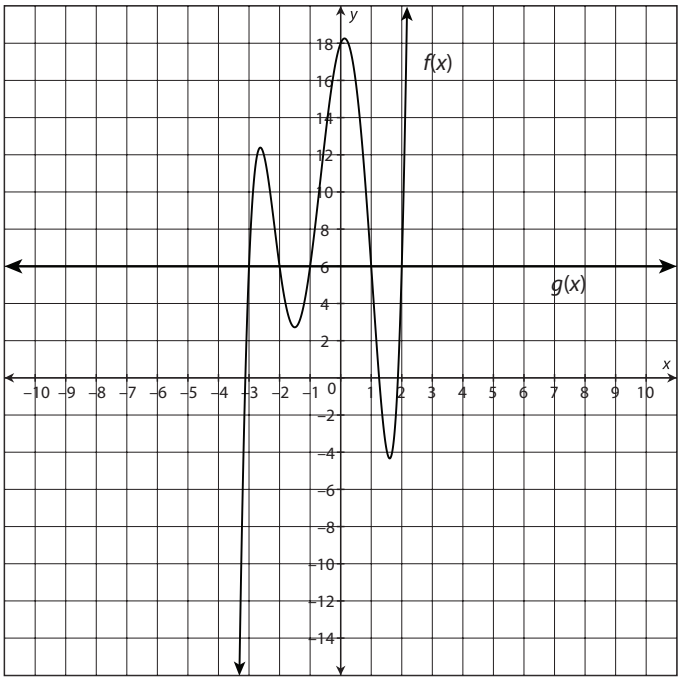
Practice 2A.4.1: Solving Systems of Equations Graphically

For problems 1–3, use the graphs to estimate the solution(s), if any, to the system of equations. Approximate solutions to the nearest tenth, if necessary.

1.



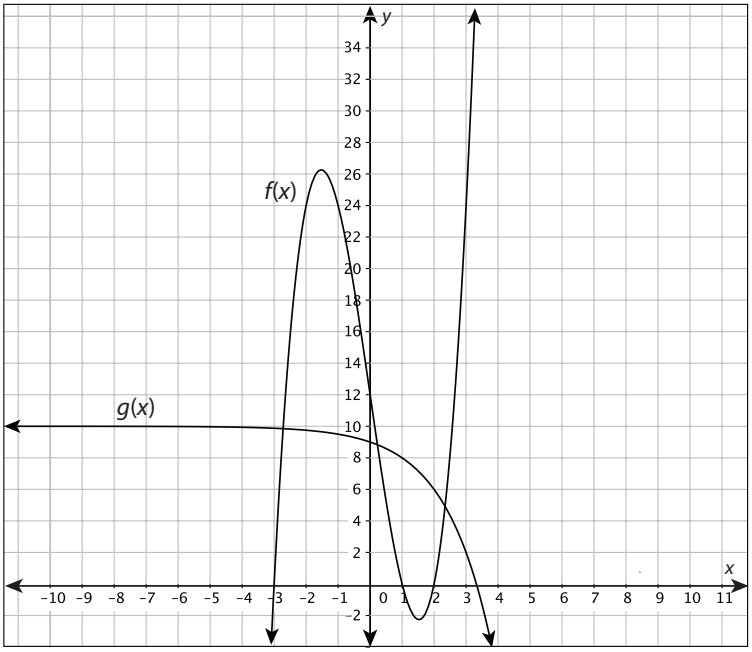
2.



continued

SWB p. 115; SRB Practice 2A.4.1 p. U2A-114

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3. 

For problems 4–7, graph each system of equations. Use the graph to estimate the real solution(s), if any exist. Round solutions to the nearest tenth, if necessary.

4. $\begin{cases} f(x)=2x^3-2x^2-10x-6 \\ g(x)=-3x+9 \end{cases}$

5. $\begin{cases} f(x)=\left|5x-5\right|-4 \\ g(x)=-x^4+4x^3-x^2-6x+6 \end{cases}$

6. $\begin{cases} f(x)=2^{(2x)} \\ g(x)=x^3+x^2+3x \end{cases}$

7. $\begin{cases} f(x)=-x+7 \\ g(x)=2x^3-2x^2+4x+5 \end{cases}$

continued

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For problems 8–10, each table shows values for two functions in a system of equations. Based on the table, what conclusions can you draw about the number of solutions and the x-values of any solutions?

8.

x	–3	–2	–1	0	1	2	3	4	5
$f(x)$	–120	–60	–24	–6	0	0	0	6	24
$g(x)$	–240	–140	–72	–30	–8	0	0	–2	0

9.

x	–3	–2	–1	0	1	2	3	4	5
$f(x)$	–40	–8	2	2	4	20	62	142	272
$g(x)$	5	4	3	2	2	3	4	5	6

10.

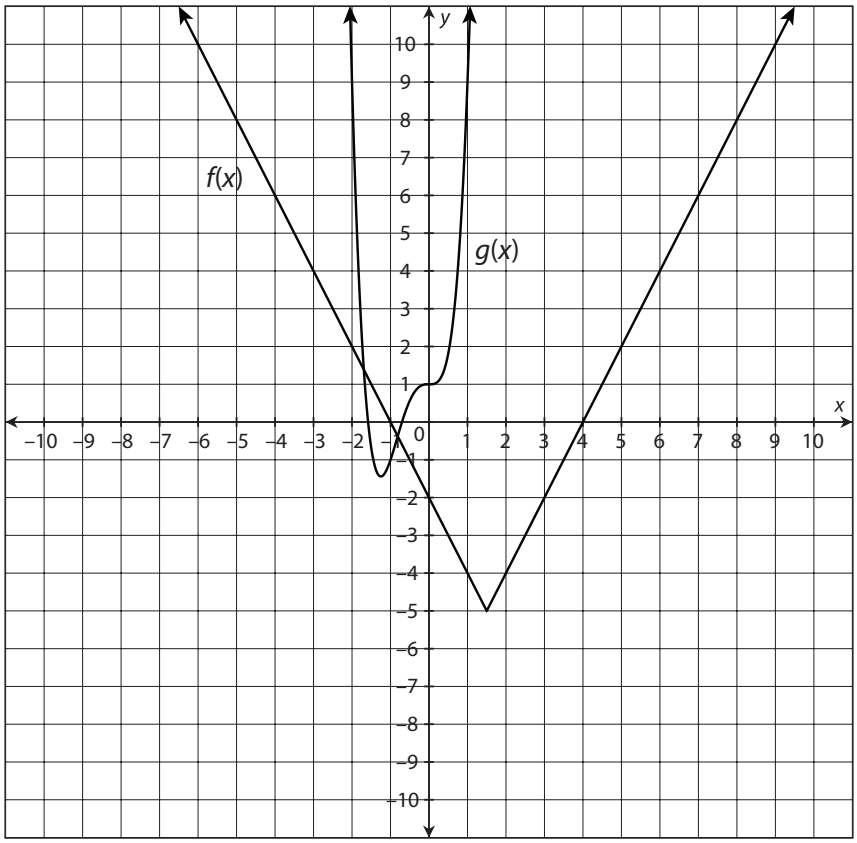
x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$	10.003	10.015	10.062	10.25	11	14	26	74	266
$g(x)$	731	179	11	11	35	11	–61	–109	11

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Progress Assessment

Circle the letter of the best answer.

1. How many real solutions does the following graphed system have?



a. no real solutions

b. one real solution

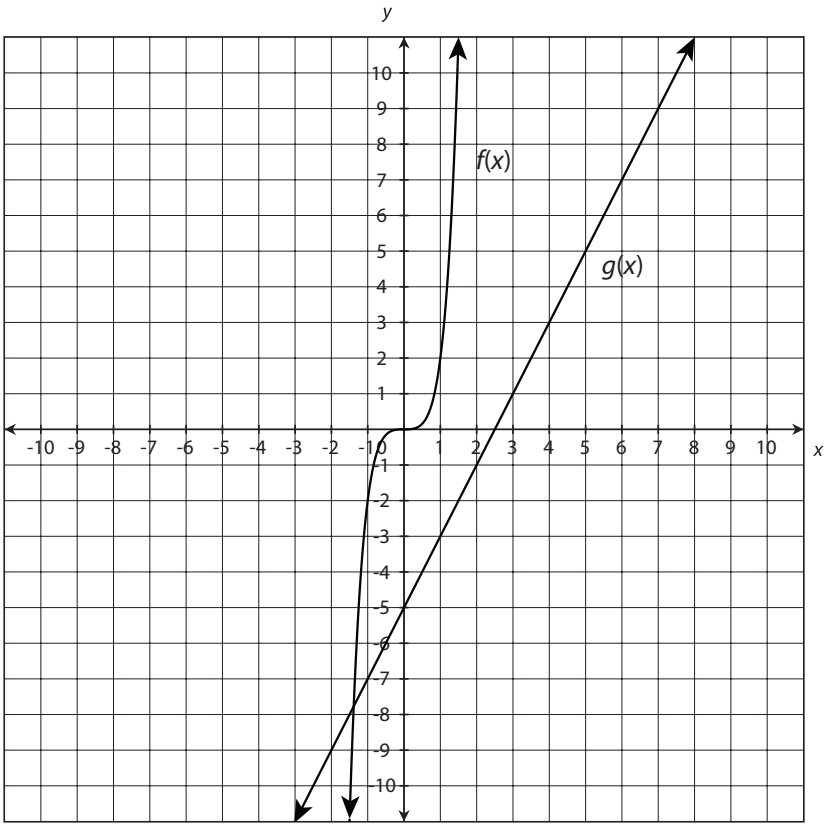
c. two real solutions

d. The number of solutions cannot be determined.

continued

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2. How many real solutions does the following graphed system have?



a. no real solutions

b. one real solution

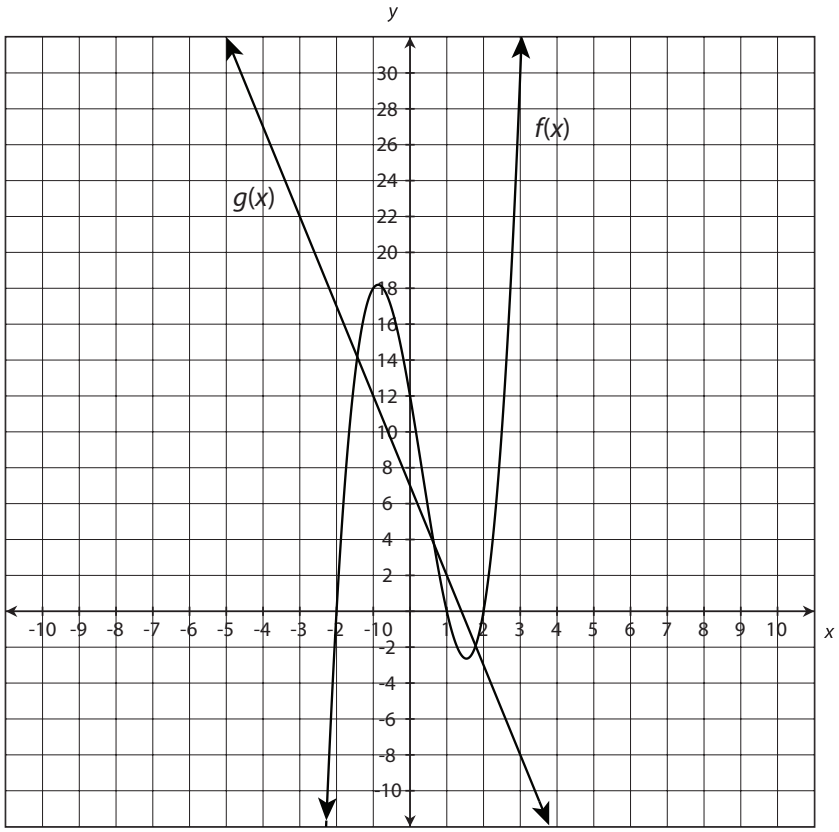
c. two real solutions

d. The number of solutions cannot be determined.

continued

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3. Use the graph to estimate the approximate solution(s) to the system of equations, if any solutions exist.



a. {(–2, 0), (1.5, 0)}

b. {(0, 6)}

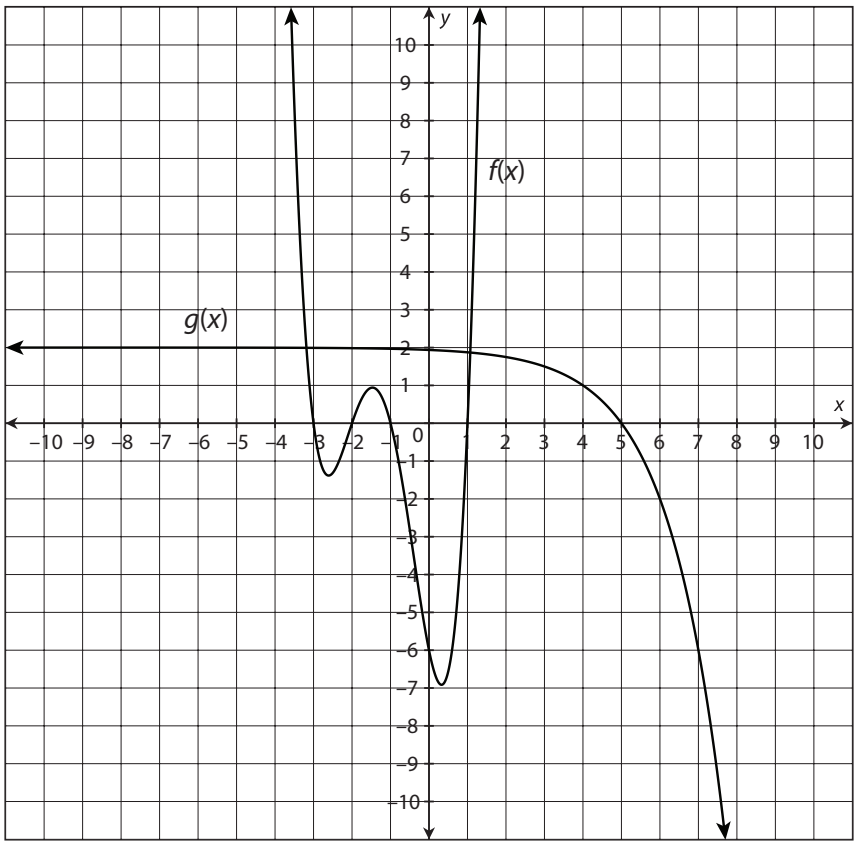
c. {(–1.4, 14), (0.6, 4), (1.8, –2)}

d. The number of solutions cannot be estimated.

continued

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4. Use the graph to estimate the approximate solution(s) to the system of equations, if any solutions exist.



a. {(–3.1, 2), (1.1, 1.9)}

b. {(–3, 0), (1, 0)}

c. {(–3, 0), (–2, 0), (–1, 0), (1, 0)}

d. The number of solutions cannot be estimated.

continued

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5. Graph the following system to estimate the real solution(s), if any.

$$\begin{cases} f(x) = x + 2 \\ g(x) = x^5 + 2x \end{cases}$$

   a. {(0, 0)}

   b. {(1, 3)}

   c. {(–2, 0), (0, 0)}

   d. There are no real solutions to this system.

6. Graph the following system to estimate the real solution(s), if any.

$$\begin{cases} f(x) = x^4 + 2x^3 + 1 \\ g(x) = |x + 3| \end{cases}$$

   a. {(–2, 1), (1, 4)}

   b. {(–3, 0), (–2, 1), (1, 4)}

   c. {(1, –2), (4, 1)}

   d. There are no real solutions to this system.

7. Graph the following system to estimate the real solution, if any.

$$\begin{cases} f(x) = -2x^3 + 4x + 1 \\ g(x) = 3^x - 3 \end{cases}$$

   a. {(1, 0)}

   b. {(0, 1)}

   c. {(1.4, 1.4)}

   d. There are no real solutions to this system.

8. Given the functions $f(x)$ and $g(x)$, if $f(a) = 3$ and $g(a) = 3$, then which of the following statements is true?

   a. $f(x)$ is the same function as $g(x)$.

   b. $(a, f(a))$ is a point of intersection of $f(x)$ and $g(x)$.

   c. $(a, f(a))$ is the only point of intersection of $f(x)$ and $g(x)$.

   d. $f(x)$ and $g(x)$ do not intersect.

continued

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9. Based on the table, what conclusion can you draw about the number of solutions to the system of equations?

x	–3	–2	–1	0	1	2	3	4
$f(x)$	–16	–13	–10	–7	–4	–1	2	5
$g(x)$	–38	–5	6	7	10	27	70	151

   a. There is at least one real solution to this system.

   b. There are exactly two real solutions to this system.

   c. There are no solutions to this system.

   d. It is not possible to determine the number of solutions to this system.

10. Based on the table, what conclusion can you draw about the number of solutions to the system of equations?

x	–1	0	1	2	3	4	5	6	7	8
$f(x)$	50.25	51	54	66	114	306	1074	4146	16,434	65,586
$g(x)$	–11	–20	–9	52	313	1164	3355	8116	17,277	33,338

   a. There is exactly one real solution to this system.

   b. There are at least two real solutions to this system.

   c. There are no solutions to this system.

   d. It is not possible to determine the number of solutions to this system.

Use what you have learned about systems of equations to complete all parts of the following problem.

11. Yolanda believes that one solution to the system of equations defined by the functions $f(x) = 13x + 4$ and $g(x) = x^3 + 2x^2 + 14$ is at the point (1, 17).

   a. Verify that the point (1, 17) is indeed a solution to the defined system.

   b. Graph the system of equations on the same coordinate plane.

   c. Identify any other points of intersection.

   d. Explain what the points of intersection, if any, mean in terms of the functions.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. A geometric series is defined such that $a_1 = 8$, $r = 3$, and $n = 7$. What is the sum of the series?

   a. 17,496
   b. 168
   c. 3,584
   d. 8,744

2. What is the sum of the following series?

$$\sum_{k=1}^{6} 1056\left(\frac{1}{4}\right)^{k-1}$$

   a. 1,057.03125
   b. 1.03125
   c. 1,407.65625
   d. No sum exists.

3. What is the value of the first term of the geometric series described as follows?

   $S_n = 1785$, $r = 2$, $n = 8$

   a. 7
   b. 56
   c. 3.5
   d. 14

4. What is the sum of the following series, if a sum exists?

$$\sum_{k=1}^{\infty} 0.3\left(\frac{3}{4}\right)^{k-1}$$

   a. $\frac{2}{5}$
   b. $\frac{6}{5}$
   c. $\frac{3}{40}$
   d. No sum exists.

5. What is the sum of the following series, if a sum exists?

$$\frac{1}{5}+\frac{3}{10}+\frac{9}{20}+\ldots$$

   a. $\frac{1}{10}$
   b. $-\frac{1}{10}$
   c. $\frac{19}{20}$
   d. No sum exists.

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Lesson 5: Geometric Series

Instruction

Common Core State Standard

A–SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★

SMP	
1 ✓	2 ✓
3 ✓	4 ✓
5	6
7 ✓	8

Essential Questions

1. What is a geometric sequence?
2. How are geometric sequences related to geometric series?
3. How are geometric series used in the real world?

WORDS TO KNOW

common ratio — the ratio of a term in a geometric sequence to the previous term in that sequence; indicated by the variable *r* and given by the formula $r = \frac{a_n}{a_{n-1}}$

converge — to approach a finite limit. If the sequence of the partial sums of a series approaches the value of a given number (the limit), then the entire series converges to that limit; that is, the series has a sum. An infinite series converges when the absolute value of the common ratio *r* is less than 1 $(|r| < 1)$.

discrete function — a function in which every element of the domain is individually separate and distinct; a geometric sequence is a discrete function

diverge — to *not* approach a finite limit. If a series does not have a sum (that is, the sequence of its partial sums does not approach a given number), then the series diverges. An infinite series diverges when the absolute value of the common ratio *r* is greater than 1 $(|r| > 1)$.

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explicit formula for a geometric sequence — a formula used to find any term in a sequence. The formula is $a_n = a_1 \bullet r^{n-1}$, where n is a positive integer that represents the number of terms in the sequence, r is the common ratio, a_1 is the value of the first term in the sequence, and a_n is the value of the nth term of the sequence.

finite — limited in number

finite geometric series — a series that has a limited or definite number of terms; can be written as $\sum_{k=1}^{n} a_1 r^{k-1}$, where n is a positive integer that represents the number of terms in the series, a_1 is the first term, r is the common ratio, and k is the number of the term

geometric sequence — an ordered list of terms in which each new term is the product of the preceding term and a common ratio, r. For a sequence to be geometric, r cannot be equal to 1 ($r \neq 1$).

geometric series — the sum of a specified number of terms from a geometric sequence

infinite — having no limit; represented by the infinity symbol, ∞

infinite geometric series — a series that has an unlimited number of terms $(n = \infty)$; can be written as $\sum_{k=1}^{\infty} a_1 r^{k-1}$, where a_1 is the first term, r is the common ratio, and k is the number of the term

partial sum — the sum of part of a series

recursive formula for a geometric sequence — a formula used to find the next term in a sequence. The formula is $a_n = a_{n-1} \bullet r$, where n is a positive integer that represents the number of terms in the sequence and r is the common ratio.

sequence — an ordered list of numbers or elements

series — the sum of the terms of a sequence

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***sigma* (uppercase), Σ**	a Greek letter used to represent the summation of values
sum formula for a finite geometric series	$S_n = \frac{a_1(1-r^n)}{1-r}$, where S_n is the sum, a_1 is the first term, r is the common ratio, and n is the number of terms
sum formula for an infinite geometric series	$S_n = \frac{a_1}{1-r}$, where S_n is the sum, a_1 is the first term, and r is the common ratio
summation notation	a symbolic way to represent a series (the sum of a sequence) using the uppercase Greek letter *sigma*, Σ
term	an element in a sequence. In the sequence $\{a_1, a_2, a_3, ..., a_n\}$, a_1 is the first term, a_2 is the second term, a_3 is the third term, and a_n is the nth term.

Recommended Resources

- Amortization-Calc.com. "Amortization Schedule Calculator."

  http://www.walch.com/rr/00172

  This site shows a real-world example of sums of finite series. Users can see exactly how much they can expect to pay for a home when taking out a mortgage. Enter values for the loan amount, number of years, interest rate, ZIP code, and the starting month of the loan, then click "Calculate." The next screen reveals the monthly payment, the total amount paid over the life of the loan, the total amount of interest paid, and the pay-off date.

- Gries, Dan. "Fractal Maker."

  http://www.walch.com/rr/00173

  Users of this applet can draw fractals by choosing a number of line segments and manipulating the points on the line. Clicking the "go" button shows the result of the fractal generated following four steps of the self-similar shape the user creates. The applet also allows users to see the progression by choosing which steps will display on the canvas.

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- Interactivate. "Sequencer."

  http://www.walch.com/rr/00174

  This site allows the user to generate a sequence and graph. Users enter a starting value, a multiplier, and a number of steps for the sequence. Clicking "Calculate Sequence" reveals the numbers in the sequence, along with a graph displaying the value of each term at each numbered step in the sequence.

- NRICH: Enriching Mathematics. "Proof Sorter—Geometric Series."

  http://www.walch.com/rr/00175

  On this site, users are shown the steps of the proof for the sum formula for a geometric series, but out of order. Users must rearrange the steps in the correct order; drag a step to the "Spare" space to free up some room before starting. The graphic on the left-hand side of the page moves with each rearrangement of the steps. Each correctly placed step moves the graphic toward 1; incorrectly placed steps move the graphic toward –1.

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Lesson 2A.5.1: Geometric Sequences

Warm-Up 2A.5.1

Alicia's little sister found a tennis ball at the playground. Her sister then climbed to the top of the jungle gym and dropped the ball from a height of 10 feet. With each new bounce, the ball reached only 40% of the height of the previous bounce.

1. What was the height of the ball after the first bounce?

2. What was the height of the ball after the second bounce?

3. What was the height of the ball after the third bounce?

4. What was the height of the ball after the fourth bounce?

5. Graph the number of bounces versus the height of each bounce in feet.

6. Does the graph show a pattern? Explain.

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Instruction

Lesson 2A.5.1: Geometric Sequences

Common Core State Standard

A–SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★

Warm-Up 2A.5.1 Debrief

1. What was the height of the ball after the first bounce?

   To determine the height after the first bounce, find 40% of 10 feet.

   $0.40(10) = 4$

   The ball was 4 feet high after the first bounce.

2. What was the height of the ball after the second bounce?

   To determine the height after the second bounce, find 40% of the previous bounce height, 4 feet.

   $0.40(4) = 1.6$

   The ball was 1.6 feet high after the second bounce.

3. What was the height of the ball after the third bounce?

   To determine the height after the third bounce, find 40% of the previous bounce height, 1.6 feet.

   $0.40(1.6) = 0.64$

   The ball was 0.64 feet high after the third bounce.

4. What was the height of the ball after the fourth bounce?

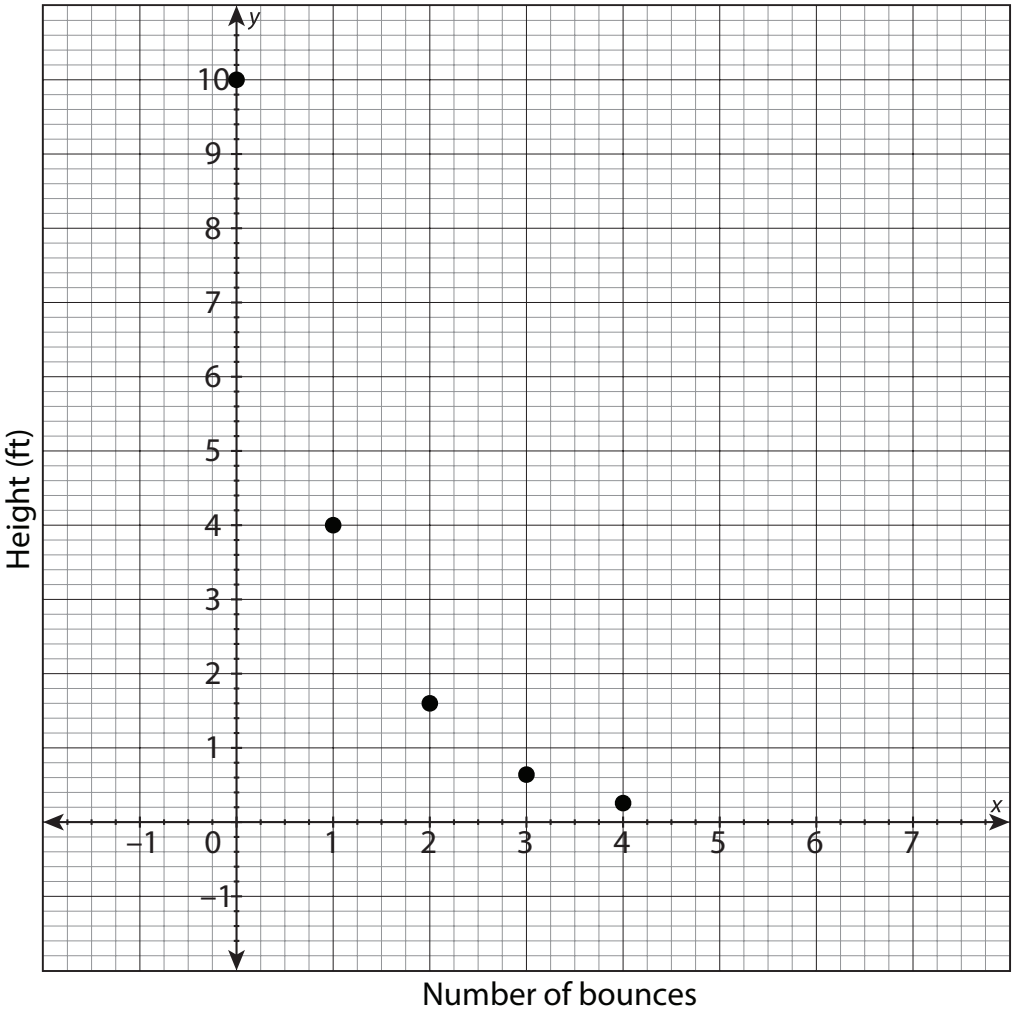
   To determine the height after the fourth bounce, find 40% of the previous bounce height, 0.64 feet.

   $0.40(0.64) = 0.256$

   The ball was 0.256 feet high after the fourth bounce.

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5. Graph the number of bounces versus the height of each bounce in feet.



6. Does the graph show a pattern? Explain.

   Yes, the graph shows that the bounce height decreases by 40% with each successive bounce.

Connection to the Lesson

- Students will continue to work with exponential functions as geometric sequences.
- Students will review what a geometric sequence is and how to write sequences using formulas.

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Prerequisite Skills

This lesson requires the use of the following skills:

- multiplying and dividing fractions (5–NF.4)
- rearranging formulas (literal equations) (A–CED.4)
- applying the order of operations (6.EE.3)
- applying exponents (6.EE.1)

Introduction

A **sequence** is an ordered list of numbers or elements. Each element in the list is called a **term**. For example, in the sequence {4, 8, 16, 32, ...}, the first term is 4, the second term is 8, and so on. In a **geometric sequence**, each new term is the product of the preceding term and a common ratio. The **common ratio**, *r*, is the ratio of a term in the sequence to the previous term; it is the same between each term in a geometric sequence. For example, in the sequence {4, 8, 16, 32, ...}, the common ratio between each term is 2, since 4 • 2 = 8, 8 • 2 = 16, and so on. Since a common ratio exists between each term, this sequence is geometric. In this lesson, we will learn how to find missing terms of geometric sequences using formulas and common ratios.

Key Concepts

- Given a geometric sequence $\{a_1, a_2, a_3, ..., a_n\}$, the first term is a_1, the second term is a_2, the third term is a_3, and the nth term is a_n.
- The domain, or input values, of a geometric sequence is at most {1, 2, 3, ..., *n*} and is made up of the positive integers 1, 2, 3, 4, and so on.
- Geometric sequences are **discrete functions** because every element of the domain is individually separate and distinct. (Note that for this to be true, *r* cannot equal 1; otherwise, each term in the sequence would be identical.)
- To determine the common ratio in a geometric sequence, find the ratio, *r*, of the second term to the first term, the third term to the second term, or, in other words, the ratio of the nth term to the $n - 1$ term: $r = \frac{a_n}{a_{n-1}}$.
- Geometric sequences can be written using either a recursive or an explicit formula.
- The **recursive formula for a geometric sequence** is $a_n = a_{n-1} \bullet r$, where n is a positive integer that represents the number of terms and r is the common ratio.

SRB p. U2A-121

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Instruction

- The recursive formula is the common ratio formula rewritten or rearranged to solve for the nth term, a_n.
- Use the recursive formula to find the next term in a sequence based on the previous term and the common ratio.
- The **explicit formula for a geometric sequence** is $a_n = a_1 \bullet r^{n-1}$, where n is a positive integer that represents the number of terms, r is the common ratio, a_1 is the value of the first term in the sequence, and a_n is the value of the nth term of the sequence.
- Use the explicit formula to find the nth term of a sequence when the first term and common ratio are known.
- The table that follows compares the terms in these two formulas.

Recursive Formula Versus Explicit Formula						
Term number (n)	1	2	3	4	...	n
Recursive term	a_1	$a_1 \bullet r$	$a_2 \bullet r$	$a_3 \bullet r$	...	$a_{n-1} \bullet r$
Explicit term	$a_1 \bullet r^0$	$a_1 \bullet r^1$	$a_1 \bullet r^2$	$a_1 \bullet r^3$	...	$a_1 \bullet r^{n-1}$

Common Errors/Misconceptions

- multiplying by the value of the previous term instead of the first term when using the explicit formula
- miscounting the number of a particular term in a sequence
- inverting the ratio when calculating the common ratio
- rounding a term before multiplying it by the common ratio to determine the next term

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Instruction

Guided Practice 2A.5.1

Example 1

Find the missing terms in the sequence that follows.

{1.2, 3.6, 10.8, 32.4, a_5, a_6, a_7, ...}

1. Determine whether there exists in the sequence a common ratio, r.

   The common ratio is determined using the formula $r = \frac{a_n}{a_{n-1}}$.

   Start by labeling each known term of the sequence in the format a_n ($a_1 = 1.2$, etc.). Then substitute pairs of terms into the common ratio formula to determine whether a common ratio exists among the pairs.

   $a_1 = 1.2$, $a_2 = 3.6$, $a_3 = 10.8$, $a_4 = 32.4$ Label the known terms as a_1, etc.

   Determine the ratio between the first pair of numbers in the sequence.

   $r_1 = \frac{a_2}{a_1} = \frac{(3.6)}{(1.2)} = 3$ Let the numerator be a_2 and the denominator be a_1. Substitute values, then evaluate.

   Determine the ratio between the second pair of numbers in the sequence.

   $r_2 = \frac{a_3}{a_2} = \frac{(10.8)}{(3.6)} = 3$ Let the numerator be a_3 and the denominator be a_2. Substitute values, then evaluate.

   Determine the ratio between the third pair of numbers in the sequence.

   $r_3 = \frac{a_4}{a_3} = \frac{(32.4)}{(10.8)} = 3$ Let the numerator be a_4 and the denominator be a_3. Substitute values, then evaluate.

   Compare the ratios.

   Since $\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = 3$, the ratios are common and the common ratio, r, is 3. A common ratio exists, so the sequence is geometric.

SWB p. 123; SRB p. U2A-123

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Instruction

2. Use the common ratio and the fourth term to find the next term in the sequence.

   The term we need to find is the fifth term, a_5.

   Since this is a geometric sequence, the recursive formula can be used to calculate the fifth term. That is, the previous term can be multiplied by the common ratio to calculate the next term. Therefore, multiply a_4 by the common ratio to obtain the fifth term, a_5.

   We are given that $a_4 = 32.4$.

   We have already determined the common ratio, $r = 3$.

   Set up the equation and evaluate.

$a_5 = a_4 \bullet r$	Recursive formula to find the fifth term
$a_5 = (32.4) \bullet (3)$	Substitute known values.
$a_5 = 97.2$	Multiply to obtain the next term in the sequence.

   The fifth term, a_5, is 97.2.

3. Use the common ratio and the fifth term to find the next term in the sequence.

   The next term we need to find is the sixth term, a_6.

   Use the recursive formula to multiply a_5 by the common ratio to obtain the sixth term, a_6.

   From the previous step, we know that $a_5 = 97.2$.

   We have already determined the common ratio, $r = 3$.

   Set up the equation and evaluate.

$a_6 = a_5 \bullet r$	Recursive formula to find the sixth term
$a_6 = (97.2) \bullet (3)$	Substitute known values.
$a_6 = 291.6$	Multiply to obtain the next term in the sequence.

   The sixth term, a_6, is 291.6.

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Instruction

4. Use the common ratio and the sixth term to find the next term in the sequence.

   The next term we need to find is the seventh term, a_7.

   Use the recursive formula to multiply a_6 by the common ratio to obtain the seventh term, a_7.

   From the previous step, we know that $a_6 = 291.6$.

   We have already determined the common ratio, $r = 3$.

   Set up the equation and evaluate.

$a_7 = a_6 \bullet r$	Recursive formula to find the seventh term
$a_7 = (291.6) \bullet (3)$	Substitute known values.
$a_7 = 874.8$	Multiply to obtain the next term in the sequence.

   The seventh term, a_7, is 874.8.

5. Summarize your results.

   Write the first seven terms of the sequence.

   Terms of a sequence are written inside a set of braces {}, separated by commas and followed by an ellipsis (...) to show that the sequence continues.

   {1.2, 3.6, 10.8, 32.4, 97.2, 291.6, 874.8, ...}

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Example 2

Find the specified term, a_n, given the first term and the common ratio.

$$a_1 = 128,\ r = \frac{1}{2},\ n = 8$$

1. Determine which geometric sequence formula to use: the recursive formula or the explicit formula.

   Since we are given the first term, use the explicit formula.

   $a_n = a_1 \bullet r^{n-1}$

2. Use this formula and the given information to find the value of the given term.

   We are given that $a_1 = 128$ and $r = \frac{1}{2}$.

$a_n = a_1 \bullet r^{n-1}$	Explicit formula
$a_{(8)} = (128) \bullet \left(\frac{1}{2}\right)^{(8)-1}$	Substitute 128 for a_1, $\frac{1}{2}$ for r, and 8 for n.
$a_8 = 128 \bullet \left(\frac{1}{2}\right)^7$	Simplify the exponent.
$a_8 = 128 \bullet 0.0078125$	Apply the exponent.
$a_8 = 1$	Multiply.

   The eighth term of the sequence, a_8, is 1.

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Example 3

Write the recursive formula for the geometric sequence that follows, then determine the missing terms in the sequence.

{0.32, 0.48, 0.72, 1.08, a_5, a_6, a_7, ...}

1. Determine the common ratio, *r*.

   The common ratio is determined using the formula $r = \frac{a_n}{a_{n-1}}$.

   Label the known terms of the sequence in the format a_n.

   $a_1 = 0.32$, $a_2 = 0.48$, $a_3 = 0.72$, $a_4 = 1.08$

   Substitute pairs of known terms into the common ratio formula, then evaluate to find the ratio for each pair.

   $$r_1 = \frac{a_2}{a_1} = \frac{(0.48)}{(0.32)} = 1.5$$

   $$r_2 = \frac{a_3}{a_2} = \frac{(0.72)}{(0.48)} = 1.5$$

   $$r_3 = \frac{a_4}{a_3} = \frac{(1.08)}{(0.72)} = 1.5$$

   Compare the ratios.

   Since $\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = 1.5$, the ratios are common and the common ratio, *r*, is 1.5.

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Instruction

2. Write the recursive formula for the *n*th term.

Since the previous terms are known and we are extending the sequence, determine the recursive formula for the sequence.

The recursive formula is the ratio formula, $r = \frac{a_n}{a_{n-1}}$, rewritten to solve for a_n.

Rewrite the common ratio formula, then substitute known information to find the recursive formula for this sequence.

$r = \frac{a_n}{a_{n-1}}$	Common ratio formula
$r \bullet a_{n-1} = a_n$	Multiply both sides by a_{n-1}.
$a_n = r \bullet a_{n-1}$	Use the symmetric property of equality to rearrange the equation so that a_n is on the left side.
$a_n = (1.5) \bullet a_{n-1}$	Substitute 1.5 for r.

The recursive formula for the *n*th term of this sequence is $a_n = 1.5 \bullet a_{n-1}$.

3. Use the recursive formula and the known information to calculate the missing terms in the sequence.

The first four terms are given:

$a_1 = 0.32, a_2 = 0.48, a_3 = 0.72, a_4 = 1.08$

To determine the next term in the sequence, use the fourth term, a_4, to find the fifth term, a_5.

We are given that $a_4 = 1.08$.

$a_n = 1.5 \bullet a_{n-1}$	Recursive formula for the given sequence
$a_{(5)} = 1.5 \bullet a_{(5)-1}$	Substitute 5 for n.
$a_5 = 1.5 \bullet a_4$	Simplify.
$a_5 = 1.5 \bullet (1.08)$	Substitute 1.08 for a_4.
$a_5 = 1.62$	Simplify.

The fifth term, a_5, is 1.62.

(*continued*)

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Instruction

Since we now know $a_5 = 1.62$, use this information to find the sixth term, a_6.

$a_n = 1.5 \bullet a_{n-1}$	Recursive formula for the given sequence
$a_{(6)} = 1.5 \bullet a_{(6)-1}$	Substitute 6 for n.
$a_6 = 1.5 \bullet a_5$	Simplify.
$a_6 = 1.5 \bullet (1.62)$	Substitute 1.62 for a_5.
$a_6 = 2.43$	Simplify.

The sixth term, a_6, is 2.43.

Since we now know $a_6 = 2.43$, use this information to find the seventh term, a_7.

$a_n = 1.5 \bullet a_{n-1}$	Recursive formula for the given sequence
$a_{(7)} = 1.5 \bullet a_{(7)-1}$	Substitute 7 for n.
$a_7 = 1.5 \bullet a_6$	Simplify.
$a_7 = 1.5 \bullet (2.43)$	Substitute 2.43 for a_6.
$a_7 = 3.645$	Simplify.

The seventh term, a_7, is 3.645.

4. Summarize your results.

   The missing terms of the sequence are 1.62, 2.43, and 3.645.

   The sequence can be written as {0.32, 0.48, 0.72, 1.08, 1.62, 2.43, 3.645, ...}.

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Example 4

Given the geometric sequence that follows, write the explicit formula and find the ninth term.

{32,768, 8192, 2048, 512, ..., a_9, ...}

1. Determine the common ratio.

   The common ratio is determined using the formula $r = \frac{a_n}{a_{n-1}}$.

   Label the known terms of the sequence in the format a_n.

   $a_1 = 32{,}768$, $a_2 = 8192$, $a_3 = 2048$, $a_4 = 512$

   Substitute pairs of known terms into the common ratio formula, then evaluate to find the ratio for each pair.

   $$r_1 = \frac{a_2}{a_1} = \frac{(8192)}{(32{,}768)} = 0.25$$

   $$r_2 = \frac{a_3}{a_2} = \frac{(2048)}{(8192)} = 0.25$$

   $$r_3 = \frac{a_4}{a_3} = \frac{(512)}{(2048)} = 0.25$$

   Compare the ratios.

   Since $\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = 0.25$, the ratios are common and the common ratio, r, is 0.25.

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Instruction

2. Write the explicit formula for the *n*th term.

Since we are determining the value of the term that is *not* next in the sequence, and we know the values of the first term and the common ratio, use the explicit formula: $a_n = a_1 \bullet r^{n-1}$.

We are given that $a_1 = 32{,}768$.

We have already determined the common ratio, $r = 0.25$.

$a_n = a_1 \bullet r^{n-1}$	Explicit formula for a geometric sequence
$a_n = (32{,}768) \bullet (0.25)^{n-1}$	Substitute 32,768 for a_1 and 0.25 for r.

The explicit formula of the given geometric sequence is $a_n = 32{,}768 \bullet 0.25^{n-1}$.

3. Use this formula to find the ninth term, a_9.

Since we are looking for the ninth term, $n = 9$.

$a_n = 32{,}768 \bullet 0.25^{n-1}$	Explicit formula for the given geometric sequence
$a_{(9)} = 32{,}768 \bullet 0.25^{(9)-1}$	Substitute 9 for n.
$a_9 = 32{,}768 \bullet 0.25^8$	Simplify the exponent.
$a_9 = 32{,}768 \bullet 0.000015259$	Apply the exponent.
$a_9 = 0.5$	Multiply.

The ninth term of the geometric sequence, a_9, is 0.5.

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Example 5

You're planning to experiment on a certain bacteria strain. It takes 1 hour for a cell of this strain to double. If you begin the experiment with 64 cells, how many cells will the experiment have exactly 1 day later?

1. Determine the common ratio.

   The cells are doubling, which means that the common ratio, r, is 2.

2. Determine the value of a_1.

   The experiment begins with 64 bacteria cells, so $a_1 = 64$.

3. Determine which geometric sequence formula to use: the recursive formula or the explicit formula.

   We know the first term and the common ratio, so use the explicit formula.

4. Substitute values for the first term and the common ratio into the explicit formula to find the equation for the nth term.

   The explicit formula for any geometric sequence is given by the equation $a_n = a_1 \cdot r^{n-1}$.

   We have already determined that $r = 2$ and $a_1 = 64$. Use these values to find the specific explicit formula for this sequence.

$a_n = a_1 \cdot r^{n-1}$	Explicit equation for any geometric sequence
$a_n = (64) \cdot (2)^{n-1}$	Substitute 2 for r and 64 for a_1.

   The explicit equation for the nth term of the given geometric sequence is $a_n = 64 \cdot 2^{n-1}$.

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Instruction

5. Determine the value for n, then solve for the unknown term.

   The problem scenario asks for the number of cells exactly 1 day later. The scenario also says that the cells double every *hour*. This means that n is in terms of hours. There are 24 hours in a day, so $n = 24$.

$a_n = 64 \bullet 2^{n-1}$	Explicit equation for the given geometric sequence
$a_{(24)} = 64 \bullet 2^{(24)-1}$	Substitute 24 for n.
$a_{24} = 64 \bullet 2^{23}$	Simplify the exponent.
$a_{24} = 64 \bullet 8{,}388{,}608$	Apply the exponent.
$a_{24} = 536{,}870{,}912$	Multiply.

   There will be 536,870,912 bacteria cells in the experiment after 24 hours.



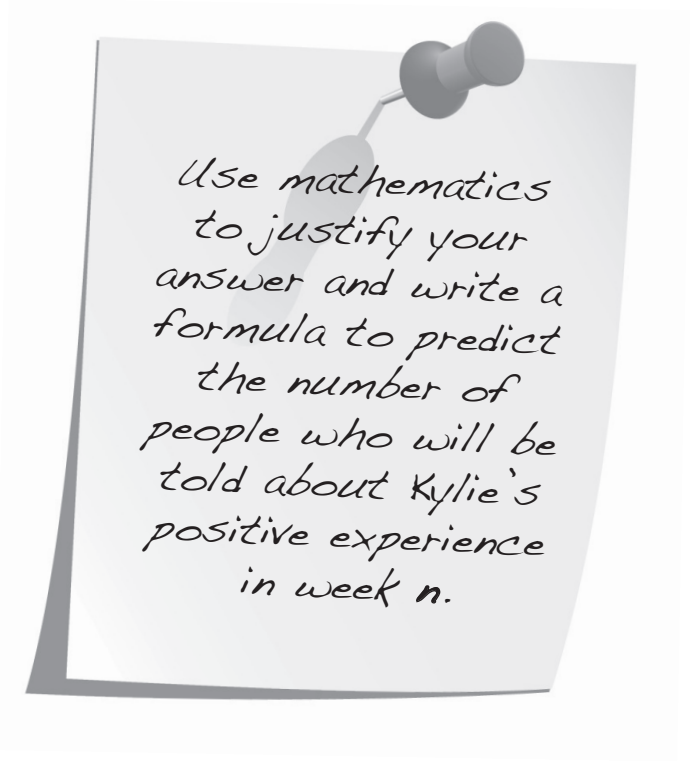
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Problem-Based Task 2A.5.1: New Restaurant Buzz

SMP	
1	2 ✓
3	4 ✓
5	6
7 ✓	8

A new restaurant has just opened in Kylie's town. Wanting to support her community, Kylie went to the restaurant for dinner. Her experience was great—not only did she have a short wait for her food, but when Kylie complimented the manager on the excellent meal and service, the manager gave her a coupon for a discount on her next meal. Kylie was so pleased, she promised she would tell her friends to try the new restaurant.

Since restaurant owners often rely on word of mouth to promote their restaurant, happy customers can be an important factor in the success of a business. Assume that Kylie will tell 4 friends about her great experience within 1 week, and that each of those friends will tell 4 other people within the next week, continuing on with that pattern of 4 new people being told about the experience each week. Use this pattern to explain why restaurant owners might rely on the compliments of customers to help promote business. Use mathematics to justify your answer and write a formula to predict the number of people who will be told about Kylie's positive experience in week n.



SWB p. 125

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Problem-Based Task 2A.5.1: New Restaurant Buzz

Coaching

a. Using the pattern cited in the problem scenario, how many people can you predict Kylie will tell about her positive experience within 1 week?

b. Predict how many new people will be told about Kylie's experience in week 2.

c. Predict how many new people will be told about Kylie's experience in week 3.

d. Predict how many new people will be told about Kylie's experience in week 4.

e. What is the common ratio?

f. What is the explicit formula for determining the number of new people who will be told about Kylie's positive experience in week n?

g. How does this explain why restaurant owners might rely on customers' positive experiences to promote business?

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Instruction

Problem-Based Task 2A.5.1: New Restaurant Buzz

Coaching Sample Responses

a. Using the pattern cited in the problem scenario, how many people can you predict Kylie will tell about her positive experience within 1 week?

According to the pattern, Kylie will tell 4 people about her positive experience within 1 week.

b. Predict how many new people will be told about Kylie's experience in week 2.

According to the pattern, each of the 4 friends Kylie told about her experience in week 1 will tell 4 additional people within the next week.

$4 \cdot 4 = 16$

16 people will be told in week 2.

c. Predict how many new people will be told about Kylie's experience in week 3.

According to the pattern, each of the 16 people told in week 2 will tell 4 additional people within the next week.

$16 \cdot 4 = 64$

64 people will be told in week 3.

d. Predict how many new people will be told about Kylie's experience in week 4.

Each of the 64 people told in week 3 will tell 4 new people within the next week.

$64 \cdot 4 = 256$

256 people will be told in week 4.

e. What is the common ratio?

Each week, the number of new people who have been told about Kylie's positive experience increases by a factor of 4. The common ratio is 4.

$r = 4$

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f. What is the explicit formula for determining the number of new people who will be told about Kylie's positive experience in week n?

The number of new people who will be told in any week is based on the first term of the sequence (a_1) and the common ratio, r. If you know the first term, you can multiply it by the common ratio, raised to the number of weeks, n, that have passed minus 1. The subtraction of 1 comes from the common ratio already having been applied to the first term.

The general equation for the number of people told after any given number of weeks is $a_n = a_1 \bullet r^{n-1}$.

Substitute the value of r into this equation to determine the explicit formula for this scenario.

$a_n = a_1 \bullet 4^{n-1}$

g. How does this explain why restaurant owners might rely on customers' positive experiences to promote business?

The number of new people being told about the positive experience is growing exponentially by a factor of 4 each week. What started out as a positive experience seemingly only affecting 1 customer is passed along by word of mouth to 4 new people in the first week, 16 additional people in the second week, 64 additional people in the third week, and 256 additional people in the fourth week. The number of people telling others about Kylie's positive experience can be modeled by the equation $a_n = a_1 \bullet 4^{n-1}$. In other words, news travels fast. This explains why restaurant owners might rely on customers' positive experiences to promote business.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2A.5.1: Geometric Sequences

Find the missing terms in each sequence and state the common ratio, r.

1. $\left\{\frac{3}{64}, \frac{3}{16}, \frac{3}{4}, 3, a_5, a_6, a_7, \ldots\right\}$

2. $\{96, 48, 24, 12, a_5, a_6, a_7, \ldots\}$

Find the specified term, a_n, using the given information.

3. $a_1 = 26{,}244,\ r = \frac{1}{3},\ n = 11$

4. $a_1 = \frac{1}{32},\ r = 2,\ n = 10$

5. $a_2 = \frac{9}{8},\ r = \frac{2}{3},\ n = 7$

Write the explicit formula for the given sequence and find the specified term.

6. $\{8, 20, 50, 125, \ldots, a_8, \ldots\}$

7. $\left\{\frac{96}{27}, \frac{24}{9}, 2, \frac{3}{2}, \ldots, a_6, \ldots\right\}$

continued

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For problems 8–10, read each scenario and use the geometric sequence it describes to answer the questions.

8. With each successive bounce, new tennis balls bounce to about 53% of the previous bounce height when dropped on a hard surface. If Byron drops a tennis ball from a height of 10 feet, how high will the ball go on the fifth bounce? What is the explicit formula that shows how high the tennis ball will be on the *n*th bounce?

9. Dagmar is cooling a pot of boiling water in the freezer as part of a science experiment. Water boils at 212°F. There's a thermometer in the pot, so Dagmar can see that the temperature decreases by a different number of degrees every minute. After 1 minute, the temperature is 190.8°F; after 2 minutes it is 171.72°F; and after 3 minutes it is 154.548°F. If this pattern continues, what would be the temperature of the water after 10 minutes? What is the explicit formula to determine the temperature after *n* minutes?

10. Sadhana is analyzing the salary she's been offered for a new job. Sadhana's prospective employer is offering her $30,000 the first year, with a guaranteed minimum raise of 2% each year. What will be Sadhana's minimum salary after 4 years, 5 years, and 7 years? What is the explicit formula to show her minimum salary in *n* years?

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Lesson 2A.5.2: Sum of a Finite Geometric Series

Warm-Up 2A.5.2

Farid is a runner who recently had knee surgery. His doctor has recommended that he slowly build up to his goal of running at least 5 miles per day. The doctor told Farid to start by running a half mile each day, then increase his distance by 20% each week.

1. What is the explicit formula for Farid's running program?

2. How many miles per day will Farid be running by the end of week 5?

3. How many weeks will it take Farid to get back to his goal of running at least 5 miles each day?

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Lesson 2A.5.2: Sum of a Finite Geometric Series

Common Core State Standard

A–SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★

Warm-Up 2A.5.2 Debrief

1. What is the explicit formula for Farid's running program?

   This is a geometric sequence. The explicit formula is given by $a_n = a_1 \bullet r^{n-1}$. $a_1 = 0.5$ because the first time Farid runs, he will run a half mile; therefore, the first term in the sequence is 0.5. Farid increases his distance by 20% each week, which means that the common ratio, r, is 1.2.

   Use this information to write the explicit formula for the nth term of this geometric sequence.

   $a_n = a_1 \bullet r^{n-1}$

   $a_n = (0.5) \bullet (1.2)^{n-1}$

   The explicit formula for Farid's running program is $a_n = 0.5 \bullet 1.2^{n-1}$.

2. How many miles per day will Farid be running by the end of week 5?

   In this problem, $n = 5$. Substitute 5 into the formula for Farid's running program and solve.

$a_n = 0.5 \bullet 1.2^{n-1}$	Explicit formula for Farid's running program
$a_{(5)} = 0.5 \bullet 1.2^{(5)-1}$	Substitute 5 for n.
$a_5 = 0.5 \bullet 1.2^4$	Simplify the exponent.
$a_5 = 0.5 \bullet 2.0736$	Apply the exponent.
$a_5 = 1.0368 \approx 1.0$	Simplify.

   Farid will be running about 1 mile each day by the end of week 5.

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3. How many weeks will it take Farid to get back to his goal of running at least 5 miles each day?

   Using trial and error, substitute values of n into the equation until your answer is greater than or equal to 5. Since it took 5 weeks for Farid to build up to 1 mile, it makes sense to start with a high number, such as 13.

$n = 13$	$n = 14$
$a_{(13)} = 0.5 \bullet 1.2^{(13)-1}$	$a_{(14)} = 0.5 \bullet 1.2^{(14)-1}$
$a_{13} \approx 0.5 \bullet 1.2^{12}$	$a_{14} \approx 0.5 \bullet 1.2^{13}$
$a_{13} \approx 0.5 \bullet 8.916$	$a_{14} \approx 0.5 \bullet 10.699$
$a_{13} \approx 4.458 \approx 4.5$ miles	$a_{14} \approx 5.350 \approx 5.4$ miles

   Farid will reach his goal of running at least 5 miles per day during week 14.

Connection to the Lesson

- Students will expand their understanding of geometric sequences to include geometric series.

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Prerequisite Skills

This lesson requires the use of the following skills:

- performing substitutions (6.EE.2c)
- applying the order of operations (6.EE.3)
- recognizing geometric sequences (F–BF.2)

Introduction

Terms of geometric sequences can be added together if needed, such as when calculating the total amount of money you will pay over the life of a loan that charges interest. When a group of some number of terms from a geometric sequence are added together, that group is called a **geometric series**. The number of terms in that group could go on forever, or the terms could be **finite**—that is, limited to a certain number. A **finite geometric series** is the sum of a specified number of terms of a geometric sequence. In this lesson, we will learn different ways to write and evaluate a finite geometric series.

Key Concepts

- A geometric series is the sum of some number of terms from a geometric sequence.
- Geometric sequences and series are related:
  - Both have a common ratio.
  - A sequence is a listing of the terms in a geometric progression.
  - A **series** is the sum of part of that list.
- A finite series finds the sum of a specified number of the terms from the sequence.
- Compare the first five terms of the geometric sequence {1, 2, 4, 8, 16, ...} with the same terms written as a geometric series:

  Sequence: 1, 2, 4, 8, 16

  Series: 1 + 2 + 4 + 8 + 16
- **Summation notation** is a symbolic way to represent the sum of a sequence.
- A common way to write summation notation for a geometric series is $\sum_{k=1}^{n} a_1 r^{k-1}$,

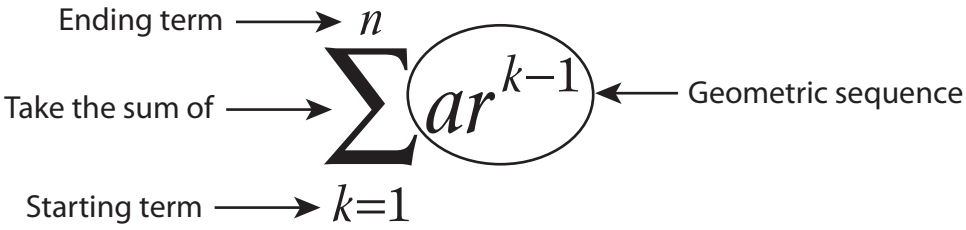
  where n is a positive integer that represents the number of terms in the series, a_1 is the first term, r is the common ratio, and k is the number of the term.

SRB p. U2A-136

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- This notation is read as "take the sum from 1 to n of the series a_1 times r to the $k - 1$."
- The symbol $\sum$ is the uppercase Greek letter *sigma*. In this context, it means "to take the sum."
- The n represents the number of iterations to complete in the sum, or the final term in the sum. For example, if $n = 3$, you will add 3 of the terms in the series.
- $k = 1$ shows the starting value of the iterations, or the number of the term to start with in the sum.
- The diagram that follows illustrates the parts of this summation notation formula.



- Finite series have a definite ending term—they do not go on forever.
- Infinite series go to infinity. These will be explored later.
- Summing terms can be tedious. To make this process easier, mathematicians developed a formula for finding the sum of terms in a series; you will see the steps of how this formula is derived in Guided Practice Example 3.
- The **sum formula for a finite geometric series** from 1 to n is $S_n = \frac{a_1(1-r^n)}{1-r}$, where S_n is the sum, a_1 is the first term, r is the common ratio, and n is the number of terms.
- Notice that the right side of the formula is a fraction. The denominator of a fraction cannot equal 0. Therefore, in this formula, $r \neq 1$.

Loan Payments and Sums of Finite Geometric Series

- The sum formula of a finite geometric series affects everyone who makes payments on a loan, whether they know it or not.
- When you take out a loan for a home or car, and pay interest on that loan, the total amount you will have paid by the time you make your last payment is actually the sum of a finite geometric series.

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- Let's say you take out a loan for a home. You will take out a loan for the price of the home, and make monthly payments to the lender. The bank that loaned you the money will charge interest each month until the loan is paid off. At the beginning of your loan, most of your payment goes to interest, because your principal amount is large. As the years pass, more of your payment goes to the principal and less goes to interest. This kind of loan is called an amortized loan.
- The formula for the sum of an amortized loan is a geometric series. The formula is used to calculate how much money will have been paid in principal and interest by the time a loan is paid off.
- The formula is given by $P=\sum_{k=1}^{n} A\left(\frac{1}{1+i}\right)^{k-1}$, where P is the principal (that is, the loan amount), A is the monthly payment amount, and i is the monthly interest based on the annual percentage rate (APR) divided by 12.
- The following table shows how this amortization formula relates to the general summation notation shown earlier.

Comparing Summation Notation to the Amortization Formula

Complete formula	Sum of the series	First term	Common ratio	Placement of the term in the sequence
$\sum_{k=1}^{n} ar^{k-1}$	$\sum_{k=1}^{n}$	a	r	$k-1$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$P=\sum_{k=1}^{n} A\left(\frac{1}{1+i}\right)^{k-1}$	$\sum_{k=1}^{n}$	A	$\frac{1}{1+i}$	$k-1$

Common Errors/Misconceptions

- misinterpreting the parts of the summation notation
- forgetting that anything raised to the 0 power is equal to 1
- skipping iterations
- substituting numbers into the summation formula incorrectly

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Guided Practice 2A.5.2

Example 1

Expand the series shown in the given summation notation.

$$\sum_{k=1}^{6} 4\left(\frac{1}{2}\right)^{k-1}$$

1. Determine the number of terms in the expanded series.

   The number shown above the *sigma* symbol represents the final term of the sum (n).

   Here, $n = 6$.

   Therefore, the expanded series will have 6 terms.

2. Substitute values for k.

   The portion of the summation notation that shows the value of each term in the sequence is $4\left(\frac{1}{2}\right)^{k-1}$.

   Since $n = 6$, write the first 6 terms of the series, each time substituting one of the counting numbers 1 through 6 for k in the exponent. Set the expansion up as an equation equal to the original notation, adding the rewritten terms together to the right of the equal sign:

$$\sum_{k=1}^{6} 4\left(\frac{1}{2}\right)^{k-1} = 4\left(\frac{1}{2}\right)^{(1)-1} + 4\left(\frac{1}{2}\right)^{(2)-1} + 4\left(\frac{1}{2}\right)^{(3)-1} + 4\left(\frac{1}{2}\right)^{(4)-1} + 4\left(\frac{1}{2}\right)^{(5)-1} + 4\left(\frac{1}{2}\right)^{(6)-1}$$

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3. Simplify and apply the exponents.

Start by simplifying each exponent.

$$\sum_{k=1}^{6} 4\left(\frac{1}{2}\right)^{k-1} = 4\left(\frac{1}{2}\right)^{0} + 4\left(\frac{1}{2}\right)^{1} + 4\left(\frac{1}{2}\right)^{2} + 4\left(\frac{1}{2}\right)^{3} + 4\left(\frac{1}{2}\right)^{4} + 4\left(\frac{1}{2}\right)^{5}$$

Then, apply each exponent to the fraction in parentheses, as shown:

$$\sum_{k=1}^{6} 4\left(\frac{1}{2}\right)^{k-1} = 4(1) + 4\left(\frac{1}{2}\right) + 4\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + 4\left(\frac{1}{16}\right) + 4\left(\frac{1}{32}\right)$$

4. Simplify the terms.

$$\sum_{k=1}^{6} 4\left(\frac{1}{2}\right)^{k-1} = 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

The expanded series is $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$.



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Example 2

Sum the series by expansion.

$$\sum_{k=1}^{5} 2(3)^{k-1}$$

1. Expand the series.

   $k = 1$ and $n = 5$; therefore, the series starts with the first term of the sequence and ends with the fifth term.

   Write the first 5 terms of the series, each time substituting one of the counting numbers 1 through 5 for k in the exponent as shown:

   $$\sum_{k=1}^{5} 2(3)^{k-1} = 2(3)^{(1)-1} + 2(3)^{(2)-1} + 2(3)^{(3)-1} + 2(3)^{(4)-1} + 2(3)^{(5)-1}$$

   Simplify and apply the exponents to each term.

   $$\sum_{k=1}^{5} 2(3)^{k-1} = 2(3)^{0} + 2(3)^{1} + 2(3)^{2} + 2(3)^{3} + 2(3)^{4}$$ Simplify the exponents.

   $$\sum_{k=1}^{5} 2(3)^{k-1} = 2(1) + 2(3) + 2(9) + 2(27) + 2(81)$$ Apply the exponents.

   Notice that each term has a factor of 2.

2. Factor the expression and simplify.

   $$\sum_{k=1}^{5} 2(3)^{k-1} = 2(1) + 2(3) + 2(9) + 2(27) + 2(81)$$ Series with exponents applied

   $$\sum_{k=1}^{5} 2(3)^{k-1} = 2(1 + 3 + 9 + 27 + 81)$$ Factor out 2 from the terms in the series.

   $$\sum_{k=1}^{5} 2(3)^{k-1} = 2(121)$$ Sum the terms.

   $$\sum_{k=1}^{5} 2(3)^{k-1} = 242$$ Multiply.

   The sum of the series containing the first 5 terms of the sequence is 242.

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Example 3

Derive the formula for the sum of a finite geometric series, which is given by $S_n = \sum_{k=1}^{n} ar^{k-1} = \frac{a_1\left(1-r^n\right)}{1-r}$.

1. Write out the terms for the series.

   The portion of the given formula that describes the series is $S_n = \sum_{k=1}^{n} ar^{k-1}$. S_n is the sum, and each term is equal to ar^{k-1}.

   No numerical value is given for n; therefore, the series extends to the nth term. Expand the terms following the steps in Examples 1 and 2, replacing "k" with counting numbers and n.

   $$S_n = \sum_{k=1}^{n} ar^{k-1} = a + ar^1 + ar^2 + ar^3 + \cdots + ar^{n-1}$$

2. Multiply the series by r.

   In multiplying both sides of the equation by r, we are not changing the integrity of the equation. Think about when you multiply an equation to isolate x. You multiply both sides of the equation without changing the solution.

   $$rS_n = \sum_{k=1}^{n} ar^{k-1} = ar^1 + ar^2 + ar^3 + \cdots + ar^n$$

   Multiply S_n and each term of the sum by r.

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3. Subtract rS_n from S_n and factor the resulting equation.

Think about subtracting two equations in a system of equations. This action does not change the solution of the equation, but allows you to manipulate the equation to get the structure you need.

$$S_n = a + ar^1 + ar^2 + ar^3 + \cdots + ar^{n-1}$$
$$-rS_n = ar^1 + ar^2 + ar^3 + \cdots + ar^n \qquad \text{Subtract } rS_n \text{ from } S_n.$$
$$S_n - rS_n = a - ar^n$$

The result of subtracting the two equations is $S_n - rS_n = a - ar^n$.

Note that the terms on either side of the equation have common factors: S_n on the left and a on the right. These terms need to be simplified.

Factor out S_n from the left side of the equation.

Think about factoring a polynomial. This follows the same process.

$S_n - rS_n = a - ar^n$ Equation found previously

$S_n(1-r) = a - ar^n$ Factor out S_n on the left side.

The result after factoring S_n is $S_n(1-r) = a - ar^n$.

4. Factor out a from the right side of the equation and solve for S_n.

$S_n(1-r) = a - ar^n$ Factored equation from step 3

$S_n(1-r) = a(1-r^n)$ Factor out a on the right side.

The result after factoring both sides of the equation is $S_n(1-r) = a(1-r^n)$.

Now you have a formula that can be rearranged to solve for S_n.

$S_n(1-r) = a(1-r^n)$ Factored equation

$S_n = \dfrac{a(1-r^n)}{(1-r)}$ Divide both sides by $(1-r)$.

The equation solved for S_n is the sum formula for a finite geometric series, $S_n = \dfrac{a(1-r^n)}{(1-r)}$, where S_n is the sum of the first n terms, a is the value of the first term, r is the common ratio, and n is the number of terms in the series.

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Example 4

Sum the given series using the sum formula for a finite geometric series.

$$\sum_{k=1}^{6} 3(2)^{k-1}$$

1. Write the sum formula and identify the variables that need to be substituted.

   $$S_n = \frac{a_1(1-r^n)}{(1-r)}$$

   The following variables need to be substituted:

   - a_1 (the first term in the series)
   - r (the common ratio)
   - n (the number of terms in the series)

   Examine the summation notation to determine values for a_1, r, and n.

2. Calculate the value of the first term, a_1.

   In the given series $\sum_{k=1}^{6} 3(2)^{k-1}$, substitute the counting number 1 for k and evaluate to find the value of the first term.

   $a_1 = 3(2)^{k-1} = 3(2)^{(1)-1} = 3(2)^0 = 3(1) = 3$

   The value of the first term, a_1, is 3.

3. Determine the common ratio, r.

   Remember the formula for a geometric sequence, $a_n = a \bullet r^{n-1}$, where r is the common ratio. Each term in this series is equal to $3(2)^{k-1}$. Therefore, the common ratio is 2.

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4. Determine the number of terms.

   The summation notation $\sum_{k=1}^{6}$ tells us the number of terms. Look at the numbers below and above *sigma*, $\sum$. The series starts with the first term, as indicated by $k = 1$. The series ends with the sixth term, as indicated by the 6 above the $\sum$. Therefore, there are 6 terms in the series and $n = 6$.

5. Substitute the values for a_1, r, and n into the sum formula for the given series.

   We have determined that $a_1 = 3$, $r = 2$, and $n = 6$.

$$S_n = \sum_{k=1}^{6} 3(2)^{k-1} = \frac{a(1-r^n)}{(1-r)}$$ Sum formula for the given geometric series

$$S_n = \frac{(3)\left[1-(2)^{(6)}\right]}{\left[1-(2)\right]}$$ Substitute 3 for a_1, 2 for r, and 6 for n.

$$S_n = \frac{3(1-64)}{(1-2)}$$ Apply the exponent.

$$S_n = \frac{3(-63)}{(-1)} = \frac{-189}{-1} = 189$$ Simplify.

$$S_n = \sum_{k=1}^{6} 3(2)^{k-1} = 189$$

The sum of the series is 189.

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Example 5

Seline is applying for a $10,000 college loan to be repaid over 10 years. The annual percentage rate (APR) on the loan is 3%. However, the interest on the loan is compounded monthly. The amount Seline will owe can be found using the formula for the sum of an amortized loan, $P=\sum_{k=1}^{n} A\left(\frac{1}{1+i}\right)^{k-1}$, where:

P = loan amount (principal)

A = monthly payment amount

i = monthly interest rate of the loan

n = number of monthly payments

Use the formula to determine Seline's monthly payment amount, A.

1. Use the given information to determine values for P, i, and n.

   The problem statement tells us that the loan is for $10,000, so $P = 10{,}000$.

   The yearly interest rate is 3%, compounded monthly. Divide 3% (0.03) by 12 to find the monthly interest rate, i.

   $$i=\frac{0.03}{12}=0.0025$$

   Seline will make 12 payments each year for 10 years. Multiply 12 by 10 to determine the total number of monthly payments, n.

   $$n=10\text{ years}\bullet\frac{12\text{ months}}{1\text{ year}}=120\text{ months}$$

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2. Apply the sum formula for a finite geometric series.

The formula for the sum of a finite geometric series is $S_n = \frac{a_1(1-r^n)}{1-r}$.

Set the amortization formula, $P = \sum_{k=1}^{n} A\left(\frac{1}{1+i}\right)^{k-1}$, equal to the sum formula.

$$P = \sum_{k=1}^{n} A\left(\frac{1}{1+i}\right)^{k-1} = \frac{a_1(1-r^n)}{(1-r)}$$

This formula simplifies to $P = \frac{a_1(1-r^n)}{(1-r)}$.

In this formula, we know the following variables:

$P = 10{,}000$

$n = 120$

We also know that $\frac{1}{1+i}$ in the amortization formula corresponds to r in summation notation, so $r = \frac{1}{1+i}$.

We don't know the value of the first term, a_1.

Recall that the variable a in summation notation, $\sum_{k=1}^{n} ar^{k-1}$,

corresponds to A in the amortization formula, $P = \sum_{k=1}^{n} A\left(\frac{1}{1+i}\right)^{k-1}$.

To find A, we must first solve the formula for the sum of this loan, $P = \frac{a_1(1-r^n)}{(1-r)}$, for a_1.

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3. Rearrange the sum formula to solve for a_1.

$P=\frac{a_1(1-r^n)}{(1-r)}$	Formula for the sum of the loan
$P\bullet\frac{(1-r)}{(1-r^n)}=a_1\frac{(1-r^n)}{(1-r)}\bullet\frac{(1-r)}{(1-r^n)}$	Multiply both sides by the reciprocal of the fraction.
$P\frac{(1-r)}{(1-r^n)}=a_1$	Simplify.
$a_1=P\frac{(1-r)}{(1-r^n)}$	Use the symmetric property of equality to switch sides.

The formula is now solved for the value of the first term, a_1.

4. Calculate r.

From step 2, we know that $r=\frac{1}{1+i}$. From step 1, we know that $i = 0.0025$.

Substitute this value for i and solve.

$$r=\frac{1}{1+i}=\frac{1}{1+0.0025}=\frac{1}{1.0025}\approx 0.998$$

Now we have the values we need to determine Seline's monthly payment:

$a_1 = A$ = monthly payment amount

$P = 10{,}000$

$r = 0.998$

$n = 120$

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5. Substitute the values into the formula $a_1 = P\dfrac{(1-r)}{(1-r^n)}$.

$a_1 = P\dfrac{(1-r)}{(1-r^n)}$	Formula
$A = 10{,}000\dfrac{(1-0.998)}{(1-0.998^{120})}$	Substitute A for a_1, 10,000 for P, 0.998 for r, and 120 for n.
$A = 10{,}000\dfrac{(0.002)}{(1-0.786)}$	Simplify.
$A = 10{,}000\left(\dfrac{0.002}{0.214}\right)$	Continue to simplify.
$A \approx 10{,}000(0.00935)$	
$A \approx 93.458$	

Seline's monthly payment will be about $93.46.



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Problem-Based Task 2A.5.2: Which Refinancing Option Is Best?

SMP	
1 ✓	2 ✓
3 ✓	4 ✓
5	6
7 ✓	8

Ten years ago, your aunt bought a house when the annual percentage rate (APR) was 6%. She took out a 30-year fixed-rate mortgage for $250,000. Now, interest rates have gone down to 4.4% for a 30-year fixed-rate mortgage and 3.5% for a 15-year fixed-rate mortgage. Your aunt wants to refinance her mortgage so that she pays a lower interest rate on the rest of what she owes. She estimates that she owes about $210,000 on her original mortgage. For a refinanced mortgage, she can choose from the 15-year loan at 3.5% interest, or the 30-year loan at 4.4% interest. Which option would you recommend to your aunt? Justify your recommendation using mathematics.

Recall that a mortgage is an amortized loan, in which monthly payments are split between principal and interest. The formula for the sum of an amortized loan is given by $P = \sum_{k=1}^{n} A\left(\frac{1}{1+i}\right)^{k-1}$.



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Problem-Based Task 2A.5.2: Which Refinancing Option Is Best?

Coaching

a. How much was the original mortgage?

b. What was the original APR?

c. How can you use the APR to calculate i? What is i?

d. What is n, the length of the original mortgage in months?

e. How can you rearrange the sum formula for a finite geometric series to solve for the first term? What is the rearranged formula?

f. How do you calculate r and what is its value?

g. How much is your aunt's monthly mortgage payment now?

h. Based on your aunt's estimate of how much she owes on the house, what is the amount she will be refinancing?

i. What is the APR for the 15-year refinanced mortgage, broken down into a monthly interest rate, i?

j. What is the value of r for the 15-year refinanced mortgage to be substituted into the sum formula?

k. How many months are in 15 years?

continued

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l. What would be the monthly payment for a 15-year refinanced mortgage?

m. How does the monthly payment for the 15-year refinanced mortgage compare to the original monthly mortgage payment?

n. What is the APR for the 30-year refinanced mortgage, broken down into a monthly interest rate, i?

o. What is the value of r for the 30-year refinanced mortgage to be substituted into the sum formula?

p. How many months are in 30 years?

q. What would be the monthly payment for a 30-year refinanced mortgage?

r. How does the monthly payment for the 30-year refinanced mortgage compare to the original monthly mortgage payment?

s. If your aunt decides *not* to refinance, what is the total amount she will have paid to the bank by the time she pays off her house?

t. Determine the total amount your aunt would pay over the life of the loan for the 15-year mortgage at an APR of 3.5% and for the 30-year mortgage at an APR of 4.4%.

u. Compare the costs of your aunt's original mortgage with the costs of the two refinancing options, including the interest rates, the number and dollar amount of the monthly payments, and the total amount paid over the life of the loan. It might be helpful to make a table.

v. Assuming your aunt chooses to refinance, which option would you recommend?

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Problem-Based Task 2A.5.2: Which Refinancing Option Is Best?

Coaching Sample Responses

a. How much was the original mortgage?

The original loan amount, *P*, was for $250,000.

b. What was the original APR?

The original APR was 6%.

c. How can you use the APR to calculate *i*? What is *i*?

APR stands for "annual percentage rate," but for a mortgage the interest is compounded monthly. In the amortization formula, *i* is the amount of monthly interest. To find *i*, take the APR and divide it by 12, the number of months in a year.

$$i = \frac{\text{APR}}{12} = \frac{0.06}{12} = 0.005$$

d. What is *n*, the length of the loan in months?

The number of months of the loan is the number of years multiplied by 12.

$n = 30 \bullet 12 = 360$

e. How can you rearrange the sum formula for a finite geometric series to solve for the first term? What is the rearranged formula?

In the sum formula, the first term a_1 is multiplied by a fraction, $\frac{(1-r^n)}{(1-r)}$. To rearrange the formula, isolate a_1 by multiplying both sides of the sum formula by the reciprocal of this fraction, then rearrange so that a_1 is on the left.

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$$S_n = a_1 \frac{(1-r^n)}{(1-r)}$$

$$S_n \bullet \frac{(1-r)}{(1-r^n)} = a_1 \frac{(1-r^n)}{(1-r)} \bullet \frac{(1-r)}{(1-r^n)}$$

$$S_n \bullet \frac{(1-r)}{(1-r^n)} = a_1$$

$$a_1 = S_n \frac{(1-r)}{(1-r^n)}$$

In this formula, a_1 is the monthly mortgage payment, S_n is the value of the original loan, n is the number of months for the loan, and r is the common ratio.

f. How do you calculate r and what is its value?

The common ratio, r, comes from the amortized loan formula. The amortized loan formula is a geometric series given by $P = \sum_{k=1}^{n} A\left(\frac{1}{1+i}\right)^{k-1}$. Here, $r = \frac{1}{1+i}$ is the common ratio. Substitute the value of i found in part c to solve for r.

$$r = \frac{1}{1+i} = \frac{1}{1+0.005} = 0.995024876$$

Round the result to the nearest thousandth.

$r = 0.995$

g. How much is your aunt's monthly mortgage payment now?

Substitute the known values into the rearranged equation for the sum of a series.

$P = S_n = 250{,}000$

$r = 0.995$

$n = 360$

$$a_1 = S_n \frac{(1-r)}{(1-r^n)} \approx 250{,}000 \frac{(1-0.995)}{(1-0.995^{360})} \approx 1496.20724$$

Rounding to the nearest penny, your aunt's monthly mortgage payments are about $1,496.21.

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h. Based on your aunt's estimate of how much she owes on the house, what is the amount she will be refinancing?

She still owes about $210,000 on the house, so that's how much she will be refinancing.

i. What is the APR for the 15-year refinanced mortgage, broken down into a monthly interest rate, i?

The APR is the annual percentage rate and needs to be divided by 12 to determine the monthly rate.

$$i = \frac{\text{APR}}{12} = \frac{0.035}{12} \approx 0.00292$$

The monthly interest rate i is approximately 0.00292, or about 0.292%.

j. What is the value of r for the 15-year refinanced mortgage to be substituted into the sum formula?

The common ratio, r, comes from the amortized loan formula. The amortized loan formula is a geometric series given by $P = \sum_{k=1}^{n} A\left(\frac{1}{1+i}\right)^{k-1}$. Here, $r = \frac{1}{1+i}$ is the common ratio. Substitute 0.00292 for i and solve for r.

$$r = \frac{1}{1+i} \approx \frac{1}{1+0.00292} \approx 0.997088502$$

Round the result to the nearest thousandth.

$r = 0.997$

k. How many months are in 15 years?

Multiply the number of years by the number of months in a year, 12.

$n = 15 \bullet 12 = 180$ months

l. What would be the monthly payment for a 15-year refinanced mortgage?

Substitute the known values into the rearranged equation for the sum of a series.

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$P = S_n = 210{,}000$

$r = 0.997$

$n = 180$

$$a_1 = S_n \frac{(1-r)}{(1-r^n)} \approx 210{,}000 \frac{(1-0.997)}{(1-0.997^{180})} \approx 1508.170954$$

Rounding to the nearest penny, your aunt's monthly mortgage payments for the 15-year refinanced mortgage would be about $1,508.17.

m. How does the monthly payment for the 15-year refinanced mortgage compare to the original monthly mortgage payment?

Find the difference in the mortgage payments.

1496.21 – 1508.17 = –11.96

The refinanced mortgage payment is $11.96 more than the original payment.

n. What is the APR for the 30-year refinanced mortgage, broken down into a monthly interest rate, *i*?

The APR is the annual percentage rate and needs to be divided by 12 to determine the monthly interest rate.

$$i = \frac{\text{APR}}{12} = \frac{0.044}{12} \approx 0.00367$$

The monthly interest rate, *i*, is approximately 0.00367, or about 0.367%.

o. What is the value of *r* for the 30-year refinanced mortgage to be substituted into the sum formula?

The common ratio, *r*, comes from the amortized loan formula. The amortized loan formula is a geometric series given by $P = \sum_{k=1}^{n} A\left(\frac{1}{1+i}\right)^{k-1}$. Here, $r = \frac{1}{1+i}$ is the common ratio. Substitute 0.00367 for *i* and solve for *r*.

$$r = \frac{1}{1+i} \approx \frac{1}{1+0.00367} \approx 0.99634342$$

Round the result to the nearest thousandth.

$r = 0.996$

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p. How many months are in 30 years?

Multiply the number of years by the number of months in a year, 12.

$n = 30 \bullet 12 = 360$ months

q. What would be the monthly payment for a 30-year refinanced mortgage?

Substitute the known values into the rearranged equation for the sum of a series.

$P = S_n = 210{,}000$

$r = 0.996$

$n = 360$

$$a_1 = S_n \frac{(1-r)}{(1-r^n)} \approx 210{,}000 \frac{(1-0.996)}{(1-0.996^{360})} \approx 1099.828512$$

Rounding to the nearest penny, your aunt's monthly mortgage payments for the 30-year refinanced mortgage would be about $1,099.83.

r. How does the monthly payment for the 30-year refinanced mortgage compare to the original monthly mortgage payment?

Find the difference in the mortgage payments.

$1496.21 - 1099.83 = 396.38$

The refinanced mortgage payment is $396.38 less than the original payment.

s. If your aunt decides *not* to refinance, what is the total amount she will have paid to the bank by the time she pays off her house?

This total is the amount of each monthly payment for the original mortgage multiplied by the number of payments.

$1496.21 \bullet 360 = 538{,}635.60$

By the time your aunt pays off the original 30-year mortgage at 6% APR, she will have paid about $538,635.60.

t. Determine the total amount your aunt would pay over the life of the loan for the 15-year mortgage at an APR of 3.5% and for the 30-year mortgage at an APR of 4.4%.

Multiply the amount of each monthly payment by the number of months in the loan.

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- 15-year mortgage at 3.5%:

  1508.17 • 180 = 271,470.60

  For the 15-year loan at 3.5% APR, the total paid over 180 months would be $271,470.60.

- 30-year mortgage at 4.4%:

  1099.83 • 360 = 395,938.80

  For the 30-year loan at 4.4% APR, the total paid over 360 months would be $395,938.80.

u. Compare the costs of your aunt's original mortgage with the costs of the two refinancing options, including the interest rates, the number and dollar amount of the monthly payments, and the total amount paid over the life of the loan. It might be helpful to make a table.

Mortgage	APR	Months	Monthly payment amount ($)	Total payment amount ($)
Original 30-year	6%	360	1,496.21	538,635.60
Refinanced 15-year	3.5%	180	1,508.17	271,470.60
Refinanced 30-year	4.4%	360	1,099.83	395,938.80

The 30-year refinanced mortgage has the lowest monthly payment, but your aunt would pay nearly $124,500 more over the course of this loan than she would if she chose the 15-year mortgage. However, the original mortgage results in the highest total payment.

v. Assuming your aunt chooses to refinance, which option would you recommend?

There are two reasonable options.

- Option 1: If your aunt's goal is to have lower monthly payments, then she should refinance with the 30-year mortgage at 4.4% APR. This would still save her money in the long term compared with how much she would pay overall if she didn't refinance.
- Option 2: If your aunt's goal is to save the most money in the long term and the additional $12 per month is within her budget, then she should choose the 15-year mortgage at 3.5% APR.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2A.5.2: Sum of a Finite Geometric Series

Expand the series.

1. $\sum_{k=1}^{5} 24\left(\frac{1}{4}\right)^{k-1}$

Evaluate the series using expansion.

2. $\sum_{k=1}^{4} 2(5)^{k-1}$

Evaluate the series using the sum formula.

3. $\sum_{k=1}^{8} 12\left(\frac{1}{3}\right)^{k-1}$

Find the first term of the series given the sum, the common ratio, and the number of terms.

4. $S_n = 20{,}000$, $r = 0.996$, $n = 48$

Read each scenario and use the given information to answer the questions.

5. You found out that one of your favorite TV shows has been canceled. That day, you post the news to 5 friends on your social media page. The next day, those 5 friends each post to 5 of their friends' pages. How many people will have shared the news after 6 days if this pattern continues?

6. Santo catches a contagious cough at the beginning of the school day. Within 1 hour, he has infected 4 other people in the school. If this pattern continues, how many people will be infected by the end of the school day, which is 5 hours after Santo arrived?

continued

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7. Nhu is a biologist who is conducting a study about a species of butterfly called the Common Buckeye. She estimates that in the spring, the number of butterflies living in her study area will increase rapidly. For every butterfly in the area, 2 new butterflies hatch each week. If approximately 50 butterflies were counted during the first week of the season, how many butterflies will there be by the twelfth week?

8. Lydia is a childbirth assistant. The first year of her career, she attended the births of 2 babies. Every year after that, the number of births Lydia attended has been double that of the previous year. After 6 years of working as a childbirth assistant, how many births has she attended in total?

9. Your sister just graduated from college and wants to buy a new car. She found a car for $14,000. The annual percentage rate is 4%, with the interest compounded monthly. What will be her monthly car payment if she takes out a 3-year loan for that amount?

10. TaNisha is purchasing a new home. She wants to borrow $150,000 for 15 years at an annual percentage rate of 4.5%, with the interest compounded monthly. How much will she pay each month for this mortgage?

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Lesson 2A.5.3: Sum of an Infinite Geometric Series

Warm-Up 2A.5.3

Mario is a land surveyor and stakes out property markers using a rubber mallet and wooden stakes. In one spot, Mario was able to pound the stake 3 inches into the ground on the first swing of his mallet. However, each of his next 5 swings only drove the stake 20% deeper into the ground.

1. Write an explicit formula of the geometric sequence for the distance Mario drove the stake into the ground on each swing.

2. Write the related series for the total distance Mario drove the stake into the ground.

3. Evaluate the series you wrote for problem 2.

4. Mario needs to drive the stake 4 inches into the ground. Do you think he'll be able to accomplish that if he keeps hitting the stake? Explain your reasoning.

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Lesson 2A.5.3: Sum of an Infinite Geometric Series

Common Core State Standard

A–SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★

Warm-Up 2A.5.3 Debrief

Mario is a land surveyor and stakes out property markers using a rubber mallet and wooden stakes. In one spot, Mario was able to pound the stake 3 inches into the ground on the first swing of his mallet. However, each of his next 5 swings only drove the stake 20% deeper into the ground.

1. Write an explicit formula of the geometric sequence for the distance Mario drove the stake into the ground on each swing.

   The first term in the series is 3. The common ratio, *r*, is 0.2. Substitute those values into the general explicit formula for a geometric sequence.

$a_n = a_1 \bullet r^{n-1}$	General explicit formula for a geometric sequence
$a_n = 3 \bullet 0.2^{n-1}$	Substitute the known values.

   The explicit formula for this geometric sequence is $a_n = 3 \bullet 0.2^{n-1}$.

2. Write the related series for the total distance Mario drove the stake into the ground.

   First, to find the value of *n,* determine the number of times Mario hit the stake. The problem scenario states that Mario's first swing drove the stake into the ground 3 inches, and then he took 5 more swings that each drove the stake an additional 20% into the ground. This means he hit the stake 6 times, so $n = 6$.

   Rewrite the explicit formula for this sequence $(3 \bullet 0.2^{n-1})$ 6 times to find the values of 6 terms, each time substituting one of the counting numbers 1 through 6 for *n* in the exponent. Use *S* to represent the sum of the terms in the series. Then, simplify and evaluate each exponent as shown:

$S_6 = (3 \bullet 0.2^{(1)-1})$	$+ (3 \bullet 0.2^{(2)-1})$	$+ (3 \bullet 0.2^{(3)-1})$	$+ (3 \bullet 0.2^{(4)-1})$	$+ (3 \bullet 0.2^{(5)-1})$	$+ (3 \bullet 0.2^{(6)-1})$
$S_6 = (3 \bullet 0.2^{0})$	$+ (3 \bullet 0.2^{1})$	$+ (3 \bullet 0.2^{2})$	$+ (3 \bullet 0.2^{3})$	$+ (3 \bullet 0.2^{4})$	$+ (3 \bullet 0.2^{5})$
$S_6 = (3 \bullet 1)$	$+ (3 \bullet 0.2)$	$+ (3 \bullet 0.04)$	$+ (3 \bullet 0.008)$	$+ (3 \bullet 0.0016)$	$+ (3 \bullet 0.00032)$
$S_6 = 3$	$+ 0.6$	$+ 0.12$	$+ 0.0024$	$+ 0.0048$	$+ 0.00096$

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3. Evaluate the series you wrote for problem 2.

   Sum the terms.

   $S_6 = 3 + 0.6 + 0.12 + 0.0024 + 0.0048 + 0.00096 = 3.74976$

   The sum of the series is 3.74976.

4. Mario needs to drive the stake 4 inches into the ground. Do you think he'll be able to accomplish that if he keeps hitting the stake? Explain your reasoning.

   It's probably physically impossible for Mario to drive the stake into the ground the required 4 inches at this rate. The terms in the series are growing geometrically smaller; therefore, the amount of progress per swing is approaching 0. It could take dozens or perhaps hundreds of additional swings to even get close to a depth of 4 inches, and Mario would most likely tire out before then.

Connection to the Lesson

- Students will extend their understanding of finite series to infinite geometric series.
- Students will explore how a series can converge even though it continues indefinitely toward infinity.
- Students will derive the sum formula for geometric series that do converge.

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Prerequisite Skills

This lesson requires the use of the following skills:

- expanding a geometric series given in summation notation (A–SSE.4)
- determining the common ratio (F–BF.2)
- simplifying complex fractions (5–NF.7)

Introduction

We have seen series that are finite, meaning they have a limited number of terms, but what happens to a series that has infinite terms? A geometric series that has **infinite** terms (that is, where $n=\infty$), is called an **infinite geometric series**. In an infinite geometric series, summing all the terms in the series is not an easy task, because the series starts with $n = 1$ and goes on until infinity. Do such sums exist? As you will see in this lesson, for infinite geometric series with certain values of the common ratio r, a sum does exist.

Key Concepts

- Whether an infinite geometric series has a sum depends on whether the absolute value of its common ratio, r, is greater or less than 1, and on the behavior of the partial sums of the series.
- **Partial sums** are the sums of part of a series. They are the result of adding pairs of terms. Partial sums are used to determine if a series converges or diverges.
- In a series, to **converge** means to approach a finite limit. If the sequence of the partial sums of a series approaches the value of a given number (the limit), then the entire series converges to that limit. In other words, the series has a sum.
- An infinite series converges when the absolute value of the common ratio r is less than 1 $(|r|<1)$.
- If the series does not have a sum—that is, the sequence of its partial sums does *not* approach a finite limit—then the series is said to **diverge**.
- In a series that diverges, the absolute value of the common ratio, r, is greater than 1 $(|r|>1)$.
- Recall that a series cannot have an r value equal to 1 ($r \neq 1$), because this would result in a 0 in the denominator of the sum formula. Fractions with a denominator of 0 are undefined.

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- Consider the sequence $a_n = \frac{1}{3} \bullet \left(\frac{1}{3}\right)^{n-1}$. The terms and the partial sums of the series from $n = 1$ to $n = 8$ are calculated in the table that follows.

Term	Value of n	Term evaluated for the power $n - 1$	Approx. value	Partial sum (term + previous sum)
a_n	n	$\frac{1}{3} \bullet \left(\frac{1}{3}\right)^{n-1}$		
a_1	1	$\frac{1}{3} \bullet \left(\frac{1}{3}\right)^{0} = \frac{1}{3} \bullet 1 = \frac{1}{3}$	0.333333333	0.333333333
a_2	2	$\frac{1}{3} \bullet \left(\frac{1}{3}\right)^{1} = \frac{1}{3} \bullet \frac{1}{3} = \frac{1}{9}$	0.111111111	0.444444444
a_3	3	$\frac{1}{3} \bullet \left(\frac{1}{3}\right)^{2} = \frac{1}{3} \bullet \frac{1}{9} = \frac{1}{27}$	0.037037037	0.481481481
a_4	4	$\frac{1}{3} \bullet \left(\frac{1}{3}\right)^{3} = \frac{1}{3} \bullet \frac{1}{27} = \frac{1}{81}$	0.012345679	0.49382716
a_5	5	$\frac{1}{3} \bullet \left(\frac{1}{3}\right)^{4} = \frac{1}{3} \bullet \frac{1}{81} = \frac{1}{243}$	0.004115226	0.497942386
a_6	6	$\frac{1}{3} \bullet \left(\frac{1}{3}\right)^{5} = \frac{1}{3} \bullet \frac{1}{243} = \frac{1}{729}$	0.001371742	0.499314128
a_7	7	$\frac{1}{3} \bullet \left(\frac{1}{3}\right)^{6} = \frac{1}{3} \bullet \frac{1}{729} = \frac{1}{2187}$	0.000457247	0.499771375
a_8	8	$\frac{1}{3} \bullet \left(\frac{1}{3}\right)^{7} = \frac{1}{3} \bullet \frac{1}{2187} = \frac{1}{6561}$	0.000152416	0.499923791

- Each term in the sequence is evaluated, then added to the value of the previous term. Notice that the partial sum of each pair of terms grows closer to 0.5 with each additional term.
- However, even as the partial sums approach 0.5, the values of the specific terms being added are growing smaller: a_8 is less than a_7, which is less than a_6, and so on. Since the values being added keep shrinking, it is impossible to reach the exact value 0.5.

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- However, if it were possible to sum this series to infinity, presumably the series would sum to 0.5.
- The **sum formula for an infinite geometric series**, if the sum exists, is given by $S_n = \frac{a_1}{1-r}$, where S_n is the sum, a_1 is the first term, and r is the common ratio. Guided Practice Example 1 shows how this formula is derived from the sum formula for a finite geometric series.
- If the series converges—that is, if $|r| < 1$—then the sum exists and this formula can be used to find it.
- Recall that the summation notation for a finite geometric series is $\sum_{k=1}^{n} ar^{k-1}$, where n is the number of terms in the sequence.
- An infinite series in summation notation has an infinity symbol (∞) above the *sigma* instead of n to indicate an unlimited number of terms:

$$\sum_{k=1}^{\infty} ar^{k-1}$$

Common Errors/Misconceptions

- mistaking a series that diverges for a series that converges
- forgetting to multiply by the reciprocal when dividing fractions

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Guided Practice 2A.5.3

Example 1

Derive the formula for the sum of infinite series, $S_n = \frac{a_1}{1-r}$, if $|r| < 1$. Base the derivation on the sum formula for a finite series.

1. Write the formula for the sum of a finite series.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

Here, S_n is the sum, a_1 is the first term of the series, r is the common ratio, and n is the number of terms.

2. Distribute the first term in the numerator.

$$S_n = \frac{a_1 - a_1 r^n}{1-r}$$

3. Rewrite the fraction as the difference of two fractions.

$$S_n = \frac{a_1}{1-r} - \frac{a_1 r^n}{1-r}$$

4. Substitute an infinity symbol for the exponent, n.

The infinity symbol, ∞, indicates an unlimited number of terms.

$$S_n = \frac{a_1}{1-r} - \frac{a_1 r^{\infty}}{1-r}$$

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5. Analyze the effect of raising a value of $|r|<1$ to an infinite power.

Recall the table of partial sums of the sequence $a_n=\frac{1}{3}\bullet\left(\frac{1}{3}\right)^{n-1}$ shown in the Key Concepts. In that sequence, $r=\frac{1}{3}$, the absolute value of which is less than 1. The table showed that the value of each new term in the series was smaller than the last, approaching 0. This is true for any value of r when the absolute value of r is less than 1.

Therefore, as the power applied to r approaches infinity, r approaches 0.

6. Simplify the formula using the process of raising r, where $|r|<1$, to a power of ∞.

$S_n=\frac{a_1}{1-r}-\frac{a_1r^{\infty}}{1-r}$ Sum formula from step 4

$S_n=\frac{a_1}{1-r}-\frac{a_1(0)}{1-r}$ Substitute 0 for r^{∞}.

$S_n=\frac{a_1}{1-r}-\frac{0}{1-r}$ Simplify by subtracting the numerators.

$S_n=\frac{a_1}{1-r}$

The formula for the sum of infinite geometric series, where $|r|<1$, is $S_n=\frac{a_1}{1-r}$.

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Example 2

Determine the sum, if a sum exists, of the following geometric series.

$$\sum_{k=1}^{\infty} 4\left(\frac{1}{2}\right)^{k-1}$$

1. Determine if the series has a sum.

   Examine the value of r. If $|r|<1$, then the series has a sum. In this series, $r=\frac{1}{2}$. We can see that the absolute value of r is less than 1: $\left|\frac{1}{2}\right|=\frac{1}{2}$. Therefore, the series has a sum.

2. Determine what variables found in the sum formula for an infinite geometric series are known and unknown.

   The sum of an infinite geometric series is given by $S_n=\frac{a_1}{1-r}$.

   Known variable: $r=\frac{1}{2}$

   Unknown variable: a_1

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3. Determine the value of a_1 in the series.

$\sum_{k=1}^{\infty} 4\left(\frac{1}{2}\right)^{k-1}$ Given series

$\sum_{k=1}^{\infty} 4\left(\frac{1}{2}\right)^{1-1} = 4\left(\frac{1}{2}\right)^{0} + \ldots$ Write out the first term and substitute 1 for k.

$4\left(\frac{1}{2}\right)^{0} = 4(1) = 4$ Simplify.

The first term of the series is 4.

The geometric series is given by $a \bullet r^{n-1}$. The first term has an n value of 1, which makes the power 0. Anything raised to a power of 0 is 1. This means that you will be multiplying the value of a by 1, which just results in a.

4. Substitute the variables into the sum formula for an infinite geometric series.

Known values: $r = \frac{1}{2}$, $a_1 = 4$

$S_n = \frac{a_1}{1-r}$ Sum formula for an infinite geometric series

$S_n = \frac{4}{1-\frac{1}{2}}$ Substitute $\frac{1}{2}$ for r.

$S_n = \frac{4}{\frac{1}{2}} = 4 \bullet 2 = 8$ Simplify.

The infinite series $\sum_{k=1}^{\infty} 4\left(\frac{1}{2}\right)^{k-1}$ sums to 8.

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Example 3

Determine the sum, if a sum exists, of the following geometric series. Justify the result.

$$\sum_{k=1}^{\infty} 6\left(\frac{3}{2}\right)^{k-1}$$

1. Determine if the series has a sum.

   Examine the value of r. If $|r|<1$, then the series has a sum. In this series, $r=\frac{3}{2}$, which simplifies to $1\frac{1}{2}$. We can see that the absolute value is greater than 1; therefore, this series does not have a sum.

2. Justify the result.

   Since the value of n in the series $\sum_{k=1}^{\infty} 6\left(\frac{3}{2}\right)^{k-1}$ goes to infinity, r^{n-1} grows without bound. To see this, make a table of values.

(*continued*)

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Term	Value of n	Term evaluated for the power $n-1$	Approx. value	Partial sum (term + previous sum)
a_n	n	$6\left(\frac{3}{2}\right)^{n-1}$		
a_1	1	$6\left(\frac{3}{2}\right)^{0}=6(1)$	6	6
a_2	2	$6\left(\frac{3}{2}\right)^{1}=6\left(\frac{3}{2}\right)=\frac{18}{2}=9$	9	15
a_3	3	$6\left(\frac{3}{2}\right)^{2}=6\left(\frac{9}{4}\right)=\frac{54}{4}=13.5$	13.5	28.5
a_4	4	$6\left(\frac{3}{2}\right)^{3}=6\left(\frac{27}{8}\right)=\frac{162}{8}=20.25$	20.25	48.75
a_5	5	$6\left(\frac{3}{2}\right)^{4}=6\left(\frac{81}{16}\right)=\frac{486}{16}=30.375$	30.375	79.125
a_6	6	$6\left(\frac{3}{2}\right)^{5}=6\left(\frac{243}{32}\right)=\frac{1458}{32}=45.5625$	45.5625	124.6875
a_7	7	$6\left(\frac{3}{2}\right)^{6}=6\left(\frac{729}{64}\right)=\frac{4374}{64}=68.34375$	68.34375	193.03125
a_8	8	$6\left(\frac{3}{2}\right)^{8}=6\left(\frac{2187}{128}\right)=\frac{13{,}122}{128}=102.515625$	102.515625	295.546875

We can see that the partial sums do not converge to a limit; they increase in value infinitely. Since the partial sums do not converge, this series is diverging and has no sum. The result found in step 1 is justified.

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Instruction

Example 4

Use the formula for an infinite geometric series to rewrite $0.\overline{6}$ as a fraction.

1. Write the repeating decimal as a sum.

   $0.\overline{6} = 0.6 + 0.06 + 0.006 + ...$

   Notice that this is a geometric sequence because there is a common ratio r between each term and the next.

2. Determine the value of r.

   Substitute the values for a_2 and a_1 into the formula for the common ratio r.

   $$r = \frac{a_n}{a_{n-1}} = \frac{(0.06)}{(0.6)} = 0.1 = \frac{1}{10}$$

   The value of r is $\frac{1}{10}$.

3. Write the sum of the series using summation notation.

   Known values: $a_1 = 0.6,\ r = \frac{1}{10}$

   $\sum_{k=1}^{\infty} ar^{k-1}$ General summation notation for an infinite geometric series

   $\sum_{k=1}^{\infty} (0.6)\left[\left(\frac{1}{10}\right)\right]^{k-1}$ Substitute 0.6 for a_1 and $\frac{1}{10}$ for r.

   The repeating decimal $0.\overline{6}$ written using summation notation is $\sum_{k=1}^{\infty} 0.6\left(\frac{1}{10}\right)^{k-1}$.

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Instruction

4. Evaluate the series using the sum formula.

$S_n = \frac{a_1}{1-r}$ — Sum formula for an infinite geometric series

$S_n = \frac{0.6}{1-\frac{1}{10}}$ — Substitute 0.6 for a_1 and $\frac{1}{10}$ for r.

$S_n = \frac{0.6}{\frac{9}{10}} = \frac{\frac{6}{10}}{\frac{9}{10}}$ — Write 0.6 as a fraction in the numerator.

$S_n = \frac{\frac{6}{10}}{\frac{9}{10}} = \frac{6}{10} \bullet \frac{10}{9} = \frac{6}{9} = \frac{2}{3}$ — Simplify.

$0.\overline{6}$ written as a fraction is $\frac{2}{3}$.



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Problem-Based Task 2A.5.3: Will It Overflow or Not?

SMP	
1 ✓	2
3 ✓	4
5	6
7 ✓	8

Luca bought a water dispenser that was advertised as holding 5 gallons, but he's not sure if that's true. Luca experiments with the container's capacity by first pouring 200 fluid ounces into it. Then he pours 100 ounces more, 50 ounces more, and so on. At what point, if any, will the container overflow? Recall that 1 gal = 128 fl oz. Explain your reasoning.



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Problem-Based Task 2A.5.3: Will It Overflow or Not?

Coaching

a. As Luca continues his experiment, what happens to the amount of water that he adds to the container?

b. As Luca continues with his experiment, what happens to the total amount of water in the container?

c. Is this a series? If so, what type? Explain.

d. What formula can you use to determine the sum of the amount of water in the container if Luca continues this experiment indefinitely?

e. What are the values of the variables that are to be substituted into this formula?

f. If this experiment is a series, what is its sum?

g. Will the container ever overflow in this experiment? Explain.

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Instruction

Problem-Based Task 2A.5.3: Will It Overflow or Not?

Coaching Sample Responses

a. As Luca continues his experiment, what happens to the amount of water that he adds to the container?

With each addition of water, Luca pours half of the previous amount into the container. Therefore, the amount of water being added is decreasing.

b. As Luca continues with his experiment, what happens to the total amount of water in the container?

As Luca adds water, the total amount of water in the container increases.

c. Is this a series? If so, what type? Explain.

This is a geometric series because it has a common ratio of $\frac{1}{2}$. Specifically, it is an infinite geometric series because there is no specified limit to the number of times Luca will add the water.

d. What formula can you use to determine the sum of the amount of water in the container if Luca continues this experiment indefinitely?

Use the formula for the sum of an infinite geometric series, given by $S_n = \frac{a_1}{1-r}$.

e. What are the values of the variables that are to be substituted into this formula?

The first time Luca added water to the container, he added 200 fluid ounces. Therefore, $a_1 = 200$. The ratio has already been established as $r = \frac{1}{2}$.

f. If this experiment is a series, what is its sum?

Substitute values for a_1 and r into the formula $S_n = \frac{a_1}{1-r}$.

$$S_n = \frac{a_1}{1-r}$$

$$S_n = \frac{200}{1-\frac{1}{2}} = \frac{200}{\frac{1}{2}} = 200 \bullet 2 = 400$$

The sum of the series is 400 fluid ounces.

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g. Will the container ever overflow in this experiment? Explain.

The container holds 5 gallons. Since 1 gal = 128 fl oz, the container can hold a total of 128 • 5 = 640 fl oz. Luca starts his experiment by pouring 200 fluid ounces into the container, and then he decreases that amount by half for subsequent pours, so that each pour contains half the liquid of the previous pour. This results in a series that converges at 400 fluid ounces. Since 400 < 640, the container will never overflow, even if Luca continues his experiment indefinitely.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2A.5.3: Sum of an Infinite Geometric Series

Determine the sum of each series, if a sum exists. If no sum exists, write "No sum exists."

1. $\sum_{k=1}^{\infty} \frac{3}{4}\left(\frac{1}{2}\right)^{k-1}$

2. $\sum_{k=1}^{\infty} \frac{5}{6}\left(\frac{3}{5}\right)^{k-1}$

3. $\frac{1}{10}+\frac{1}{9}+\frac{10}{81}+\ldots$

4. $0.37 + 0.0037 + 0.000037 + ...$

Use the formula for an infinite geometric series to rewrite each repeating decimal as a fraction.

5. $0.\overline{7}$

6. $0.\overline{345}$

Use the following information to solve problems 7 and 8.

> An experiment to determine the rate of decay for a certain radioactive isotope starts out with 1,000 mg of the isotope. After 1 hour, only 995 mg remain.

7. Write an infinite geometric series to model the rate of decay of the isotope over time.

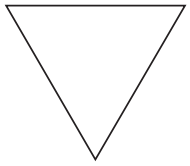
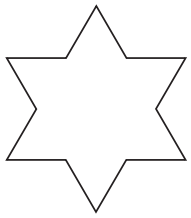
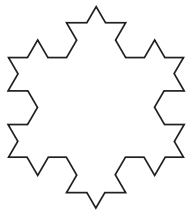
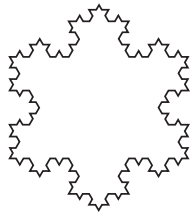
8. Evaluate the series from problem 7.

continued

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Use the information that follows to solve problems 9 and 10.

Fractals are very detailed geometric figures made up of repeated, self-similar patterns. Each small section (or stage) of the figure is the same shape as the entire figure. For example, the following illustration shows stages 0 through 4 of the Koch snowflake. The original stage, stage 0, is constructed of an equilateral triangle. Each segment of the triangle is divided into thirds and new equilateral triangles are constructed to create stage 1. This is repeated over and over again. For problems 9–10, assume that the length of each segment of the equilateral triangle in stage 0 is 3 cm long. (Drawings are not to scale.)

Stage 0	Stage 1	Stage 2	Stage 3
			

9. Write a geometric series for the area of the stages of the Koch snowflake. (*Hint*: Use the area formula for an equilateral triangle: $A = \frac{s^2\sqrt{3}}{4}$.)

10. Evaluate the series from problem 9.

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Progress Assessment

Circle the letter of the best answer.

1. A geometric series is defined such that $a_1 = 8$, $r = 3$, and $n = 6$. What is the sum of the series?

   a. 2,912
   c. 72
   b. 108
   d. 1,944

2. What is the next term in the sequence?

   {1000, 800, 640, 512, ...}

   a. 312
   c. 409.6
   b. 384
   d. 400.5

3. Each year, Javier's salary increases by 2.4% to account for inflation. If his starting salary was $30,000, what was his salary after 5 years?

   a. $33,777
   c. $34,587.65
   b. $32,985.35
   d. $32,212.25

4. What is the expansion of the following series?

   $$\sum_{k=1}^{4} 0.01(1.5)^{k-1}$$

   a. 0.01 + 0.015 + 0.0225 + 0.03375
   c. 0.015 + 0.0225 + 0.03375 + 0.050625
   b. 0.01 + 0.015 + 0.03 + 0.045
   d. 0.015 + 0.03 + 0.045 + 0.60

5. What is the sum of the series?

   $$\sum_{k=1}^{5} 20\left(\frac{1}{5}\right)^{k-1}$$

   a. 24.992
   c. 0.032
   b. 27.875
   d. 16,384

continued

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6. What is the value of the first term of the geometric series described as follows?

$S_n = \frac{26{,}600}{9},\ r = \frac{2}{3},\ n = 6$

a. 720

b. 1,080

c. $\frac{26{,}600}{3}$

d. 13,300

7. Soriah posted a video on a social media site about raising awareness for cancer research. Each person who watched the video shared it with 4 people the same day, and each of those 4 people shared it with 4 additional people the next day, and so on. After 7 days, the video had a total of 1,529,080 views. How many people watched the video on the first day?

a. 4

b. 280

c. 54,610

d. 520

8. Determine the sum of the series, if a sum exists.

$\sum_{k=1}^{\infty} 9\left(\frac{1}{3}\right)^{k-1}$

a. 15

b. 27

c. 13.5

d. No sum exists.

9. Determine the sum of the series, if a sum exists.

$\sum_{k=1}^{\infty} 5\left(\frac{4}{3}\right)^{k-1}$

a. –15

b. 15

c. 5

d. No sum exists.

10. What is the expanded series of the decimal $0.\overline{34}$?

a. $\sum_{k=1}^{\infty} 0.34\left(\frac{1}{10}\right)^{k-1}$

b. 0.34 + 0.034 + 0.0034 + ...

c. $\sum_{k=1}^{\infty} 0.34\left(\frac{1}{1000}\right)^{k-1}$

d. 0.34 + 0.0034 + 0.000034 + ...

continued

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Assessment

For problem 11, use the given information to answer the questions that follow.

11. Sasha is purchasing a new home. Her loan is for $260,000 for 30 years with an annual percentage rate of 4.5%, compounded monthly. The amount of Sasha's monthly mortgage payment can be calculated using the following formula:

$$P=\sum_{k=1}^{n} A\left(\frac{1}{1+i}\right)^{k-1}$$

a. What is the value of i?

b. What is the value of r?

c. For how many months is the loan?

d. What is Sasha's approximate monthly mortgage payment, A?

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Unit 2A Assessment

Circle the letter of the best answer.

1. What is the degree of the polynomial $-4x^7+3x^5+7x+6$?

   a. –4
   c. 14
   b. 13
   d. 7

2. What is the result of $\left(8x^4-12x^3+6\right)-\left(4x^3+2x-9\right)$?

   a. $4x+14x+15$
   c. $8x^4-16x^3-2x+15$
   b. $8x^4-8x^3+2x-9$
   d. $8x^4-16x^3+2x-3$

3. What is the result of $\left(4x^3+5x^2+3\right)\left(4x^2-9\right)$?

   a. $16x^5+20x^4-36x^3-33x^2-27$
   c. $16x^5+20x^4+36x^3+57x^2+27$
   b. $16x^6+5x^4-36x^3-33x^2-27$
   d. $16x^5+20x^4+12x^2$

4. Which expression represents a factored form of x^4-16?

   a. $(x+4)(x-4)$
   c. $(x^2+8)(x^2-8)$
   b. $(x+8)(x-8)$
   d. $(x^2+4)(x^2-4)$

5. What is term 5 in row 9 of Pascal's Triangle?

   a. 126
   c. 56
   b. 84
   d. 36

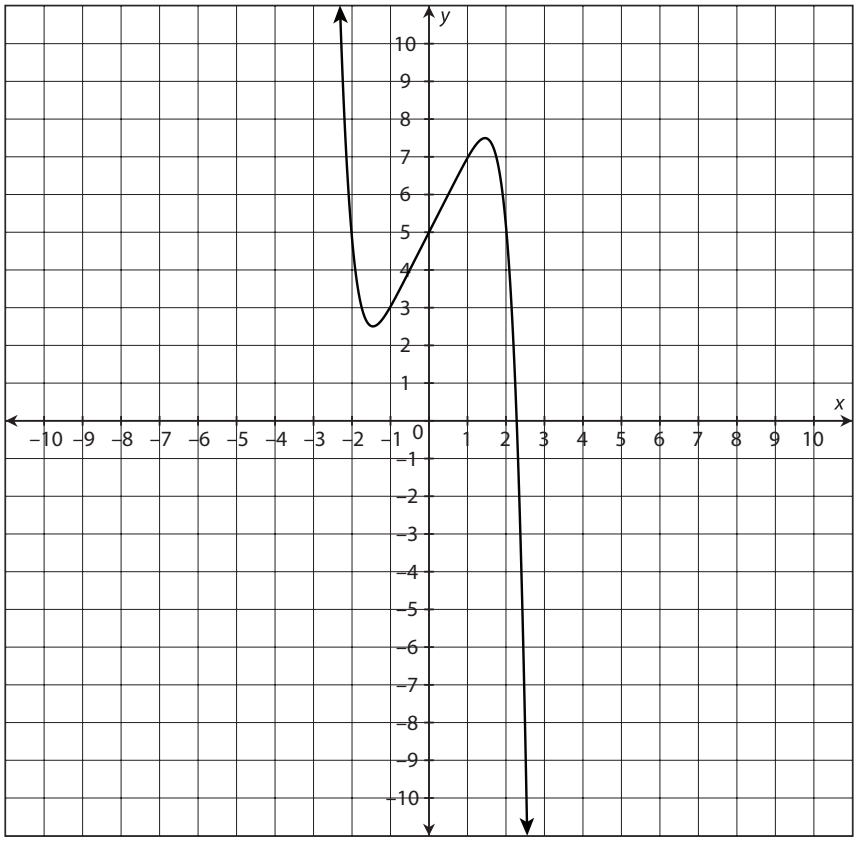
6. Which expression represents the expanded form of $(-2x+y)^3$?

   a. $8x^3+12x^2y+6xy^2+y^3$
   c. $-8x^3-4x^2y+2xy^2+y^3$
   b. $-8x^3+12x^2y-6xy^2+y^3$
   d. $8x^3-4x^2y-2xy^2+y^3$

continued

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7. What must be true about the degree and the leading coefficient of the graphed polynomial?



a. The polynomial is an odd-degree polynomial and has a positive coefficient.

b. The polynomial is an odd-degree polynomial and has a negative coefficient.

c. The polynomial is an even-degree polynomial and has a positive coefficient.

d. The polynomial is an even-degree polynomial and has a negative coefficient.

8. Which polynomial function has zeros of –3, –1, and 2?

a. $f(x) = x^3 - 7x - 6$

b. $f(x) = x^3 + 4x^2 + x - 6$

c. $f(x) = x^3 - 2x^2 - 5x + 6$

d. $f(x) = x^3 + 2x^2 - 5x - 6$

continued

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9. What are all the possible rational roots for the polynomial equation $5x^3 - 3x^2 + 6x - 20 = 0$?

   a. $\pm\frac{1}{4}, \pm\frac{1}{5}, \pm\frac{1}{20}, \pm\frac{5}{4}, \pm 1, \pm 5$

   b. $\pm 1, \pm 4, \pm 5, \pm 20$

   c. $\pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{4}{5}, \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

   d. $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

10. Graph the following system to determine the real solution(s), if any.

$$\begin{cases} f(x) = -x^5 + 2x^3 + 7 \\ g(x) = 6x + 2 \end{cases}$$

   a. {(1, 8)}

   b. {(1, 0)}

   c. {(–0.5, 0), (1.8, 0)}

   d. There are no real solutions to this system.

11. What is the sum of the following series, if a sum exists?

$$\sum_{k=1}^{\infty} 3\left(\frac{5}{4}\right)^{k-1}$$

   a. $\frac{5}{4}$

   b. $\frac{15}{4}$

   c. $\frac{12}{5}$

   d. No sum exists.

12. What is the expansion of the following series?

$$\sum_{k=1}^{4} 0.02(1.2)^{k-1}$$

   a. 0.02 + 0.024 + 0.029 + 0.0346

   b. 0.02 + 0.024 + 0.0288 + 0.03456

   c. 0.024 + 0.029 + 0.0346 + 0.0415

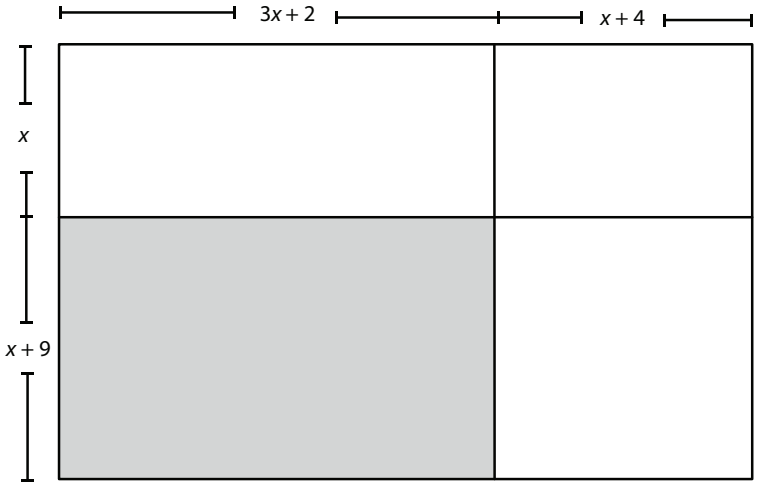
   d. 0.024 + 0.0288 + 0.03456 + 0.041472

continued

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Use what you have learned about polynomial relationships to complete all parts of problems 13–15.

13. Lia is creating a banner for her brother's surprise birthday party. The banner will be a large rectangle, divided into sections for text and pictures. The figure below represents Lia's design for the banner. Use the diagram to answer the questions that follow. All units are in inches.



a. What is the perimeter of the shaded portion of the banner?

b. What is the area of the entire rectangular banner?

14. A third-degree polynomial with rational coefficients has the roots 4 and $(3 + i)$.

a. What are the additional roots?

b. What is this polynomial?

c. What is the degree of this polynomial?

d. Identify the end behavior and maximum number of turning points of the graph of the polynomial.

15. Troy is purchasing a new home. His loan is for \$192,000 for 20 years with an annual percentage rate of 3%, compounded monthly. The amount of Troy's monthly mortgage payment can be calculated using the following formula:

$$P = \sum_{k=1}^{n} A\left(\frac{1}{1+i}\right)^{k-i}$$

a. What is the value of i?

b. What is the value of r?

c. For how many months is the loan?

d. What is Troy's approximate monthly mortgage payment, A?

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Answer Key

Lesson 1: Polynomial Structures and Operating with Polynomials

Pre-Assessment, p. U2A-1

1. c
2. d
3. a
4. b
5. b

Warm-Up 2A.1.1, p. U2A-4

1. 10 total seats
2. 21 total seats
3. Tara will not be able to buy enough seats for her group.

Practice 2A.1.1: Structures of Expressions, p. U2A-14

1. Terms: y^4 and 13.
   y^4 has a coefficient of 1, a variable of y, and a power of 4.
   13 is the constant term.
2. Terms: $8c^3$, $-c^2$, and $8c$.
   $8c^3$ has a coefficient of 8, a variable of c, and a power of 3.
   $-c^2$ has a coefficient of –1, a variable of c, and a power of 2.
   $8c$ has a coefficient of 8, a variable of c, and a power of 1.
3. Terms: $12z^5$, $9z^2$, $-z$, and –7.
   $12z^5$ has a coefficient of 12, a variable of z, and a power of 5.
   $9z^2$ has a coefficient of 9, a variable of z, and a power of 2.
   $-z$ has a coefficient of –1, a variable of z, and a power of 1.
   –7 is the constant term.
4. Terms: $-5m^{10}$, m^8, $5m^6$, and $-m^4$.
   $-5m^{10}$ has a coefficient of –5, a variable of m, and a power of 10.
   m^8 has a coefficient of 1, a variable of m, and a power of 8.
   $5m^6$ has a coefficient of 5, a variable of m, and a power of 6.
   $-m^4$ has a coefficient of –1, a variable of m, and a power of 4.
5. $f(x) = x^8 - 3x^3 + 30x$; degree: 8
6. $f(x) = -x^6 - 6x^3 + 14x^2 + 10$; degree: 6
7. $f(x) = x^7 - x^5 + 7x^3 + 2x + 22$; degree: 7
8. $(9x^2 + 12x)$ in^2
9. $(3x^2 + 5x + 8)$ in^2
10. $(3x^3 + 6x^2 + 2x + 8)$ in^2

Warm-Up 2A.1.2, p. U2A-16

1. 228 inches
2. $42 - x - x$, or $42 - 2x$
3. $72 - 2x - 2x$, or $72 - 4x$

Practice 2A.1.2: Adding and Subtracting Polynomials, p. U2A-25

1. $-x^6 + 22x^3 - 3x + 13$
2. $14y^3 + 26y^2 - 3y - 33$
3. $17x^4 - 13x^3 - 11x^2 + 16$
4. $11x^5 - 19x^4 - 13x^3 + 25x - 10$
5. $-10z^4 + 7z^3 - 3z^2 - 13z + 23$
6. $13x^6 - 12x^5 + 9x^4 + 20$
7. $-3h^3 + 5h^2 - 17h - 25$
8. $(4x + 10)$ cm
9. $(2x^2 + 8x + 2)$ cm
10. $(-12x + 54)$ cm

Warm-Up 2A.1.3, p. U2A-26

1. $6x$ feet
2. $30x$
3. \$450

Practice 2A.1.3: Multiplying Polynomials, p. U2A-36

1. $-11x^3 - 3x^2 + 77x + 21$
2. $12x^6 + 10x^5 + 22x^3 + 20x^2 - 4$
3. $-8y^5 + 78y^3 + 20y$
4. $20x^7 - 60x^6 + 30x^5 + 8x^4 - 24x^3 + 12x^2$
5. $-3x^7 + 5x^6 - x^5 + 18x^4 - 30x^3 + 6x^2$
6. $28y^9 + 7y^8 - 36y^7 - 16y^6 + 9y^4 + 8y^3 + 2y^2 - 2$
7. $35x^5 + 28x^4 - 4x^3 - 30x^2 - 23x - 6$
8. $(3x^2 + 22x - 16)$ m^2
9. $(4x^3 + 10x^2 + 4x + 10)$ m^2
10. $(12x^3 + 10x^2 - 18x - 15)$ m^2

Progress Assessment, p. U2A-37

1. a
2. c
3. b
4. a
5. d
6. b
7. a
8. b
9. c
10. a
11. a. $2(-2x + 20) + 2(12) = (-4x + 64)$ cm
    b. $(-2x + 20) \bullet (12) = (-24x + 240)$ cm^2

Lesson 2: Proving Identities

Pre-Assessment, p. U2A-39

1. a
2. c
3. b
4. a
5. b

Warm-Up 2A.2.1, p. U2A-43

1. 550 ft^2
2. 1,425 ft^2
3. 2,500 ft^2

Practice 2A.2.1: Polynomial Identities, p. U2A-58

1. $x^2 - 32x + 256$
2. $16x^2 + 168x + 441$
3. $4(5x + 1)(5x - 1)$
4. $(x + 9)(x - 9)$
5. $(3x + 7)(9x^2 - 21x + 49)$
6. $8(4x - 1)(16x^2 + 4x + 1)$
7. $x^4 + 2x^3 + 7x^2 + 6x + 9$
8. 280,900 ft^2
9. 220,900 m^2
10. 96,100 cm^2

Warm-Up 2A.2.2, p. U2A-59

1. 6,400 in^3
2. 6,912 in^3
3. 5,832 in^3

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Practice 2A.2.2: Complex Polynomial Identities, p. U2A-71

1. 401
2. 25
3. $49x^2 + 1$
4. $(x + 8i)(x - 8i)$
5. $(5x + 22i)(5x - 22i)$
6. $81x^2 + 225$
7. $36(x + 2i)(x - 2i)$
8. $\frac{1488 - 270i}{6868} = \frac{744 - 135i}{3434}$
9. $\frac{3489 - 417i}{21{,}106}$
10. $\frac{3147 - 519i}{18{,}666} = \frac{1049 - 173i}{6222}$

Warm-Up 2A.2.3, p. U2A-72

1. 65 miles per hour
2. 325 miles
3. 422.5 miles

Practice 2A.2.3: The Binomial Theorem, p. U2A-87

1. 4
2. 120
3. $-512x^3 + 768x^2 - 384x + 64$
4. $32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5$
5. $900x^2$
6. $1760x^9y^3$
7. $56xy^{13}$
8. 455
9. 3,003
10. 6,435

Progress Assessment, p. U2A-88

1. b
2. a
3. c
4. c
5. a
6. d
7. b
8. a
9. b
10. a
11. a. $21x^6y$
    b. $-189x^5y^2$
    c. $5103x^2y^5$
    d. $2187y^7$

Lesson 3: Graphing Polynomial Functions

Pre-Assessment, p. U2A-90

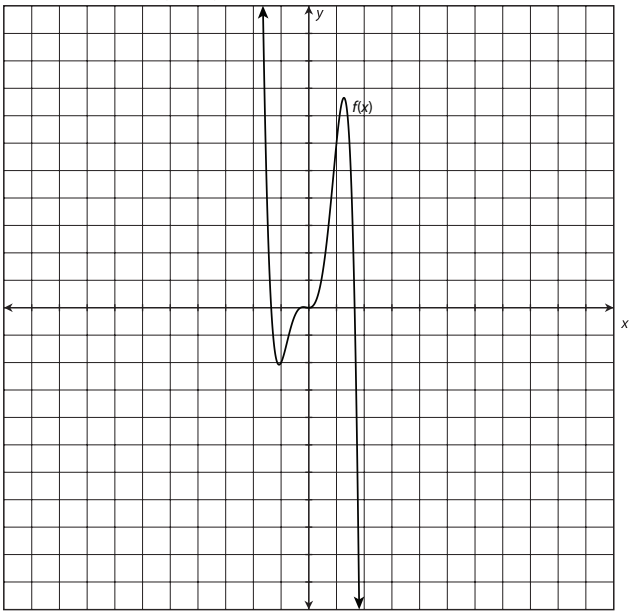
1. d
2. b
3. a
4. c
5. a

Warm-Up 2A.3.1, p. U2A-96

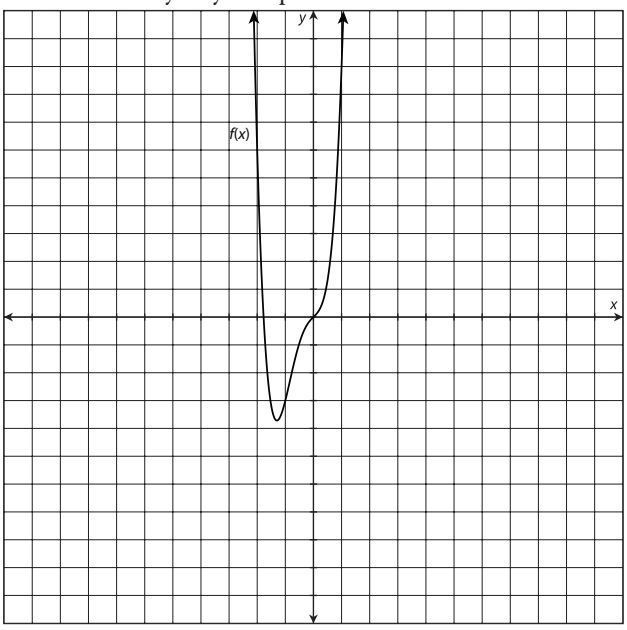
1. increasing: $t < 20$; decreasing: $t > 20$
2. (20, 22). The concentration is at its highest, 22 parts per million, 20 hours after a person takes the dose.
3. (41, 0). The concentration is 0 parts per million after 41 hours.
4. (0, 2). The concentration is 2 parts per million at the time of the dose.

Practice 2A.3.1: Describing End Behavior and Turns, p. U2A-113

1. $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$; turning points: 3; real roots: 4
2. $g(x) \to -\infty$ as $x \to -\infty$ and $g(x) \to +\infty$ as $x \to +\infty$; turning points: 2; real roots: 3
3. $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$; odd-degree; 1 real root
4. $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$; even-degree; 2 real roots
5. Answers may vary. Sample answer:



6. Answers may vary. Sample answer:



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7. (4, 375), (14, 150), (29, 550)
8. (4, 375) is a local (relative) maximum; (14, 150) is a local (relative) minimum; (29, 550) is a local (relative) maximum. The price of the stock increases until day 4, as shown by turning point (4, 375); then the price decreases until day 14, as shown by turning point (14, 150); next, the price increases until day 29, as shown by turning point (29, 550); after this point, the price decreases.
9. It appears as though $P(t) \to -\infty$ as $t \to -\infty$ and $P(t) \to -\infty$ as $t \to -\infty$, but this is difficult to tell for sure since the domain is restricted to $0 \le t \le 31$.
10. The least degree is 4; there are 3 turning points, and the degree must be 1 more than the number of turning points.

Warm-Up 2A.3.2, p. U2A-115

1. 464 boxes
2. 4 sweatshirts
3. $\frac{4}{20} = \frac{1}{5}$

Practice 2A.3.2: The Remainder Theorem, p. U2A-134

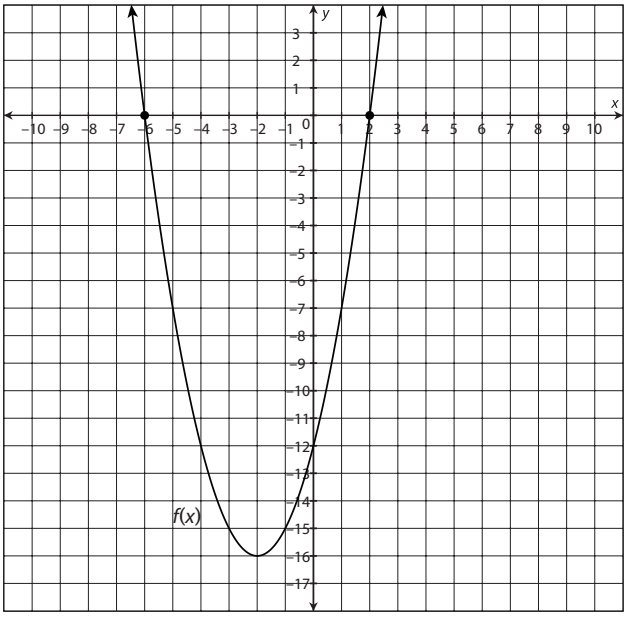
1. $x^2 - 4x - 12$
2. $2x^3 - 4x^2 + 2x + 4 - \frac{6}{x+2}$
3. $p(-4) = 34$
4. $p(2) = 52$
5. $k = -7$
6. $k = 3$
7. $(3x + 7)$ ft
8. $(7x - 6)$ m
9. $0.5t^2 + 6t$
10. $0.2t^2 + 8t$

Warm-Up 2A.3.3, p. U2A-135

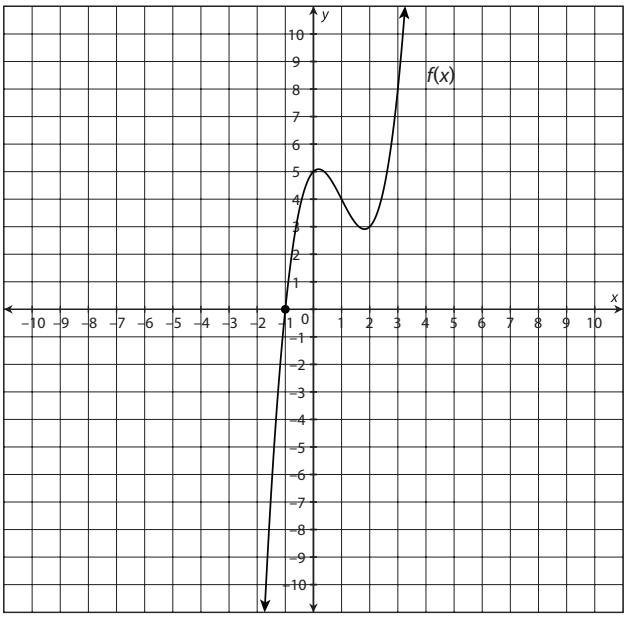
1. $(x - 31)$ feet
2. $2x^2 - 62x - 720 = 0$
3. $(x - 40)$ and $(x + 9)$
4. 180 ft^2

Practice 2A.3.3: Finding Zeros, p. U2A-152

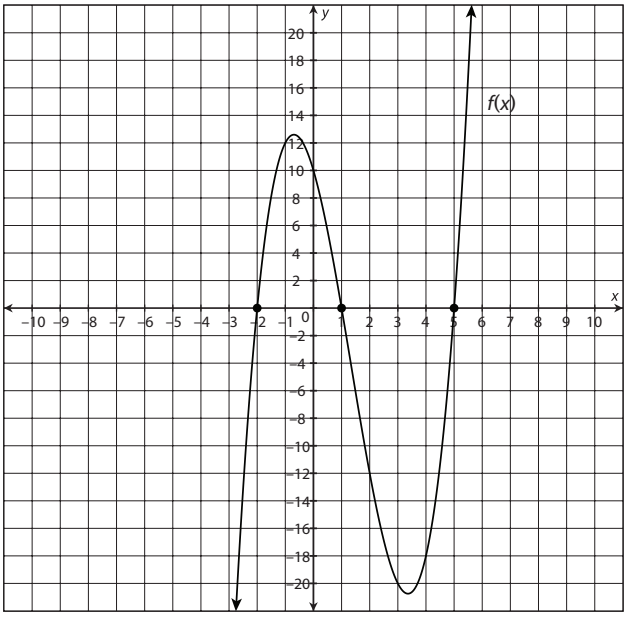
1. 3 real roots; –3, 1, 2
2. 1 real root, 2 imaginary roots; $-1, 2 - 5i, 2 + 5i$
3. 3 real roots; –2, 3
4. $x = -6, x = 2$



5. $x = -1, x = 2 + i, x = 2 - i$



6. $x = -2, x = 1, x = 5$



7. $f(x) = (x + 5)(x + 1)(x - 3)(x - 7) = x^4 - 4x^3 - 34x^2 + 76x + 105$
8. $f(x) = (x - 4)[x - (2 + 3i)][x - (2 - 3i)] = x^3 - 8x^2 + 29x - 52$
9. $f(x) = x(x + 5)(x + 29) = 33{,}488$ or $f(x) = x^3 + 34x^2 + 145x = 33{,}488$
10. 0, –29, –5; width: 23 ft; length: 28 ft; depth: 52 ft

Warm-Up 2A.3.4, p. U2A-153

1. 30 MP3 players
2. 24 shipping boxes
3. 60 MP3 players

Practice 2A.3.4: The Rational Root Theorem, p. U2A-168

1. ±1, ±5
2. ±1, ±3, ±9

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3. $\pm\frac{1}{4}, \pm\frac{1}{2}, \pm 1$

4. $\pm\frac{1}{2}, \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

5. $-\frac{7}{2}, -3, 1$

6. $-3, i\sqrt{3}, -i\sqrt{3}$

7. $5, 2-\sqrt{2}, 2+\sqrt{2}$

8. $-1, 3, \frac{1}{3}\left(1-\sqrt{2}\right), \frac{1}{3}\left(1+\sqrt{2}\right)$

9. $x^3 + 3x^2 + 4x + 12$

10. $x^3 - 10x^2 + 31x - 28$

Progress Assessment, p. U2A-169

1. b
2. c
3. a
4. b
5. d
6. d
7. b
8. b
9. c
10. d
11. a. $2 - i$
    b. $x^3 - 7x^2 + 17x - 15$
    c. 3
    d. $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$; turning points: 2

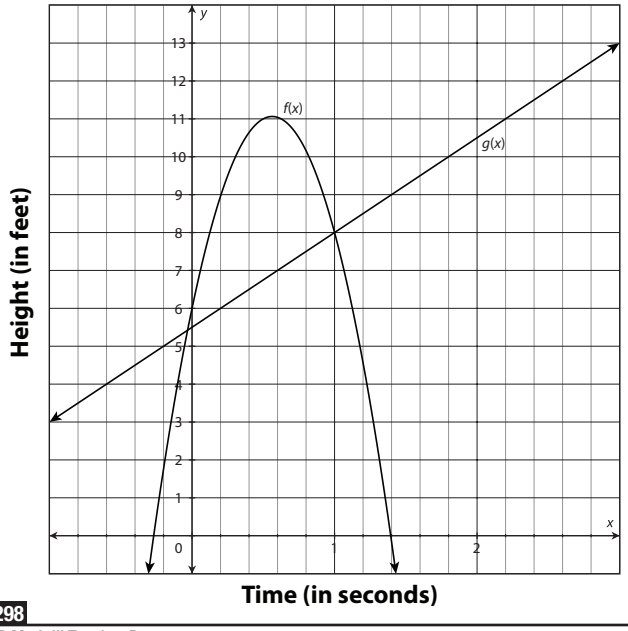
Lesson 4: Solving Systems of Equations with Polynomials

Pre-Assessment, p. U2A-173

1. b
2. c
3. a
4. d
5. b

Warm-Up 2A.4.1, p. U2A-178

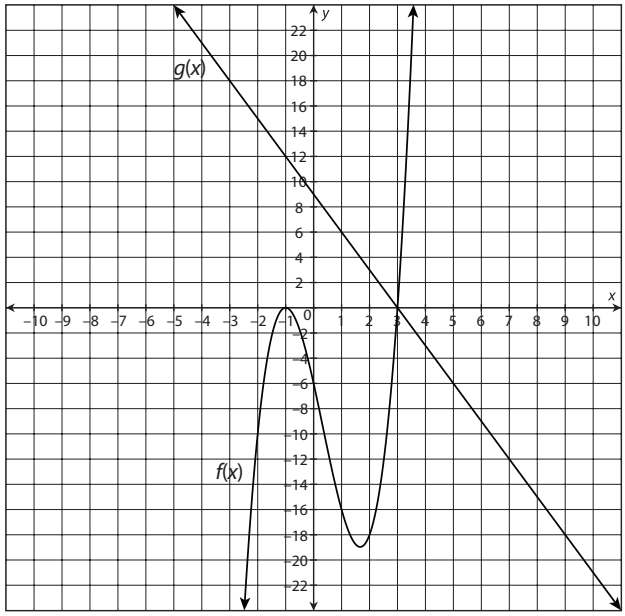
1.



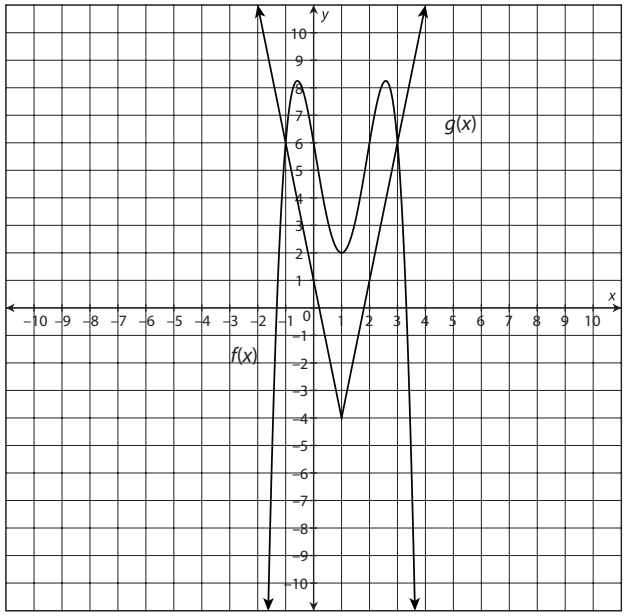
2. two points of intersection: approximately (–0.03, 5.4) and (1, 8)
3. 1 second

Practice 2A.4.1: Solving Systems of Equations Graphically, p. U2A-203

1. {(2, 0), (6, 0), (–1.2, 19)}
2. {(–3, 6), (–2, 6), (–1, 6), (1, 6), (2, 6)}
3. {(–2.7, 10), (0.2, 9), (2.3, 5)}
4. {(3, 0)}

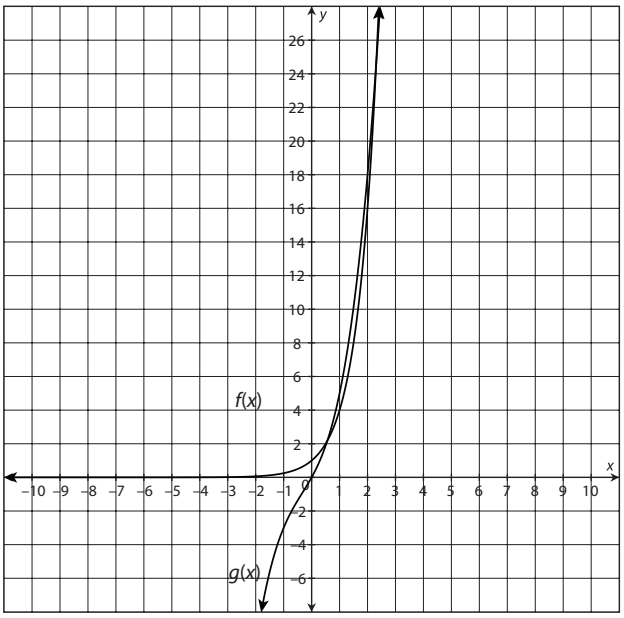


5. {(–1, 6) and (3, 6)}

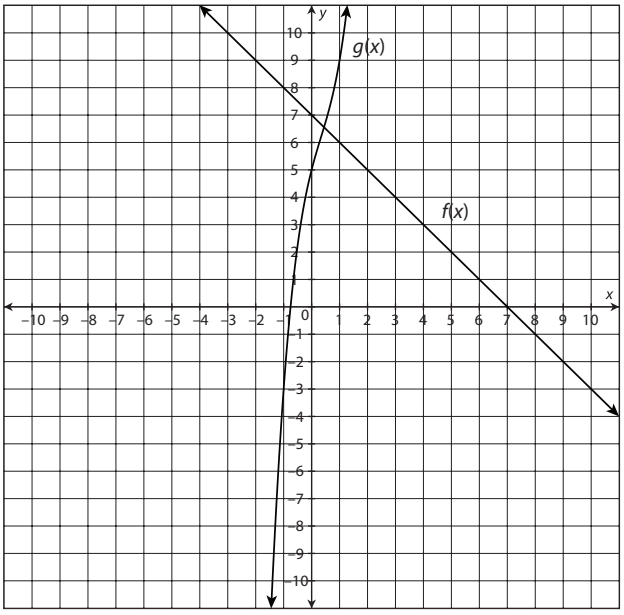


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6. {(0.5, 2.0), (2.3, 24.3)}



7. {(0.4, 6.6)}

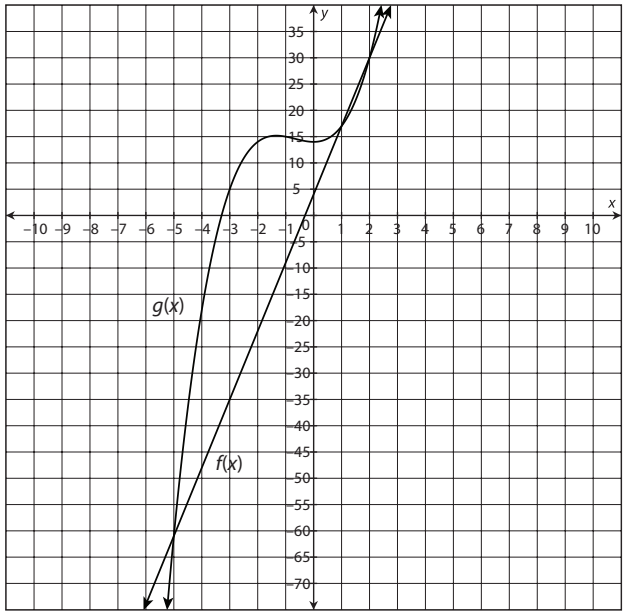


8. There appear to be two solutions: $x = 2$ and $x = 3$.
9. There appears to be one solution: $x = 0$.
10. There appears to be one solution between the x-values of 0 and 1.

Progress Assessment, p. U2A-206

1. c
2. b
3. c
4. a
5. b
6. a
7. c
8. b
9. a
10. b

11. a. $f(1) = 13(1) + 4 = 17$; $g(1) = (1)^3 + 2(1)^2 + 14 = 17$
    b.



    c. (–5, –61) and (2, 30)
    d. The points of intersection represent the solutions to the defined system of equations.

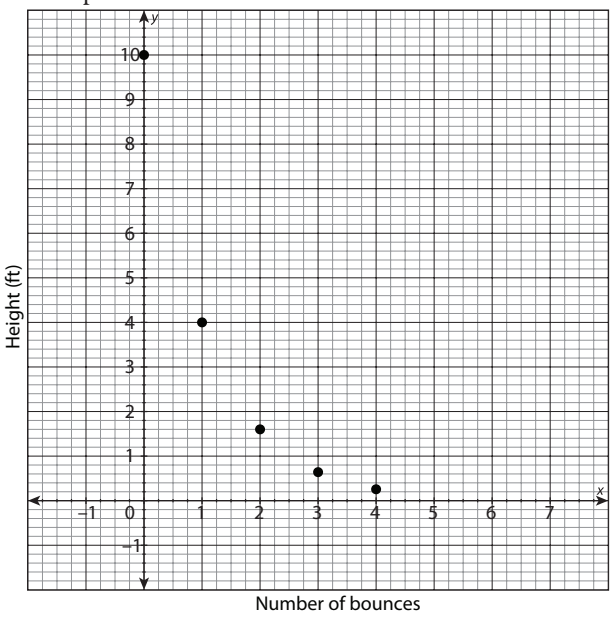
Lesson 5: Geometric Series

Pre-Assessment, p. U2A-212

1. d
2. c
3. a
4. b
5. d

Warm-Up 2A.5.1, p. U2A-217

1. 4 ft
2. 1.6 ft
3. 0.64 ft
4. 0.256 ft
5. Graph:



6. Yes, the graph shows that the bounce height decreases by 40% with each successive bounce.

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Practice 2A.5.1: Geometric Sequences, p. U2A-237

1. 12, 48, 192; $r = 4$
2. 6, 3, 1.5; $r = 0.5$
3. $a_{10} = 0.\overline{4}$
4. $a_{10} = 16$
5. $a_7 = 0.\overline{148}$
6. $a_n = 8 \bullet (5/2)^{n-1}$; $a_8 = 4882.8125$
7. $a_n = (96/27) \bullet (3/4)^{n-1}$; $a_6 = 0.84375$
8. 0.7890481 ft; $a_n = 10 \bullet 0.53^{n-1}$
9. about 82.13°F, $a_n = 212 \bullet 0.9^{n-1}$
10. \$31,836.24, \$32,472.96, \$33,784.87; $a_n = 30{,}000 \bullet 1.02^{n-1}$

Warm-Up 2A.5.2, p. U2A-239

1. $a_n = 0.5 \bullet (1.2)^{n-1}$
2. about 1 mile
3. about 14 weeks

Practice 2A.5.2: Sum of a Finite Geometric Series, p. U2A-265

1. $24 + 6 + 1.5 + 0.375 + 0.09375$
2. 312
3. 17.997
4. 457.115
5. 3,906 people
6. 341 people
7. 204,750 butterflies
8. 126 births
9. about \$409.68
10. about \$1,167.43

Warm-Up 2A.5.3, p. U2A-267

1. $a_n = 0.5 \bullet 1.2^{n-1}$
2. $S_6 = 3 + 0.6 + 0.12 + 0.0024 + 0.0048 + 0.00096$
3. 3.749 inches
4. It's probably physically impossible for Mario to drive the stake into the ground the required 4 inches at this rate. The terms in the series are growing geometrically so much smaller that the amount of progress per swing is approaching 0. It could take dozens or perhaps hundreds of additional swings to even get close to a depth of 4 inches, and Mario would tire out before then.

Practice 2A.5.3: Sum of an Infinite Geometric Series, p. U2A-285

1. 1.5
2. 25/12
3. No sum exists.
4. 37/99
5. 7/9
6. 123/999
7. $\sum_{k=1}^{\infty} 5(0.995)^{k-1}$
8. 1,000
9. $\sum_{k=1}^{\infty} \frac{9\sqrt{3}}{4}\left(\frac{4}{9}\right)^{k-1}$
10. $\frac{81\sqrt{3}}{20}$ cm^2

Progress Assessment, p. U2A-287

1. a
2. c
3. a
4. a
5. a
6. b
7. b
8. c
9. d
10. d
11. a. $i = \frac{0.045}{12} = 0.00375$
    b. $r = \frac{1}{1+i} \approx 0.996$
    c. $n = 12 \bullet 30 = 360$ months
    d. approximately \$1,361.69

Unit Assessment
p. U2A-290

1. d
2. c
3. a
4. d
5. a
6. b
7. b
8. d
9. c
10. a
11. d
12. b
13. a. $8x + 22$ in
    b. $8x^2 + 48x + 54$ in^2
14. a. $3 - i$
    b. $x^3 - 10x^2 + 34x - 40$
    c. 3
    d. $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$; two turning points
15. a. $i = 0.0025$
    b. $r = \frac{1}{1+i} \approx 0.998$
    c. 240 months
    d. approximately \$1,006.52

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Student Book Answer Key

Lesson 1: Polynomial Structures and Operating with Polynomials

Practice 2A.1.1: Structures of Expressions, p. U2A-8

1. Terms: $-k$ and -1.
   $-k$ has a coefficient of -1, a variable of k, and a power of 1.
   -1 is the constant term.
2. Terms: $2p^3$, p^2, and 30.
   $2p^3$ has a coefficient of 2, a variable of p, and a power of 3.
   p^2 has a coefficient of 1, a variable of p, and a power of 2.
   30 is the constant term.
3. Terms: $-4b^4$, $3b^3$, $2b^2$, and 1.
   $-4b^4$ has a coefficient of -4, a variable of b, and a power of 4.
   $3b^3$ has a coefficient of 3, a variable of b, and a power of 3.
   $2b^2$ has a coefficient of 2, a variable of b, and a power of 2.
   1 is the constant term.
4. Terms: $8x^{12}$, $-7x^2$, $6x$, and 2.
   $8x^{12}$ has a coefficient of 8, a variable of x, and a power of 12.
   $-7x^2$ has a coefficient of -7, a variable of x, and a power of 2.
   $6x$ has a coefficient of 6, a variable of x, and a power of 1.
   2 is the constant term.
5. $f(x) = -3x^5 - 6x^3 - 5x^2$; degree: 5
6. $f(x) = x^4 - x^2 + 15x + 9$; degree: 4
7. $f(x) = -2x^6 - x^5 + 10x^3 + 4x + 20$; degree: 6
8. $(10x + 13)$ in^2
9. $(5x^2 + 14x + 30)$ in^2
10. $(2x^3 + 25x^2 + 12x + 50)$ in^2

Practice 2A.1.2: Adding and Subtracting Polynomials, p. U2A-14

1. $13x^6 + 7x^2 + 2x + 30$
2. $14x^5 + x^4 + 7x^3 + 12x$
3. $16y^4 + 7y^3 + 14y^2 + 8y - 1$
4. $-9x^5 - 10x^4 + 5x^2 + 5x - 18$
5. $7z^2 - 2z + 13$
6. $9x^6 - 20x^3 - 6x^2 - x + 25$
7. $-11x^6 + 27x^4 + 19x^3 - 6x^2 + 7$
8. $(8x - 20)$ cm
9. $(-2x^2 + 2x + 72)$ cm
10. $(18x - 54)$ cm

Practice 2A.1.3: Multiplying Polynomials, p. U2A-20

1. $5x^5 - 2x^3 + 20x^2 - 8$
2. $-3y^7 - y^5 + 30y^4 + 10y^2$
3. $-24z^5 + 12z^4 - 12z^3 + 10z^2 - 2z$
4. $-7x^6 + 37x^5 - 18x^4 + 40x^3$
5. $y^{10} - 4y^7 + 2y^5 + 4y^2 - 3$
6. $6x^5 - 12x^4 + 13x^3 - 3x^2 - x + 5$
7. $8x^7 - 3x^6 + 4x^4 - 45x^3 - 18x^2 - 4x - 24$
8. $(x^2 + 10x - 56)$ m^2
9. $(4x^3 - 2x^2 + 4x - 2)$ m^2
10. $(24x^3 + 21x^2 - 80x - 70)$ m^2

Lesson 2: Proving Identities

Practice 2A.2.1: Polynomial Identities, p. U2A-32

1. $4x^2 - 56x + 196$
2. $x^2 + 30x + 225$
3. $(x - 19)(x + 19)$
4. $(11x + 8)(11x - 8)$
5. $(7x - 1)(49x^2 + 7x + 1)$
6. $3(-2x + 1)(12x^2 + 6x + 3)$
7. $25x^4 - 36x^2 - 20x^3 + 16x + 16$
8. 36,100 ft^2
9. 115,600 in^2
10. 422,500 cm^2

Practice 2A.2.2: Complex Polynomial Identities, p. U2A-37

1. 362
2. 89
3. $x^2 + 256$
4. $(15x + 4i)(15x - 4i)$
5. $(10x + 11i)(10x - 11i)$
6. $4x^2 + 196$
7. $(3x + 13i)(3x - 13i)$
8. $\frac{3264 - 308i}{18{,}532} = \frac{816 - 77i}{4633}$
9. $\frac{2142 - 506i}{10{,}600} = \frac{1071 - 253i}{5300}$
10. $\frac{2152 - 334i}{9620} = \frac{1076 - 167i}{4810}$

Practice 2A.2.3: The Binomial Theorem, p. U2A-47

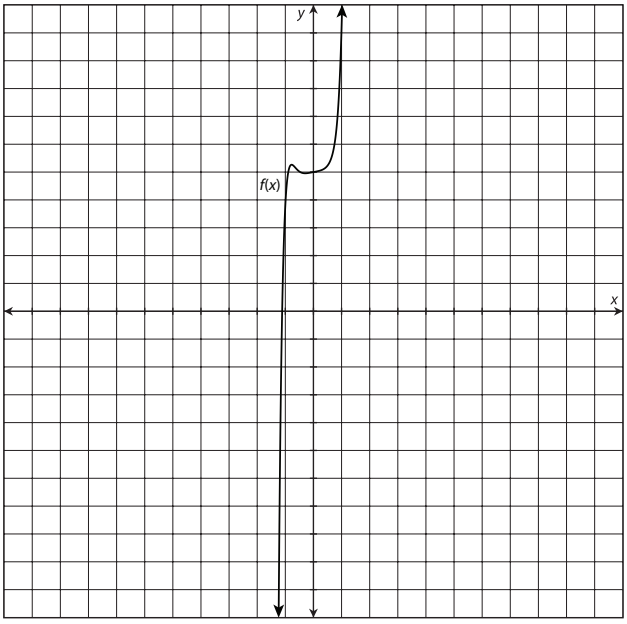
1. 3
2. 252
3. $-x^3 + 12x^2 - 48x + 64$
4. $16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$
5. $589{,}824x^7$
6. $-24x^{11}y$
7. $-2912x^3y^{11}$
8. 715
9. 1,716
10. 286

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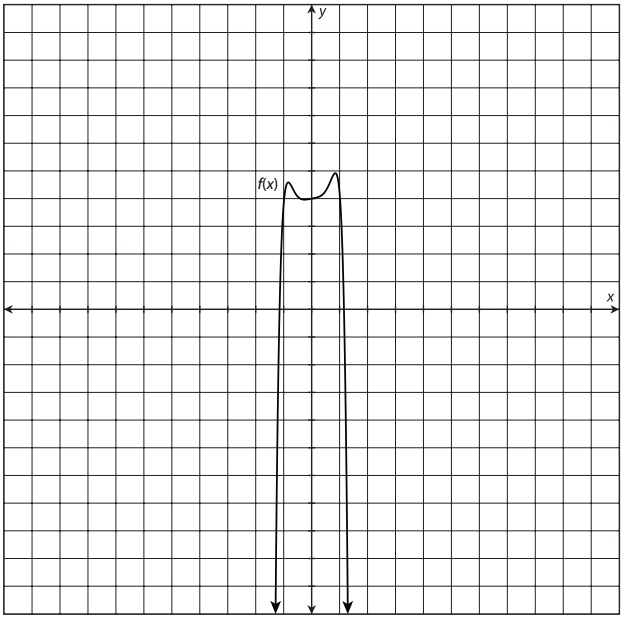
Lesson 3: Graphing Polynomial Functions

Practice 2A.3.1: Describing End Behavior and Turns, p. U2A-64

1. $g(x) \to +\infty$ as $x \to -\infty$ and $g(x) \to -\infty$ as $x \to +\infty$; turning points: 6; real roots: 7
2. $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$; turning points: 7; real roots: 8
3. $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$; even-degree; 2 real roots
4. $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$; odd-degree; 5 real roots
5. Answers may vary. Sample answer:



6. Answers may vary. Sample answer:



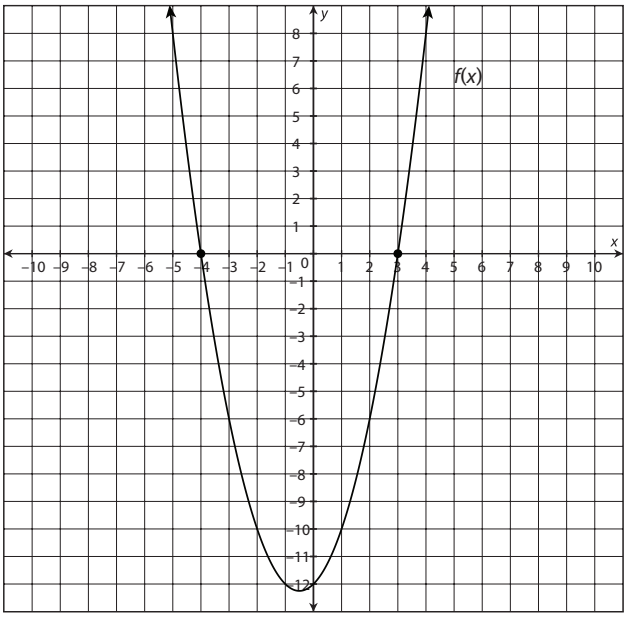
7. (50, 220), (70, 125)
8. (0, 0) is a local (relative) maximum; (50, 220) is a local (relative) maximum; (70, 125) is a local (relative) minimum. The volume increases until 1950, as shown by turning point (50, 220); then the volume decreases until 1970, as shown by turning point (70, 125); next, the volume increases until 2010, as shown by the point (110, 400).
9. It appears as though $V(t) \to +\infty$ as $t \to +\infty$ and $V(t) \to +\infty$ as $t \to -\infty$, but this is difficult to tell for sure since the domain is restricted to $0 \le t \le 110$.
10. The least degree is 3; there are 2 turning points, and the degree must be 1 more than the number of turning points.

Practice 2A.3.2: The Remainder Theorem, p. U2A-78

1. $x + 10 + \dfrac{30}{x-2}$
2. $3x^4 - 3x^3 - x^2 + x + 7$
3. $p(-2) = 4$
4. $p(1) = 20$
5. $k = -6$
6. $k = 8$
7. $(4x - 6)$ in
8. $(5x + 9)$ cm
9. $0.5t^2 + 8t$
10. $0.2t^2 + 8t$

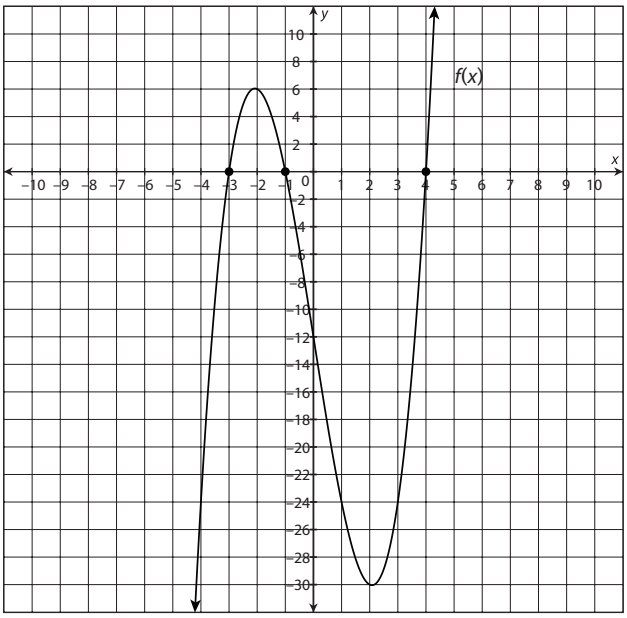
Practice 2A.3.3: Finding Zeros, p. U2A-88

1. 3 real roots; –4, –2, 2
2. 3 real roots; –1, 3
3. 1 real root, 2 imaginary roots; 2, $1 - 2i$, $1 + 2i$
4. $x = -4$, $x = 3$

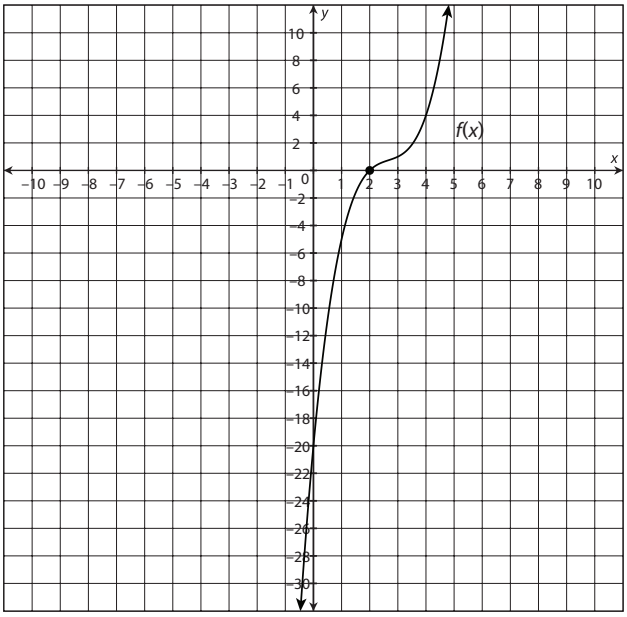


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5. $x = -3, x = -1, x = 4$



6. $x = 2, x = 3 + i, x = 3 - i$



7. $f(x) = (x + 3)(x + 2)(x - 1)(x - 5) = x^4 - x^3 - 19x^2 - 11x + 30$
8. $f(x) = (x + 2)[x - (5 - 2i)][x - (5 + 2i)] = x^3 - 8x^2 + 9x + 58$
9. $f(x) = (x + 8)(x - 4)x = 4608$ or $f(x) = x^3 + 4x^2 - 32x = 4608$
10. –8, 0, 4; length = 24 in; width = 12 in; height = 16 in

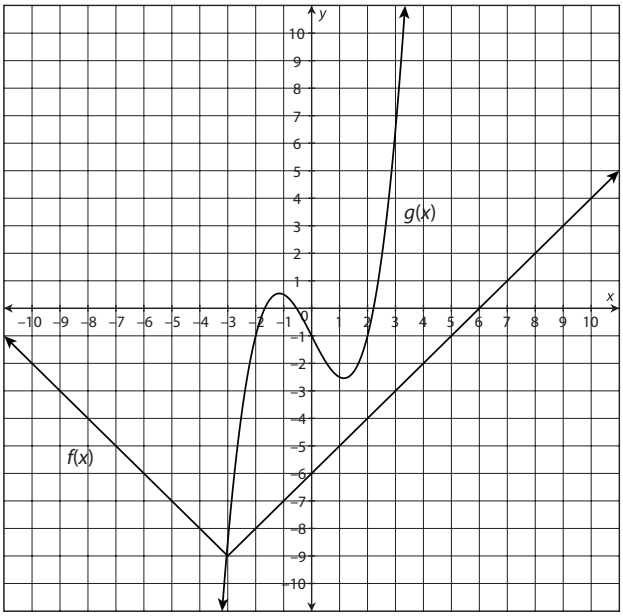
Practice 2A.3.4: The Rational Root Theorem, p. U2A-95

1. $\pm 1, \pm 7$
2. $\pm 1, \pm 2, \pm 4, \pm 8$
3. $\pm\frac{1}{6}, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm 1$
4. $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$
5. $-2, -1, \frac{5}{2}$
6. $3, -i\sqrt{2}, i\sqrt{2}$
7. $3, 1-\sqrt{3}, 1+\sqrt{3}$
8. $-2, 1, \frac{1}{2}-\frac{1}{\sqrt{2}}, \frac{1}{2}+\frac{1}{\sqrt{2}}$
9. $x^3 - 5x^2 - 3x + 15$
10. $x^3 - 6x^2 + x + 34$

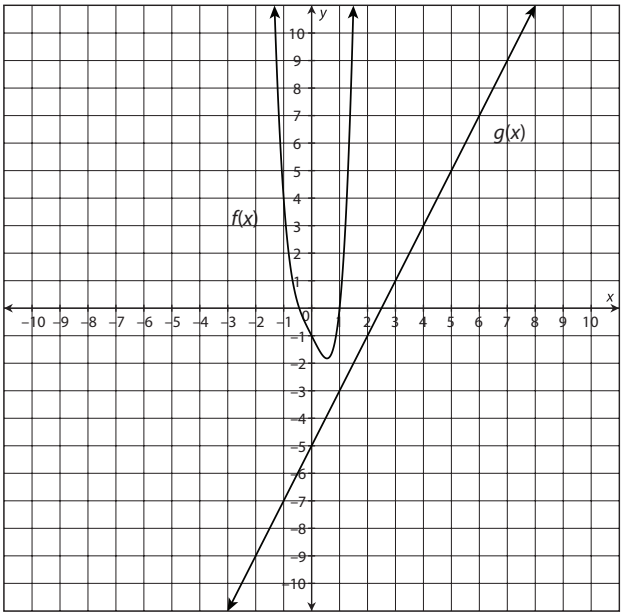
Lesson 4: Solving Systems of Equations with Polynomials

Practice 2A.4.1: Solving Systems of Equations Graphically, p. U2A-114

1. {(–3.6, –35), (2, 0)}
2. {(2, 9)}
3. {(–3, 5), (0, –2)}
4. {(–3, –9)}

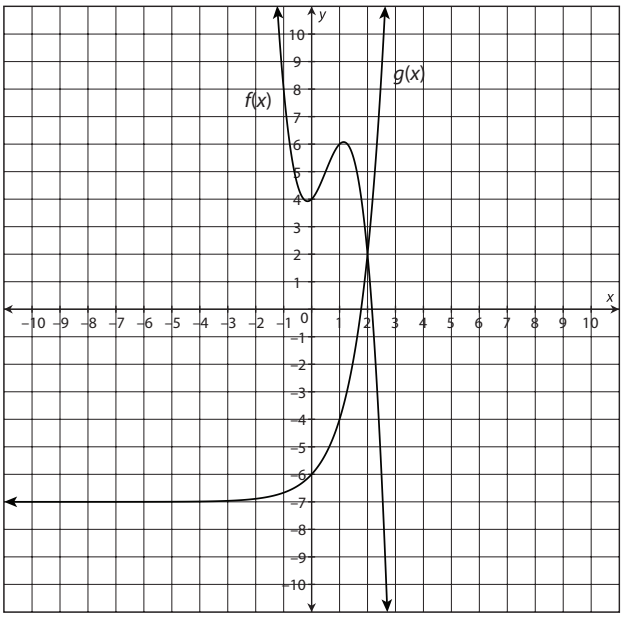


5. There are no real solutions.

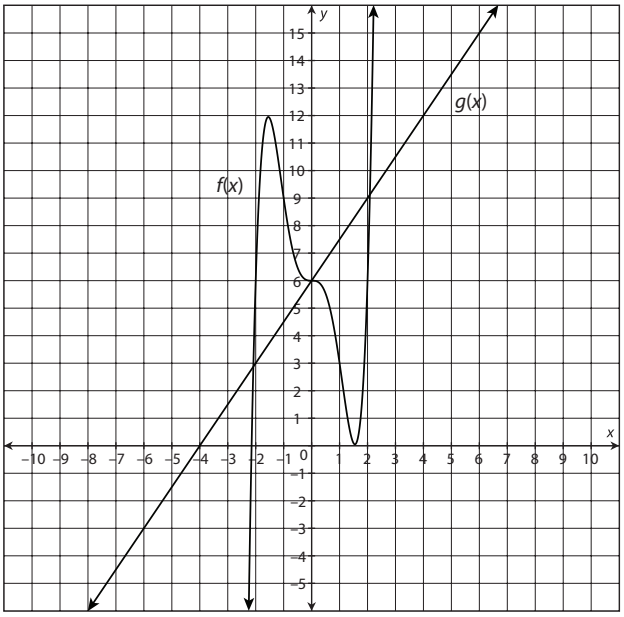


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6. {(2, 2)}



7. {(–2.1, 2.9), (0, 6), (2.1, 9.1)}



8. There appears to be one solution: $x = 1$.
9. There appear to be three solutions: between the x-values of –3 and –2, between the x-values of –1 and 0, and when $x = 3$.
10. There appear to be three solutions: when $x = -1$, between the x-values of –1 and 0, and between the x-values of 3 and 4.

Lesson 5: Geometric Series

Practice 2A.5.1: Geometric Sequences, p. U2A-134

1. 22, 11, 5.5; $r = 0.5$
2. 189, 567, 1,701; $r = 3$
3. $a_9 = 35.84$
4. $a_7 = 0.0625$
5. $a_5 = 500/27$
6. $a_n = (10/9) \bullet (3/5)^{n-1}$; $a_7 = 0.864$
7. $a_n = 6 \bullet (3/2)^{n-1}$; $a_6 = 68.34375$
8. 6,400 cells; $a_n = 2 \bullet 50^{n-1}$
9. 0.469 mg; $a_n = 120 \bullet \left(\frac{1}{2}\right)^{4n} = 120 \bullet \left(\frac{1}{16}\right)^{n}$
10. 15 swings

Practice 2A.5.2: Sum of a Finite Geometric Series, p. U2A-150

1. $48 + 24 + 12 + 6 + 4 + 1.5 + 0.75$
2. $86.\overline{7}$
3. 2.39969136
4. 939.834196
5. 318.75 mg
6. $243,644.62
7. It would be better for Joaquin to take the first payment option his mother offered, because that option would pay out $163.83, which is more than a flat fee of $100.
8. 1,364 people
9. about $127.90
10. about $1,346.59

Practice 2A.5.3: Sum of an Infinite Geometric Series, p. U2A-163

1. 3/8
2. 20/3
3. No sum exists.
4. 39/80
5. 27
6. No sum exists.
7. $\sum_{k=1}^{\infty} 4\left(\frac{32}{100}\right)^{k-1}$
8. The sum of the series is $100/17 \approx 5.88$ inches. Randall will not be able to drive the stake into the ground far enough because the amount of progress he makes with each new strike is a small fraction of the progress he made with the previous strike. Adding up the depth of each strike, the total depth of the stake will never reach 6 inches.
9. $\sum_{k=1}^{\infty} \frac{\left[9\left(\frac{1}{2}\right)^{n-1}\right]^2 \sqrt{3}}{4}$
10. $27\sqrt{3}$

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Instruction

Goal: To guide students to facility with evaluating, factoring, and graphing polynomials, both by hand and with technology

Common Core State Standards

N–CN.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

A–SSE.2 Use the structure of an expression to identify ways to rewrite it.

A–APR.2 Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x – a$ is $p(a)$, so $p(a) = 0$ if and only if $(x – a)$ is a factor of $p(x)$.

A–APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

F–IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Student Activities Overview and Answer Key

Station 1

Working with groups, students use the Rational Root Theorem, the Fundamental Theorem of Algebra, and the Remainder Theorem to evaluate polynomials and find their factors.

Answers

1. a. 5
   b. no
   c. $\pm\frac{1,7}{1,3}$
   d. –5
2. a. 4
   b. $\pm\frac{1}{1,3}$
   c. no
   d. no
   e. no
3. a. 3
   b. $\pm\frac{1,2,5,10}{1,2}$
   c. no
   d. –290
4. a. 6
   b. ±1
   c. –2
   d. 6

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Instruction

Station 2

Working with groups, students use technology to evaluate polynomials.

Answers

1. –12
2. 59,452
3. –1,354
4. 8
5. 40,905
6. 55,017
7. 36

Station 3

Working with groups, students work to factor polynomial expressions and functions, including polynomials with radical and complex roots.

Answers

1. a. 7
   b. $g(1) = 6$
   c. $g(-1) = 2$
2. a. 6
   b. $f(6) = 131{,}509$
   c. $f(1) = 3$
3. $(x - 9)(x + 6)(x - 2)(x + 1)$
4. $(x - 2)(x + 2)(x + 3)(x - 1)(x - 4)$
5. $\left(x - \sqrt{2}\right)\left(x - 4\right)(3x + 2)$
6. $-3\sqrt{5}, -\frac{1}{4}, 2$
7. $1 + 2i, 3$
8. $2 - 2i, 2, -\frac{1}{2}$

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Instruction

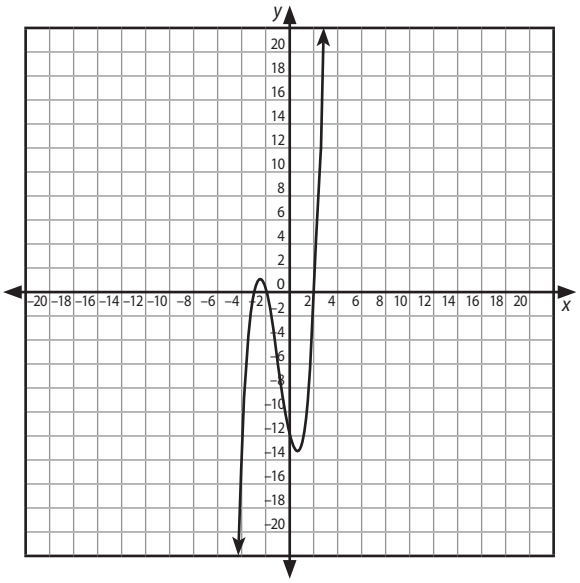
Station 4

Working with groups, students factor polynomial functions, find the zeros, and use technology to graph the functions. Students also use the Rational Root Theorem to find possible roots.

Answers

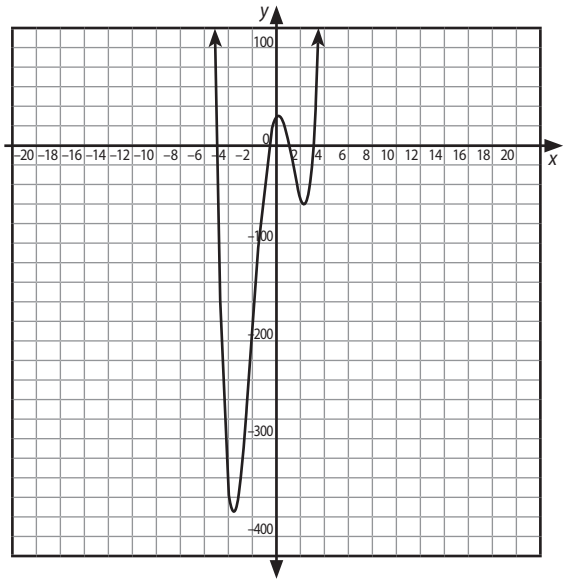
1. a. $f(x) = (x - 2)(x + 3)(x + 2)$; zeros are –3, –2, and 2

   b.



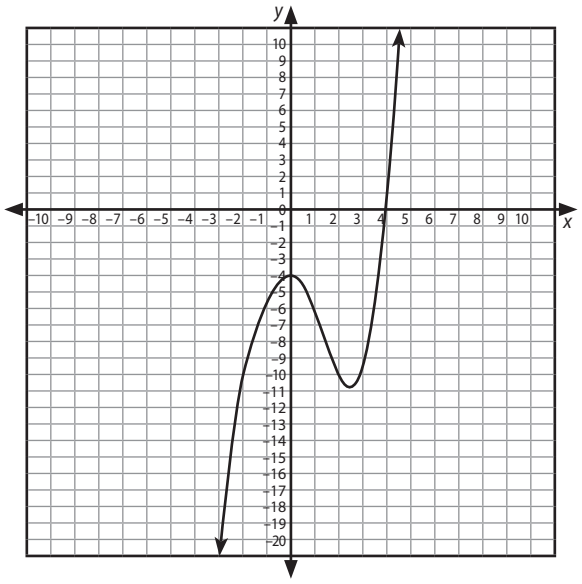
2. a. $f(x) = (x + 5)(x - 1)(3x + 2)(x - 3)$; zeros are –5, $-\frac{2}{3}$, 1, and 3

   b.

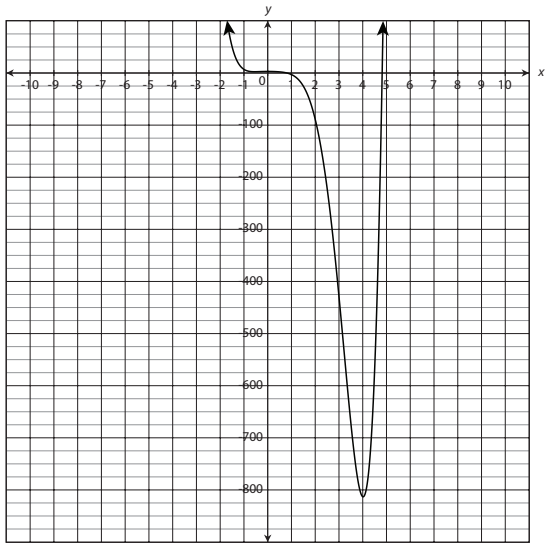


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3. a. $(x + i)(x - i)(x - 4)$, zeros are $-i$, i, and 4

   b. 

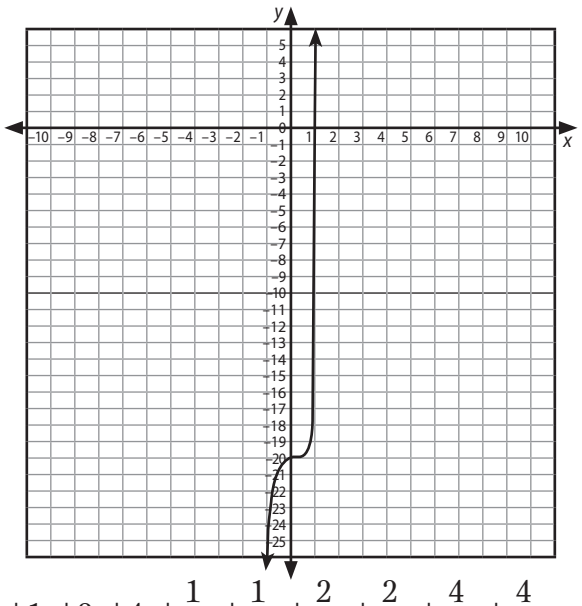
4. a. ±1, ±3; there are no rational zeros for this function, only irrational and complex.

   b. 

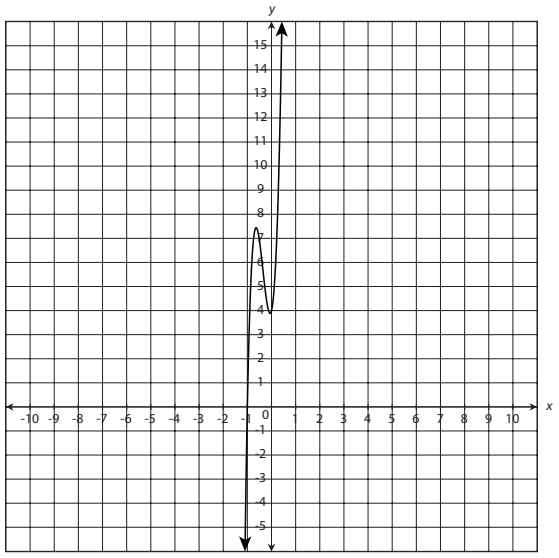
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Instruction

5. a. $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20, \pm \frac{1}{2}, \pm \frac{5}{2}, \pm \frac{1}{5}, \pm \frac{2}{5}, \pm \frac{4}{5}, \pm \frac{1}{10}$; $x = 1$. The other zeros are complex.

b. 

6. a. $\pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{1}{9}, \pm \frac{2}{3}, \pm \frac{2}{9}, \pm \frac{4}{3}, \pm \frac{4}{9}$; $x = -1, \pm \frac{1}{3}i, \pm 2i$

b. 

Materials List/Setup

Station 1 calculator

Station 2 calculator

Station 3 calculator

Station 4 graphing calculator

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. What is the degree of a polynomial?
2. What is the Fundamental Theorem of Algebra?
3. What is a complex conjugate? What is a radical conjugate?
4. How are conjugates useful when you are factoring?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. The degree of a polynomial is the highest power to which x is raised.
2. Any polynomial of degree x^n will have n roots.
3. If you have a complex number in the form $a + bi$, the complex conjugate is $a - bi$. If you have a radical in the form $a + b\sqrt{c}$, the radical conjugate is $a - b\sqrt{c}$.
4. Complex conjugates allow you to treat $x^2 - n$ as the difference of two squares, even if n is not a perfect square. Additionally, if any polynomial equation has a complex or radical root, the root's conjugate will also be a root.

Possible Misunderstandings/Mistakes

- Incorrectly calculating the value of functions
- Incorrectly manipulating exponents
- Incorrectly factoring
- Not understanding the Rational Root Theorem
- Not extending the Rational Root Theorem far enough (i.e., not listing all possible factors of the constant and coefficient)

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- Not recognizing redundant factors in the list of possibilities from the Rational Root Theorem
- Incorrectly using graphing technology
- Attempting to find zeros from the polynomial's graph
- Not understanding or misusing polynomial long division and synthetic long division
- Not understanding the Remainder Theorem
- Making arithmetical errors

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Station 1

Work with your group to answer the questions about each polynomial function.

1. $f(x) = 3x^5 + x^3 - 2x^2 + 6x + 7$
   a. How many zeros does this function have?
   b. Is 3 a zero of this function?
   c. What are all possible rational zeros?
   d. What is $f(-1)$?

2. $f(x) = 3x^4 + 2x + 1$
   a. How many zeros does this function have?
   b. What are all possible rational zeros?
   c. Is $\frac{1}{3}$ a zero of this function?
   d. Is $-\frac{1}{3}$ a zero of this function?
   e. Is –1 a zero of this function?

3. $g(x) = -2x^3 - 3x^2 + 5x + 10$
   a. How many zeros does this function have?
   b. What are all possible rational zeros?
   c. Is 1 a zero of this function?
   d. What is $g(5)$?

4. $g(x) = x^6 + 4x^5 + 2x^4 - 1$
   a. How many zeros does this function have?
   b. What are all possible rational zeros?
   c. What is $g(-1)$?
   d. What is $g(1)$?

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Station 2

Work with your group to evaluate each polynomial function for the given value. Use a calculator to compute values.

1. $g(x) = -2x^3 + 2x^2 - 7x + 10$

   $g(2) =$

2. $f(x) = 4x^7 - 6x^5 + x^4 - 3x^3 - 4$

   $f(4) =$

3. $f(x) = -2x^8 - 7x^7 + x^6 - 2x^5 + x^4 + 5x^3 - x$

   $f(2) =$

4. $g(x) = x^4 - 5x + 2$

   $g(2) =$

5. $f(x) = 3x^6 - 2x^5 + x^4 - 2x^3 - 4x^2 + 5$

   $f(5) =$

6. $g(x) = 18x^5 - 10x^3 + 2x^2 - 6x - 3$

   $g(5) =$

7. $h(x) = 4x^4 - 3x^3 + 2x^2 - x - 10$

   $h(2) =$

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Station 3

Work with your group to answer the questions about each polynomial function. Show all your work.

1. $g(x) = 4x^7 - 2x + 4$
   a. How many zeros does the function have?
   b. What is $g(1)$?
   c. What is $g(-1)$?

2. $f(x) = 2x^6 + 5x^5 - 3x^3 - x^2 - x + 1$
   a. How many zeros does the function have?
   b. What is $f(6)$?
   c. What is $f(1)$?

Factor the following polynomials. Use any method.

3. $x^4 - 4x^3 - 53x^2 + 60x + 108$

4. $x^5 - 2x^4 - 15x^3 + 20x^2 + 44x - 48$

5. If $\left(x + \sqrt{2}\right)$ is a factor of $3x^4 - 10x^3 - 14x^2 + 20x + 16$, what are the other factors?

6. If $3\sqrt{5}$ is a zero of $f(x) = 4x^4 - 7x^3 - 182x^2 + 315x + 90$, what are the other zeros?

7. If $(1 - 2i)$ is a zero of $f(x) = x^3 - 5x^2 - 11x - 15$, what are the other zeros?

8. If $(2 + 2i)$ is a zero of $f(x) = 2x^4 - 11x^3 + 26x^2 - 16x - 16$, what are the other zeros?

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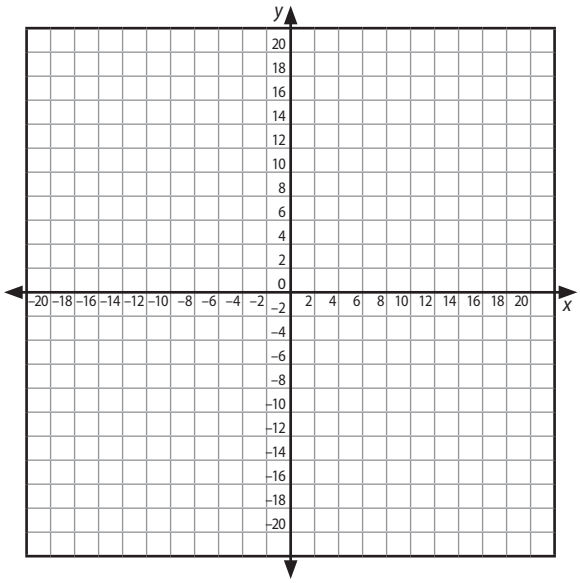
Station 4

Work with your group to answer the questions about each polynomial function. Then use a graphing calculator to find the graph of the function. Sketch the graphs.

1. $f(x) = x^3 + 3x^2 - 4x - 12$

   a. Factor the polynomial to find the zeros of the function.

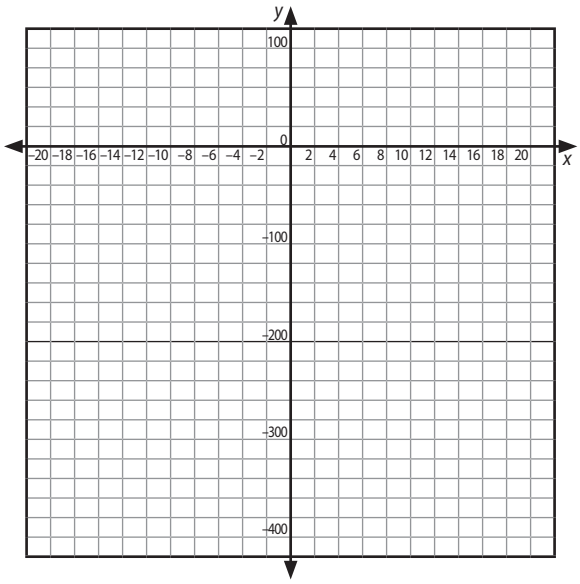
   b.



2. $f(x) = 3x^4 + 5x^3 - 49x^2 + 11x + 30$

   a. Factor the polynomial to find the zeros of the function.

   b.



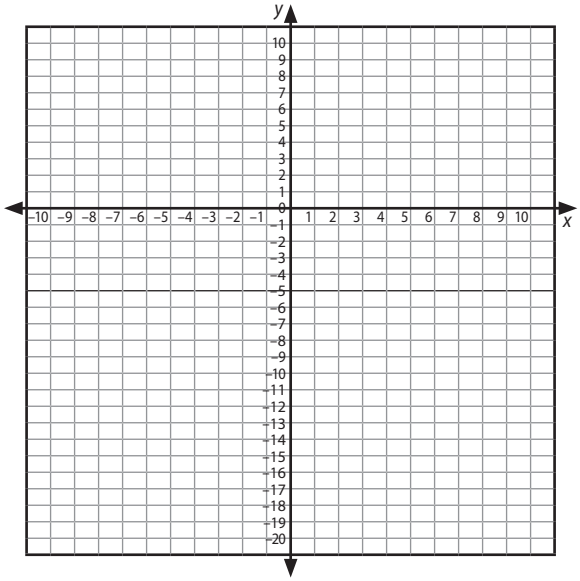
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3. $f(x) = x^3 - 4x^2 + x - 4$

   a. Factor the polynomial to find the zeros of the function given that one of the zeros is $-i$.

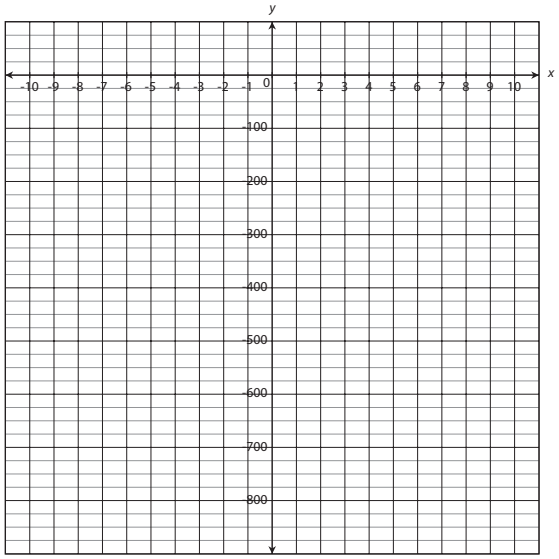
   b.



4. $f(x) = x^6 - 5x^5 + x^4 - 3x^2 + 3$

   a. State all possible rational zeros and find all rational zeros.

   b.



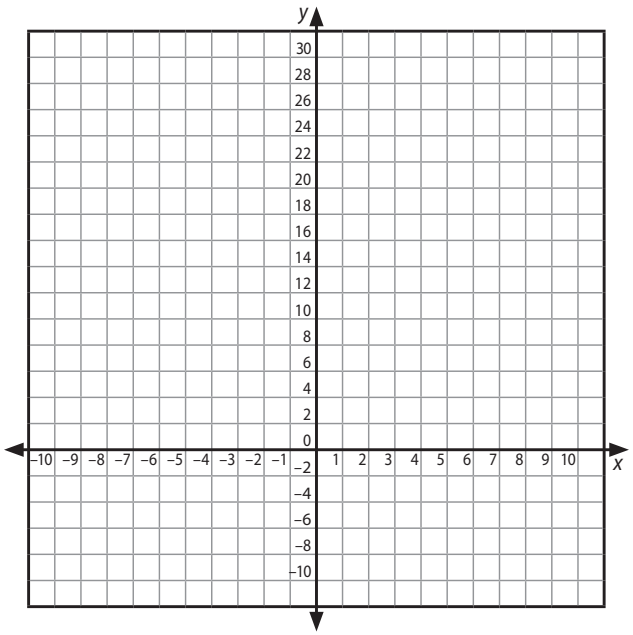
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5. $f(x) = 10x^5 + 19x^3 - 10x^2 + x - 20$

   a. State all possible rational zeros and find all rational zeros.

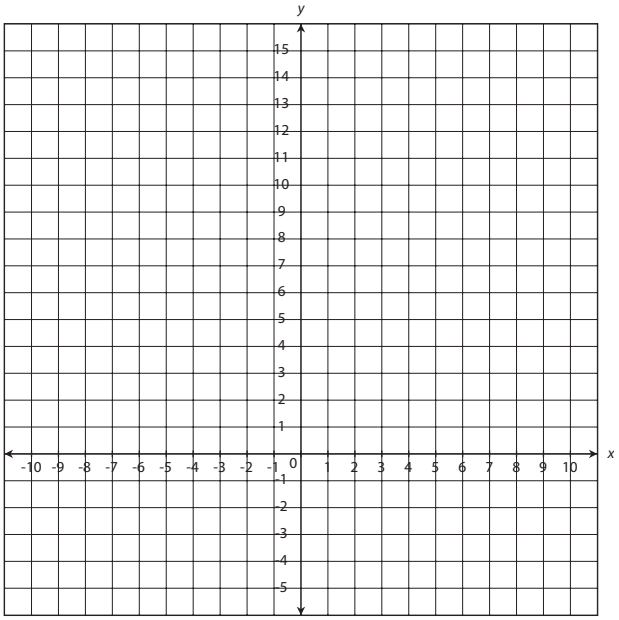
   b.



6. $f(x) = 9x^5 + 9x^4 + 37x^3 + 37x^2 + 4x + 4$

   a. State all possible rational zeros and find ALL zeros. (*Hint*: Use the quadratic formula after finding a rational zero.)

   b.



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Instruction

Goal: To provide opportunities for students to apply their knowledge of sequences in real-world situations

Common Core State Standard

A–SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★

Student Activities Overview and Answer Key

Station 1

Working in pairs, students will practice reading summation notation and writing out the terms of a series.

Answers

1. $3(1) + 3(2) + 3(3) + \cdots 3(n)$
2. $4^1 + 4^2 + 4^3 + \cdots + 4^n$
3. $(a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + \cdots + (a_1 + nd)$
4. $Ar + Ar^2 + Ar^3 + \ldots$
5. $6(3 - 1) + 6(4 - 1) + 6(5 - 1) + \cdots + 6(8 - 1)$
6. $(2^2 + 2^2) + (3^2 + 2^3) + (4^2 + 2^4) + (5^2 + 2^5)$

Station 2

Working in pairs, students develop a formula to find the partial value of a geometric series; that is, when to find the sum of a few consecutive terms but not starting with the first term: $\sum_{k=m}^{n} r^{k-1}$, where $1 < m < n$.

Answers

1. $\sum_{k=1}^{n} r^{k-1} = \frac{1 - r^n}{1 - r}$
2. $\sum_{k=1}^{n} r^{k-1} = 1 + r^1 + r^2 + \cdots + r^{n-2} + r^{n-1}$

   $\sum_{k=m}^{n} r^{k-1} = r^m + r^{m+1} + \cdots + r^{n-2} + r^{n-1}$

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3. We know that $n > m > 1$, so if we start from $k = 1$ to $k = n$, the series will have more terms than if we start from $k = m$. In fact, if $m > 1$ and the second series starts with r^m, then that second series must be contained within the first one. We can rewrite the first, longer series as:

$$\sum_{k=1}^{n} r^{k-1} = 1 + r^1 + \cdots + r^{m-1} + r^m + r^{m+1} + \cdots + r^{n-2} + r^{n-1}$$

The series from $k = m$ (shorter series) is contained in the series from $k = 1$ (longer series).

4. Now we replace the terms that start with r^m with the summation notation for the shorter series.

$$\sum_{k=1}^{n} r^{k-1} = 1 + r^1 + \cdots + r^{m-1} + \sum_{k=m}^{n} r^{k-1}$$

If we look closely, we'll notice that the terms that remain, $1 + r^1 + \cdots + r^{m-1}$, can also be replaced. They are the same series from $k = 1$ to $k = m$.

$$\sum_{k=1}^{n} r^{k-1} = \sum_{k=1}^{m} r^{k-1} + \sum_{k=m}^{n} r^{k-1}$$

Now, we solve for the series whose formula we want to find.

$$\sum_{k=m}^{n} r^{k-1} = \sum_{k=1}^{n} r^{k-1} - \sum_{k=1}^{m} r^{k-1}$$

And, finally, we replace the formulas that we know and then simplify.

$$\sum_{k=m}^{n} r^{k-1} = \frac{1-r^n}{1-r} - \frac{1-r^m}{1-r} = \frac{1-r^n-1+r^m}{1-r} = \frac{r^m-r^n}{1-r}$$

Station 3

Students will be given ten note cards with odd numbers from 1 to 19 (one number per card). Then they will start pulling the cards in order according to the directions, and answer the questions.

Answers

1. 1
2. 4
3. 9
4. 16
5. 25, 36, 49, 64, 81, 100
6. Each sum is a perfect square.
7. $1 + 3 + 5 + \cdots + 19 = \sum_{k=1}^{10} 2k - 1$
8. $\sum_{k=1}^{n} 2k - 1 = n^2$

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Station 4

Each group of students will be given two sets of five note cards each. On the first set of note cards, students will write the numbers 1 through 5. On the second set, they will write the cube of each number from the first set of cards. Students will explore and find the general formula for the series of the first n cube numbers.

Answers

1. The sum of the original numbers is 1. The sum of the cubed numbers is 1.
2. The sum of the original numbers is 3. The sum of the cubed numbers is 9.
3. The sum of the original numbers is 6. The sum of the cubed numbers is 36.
4. The sum of the original numbers is 10. The sum of the cubed numbers is 100.
5. The sum of the original numbers is 15. The sum of the cubed numbers is 225.
6. The sum of the cubed numbers is always the square of the sum of the original numbers.
7. $1^3 + 2^3 + \cdots + 5^3 = \sum_{k=1}^{5} k^3$
8. $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

Materials List/Setup

Station 1 none

Station 2 none

Station 3 10 note cards numbered 1 to 19 (odd numbers only)

Station 4 10 note cards, preferably 2 sets of 5 in different colors, for each group

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. How do you interpret the following formula? $\sum_{k=1}^{9} a_k$
2. What is the difference between a sequence and a series?
3. What is the process of working with formulas you already know to create new formulas?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. Sum all the terms in a sequence beginning with the first term and ending with the ninth term.
2. A sequence is a list of numbers that follow a particular pattern or order. A series is a sum of a sequence.
3. Begin by writing the terms of the series. Then write the terms of the series using the formulas you know. Compare the two sequences. Look for terms that can be replaced, or add the two sequences together and then simplify.

Possible Misunderstandings/Mistakes

- Incorrectly determining explicit formulas
- Using a recursive instead of an explicit formula to represent general terms in a series
- Incorrectly reading summation notation
- Incorrectly determining whether a sequence is arithmetic or geometric
- Incorrectly associating terms
- Incorrectly factoring
- Making arithmetic or algebraic errors

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Station 1

Working in pairs, practice reading summation notation and writing out the terms of a series. List the terms of each series.

1. $\sum_{k=1}^{n} 3k$

2. $\sum_{k=1}^{n} 4^k$

3. $\sum_{k=1}^{n} a_1 + dk$

4. $\sum_{k=1}^{\infty} Ar^k$

5. $\sum_{k=3}^{8} 6(k-1)$

6. $\sum_{k=2}^{5} k^2 + 2^k$

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Station 2

With a partner, discuss and develop a formula to calculate the value of $\sum_{k=m}^{n} r^{k-1}$ easily when $1 < m < n$.

1. What similar formulas do you already know?

2. Write out the terms of the series given above and of similar ones that you know.

3. Compare the terms in the two sequences.

4. Based on the formula you already know, find a new one to calculate $\sum_{k=m}^{n} r^{k-1}$.

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Station 3

At this station, you will find a set of cards labeled with odd numbers from 1 to 19. Pull the notecards as directed below to answer the questions.

1. First, pull out the card that has the number 1 on it. If this is the only term in a series, what is the total sum?

2. Pull out the card that has the number 3 on it and place it next to the "1" card. What is the total sum?

3. Pull out the card that has the number 5 on it and place it next to the "3" card. What is the total sum?

4. Pull out the card that has the number 7 on it and place it next to the "5" card. What is the total sum?

5. Repeat the process until you have added all the cards. Write the new total sum each time that you add a new card.

6. What do you notice about the numbers in each sum?

7. Write out the series in summation notation.

8. Write a formula to find the value of the series of the first n odd numbers.

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Station 4

At this station, you will find two sets of cards. On one set of cards, write the numbers 1 through 5. On the second set of cards, write the cube of each number from the first set of cards. Follow the instructions to form two rows of cards.

1. Place the card with the number 1 on the table. Below it, place its cube. If those are the only terms of two different series, what is each total sum? (Determine the sum for each series separately.)

2. Place the card with the number 2 on the table next to the "1" card. Below it, place its cube. What is the total sum of each series?

3. Repeat the process with the number 3 and its cube. What is the total sum of each series?

4. Repeat the process with the number 4 and its cube. What is the total sum of each series?

5. Repeat the process with the number 5 and its cube. What is the total sum of each series?

6. Compare each sum of original numbers with the corresponding sum of cubed numbers. What do you notice?

7. Write out the series of cubed numbers in summation notation.

8. Write a formula to find the value of the series of the first n cubed numbers.

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APPENDIX

Problem-Based Task Implementation Guides

Introduction

This addendum contains a collection of Implementation Guides (IGs) designed to support the use of the Problem-Based Tasks found throughout this unit.

Each Implementation Guide includes an overview and suggestions for introducing and facilitating the task. These contain specific recommendations for selected Standards for Mathematical Practice (SMPs) to highlight, guidance on addressing common errors and misconceptions associated with the task, and ideas for monitoring and coaching students as they work through the task. Finally, there is guidance for debriefing the task, with special attention to the SMPs and relevant Key Concepts from the instruction, as well as information on suitable extensions and alternate strategies.

Throughout the Implementation Guides, suggestions are provided in a question/prompt format to support your efforts to elicit student thinking and facilitate classroom discourse.

We developed these Implementation Guides for our Problem-Based Tasks at the request of math educators, and with advice and feedback from mathematics supervisors and teachers. Please let us know if they are helpful. Visit us at www.walch.com, follow us on Twitter (@WalchEd), or e-mail suggestions to customerservice@walch.com.

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Implementation Guide

Problem-Based Task 2A.1.1: Laying Tile

Common Core State Standard

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

Task Overview

Focus

How can a polynomial expression be written to represent a real-world scenario? What is the general format of a polynomial expression? In this task, students will write a polynomial expression in the correct format that represents the area of a strip of tile that is being designed by a contractor. The task will reinforce the structure of polynomials and allow an opportunity to review associated vocabulary.

This activity will provide practice with:

- analyzing terms in a diagram representing a real-world scenario
- classifying terms in an expression
- finding the sum of terms in order to represent the area
- writing a polynomial expression
- putting a polynomial in standard form

Introduction

This task should be used to explore or apply the skill of interpreting expressions that represent a quantity in terms of its context, as well as ordering the terms of an expression.

Begin by reading the problem and clarifying the meaning of *coefficient*, *constant term*, *degree*, *descending order*, *exponential expression*, *leading coefficient*, *polynomial*, and *term*.

coefficient	the number multiplied by a variable in an algebraic expression
constant term	a term whose value does not change
degree	a unit used to measure angles. One degree represents $\frac{1}{360}$ of a full rotation.
descending order	polynomials ordered by the power of the variables, with the largest power listed first and the constant last
exponential expression	an expression that contains a base and a power/exponent
leading coefficient	the coefficient of the term with the highest power. For a quadratic equation in standard form ($y = ax^2 + bx + c$), the leading coefficient is a.

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polynomial	an expression that contains variables, numeric quantities, or both, where variables are raised to integer powers greater than or equal to 0
term	a number, a variable, or the product of a number and variable(s)

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 4:** Model with mathematics.

  Students will identify the relationships between the individual tiles and the entire strip of tile. They will model the combined area of the three tiles by adding the expressions that represent the areas of the tiles.

- **SMP 6:** Attend to precision.

  Make sure students are using precise mathematical language when describing their algebraic expression, and ask them questions to elicit this mathematical language. When describing their polynomial, students should use terms such as *polynomial*, *term*, *factor*, *constant term*, *leading coefficient*, *descending order*, *exponential expression*, and *degree*.

- **SMP 7:** Look for and make use of structure.

  Students should recognize that the expression that represents the combined area of the tiles is a polynomial or, more specifically, a quadratic, and that the structure of a quadratic expression in standard form has the exponents in descending order.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly ordering terms with different powers

  Remind students that the final form of their answer should have terms ordered so that the exponents are in descending order. Also remind students that there is an understood exponent of 1 if the variable does not have an exponent.

- combining terms in an expression that have variables raised to different powers

  Remind students that like terms can only be combined if they contain the same variables raised to the same power; therefore, the terms in the expression created in this problem cannot be combined.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task:
  - Ask, "What is the difference between *ascending* and *descending*?" (**Answer:** The word *ascending* means to 'go up' or increase, and *descending* means to 'go down' or decrease.)
  - Ask, "What order should the terms be in for the final answer?" (**Answer:** The terms should be ordered such that the exponents are in descending order and the constant term is at the end.)
- Ask students, "How do you know which operation you must use to find the area?" (**Answer:** To find the total area, the three areas must be added together, so the operation is addition.)
- Ask, "What are the units for the answer?" (**Answer:** Square inches, or in^2)
- As students begin to write the final expression, ask, "How can you determine which term is written first, $3x$ or x^2?" (**Answer:** Because both terms have the same variable, the next step is to look at the exponents. The term $3x$ has an understood exponent of 1, and the term x^2 has an exponent of 2, so x^2 should be written first, followed by $3x$.)
- Reinforce the lesson's vocabulary by asking the following questions:
  - "How many terms are in the answer?" (**Answer:** 3)
  - "Which terms in the answer are products? What are the factors of one of these terms?" (**Answer:** x^2 and $3x$; x and x, or 3 and x)
  - "What is the degree of the polynomial?" (**Answer:** 2)
  - "What is/are the coefficient(s) in the answer?" (**Answer:** 1 and 3)
  - "What is the leading coefficient in the answer?" (**Answer:** 1)
  - "What is the constant in the answer?" (**Answer:** 36)
  - "What is the exponential expression in the answer?" (**Answer:** x^2)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Ask students to discuss their thought processes as they determined the order of the polynomial expression. Be sure to reinforce the vocabulary of the lesson in the discussion.

Connecting to Key Concepts

Make explicit connections to key concepts:

- A polynomial is an expression that contains variables and/or numeric quantities where the variables are raised to integer powers greater than or equal to 0.

  In this task, students write the polynomial expression $x^2 + 3x + 36$.

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- A term is a number, a variable, or the product of a number and variable(s).

  The terms in the expression in this task are x^2, $3x$, and 36.

- The terms in a polynomial are ordered by the power of the variables, with the largest exponent listed first. This is known as descending order.

  The polynomial expression developed in this task, written in descending order, is $x^2 + 3x + 36$.

Extending the Task

To extend the task:

- Ask students to find the area of the tiles if there are to be x rows of the three given tiles. (**Answer:** $x^3 + 3x^2 + 36x$)
- Ask students, "If two of the given tiles each have an area of 36 in^2, what is the area of the third tile?" (**Answer:** One of the given tiles is known to have an area of 36. If the second tile with an area of 36 is the tile labeled x^2, then $x = 6$, so the third tile would be $3 \cdot 6 = 18$ in^2. Or, if the second tile with an area of 36 is the tile labeled $3x$, then $x = 12$, so the other tile would be $12^2 = 144$ in^2. Thus, the two possible answers are 18 in^2 and 144 in^2.)
- Ask students to make up other problems that are an extension or adaptation of the given problem, and have students solve each other's problems.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I modeled this scenario by determining that the combined area of the tiles is the sum of the areas of the individual tiles, and then I used my knowledge of terms, like terms, and descending order to write a polynomial expression that represents the total area of the strip of tile.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I made sure to use correct and precise math vocabulary when describing the polynomial that is the answer to this problem.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I recognized the general structure of a polynomial expression, and I used this structure to write a polynomial expression representing the total area of the strip of tile.)

Alternate Strategies or Solutions

Students may choose to redraw the tiles in the diagram such that they are in descending order. Remind them to label the tiles carefully if they redraw them.

Technology

No technology is needed for this task.

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Problem-Based Task 2A.1.3: Not Quite Set in Stone

Common Core State Standard

A–APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Task Overview

Focus

How can polynomials be multiplied? How can the sum of two products of polynomials be written to represent the area of a figure in the context of a real-world scenario? In this task, students will create polynomial expressions that represent the dimensions of a walkway, and they will use polynomial multiplication and addition to determine the total area of the walkway.

This activity will provide practice with:

- analyzing dimensions in a diagram representing a real-world scenario
- writing a polynomial expression
- multiplying polynomial expressions
- adding polynomial expressions
- combining like terms in an expression
- determining an expression for the area of a figure

Introduction

This task should be used to explore or apply multiplication and addition of polynomials, and to explore the idea that polynomials are closed under the operations of addition, subtraction, and multiplication. Students should already be familiar with combining like terms to simplify expressions, and with calculating the area of rectangles.

Begin by reading the problem and clarifying the meaning of *area*, *like terms*, and *polynomial*.

area	the amount of space inside the boundary of a two-dimensional figure
like terms	terms that contain the same variables raised to the same power
polynomial	an expression that contains variables, numeric quantities, or both, where variables are raised to integer powers greater than or equal to 0

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  There are several ways to arrive at the solution to the problem presented in this task. Encourage students to describe their own solution methods, listen to and critique their classmates' methods, and discuss which method, if any, is best.

- **SMP 4:** Model with mathematics.

  The real-life scenario presented in this task was given as a geometric model, and this can be used to find an algebraic model. Students use the dimensions presented in the diagram to develop and solve an algebraic expression that represents the entire area of the walkway.

- **SMP 6:** Attend to precision.

  As students are working through the task, they should be using math terms and definitions clearly and accurately to explain their reasoning. Strive toward precise mathematical language, such as using the terms *area* and *expression* correctly, and using the correct language to refer to solution methods (e.g., combining like terms, finding the product of two polynomial expressions, finding the sum of two polynomial expressions).

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- forgetting to multiply all terms in each polynomial

  Remind students that in order to find the areas of the various sections of the walkway, each term in the first polynomial must be multiplied by each term in the second polynomial.

- incorrectly simplifying the resulting product

  Remind students that a product can be simplified by combining like terms, which are terms that contain the same variable(s) raised to the same power.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask, "What is the difference between finding the perimeter of a rectangle and finding the area of a rectangle?" (**Answer:** The perimeter of a rectangle is the total distance around the outside, and it is found by adding the lengths of the four sides together. The area of a rectangle is the total amount of space inside the rectangle, and it is found by multiplying the length and width.)
- If students have trouble determining which quadrilateral(s) comprise the "top portion" of the walkway, ask, "What are the different ways that the stone walkway can be separated into individual quadrilaterals?" (**Answer:** One way is to separate the diagram into three separate quadrilaterals; specifically, two rectangles and one square. Another way is to divide the figure into two rectangles: the lower right rectangle, and a second rectangle formed by combining the upper left rectangle with the square to form one quadrilateral. A third way also divides the walkway into two rectangles: the upper left rectangle, and a second rectangle formed by the square and the lower right rectangle.)
- If students have different opinions about how they should divide up the walkway in order to find the expression, ask, "Is there just one method of finding the expression for the area of the walkway?" (**Answer:** No, there are several different methods.) Encourage students to use the method that they prefer, rather than guiding them to all use the same method.
- Once students have divided the walkway into separate quadrilaterals:

  For students who divide the walkway into three quadrilaterals, ask, "What are the dimensions of the two rectangles and the square portion of the walkway?" (**Answer:** The upper left rectangle has a width of x and a length of 25, the lower right rectangle has a width of 10 and a length of x, and the square has a length and width of x.)

  For students who decide to find the area by combining the upper left rectangle and the square into one quadrilateral, ask, "What are the dimensions of the top portion of the walkway, and what property can be used to simplify the area of this rectangle?" (**Answer:** The length of the top portion of the walkway is $(25 + x)$ and the width is x. The Distributive Property will be used to simplify the area of this rectangle: $x(25 + x) = 25x + x^2$.)

  For students who decide to combine the lower right rectangle and the square into one quadrilateral, ask, "What are the length and width of the right-hand portion of the walkway, and what property can be used to simplify the area of this rectangle?" (**Answer:** The length of the right-hand portion of the walkway is $(10 + x)$ and the width is x. The Distributive Property will be used to simplify the area of this rectangle: $x(10 + x) = 10x + x^2$.)

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- After students have found the areas for the separate rectangles, ask, "What must be done in order to simplify the expression that represents the sum of the two rectangles?" (**Answer:** The expression that is found by adding the two areas is $x^2 + 25x + 10x$. To simplify it, we need to combine like terms. The terms "$25x$" and "$10x$" are like terms, so they can be combined to get $35x$.)
- Ask students, "How is the final expression representing the total area of the stone walkway written in the correct order and with correct units?" (**Answer:** The final expression representing the total area of the stone walkway is written in descending order, which is $x^2 + 35x$, and the units are square feet, or ft^2.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask students to explain their thought processes as they constructed their solution plan to find the expression for the total area of the stone walkway. Encourage them to justify and defend their processes to each other if there are disagreements.
- As students are discussing their processes for working through the task, focus on the use of precise mathematical language, specifically when discussing vocabulary such as *area*, *like terms*, and *polynomial expression*.

Connecting to Key Concepts

Make explicit connections to key concepts:

- To multiply two polynomials, multiply each term in the first polynomial by each term in the second polynomial.

  In this task, students will multiply the expression for the length of a rectangle by the expression for its width. (*Note*: In the Coaching Sample Responses, the length of the "top" rectangle is described as $25 + x$, and the width is described as x. Different student methods may initially result in different dimensions.)

- The Distributive Property can be used to simplify the product of two polynomials.

  When students multiply x by $(25 + x)$ as described in the Coaching Sample Responses, they will use the Distributive Property by multiplying the first expression, x, by both 25 and x to get a product of $25x + x^2$.

- The product of two polynomials is a polynomial, so the system of polynomials is closed under multiplication.

  Students will observe that when they multiply two polynomials, the result is also a polynomial; specifically, $25x + x^2$. This demonstrates closure.

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Extending the Task

To extend the task, ask students to find the area of the walkway if each side length is twice as long as the dimensions in the diagram. Have students determine if doubling the side lengths will also double the area of the walkway. (**Answer:** If the side lengths are doubled, the area is multiplied by 4, not 2.)

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

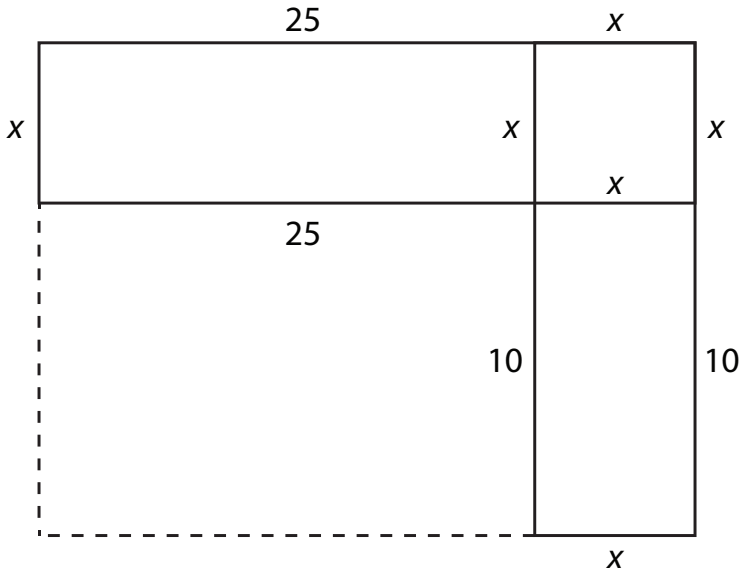
- **For SMP 3, ASK:** "Did you construct viable arguments and did you critique the reasoning of others?" (**Answer:** I explained my method to other students and I listened to their ways of arriving at the solution. We discussed which solution method was better or easier, and we agreed that there was no best method because all of our methods worked.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this scenario by creating an algebraic expression that represents the area of the walkway by using the information given in the visual model.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I made sure I used correct mathematical language, and I gave precise explanations of my reasoning when working through the steps to find the area of the stone walkway.)

Alternate Strategies or Solutions

- Before they begin the task, students may choose to redraw the diagram of the stone walkway such that the "top portion" of the walkway (combining the left-hand rectangle and square) is separate from the remaining rectangle of the walkway. Encourage students to carefully label the sides of each rectangle if they draw them separately. Remind students to redraw the rectangles such that their parts do not overlap, to ensure that the parts are counted twice in terms of dimensions.
- Instead of considering the "top portion" of the walkway to be formed by the left-hand rectangle and square (with resulting dimensions of $25 + x$ and x), students may divide the walkway into two parts based on using only the leftmost rectangle as the "top portion," with dimensions of 25 and x. In this case, the dimensions of the remaining rectangle of the walkway would have dimensions of $10 + x$ and x. Remind students that although this process will still have the same outcome as the process used in the task, it is helpful to verify their results both ways.

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- Alternatively, students may find the area of each of the three sections separately, and then add them together. The three separate areas are $25x$, $10x$, and x^2. Remind students that if they choose this method, they will have three parts to sum together to find the total area. Again, the resulting expression would be the same.
- Another way to find the area of the walkway is to draw two segments on the diagram to create a large rectangle, like this:



The area of the walkway is equal to the area of the large rectangle, $(x + 10)(x + 25)$, minus the area of the rectangle on the lower left, $(25)(10)$. Thus, the expression for this method is $(x + 10)(x + 25) - (25)(10)$. Simplifying this expression produces the same result as the other methods: $(x^2 + 35x)$ ft^2.

Technology

No technology is needed for this task.

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Implementation Guide

Problem-Based Task 2A.2.1: How Big Is the Dog Park?

Common Core State Standards

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.*

A–APR.4 Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 – y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.*

Task Overview

Focus

How can polynomial identities be used to rewrite numerical quantities? How is the structure of an expression used to identify a specific polynomial identity? In this task, students will rewrite an expression for the area of a dog park using polynomial identities, and they will use the rewritten expression and apply the polynomial identity to calculate the area of the dog park.

This activity will provide practice with:

- rewriting a numerical value as a difference of two numbers
- calculating the area of a square without the use of a calculator
- identifying a polynomial identity used to find the square of a value
- applying the structure of a polynomial identity
- evaluating terms in a numerical expression

Introduction

This task should be used to apply the skill of using the structure of an expression to identify ways to rewrite it. Students should already be familiar with finding sums, products, and differences of polynomial terms, and simplifying expressions by combining like terms.

Begin by reading the problem and reviewing and clarifying the meaning of *area of a square* and *polynomial identity*.

area of a square	the amount of space inside a square, determined by the square of the length of a side of the square
polynomial identity	a true equation that is often generalized so it can apply to more than one example

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to calculate the area of a dog park by first rewriting the side length of the dog park as a sum or difference of two numbers, and then identifying and applying a polynomial identity to create an equivalent expression that represents the area. They will recognize that the length of a side of the dog park, 280 feet, can be rewritten as a difference of two values, specifically 300 – 20. They will examine the list of polynomial identities and choose the Square of Differences Identity to rewrite this numerical expression to represent the area of the park, and then they will evaluate each term in the expression in order to determine the area of the dog park, in units of square feet.

- **SMP 2:** Reason abstractly and quantitatively.

  Students will recognize that they need to determine how to rewrite a numerical value so that it can be represented as the difference of two values. They will also need to identify the polynomial identities that can be used in order to find the square of a value, and then decide which identity can be applied in order to rewrite the expression to represent the area. They will reason quantitatively as they evaluate each term in the rewritten expression from the polynomial identity and perform the calculations necessary to find the total area of the dog park in units of square feet.

- **SMP 6:** Attend to precision.

  Students will use precision as they carefully follow and apply the structure of a polynomial identity to the rewritten expression representing the area of the dog park. They will evaluate the numerical expression created by applying the specific polynomial identity - Square of Differences Identity. They will evaluate exponents, find the product of three numbers, and use order of operations to find the numerical value representing the area of the dog park. Encourage them to use the correct terminology throughout the task, specifically when referring to the Square of Differences Identity, as well as when identifying the appropriate units to represent the area.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- confusing identities, such as applying the Square of Differences Identity instead of the Difference of Two Squares Identity

  Have students make a table of the various polynomial identities to keep with them as they work through the task. Remind them to pay close attention to the parentheses and signs for each specific identity.

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- incorrectly applying an identity to find an equivalent expression

  Remind students to refer to their table of polynomial identities and pay attention to the structure of each identity. Encourage them to make sure the structure of the identity that they choose for the task matches the structure of the format of the corresponding identity in the table.

- incorrectly evaluating the expression for area and subsequently getting an incorrect value for the area of the dog park

  Remind students of order of operations (PEMDAS), and encourage them to apply it when evaluating the expression. Also encourage them to double check their answer by performing the calculations twice.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, briefly review the concept of area with them, ensuring they understand the connection between the units of area and applying certain polynomial identities. Ask them, "Without using a formula, explain what area means. How are the units representing area related to the use of certain polynomial identities?" (**Answer:** Area is the space inside of a figure. Since area is represented in square units, the polynomial identities including the Square of Differences Identity, the Square of Sums Identity, and the Difference of Two Squares Identity could be applied since they all involve quantities squared.)
- As students begin working on the task, ask them to consider other ways to rewrite 280 as a sum of two numbers. Ask them, "What is a way to write 280 as a sum of two numbers, in which the two numbers are "easy" to work with?" (**Answer:** One possible way to rewrite 280 as a sum of two numbers in which the two numbers are simple to work with is 200 + 80.)
- As an extension to the previous question, ask students, "Why is it more practical to rewrite 280 as a sum or difference of two numbers which end in 0? Explain why rewriting 280 as 227 + 53 would not be as practical as rewriting it as 200 + 80." (**Answer:** Since the Square of Sums Identity requires the values of a squared and b squared, squaring both 227 and 53 would be more involved and would take more time to calculate by hand, without the use of a calculator, than squaring 200 and 80 would be.)
- As students are working through the task, ask them, "Why is it not practical to consider using the Difference of Two Squares Identity in this task? Explain." (**Answer:** The Difference of Two Squares Identity is practical when the two values for a and b are perfect square numbers, such as 4, 9, 16, 25, etc. The number 280 cannot be rewritten as a sum or difference of two numbers which are perfect squares.)

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- As students are working through the task, review order of operations with them to ensure that they evaluate the terms in the expression correctly. Ask them, "What does the acronym PEMDAS stand for? (**Answer:** The acronym PEMDAS is the order that operations are performed when simplifying and/or evaluating an expression. Operations within **P**arentheses are performed first, then **E**xponents, then **M**ultiplication and **D**ivision (in order from left to right), and finally **A**ddition and **S**ubtraction (in order from left to right.)
- As an extension to the previous question, ask students, "How is order of operations specifically applied when evaluating the expression $300^2 - 2(300)(20) + 20^2$? Explain your steps and how you reached the final value for the area." (**Answer:** In the expression $300^2 - 2(300\ (20) + 20^2$, since there are no operations within parentheses that need to be performed, the next step is to evaluate the terms which contain exponents, 300^2 and 20^2: $300^2 = 90{,}000$, and $20^2 = 400$. The next line in the expression would be: $90{,}000 - 2(300)(20) + 400$. The next step is to perform the multiplication: $2(300)(20) = (600)(20) = 12{,}000$. The next line in the expression would be: $90{,}000 - 12{,}000 + 400$. Finally, addition and subtraction is performed from left to right: $90{,}000 - 12{,}000 + 400 = 78{,}000 + 400 = 78{,}400$.)
- As students are completing the task, discuss the purpose of applying polynomial identities as an alternative to finding the product of a number. Ask students, "Although the area of the dog park could have been determined by multiplying 280 by 280, what is an advantage of using a polynomial identity to find the area?" (**Answer:** The polynomial identities are useful because it allows for a number to be rewritten as a sum or difference of two smaller numbers, which can be easier to find sums and products for, as opposed to squaring a number that does not have "easy" digits and will require more complex calculations.)
- To ensure students' understanding of applying polynomial identities, ask them to work through the steps for using the Square of Sums Identity. Ask them, "Explain the steps for applying a polynomial identity and calculating the area of the dog park if 280 were rewritten as 200 + 80." (**Answer:** If 280 was rewritten as 200 + 80, the steps for applying the Square of Sums Identity and calculating the area of the dog park are as follows:

$(a + b)^2 = a^2 + 2ab + b^2$	Square of Sums Identity
$(200 + 80)^2 = 200^2 + 2(200)(80) + 80^2$	Substitute values for a and b into the identity.
$40{,}000 + 2(200)(80) + 6400$	Evaluate exponents.
$40{,}000 + 32{,}000 + 6400$	Multiply.
$72{,}000 + 6400$	Add.
$78{,}400$	The area of the dog park is $78{,}400$ feet^2.

- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask for volunteers to discuss their initial thoughts about how find the area of the dog park with a side length of 280, without the use of a calculator. Ask them to share their ideas on the possible ways to rewrite 280 as a sum or difference of two numbers. Discuss why the numbers 300 and 20 were used, in terms of working with numbers ending in 0 that will lead to more ease in finding products and squares of numbers. Ask students to explain how they applied the structure of the Square of Differences Identity in order to accurately represent the rewritten expression for the area of the dog park. Also ask them to explain how they applied order of operations to evaluate each term of the expression and calculate the numerical value representing the area. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process for rewriting 280 as 300 – 20. Also discuss other ways to rewrite 280 as another difference, or as a sum of two numbers. Ask students to justify their choices of their alternate rewritten sum or difference by explaining how the terms of the expression of the polynomial identity will lead to more simple calculations. Encourage students to discuss the advantages of applying polynomial identities in a context such as the one presented in the task, especially if the use of a calculator is not possible. Focus on the use of precise mathematical language and clarity, specifically when discussing the specific polynomial identity used in the task. Also focus on the use of correct terminology when discussing the evaluation of the terms and the calculation of the area.

Connecting to Key Concepts

- Identities can be used to expand or factor polynomial expressions.

  In this task, students will use an identity to expand an expression which represents the area of a dog park.

- A **polynomial identity** is a true equation that is often generalized so it can apply to more than one example.

  Students will choose a specific polynomial identity, the Square of Differences Identity, in order to rewrite the expression given for the area of a dog park.

- When dealing with large numbers, the use of identities often results in simpler calculations.

  Students will perform the necessary steps in order to calculate the area of a dog park using a polynomial identity. They will recognize that the parts of the expression allow for calculations with smaller numbers.

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Extending the Task

To extend the task, ask students to work with a partner. Ask each person in the pair to create 3 different values for the side length of the dog park. Have each person give their 3 values to their partner and ask the partner to calculate the area of the dog park by rewriting the length of a side as a sum or difference of two numbers and using at least 2 different polynomial identities. After each student has had a chance to work through three scenarios of different side lengths, have them show their partner their work, and ask each pair of students to discuss the results. Specifically, ask them to evaluate the accuracy and precision of each step, as well as if the polynomial identity was applied correctly. Encourage them to troubleshoot and work together to fix any errors. Ask each pair of students to share one of their examples with the class.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by recognizing that the goal of the task is to calculate the area of a dog park by first rewriting the side length of the dog park as a sum or difference of two numbers, and then identifying and applying a polynomial identity to create an equivalent expression that represents the area. I recognized that the length of a side of the square-shaped dog park, 280 feet, could be rewritten as a difference of two values, specifically 300 – 20. I examined the list of polynomial identities and chose the Square of Differences Identity to rewrite this numerical expression to represent the area of the park, and then I evaluated each term in the expression in order to determine the area of the dog park, in units of square feet.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first made sense of the goal of the task by recognizing that I needed to determine how to rewrite a numerical value so that it could be represented as the difference of two values. I also identified the polynomial identities that could be used in order to find the square of a value, and then I decided which identity could be applied in order to rewrite the expression to represent the area.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning as I evaluated each term in the rewritten expression from the polynomial identity and performed the calculations necessary to find the total area of the dog park in units of square feet.)

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- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision by carefully following and applying the structure of a polynomial identity to the rewritten expression representing the area of the dog park. I evaluated the numerical expression created by applying the specific polynomial identity - Square of Differences Identity. Specifically, I evaluated exponents, found the product of three numbers, and used order of operations to find the numerical value representing the area of the dog park. I was careful to use the correct terminology throughout the task, specifically when referring to the Square of Differences Identity, as well as when identifying the appropriate units to represent the area.)

Alternate Strategies or Solutions

Some students may choose alternate ways to rewrite 280 as a sum or a difference of two numbers. Some students may choose to rewrite it as a sum of 200 + 80 and apply the Square of Sums Identity. Encourage students who choose an alternate path to work through the method of applying the Square of Differences Identity as presented in the task, as well as using 200 + 80 and applying the Square of Sums Identity, and verify that both methods lead to the same results. For students who might choose to rewrite 280 as a sum or difference of numbers which do not end in 0, discuss how the calculations are more difficult and time consuming, as opposed to using numbers which end in 0.

Technology

No technology is needed for this task.

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Problem-Based Task 2A.2.3: Combinations of Candidates

Common Core State Standards

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

A–APR.4 Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.*

A–APR.5 (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[1]

Task Overview

Focus

How can patterns be used to find specific terms in the expansion of binomials? How can Pascal's Triangle be applied to determine the number of combinations of objects that can be made from a larger group of objects? In this task, students will use Pascal's Triangle to find the number of different combinations of 4 students that can be elected to be on the student council from a group of 9 students who are running for the four positions available.

This activity will provide practice with:

- analyzing verbal information given in a task
- creating Pascal's Triangle to a specified row number
- identifying the correct term number in a row in Pascal's Triangle
- interpreting results in the context of a real-world scenario

[1] The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.

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Implementation Guide

Introduction

This task should be used to explore or to apply the skill of knowing and applying the Binomial Theorem for the expansion of $(a + b)^n$ in powers of a and b for a positive integer n, where a and b are any numbers, with coefficients determined for example by Pascal's Triangle. Students should already be familiar with using terminology related to polynomials, such as *terms* and *powers*.

Begin by reading the problem and reviewing and clarifying the meaning of *combination* and *Pascal's Triangle*.

combination	a selection of all or part of a set of objects, without regard to the order in which objects are selected
Pascal's Triangle	a triangle displaying a pattern of numbers in which the terms in additional rows are found by adding pairs of terms in previous rows, so that any given term is the sum of the two terms directly above it. The number 1 is the top number of the triangle, and is also the first and last number of each row.

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they recognize that they will need to use Pascal's Triangle in order to accomplish the goal of the task. They will reason that they will first need to find row 9 in the triangle, since there are 9 total students who are running for student council, as 9 represents the total number of objects in the larger group. Then, they will reason that since only 4 positions are available, only 4 students can be elected from the group of 9 students. They will identify the appropriate numbered term in row 9 of the triangle, which they will interpret as the total number of ways that 4 students can be elected to the student council. They will reason quantitatively as they carefully interpret and analyze the appropriate row and term in Pascal's Triangle. The they will interpret their results in the context of the scenario of the task.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled by applying Pascal's Triangle, which represents a pattern of numbers and can be used to find the number of combinations from a larger group. They will interpret the information given in the task and apply it to the triangle to determine the number of ways 4 students can be elected from a group of 9 students who are running for student council. They will recognize and identify that the rows and terms in Pascal's Triangle represent the total number of students as well as the number of students chosen from the total number.

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- **SMP 8:** Look for and express regularity in repeated reasoning.

  Students will express regularity in repeated reasoning as they explain and justify their conclusions about the number of ways 4 students can be elected to the student council out of 9 total students. They will follow the pattern displayed in Pascal's Triangle in order to find the ninth row and fourth term in the triangle. They will apply the repeated reasoning from the triangle, in regards to how each subsequent row is created from its preceding row.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- misidentifying the coefficients from Pascal's Triangle

  Have students keep the diagram of Pascal's Triangle next to them as they work through the task. Review with them how to determine the correct row and term to look for.

- mistakenly choosing the fourth term in the ninth row, instead of the fifth term

  Remind students that the first term in each row is named "term 0", so term 4 is actually the fifth term in the row.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, discuss their prior knowledge about combinations. Ask them, "If four students are chosen out of nine students to have positions on the student council, does it matter what order the four students are chosen in, since each student can only hold one position at a time? Explain using letters to represent 4 sample students." (**Answer:** No, the order does not matter. If students A, B, C, and D were chosen out of students A, B, C, D, E, F, G, H, I, it would not matter if the grouping was ABCD, or BACD, or DCAB, etc. Since specific positions are not mentioned, the group of four students can be chosen in any order.)
- As students begin, ask them to consider how the problem could be written in terms of a binomial expression. Ask them, "In the context of the scenario presented, which value does the variable n represent in the binomial form $(a + b)^n$?" (**Answer:** The variable n would be represented by the number 9 in the task, which is the total number of students to choose from who are running for the student council.)
- As students begin working, ensure their understanding of Pascal's Triangle. Ask them, "When expanding binomials, why is Pascal's Triangle helpful?" (**Answer:** Pascal's Triangle is helpful because instead of having to expand a binomial raised to a larger power by hand, the triangle displays the coefficients of each term based on a pattern.)

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- As an extension to the previous question, ask students, "If Pascal's Triangle was not available, how would you have to find the expanded form of $(a + b)^5$? What is the expansion of $(a + b)^5$, using Pascal's Triangle?" (**Answer:** Without the use of Pascal's Triangle, you would have to multiply the binomial $(a + b)$ out five times: $(a + b) \bullet (a + b) \bullet (a + b) \bullet (a + b) \bullet (a + b)$. This would require time and careful calculations of operations with polynomials. Using Pascal's Triangle, the expansion of $(a + b)^5$ is $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.)
- As students are working on the task, ask them, "Why does the very top of Pascal's Triangle, row 0, have a value of 1? Explain." (**Answer:** The top of Pascal's Triangle represents the binomial $(a + b)$ raised to the 0 power, or $(a + b)^0$. One rule of exponents states that any base raised to the zero power has a value of 1. Therefore, $(a + b)^0 = 1$, where the zero exponent represents the first row of the triangle, and the only term in the binomial expansion is 1.)
- To ensure further understanding of the structure of Pascal's Triangle, ask students, "How are the terms in each successive row determined?" (**Answer:** The terms in each successive row are found by adding pairs of terms in the previous rows, so that any given term is the sum of the two terms directly above it.)
- Once students have located the correct term in the ninth row of the triangle, ask them, "What does the value 126 mean in the context of the scenario? Is it possible to have found this value without the use of Pascal's Triangle? Explain other possible methods for finding the value of 126." (**Answer:** The value 126 represents the number of ways that 4 students can be elected to be in the student council out of 9 total students. Without the use of Pascal's triangle, it is possible to find this value by writing down all the possible combinations of choosing 4 students from 9 students.Another method is by using the formula for combinations, which is $\frac{n!}{r!(n-r)!}$, where r represents the number of objects chosen from a total of n objects.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss why it would be difficult and time consuming to attempt to expand a binomial out to the ninth power. Also discuss how the task could be accomplished by applying the formula for combinations, and discuss the benefits and drawbacks of this method as opposed to the method of applying Pascal's Triangle, as presented in the task. Ask students to explain how they carefully identified the correct row and term in Pascal's triangle, using the information presented in the task. Also ask them to explain how they interpreted the results in the context of the scenario. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and analysis for ensuring they identified the correct row and term in the triangle. Discuss the possible errors that could have occurred in the row and term choices, as well as how the structure and format of the triangle could have led to these errors. Ask students to explain why Pascal's triangle is a more efficient method of accomplishing a task such as the one presented, as opposed to using other methods. Focus on the use of precise mathematical language and clarity, specifically when discussing the connection between the number of total students and the row number in the triangle, and also the number of openings for student council and the term number in that row.

Connecting to Key Concepts

- **Pascal's Triangle** is a triangle displaying a pattern of numbers in which the terms in additional rows are found by adding pairs of terms in the previous rows, so that any given term is the sum of the two terms directly above it.

  In this task, students will apply Pascal's Triangle in order to determine the number of ways 4 students can be elected for student council out of a group of 9 students.

- The top row, containing only the number 1, is called "row 0." The next row is "row 1" and so on.

  Students will recognize that although the first row in the triangle is "row 0", they will use row 9 of the triangle, since there are 9 total students.

- The coefficients of $(a + b)^n$ can be found in the nth row of Pascal's Triangle.

  Students will look in the 9th row of the triangle, since 9 represents the number of terms, n.

- Pascal's Triangle can also be used to find the number of combinations, or unique groups of objects, from a larger group.

  Students will recognize that Pascal's Triangle can be applied in this task in order to find the number of ways that 4 students can be chosen from a larger group of 9 students.

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- The row number of the triangle is the total number of objects in the group, and the term number in the row is the number of objects being selected from the group. The coefficient for that term number is the number of combinations.

  Students will identify 9 as the total number of objects in the group, so they will look for the 9th row in the triangle. Then, they will look at the 4th term in the row, which they will identify as the fifth term in the row, since the first term in each row is designated as "term 0". They will then identify the coefficient for that term number as the number of combinations of students.

Extending the Task

To extend the task, ask students to work with a partner to create a similar scenario to the one presented in the task but change the total number of objects as well as the number of objects in the smaller group. Ask them to write down a complete real-world scenario, and then ask them to write down detailed steps in the procedure of how they used Pascal's triangle to determine their desired term. Ask for volunteers to share their scenario, and explain their detailed procedure for using the triangle, to the class.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first made sense of the goal of the task by recognizing I needed to use Pascal's Triangle in order to accomplish the goal of the task. I reasoned that I first needed to find row 9 in the triangle, since there were 9 total students who were running for student council, as 9 represents the total number of objects in the larger group. Then, I reasoned that since only 4 positions were available, only 4 students can be elected from the group of 9 students. I identified the appropriate numbered term in row 9 of the triangle, which I interpreted as the total number of ways that 4 students could be elected to the student council.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning as I carefully interpreted and analyzed the appropriate row and term in Pascal's Triangle. Then I interpreted my results in the context of the scenario of the task.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by recognizing that this scenario could be modeled by applying Pascal's Triangle, which represents a pattern of numbers and can be used to find the number of combinations from a larger group. I interpreted the information given in the task and applied it to the triangle to determine the number of ways 4 students could be elected from a group of 9 students who are running for student council. I recognized that the rows and terms in Pascal's Triangle represented the total number of students as well as the number of students chosen from the total number.)

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- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** I looked for and expressed regularity in repeated reasoning as I explained and justified my conclusions about the number of ways 4 students can be elected to the student council out of 9 total students. I followed the pattern displayed in Pascal's Triangle in order to find the ninth row and fourth term in the triangle. I applied the repeated reasoning from the triangle, in regards to how each subsequent row is created from its preceding row.)

Alternate Strategies or Solutions

- Some students may choose to use the formula for the number of possible combinations of r objects from a set of n objects, i.e., $\frac{n!}{r!(n-r)!}$, in order to determine the number of ways 4 students can be elected to the student council from 9 total students. Discuss with students that although this is a correct alternative, since they recognized that this is a problem in which the order does not matter, this method requires more time and calculations. In contrast, the use of Pascal's Triangle only requires choosing the correct row and term number. Encourage students to use both methods in order to verify their results.
- Some students may choose to determine the number of possible combinations by creating a table and writing down all possible combinations of choosing 4 students from 9. Remind students of the time involved in this process, and remind them that it will be difficult to ensure that they identified all possible combinations.

Technology

No technology is needed for this task.

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Problem-Based Task 2A.3.1: It's Electric!

Common Core State Standard

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Task Overview

Focus

How are key characteristics of a polynomial function identified from a graph of the function? How can a graph of a polynomial function provide information about a real-world scenario? In this task, students will analyze the graph of a polynomial function and identify characteristics of the function that will help make predictions about the growth of demand for electricity in the United States from 2040 to 2050.

This activity will provide practice with:

- analyzing a graph of a polynomial function
- identifying turning points of a graph
- classifying turning points as local maximum or local minimum
- representing the roots of a function as ordered pairs
- describing end behavior of a function
- making predictions about the behavior of a function based on its characteristics

Introduction

This task should be used to explore or to apply the skill of graphing polynomial functions, as well as analyzing the graphs of polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. Students should already be familiar with recognizing the format of a polynomial function, identifying intercepts of many types of functions, and identifying ordered pairs on a graph.

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Begin by reading the problem and clarifying the meaning of *end behavior*, *local maximum*, *local minimum*, *root*, and *turning point*.

end behavior	the behavior of the graph as x approaches positive or negative infinity
local maximum	the greatest value of a function for a particular interval of the function; also known as a *relative maximum*
local minimum	the least value of a function for a particular interval of the function; also known as a *relative minimum*
root	the x-intercept of a function; also known as *zero*
turning point	a point where the graph of the function changes direction, from sloping upard to sloping downward or vice versa

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal is to analyze a graph of a polynomial function representing information about the growth of demand for electricity in the United States, and predict whether they expect the growth of electricity to increase or decrease between 2040 and 2050. Specifically, they will look at the end behavior, turning points, and the roots of the polynomial function on the graph in order to make a valid prediction about what the graph will look like in the interval that represents the years between 2040 and 2050. They will also describe the behavior of the graph using specific points on the graph that represent the turning points and roots. Their predictions about the graph's behavior between 2040 and 2050 will be based their analysis of these characteristics of the graph.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for predicting whether or not they expect the growth of demand for electricity in the United States to increase or decrease between the years 2040 and 2050. They will examine and analyze the given graph of the polynomial function which represents the approximate growth since 1950 and includes projections through 2040. When presenting their arguments to others, they will explain key characteristics of the graph, specifically the turning points, roots, and end behavior, and they will justify their expectations about the growth between 2040 and 2050 based these characteristics.

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- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled with the graph of a polynomial function representing the growth of demand for electricity in the United States from 1950 and projected through 2040. They will analyze the specific characteristics of the graph which will provide information about the future behavior of the graph. This analysis will allow them to make a prediction about their expectations for the growth of demand for electricity from 2040 to 2050.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- confusing the roots of the function for the turning points

  Review the definition of roots and turning points of a function with students. Remind them that the roots are also the *x*-intercepts of a function, and these are the points where the graph intersects the *x*-axis. Also remind them that turning points are the points where the graph changes direction.

- incorrectly classifying local minimum points for local maximum points and vice versa

  Have students write the definitions for local minimum and maximum points. Discuss the meanings of the words "minimum" and "maximum" and how they relate to the points on the graph where the slope of the graph changes direction.

- mistakenly identifying the roots of the function as (0, 2003) and (0, 2011) instead of (2003, 0) and (2011, 0)

  Review the process for writing an ordered pair to represent a point on a graph. Remind students that the *x*-coordinate is first in an ordered pair, and it is found by moving left or right on the *x*-axis. Then the *y*-coordinate is found. On the graph, the roots lie directly on the *x*-axis, therefore the *x*-coordinate is represented by the point where the graph crosses the *x*-axis.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, discuss their initial thoughts about accomplishing the goal of the task. Ask them, "Based solely on briefly examining the given graph of the polynomial function representing the growth of demand for electricity since 1950, what can you predict about the growth of demand for 2040 through 2050?" (**Answer:** Answers may vary. Some students may think the demand will increase since the graph is already sloping upwards at the year 2040. Some students may think the demand will decrease, because there could be another turning point at the year 2040, since are several turning points from the previous years.)

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- Ask students, "Why do you think the growth of demand for electricity in the United States was much higher from 1950 through 1970, compared to the overall trend of decreasing growth after 1970?" (**Answer:** Answers may vary. In general, electricity was the most well-known form of power in those earlier decades. After 1970, alternate power sources were being discovered and implemented around the United States.)
- Before students begin the task, ask them to classify the type of function shown on the graph. Ask them, "Based on the graph of the polynomial function, is this an even or an odd-degree polynomial? How can you tell?" (**Answer:** The function represented in the graph is an even-degree polynomial, because both ends of the function extend upward.)
- Ask students, "Why is only the first quadrant represented in this problem? What does each axis represent in the context of the scenario?" (**Answer:** Since the scenario represents the percent of growth of demand for electricity, the values representing the independent variable, t for time, are positive. Therefore the first quadrant is used, since time is not negative. The x-axis represents the time in years, and the y-axis represents the percentage of growth of electricity demand.)
- As students are working through the task and analyzing the turning points, ask them, "Based on what you know about the relationship between the number of turning points and the degree of the polynomial, what is the smallest possible degree for this polynomial function? Explain." (**Answer:** Since there are seven turning points, the smallest possible degree of the function is 8, because the maximum number of turning points is one less than the degree of a polynomial. The polynomial could have a higher degree, but it is at least an eighth-degree polynomial since there are seven turning points.)
- Ask students, "Explain the turning points in terms of the slope of the graph. Use a specific point from the graph to support your explanation." (**Answer:** A turning point on the graph indicates a change in slope. If the graph was sloping upwards, then it will be sloping downwards at a turning point. For example, from 1960 to 1970, the growth of demand was sloping upwards. At the point (1970, 8), the graph now slopes downwards.)
- As students are working through the task, ask them to examine the graph in terms of intervals. Ask them, "From the years 1950 through 2000, what pattern do you notice about the turning points in this time interval? Explain what this means in the context of the scenario." (**Answer:** From 1950 through 2000, there are four turning points which occur approximately every ten years. This means that about every ten years, the growth of demand for electricity changes from either increasing to decreasing, or the demand changes from decreasing to increasing.)
- Ask students to analyze the behavior of the graph at the year 2010. Ask them, "What value is represented when the graph went below zero in the year 2010?" (**Answer:** In 2010, the growth of demand for electricity was approximately –1%.)

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- Ensure students' understanding of the roots of an equation. Ask them, "What does it mean that this polynomial function has two roots? Explain in the context of the scenario and also using proper function terminology." (**Answer:** There are two points on the graph where the graph intersects the *x*-axis, which are called the zeros or roots of the function. In the context of the scenario, this represents the years in which the growth of demand for electricity was 0. In function terminology, the roots of a function represent the values that are solutions to the equation of the function, or in other words, the values that satisfy the equation.)
- As students are working to describe the end behavior of the function, ask them, "Why do you think it is difficult to accurately predict the end behavior of the function?" (**Answer:** One reason that it is difficult to accurately predict the end behavior of the function is that there are seven turning points represented on the graph, which means that the growth for demand of electricity has changed direction several times over the course of the years since 1950. One decade the growth increases, and then the next decade the growth decreases. Based on this behavior, it is difficult to predict if the growth will increase or decrease after the year 2040.)
- Ask students to share their thoughts about alternate power sources. Ask them, "What other types of power sources are available today that might cause the growth of demand for electricity to decrease as the years go by?" (**Answer:** Some common alternate power sources are wind power and solar power. Some students may discuss biofuels, tidal and wave energy, and hydro power.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts and predictions about if they expect the growth of electricity demand in the United States to increase or decrease between 2040 and 2050. Discuss their general observations about the shape of the graph, and the general trends of the graph. Ask them to explain how they identified the turning points and classified which ones were local minimum and which ones were local maximum points. Also ask them to explain how they identified the ordered pairs which represented the roots of the function. Ask for volunteers to explain how the compilation of the characteristics of the graph of the function led them to their predicted expectations about the growth of demand for the years of 2040 to 2050. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.

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- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and methods of analyzing the turning points, roots, and end behavior of the polynomial function. Encourage them to discuss which characteristics of the graph had more weight in their overall expectation for the growth of demand for 2040 – 2050. Focus on the use of precise mathematical language and clarity, specifically when discussing the behavior of the function, such as the local minimum and local maximum turning points, as well as the roots of the function.

Connecting to Key Concepts

- To determine the **end behavior** of a polynomial function, or the behavior of the graph as x approaches positive or negative infinity, consider the highest degree of the polynomial and its coefficient, ax^n.

  In this task, students will be examining the graph of a polynomial function and making predictions about its end behavior based on the shape of the graph as the amount of time, in years, increases.

- A **turning point** of a function is a point where the graph of the function changes from sloping upward to sloping downward or, alternatively, from sloping downward to sloping upward.

  Students will identify the turning points on the graph, and they will specify these points as ordered pairs. Specifically, they will identify 7 turning points on the graph.

- A turning point corresponds to a **local maximum**, the greatest value of a function for a particular interval of the function, or a **local minimum,** the least value of a function for a particular interval of the function.

  Students will examine each of the seven turning points and classify each one as either a local maximum, or a local minimum, based on whether the graph increases or decreases after the point.

- The highest degree of the polynomial determines the maximum number of **roots**, or x-intercepts of a function.

  Students will examine the graph of the polynomial function and recognize that there are two roots, or x-intercepts on the graph. They will specify these roots with ordered pairs which each have a zero for the y-coordinate, indicating that the graph intersects the x-axis.

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Extending the Task

- To extend the task, ask students to work with a partner and create an eighth-degree polynomial function with a positive leading coefficient. Encourage them to use coefficients which are small. After each pair of students has written an eighth-degree polynomial, ask them to graph the function using their graphing calculators or other graphing software. Ask them to identify and write down all turning points and roots of their function. Ask for volunteers to share the graphs of their functions, and note if any pair of students was able to graph a function similar to the one in the task. If so, discuss the similarities and differences of their graphs to the one in the task.
- Another option for extending the task is to ask students to work together and use graph paper to sketch three graphs: an eighth-degree polynomial function with a negative leading coefficient, an odd-degree function with a positive leading coefficient, and an odd-degree function with a negative leading coefficient. Ask for volunteers to share the sketches of their three graphs, and discuss the ends of each graph and how the direction of these ends determines which type of polynomial function is represented.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the task is to analyze a graph of a polynomial function representing information about the growth of demand for electricity in the United States, and predict whether I expect the growth of electricity to increase or decrease between 2040 and 2050. Specifically, I looked at the end behavior, turning points, and the roots of the polynomial function on the graph so I could make a valid prediction about what the graph will look like in the interval which represents the years between 2040 and 2050. I described the behavior of the graph using specific points on the graph which represent the turning points and roots. My prediction about the graph's behavior between 2040 and 2050 was based on my analysis of these characteristics of the graph.)
- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining my choices for predicting whether or not I expect the growth of demand for electricity in the United States to increase or decrease between the years 2040 and 2050. First I examined and analyzed the given graph of the polynomial function which represents the approximate growth since 1950 and includes projections through 2040. Then I explained the specific characteristics of the graph, specifically the turning points, roots, and end behavior, and then I justified my expectations about the growth between 2040 and 2050 based on my analysis of these characteristics.)

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- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by recognizing that this scenario could be modeled with the graph of a polynomial function representing the growth of demand for electricity in the United States from 1950 and projected through 2040. I analyzed the specific characteristics of the graph which provided information about the future behavior of the graph. This analysis allowed me to make a prediction about my expectations for the growth of demand for electricity from 2040 to 2050.)

Alternate Strategies or Solutions

Some students may choose to sketch the given graph on their own graph paper and extend the *x*-axis out to include the year 2050. This may allow them to sketch their prediction for the behavior of the graph after 2040, and they may try to determine a pattern in the turning points from 1950 to 2040 and use this pattern to predict their expectation for the growth of demand for electricity from 2040 to 2050.

Technology

Students can use graphing calculators or other graphing software in this task.

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Problem-Based Task 2A.3.3: In Hot Water

Common Core State Standards

A–APR.3	Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.
N–CN.9	(+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.
F–IF.7	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
	c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Task Overview

Focus

How can the graph of a polynomial function be used to identify the zeros, or roots, of the function? How can the equation of a polynomial function provide information in a real-world scenario? In this task, students will use the equation representing the volume of a hot-water tank to determine if there is enough space for the hot-water tank to fit in the space allotted in a basement.

This activity will provide practice with:

- writing an equation in function notation
- rewriting an equation in standard form
- graphing an equation using a graphing calculator or other graphing software
- identifying valid roots of an equation
- performing synthetic division
- interpreting results of synthetic division in the context of the scenario
- calculating the diameter of an object based on its radius

Introduction

This task should be used to explore or to apply the skill of identifying zeros of polynomials when suitable factorizations are available and graphing polynomial functions. Students should be familiar with solving quadratic equations using a variety of methods and simplifying polynomial expressions using synthetic division.

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Begin by reading the problem and clarifying the meaning of *depressed polynomial*, *root*, and *synthetic division*.

depressed polynomial	the result of dividing a polynomial by one of its binomial factors
root	the *x*-intercept of a function; also known as *zero*
synthetic division	a shorthand way of dividing a polynomial by a linear binomial

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine if Luz has enough space in his basement to put a hot-water tank with specific dimensions. To accomplish this task, students will use the formula given for the volume of the hot-water tank to create a polynomial function representing the volume formula. They will graph the equation of the function and identify the roots, or zeros, of the equation in order to find the radius of the hot-water tank. Finally, they will use the value of the radius to determine the diameter, and then they will decide if the hot-water tank will fit in the allotted space that Luz has in his basement.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled with a polynomial function representing the formula for the volume of the hot-water tank. They will graph the equation of this function in order to identify key characteristics of the function that will allow them to determine the roots of the function. They will also interpret the roots of the function in terms of finding the root that is valid in the context of the scenario. They will recognize that the positive root of the function represents the radius of the hot-water tank, and they will calculate the diameter based on this value.

- **SMP 5:** Use appropriate tools strategically.

  Students will use a graphing calculator, or other graphing software, in order to accurately graph a polynomial function which represents the volume of a hot-water tank. They will identify the approximate roots of the equation given by the graph, and they analyze these roots to determine which one is applicable in the context of the scenario.

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Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- making sign errors when performing synthetic division

  Review the process of synthetic division with students before they begin the task, possibly even going through an example. Remind students to pay careful attention to the signs as they are going through the division.

- incorrectly interpreting the roots of the equation as possible values for the radius

  Remind students that the graph of the function will display all possible roots for the equation. Reread the problem with them, and discuss how the radius of the hot-water tank can only be a positive value, and as a result, the negative roots are not possible values for the radius.

- making decisions about the hot-water tank fitting into the basement based on the value of the radius and not the diameter

  Have students sketch a drawing of the hot-water tank. Remind them that in the formula for the volume of the hot-water tank, the variable *r* represents the radius of the tank. However, in order to determine if the tank will fit in the allotted space in the basement, the diameter of the tank is the value that is needed, since it represents the entire width of the tank.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, review the volume formula given in the task. Ask them, "The formula for the volume of the cylindrical-shaped hot-water tank with a hemisphere on top is $V = \frac{2}{3}\pi r^3 + \pi r^2 h$. Explain the components of this formula." (**Answer:** Since the hot-water tank is composed of a cylinder with a hemisphere on the top, there are two parts of the volume formula. The first part of the formula, $\frac{2}{3}\pi r^3$, is the formula for the volume of a hemisphere. The second part of the formula, $\pi r^2 h$, is the formula for the volume of a cylinder. The total volume of the hot-water tank is the sum of the volume of the hemisphere and the volume of the cylinder.)
- After students have read through the problem and are ready to start the task, ask them, "Which dimension of the cylinder indicates how wide the hot-water tank is?" (**Answer:** The width of the tank is indicated by the diameter of the tank.)

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- As an extension to the previous question, ask students, "Since the value for the diameter is not represented in the formula for the volume of the tank, how can you determine the value of the diameter?" (**Answer:** The diameter is twice the length of the radius; it can be determined by finding the value of the radius and then doubling that value.)
- As students begin working on the task, ask them, "Based on the formula for the volume of the hot-water tank, what do you know about the possible number of roots for the equation?" (**Answer:** Since the formula for the volume of the hot-water tank is a polynomial function with a degree of 3, then the maximum number of roots is 3. The function could also have less than 3 roots, but 3 is the maximum number possible.)
- As students are writing the equation of the volume of the hot-water tank in standard form, ask them, "Why are you able to eliminate the value of π in each term? Also, if you wanted to eliminate the fractional coefficient of the first term, how could you do it?" (**Answer:** The value of π can be eliminated because it is contained in each of the terms of the equation. If the coefficient of $\frac{2}{3}$ was to be eliminated, each term in the equation could be multiplied by the denominator, 3."
- After students have used their graphing calculators to graph the equation of the function, ensure their understanding of the roots, or zeros, of an equation. Ask them, "Based on your graph, how can you identify the roots of the equation? What is another name for the roots of an equation?" (**Answer:** On a graph, the roots of an equation are represented by the points where the graph crosses the *x*-axis, which are also called the *x*-intercepts. The roots are also known as the zeros of an equation.)
- As students are analyzing the graph, ask them, "In the context of the scenario, how do you know which roots indicate possible values for the radius of the hot-water tank? Explain." (**Answer:** Since the values for the radius of the hot-water tank must be positive, the only roots which are valid in the context of the scenario are the positive roots. So the only possible root is 30, since the other two roots of –125 and –40, are negative values.)
- Ensure students' understanding of how to algebraically determine if a value is a root, or solution, of an equation. Ask them, "Without the use of the graph, how could you check to make sure that 30 is a solution, or root, of the equation representing the formula for the volume of the water tank?" (**Answer:** If 30 is a valid solution of the equation, then when 30 is substituted in for the value of *r* in the formula, the equation will be satisfied and the result will be 0 on both sides of the equation.)

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- Ask students, "What is the purpose of synthetic division, and what is the result of performing it on the equation for the volume of the hot-water tank?" (**Answer:** Synthetic division is a shorthand way of dividing a polynomial by a linear binomial. It results in a depressed polynomial, which is a polynomial that is one degree less than the original polynomial.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss which value represented in the formula for the volume of the hot-water tank is necessary to find in order to accomplish this goal. Discuss the algebraic manipulations that were performed in order to rewrite the formula in standard form. Also discuss the purpose for graphing the equation in order to identify its roots. Ask students to explain how they determined the roots of the equation, as well as how they applied these roots in the context of the scenario. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their process and analysis for determining if the hot-water tank was able to fit in the allotted space in Luz's basement or not. Discuss the steps and procedures they used to find the roots of the equation, as well as their reasoning about which roots were possible values for the radius and which roots were not. Also discuss how they used the value of the radius to find the diameter of the hot-water tank. Focus on the use of precise mathematical language and clarity, specifically when discussing the standard form of the equation representing the volume of the hot-water tank, as well as the roots, zeros, and/or x-intercepts of the equation.

Connecting to Key Concepts

- Recall that roots are the x-intercepts of a function. In other words, these are the x-values for which a function equals 0. These **zeros** are another way of referring to the roots of a function.

  In this task, students will graph the polynomial function, and then they will identify the roots of the function from the graph.
- For a polynomial function $p(x)$, the **factor of a polynomial** is any polynomial that divides evenly into that function. Recall that when a polynomial is divided by one of its factors, there is a remainder of 0 and the result is a depressed polynomial.

  Students will use the factor of 30 and perform synthetic division to find the resulting depressed polynomial.

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- The zeros of a function are related to the factors of the polynomial. The graph of a polynomial function shows the zeros of the function, which are the *x*-intercepts of the graph.

  Students will graph the given function and identify the zeros of the function from the graph. They will recognize that these zeros are the *x*-intercepts of the graph, and they will analyze the zeros to determine which one is applicable in the context of the scenario.

Extending the Task

To extend the task, ask students to work with a partner to create a new problem, simila to the one presented in the task. Ask them to create a new formula for the volume of the hot-water tank, with the hot-water tank shaped like a cylinder with a cone-shape on top (as opposed to the hemisphere). Ask them to state the value of the radius of the hot-water tank, with the height of the tank as the unknown value this time. Ask them to write down the steps and procedures necessary for solving the equation for the height. Remind them to consider rounding to the nearest tenth of a centimeter. Also ask them to use their graphing calculators to graph their polynomial function and determine the roots of the function. Ask for volunteers to share their problems and discuss their procedures for solving for the height, as well as their procedures for graphing and finding the roots.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the task is to determine if Luz has enough space in his basement to put a hot-water tank with specific dimensions. To accomplish this task, I used the formula given for the volume of the hot-water tank to create a polynomial function representing the volume formula. I graphed the equation of the function and identified the roots, or zeros, of the equation in order to find the radius of the hot-water tank. Finally, I used the value of the radius to determine the diameter, and then I decided if the hot-water tank will fit in the allotted space that Luz has in his basement.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by recognizing that this scenario could be modeled with a polynomial function representing the formula for the volume of the hot-water tank. I graphed the equation of this function in order to identify key characteristics of the function which allowed me to determine the roots of the function. I also interpreted the roots of the function in terms of finding the root that was valid in the context of the scenario. I recognized that the positive root of the function represents the radius of the hot-water tank, and then I calculated the diameter based on the value of the radius.)

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- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** I used appropriate tools strategically as I used a graphing calculator to accurately graph a polynomial function which represented the volume of a hot-water tank. I identified the approximate roots of the equation given by the graph, and I analyzed these roots to determine which one was applicable in the context of the scenario.)

Alternate Strategies or Solutions

Instead of graphing the equation using graphing technology, some students may choose to plot points on graph paper and sketch a graph of the equation in this manner. Encourage students to choose enough points so that they will get a representative picture of the equation. One method of choosing points is to find the x- and y-intercepts and graph those points. Remind students to choose simple, whole number values, and also remind them to check the scale on their graph to accommodate the larger y-values. Encourage students to check their graphing with that of a classmate who used graphing technology.

Technology

Students can use graphing calculators or other graphing software for this task.

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Problem-Based Task 2A.4.1: Advertise Here!

Common Core State Standard

A–REI.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★

Task Overview

Focus

How can the solution(s) to a system of equations be found graphically? What do the points of intersections of two equations indicate about the solution(s)? In this task, students will graph two equations that describe the location of the S-curve in a racetrack and the location of a footbridge, and they will determine the coordinates of the points of intersection of these two equations, or solutions of the system of equations, which represent the locations of advertisement signs.

This activity will provide practice with:

- graphing linear and polynomial equations using graphing technology
- identifying points of intersection on a graph
- interpreting points of intersection in the context of a real-world scenario
- estimating coordinates for ordered pairs
- substituting values into equations
- verifying solutions of a system of equations

Introduction

This task should be used to explore or to apply the skill of explaining why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; finding the solutions approximately, e.g., using technology to graph the functions, and making tables of values, or finding successive approximations. Students should already be familiar with creating a table of values for an equation, graphing various forms of equations, and graphing multiple equations on a graphing calculator.

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Begin by reading the problem and clarifying the meaning of *point(s) of intersection* and *system of equations*.

point(s) of intersection	the ordered pair(s) where graphed functions intersect on a coordinate plane; these are also the solutions to systems of equations
system of equations	a set of equations with the same unknowns

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to find the coordinates of the proper positions of the advertisement signs along a footbridge at a new racetrack. They will accomplish this goal by first graphing the equations which describe the locations of the footbridge and the S-curve in the racetrack. Then, they will identify the points of intersection of the two equations, as these intersection points represent the coordinates of the positions in which the advertising signs should be placed. They will either create a table of values or use a graphing calculator to graph the equations, and they will approximate the points of intersection. Finally, they will verify that the approximated points represent the proper locations of the signs by substituting each x-coordinate into both equations and evaluating the equations to see that the resulting value is the same as the corresponding y-value.

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they recognize that they will need to graph the given equations representing the locations of the footbridge and the S-curve in the racetrack. They will reason that the points of intersection of the two equations represent the locations where the advertising signs should be placed. They will use their graphing calculators to find the best approximation for each point of intersection, by either analyzing the table of values, or by tracing over the points of intersection. They will reason quantitatively as they carefully substitute the x-coordinates for the points of intersection into both equations, evaluate and simplify the equations, and determine if the resulting value is the same as the corresponding y-coordinate for each point.

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- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled by a system of equations that can be graphed in order to find the solutions. They will recognize that the equation describing the location of the S-curve on the racetrack is a polynomial, and the equation describing the location of the footbridge is linear. They will examine and analyze the graph of both equations on the same coordinate plane in order to identify the points of intersection of the equations. They will interpret these points of intersection as the locations of the positions of the advertisement signs that will allow television viewers to see the signs as they are watching the cars race.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- not identifying all points of intersection

  Remind students that there may be several points of intersection between two equations on a graph. Encourage them to look carefully at the graph of the system of equations to ensure they have found all points of intersection.

- incorrectly graphing each equation or incorrectly entering the equations into the calculator

  Encourage students to double-check their work when entering the equations on their calculator by performing the steps twice. Also encourage them to create a table of values and sketch the graphs by hand as another method to verify their results with a calculator.

- not understanding that solutions must satisfy every function in a system

  Remind students to check the coordinates of the points of intersection in both equations. Remind them that they must substitute the coordinates for x and y in both of the original equations in order to verify that each point of intersection is indeed a solution.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them to identify characteristics of the equation describing the location of the S-curve in the racetrack. Ask them, "For the equation describing the location of the S-curve in the racetrack, what type of equation is it, and what can you identify about its degree and shape?" (**Answer:** The equation describing the location of the S-curve in the racetrack is a third-degree polynomial equation. This is an odd-degree polynomial, and since the leading coefficient is positive, the left "end" of the equation will extend down to the left towards negative infinity, and the right "end" will extend up to the right towards positive infinity.)

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- Ask students to consider a way to rewrite the polynomial representing the location of the S-curve in the racetrack. Ask them, "The equation given in the task for the S-curve is $f(x) = \frac{1}{30}\left(x^3 - 3x^2 - 270x\right)$. How can this equation be rewritten to eliminate the parentheses? What is the resulting new equation?" (**Answer:** The fraction, $\frac{1}{30}$, could be distributed throughout the terms in the parentheses: $f(x) = \frac{1}{30}(x^3) - \frac{1}{30}(3x^2) - \frac{1}{30}(270x)$. This equation simplifies to $f(x) = \frac{1}{30}x^3 - \frac{1}{10}x^2 - 9x$.)
- Ask students to identify characteristics about the equation describing the location of the footbridge. Ask them, "How can you rewrite the equation describing the location of the footbridge in slope-intercept form, and what can you tell about the slope and the intercept of the equation, without graphing it?" (**Answer:** The equation for the location of the footbridge can be rewritten as $g(x) = -3x + 2$. The slope of the equation is a negative 3, so the line is graphed in a downward direction. The y-intercept is 2, so the graph will intercept the y-axis at the point (0, 2).)
- Ensure students' understanding of the possible types of solutions for any system of equations. Ask them, "When two or more equations are graphed on the same coordinate plane, what are the possibilities for the types of solutions to the system of equations? Explain each type." (**Answer:** One type of solution a system can have is an infinite number of solutions. This is when the equations overlap each other, and every point on the equations is a point of intersection. A system can have no solutions, which means that the equations will never intersect each other. Finally, a system can have a finite number of solutions, where the equations will intersect at a specific number of points.)
- Before students graph the two equations on the same coordinate plane, ask them, "Without graphing the equations, how can you verify algebraically that the point (0, 2) is not a solution to the system of equations?" (**Answer:** In order for the point (0, 2) to be a solution to the system of the two equations, then when 0 is substituted in for x in both equations, the result would have to be 2. When 0 is substituted in for the equation describing the location of the S-curve on the track, the result is 0: $f(0) = \frac{1}{30}\left((0)^3 - 3(0)^2 - 270(0)\right); f(0) = \frac{1}{30}(0 - 0 - 0); f(0) = 0$. When 0 is substituted in for the equation describing the location of the footbridge, the result is 2: $f(0) = 2 - 3(0); f(0) = 2 - 0; f(0) = 2$. Therefore, (0, 2) is not a solution to this system.)

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- As students are working through the task, ask them, "Why do you think it is advantageous to graph the equations using a graphing calculator, as opposed to creating a table of values and sketching the graph of the equations by hand?" (**Answer:** Although the linear equation describing the location of the footbridge is an equation which has whole number coefficients and would be fairly simple to create a table of values for graphing purposes, the equation describing the location of the S-curve in the race track has exponents and fractions, and therefore creating a table of values by hand would require more complex calculations.)
- Discuss with students the values chosen from the calculator for the x-values representing the points of intersection. Ask students, "Why do you think the x-values given in the task for the intersection points are all rounded to the hundredths place value, and not rounded to the nearest whole number?" (**Answer:** If the x-values had been rounded to the nearest whole number, then the values for the coordinates representing each point of intersection of the equations would be less precise and less accurate than when more decimal places are used to approximate the values.)
- Ensure students' understanding of algebraically verifying that the estimated coordinates for the points of intersection are solutions to the system of equations. Ask them, "What is the goal of substituting in the approximated values of the x-coordinates for each point of intersection into each equation?" (**Answer:** The goal of substituting the approximated x-values of the coordinates for each point of intersection is to confirm that the y-values, or $f(x)$ and $g(x)$ values, are the same as shown on the calculator. Once these values are shown to be closely equal, then it can be concluded that the given estimated coordinates represent the points of intersection of the system of equations.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss how the scenario could be modeled by graphing a system of equations, and discuss what each equation represents in the context of the scenario. Ask students to explain their thoughts on the two equations in terms of why it was better to graph them with a calculator as opposed to creating a table of values for each equation to find the points of intersection. Discuss how the tracing feature of the graphing calculator is helpful in terms of approximating the coordinates of the intersection points. Also discuss the choice of rounding the values of the coordinates to the nearest hundredth, as opposed to rounding to the nearest whole numbers. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.

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- Compare students' strategies and ways of justifying responses. Ask students to share why they either created a table of values for each equation, or graphed the equations instead. Discuss the advantages of approximating the locations of the points of intersection with a graphing calculator, and discuss how rounding affects the outcome of the values of the coordinates. Ask students to explain their steps for substituting and evaluating the equations for the specific x-coordinates, and ensure their understanding of how this process verifies that the points are indeed solutions to the system of equations. Focus on the use of precise mathematical language and clarity, specifically when discussing the types of equations used to represent the two objects in the task, as well as how the points of intersection relate to the solutions of the system of the two equations.

Connecting to Key Concepts

- The real solutions of a system of equations can often be found by graphing. Recall that the graph of an equation is the set of all the solutions of the equation plotted on a coordinate plane.

  In this task, students will graph a linear equation and a polynomial equation that represent the locations of a footbridge and a racetrack in a real-world scenario. They will either use graphing technology, or they will create a table of values in order to represent the equations on a coordinate plane.

- The **points of intersection** of the graphed system of equations are the ordered pairs where graphed functions intersect on a coordinate plane. These are also the solutions to the system. The solution or solutions to a system of equations with $f(x)$ and $g(x)$ occurs where $f(x) = g(x)$ and the difference between these two values is 0.

  Students will identify the points of intersection of the two graphed equations by determining the coordinates of the points where the two graphs intersect each other. They will also verify that these points are solutions to the system of equations by substituting each x-coordinate into both equations and verifying that both equations result in the same corresponding y-coordinate.

- A system may have an infinite number of solutions, a finite number of solutions, or no realsolutions.

  Students will recognize that the given system has a finite number of solutions, as they will identify three finite points of intersection of the two equations on the graph.

- Using a graphing calculator to graph both equations is often helpful. Most calculators have a trace feature to approximate where the equations intersect, or can display a table of values for each equation so you can analyze the values.

  Students will use graphing calculators to graph both equations on the same coordinate plane. They will use the trace feature to approximate the points of intersections. They may also analyze the table of values given by the graphing calculator to determine closer approximations.

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- To confirm a solution to a system, substitute the coordinates for x and y in both of the original equations. If the intersection point is approximated, the results will be nearly equal.

  Students will substitute the three pairs of coordinates into both equations in order to verify that they are solutions to the system. They will observe that the results are approximately the same, and therefore they can conclude that the points of intersection are indeed solutions to the equations.

Extending the Task

To extend the task, ask students to work with a partner to recreate the task, but this time with the use of a quadratic equation to describe the location of the curve on the racetrack. Allow students to use the same linear equation to represent the footbridge. Encourage them to create their quadratic equation so that it will intersect the equation for the footbridge. Discuss the number of possible points of intersection, based on the type of equation. Ask for volunteers to share their new problems and discuss their graphs of the two equations, including the points of intersection and how they were able to verify that the intersection points were solutions to the system of equations.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the task is to find the coordinates of the proper positions of the advertisement signs along a footbridge at a new racetrack. I accomplished this goal by first graphing the equations which described the locations of the footbridge and the S-curve in the racetrack. Then, I identified the points of intersection of the two equations, as these intersection points represented the coordinates of the positions in which the advertising signs should be placed. I used a graphing calculator to graph the equations, and I used the table of values given by my calculator to approximate the points of intersection. Finally, I verified that the approximated points represented the proper locations of the signs by substituting each x-coordinate into both equations and evaluating the equations to see that the resulting value was the same as the corresponding y-value.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I recognized that I needed to graph the given equations representing the locations of the footbridge and the S-curve in the racetrack. I reasoned that the points of intersection of the two equations represented the locations where the advertising signs should be placed. I used my graphing calculator to find the best approximation for each point of intersection, by analyzing the table of values and by tracing over the points of intersection.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning as I carefully substituted the x-coordinates for the points of intersection into both equations, evaluated and simplified the equations, and determined if the resulting value was the same as the corresponding y-coordinate for each point.)

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- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by recognizing that this scenario could be modeled by a system of equations that can be graphed in order to find the solutions. I recognized that the equation describing the location of the S-curve on the racetrack was a polynomial equation, and the equation describing the location of the footbridge was a linear equation. I examined and analyzed the graph of both equations on the same coordinate plane in order to identify the points of intersection of the equations. I interpreted these points of intersection as the locations of the positions of the advertisement signs that will allow television viewers to see the signs as they are watching the cars race.)

Alternate Strategies or Solutions

- Some students may sketch a drawing of a racetrack with an S-shaped curve, as well as a footbridge that passes over the track. Encourage students to sketch their drawing from an overhead view, such that they are looking down on the track and the footbridge. A sketch of the scenario may help students better visualize the task.
- Some students may choose not to use any graphing technology, but instead they may decide to create a table of values for both equations in order to graph them. Encourage students to carefully choose x-values such that the calculations for the corresponding y-values are not too tedious and time-consuming. Also remind them to ensure that they have chosen an appropriate number of values so that they will be able to sketch an accurate representation of the equations on the graph. Encourage students to verify their graphs with students who have used calculators to graph the equations.

Technology

Students can use graphing calculators or other graphing technology in this task.

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Implementation Guide

Problem-Based Task 2A.5.1: New Restaurant Buzz

Common Core State Standard

A–SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★

Task Overview

Focus

How can a geometric sequence be applied to a real-world scenario? How can formulas used in a geometric sequence be used to predict future terms in a sequence? In this task, students will determine and apply the explicit formula for a geometric sequence which represents the number of new people who will be told about a positive restaurant experience each week.

This activity will provide practice with:

- interpreting a pattern described from written information
- predicting the next term in a sequence based on information given in a scenario
- identifying the common ratio of a geometric sequence
- creating an explicit formula for a geometric sequence
- applying a geometric sequence in the context of a real-world scenario

Introduction

This task should be used to explore or to apply the skill of using a geometric sequence and its explicit formula to solve problems. Students should already be familiar with rearranging formulas, applying order of operations, and applying exponents.

Begin by reading the problem and clarifying the meaning of *common ratio*, *explicit formula for a geometric sequence*, *geometric sequence*, *sequence*, and *term*.

common ratio — the ratio of a term in a geometric sequence to the previous term in that sequence; indicated by the variable r and given by the formula $r = \frac{a_n}{a_{n-1}}$

explicit formula for a geometric sequence — a formula used to find any term in a sequence. The formula is $a_n = a_1 \bullet r^{n-1}$, where n is a positive integer that represents the number of terms in the sequence, r is the common ratio, a_1 is the value of the first term in the sequence, and a_n is the value of the nth term of the sequence.

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geometric sequence	an ordered list of terms in which each new term is the product of the preceding term and a common ratio, r. For a sequence to be geometric, r cannot be equal to 1 ($r \neq 1$).
sequence	an ordered list of numbers or elements
term	an element in a sequence

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they recognize that the scenario presented in the task represents a geometric sequence, in that the number of people who will be told each week about the restaurant is found by multiplying the number of people from the previous week by a factor of 4. They will reason that 4 is the common ratio of the sequence, and an explicit formula for a geometric sequence can be created to determine the number of people told about the restaurant for any specific number of weeks. They will reason quantitatively as they first determine the common ratio of the sequence by dividing each term by the previous term, to achieve the result of 4. Then they will substitute the value for the common ratio into the explicit formula.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled with a geometric sequence. They will determine this by identifying that each week the number of people told about the positive experience from the restaurant increases by a factor of four. They will identify this factor of four as the common ratio of the sequence. They will also recognize that an explicit formula can be created in order to determine the number of people who are told about the positive restaurant experience for any specific number of weeks. They will also interpret and analyze the results of this sequence in the context of the scenario and conclude that the restaurant will receive a lot of positive exposure as a result of the large number of people who will know about it.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure and procedures involved in calculating the number of people who are told about the positive restaurant experience each week. They will recognize that the number of people increases by a factor of four each week, and they will identify this factor as the common ratio of a geometric sequence. They will also use the structure of the explicit formula of a geometric sequence to create a formula which will allow them to determine the number of people told about the restaurant given any specific number of weeks.

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Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- inverting the ratio when calculating the common ratio

  Remind students that the common ratio is the ratio of a term to the previous term in the sequence. Remind them that the number of people from one term to the next is increasing exponentially, so each term is larger than the previous one by the same common factor.

- multiplying by the value of the previous term instead of the first term when using the explicit formula

  Have students write down the general format for the explicit formula on the top of their papers. Also encourage them to write down what each variable represents in the formula and review the parts of the formula with them before they begin the task.

- incorrectly determining the number of people told about the restaurant in the second week as 8 instead of 16 because 4 was multiplied by the term number, 2, instead of the common ratio

  Remind students that Kylie tells 4 people about her positive restaurant experience, so 4 represents the first term in the sequence. Since each of these 4 people then tell 4 people, then the first term is multiplied by 4, and not multiplied by 2. So, the number of people who are told about the restaurant in week 2 is 16.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them to consider other possible factors which can lead to a positive experience in a restaurant. Ask them, "Besides a short wait time for your food, what factors could contribute to you having a great experience in a restaurant, such that you want to tell your friends about the restaurant?" (**Answer:** Answers may vary. Some other factors may include friendly and helpful wait staff, excellent quality of food, aesthetically pleasing appearance of the food, comfortable chairs, a pleasant environment, and reasonable prices.)

- As students begin working on the task, ask them, "Why do you think it would benefit a restaurant owner to use a geometric sequence, in terms of trying to generate more customers in their establishment?" (**Answer:** Knowing how news travels fast by word of mouth, a restaurant owner could use a geometric sequence to try and predict the number of possible customers, based on having positive customer feedback, and assuming that satisfied customers will share their experiences with other people. A restaurant owner could use a model such as presented in this task, in that it is possible for a happy customer to tell 4 friends a week about their experience. The owner could hope that these 4 friends would come into the restaurant, and then each tell 4 of their friends, and so on.)

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- As students begin working on the task, ask them, "What is the difference between a recursive formula and an explicit formula for a geometric sequence?" (**Answer:** A recursive formula allows you to find the next term in a sequence, and an explicit formula allows you to find any specific term of a sequence.)
- As students are working on the task, ensure their understanding of the components of the explicit formula. Ask them, "What values, or parts of the sequence, must be known in order to apply the explicit formula of a geometric sequence?" (**Answer:** The common ratio and the value of the first term of the sequence must be known in order to apply the explicit formula.)
- Ask students, "When applying the explicit formula, why is it important to remember how to apply order of operations?" (**Answer:** Since the explicit formula contains multiplication and exponents, it is important to remember that evaluating the exponents in the formula is done before the multiplication in the formula.)
- As students are working through the task, ask them to apply the explicit formula to ensure their understanding of the formula. Ask them, "Based on the geometric sequence used to represent the scenario given in the task, how many people will have been told about the restaurant in week 7? Explain your answer using the explicit formula." (**Answer:** Week 7 would represent the seventh term of the sequence, a_7. Using the explicit formula, with 4 as the common ratio, r, and 4 as the first term, a_1, the number of people who will have been told about the restaurant in week 7 is $a_7 = 4 \bullet 4^6$; $a_7 = 4 \bullet 4096$; $a_7 = 16{,}384$.)
- As students are completing the task, ask them to reflect on the reality of the results. Ask them, "Do you think it is realistic that after 7 weeks there are 16,384 people who have been told about the restaurant? Why or why not?" (**Answer:** Answers will vary. Some students may conclude that it is not very realistic for each person who hears about the positive restaurant experience to tell 4 people about it. However, even if each person told only one person about the restaurant each week, it could lead to a significant number of new customers. Another student might note that soon almost everyone in the area would already know about the restaurant, so that the growth rate becomes unrealistic.)
- Ask students, "What are some other ways that the restaurant could advertise their positive feedback from their customers that could allow them to generate more business? Which method do you think would cause the most number of people to know about the restaurant?" (**Answer:** The restaurant could ask satisfied customers to fill out a survey either in person or online. Filling out a review online could possibly allow for the most number of people to see the positive feedback.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss how the scenario could be modeled with a geometric sequence, and discuss which parts of the scenario can be represented by terms of a geometric sequence. Ask students to explain how the common ratio was determined, and ask them to explain how this ratio could be used to help determine the number of people who are told about the positive restaurant experience each week. Discuss how the explicit formula is used to determine any term of the sequence and how this relates to the context of the scenario, specifically how to determine the number of people for any given week. Ask students to explain their thoughts about how a model such as this could help a restaurant business in terms of predicting the possible number of future customers. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process for determining that four was the common ratio for the geometric sequence representing the number of people who are told about the restaurant each week. Discuss the advantages of using a geometric sequence as a model for this particular scenario, as well as how a recursive formula could be created in order to determine the number of people told about the restaurant for each consecutive week. Focus on the use of precise mathematical language and clarity, specifically when discussing using a geometric sequence to model the scenario, as well as when identifying the factor of four as the common ratio of the sequence. Also ensure that students apply the correct terminology for the components of the explicit formula, specifically the denotations of the term numbers used in the formula.

Connecting to Key Concepts

- Given a geometric sequence $\{a_1, a_2, a_3, \ldots a_n\}$, the first term is a_1, the second term is a_2, the third term is a_3, and the nth term is a_n.

  In this task, students will recognize that the number of people that will be told about the positive restaurant experience each week represent terms of a geometric sequence. For example, the first term, a_1, is 4 people, the second term, a_2, is 16 people, the third term, a_3, is 64 people, and so on.

- To determine the common ratio in a geometric sequence, find the ratio, r, of the second term to the first term, the third term to the second term, or, in other words, the ratio of the nth term to the $n-1$ term: $r = \frac{a_n}{a_{n-1}}$.

  Students will determine the common ratio for the scenario given in the task by recognizing that the number of people who are told about the positive restaurant experience increases by a factor of four each week. They will find this by dividing the number of people in a specific number of weeks by the number of people from the previous week.

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- The **explicit formula for a geometric sequence** is $a_n = a_1 \bullet r^{n-1}$, where n is a positive integer that represents the number of terms, r is the common ratio, a_1 is the value of the first term, and a_n is the value of the nth term of the sequence.

  Students will develop the explicit formula for the geometric sequence given in the task by identifying 4 as the common ratio, r, and 4 as the first term. They will recognize that this formula can be used to find the nth term in the sequence.

- Use the explicit formula to find the nth term of a sequence when the first term and common ratio are known.

  Students will recognize that the explicit formula they created can be used to find a specific term in the sequence, as they will have identified the common ratio as 4, and the first term in the sequence as 4.

Extending the Task

- To extend the task, have students determine the number of people who will have been told about the positive restaurant experience after 8 weeks. Ask them to explain how they used the explicit formula to find this value. Then, ask them to predict how many weeks it will take for over 10,000 people to be told about the restaurant. Ask them to explain and discuss how they determined which values to use in the formula, as well as which parts were unknown. Ask for volunteers to explain their processes used when determining the number of weeks.
- Another option for extending the task is to ask students to recreate the task using the same scenario, but this time with Kylie only telling 3 people about her positive restaurant experience, and then assuming those three people tell 3 new people each week. Ask students to find the first 10 terms of the sequence, and then ask them to find how many people will have been told about the restaurant after a year. Ask for volunteers to share their results and explain their steps and procedures when applying the explicit formula.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first recognize that the scenario presented in the task represented a geometric sequence, in that the number of people who will be told each week about the restaurant was found by multiplying the number of people from the previous week by a factor of 4. I reasoned that 4 is the common ratio of the sequence, and an explicit formula for a geometric sequence could be created to determine the number of people told about the restaurant for any specific number of weeks.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning as I first determined the common ratio of the sequence by dividing each term by the previous term, to achieve the result of 4. Then, I substituted the value for the common ratio into the explicit formula.)

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- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by first recognizing that this scenario could be modeled with a geometric sequence. I determined this by identifying that each week the number of people told about the positive experience from the restaurant increased by a factor of four. I identified this factor of four as the common ratio of the sequence. I also recognized that an explicit formula could be created in order to determine the number of people who are told about the positive restaurant experience for any specific number of weeks. I also interpreted and analyzed the results of this sequence in the context of the scenario and concluded that the restaurant will receive a lot of positive exposure as a result of the large number of people who will know about them.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I looked for and made use of structure when solving this problem by applying the structure and procedures involved in calculating the number of people who are told about the positive restaurant experience each week. I recognized that the number of people increased by a factor of four each week, and I identified this factor as the common ratio of a geometric sequence. I also used the structure of the explicit formula of a geometric sequence to create a formula which allowed me to determine the number of people told about the restaurant given any number of weeks.)

Alternate Strategies or Solutions

Instead of only creating the explicit formula, some students may choose to use the recursive formula for a geometric sequence in order to find the first several terms of the sequence. They may also choose to draw a tree diagram in order to visualize how many people are told about the restaurant each week. Or, they may also create another type of organizational chart representing the number of people for each week. Encourage them to verify their results for the first several terms of the sequence with the results given in the task.

Technology

Students may use scientific calculators for this task.

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Implementation Guide

Problem-Based Task 2A.5.3: Will It Overflow or Not?

Common Core State Standard

A–SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★

Task Overview

Focus

How can a geometric series be applied to a real-world scenario? How can the formula for the sum of an infinite geometric series be used to determine a sum when an experiment is performed indefinitely? In this task, students will apply the formula for the sum of an infinite geometric series in order to determine the point, if any, that a water dispenser container will overflow when water is continually added at a specific rate.

This activity will provide practice with:

- interpreting written information in the context of a scenario
- determining that a scenario can be represented by a geometric series
- identifying the common ratio and first term of a geometric series
- substituting values into the formula for the sum of an infinite geometric series
- calculating the sum of an infinite geometric series
- analyzing results of the sum of an infinite geometric series and drawing conclusions based on these results

Introduction

This task should be used to explore or to apply the skill of using the formula for the sum of an infinite geometric series to solve problems. Students should already be familiar with determining the common ratio and expanding a geometric series given in summation notation.

Begin by reading the problem and clarifying the meaning of *converge*, *geometric series*, *infinite geometric series*, *series*, and *sum formula for an infinite geometric series*.

converge to approach a finite limit. If the sequence of the partial sums of a series approaches the value of a given number (the limit), then the entire series converges to that limit; that is, the series has a sum

geometric series the sum of a specified number of terms from a geometric sequence

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infinite geometric series	a series that has an unlimited number of terms $(n = \infty)$; can be written as $\sum_{k=1}^{\infty} a_1 r^{k-1}$, where a_1 is the first term, r is the common ratio, and k is the number of the term
series	the sum of the terms of a sequence
sum formula for an infinite geometric series	$S_n = \frac{a_1}{1-r}$, where S_n is the sum, a_1 is the first term, and r is the common ratio

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine whether or not the water dispenser container will overflow when water is continually added at a specific rate. To accomplish this task, they will recognize that the scenario can be modeled with an infinite geometric series. First, they will identify the common ratio of the series, as well as the first term of the series. Then, they will apply the formula for the sum of an infinite geometric series to determine the amount of water, in fluid ounces, that can be added to the container. They will also convert the capacity of the water tank in gallons to fluid ounces in order to compare the amount of water the container can hold to the sum of the series in order to make a final determination about whether or not the container will overflow or not when the amount of water added decreases by half each time.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for determining whether or not the water dispenser container will overflow if Luca continues to add water to it. They will base their arguments on the information given that each time he adds water, he is adding half of the previous amount of water to the container. They will recognize that this scenario can be represented with an infinite geometric series because there is a common ratio of $\frac{1}{2}$, and there is no specified limit for the number of times that Luca will add water to the container. They will apply the formula for the sum of an infinite geometric series in order to find the value of

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the sum of the amount of water that will be in the container if Luca continues the experiment indefinitely. Once this sum is determined, they will compare it to the total capacity of the container to see if the container will overflow or not. Then, they will present their arguments to others and explain their decision based on the value of the sum of the series.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure and procedures involved in using a geometric series in order to find a sum. They will first recognize that the amount of water added to a container each time decreases by one-half of the previous amount. Then, they will identify the common ratio as $\frac{1}{2}$, and they will use this value and the value of the first term of the series and substitute them into the formula for the sum of an infinite geometric series. Finally, they will apply the structure of this formula and determine the amount of water that can be added to the water container, and they will compare this amount to the capacity of the water container to conclude if the water tank will overflow or not.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- mistakenly thinking that the series in the task diverges instead of converges

  Remind students that a series that diverges will not have a sum, and that a series that converges will have a sum. Also remind them that the absolute value of the common ratio in the task is less than 1, so it is a series that converges.

- forgetting to multiply by the reciprocal when dividing fractions

  Remind students that they will be working with complex fractions when they use the formula for the sum of an infinite geometric series, because the common ratio of the series is $\frac{1}{2}$, and when this value is subtracted from 1 in the formula, the remaining value in the denominator is $\frac{1}{2}$. To simplify the remaining complex fraction, the reciprocal of $\frac{1}{2}$ must be used and multiplied by the value in the denominator. Review other examples with complex fractions if necessary.

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- incorrectly concluding that the container will overflow because the wrong units were compared

  Remind students that the capacity of the water dispenser container is given in units of gallons, but the amount of water that Luca adds is given in units of fluid ounces. Therefore, in order to compare the capacity of the container to the sum of the infinite geometric series, 5 gallons must be converted into fluid ounces so that the same units of measurement can be compared.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin working on the task, discuss their initial thoughts and predictions about whether or not the container will overflow. Ask them, "Based on the information given in the task, do you think the water dispenser container will overflow if Luca continually adds water at the rate described in the task?" (**Answer:** Answers will vary. Some students may think that the container will overflow since there is no given limit to the number of times that Luca adds water to it. Some students may recognize how much smaller the amounts of water added becomes and think it will never overflow.)
- As students begin working on the task, ensure their understanding of the choice for the type of series used in this task. Ask them, "What information are you given that tells you that the scenario can be represented by an infinite geometric series?" (**Answer:** Since it can be determined that the common ratio is $\frac{1}{2}$, a geometric series can be used. Also, since there is no limit to the number of times that Luca will add water to the container, an infinite geometric series can be used.)
- As students begin working on the task, ask them, "What can you do with the given values in the task in order to better visualize the amounts of water that are added to the container? What are the first 10 amounts of water that Luca adds, i.e. the first 10 terms of the series?" (**Answer:** It might be helpful to list out the values in the series in order to better visualize the amounts of water. The first 10 terms of the series are: 200, 100, 50, 25, 12.5, 6.25, 3.125, 1.5625, 0.78125, 0.390625.)
- As an extension to the previous question, ask students, "What do you notice about the number of decimal places with each consecutive term, beginning with the fifth term?" (**Answer:** When the amounts are decreased by a factor of $\frac{1}{2}$ each time, beginning with the fifth term, there is an additional decimal place added onto each consecutive term.)

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- Ensure students' understanding of the difference between a series that converges and a series that diverges. Ask them, "What is the difference between the word 'converge' and the word 'diverge' in the context of a geometric series?" (**Answer:** The word 'converge' means to meet at one point, or toward a definite limit as more terms are added. The word 'diverge' means to go a different direction, or increase indefinitely as more terms are added.)
- As an extension to the previous question, ask students, "Why does the geometric series in this task converge? To what point do the terms seem to be converging?" (**Answer:** The geometric series in this task converges because its common ratio is less than one. As more and more terms are added, the values of the terms get closer and closer to zero, but they will never reach 0.)
- Discuss the importance of comparing values that have the same units of measurements. Ask students, "Why is it necessary to convert gallons to fluid ounces in the task?" (**Answer:** In the original information, the capacity of the water dispenser container is given as 5 gallons. However, the amount of water that Luca continually adds to the container is given in fluid ounces. Once the sum of the series is found as 400 fluid ounces, then the capacity of the container must be represented in fluid ounces in order to compare it with the sum of 400 fluid ounces.)
- As students are completing the task, ask them, "What are the practical limitations to performing an experiment such as this one, in terms of continually adding water to a container at half the amount each time as the previous time?" (**Answer:** In reality, there will be a limit to the number of times that water could be added to a container, because eventually it will be nearly impossible to measure such small amounts of water to put into the container.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their thoughts on how to accomplish the goal of the task. Discuss how the scenario could be modeled with a geometric series, and discuss which parts of the scenario can be represented by terms of a geometric series. Ask students to explain how the common ratio was determined, given the amounts of water that are being added to the container each time. Discuss why the geometric series is an infinite geometric series, and ask students to explain how the common ratio and first term are substituted into the formula for an infinite geometric series. Also ask students to explain how the resulting sum is used to compare the total capacity of the water container. Encourage students to discuss any difficulties or confusion they experienced when working through the task.

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- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process for determining that the series modeled in the scenario is an infinite geometric series. Discuss the practical and realistic limitations of continually adding water for an indefinite period of time. Ensure students' understanding of why the tank will never overflow, in terms of interpreting the sum of the infinite geometric series. Also ensure their understanding of the necessity to convert the 5-gallon capacity to units of fluid ounces in order to make a valid comparison. Focus on the use of precise mathematical language and clarity, specifically when discussing using a geometric series to model the scenario, as well as when identifying the common ratio and first term of the series. Also ensure that students apply the correct terminology for the components of the formula for the sum of an infinite geometric series.

Connecting to Key Concepts

- A geometric series that has **infinite** terms (that is, where $n = \infty$), is called an **infinite geometric series**.

  In this task, students will identify the type of series represented in the scenario of the task as an infinite geometric series, because the information indicates that Luca could add water to the container indefinitely.

- In a series, to **converge** means to approach a finite limit. If the sequence of the partial sums of a series approaches the value of a given number (the limit), then the entire series converges to that limit. In other words, the series has a sum.

  Students will recognize that the scenario represents a series that will converge, because the series will have a sum.

- An infinite series converges when the absolute value of the common ratio r is less than 1; $(|r| < 1)$.

  Students will identify the common ratio in the series as $\frac{1}{2}$, and they will recognize that the absolute value of $\frac{1}{2}$ is less than 1. Therefore, they will conclude that the series converges.

- The **sum formula for an infinite geometric series**, if the sum exists, is given by $S_n = \frac{a_1}{1-r}$, where S_n is the sum, a_1 is the first term, and r is the common ratio.

  Students will apply the sum formula for an infinite geometric series and find the sum of the series, which represents the amount of water that Luca adds to the container.

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Extending the Task

- To extend the task, have students work through the task again, but this time using 300 fluid ounces as the initial amount of water that Luca adds to the container. Using the same common ratio of $\frac{1}{2}$, ask them to determine if the water tank will overflow given this scenario. Ask for volunteers to share the procedures and steps they used in order to make a conclusion, specifically their calculations using the formula for the sum of the infinite geometric series.

- Once it has been determined that the water tank will not overflow if the initial amount of water added is 300 fluid ounces, because the sum of the infinite geometric series is now 600, and 600 is less than 640, ask students to work together to try and determine the amount of water that would need to be added initially, in order to cause the water container to overflow. Have them use $\frac{1}{2}$ as the common ratio again. Remind them to see how close they can get to a value that is just over 640, with the understanding that they might not be able to get an exact value. Ask for volunteers to share their results and explain their thoughts and procedures.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the task is to determine whether or not the water dispenser container will overflow when water is continually added at a specific rate. To accomplish this task, I recognized that the scenario could be modeled with an infinite geometric series. First, I identified the common ratio of the series, as well as the first term of the series. Then I applied the formula for the sum of an infinite geometric series to determine the amount of water, in fluid ounces, that can be added to the container. I also converted the capacity of the water tank in gallons to fluid ounces in order to compare the amount of water the container can hold to the sum of the series, in order to make my final determination about whether or not the container will overflow or not when the amount of water added decreases by half each time.)

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- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining my choice for determining whether or not the water dispenser container will overflow, if Luca continues to add water to it. I based my arguments on the information given that each time Luca added water, he added half of the previous amount of water to the container. I recognized that this scenario could be represented with an infinite geometric series because there is a common ratio of $\frac{1}{2}$, and there is no specified limit for the number of times that Luca will add water to the container. I applied the formula for the sum of an infinite geometric series in order to find the value of the sum of the amount of water that will be in the container if Luca continues the experiment indefinitely. After I found this sum, I compared it to the total capacity of the container to determine if the container will overflow or not. I presented my arguments to others and explained my decision based on the value of the sum of the series.)

- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I looked for and made use of structure when solving this problem by applying the structure and procedures involved in using a geometric series in order to find a sum. I first recognized that the amount of water added to a container each time decreased by one-half of the previous amount. Then, I identified the common ratio as $\frac{1}{2}$, and I used this value and the value of the first term of the series and substituted them into the formula for the sum of an infinite geometric series. Finally, I applied the structure of this formula and determined the amount of water that could be added to the water container, and I compared this amount to the capacity of the water container and concluded that the water tank will not overflow.)

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Alternate Strategies or Solutions

Some students may choose to list many of the terms in the series and manually find the sum of these terms. For example, they may choose to list out the first 10 terms of the series and find the sum, which is 399.609375. If they continue to add more terms, they may be able to conclude that the sum of the series is approximately 400, as they may see that the sum will never reach 400. Encourage students to also use the formula for the sum of an infinite series in order to verify that their calculations are approximately the same as the resulting sum when the formula is used.

Technology

Students can use scientific calculators for this task.

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Common Core State Standards

Mathematics III

Integrated Pathway

Teacher Resource

Unit 2B

WALCH

INTEGRATED MATH

This program was developed and reviewed by experienced math educators who have both academic and professional backgrounds in mathematics. This ensures: freedom from mathematical errors, grade level appropriateness, freedom from bias, and freedom from unnecessary language complexity.

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Standards Correlations

Each lesson in this Integrated Pathway: Mathematics III program was written specifically to address the Common Core State Standards. Each lesson lists the standards covered in all the sub-lessons, and each sub-lesson lists the standards addressed in that particular section. In this section, you'll find a comprehensive list mapping the sub-lessons to the CCSS.

Guide to Common Core State Standards Annotation

As you use this program, you will come across a symbol included with the Common Core standards for some of the lessons and activities. These symbols are explained below.

Symbol: ⋆

Denotes: Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (⋆).

From http://www.walch.com/CCSS/00003

Symbol: (+)

Denotes: College and Career Readiness Standards

Advanced mathematics standards that are required in higher-level courses such as advanced statistics may also be included in lower-level courses. These additional standards are denoted by (+). According to the Common Core State Standards Initiative, "the evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6–8."

From http://www.walch.com/CCSS/00004

Connections to Future Courses

This section provides a map between topics introduced in each unit of this course and subsequent courses where each topic is revisited and built upon.

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CCSS INTEGRATED PATHWAY: MATHEMATICS III STANDARDS CORRELATIONS

Unit 2B: Rational and Radical Relationships				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Operating with Rational Expressions**			
	2B.1.1	Structures of Rational Expressions	A–SSE.1a★ A–SSE.1b★ A–SSE.2	U2B-5
	2B.1.2	Adding and Subtracting Rational Expressions	A–APR.7 (+) A–SSE.2	U2B-23
	2B.1.3	Multiplying Rational Expressions	A–APR.7 (+) A–SSE.2	U2B-45
	2B.1.4	Dividing Rational Expressions	A–APR.6 A–APR.7 (+) A–SSE.2	U2B-61
Lesson 2	**Solving Rational and Radical Equations**			
	2B.2.1	Solving Rational Equations	A–REI.2	U2B-90
	2B.2.2	Solving Radical Equations	A–REI.2	U2B-110
	2B.2.3	Solving Systems of Equations	A–REI.11★	U2B-130

TOPICS FROM FUTURE COURSES: CCSS INTEGRATED PATHWAY MATHEMATICS III

Math III	Topic introduced	Course/unit where addressed	How addressed
Unit 2B	Rational expressions; solving rational and radical equations; linear-rational and linear-radical systems	Precalculus, Calculus, or Math IV	solving systems that can be extended to larger, more complex systems (e.g., 3×3 and larger) using vectors and matrices; graphing more complex rational functions and exploring features of the graphs

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Conceptual Activities

Use these interactive open education and/or Desmos resources to build conceptual understanding of mathematical ideas. (*Note*: Activity links will be monitored and repaired or replaced as necessary.)

- Desmos. "Marbleslides: Rationals."

  http://www.walch.com/ca/01057

  In this activity, students will transform rational functions to send marbles through stars.

- Desmos. "Polygraph: Rational Functions."

  http://www.walch.com/ca/01058

  This activity is designed to spark vocabulary-rich conversations about rational functions. Key vocabulary terms that may appear in student questions include *asymptote*, *vertical*, *horizontal*, *quadrant*, *axis*, *increasing*, *decreasing*, *discontinuity*, and *hole*.

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Pre-Assessment

Circle the letter of the best answer.

1. Which simplified rational expression is equivalent to $\frac{x^2+3x-10}{2-x}$?

   a. $-x-5$
   b. $x+5$
   c. $-x+5$
   d. $5-x$

2. Which simplified rational expression is equivalent to $\frac{x+1}{2x}+5x$?

   a. $\frac{5x^2+x+1}{2}$
   b. $\frac{10x^2+x+1}{2x}$
   c. $5x+1$
   d. $\frac{6x+1}{2x}$

3. Which simplified rational expression is equivalent to $\frac{2x^2+4x-6}{x+3}\bullet\frac{5+2x}{x-1}$?

   a. $\frac{2x^2-6x-1}{x+3}$
   b. $\frac{2x^2+6x-1}{x^2+2x-3}$
   c. $4x+10$
   d. $\frac{4x^3+18x^2+8x-30}{x^2+2x-3}$

4. Which simplified rational expression is equivalent to $\frac{x^2+1}{x-1}-\frac{x-6}{x+2}$?

   a. $\frac{10x-5}{x^2+x-2}$
   b. $\frac{x^3+2x^2-6x+8}{x^2+x-2}$
   c. $\frac{x^3+x^2+8x-4}{x^2+x-2}$
   d. $\frac{x^2+x-7}{x^2+x-2}$

5. Which simplified rational expression is equivalent to $\frac{2x^2+13x+15}{x+5}\div\frac{2x+3}{3x-1}$?

   a. $3x-1$
   b. $6x^2+7x-3$
   c. $\frac{4x^3+32x^2+57x+27}{3x^2-14x-5}$
   d. $\frac{4x^2+12x+9}{3x-1}$

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Lesson 1: Operating with Rational Expressions

Instruction

SMP
1 ✓ 2 ✓
3 ✓ 4 ✓
5 6 ✓
7 ✓ 8 ✓

Common Core State Standards

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.*

A–APR.6 Rewrite simple rational expressions in different forms; write $^{a(x)}/_{b(x)}$ in the form $q(x) + {}^{r(x)}/_{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

A–APR.7 (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Essential Questions

1. What does a rational expression look like?
2. How can a rational expression be rewritten into an equivalent form?
3. How is it possible to find the sum or difference of rational expressions?
4. How is it possible to find the product of rational expressions?
5. How is it possible to find the quotient of rational expressions?

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Instruction

WORDS TO KNOW

common denominator	a quantity that is a shared multiple of the denominators of two or more fractions
denominator	the value located below the line of a rational expression or fraction; the divisor
fraction	a ratio of two expressions or quantities
least common denominator (LCD)	the least common multiple of the denominators of two or more fractions
numerator	the value located above the line of a rational expression or fraction; the dividend
quadratic equation	an equation that can be written in the form $ax^2 + bx + c = 0$, where x is the variable, a, b, and c are constants, and $a \neq 0$
quadratic expression	an algebraic expression that can be written in the form $ax^2 + bx + c$, where x is the variable, a, b, and c are constants, and $a \neq 0$
quadratic formula	a formula that states the solutions of a quadratic equation of the form $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. A quadratic equation in this form can have no real solutions, one real solution, or two real solutions.
rational expression	an expression made of the ratio of two polynomials, in which a variable appears in the denominator
reciprocal	a number that, when multiplied by the original number, has a product of 1

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Instruction

Recommended Resources

- Math.com. "Quadratic Equation Solver."

  http://www.walch.com/rr/00198

  After entering the coefficients of a quadratic equation into this straightforward site, users can press the "Solve" button to generate appropriate roots. Detailed factors are not provided, and users must use the zeros to determine their own factors.

- Mathway. "Math Problem Solver."

  http://www.walch.com/rr/00199

  This resource is an online symbolic algebraic manipulator, capable of simplifying expressions. It can be a useful tool for tech-savvy individuals.

- WolframAlpha. "Simplify Rational Expressions."

  http://www.walch.com/rr/00200

  A powerful online symbolic manipulator, this resource can simplify complex rational expressions. It requires precise data entry to achieve valuable results.

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Lesson 2B.1.1: Structures of Rational Expressions

Warm-Up 2B.1.1

Ms. Appleton teaches high school math. This year, she has 24 students in her afternoon geometry class.

1. If $\frac{1}{8}$ of Ms. Appleton's students are absent one day, how many students are present?

2. To celebrate Sarah's birthday, Ms. Appleton brought in a cake. If the girls in the class ate $\frac{1}{2}$ of the cake, and the boys ate another $\frac{1}{3}$, how much of the cake remains for Ms. Appleton to share with the other teachers?

3. It's early in the school year, and the expression $x^4 - 4x^2$ describes the number of days the students have been in school. How can this expression be rewritten as an equivalent expression?

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Instruction

Lesson 2B.1.1: Structures of Rational Expressions

Common Core State Standards

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.*

Warm-Up 2B.1.1 Debrief

1. If $\frac{1}{8}$ of Ms. Appleton's students are absent one day, how many students are present?

   If $\frac{1}{8}$ of Ms. Appleton's students are absent, then we can determine the fraction of students who are present by subtracting $\frac{1}{8}$ from 1.

   $$1-\frac{1}{8}=\frac{8}{8}-\frac{1}{8}=\frac{7}{8}$$

   We can find the number of students present by multiplying this fraction by the total number of students, 24.

   $$24\cdot\frac{7}{8}=\cancel{24}^{3}\cdot\frac{7}{\cancel{8}^{1}}=\frac{21}{1}=21$$

   There are 21 students present in Ms. Appleton's class.

2. To celebrate Sarah's birthday, Ms. Appleton brought in a cake. If the girls in the class ate $\frac{1}{2}$ of the cake, and the boys ate another $\frac{1}{3}$, how much of the cake remains for Ms. Appleton to share with the other teachers?

   We can find the fraction of cake that remains by adding the fractions of cake eaten, and then subtracting this amount from 1.

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Instruction

The fraction of cake the girls ate is $\frac{1}{2}$.

The fraction of cake the boys ate is $\frac{1}{3}$.

Because these fractions have different denominators, before adding, we must find a common multiple of 2 and 3; 6 is a good choice since 2 • 3 = 6.

Rewrite the fractions using the common multiple, 6, as the denominator, and then add them.

$$\frac{1}{2}+\frac{1}{3}=\frac{3}{3}\bullet\frac{1}{2}+\frac{2}{2}\bullet\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$

The fraction of cake that has already been eaten is $\frac{5}{6}$.

Subtract this result from 1 to determine the fraction of cake that still remains.

$$1-\frac{5}{6}=\frac{6}{6}-\frac{5}{6}=\frac{1}{6}$$

If the class ate $\frac{5}{6}$ of the cake, then $\frac{1}{6}$ of it remains for Ms. Appleton to share with her fellow teachers.

3. It's early in the school year, and the expression $x^4 - 4x^2$ describes the number of days the students have been in school. How can this expression be rewritten as an equivalent expression?

There are an infinite number of ways to rewrite this expression.

This expression can be rewritten as $\frac{x^4-4x^2}{1}$. It's possible to make an equivalent expression by multiplying both the numerator and the denominator by the same number; in effect, by multiplying by a fraction equivalent to 1. An example of this is shown as follows, where $\frac{4}{4}=1$.

$$\frac{4}{4}(x^4-4x^2)=\frac{4}{4}\bullet\frac{x^4-4x^2}{1}=\frac{4x^4-16x^2}{4}$$

The given expression can also be factored to show an equivalent expression.

$x^4 - 4x^2$	Given expression
$= x^2(x^2 - 4)$	Factor out x^2.
$= x^2(x + 2)(x - 2)$	Rewrite $(x^2 - 4)$ as the difference of two squares.

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Instruction

Since x^4 is also a perfect square, $(x^2)^2$, and so are 4 and x^2, the expression $x^4 - 4x^2$ is also the difference of two squares, and can be factored as $(x^2 - 2x)(x^2 + 2x)$.

The expression $x^4 - 4x^2$ can be rewritten as an equivalent expression in multiple ways, such as $\frac{4x^4 - 16x^2}{4}$, $x^2(x + 2)(x - 2)$, and $(x^2 - 2x)(x^2 + 2x)$.

Connection to the Lesson

- Students will find that techniques useful in working with fractions may also be useful in working with rational expressions.
- Students will use factoring to make complicated expressions simpler.
- Students will discover that rational expressions may be broken down into smaller, more familiar patterns.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- factoring polynomial expressions (A-SSE.2)
- simplifying fractions (4.NF.1)
- recognizing and factoring quadratic equations (A-SSE.3a)

Introduction

Expressions come in a variety of types, including rational expressions. A **rational expression** is a ratio of two polynomials, in which a variable appears in the denominator; for example, $\frac{3x}{x+1}$ is a rational expression. Working with rational expressions can often be made easier by analyzing them to uncover more familiar (and sometimes less complex) structures within them. In this section, you will explore some of those structures.

Key Concepts

- A **fraction** is a ratio of two expressions or quantities. Rational expressions are also expressed as a ratio of one quantity to another.
- While a fraction usually involves integers, such as $\frac{13}{29}$, a rational expression may involve some integers, but will also include at least one variable.
- A rational expression can be rewritten in many equivalent forms.
- Factoring the expressions in the numerator and the denominator can reveal such forms.
- Recall that the **numerator** is the value located above the line of a rational expression or fraction; it is the dividend.
- The **denominator** is the value located below the line of a rational expression or fraction; it is the divisor.
- Rules learned for factoring polynomial expressions (including the quadratic formula) can be useful in finding factors within rational expressions.

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Instruction

- Recall that the **quadratic formula** states the solutions of a quadratic equation of the form $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a \neq 0$.
- A **quadratic equation** is an equation that can be written in the form $ax^2 + bx + c = 0$, where x is the variable, a, b, and c are constants, and $a \neq 0$.
- A **quadratic expression** is an algebraic expression that can be written in the form $ax^2 + bx + c$, where x is the variable, a, b, and c are constants, and $a \neq 0$.
- Rewriting rational expressions can more easily reveal patterns, or make rational expressions easier to use within more complex operations.
- Because division by 0 is undefined, a rational expression cannot include a value equal to 0 in the denominator.

- Any values that make the denominator equal to 0 must be identified and removed.

  For example, the value $x = 1$ must be excluded from the rational expression $\frac{2x}{x-1}$, since that value makes the rational expression equivalent to $\frac{2x}{x-1} = \frac{2(1)}{(1)-1} = \frac{2}{0}$, which is undefined.

Common Errors/Misconceptions

- canceling individual terms of the numerator or denominator, rather than factors (for example, attempting to cancel the $3x$ term in the expression $\frac{3x+5}{3x-2}$, resulting in the incorrect conclusion that $\frac{3x+5}{3x-2} = \frac{\cancel{3x}+5}{\cancel{3x}-2} = -\frac{5}{2}$)
- forgetting to check for invalid values in the denominator

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Guided Practice 2B.1.1

Example 1

Simplify the rational expression $\frac{2x}{8}$ by eliminating any common factors found in both the numerator and denominator.

1. Rewrite the original rational expression as a product of two expressions.

   We can rewrite the given expression, $\frac{2x}{8}$, as follows.

   $$\frac{2x}{8} = \frac{2}{8} \bullet \frac{x}{1}$$

   The expression $\frac{2}{8}$ is a fraction containing only constants (2 and 8).

   The expression $\frac{x}{1}$ contains a variable (x).

2. Identify any factors common to the numerator and denominator.

   We can see that 2 is a factor of both the numerator and denominator of $\frac{2}{8}$.

   Rewrite the expression to factor out 2.

   $$\frac{2}{8} \bullet \frac{x}{1} = \left(\frac{2 \bullet 1}{2 \bullet 4}\right) \bullet \frac{x}{1}$$

3. Simplify the resulting expression.

   Cancel out the common factor of 2, then multiply.

   $$\frac{2}{8} \bullet \frac{x}{1} = \left(\frac{\cancel{2} \bullet 1}{\cancel{2} \bullet 4}\right) \bullet \frac{x}{1} = \frac{1}{4} \bullet \frac{x}{1} = \frac{x}{4}$$

   The rational expression $\frac{2x}{8}$ simplifies to $\frac{x}{4}$.

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Instruction

Example 2

Simplify the rational expression $\frac{2}{8x}$ by eliminating any common factors.

1. Rewrite the original rational expression as a product of two expressions, one with a variable, and one without.

   As in Example 1, the first expression will be a fraction made up of whole numbers in both the numerator and denominator, and the second expression will be a rational expression containing the variable x.

   $$\frac{2}{8x}=\frac{2}{8}\cdot\frac{1}{x}$$

2. Identify any invalid values of the expression.

   As in Example 1, the first fraction in the expression $\frac{2}{8}\cdot\frac{1}{x}$ is composed only of constants: it has no invalid values. However, in the second expression, there is a variable in the denominator. Dividing by 0 is undefined in mathematics. Because we cannot have a 0 in a denominator, we first must find any values that would invalidate the denominator. In this example, $x \neq 0$.

3. Identify any common factors.

   The common factor for the numerator and denominator of $\frac{2}{8}$ is 2. Rewrite the expression to factor out 2.

   $$\frac{2}{8}\cdot\frac{1}{x}=\left(\frac{2\cdot1}{2\cdot4}\right)\cdot\frac{1}{x}$$

4. Simplify the resulting expression.

   Cancel out the common factor of 2, then multiply.

   $$\frac{2}{8}\cdot\frac{1}{x}=\left(\frac{\cancel{2}\cdot1}{\cancel{2}\cdot4}\right)\cdot\frac{1}{x}=\frac{1}{4}\cdot\frac{1}{x}=\frac{1}{4x}$$

   The rational expression $\frac{2}{8x}$ simplifies to $\frac{1}{4x}$ when $x \neq 0$.

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Instruction

Example 3

Simplify the rational expression $\frac{x-5}{x^2-3x-10}$.

1. Identify any invalid values of the expression.

   This expression has terms with variables in the denominator. Since the denominator cannot equal 0, we have to eliminate any values that would result in 0 from the possible values for x. In other words, we need to set the denominator equal to 0 and solve the resulting equation for x.

   $x^2 - 3x - 10 = 0$

   This is a quadratic equation. While it can be factored using the quadratic formula, it can also be factored directly.

   $x^2 - 3x - 10 = 0$

   $x^2 - 5x + 2x - 10 = 0$

   $(x - 5)(x + 2) = 0$

   Set each factor equal to 0, then solve for x. Determining the value of x when each factor equals 0 indicates the values of x that are undefined.

   $x - 5 = 0$ $\qquad$ $x + 2 = 0$

   $x = 5$ $\qquad$ $x = -2$

   The factors equal 0 when $x = 5$ and $x = -2$. Since these values would invalidate the denominator of the original expression, the domain of the expression is restricted to $x \neq 5$ and $x \neq -2$.

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2. Rewrite the original expression using the factored form.

   With the denominator factored, we can rewrite the original expression as $\frac{x-5}{x^2-3x-10}=\frac{x-5}{(x-5)(x+2)}$.

   In this format, it becomes clear that we have a common factor in both the numerator and the denominator: $x-5$. Therefore, we can simplify.

3. Simplify the resulting expression.

   Cancel out common factors.

$$\frac{x-5}{x^2-3x-10}=\frac{\cancel{x-5}}{\cancel{(x-5)}(x+2)}=\frac{1}{x+2}$$

   The rational expression $\frac{x-5}{x^2-3x-10}$ simplifies to $\frac{1}{x+2}$ when $x\neq 5$. When $x=-2$, neither expression is valid.

Example 4

Simplify the rational expression $\frac{x-3}{3-x}$.

1. Identify any invalid values of the expression.

   This rational expression contains a variable in the denominator. Because $3-x\neq 0$, the expression is not valid when $x=3$. Therefore, this expression is only valid when $x\neq 3$.

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2. Rewrite the numerator with –1 as a factor.

At first glance, it may seem that the original expression equals 1, based on the idea that the 3 terms would cancel out, as would the x terms. However, in simplifying a rational expression, it is not permitted to cancel terms—only factors. Nevertheless, the similarity between the numerator and denominator is useful. Since $3 - x = -x + 3 = -1(x - 3)$, multiplying the numerator by –1 will make the expression less complex.

$\frac{x-3}{3-x}$	Original expression
$= \frac{-3+x}{3-x}$	Apply the Commutative Property of Addition to the numerator.
$= \frac{-1(3-x)}{3-x}$	Factor out –1 from both terms in the numerator.
$= -1 \bullet \frac{3-x}{3-x}$	Rewrite the expression.

3. Simplify the result to find an equivalent expression.

Because the numerator and denominator of the expression $\frac{3-x}{3-x}$ both have the same factor, $3 - x$, the factors cancel out.

$$-1 \bullet \frac{\cancel{3-x}}{\cancel{3-x}} = -1 \bullet 1 = -1$$

The rational expression $\frac{x-3}{3-x}$ simplifies to –1 when $x \neq 3$.

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Instruction

Example 5

Simplify the rational expression $\frac{12x^2y^5}{4xy^2}$.

1. Identify any invalid values of the expression.

   This rational expression contains variables in the denominator. If either x or y is equal to 0, the expression is not valid. This expression is valid only when $x \neq 0$ and $y \neq 0$.

2. Rewrite the numerator as a series of factors.

   This expression contains many individual factors and exponents. To better see their relationships, it may be helpful to separate them. Both the numerator and denominator are composed of a constant factor, followed by factors of x and y.

   $$\frac{12x^2y^5}{4xy^2} = \frac{12 \bullet x^2 \bullet y^5}{4 \bullet x \bullet y^2} = \frac{12}{4} \bullet \frac{x^2}{x} \bullet \frac{y^5}{y^2}$$

3. Simplify the resulting expression.

   Cancel out common factors for each of the three terms.

   $\frac{12}{4} \bullet \frac{x^2}{x} \bullet \frac{y^5}{y^2}$ — Rewritten expression from the previous step

   $= \frac{4 \bullet 3}{4} \bullet \frac{x \bullet x}{x} \bullet \frac{y^2 \bullet y^3}{y^2}$ — Rewrite each numerator to show common factors.

   $= \frac{\cancel{4} \bullet 3}{\cancel{4}} \bullet \frac{\cancel{x} \bullet x}{\cancel{x}} \bullet \frac{\cancel{y^2} \bullet y^3}{\cancel{y^2}}$ — Cancel out common factors in each term.

   $= \frac{3}{1} \bullet \frac{x}{1} \bullet \frac{y^3}{1}$ — Simplify.

   $= 3xy^3$

   The rational expression $\frac{12x^2y^5}{4xy^2}$ simplifies to $3xy^3$ when $x \neq 0$ and $y \neq 0$.

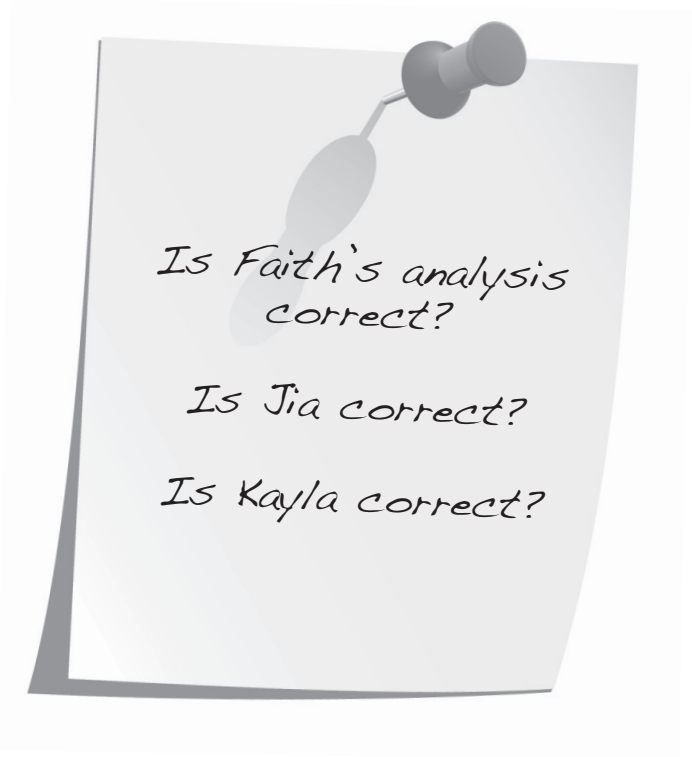
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Problem-Based Task 2B.1.1: Rewriting a Rational Expression

SMP	
1	2
3 ✓	4
5	6 ✓
7 ✓	8

As part of their preparations for an upcoming math test, Faith, Jia, and Kayla work together to simplify the rational expression $\frac{t^4-16}{t^2-t-6}$.

- Faith reasons that, because there is a factor of t^4 in the numerator but a factor of only t^2 in the denominator, there is no way to simplify this expression. She says it is already as simple as it can be. Is Faith's analysis correct?
- After factoring the numerator, Jia says that the expression is equivalent to $\frac{(t-2)(t^2+4)}{t-3}$. Is Jia correct?
- Kayla tries a different method. After much factoring, she arrives at the expression $\frac{8-4t+2t^2-t^3}{3-t}$. Is Kayla correct?



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Problem-Based Task 2B.1.1: Rewriting a Rational Expression

Coaching

a. Do you see any familiar pattern in the denominator of the rational expression $\frac{t^4-16}{t^2-t-6}$?

b. If the expression looks like a quadratic, it might be possible to factor it. How could you do that?

c. What is the factored form of the denominator, $t^2 - t - 6$?

d. What is the square of t^2?

e. Is 16 a perfect square?

f. What do the results of parts d and e imply about the numerator of the rational expression $\frac{t^4-16}{t^2-t-6}$? Is this a familiar pattern?

g. What is the factored form of the numerator, $t^4 - 16$?

h. How could you write an equivalent expression using the factored forms of the numerator and denominator?

i. In this factored form, do any factors cancel? If so, what is the resulting expression?

j. What does this result reveal about Faith's answer?

k. How does this result compare to Jia's answer?

l. Does this result have anything in common with Kayla's answer?

m. What happens if you expand the terms of the numerator in the expression from part i—that is, if you rewrite the terms to eliminate the parentheses?

n. How could you adjust the denominator in the factored expression to match the denominator in Kayla's answer?

o. The action taken in part n must also be applied to the numerator in order to maintain an equivalent expression. When you do so, what is the result?

p. What does this result imply about Kayla's answer?

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Instruction

Problem-Based Task 2B.1.1: Rewriting a Rational Expression

Coaching Sample Responses

a. Do you see any familiar pattern in the denominator of the rational expression $\frac{t^4-16}{t^2-t-6}$?

The denominator, $t^2 - t - 6$, looks like a quadratic expression.

b. If the expression looks like a quadratic, it might be possible to factor it. How could you do that?

To factor the quadratic, you could use the quadratic formula, try a table or a graph, or try to find factors that work through trial and error.

c. What is the factored form of the denominator, $t^2 - t - 6$?

The factors of $t^2 - t - 6$ are $(t - 3)$ and $(t + 2)$; therefore, the factored form is $(t - 3)(t + 2)$.

d. What is the square of t^2?

$(t^2)^2 = t^4$

e. Is 16 a perfect square?

Yes; $16 = 4^2$.

f. What do the results of parts d and e imply about the numerator of the rational expression $\frac{t^4-16}{t^2+t-6}$? Is this a familiar pattern?

The results of parts d and e imply that $t^4 - 16$ is the difference of two squares. This is a familiar quadratic form.

g. What is the factored form of the numerator, $t^4 - 16$?

As the difference of two squares, $t^4 - 16 = (t^2 - 4)(t^2 + 4)$. The first term is, itself, a difference of two squares and can be factored further. Therefore, $t^4 - 16 = (t - 2)(t + 2)(t^2 + 4)$.

h. How could you write an equivalent expression using the factored forms of the numerator and denominator?

The original expression can be rewritten in factored form as follows.

$$\frac{t^4-16}{t^2-t-6}=\frac{(t-2)(t+2)(t^2+4)}{(t-3)(t+2)}$$

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Instruction

i. In this factored form, do any factors cancel? If so, what is the resulting expression?

Both the numerator and denominator have the factor $t + 2$, which can be factored out as follows.

$$\frac{t^4-16}{t^2-t-6}=\frac{(t-2)\cancel{(t+2)}(t^2+4)}{(t-3)\cancel{(t+2)}}=\frac{(t-2)(t^2+4)}{(t-3)}$$

j. What does this result reveal about Faith's answer?

We were able to factor the expression into a simpler form, so Faith's answer is incorrect.

k. How does this result compare to Jia's answer?

This result matches the expression given by Jia; therefore, Jia is correct.

l. Does this result have anything in common with Kayla's answer?

The expression from part i, $\frac{(t-2)(t^2+4)}{(t-3)}$, looks a little like Kayla's answer of $\frac{8-4t+2t^2-t^3}{3-t}$. There is an expression in the denominator (for which $t \neq 3$). However, the expression from part i is in factored form, whereas Kayla's answer is in expanded form.

m. What happens if you expand the terms of the numerator in the expression from part i—that is, if you rewrite the terms to eliminate the parentheses?

Expanding the numerator results in the following expression.

$(t - 2)(t^2 + 4)$

$= t(t^2 + 4) - 2(t^2 + 4)$

$= t^3 + 4t - 2t^2 - 8$

$= t^3 - 2t^2 + 4t - 8$

This resembles the numerator in Kayla's answer, except it is reversed. Also, compared with Kayla's numerator, every positive term is negative and the negative terms are positive.

n. How could you adjust the denominator in the factored expression to match the denominator in Kayla's answer?

The denominator of Kayla's answer is $3 - t$.

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The denominator of the factored expression is $t - 3$.

Multiply $t - 3$ by -1.

$-1(t - 3)$

$= -t + 3$

$= 3 - t$

This matches the denominator of Kayla's expression.

o. The action taken in part n must also be applied to the numerator of the factored expression in order to maintain an equivalent expression. When you do so, what is the result?

From part m, we determined the numerator of the factored expression, when written in expanded form, is $t^3 - 2t^2 + 4t - 8$.

Multiply $t^3 - 2t^2 + 4t - 8$ by -1.

$-1(t^3 - 2t^2 + 4t - 8)$

$= -t^3 + 2t^2 - 4t + 8$

$= 8 - 4t + 2t^2 - t^3$

This is an exact match between our result and Kayla's factored form. Therefore, we can finally see the following:

$$\frac{t^3-2t^2+4t-8}{t-3}=\frac{t^3-2t^2+4t-8}{t-3}\bullet\frac{-1}{-1}=\frac{-(t^3-2t^2+4t-8)}{-1(t-3)}=\frac{8-4t+2t^2-t^3}{3-t}$$

p. What does this result imply about Kayla's answer?

Kayla's answer is correct.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2B.1.1: Structures of Rational Expressions

For problems 1–6, simplify each rational expression. State any restrictions on x.

1. $\dfrac{35x}{7x}$

2. $\dfrac{3x+12}{2x+8}$

3. $\dfrac{x^2+7x}{x}$

4. $\dfrac{18x^4-6x^2+9x}{3x}$

5. $\dfrac{8+2x}{2x^2+10x+8}$

6. $\dfrac{2x^3+13x^2-7x}{x^2+7x}$

Use your knowledge of rational expressions to complete problems 7–10. State any restrictions on x.

7. Show that the expression $\dfrac{x^3-6x^2-7x}{x^2+4x+3}$ and the expression $\dfrac{x^2-7x}{x+3}$ are equivalent.

8. What simplest rational expression represents the depth of a pond that is $\dfrac{3x^2-8x}{5x}$ meters deep?

9. What simplest rational expression represents the diameter of a circle that has a radius of $\dfrac{4x-8}{3x-6}$ centimeters?

10. What simplest rational expressions can be used to represent the length and width, in inches, of a rectangle with sides that are $\dfrac{x^2-x-6}{x+2}$, $\dfrac{x^2+x-20}{2x+10}$, $\dfrac{6x^2-96}{48+12x}$, and $\dfrac{2x-6}{2}$?

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Lesson 2B.1.2: Adding and Subtracting Rational Expressions

Warm-Up 2B.1.2

Every pizza at Percy's Pizza Palace is cut into 12 slices, regardless of the pizza's size. Use this information to answer the following questions.

1. If Caiden eats $\frac{1}{3}$ of a pizza, and Nate eats another $\frac{1}{4}$ of the same pizza, what fraction of the pizza have they eaten together?

2. Mrs. Booker ordered a pizza at Percy's. She ate $\frac{1}{2}$ of it, and saved $\frac{1}{4}$ of the pizza for lunch the next day. What fraction remains?

3. The Shepard family—Jake, Jena, Jamie, and their mother—ordered 2 pizzas. Jake ate $\frac{2}{3}$ of 1 pizza and Jena ate $\frac{1}{2}$ of 1 pizza. If Jamie ate $\frac{3}{4}$ of a pizza, what was left for Mrs. Shepard?

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Lesson 2B.1.2: Adding and Subtracting Rational Expressions

Common Core State Standards

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

A–APR.7 (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Warm-Up 2B.1.2 Debrief

1. If Caiden eats $\frac{1}{3}$ of a pizza, and Nate eats another $\frac{1}{4}$ of the same pizza, what fraction of the pizza have they eaten together?

   We know that Caiden ate $\frac{1}{3}$ of the pizza.

   Nate ate $\frac{1}{4}$ of the same pizza.

   Together, Caiden and Nate have eaten $\frac{1}{3}+\frac{1}{4}$ of the pizza. To find the sum of these fractions, a common denominator needs to be found. Since 3 • 4 = 12, 12 will be a good choice for the common denominator.

   $$\frac{1}{3}+\frac{1}{4}=\frac{4}{4}\bullet\frac{1}{3}+\frac{3}{3}\bullet\frac{1}{4}=\frac{4\bullet1}{4\bullet3}+\frac{3\bullet1}{3\bullet4}=\frac{4}{12}+\frac{3}{12}=\frac{7}{12}$$

   Caiden and Nate have eaten $\frac{7}{12}$ of the pizza.

2. Mrs. Booker ordered a pizza at Percy's. She ate $\frac{1}{2}$ of it, and saved $\frac{1}{4}$ of the pizza for lunch the next day. What fraction remains?

   One entire pizza is represented by the fraction $\frac{1}{1}$. The remaining pizza is the pizza Mrs. Booker starts with, minus the part she ate and minus the part she saves for lunch.

   She started with $\frac{1}{1}$ of a pizza.

   She ate $\frac{1}{2}$ of it.

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She saved $\frac{1}{4}$ of it.

Numerically, we can represent this as follows.

$$\frac{1}{1}-\frac{1}{2}-\frac{1}{4}$$

To simplify this into a single fraction, we need to find a common denominator. 1, 2, and 4 are all factors of 4. Therefore, use 4 as the common denominator and solve for the remaining portion of the pizza as follows.

$$\frac{1}{1}-\frac{1}{2}-\frac{1}{4}=\frac{1}{1}\bullet\frac{4}{4}-\frac{1}{2}\bullet\frac{2}{2}-\frac{1}{4}=\frac{4}{4}-\frac{2}{4}-\frac{1}{4}=\frac{1}{4}$$

There is $\frac{1}{4}$ of the pizza remaining.

3. The Shepard family—Jake, Jena, Jamie, and their mother—ordered 2 pizzas. Jake ate $\frac{2}{3}$ of 1 pizza and Jena ate $\frac{1}{2}$ of 1 pizza. If Jamie ate $\frac{3}{4}$ of a pizza, what was left for Mrs. Shepard?

This is similar to the previous examples, except it involves 2 pizzas rather than 1. Instead of $\frac{1}{1}$ pizza, Mrs. Shepard has purchased $2\bullet\frac{1}{1}=\frac{2}{1}$ pizzas. Subtracting the fractions eaten by each of her children results in the following expression.

$$\frac{2}{1}-\frac{2}{3}-\frac{1}{2}-\frac{3}{4}$$

Since the denominators are 1, 3, 2, and 4, a good candidate for a common denominator is 12.

$$\frac{2}{1}-\frac{2}{3}-\frac{1}{2}-\frac{3}{4}=\left(\frac{2}{1}\bullet\frac{12}{12}\right)-\left(\frac{2}{3}\bullet\frac{4}{4}\right)-\left(\frac{1}{2}\bullet\frac{6}{6}\right)-\left(\frac{3}{4}\bullet\frac{3}{3}\right)$$

$$=\left(\frac{24}{12}\right)-\left(\frac{8}{12}\right)-\left(\frac{6}{12}\right)-\left(\frac{9}{12}\right)$$

$$=\frac{24-8-6-9}{12}=\frac{24-23}{12}=\frac{1}{12}$$

After all three children have eaten, $\frac{1}{12}$ of a pizza or one slice remains for Mrs. Shepard.

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Instruction

Connection to the Lesson

- Students will learn that adding and subtracting rational expressions is closely related to adding and subtracting fractions.
- Just as adding or subtracting fractions often results in another fraction, students will discover that adding or subtracting rational expressions often results in another rational expression.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- finding a common denominator (5.NF.1)
- simplifying a rational expression (A-APR.6)
- determining values that make a rational expression invalid (A-APR.3)
- adding and subtracting fractions (5.NF.1)

Introduction

Where one rational expression exists, another may as well. Extracting meaning from the context may require these expressions to be combined in order to determine a sum or difference. Combining rational expressions through addition or subtraction is not complex, though it does demand attention to detail. Rewriting equivalent fractions can often simplify the solution process.

Key Concepts

- Before adding or subtracting rational expressions, you must find a **common denominator**. A common denominator is a quantity that is a shared multiple of the denominators of two or more fractions.

- A common denominator can be determined by finding the product of the denominators. For example, a common denominator of the rational expression $\frac{3x}{x-1}+\frac{2x+1}{x^2+x-2}$ is found by multiplying the two denominators, $x-1$ and x^2+x-2: $(x-1)(x^2+x-2)=x^3-3x+2$.
- Once a common denominator has been found, it can be used to write equivalent rational expressions for each term of the sum (or difference, if subtracting).
- Using common denominators, the same rational expression as before, $\frac{3x}{x-1}+\frac{2x+1}{x^2+x-2}$, can be rewritten in an equivalent form:

$$\frac{3x}{x-1}+\frac{2x+1}{x^2+x-2}=\frac{3x}{x-1}\bullet\frac{x^2+x-2}{x^2+x-2}+\frac{2x+1}{x^2+x-2}\bullet\frac{x-1}{x-1}$$
$$=\frac{3x^3+3x^2-6x}{x^3-3x+2}+\frac{2x^2-x-1}{x^3-3x+2}$$

SRB p. U2B-13

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Instruction

- Expressed with a common denominator, the sum of rational expressions can be found by adding the numerators: $\frac{x}{z}+\frac{y}{z}=\frac{x+y}{z}$. Thus, we can rewrite the expressions over a single denominator, as shown.

$$\frac{3x^3+3x^2-6x}{x^3-3x+2}+\frac{2x^2-x-1}{x^3-3x+2}=\frac{3x^3+3x^2-6x+2x^2-x-1}{x^3-3x+2}$$

- Expressed with a common denominator, the difference of rational expressions can be found by subtracting the numerators: $\frac{x}{z}-\frac{y}{z}=\frac{x-y}{z}$.
- The **least common denominator (LCD)** is the least common multiple of the denominators of two or more fractions. In other words, it's the smallest possible common denominator. The LCD can be determined by finding the product of all the unique factors of the denominator. Our sample rational expression, $\frac{3x}{x-1}+\frac{2x+1}{x^2+x-2}$, is equivalent to the rational expression $\frac{3x}{x-1}+\frac{2x+1}{(x-1)(x+2)}$. The LCD is $(x-1)(x+2)$, or x^2-x-2.
- Recall that a rational expression cannot include a value in the denominator that causes it to equal 0, since 0 is undefined in the denominator.

Common Errors/Misconceptions

- canceling individual terms of the numerator or denominator, rather than factors
- distributing values incorrectly, especially negative numbers
- forgetting to check for invalid values in the denominator
- neglecting to keep track of signs (particularly when subtracting terms that must first be properly distributed)

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Guided Practice 2B.1.2

Example 1

Simplify the rational expression $\frac{3}{5}+\frac{2}{x}$.

1. Identify any invalid values of the expression.

   The first term, $\frac{3}{5}$, is composed entirely of constants; there are no invalid values. With an x in the denominator of the second term, $\frac{2}{x}$, we have to ensure that $x \neq 0$ to keep this expression valid.

2. Find a common denominator.

   The two denominators are 5 and x. Their product, $5x$, is a useful common denominator.

3. Rewrite each term of the expression using the new denominator.

   If the fraction's denominator is not the common denominator, then multiply each fraction's numerator and denominator by the missing factor of the common denominator.

   Remember that multiplication is commutative: that means $x \cdot 3 = 3 \cdot x$, and $x \cdot 5 = 5 \cdot x$.

$\frac{3}{5}+\frac{2}{x}$	Original expression
$=\left(\frac{x}{x}\right)\cdot\frac{3}{5}+\left(\frac{5}{5}\right)\cdot\frac{2}{x}$	Multiply $\frac{3}{5}$ by $\frac{x}{x}$ and $\frac{2}{x}$ by $\frac{5}{5}$.
$=\frac{x\cdot 3}{x\cdot 5}+\frac{5\cdot 2}{5\cdot x}$	Simplify each term.
$=\frac{3x}{5x}+\frac{10}{5x}$	Multiply.
$=\frac{3x+10}{5x}$	Rewrite the expression by adding the numerators over the common denominator.

The rewritten expression is $\frac{3x+10}{5x}$.

SWB p. 19; SRB p. U2B-15

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4. Check to see if the result can be written in a simpler form.

   The numerator does not have either 5 or x as a factor; therefore, it cannot be written in any other format that will further simplify the result. Thus $\frac{3}{5}+\frac{2}{x}=\frac{3x+10}{5x}$, where $x \neq 0$, is the simplest way to rewrite the original expression, $\frac{3}{5}+\frac{2}{x}$.



Example 2

Simplify the rational expression $\frac{5}{x+1}+3$.

1. Identify any invalid values of the expression.

   With an x in the denominator of the first term, $\frac{5}{x+1}$, we have to ensure that $x + 1 \neq 0$ in order to keep this expression valid. Set the denominator equal to 0 and solve this resulting equation for x to determine values of x that make the expression invalid.

   $x + 1 = 0$

   $x = -1$

   An x-value of -1 invalidates the expression, so $x \neq -1$.

2. Find a common denominator.

   The second term, 3, does not have an apparent denominator. Yet any number can be rewritten as a ratio over 1; therefore, the second term is equivalent to $\frac{3}{1}$. Thus, if we rewrite the rational expression as $\frac{5}{x+1}+\frac{3}{1}$, it can be seen that the two denominators are $x + 1$ and 1. Their product, $x + 1$, is a useful common denominator.

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Instruction

3. Rewrite each term of the expression using the new denominator.

   The denominator of the first term is already $x + 1$, so it doesn't need to be multiplied by anything.

$\frac{5}{x+1}+\frac{3}{1}$	Rewritten expression from the previous step
$=\frac{5}{x+1}+\left(\frac{3}{1}\bullet\frac{x+1}{x+1}\right)$	Multiply $\frac{3}{1}$ by $\frac{x+1}{x+1}$.
$=\frac{5}{x+1}+\frac{3(x+1)}{x+1}$	Simplify each term.
$=\frac{5+3(x+1)}{x+1}$	Rewrite the expression by adding the numerators over the common denominator.
$=\frac{5+3x+3}{x+1}$	Simplify.
$=\frac{3x+8}{x+1}$	

   The rewritten expression is $\frac{3x+8}{x+1}$.

4. Check to see if the result can be written in a simpler form.

   There are no common factors of the numerator or denominator.

   This expression cannot be written in any other format that will simplify the result. Thus $\frac{3x+8}{x+1}$, where $x \neq -1$, is the simplest way to rewrite the original expression, $\frac{5}{x+1}+3$.

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Instruction

Example 3

Simplify the rational expression $\frac{5}{x^2}+\frac{3}{x}+\frac{1}{4x}$.

1. Identify any invalid values of the expression.

   In the first rational term, $x^2 \neq 0$. Taking the square root of each side results in $x \neq 0$. This same value also makes the other two denominators invalid. We only need to avoid this one value, so the domain of the expression is $x \neq 0$.

2. Find a common denominator.

   As in previous examples, we can determine a common denominator by finding the product of the denominators. That gives us $x^2 \bullet x \bullet 4x = 4x^4$. This would work just fine (it is most certainly a common denominator), however, it does seem large. We can likely determine a smaller common denominator.

   Each of the three denominators in the original expression has x as a factor, so we only need to include it once. The first term has an additional factor of x, and the last term has an additional term of 4. The product of these three values—the common x, another x from the first term, and the 4 from the last term—is $4x^2$. This is a smaller and perhaps easier to handle denominator.

   Thus, we can use $4x^2$ as the common denominator.

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3. Rewrite each term of the expression using the new denominator.

$\frac{5}{x^2}+\frac{3}{x}+\frac{1}{4x}$ — Original expression

$=\left(\frac{4}{4}\bullet\frac{5}{x^2}\right)+\left(\frac{4x}{4x}\bullet\frac{3}{x}\right)+\left(\frac{x}{x}\bullet\frac{1}{4x}\right)$ — Multiply $\frac{5}{x^2}$ by $\frac{4}{4}$, $\frac{3}{x}$ by $\frac{4x}{4x}$, and $\frac{1}{4x}$ by $\frac{x}{x}$.

$=\frac{20}{4x^2}+\frac{12x}{4x^2}+\frac{x}{4x^2}$ — Simplify each term.

$=\frac{20+12x+x}{4x^2}$ — Rewrite the expression by adding the numerators over the common denominator.

$=\frac{20+13x}{4x^2}$ — Combine like terms.

The rewritten expression is $\frac{20+13x}{4x^2}$.

4. Check to see if the result can be written in a simpler form.

There are no common factors of the numerator or denominator. This expression cannot be written in any other format that will simplify the result. Therefore, $\frac{20+13x}{4x^2}$, where $x \neq 0$, is the simplest way to rewrite the original expression.

You can use this strategy to combine many different rational expressions, including those that appear to be more complex. The expressions may involve more terms, or perhaps subtraction here and there. Yet the process outlined in this example remains the same. Carefully following these same steps will yield successful results.

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Example 4

Simplify the rational expression $\frac{3x}{2x+1}+\frac{8}{x-3}$.

1. Identify any invalid values of the expression.

   The first rational term has a denominator of $2x + 1$. Set the denominator equal to 0 and then solve the resulting equation for x to determine values that make this expression invalid.

   $2x + 1 = 0$

   $2x = -1$

   $x=-\frac{1}{2}$

   An x-value of $-\frac{1}{2}$ invalidates the expression, so $x \neq -\frac{1}{2}$.

   The second rational term has a denominator of $x - 3$. Again, set the denominator equal to 0 and then solve the resulting equation for x to determine values that make this expression invalid.

   $x - 3 = 0$

   $x = 3$

   An x-value of 3 invalidates the expression, so $x \neq 3$.

   The rational expression $\frac{3x}{2x+1}+\frac{8}{x-3}$ is only valid when $x \neq -\frac{1}{2}$ and $x \neq 3$.

2. Find a common denominator.

   The two denominators are $2x + 1$ and $x - 3$. They share no factors, so their product will be a good common denominator.

   $(2x + 1)(x - 3) = 2x(x - 3) + 1(x - 3)$

   $= 2x^2 - 6x + x - 3$

   $= 2x^2 - 5x - 3$

   The common denominator is $2x^2 - 5x - 3$.

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3. Rewrite each term of the expression using the new denominator.

$$\frac{3x}{2x+1}+\frac{8}{x-3}$$ Original expression

$$=\frac{x-3}{x-3}\cdot\frac{3x}{2x+1}+\frac{2x+1}{2x+1}\cdot\frac{8}{x-3}$$ Multiply $\frac{3x}{2x+1}$ by $\frac{x-3}{x-3}$ and $\frac{8}{x-3}$ by $\frac{2x+1}{2x+1}$.

$$=\frac{3x(x-3)}{(x-3)(2x+1)}+\frac{8(2x+1)}{(2x+1)(x-3)}$$ Simplify each term.

$$=\frac{3x^2-9x}{2x^2-5x-3}+\frac{16x+8}{2x^2-5x-3}$$ Multiply.

$$=\frac{3x^2-9x+(16x+8)}{2x^2-5x-3}$$ Rewrite the expression by adding the numerators over the common denominator.

$$=\frac{3x^2+7x+8}{2x^2-5x-3}$$ Combine like terms.

4. Check to see if the result can be written in a simpler form.

We already know the denominator has $x-3$ and $2x+1$ as factors. The numerator, however, has no integer factors. This expression cannot be written in any other format that will simplify the result.

Thus $\frac{3x^2+7x+8}{2x^2-5x-3}$, where $x\neq-\frac{1}{2}$ and $x\neq 3$, is the simplest way to rewrite the original expression, $\frac{3x}{2x+1}+\frac{8}{x-3}$.

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Example 5

Simplify the rational expression $\frac{1}{3-x}+\frac{10x-6}{x^2-2x-3}-\frac{4}{x+1}$.

1. Identify any invalid values of the expression.

   There are three terms in the expression and they each have a denominator.

   Because it appears to be the most complex, start with the second denominator, $x^2 - 2x - 3$. This expression is quadratic, so try to factor it.

   $x^2 - 2x - 3 = (x - 3)(x + 1)$

   The factors are $(x - 3)$ and $(x + 1)$. Set each of these equal to 0 and solve for x to determine values that make the expression invalid.

   $x - 3 = 0$ $\qquad$ $x + 1 = 0$

   $x = 3$ $\qquad$ $x = -1$

   An x-value of 3 invalidates the expression, as does an x-value of -1. Therefore, we can now more clearly see that $x \neq 3$ and $x \neq -1$ for the second term's denominator. That takes care of the third term's denominator, $x + 1$, since it is one of the factors of the second denominator; in other words, $x \neq -1$ is likewise a restriction of the third term in the rational expression. We can verify this by setting the denominator of the third term equal to 0 and solving for x.

   $x + 1 = 0$

   $x = -1$

   Thus, an x-value of -1 invalidates the expression, and $x \neq -1$ for the third term's denominator.

   Now we need to determine invalid values for the first term's denominator, $3 - x$.

   Because $3 - x = -(x - 3)$, we can rewrite the first term as follows.

   $$\frac{1}{3-x}=\frac{-1}{x-3}$$

   Now we can see that the first term's denominator, rewritten as $x - 3$, is a factor of the second denominator as well. That means that $x \neq 3$ and $x \neq -1$ for all three terms of the expression.

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2. Find a common denominator.

   Because the second term's denominator, $x^2 - 2x - 3$, has the denominators of the first term and third term as factors, the second term's denominator is our common denominator.

3. Rewrite each term of the expression using the new denominator.

   The middle term needs no adjustment, since it already includes the common denominator.

$$\frac{1}{3-x}+\frac{10x-6}{x^2-2x-3}-\frac{4}{x+1}$$ Original expression

$$=\frac{-1}{x-3}\bullet\frac{x+1}{x+1}+\frac{10x-6}{x^2-2x-3}-\frac{4}{x+1}\bullet\frac{x-3}{x-3}$$ Multiply $\frac{-1}{x-3}$ by $\frac{x+1}{x+1}$ and $\frac{4}{x+1}$ by $\frac{x-3}{x-3}$.

$$=\frac{-1(x+1)}{(x-3)(x+1)}+\frac{10x-6}{x^2-2x-3}-\frac{4(x-3)}{(x+1)(x-3)}$$ Simplify each term.

$$=\frac{-1x-1}{x^2-2x-3}+\frac{10x-6}{x^2-2x-3}-\frac{4x-12}{x^2-2x-3}$$ Multiply.

$$=\frac{(-1x-1)+(10x-6)-(4x-12)}{x^2-2x-3}$$ Rewrite the expression by adding the numerators over the common denominator.

$$=\frac{-1x-1+10x-6-4x+12}{x^2-2x-3}$$ Simplify.

$$=\frac{5x+5}{x^2-2x-3}$$ Combine like terms.

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4. Check to see if the result can be written in a simpler form.

   We already know the denominator has $x - 3$ and $x + 1$ as factors. The numerator can be factored, because both terms include a factor of 5.

   Factor the expression to simplify it even further.

$\frac{5x+5}{x^2-2x-3}$	Expression from the previous step
$=\frac{5(x+1)}{(x+1)(x-3)}$	Factor the numerator and denominator.
$=\frac{5\cancel{(x+1)}}{\cancel{(x+1)}(x-3)}$	Cancel out common factors.
$=\frac{5}{x-3}$	Simplify.

   The factor $x + 1$ cancels out of the numerator and the denominator.

   Therefore, the simplest way to rewrite the expression $\frac{1}{3-x}+\frac{10x-6}{x^2-2x-3}-\frac{4}{x+1}$ is $\frac{5}{x-3}$, where $x \neq 3$.



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Problem-Based Task 2B.1.2: Who Is Right?

SMP	
1 ✓	2
3	4
5	6 ✓
7 ✓	8

Three students in Mr. Kunal's class were given the task of determining the result when the rational expression $\frac{3}{x-7}$ is subtracted from $\frac{4x-9}{x^2-10x+21}$. Their findings are below. Which students, if any, are correct?

- Deion calculates the result as $\frac{x}{x^2-10x+21}$.
- Daniel thinks the answer is $\frac{x-18}{x^2-10x+21}$.
- Dylan is pretty sure his response of $\frac{x^2}{x^3-10x^2+21x}$ is absolutely correct.



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Problem-Based Task 2B.1.2: Who Is Right?

Coaching

a. What is the expression the students are trying to simplify?

b. How would simplifying the expression help determine who has a correct answer?

c. Do you see any patterns in the original problem the students were asked to solve?

d. If the expression looks like a quadratic, it might be possible to factor it. Does the expression have integer factors we can use?

e. Are there any values for x that will make the denominators in the expression invalid?

f. What is the common denominator of the expression the students were asked to simplify?

g. If the expression is rewritten using this common denominator, is it possible to then simplify the expression? If so, what is the result?

h. What does this result reveal about Deion's response?

i. What does this result reveal about Daniel's response?

j. Do you see any similarities between the result found in part g and Dylan's response?

k. How can the expression from part g be adjusted to look like Dylan's response?

l. Which students, if any, correctly determined the result?

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Problem-Based Task 2B.1.2: Who Is Right?

Coaching Sample Responses

a. What is the expression the students are trying to simplify?

The students are trying to simplify the rational expression $\frac{4x-9}{x^2-10x+21}-\frac{3}{x-7}$.

b. How would simplifying the expression help determine who has a correct answer?

If the result we get when rewriting the expression matches that of Deion, Daniel, and/or Dylan, we will be able to directly verify their results. Even if the results are not exact matches, they might still help us see a connection that will allow us to answer the question.

c. Do you see any patterns in the original problem the students were asked to solve?

The expression $\frac{3}{x-7}$ is already completely simplified. The expression $\frac{4x-9}{x^2-10x+21}$ has a quadratic for a denominator; therefore, we might be able to factor the denominator.

d. If the expression looks like a quadratic, it might be possible to factor it. Does the expression have integer factors we can use?

We can use the quadratic formula, a table, a graph, or factoring to determine an equivalent form of the quadratic. By factoring, we can determine that the quadratic $x^2 - 10x + 21 = (x - 7)(x - 3)$.

e. Are there any values for x that will make the denominators in the expression invalid?

Set each denominator or factor equal to 0 and then solve for x to determine the values that make the expression invalid. The factors of the quadratic expression are $x - 7$ and $x - 3$. Since the factor $x - 7$ is also the denominator of the expression $\frac{3}{x-7}$, we only need to set each factor equal to 0 and solve.

$x - 7 = 0$ $\qquad$ $x - 3 = 0$

$x = 7$ $\qquad$ $x = 3$

If $x = 7$ or $x = 3$, then the denominator of $\frac{4x-9}{x^2-10x+21}$ equals 0. The value $x = 7$ also makes $\frac{3}{x-7}$ not valid. Therefore, the rational expression is only valid when $x \neq 3$ and $x \neq 7$.

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f. What is the common denominator of the expression the students were asked to simplify?

Because the factor $x - 7$ is also the denominator of $\frac{3}{x-7}$, the common denominator is $x^2 - 10x + 21$, or $(x - 7)(x - 3)$.

g. If the expression is rewritten using this common denominator, is it possible to then simplify the expression? If so, what is the result?

Yes, it is possible. Since one denominator is already written using the common denominator, it is only necessary to multiply the second denominator to achieve a common denominator.

$$\frac{4x-9}{x^2-10x+21}-\frac{3}{x-7}=\frac{4x-9}{x^2-10x+21}-\frac{3}{x-7}\cdot\frac{x-3}{x-3}$$
$$=\frac{4x-9}{x^2-10x+21}-\frac{3(x-3)}{x^2-10x+21}$$
$$=\frac{4x-9}{x^2-10x+21}-\frac{3x-9}{x^2-10x+21}$$
$$=\frac{4x-9-(3x-9)}{x^2-10x+21}$$
$$=\frac{4x-9-3x+9}{x^2-10x+21}$$
$$=\frac{x}{x^2-10x+21}$$

h. What does this result reveal about Deion's response?

This result is exactly the same as Deion's response, so Deion's answer is correct.

i. What does this result reveal about Daniel's response?

The denominator in Daniel's response is the same as the result from part g, but the numerators are different. For Daniel to be correct, the two numerators would have to be equal. That would mean that $x = x - 18$. Subtracting x from both sides gives $0 = -18$, which is not correct. Therefore, Daniel's answer is not correct. (A closer look at Daniel's solution shows that he may not have properly distributed the negative sign in front of the $3x - 9$ term.)

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j. Do you see any similarities between the result found in part g and Dylan's response?

Dylan's answer is $\frac{x^2}{x^3-10x^2+21x}$. Both the numerator and the denominator of this response are equal to x times the numerator and denominator of the expression found in part g.

k. How can the expression from part g be adjusted to look like Dylan's response?

Multiplying the expression from part g by 1 in the form of $\frac{x}{x}$ will make it identical to Dylan's response.

$$\frac{x}{x^2-10x+21} \bullet \frac{x}{x} = \frac{x^2}{x^3-10x^2+21x}$$

That means Dylan is correct, because his response is equivalent to the answer for the original problem. However, because Dylan's response can be further simplified (by factoring out an x), it cannot be considered a simplified result.

l. Which students, if any, correctly determined the result?

While Deion and Dylan have correctly rewritten the expression, Daniel has not.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2B.1.2: Adding and Subtracting Rational Expressions

For problems 1–7, simplify each rational expression. Whenever possible, reduce the expression to its lowest terms. State any restrictions on x.

1. $\frac{3}{x}+\frac{7}{5x}$
2. $\frac{11}{x}-\frac{x}{x-1}$
3. $\frac{x^2+3}{8}-\frac{4}{x}$
4. $\frac{x+1}{5}+\frac{1}{x}$
5. $\frac{9x}{x+1}+5x$
6. $\frac{5x}{x+1}+\frac{3x+1}{x^2-x-2}$
7. $\frac{8x+1}{6x^2-7x-3}-\frac{4}{3x+1}+\frac{11x}{2x-3}$

Use your knowledge of rational expressions to complete problems 8–10. State any restrictions on x.

8. Sam walks $\frac{3}{8}$ km to school. After school, he walks another $\frac{x+1}{3}$ km to get to work. What simplified rational expression describes the total length of both of his walks?

9. Emma's strawberry farm has a total of $5x$ hectares of arable land. This past season, Emma planted $\frac{x^2+3}{x+7}$ hectares of strawberries. What simplified rational expression represents the total amount of land, in hectares, that was not planted last season?

10. The width of a rectangle is $\frac{x+2}{5}$ cm. The rectangle's length is $\frac{x^2+3x+2}{x+3}$ cm. What expression represents the perimeter of this rectangle?

SWB p. 23; SRB Practice 2B.1.2 p. U2B-25

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Lesson 2B.1.3: Multiplying Rational Expressions

Warm-Up 2B.1.3

The area of a rectangle is the product of its length and width. Use this relationship to answer the following questions.

1. A rectangle has a length of $\frac{7}{8}$ cm and a width of $\frac{2}{3}$ cm. What expression represents the rectangle's area?

2. A square has a perimeter of 17 cm. What expression represents the square's area?

3. Another rectangle has a width of $\frac{x}{2}$ m and a height of $\frac{3}{4}$ m. What expression represents this rectangle's area?

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Lesson 2B.1.3: Multiplying Rational Expressions

Common Core State Standards

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

A–APR.7 (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Warm-Up 2B.1.3 Debrief

1. A rectangle has a length of $\frac{7}{8}$ cm and a width of $\frac{2}{3}$ cm. What expression represents the rectangle's area?

   The area is the product of the length and width, or *area* = *length* • *width*.

   The rectangle's length is $\frac{7}{8}$ cm.

   The rectangle's width is $\frac{2}{3}$ cm.

   Multiply these two quantities to determine the area.

   $\text{area} = \text{length} \bullet \text{width}$ Formula for area of a rectangle

   $\text{area} = \left(\frac{7}{8}\right) \bullet \left(\frac{2}{3}\right)$ Substitute known values.

   $\text{area} = \frac{14}{24}$ Multiply.

   The fraction $\frac{14}{24}$ can be reduced to lowest terms, as follows.

   $\frac{14}{24} = \frac{7}{12}$

   Therefore, the area is $\frac{14}{24}$ square centimeters, which reduces to $\frac{7}{12}$ cm^2.

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2. A square has a perimeter of 17 cm. What expression represents the square's area?

The perimeter is the sum of all the sides, and a square has four equal sides. That means the length of one side of the square is $\frac{17}{4}$ cm.
The area is the product of the length and width; since the sides of a square are all equal, the length and width are the same: *area* = *side*^{2}. Square the side length to determine the area.

$\text{area} = \text{side}^2$	Formula for area of a square
$\text{area} = \left(\frac{17}{4}\right)^2$	Substitute the known value.
$\text{area} = \frac{17^2}{4^2}$	Apply the exponent to the numerator and denominator.
$\text{area} = \frac{289}{16}$	Simplify.
$\text{area} = 18\frac{1}{16}$	Rewrite the improper fraction as a mixed fraction.

For this square, the area is $18\frac{1}{16}$ cm^2.

3. Another rectangle has a width of $\frac{x}{2}$ m and a height of $\frac{3}{4}$ m. What expression represents this rectangle's area?

The area is the product of the length and width, or *area* = *length* • *width*.
The rectangle's width is $\frac{x}{2}$ m. The rectangle's length (or height) is $\frac{3}{4}$ m.
Multiply these two quantities to determine the area.

$\text{area} = \text{length} \bullet \text{width}$	Formula for area of a rectangle
$\text{area} = \left(\frac{3}{4}\right) \bullet \left(\frac{x}{2}\right)$	Substitute known values.
$\text{area} = \frac{3x}{8}$	Multiply.

The area of this rectangle is $\frac{3x}{8}$ m^2.

Connection to the Lesson

- Students will learn that multiplying rational expressions is similar to multiplying fractions.

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Prerequisite Skills

This lesson requires the use of the following skills:

- simplifying rational expressions (A-APR.6)
- determining values that make a rational expression invalid (A-APR.3)
- multiplying fractions (5.NF.4)

Introduction

Like with multiplying fractions, finding the product of rational expressions requires that the values in the numerator be handled separately from the values in the denominator. As the expressions grow more complex, manipulating these values is similar to steps taken previously when working with polynomials. Looking for possible invalid values and opportunities to simplify the expression can make the task less complex.

Key Concepts

- Multiplication of rational expressions does not require a common denominator.
- The product of a collection of rational expressions is the product of the numerators divided by the product of the denominators. Written symbolically, $\frac{a(x)}{z(x)} \bullet \frac{b(x)}{y(x)} \bullet \frac{c(x)}{w(x)} = \frac{a(x) \bullet b(x) \bullet c(x)}{z(x) \bullet y(x) \bullet w(x)}$.
- As previously encountered, any values that make the denominator equal to 0 must be identified and removed. For example, the value $x = 1$ must be excluded from the rational expression $\frac{2x}{x-1}$, since that value makes the rational expression equivalent to $\frac{2x}{x-1} = \frac{2(1)}{(1)-1} = \frac{2}{0}$, an undefined form.
- Similar to fractions, rational expressions can be reduced by eliminating common factors from both the numerator and denominator.

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Common Errors/Misconceptions

- canceling individual terms of the numerator or denominator, rather than factors
- distributing values incorrectly, especially negative numbers
- forgetting to check for invalid values in the denominator

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Guided Practice 2B.1.3

Example 1

Simplify the rational expression $\frac{3}{x+1} \bullet 8$.

1. Identify any invalid values of the expression.

   The denominator of the first term, $x + 1$, cannot equal 0. Set this expression equal to 0 and then solve for x to determine values that make this expression invalid.

   $x + 1 = 0$

   $x = -1$

   An x-value of -1 invalidates the expression, so $x \neq -1$.

2. Rewrite the second term as a fraction.

   The number 8 can be rewritten in equivalent form as $\frac{8}{1}$.

3. Multiply the expressions to form a single rational expression.

   $$\frac{3}{x+1} \bullet \frac{8}{1} = \frac{3 \bullet 8}{1(x+1)} = \frac{24}{x+1}$$

4. Check for any factors that might make it possible to further simplify the resulting expression.

   The individual terms from the original expression are both simple. There is no common factor.

   The final result of $\frac{3}{x+1} \bullet 8$ is $\frac{24}{x+1}$ when $x \neq -1$.

SWB p. 29; SRB p. U2B-28

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Example 2

Simplify the rational expression $\frac{x-3}{x} \bullet 5x$.

1. Identify any invalid values of the expression.

   The denominator of the first term, x, cannot equal 0. Therefore, $x \neq 0$.

2. Rewrite the second term as a rational expression.

   The expression $5x$ can be rewritten as $\frac{5x}{1}$.

3. Multiply the expressions to form a single rational expression.

   $$\frac{x-3}{x} \bullet \frac{5x}{1} = \frac{5x(x-3)}{x \bullet 1} = \frac{5x^2-15x}{x}$$

4. Check for any factors that might make it possible to further simplify the resulting expression.

   The numerator shares a factor, x, with the denominator (you might have noticed this in the last step, and adapted then). Factored, the result looks like this:

   $$\frac{5x^2-15x}{x} = \frac{\cancel{x}(5x-15)}{\cancel{x}} = 5x-15$$

   The final result of $\frac{x-3}{x} \bullet 5x$ is $5x-15$ when $x \neq 0$.



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Example 3

Simplify the rational expression $\frac{5}{x} \bullet \frac{x+1}{x^2}$.

1. Identify any invalid values of the expression.

   Both expressions are invalid when $x = 0$. Therefore, $x \neq 0$.

2. Multiply the expressions to form a single rational expression.

   $$\frac{5}{x} \bullet \frac{x+1}{x^2} = \frac{5(x+1)}{x \bullet x^2} = \frac{5x+5}{x^3}$$

3. Check for any common factors that might make it possible to further simplify the resulting expression.

   There is no common factor. Though the numerator and denominator both have an x term, x is not a common factor; we cannot factor x out of $5x + 5$.

   The final result of $\frac{5}{x} \bullet \frac{x+1}{x^2}$ is $\frac{5x+5}{x^3}$ when $x \neq 0$.

Example 4

Simplify the rational expression $\frac{x+1}{x^2} \bullet \frac{2x-3}{x-1}$.

1. Identify any invalid values of the expression.

   The denominator of the first term cannot equal 0. That means $x \neq 0$. In the second term, $x - 1 \neq 0$, which means $x \neq 1$.

   Therefore, $x \neq 0$ and $x \neq 1$.

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2. Multiply the expressions to form a single rational expression.

$$\frac{x+1}{x^2}\bullet\frac{2x-3}{x-1}=\frac{(x+1)(2x-3)}{x^2(x-1)}=\frac{2x^2-x-3}{x^3-x^2}$$

3. Check for any factors that might make it possible to further simplify the resulting expression.

   There are no common factors.

   The final result of $\frac{x+1}{x^2}\bullet\frac{2x-3}{x-1}$ is $\frac{2x^2-x-3}{x^3-x^2}$ when $x \neq 0$ and $x \neq 1$.

Example 5

Simplify the rational expression $\frac{x^2-4x-5}{x+1}\bullet\frac{x+8}{x^2+9x+8}$.

1. Identify any invalid values of the expression.

   The denominator of the first term means $x + 1 \neq 0$, or $x \neq -1$.

   The second term is a quadratic, and must be factored to determine invalid values for x.

   $x^2 + 9x + 8 = (x + 1)(x + 8)$

   The factors are $(x + 1)$ and $(x + 8)$. Set each of these equal to 0 and solve for x to determine values that make the expression invalid.

   $x + 1 = 0$ $\qquad$ $x + 8 = 0$

   $x = -1$ $\qquad$ $x = -8$

   An x-value of -1 invalidates the expression, as does an x-value of -8. Therefore, for the second denominator, $x \neq -1$ and $x \neq -8$.

   Notice that $x \neq -1$ is also a restriction on the denominator of the first term. Thus, for the entire rational expression, $x \neq -1$ and $x \neq -8$.

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2. Factor the numerator of the first term.

Since the numerator of the term $\frac{x^2-4x-5}{x+1}$ appears to be quadratic, factoring this numerator may aid in simplifying the expression.

The numerator of the first term is $x^2 - 4x - 5$.

$x^2 - 4x - 5 = (x - 5)(x + 1)$

Notice that $(x + 1)$ is one of the factors of the denominator of the second term.

3. Using the previously determined factors, multiply the expressions to form a single rational expression.

Replace the quadratic in each term of the original expression with its factored form.

$$\frac{x^2-4x-5}{x+1}\cdot\frac{x+8}{x^2+9x+8}=\frac{(x-5)(x+1)}{x+1}\cdot\frac{x+8}{(x+1)(x+8)}$$

Multiply the rewritten terms.

$$\frac{(x-5)(x+1)}{x+1}\cdot\frac{x+8}{(x+1)(x+8)}=\frac{(x-5)(x+1)(x+8)}{(x+1)(x+1)(x+8)}$$

4. Simplify the rational expression.

Cancel out common factors.

$$\frac{(x-5)(x+1)(x+8)}{(x+1)(x+1)(x+8)}=\frac{(x-5)\cancel{(x+1)}\cancel{(x+8)}}{(x+1)\cancel{(x+1)}\cancel{(x+8)}}=\frac{x-5}{x+1}$$

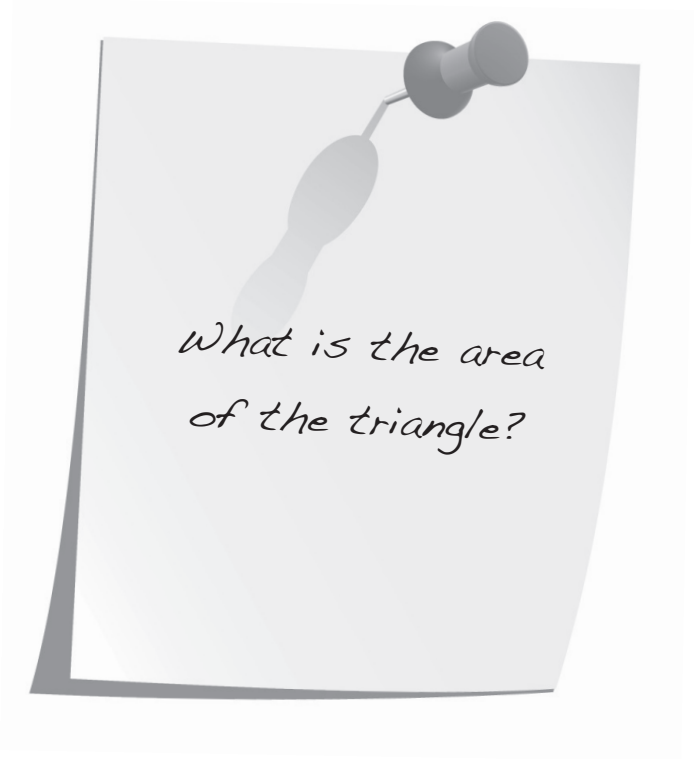
The final result of $\frac{x^2-4x-5}{x+1}\cdot\frac{x+8}{x^2+9x+8}$ is $\frac{x-5}{x+1}$ when $x \neq -1$ and $x \neq -8$.

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Problem-Based Task 2B.1.3: The Area of a Triangle

SMP	
1	2
3	4
5	6 ✓
7 ✓	8 ✓

One leg of a right triangle is represented by the expression $\frac{3x}{x+1}$. The other leg is represented by the expression $\frac{x-1}{x^2}$. What is the area of the triangle? The length of each leg is in centimeters.



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Problem-Based Task 2B.1.3: The Area of a Triangle

Coaching

a. How can you determine the area of a right triangle if you know the lengths of the legs?

b. What rational expression represents the area of the triangle?

c. Are there any values that make this relationship invalid?

d. What is $\frac{1}{2}$ the product of the two legs?

e. What is the solution to the problem?

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Problem-Based Task 2B.1.3: The Area of a Triangle

Coaching Sample Responses

a. How can you determine the area of a right triangle if you know the lengths of the legs?

The area of a right triangle is $\frac{1}{2}$ times the product of the two legs, called the base and the height: $area = \frac{1}{2}(base \bullet height)$.

b. What rational expression represents the area of the triangle?

The area of a right triangle is $\frac{1}{2}$ the product of the legs.

Set up an expression using the given lengths of each leg times $\frac{1}{2}$.

Let $\frac{3x}{x+1}$ represent the base, and let $\frac{x-1}{x^2}$ represent the height.

$$\text{area} = \frac{1}{2}(\text{base} \bullet \text{height})$$

$$\text{area} = \frac{1}{2} \bullet \left(\frac{3x}{x+1}\right) \bullet \left(\frac{x-1}{x^2}\right)$$

c. Are there any values that make this relationship invalid?

Since neither denominator of the values of the legs can equal 0, $x + 1 \neq 0$ and $x^2 \neq 0$.

Set each denominator equal to 0 and solve for x to determine values that make the relationship invalid.

$x + 1 = 0$ $\qquad$ $x^2 = 0$

$x = -1$ $\qquad$ $x = 0$

An x-value of -1 invalidates the relationship, as does an x-value of 0. Therefore, $x \neq -1$ and $x \neq 0$.

d. What is $\frac{1}{2}$ the product of the two legs?

Solve the equation determined in part b.

$$\text{area} = \frac{1}{2} \bullet \left(\frac{3x}{x+1}\right) \bullet \left(\frac{x-1}{x^2}\right)$$

$$\text{area} = \frac{1}{2} \bullet \frac{3x(x-1)}{x^2(x+1)}$$

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$$\text{area} = \frac{3(x-1)}{2x(x+1)}$$

$$\text{area} = \frac{3x-3}{2x^2+2x}$$

e. What is the solution to the problem?

The area of the right triangle with legs of length $\frac{3x}{x+1}$ cm and $\frac{x-1}{x^2}$ cm is $\frac{3x-3}{2x^2+2x}$ cm^2, when $x \neq -1$ and $x \neq 0$. Additionally, since distance must be positive, $x \geq 1$.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2B.1.3: Multiplying Rational Expressions

For problems 1–5, simplify each rational expression. Whenever possible, reduce the expression to its lowest terms. If necessary, factor the terms in each expression before multiplying. State any restrictions on x.

1. $\frac{3}{x} \bullet \frac{5x}{6x^2+9}$

2. $\frac{x}{x-3} \bullet \frac{2x+7}{x+1}$

3. $\frac{x^2+3x}{x-4} \bullet \frac{x^2+1}{x^2}$

4. $\frac{x}{12} \bullet \frac{8+4x}{5x}$

5. $\frac{2x+6}{x-6} \bullet \frac{x^2-4x-12}{30+4x-2x^2}$

continued

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Use your knowledge of rational expressions and factoring to complete problems 6–10. State any restrictions on x.

6. For what values of x is $\frac{2x-8}{x^2+7x+10}$ an invalid expression?

7. Show that the rational expression $\frac{5x+5}{x} \bullet \frac{x^3+3x^2}{x^2-1} \bullet \frac{x-1}{5x}$ is equivalent to the rational expression $x+3$.

8. What simplified rational expression represents the area of a rectangle with a width of $\frac{x}{2}$ inches and a length of $\frac{2x+1}{x-5}$ inches?

9. The momentum of an object is the product of the object's mass and its velocity. What simplified expression describes the momentum (in g • cm/s) of a moving toy car if the rational expression $\frac{x^2+4x+4}{x^2-9}$ describes the toy car's mass in grams, and the expression $\frac{x^2+5x+6}{x+2}$ describes the toy car's velocity in centimeters per second (where $x > 3$)?

10. What simplified expression describes the volume of a rectangular box with a length of $\frac{x^2+x}{x-5}$ meters, a width of $\frac{7x}{x+1}$ meters, and height of $\frac{x^2+x-30}{x^2+7x}$ meters?

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Lesson 2B.1.4: Dividing Rational Expressions

Warm-Up 2B.1.4

Yuri will be making a glass window hanging with a metal frame for his art class. He is deciding which shape to use: a hexagon, a triangle, or a square. His window hanging must have all sides the same length.

1. If Yuri decides to use a hexagon (a 6-sided figure), what is the length of one side if the hexagon's perimeter is 25 cm?

2. If he decides on a triangular shape, what is the length of one side if the perimeter of the triangle measures $\frac{8}{3}$ m?

3. Yuri's math teacher has a square frame with a perimeter of $(8x + 12)$ cm. The teacher will give Yuri the frame if Yuri can tell him what expression represents the length of one side. What is the correct expression?

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Lesson 2B.1.4: Dividing Rational Expressions

Common Core State Standards

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

A–APR.6 Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

A–APR.7 (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Warm-Up 2B.1.4 Debrief

1. If Yuri decides to use a hexagon (a 6-sided figure), what is the length of one side if the hexagon's perimeter is 25 cm?

   To determine the length of one side, divide the total perimeter, 25, by the number of sides, 6.

   $$\frac{25}{6} = 4\frac{1}{6}$$

   With a total perimeter of 25 cm, the length of one side of the hexagon is $\frac{25}{6}$ cm, or $4\frac{1}{6}$ cm.

2. If he decides on a triangular shape, what is the length of one side if the perimeter of the triangle measures $\frac{8}{3}$ m?

   To determine the length of one side, divide the total perimeter, $\frac{8}{3}$, by the number of sides, 3.

   Recall that, when dividing fractions, you must multiply by the reciprocal of the divisor; in this case, the reciprocal of 3 is $\frac{1}{3}$.

   $$\frac{8}{3} \div 3 = \frac{8}{3} \bullet \frac{1}{3} = \frac{8}{9}$$

   The length of one side of the triangle is $\frac{8}{9}$ m.

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3. Yuri's math teacher has a square frame with a perimeter of $(8x + 12)$ cm. The teacher will give Yuri the frame if Yuri can tell him what expression represents the length of one side. What is the correct expression?

   Each of the four sides of a square is the same length. To find the length of one side, we need to divide the perimeter by 4.

$$(8x+12)\div 4=\frac{8x+12}{4}=\frac{\cancel{4}(2x+3)}{\cancel{4}}=2x+3$$

   The length of one side of the square is $(2x + 3)$ cm.

Connection to the Lesson

- Students will learn that dividing rational expressions is similar to dividing fractions.

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Prerequisite Skills

This lesson requires the use of the following skills:

- simplifying rational expressions (A-APR.6)
- determining values that make a rational expression invalid (A-APR.3)
- dividing fractions (5.NF.4)

Introduction

Previously, explorations of ways to recognize and simplify rational expressions have been used to add, subtract, and multiply those expressions. In this section, these skills will be extended to explore the results of the operation of division on rational expressions.

As in previous examples, familiarity with numerical fractions will be useful in working with rational expressions. Dividing rational expressions will also be made easier using a basic fact learned in dealing with fractions; namely, that division is the equivalent to multiplying by the reciprocal of the divisor. Recall that a **reciprocal** is a number that, when multiplied by the original number, has a product of 1. For example, dividing 12 by 3 is the same as multiplying 12 by the reciprocal of 3, which is $\frac{1}{3}$. We can verify that 3 and $\frac{1}{3}$ are reciprocals by showing that $3 \bullet \frac{1}{3} = 1$.

Key Concepts

- The division expression $a(x) \div b(x)$ is equivalent to $a(x) \bullet \frac{1}{b(x)}$: the product of the divisor and the reciprocal of the dividend.
- A rational expression of the form $\frac{a(x)}{b(x)} \div \frac{c(x)}{d(x)}$ is equivalent to the rational expression $\frac{a(x)}{b(x)} \bullet \frac{1}{\frac{c(x)}{d(x)}} = \frac{a(x)}{b(x)} \bullet \frac{d(x)}{c(x)}$.
- This expression is the product of the divisor and the reciprocal of the dividend.
- A rational expression of the form $\frac{a(x)}{b(x)}$ is equivalent to the expression $a(x) \div b(x)$.
- If $b(x)$ is a factor of $a(x)$, there is no remainder.

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- If $b(x)$ is *not* a factor of $a(x)$, then the remainder can be described as $r(x)$.
- If $b(x)$ is *not* a factor of $a(x)$, $\frac{a(x)}{b(x)}$ is equivalent to the sum of the quotient $q(x)$ and the ratio $\frac{r(x)}{b(x)}$.
- We can use polynomial division to represent the original rational expression in a different form.
- For example, let $a(x) = x^2 + 9x + 14$ and $b(x) = x + 7$. If $\frac{a(x)}{b(x)} = \frac{x^2+9x+14}{x+7}$, then the rational expression $\frac{x^2+9x+14}{x+7}$ is equivalent to the expression $(x^2 + 9x + 14) \div (x + 7)$. To find the quotient and remainder, use polynomial division:

$$\begin{array}{r} x+2 \\ x+7\overline{)\,x^2+9x+14} \\ \underline{x^2+7x} \\ 2x+14 \\ \underline{2x+14} \\ 0 \end{array}$$

- In this example, the quotient, $q(x)$, is equal to $x + 2$, and the remainder, $r(x)$, is 0. The original expression $\frac{a(x)}{b(x)}$ is equivalent to $\frac{x^2+9x+14}{x+7} = (x+2) + \frac{0}{x+7}$, or, more simply, $\frac{x^2+9x+14}{x+7} = x+2$.
- Synthetic division (a shortcut for polynomial division) can be used to ease the task of rewriting rational expressions. The rational expression $\frac{x^2+8x+15}{x+5}$ results in the following synthetic division.

$$\begin{array}{r|rrr} -5 & 1 & 8 & 15 \\ & & -5 & -15 \\ \hline & 1 & 3 & \boxed{0} \end{array}$$

- This result yields a quotient of $x + 3$, with a remainder of 0. This shows that the original expression $\frac{x^2+8x+15}{x+5}$ is equivalent to $(x+3) + \frac{0}{x+5}$, or, more simply, $x + 3$.

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Common Errors/Misconceptions

- canceling individual terms of the numerator or denominator, rather than factors
- distributing values incorrectly, especially negative numbers
- using the reciprocal of the wrong rational expression
- forgetting to check for invalid values in the denominator
- neglecting to include terms when using polynomial or synthetic division; for example, a dividend of $x^3 - x + 3$ becomes $x^3 + 0x^2 - x + 3$

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Guided Practice 2B.1.4

Example 1

Simplify the rational expression $\frac{3x^2}{7} \div \frac{x}{3}$.

1. Identify any invalid values of the expression.

   The denominators of both the first and second terms are nonzero constants, and are therefore valid.

2. Rewrite the division problem as a multiplication problem, using the reciprocal of the divisor.

   The divisor is $\frac{x}{3}$; the reciprocal of $\frac{x}{3}$ is $\frac{3}{x}$.

   $\frac{3x^2}{7} \div \frac{x}{3}$ Original expression

   $= \frac{3x^2}{7} \bullet \left(\frac{3}{x}\right)$ Multiply by the reciprocal.

   The rewritten expression is $\frac{3x^2}{7} \bullet \frac{3}{x}$.

3. Identify any invalid values of the rewritten second term.

   The denominator of the first term is a constant, and is therefore valid. The second term requires that $x \neq 0$.

4. Multiply the terms of the rewritten expression.

   $\frac{3x^2}{7} \bullet \frac{3}{x} = \frac{3x^2 \bullet 3}{7 \bullet x} = \frac{9x^2}{7x}$

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5. Check for any factors that might make it possible to further simplify the resulting expression, and then factor them out.

   There is common factor of x in both the numerator and the denominator.

   $$\frac{9x^2}{7x}=\frac{9x\bullet\cancel{x}}{7\cancel{x}}=\frac{9x}{7}$$

   The expression $\frac{3x^2}{7}\div\frac{x}{3}$ simplifies to $\frac{9x}{7}$ when $x\neq 0$.



Example 2

Simplify the rational expression $\frac{x}{x^2+1}\div\frac{5}{x^2-x+1}$.

1. Identify any invalid values of the expression.

   The denominators of both expressions are quadratics, so see if they can be factored.

   They cannot be factored, but they are always nonzero, and are therefore valid.

2. Rewrite the division problem as a multiplication problem, using the reciprocal of the divisor.

   The divisor is $\frac{5}{x^2-x+1}$; the reciprocal of $\frac{5}{x^2-x+1}$ is $\frac{x^2-x+1}{5}$.

   $\frac{x}{x^2+1}\div\frac{5}{x^2-x+1}$ Original expression

   $=\frac{x}{x^2+1}\bullet\left(\frac{x^2-x+1}{5}\right)$ Multiply by the reciprocal.

   The rewritten expression is $\frac{x}{x^2+1}\bullet\frac{x^2-x+1}{5}$.

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3. Identify any invalid values of the rewritten second term.

   The denominator is 5.

   Since 5 is a constant, there are no real number values that will make the denominator equal 0.

4. Multiply the terms of the rewritten expression.

$$\frac{x}{x^2+1}\bullet\frac{x^2-x+1}{5}=\frac{x(x^2-x+1)}{5(x^2+1)}=\frac{x^3-x^2+x}{5x^2+5}$$

5. Check for any factors that might make it possible to further simplify the resulting expression, and then factor them out.

   There are no common factors.

   The expression $\frac{x}{x^2+1}\div\frac{5}{x^2-x+1}$ simplifies to $\frac{x^3-x^2+x}{5x^2+5}$.

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Example 3

Simplify the rational expression $\frac{x^2-3x}{x^2-x-6} \div \frac{x-3}{x+2}$.

1. Identify any invalid values of the expression.

   The denominators are $x^2 - x - 6$ and $x + 2$. Since $x^2 - x - 6$ is a quadratic, see if it can be factored.

   $x^2 - x - 6 = (x - 3)(x + 2)$

   The factors are $x - 3$ and $x + 2$.

   Set each factor equal to 0 and solve for x to determine the values that make the expression invalid.

   $x - 3 = 0$ $\qquad$ $x + 2 = 0$

   $x = 3$ $\qquad$ $x = -2$

   An x-value of 3 invalidates the expression, as does an x-value of -2. Therefore, for the first denominator, $x \neq 3$ and $x \neq -2$.

   Set the denominator of the second term, $x + 2$, equal to 0 to determine values that make the expression invalid.

   $x + 2 = 0$

   $x = -2$

   Therefore, $x \neq -2$, which was also a restriction determined for the first expression.

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2. Rewrite the division problem as a multiplication problem, using the reciprocal of the divisor.

The divisor is $\frac{x-3}{x+2}$; the reciprocal of $\frac{x-3}{x+2}$ is $\frac{x+2}{x-3}$.

$\frac{x^2-3x}{x^2-x-6} \div \frac{x-3}{x+2}$ Original expression

$= \frac{x^2-3x}{x^2-x-6} \bullet \left(\frac{x+2}{x-3}\right)$ Multiply by the reciprocal.

The rewritten expression is $\frac{x^2-3x}{x^2-x-6} \bullet \frac{x+2}{x-3}$.

3. Identify any invalid values of the rewritten second term.

The denominator is $x - 3$.

Set the denominator of the second term, $x - 3$, equal to 0 to determine values that make the expression invalid.

$x - 3 = 0$

$x = 3$

Therefore, $x \neq 3$, which was also a restriction determined for the first expression.

The invalid values of the rewritten expression are $x \neq 3$ and $x \neq -2$, the same as the restrictions for the original expression.

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4. Check for factors that may cancel, then multiply the terms of the rewritten expression.

In the rewritten expression, $\frac{x^2-3x}{x^2-x-6}\bullet\left(\frac{x+2}{x-3}\right)$, there are quadratic expressions in both the numerator and denominator of the first term.

Both of these quadratic expressions can be factored.

We have already determined the factors of $x^2 - x - 6$ are $x - 3$ and $x + 2$.

Factor the quadratic in the numerator of the first term, $x^2 - 3x$.

$x^2 - 3x = x(x - 3)$

The factors are x and $x - 3$.

Notice that in both the numerator and denominator of the first term, $x - 3$ is a factor.

Use this information to cancel out common factors and multiply the rewritten expression.

$\frac{x^2-3x}{x^2-x-6}\bullet\frac{x+2}{x-3}$ Rewritten expression

$=\frac{x(x-3)}{(x-3)(x+2)}\bullet\frac{x+2}{x-3}$ Factor the first term of the expression.

$=\frac{x\,\cancel{(x-3)}}{\cancel{(x-3)}\,\cancel{(x+2)}}\bullet\frac{\cancel{x+2}}{x-3}$ Cancel out common factors.

$=\frac{x}{x-3}$ Simplify.

The expression $\frac{x^2-3x}{x^2-x-6}\div\frac{x-3}{x+2}$ simplifies to $\frac{x}{x-3}$.

5. Check for any factors that might make it possible to further simplify the resulting expression, and then factor them out.

There are no additional common factors.

The expression $\frac{x^2-3x}{x^2-x-6}\div\frac{x-3}{x+2}$ simplifies to $\frac{x}{x-3}$ when $x \neq 3$ and $x \neq -2$.

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Example 4

Simplify the rational expression $\frac{x^2+9x+14}{x^2+7x} \div \frac{x^2+x-2}{4x^2-4x}$.

1. Identify any invalid values of the expression.

   The denominators are $x^2 + 7x$ and $4x^2 – 4x$. Both of these denominators are quadratic, so see if they can be factored.

   $x^2 + 7x = x(x + 7)$

   The factors of the denominator of the first term are x and $x + 7$.

   Set each factor equal to 0 and solve for x to determine the values that make the expression invalid.

   $x = 0$ $\qquad$ $x + 7 = 0$

   $\qquad\qquad\quad$ $x = –7$

   An x-value of 0 invalidates the expression, as does an x-value of –7. Therefore, for the first denominator, $x \neq 0$ and $x \neq –7$.

   Next, factor the second denominator, $4x^2 – 4x$.

   $4x^2 – 4x = 4x(x – 1)$

   The factors of the denominator of the second term are $4x$ and $x – 1$.

   Set each factor equal to 0 and solve for x to determine the values that make the expression invalid.

   $4x = 0$ $\qquad$ $x – 1 = 0$

   $x = 0$ $\qquad$ $x = 1$

   An x-value of 0 invalidates the expression, as does an x-value of 1. Therefore, for the second denominator, $x \neq 0$ and $x \neq 1$.

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2. Rewrite the division problem as a multiplication problem, using the reciprocal of the divisor.

The divisor is $\frac{x^2+x-2}{4x^2-4x}$; the reciprocal of $\frac{x^2+x-2}{4x^2-4x}$ is $\frac{4x^2-4x}{x^2+x-2}$.

$\frac{x^2+9x+14}{x^2+7x} \div \frac{x^2+x-2}{4x^2-4x}$ Original expression

$= \frac{x^2+9x+14}{x^2+7x} \bullet \left(\frac{4x^2-4x}{x^2+x-2} \right)$ Multiply by the reciprocal.

The rewritten expression is $\frac{x^2+9x+14}{x^2+7x} \bullet \frac{4x^2-4x}{x^2+x-2}$.

3. Identify any invalid values of the rewritten second term.

The denominator is $x^2 + x - 2$. The denominator is quadratic, so see if it can be factored.

$x^2 + x - 2 = (x + 2)(x - 1)$

The factors of the denominator of the second term are $x + 2$ and $x - 1$.

Set each factor equal to 0 and solve for x to determine the values that make the expression invalid.

$x + 2 = 0$ $\quad$ $x - 1 = 0$

$x = -2$ $\quad$ $x = 1$

An x-value of -2 invalidates the expression, as does an x-value of 1. Therefore, for the second denominator, $x \neq -2$ and $x \neq 1$. The list of invalid x-values already includes $x \neq 1$, but it does not include $x \neq -2$.

So, the values of x which make the rewritten expression invalid are $x \neq 0$, $x \neq -7$, $x \neq -2$, and $x \neq 1$.

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4. Check for factors that may cancel, then multiply the terms of the rewritten expression.

   In the rewritten expression, $\frac{x^2+9x+14}{x^2+7x} \bullet \frac{4x^2-4x}{x^2+x-2}$, there are quadratic expressions in both the numerator and denominator of both expressions. Rewrite each quadratic expression in factored form to determine if there are common factors that may cancel.

   We have already determined the factored form of each denominator:

   $x^2 + 7x = x(x + 7)$ $\qquad$ $x^2 + x - 2 = (x + 2)(x - 1)$

   Now, factor the numerators.

   $x^2 + 9x + 14 = (x + 2)(x + 7)$ $\qquad$ $4x^2 - 4x = 4x(x - 1)$

   Substitute the factored forms into the rewritten expression to cancel out common factors and multiply the terms.

$\frac{x^2+9x+14}{x^2+7x} \bullet \frac{4x^2-4x}{x^2+x-2}$	Rewritten expression from step 2
$= \frac{(x+2)(x+7)}{x(x+7)} \bullet \frac{4x(x-1)}{(x+2)(x-1)}$	Write the quadratic expressions in factored form to identify common factors.
$= \frac{\cancel{(x+2)}\ \cancel{(x+7)}}{\cancel{x}\ \cancel{(x+7)}} \bullet \frac{4\cancel{x}\ \cancel{(x-1)}}{\cancel{(x+2)}\ \cancel{(x-1)}}$	Cancel out common factors.
$= 4$	Simplify.

   The expression $\frac{x^2+9x+14}{x^2+7x} \div \frac{x^2+x-2}{4x^2-4x}$ simplifies to 4 when $x \neq 0$, $x \neq -7$, $x \neq -2$, and $x \neq 1$.



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Example 5

Use synthetic division to rewrite $\frac{a(x)}{b(x)}=\frac{x^3+2x^2-x+3}{x+1}$ in the form $\frac{a(x)}{b(x)}=q(x)+\frac{r(x)}{b(x)}$.

1. Identify any values that make the expression invalid.

   If $x + 1 = 0$, the denominator equals 0; this creates an invalid form. Since $x + 1$ cannot equal 0, $x \neq -1$.

2. Set up the synthetic division problem to divide the given numerator, $a(x)$, by the denominator, $b(x)$.

   The numerator is $x^3 + 2x^2 - x + 3$. The coefficients of the numerator are 1, 2, –1, and 3. Because we are dividing by $x + 1$, the number –1 must go on the "shelf" of the synthetic division. The resulting synthetic division problem is $\underline{-1}\,|\;\; 1 \quad 2 \quad -1 \quad 3$.

3. Solve the synthetic division problem.

   Perform synthetic division to obtain the following result:

$$\begin{array}{r|cccc} -1 & 1 & 2 & -1 & 3 \\ & & -1 & -1 & 2 \\ \hline & 1 & 1 & -2 & \boxed{5} \end{array}$$

   The first row, $\underline{-1}\,|\;\; 1 \quad 2 \quad -1 \quad 3$, comes from the example as shown in step 2. The next step in solving is to simply bring the first number (1) down to the bottom row. The –1 in the second column is the product of that 1 and –1 (the shelf number). The 1 in the bottom row of the second column is the sum of 2 + (–1). This process is repeated to determine the values in the third and fourth columns. This yields the bottom row: $1 \quad 1 \quad -2 \quad \boxed{5}$.

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4. Identify the remainder.

   The remainder is the last entry in the bottom row (shown with a box around it). In this case, the remainder, $r(x)$, is 5.

   Since the denominator of the original expression is $x + 1$, the remainder of the expression is $\frac{5}{x+1}$.

5. Identify the quotient.

   The quotient, $q(x)$, is described by the coefficients preceding the remainder in the bottom row. Those coefficients are 1, 1, and –2. Therefore, the quotient is $1x^2 + 1x - 2$, or $x^2 + x - 2$.

6. Use the results of the synthetic division to rewrite the original expression.

   The rational expression $\frac{x^3+2x^2-x+3}{x+1}$ simplifies to $x^2 + x - 2 + \frac{5}{x+1}$ when $x \neq -1$.

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Problem-Based Task 2B.1.4: Fuel Economy

SMP	
1	2 ✓
3	4 ✓
5	6 ✓
7	8

A car's fuel economy, measured in miles per gallon, is the ratio of the distance the car can go on a specified amount of fuel. Piper's car can travel $\frac{3x^2+7x+4}{x-10}$ miles using $\frac{16}{x}$ gallons of gas. What is her car's fuel economy?



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Problem-Based Task 2B.1.4: Fuel Economy

Coaching

a. What relationship can be used to determine fuel economy in terms of x?

b. What rational expression represents the fuel economy of Piper's car?

c. How can the division problem from part b be rewritten as a multiplication problem?

d. What is the resulting rational expression?

e. Are there any values of x that make this expression invalid?

f. Are there any common factors that can be used to simplify the result?

g. How can you rewrite the expression from part d in a simplified form?

h. What is the answer to the problem?

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Problem-Based Task 2B.1.4: Fuel Economy

Coaching Sample Responses

a. What relationship can be used to determine fuel economy in terms of x?

Fuel economy is the ratio of miles driven to gallons of fuel used.

b. What rational expression represents the fuel economy of Piper's car?

Since Piper uses $\frac{16}{x}$ gallons to go $\frac{3x^2+7x+4}{x-10}$ miles, the ratio for fuel economy, or $\frac{\text{miles driven}}{\text{gallons of fuel used}}$, is the expression $\frac{3x^2+7x+4}{x-10} \div \frac{16}{x}$.

c. How can the division problem from part b be rewritten as a multiplication problem?

The division problem can be rewritten as a multiplication problem by multiplying the dividend by the reciprocal of the divisor.

d. What is the resulting rational expression?

The reciprocal of the divisor is $\frac{x}{16}$. Therefore, the resulting rational expression is $\frac{3x^2+7x+4}{x-10} \bullet \frac{x}{16}$.

e. Are there any values of x that make this expression invalid?

In the first term's denominator, $x - 10 \neq 0$. Additionally, any nonzero x-value that is less than 10 will result in a negative value for the expression. Therefore, $x > 10$.

f. Are there any common factors that can be used to simplify the result?

While the quadratic in the first term can be written in factored form as $(x + 1)(3x + 4)$, this does not result in any common factors that can be used to simplify the result.

g. How can you rewrite the expression from part d in a simplified form?

To simplify the expression, multiply the terms.

$$\frac{3x^2+7x+4}{x-10} \bullet \frac{x}{16} = \frac{x(3x^2+7x+4)}{16(x-10)} = \frac{3x^3+7x^2+4x}{16x-160}$$

h. What is the answer to the problem?

The fuel economy of Piper's car is $\frac{3x^3+7x^2+4x}{16x-160}$ miles per gallon when $x \neq 10$.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2B.1.4: Dividing Rational Expressions

For problems 1–4, simplify each rational expression. Whenever possible, reduce the expression to its lowest terms. State any restrictions on x.

1. $\frac{2x}{7} \div \frac{1}{x}$

2. $\frac{3}{5x} \div \frac{5}{x}$

3. $\frac{8x+3}{5} \div \frac{x}{9}$

4. $\frac{2x-4}{x+1} \div \frac{x}{x+2}$

Use your knowledge of rational expressions to complete problems 5–10. State any restrictions on x.

5. Show that the expression $\frac{x^2-2x-15}{2x^2-8x} \bullet \frac{32-2x^2}{x^2-13x+40} \div \frac{x+4}{8x-x^2}$ is equivalent to the expression $x + 3$.

continued

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6. Use synthetic division to rewrite the rational expression $\frac{2x^2-7x-15}{x-5}$ in the form $q(x)+\frac{r(x)}{b(x)}$.

7. Use polynomial division to rewrite the rational expression $\frac{x^2-8x-1}{x+1}$ in the form $q(x)+\frac{r(x)}{b(x)}$.

8. One way to find an average velocity is to take the quotient of distance traveled and the time required to do so. Find the velocity (in kilometers per hour) of a car that covers a distance of $2x + 4$ kilometers in a total time, in hours, described by the expression $\frac{3x-6}{x-2}$ (where $x > 2$).

9. Density is the ratio of an object's mass to its volume. What simplified rational expression represents the density (in grams per meter^3) of an alloy ingot whose mass, in grams, is described by the expression $\frac{6x^2+x-15}{x+4}$, with a volume in cubic meters represented by the expression $\frac{2x^2-11x+12}{5x}$ (where $x > 4$)?

10. The area of a right triangle, in square meters, is represented by the rational expression $\frac{x^2-3x+5}{x-2}$, where $x > 2$. If the length of one leg, in meters, is described by the rational expression $\frac{x^2-5}{x-1}$, what simplified rational expression represents the other leg?

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Progress Assessment

Circle the letter of the best answer.

1. Which simplified rational expression is equivalent to $\frac{36x^2+24x}{6x}$?

   a. $6x+4$
   c. $6x+24$
   b. $\frac{6x^2+24x}{x}$
   d. $6x^2+4x$

2. Which simplified rational expression is equivalent to $\frac{x}{x+1} \bullet 3x$?

   a. $3x$
   c. $\frac{3}{2x+1}$
   b. $\frac{3x^2}{x^2+2x+1}$
   d. $\frac{3x^2}{x+1}$

3. Which simplified rational expression is equivalent to $\frac{x^2+6}{2x}+\frac{3x}{x+1}$?

   a. $\frac{x^3+6x^2+6x+6}{2x^2+2x}$
   c. $\frac{x^3+7x^2+6x+6}{2x^2+2x}$
   b. x^3+4x^2+3x+6
   d. $\frac{x^2+6x+6}{2x+3}$

4. Which simplified rational expression is equivalent to $\frac{2x^3+6x+8x^2}{6+2x}$?

   a. x^2+x
   c. $\frac{x^2+3x+4}{6}$
   b. x^2+2x
   d. $\frac{x^3+6x+8x^2}{3+2x}$

5. Which simplified rational expression is equivalent to $\frac{3x+1}{2x}-\frac{x-1}{x^2}$?

   a. $\frac{3x^2+x-2}{2x^2}$
   c. $\frac{3+(x+2)}{2}$
   b. $\frac{3x^2-x+2}{2x^2}$
   d. $\frac{2x}{x^2}$

continued

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Assessment

6. Which simplified rational expression is equivalent to $\frac{x^2-4}{x+1} \div \frac{x+2}{x}$?

   a. $\frac{x^2-2x}{x+1}$

   b. $\frac{x^3-4x}{x^2+3x+2}$

   c. $\frac{x^3+2x^2-4x-8}{x^2+x}$

   d. $\frac{x}{x+1}$

7. Which simplified rational expression is equivalent to $\frac{x+3}{x-5} \bullet \frac{3x^2+6x}{2x^2+7x+3} \bullet \frac{5+9x-2x^2}{x+2}$?

   a. $\frac{3x}{x-5}$

   b. $\frac{5x-3x^2}{x-5}$

   c. $3x$

   d. $-3x$

8. Which simplified rational expression is equivalent to $\frac{x^2-2x-15}{x^2+5x+6} + \frac{8x}{x+2}$?

   a. $\frac{9x-13}{x+2}$

   b. $\frac{9x-5}{x+2}$

   c. $\frac{x^2+6x-15}{x^2+5x+6}$

   d. $8x-3$

9. Which simplified rational expression is equivalent to $\frac{x^2+11x+30}{x^2-2x-35} - \frac{2x-1}{x-7}$?

   a. $\frac{x+7}{7-x}$

   b. $\frac{x-5}{x-7}$

   c. -1

   d. $\frac{5-x}{x-7}$

10. Which simplified rational expression is equivalent to $\frac{x^2-11x+24}{3x+9} \div \frac{2x-16}{x+3}$?

    a. $\frac{x-3}{2}$

    b. $\frac{x-3}{6}$

    c. $\frac{24-11x+x^2}{6x-48}$

    d. $\frac{x^2-9x-40}{3x+9}$

continued

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Assessment

Use what you have learned about operating with rational expressions to complete all parts of the following problem.

11. A rectangle has a width described by the expression $\frac{x^2+2x-15}{x}$, and a length described by the expression $\frac{3x^2+4x}{5+x}$. Both dimensions are given in centimeters.

    a. What simplified rational expression describes the perimeter of the rectangle?

    b. What simplified rational expression describes the area of the rectangle?

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Pre-Assessment

Circle the letter of the best answer.

1. What is the solution to the equation $\frac{1}{8}-\frac{3}{4x}=-\frac{1}{x}$?
   a. $x = -2$
   b. $x = 1$
   c. $x = 2$
   d. $x = 14$

2. What is one solution to the equation $\frac{4x+2}{x-2}+\frac{9}{4x}=-1$?
   a. $x=\frac{4}{3}$
   b. $x=\frac{3}{4}$
   c. $x=-\frac{3}{4}$
   d. $x=-\frac{4}{3}$

3. What is the solution to the equation $\sqrt{x}+3=12$?
   a. $x = 3$
   b. $x = 141$
   c. $x = 9$
   d. $x = 81$

4. What is the solution to the equation $\sqrt{x+3}+\sqrt{x}=3$?
   a. $x = 9$
   b. $x = 3$
   c. $x = 1$
   d. The equation has no real solution.

5. What is the solution to the system of equations defined by the equations $y=\sqrt{2x}+4$ and $4x - 2y = -4$?
   a. {(4.5, 7)}
   b. {(0, 4)}
   c. {(0, 2)}
   d. {(2, 6)}

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Lesson 2: Solving Rational and Radical Equations

Instruction

Common Core State Standards

SMP	
1 ✓	2 ✓
3	4 ✓
5	6 ✓
7	8

A–REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

A–REI.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★

Essential Questions

1. How can you find a solution to a rational equation?
2. How can you identify extraneous solutions?
3. How can you find a solution to an equation involving radicals?
4. How can you determine the point at which two lines meet?
5. What is a system of equations?
6. Does every system of equations have a unique solution?
7. What does a solution to a system of equations represent?

WORDS TO KNOW

common denominator	a quantity that is a shared multiple of the denominators of two or more fractions
dependent system	a system of equations that intersect at every point
empty set	a set that has no elements, denoted by ∅; the solution to a system of equations with no intersection points, denoted by { }
extraneous solution	a solution of an equation that arises during the solving process, but which is not a solution of the original equation

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Instruction

least common denominator (LCD)	the least common multiple of the denominators of two or more fractions
point(s) of intersection	in a graphed system of equations, the ordered pair(s) where graphed functions intersect on a coordinate plane
proportion	a statement of equality between two ratios
radical equation	an algebraic equation in which at least one term includes a radical expression
radical expression	an expression containing a root
ratio	the relation between two quantities; can be expressed in words, fractions, decimals, or as a percentage
rational equation	an algebraic equation that contains at least one rational expression
rational number	any number that can be written as $\frac{m}{n}$, where m and n are integers and $n \neq 0$; any number that can be written as a decimal that ends or repeats
solution set	the set of ordered pairs that represent all of the solutions to an equation or a system of equations
system of equations	a set of equations with the same unknowns

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Recommended Resources

- Math Portal. "Rational Equation Solver."

  http://www.walch.com/rr/00201

  This online calculator allows users to solve rational equations. Specific directions and examples for correctly entering data are included.

- Math Portal. "Simplifying Radical Expressions."

  http://www.walch.com/rr/00202

  This online calculator allows users to input and simplify radical expressions. The site includes specific directions and examples for correctly entering data.

- Math Warehouse. "How to Solve Radical Equations."

  http://www.walch.com/rr/00203

  This site first offers the user a simple algorithm for solving radical expressions. After watching a video that shows the process of solving such an equation, users can practice the solution process with several guided samples.

- Purplemath.com. "Rational Expressions: Simplifying."

  http://www.walch.com/rr/00204

  This site defines rational expressions, and includes examples. Users can access the interactive widget to simplify rational expressions.

- Quia Math. "Systems of Equations Jeopardy."

  http://www.walch.com/rr/00205

  This interactive quiz mimics the format of a popular game show, and can be played by one or two people. Players earn points by correctly answering questions about systems of linear equations.

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Lesson 2B.2.1: Solving Rational Equations

Warm-Up 2B.2.1

Jordan and his sister, Julia, have been assigned the chore of raking their lawn. Suppose it takes Jordan 90 minutes to rake the lawn by himself. If Julia rakes it by herself, it also takes her 90 minutes.

1. If Jordan and Julia work together, how long will it take them to rake the lawn? Assume that their rates are not affected by working together.

2. Suppose again that it takes Jordan 90 minutes to rake the lawn by himself. Jordan's brother, Ian, however, can rake the lawn faster than Jordan, completing the job in just 45 minutes. If Jordan and Ian work together, how long will it take them to rake the lawn?

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Lesson 2B.2.1: Solving Rational Equations

Common Core State Standard

A–REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Warm-Up 2B.2.1 Debrief

1. If Jordan and Julia work together, how long will it take them to rake the lawn? Assume that their rates are not affected by working together.

   Jordan and Julia are both raking equal proportions of the lawn in equal amounts of time.

   The sum of the proportions must equal the entire lawn, or 1 lawn.

   If the proportion Jordan is raking is p, then it is also p for Julia.

   To rake the entire lawn implies that $p + p = 1$. When simplified, $2p = 1$. Solved for p, $p = \frac{1}{2}$.

   If it takes Jordan 90 minutes to rake the entire lawn, to rake half of it will take him half that time, or 45 minutes.

   $$\frac{1}{2} \bullet 90 = 45$$

   It will take Jordan and Julia 45 minutes to rake the lawn together.

2. Suppose again that it takes Jordan 90 minutes to rake the lawn by himself. Jordan's brother, Ian, however, can rake the lawn faster than Jordan, completing the job in just 45 minutes. If Jordan and Ian work together, how long will it take them to rake the lawn?

   Ian can rake the lawn in half the time Jordan can.

   In other words, Ian can rake twice as much lawn as Jordan can in the same amount of time.

   If the proportion Jordan is raking is p, then the proportion Ian is raking, in the same amount of time, is $2p$.

   To rake the entire lawn implies that $p + 2p = 1$. When simplified, $3p = 1$. Solved for p, $p = \frac{1}{3}$.

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If it takes Jordan 90 minutes to rake the entire lawn by himself, it will take him $\frac{1}{3}$ of that time to rake his portion of the lawn if Ian is helping.

$$\frac{1}{3} \bullet 90 = 30$$

It will take Jordan and Ian 30 minutes to rake the lawn together.

Connection to the Lesson

- Students will continue to use rational representations of numbers as well as refine the process of solving proportions into a more formal algebraic method.
- Students will use rational equations to solve real-world problems.

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Prerequisite Skills

This lesson requires the use of the following skills:

- determining a common denominator (5.NF.1)
- multiplying binomial expressions (A-APR.1)
- solving linear equations (A-REI.3)
- solving quadratic equations (A-REI.4)

Introduction

Your previous experiences have helped you understand how to represent situations using algebraic equations, and how solving those equations can reveal something specific about those situations. Most of these equations have involved integer coefficients. Yet you have also learned that numbers can be expressed as a **ratio**, or the comparison of two quantities, often written as a fraction. Numbers expressed as a ratio are called **rational numbers**. More formally, a rational number is any number that can be written as $\frac{m}{n}$, where m and n are integers and $n \neq 0$. In this lesson, we will explore ways to solve rational equations, and to assess and validate the resulting solutions.

Key Concepts

- A **rational equation** is an algebraic equation that contains at least one rational expression.
- A rational equation defines a specific relationship between numbers and variables; this relationship can be analyzed to find a solution to the equation.
- Finding the solution to a rational equation may be accomplished by simplifying the rational terms. To simplify rational terms, convert the rational equation into an equivalent equation with integer coefficients.
- Finding the solution often requires multiplying both sides of this equation by a common denominator—a quantity that is a shared multiple of the denominators of two or more fractions—to eliminate any rational terms.
- Recall that the least common denominator is the least common multiple of the denominators of the fractions in the expression.
- Like other algebraic equations, a rational equation may have no solutions.

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Instruction

- An **extraneous solution** is a solution of an equation that arises during the solving process, but which is not a solution of the original equation.
- Recall that a **proportion** is a statement of equality between two ratios. Techniques for solving proportions can often be used to simplify rational equations.
- The process of solving rational equations can be applied to real-world examples in which a specific task that may be done alone is to be completed cooperatively.

Common Errors/Misconceptions

- multiplying only some terms of the rational equation when eliminating rational numbers
- incorrectly or prematurely reducing rational numbers
- eliminating terms instead of factors
- neglecting to check for extraneous solutions

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Instruction

Guided Practice 2B.2.1

Example 1

Solve the proportion $\frac{x}{4}=\frac{6}{8}$ for x.

1. Determine the least common denominator.

   The denominator of the left expression is 4, while the denominator of the right expression is 8.

   The least common denominator (LCD) is 8, the smallest multiple with both denominators as factors.

2. Multiply each ratio by the common denominator.

   $(8)\frac{x}{4}=(8)\frac{6}{8}$ Multiply each ratio by the LCD, 8.

   $\frac{8x}{4}=\frac{48}{8}$ Simplify.

3. Simplify the resulting equation.

   $\frac{8x}{4}=\frac{48}{8}$ Equation from the previous step

   $\frac{2x}{1}=\frac{6}{1}$ Simplify the fractions.

   $2x = 6$

   The result is an algebraic equation without any remaining rational numbers that can be solved using methods you have already learned.

4. Solve the resulting equation.

   $2x = 6$ Simplified equation from the previous step

   $x = 3$ Divide both sides of the equation by 2.

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5. Verify the solution(s).

While it appears that we have a solution, $x = 3$, we have to return to the original equation to confirm it.

$\frac{x}{4} = \frac{6}{8}$ Original equation

$\frac{(3)}{4} = \frac{6}{8}$ Substitute 3 for x.

$\frac{3}{4} = \frac{6}{8}$

The result is a true statement; therefore, $x = 3$ is an acceptable result.

For the proportion $\frac{x}{4} = \frac{6}{8}$, $x = 3$.



Example 2

Solve the rational equation $\frac{5}{t} - \frac{3}{12} = \frac{9}{3t}$ for t.

1. Determine the least common denominator.

The denominators in the equation are t, 12, and $3t$.

The smallest multiple having all three of these values as a factor is $12t$; therefore, the least common denominator is $12t$.



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2. Multiply each term of the equation by the common denominator and simplify.

$(12t)\frac{5}{t} - (12t)\frac{3}{12} = (12t)\frac{9}{3t}$ Multiply each term by the LCD, $12t$.

$(12)\frac{5}{1} - (t)\frac{3}{1} = (4)\frac{9}{1}$ Simplify each fraction.

$(12)5 - (t)3 = (4)9$ Continue to simplify.

$60 - 3t = 36$

The result is an algebraic equation without any remaining rational numbers that can be solved using methods you have already learned.

3. Solve the resulting equation.

$60 - 3t = 36$	Simplified equation from the previous step
$60 = 36 + 3t$	Add $3t$ to both sides.
$24 = 3t$	Subtract 36 from both sides.
$t = 8$	Divide both sides by 3.

4. Verify the solution(s).

While it appears that we have a solution, $t = 8$, we have to return to the original equation to confirm it. Because there are variables in the denominators of the first and last terms, we must determine what value(s) would make the denominator equal 0, and thereby invalidate the equation (since division by 0 yields an undefined result and is not acceptable).

The expression $\frac{5}{t}$ is undefined when $t = 0$; therefore, $t \neq 0$.

Similarly, the expression $\frac{9}{3t}$ is also undefined when $t = 0$; once again, $t \neq 0$. Our solution, $t = 8$, does not violate this condition; therefore, it is an acceptable result.

For the equation $\frac{5}{t} - \frac{3}{12} = \frac{9}{3t}$, $t = 8$.

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Example 3

Solve the rational equation $\frac{2}{y}-\frac{1}{y+2}=\frac{1}{3}$ for y.

1. Determine the least common denominator.

   The denominators are y, $y + 2$, and 3.

   The smallest multiple having all three of these values as a factor is $3y(y + 2)$, the irreducible product of all three denominators.

   The least common denominator is $3y(y + 2)$.

2. Multiply each term by the common denominator.

$$3y(y+2)\bullet\frac{2}{y}-3y(y+2)\bullet\frac{1}{y+2}=3y(y+2)\bullet\frac{1}{3}$$

3. Simplify the resulting equation.

   Start by rewriting each occurrence of the common denominator in the equation as a ratio:

$3y(y+2)=\frac{3y(y+2)}{1}$	Common denominator rewritten as a ratio
$\frac{3y(y+2)}{1}\bullet\frac{2}{y}-\frac{3y(y+2)}{1}\bullet\frac{1}{y+2}=\frac{3y(y+2)}{1}\bullet\frac{1}{3}$	Rewritten equation
$3(y + 2) \bullet 2 - 3y \bullet 1 = y(y + 2) \bullet 1$	Simplify the ratios.
$6y + 12 - 3y = y^2 + 2y$	Carry out the multiplication.
$3y + 12 = y^2 + 2y$	Subtract $3y$ from $6y$.

   The algebraic equation that results this time, $3y + 12 = y^2 + 2y$, still has only integer coefficients. However, it is no longer linear—this is a quadratic equation because y is raised to a power of 2. We will need to either factor or use the quadratic formula to determine the value of y.

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4. Solve the quadratic equation by factoring or using the quadratic formula.

   Start by setting the equation equal to 0 so that the equation is in the standard form of a quadratic equation. Then, factor the quadratic.

$3y + 12 = y^2 + 2y$	Equation found in the previous step
$12 = y^2 + (-y)$	Subtract $3y$ from both sides.
$0 = y^2 - y - 12$	Subtract 12 from both sides.
$0 = (y - 4)(y + 3)$	Factor the resulting quadratic.
$y = 4$ or $y = -3$	

5. Verify the solution(s).

   Now that we have potential solutions, $y = 4$ or $y = -3$, we must return to the original equation. We must determine what value(s) would make the denominators equal 0, and thereby invalidate the equation.

   The expression $\frac{2}{y}$ is undefined when $y = 0$; therefore, $y \neq 0$.

   The expression $\frac{1}{y+2}$ is also undefined when $y = -2$; therefore, $y \neq -2$.

   There is no value for y that invalidates the last term, $\frac{1}{3}$, because the denominator does not contain a variable.

   Our solutions, $y = 4$ or $y = -3$, do not violate this condition; therefore, they are acceptable results.

   If $\frac{2}{y} - \frac{1}{y+2} = \frac{1}{3}$, then $y = 4$ or $y = -3$.

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Example 4

Solve the rational equation $\frac{1}{p-4}+\frac{p}{p+2}=\frac{6}{p^2-2p-8}$ for p.

1. Determine the least common denominator.

   The denominators are $p-4$, $p+2$, and p^2-2p-8.

   The least common denominator is not apparent with these denominators. However, the product of the first two denominators can be determined.

$(p-4)(p+2)$	Product of the first two denominators
$p(p+2)-4(p+2)$	Rewrite the product using the Distributive Property.
$p^2+2p-4p-8$	Distribute.
p^2-2p-8	Simplify.

   Since the product of the first two denominators is equal to the third denominator, we have determined a suitable common denominator: p^2-2p-8.

2. Multiply each term by the common denominator.

$$(p^2-2p-8)\bullet\frac{1}{p-4}+(p^2-2p-8)\bullet\frac{p}{p+2}=(p^2-2p-8)\bullet\frac{6}{p^2-2p-8}$$

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3. Simplify the resulting equation.

Think of each term as a ratio, then simplify each ratio by using factors.

$\frac{p^2-2p-8}{1}\bullet\frac{1}{p-4}+\frac{p^2-2p-8}{1}\bullet\frac{p}{p+2}=\frac{p^2-2p-8}{1}\bullet\frac{6}{p^2-2p-8}$ Equation with each term written as a ratio

$\frac{(p-4)(p+2)}{1}\bullet\frac{1}{p-4}+\frac{(p-4)(p+2)}{1}\bullet\frac{p}{p+2}=\frac{p^2-2p-8}{1}\bullet\frac{6}{p^2-2p-8}$ Rewritten equation with factors

$\frac{\cancel{(p-4)}(p+2)}{1}\bullet\frac{1}{\cancel{p-4}}+\frac{(p-4)\cancel{(p+2)}}{1}\bullet\frac{p}{\cancel{p+2}}=\frac{\cancel{p^2-2p-8}}{1}\bullet\frac{6}{\cancel{p^2-2p-8}}$ Divide out like factors.

$\frac{(p+2)}{1}\bullet 1+\frac{(p-4)}{1}\bullet p=1\bullet 6$ Simplify.

$(p+2)(1)+(p-4)(p)=6$ Simplify the ratios.

$p+2+p^2-4p=6$ Simplify the equation.

$p^2-3p+2=6$ Continue to simplify.

$p^2-3p-4=0$

The simplified equation, $p^2-3p-4=0$, is quadratic.

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Instruction

4. Solve the resulting quadratic equation by factoring or using the quadratic formula.

This quadratic equation can be solved using the quadratic formula, $\frac{-b \pm \sqrt{b^2 - 4(a)(c)}}{2a}$, where a, b, and c are the coefficients of each term.

$\frac{-b \pm \sqrt{b^2 - 4(a)(c)}}{2a}$ Quadratic formula

$\frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-4)}}{2(1)}$ Substitute 1 for a, –3 for b, and –4 for c.

$\frac{3 \pm \sqrt{9+16}}{2}$ Simplify.

$\frac{3 \pm 5}{2}$

$p = 4$ or $p = -1$

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5. Verify the solution(s).

In the original equation, there are variables in the denominator of each term.

We must determine what value(s) would make those denominators equal 0, and thereby invalidate the equation.

The expression $\frac{1}{p-4}$ is undefined when $p = 4$; therefore, $p \neq 4$.

The expression $\frac{1}{p+2}$ is also undefined when $p = -2$; therefore, $p \neq -2$.

When we found our least common denominator, we determined that the expression $p^2 - 2p - 8 = (p - 4)(p + 2)$. That means $\frac{1}{p^2-2p-8} = \frac{1}{(p-4)(p+2)}$.

The same results as before are true: $p \neq 4$ and $p \neq -2$.

One of our algebraic solutions is that $p = 4$. This is not a valid solution, because it would result in a 0 in the denominator. This kind of solution is called an extraneous solution, since it satisfies our algebraic problem, but makes no sense in the original equation or problem.

However, our other solution, $p = -1$, poses no such problem. It is an acceptable result, and therefore is our only solution to the original problem.

If $\frac{1}{p-4} + \frac{p}{p+2} = \frac{6}{p^2-2p-8}$, then $p = -1$.

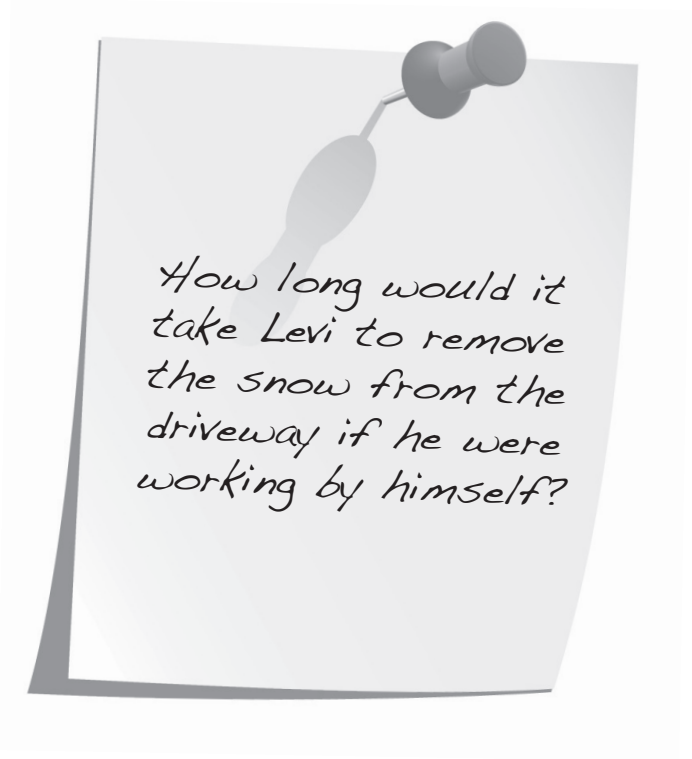


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Problem-Based Task 2B.2.1: Snow Removal

SMP	
1 ✓	2 ✓
3	4 ✓
5	6
7	8

When Scot and his younger brother Levi work together, they can remove snow from a driveway in 40 minutes. But if Scot were working alone, he could do the same job in half the time it would take Levi to do it alone. How long would it take Levi to remove the snow from the driveway if he were working by himself? Assume the rate at which each brother works is not affected by working together.



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Problem-Based Task 2B.2.1: Snow Removal

Coaching

a. What kind of a solution do we expect to find for this problem: the number of driveways, the number of people, or an amount of time? Explain your answer.

b. What would be an appropriate variable to use to represent the solution to this problem?

c. Using the variable determined in part b, what ratio represents the portion of the driveway Scot can clear in 40 minutes?

d. Using the same variable, what algebraic expression represents the time it would take Levi to finish removing snow from the driveway?

e. When the snow has been completely cleared from the driveway, what portion of the driveway has been cleared?

f. How are the results of part c and part e algebraically related?

g. What process must you follow to solve this rational equation?

h. Remember what you set out to find. Is your algebraic solution the answer to that problem?

i. In the context of the problem, does this algebraic solution make sense?

j. How long will it take Levi to remove the snow from the driveway by himself?

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Problem-Based Task 2B.2.1: Snow Removal

Coaching Sample Responses

a. What kind of a solution do we expect to find for this problem: the number of driveways, the number of people, or an amount of time? Explain your answer.

Because the problem asks, "How long would it take...?" the solution will be a time. The problem scenario asks for the total time it will take for Levi to remove the snow from the driveway by himself. The text of the problem also indicates that solution will probably be in minutes.

b. What would be an appropriate variable to use to represent the solution to this problem?

Choosing the variable t would be a useful way to remember that the solution will be an amount of time.

c. Using the variable determined in part b, what ratio represents the portion of the driveway Scot can clear in 40 minutes?

This ratio is the proportion of time spent by both brothers on the task relative to the total time Scot needs to complete the task individually. Since the two brothers can clear the driveway in 40 minutes, but we don't know how long it would take Scot to finish the job alone, the correct proportion is $\frac{40}{t}$.

d. Using the same variable, what algebraic expression represents the time it would take Levi to finish removing snow from the driveway?

Levi requires twice as much time as Scot to finish. If it takes Scot t minutes to clear the driveway, Levi needs $2t$ minutes.
In 40 minutes, Levi will clear $\frac{40}{2t}$ of the driveway.

e. When the snow has been completely cleared from the driveway, what portion of the driveway has been cleared?

When the job is done, all of the snow has been removed from the driveway. This is represented by the ratio $\frac{1}{1}$, or just 1.

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f. How are the results of part c and part e algebraically related?

After 40 minutes, Scot and Levi will have cleared the portion of the driveway represented by $\frac{40}{t}+\frac{40}{2t}$. The brothers will not be done until they have removed all the snow from the entire driveway, represented by the ratio $\frac{1}{1}$. Therefore, for the brothers to finish the entire task in 40 minutes, these two ratios must be equal so we can write a proportion. That means $\frac{40}{t}+\frac{40}{2t}=\frac{1}{1}$.

g. What process must you follow to solve this rational equation?

To solve this rational equation, first find the least common denominator. The least common denominator of t, $2t$, and 1 is $2t$.

Next, multiply each term by the least common denominator:

$$2t \bullet \frac{40}{t}+2t \bullet \frac{40}{2t}=2t \bullet \frac{1}{1}$$

Simplify the resulting equation to find t.

$$2\cancel{t} \bullet \frac{40}{\cancel{t}}+\cancel{2t} \bullet \frac{40}{\cancel{2t}}=2t \bullet \frac{1}{1}$$

$2 \bullet 40 + 40 = 2t$

$80 + 40 = 2t$

$120 = 2t$

$t = 60$

h. Remember what you set out to find. Is your algebraic solution the answer to that problem?

When we defined the variable, t represented the time it takes Scot to remove the snow from the driveway by himself. The problem is not asking for the time Scot needs to finish clearing the driveway by himself; the problem is asking for the time it takes Levi to remove the snow from the driveway when working alone. It was determined in part d that Levi's time is $2t$. If $t = 60$, then $2t = 120$.

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i. In the context of the problem, does this algebraic solution make sense?

The problem statement established that Scot can clear the driveway in half the time that it takes Levi. It also makes sense that the time Levi requires to clear the entire driveway by himself is longer than the time it takes the brothers to cooperatively clear the driveway; therefore, the answer $2t = 120$ makes sense.

Now that we have a potential solution, $t = 60$, return to the equation $\frac{40}{t}+\frac{40}{2t}=\frac{1}{1}$. We must determine what value(s) would make the denominators equal 0, and thereby invalidate the equation.

The expression $\frac{40}{t}$ is undefined when $t = 0$; therefore, $t \neq 0$.

The expression $\frac{40}{2t}$ is also undefined when $2t = 0$, or $t = 0$; therefore, $t \neq 0$.

There is no value for t that invalidates the last term, $\frac{1}{1}$, because the denominator does not contain a variable.

The solution, $t = 60$, does not violate this condition; therefore, it is an acceptable result.

j. How long will it take Levi to remove the snow from the driveway by himself?

Working alone, it will take Levi 120 minutes, or $\frac{120}{60}=2$ hours, to clear the driveway.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2B.2.1: Solving Rational Equations

Solve each rational equation for x.

1. $\frac{x+3}{3}+\frac{5}{x-4}=5$

2. $\frac{3}{2x}+2=\frac{2x}{x+1}$

3. $x-\frac{2}{x-3}=\frac{x-1}{3-x}$

Use the given information and what you know about rational equations to solve problems 4–10.

4. Gideon can rake his lawn in 3 hours. His friend Katsuro can rake a lawn the same size in just 2 hours. Suppose they worked together to rake Gideon's lawn. What rational equation represents the information presented in this problem?

5. Can $x = -2$ be a solution for $\frac{3}{x+2}-\frac{6x}{x^2-4}=\frac{1}{x-2}$? Explain.

6. It takes Lana 8 hours to paint a particular size room, but her co-worker Nila can do the same job in 7 hours. If they paint the room together, how long will it take to finish painting the room?

7. The reciprocals of two consecutive, positive odd integers have a difference of $\frac{2}{63}$. What are the integers?

8. Kris has access to two different water sources for filling up his swimming pool for summer. One source, using a rigid pipe, supplies water 50% faster than the other source, which uses a flexible hose. Using both the pipe and the hose, Kris can fill the pool in 6 hours. How long would it take to fill the pool if Kris only used the pipe?

9. Reena rode her skateboard to her uncle's house, 12 miles away. A wheel broke just as she arrived, so Reena borrowed her uncle's bicycle to return home. Traveling by bicycle, she was able to go twice as fast, and returned home in 2 hours less time than it took to go to her uncle's house. How long did it take Reena to get home on the bicycle?

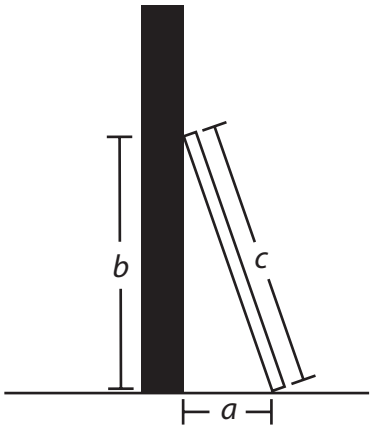
10. A passenger train is 40 mph faster than a freight train. In the time it takes the slower freight train to travel 161 miles, the passenger train can travel 301 miles. What is the speed of each train?

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Lesson 2B.2.2: Solving Radical Equations

Warm-Up 2B.2.2

A large canvas leaning against a wall forms a right triangle. The horizontal distance from the base of the canvas to the wall represents one leg of the triangle. The vertical distance from the floor to the point of contact between the canvas and the wall represents the other leg. The canvas itself represents the hypotenuse. The diagram that follows illustrates the triangle created by the canvas, floor, and wall. Use the diagram and the Pythagorean Theorem, $a^2 + b^2 = c^2$, to solve the problems that follow.



1. If the base of the canvas is 1.5 ft from the wall, and the top of the canvas touches the wall 3.6 ft from the floor, how long is the canvas?

2. Suppose the base of this same canvas were pulled away from the wall until the base was 2.34 ft from the wall. How high along the wall would the canvas now reach?

3. If a 6-foot-long canvas were set against the wall so that the angle formed by the canvas and the floor was 45°, how far from the wall would the base of the canvas be?

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Lesson 2B.2.2: Solving Radical Equations

Common Core State Standard

A–REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Warm-Up 2B.2.2 Debrief

1. If the base of the canvas is 1.5 ft from the wall, and the top of the canvas touches the wall 3.6 ft from the floor, how long is the canvas?

   Use the Pythagorean Theorem, $a^2 + b^2 = c^2$, to determine the length of the canvas.

   From the diagram, we are given that a = the horizontal leg, b = the vertical leg, and c = the hypotenuse. Solve for c, the length of the canvas.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$(1.5)^2 + (3.6)^2 = c^2$	Substitute 1.5 for a and 3.6 for b.
$2.25 + 12.96 = c^2$	Simplify.
$15.21 = c^2$	
$3.9 = c$	Take the square root of both sides.

   The canvas is 3.9 ft long.

2. Suppose the base of this same canvas were pulled away from the wall until the base was 2.34 ft from the wall. How high along the wall would the canvas now reach?

   Again, use the Pythagorean Theorem to determine how high the canvas would reach. Solve for b, the vertical distance along the wall between the floor and the canvas.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$(2.34)^2 + b^2 = (3.9)^2$	Substitute 2.34 for a and 3.9 for c.
$5.4756 + b^2 = 15.21$	Simplify.
$b^2 = 9.7344$	
$b = 3.12$	Take the square root of both sides.

   The top of the canvas reaches 3.12 ft up the wall.

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3. If a 6-foot-long canvas were set against the wall so that the angle formed by the canvas and the floor was 45°, how far from the wall would the base of the canvas be?

   With the canvas forming a 45° angle with the floor, the right triangle is also an isosceles triangle, with both legs equal: $a = b$.

   Use the Pythagorean Theorem to determine how far from the wall the base of the canvas would be.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$a^2 + (a)^2 = (6)^2$	Substitute a for b and 6 for c.
$2a^2 = 6^2$	Simplify.
$2a^2 = 36$	Apply the exponent.
$a^2 = 18$	Divide both sides by 2.
$a \approx 4.24$	Take the square root of both sides.

   The base of the canvas would be about 4.24 ft from the wall.

Connection to the Lesson

- Students will continue to use radical representations of numbers, as well as refine the process of solving radical equations into a more formal algebraic method.
- Students will solve radical equations in context.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- simplifying numerical radical expressions (N-RN.2)
- simplifying algebraic radical expressions (N-RN.2)
- applying the Pythagorean Theorem (8.G.7, 8.G.8)

Introduction

Using the Pythagorean Theorem to solve problems provides a familiar example of a relationship between variables that involve radicals (or roots). Recall that the word *root* in mathematics has several definitions. You have seen and worked with function roots, or zeros; in this lesson, you will work with a different kind of root—that is, square roots, cube roots, and other radicals. **Radical expressions**, or expressions containing a root, follow rules that may seem more complicated than those of more common numbers. While you can easily add the numbers 4 and 9 to get 13, there is no similar simple way to add the radicals $\sqrt{4}$ and $\sqrt{9}$ without simplifying each individual term first. Since radical expressions require more complex methods in order to be simplified, radical equations also involve more complex methods in order to be solved. However, some common real-life situations can be more readily solved by applying radical equations, making fluency with radical equations a useful skill to learn. In this lesson, we will explore ways to solve radical equations, as well as how to assess and validate the resulting solutions.

Key Concepts

- A **radical equation** is an algebraic equation in which at least one term includes a radical expression.
- A radical equation defines a specific relationship between numbers and variables that can be analyzed to find a solution.
- It is possible to find the solution to a radical equation by isolating the radical, then raising the expression to an appropriate power to eliminate the radical.
- Like other algebraic equations, a radical equation may have no solutions.
- Recall that an extraneous solution can occur when the solution process results in a value that satisfies the original equation, but creates an invalid equation; in the case of radical equations, extraneous solutions occur when a negative value results under the radical. An extraneous solution can also occur when a solution results in a value that is irrelevant in the context of the original equation.

SRB p. U2B-65

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Instruction

Common Errors/Misconceptions

- raising only parts of an equation to a power when trying to eliminate a radical (for example, raising only one side of an equation to a power)
- confusing $-x^2$ with $(-x)^2$, particularly on a calculator
- disregarding negative signs within radical expressions
- neglecting to check for extraneous solutions

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Instruction

Guided Practice 2B.2.2

Example 1

Solve the radical equation $\sqrt{x-5}=2$ for x.

1. Isolate the radical expression.

   The radical expression $\left(\sqrt{x-5}\right)$ is already isolated on one side of the equation.

2. Raise both sides of the equation to a power to eliminate the radical.

   Because we have a square root on the left side of the equation, we need to raise both sides to the power of 2 to eliminate the radical expression. In other words, square both sides.

$\sqrt{x-5}=2$	Original equation
$\left(\sqrt{x-5}\right)^2=2^2$	Square both sides of the equation.
$x-5=4$	Simplify.

3. Solve the resulting equation.

$x-5=4$	Equation found in the previous step
$x=9$	Add 5 to both sides.

   The solution appears to be $x = 9$. However, we need to confirm that this is not an extraneous solution.

4. Verify the solution(s).

   Substitute the potential solution into the original equation, and check the result.

$\sqrt{x-5}=2$	Original equation
$\sqrt{(9)-5}=2$	Substitute 9 for x.
$\sqrt{4}=2$	Simplify.

   The solution satisfies the original equation, and is therefore valid.

   The solution to the radical equation $\sqrt{x-5}=2$ is $x = 9$.

SWB p. 59; SRB p. U2B-66

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Instruction

Example 2

Solve the radical equation $\sqrt{x}-5=2$ for x.

1. Isolate the radical expression.

   At first glance, this looks much like the equation given in Example 1. However, "– 5" is not under the radical. This small change creates a big difference, for in this example, the radical expression $\left(\sqrt{x}\right)$ is *not* already isolated on one side of the equation. The first step is to isolate the radical.

$\sqrt{x}-5=2$	Original equation
$\sqrt{x}=7$	Add 5 to both sides.

2. Raise both sides of the equation to a power to eliminate the radical, then simplify the result.

   Square both sides of the equation to eliminate the radical expression, then solve the equation algebraically.

$\sqrt{x}=7$	Equation found in the previous step
$\left(\sqrt{x}\right)^2=7^2$	Square both sides.
$x = 49$	Simplify.

   The solution appears to be $x = 49$.

3. Verify the solution(s).

   Substitute the potential solution into the original equation, and check the result.

$\sqrt{x}-5=2$	Original equation
$\sqrt{(49)}-5=2$	Substitute 49 for x.
$7 - 5 = 2$	Simplify.
$2 = 2$	

   Since the solution satisfies the original equation, the solution is valid.

   The solution to the radical equation $\sqrt{x}-5=2$ is $x = 49$.

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Instruction

Example 3

Solve the radical equation $\sqrt{2-x}=x$ for x.

1. Isolate the radical expression.

   The radical expression $\left(\sqrt{2-x}\right)$ is already isolated on one side of the equation.

2. Raise both sides of the equation to a power to eliminate the radical, then simplify the result.

   Square both sides of the equation to eliminate the radical expression. Solve one side of the equation for 0.

$\sqrt{2-x}=x$	Original equation
$\left(\sqrt{2-x}\right)^2=(x)^2$	Square both sides.
$2-x=x^2$	Simplify.
$2=x^2+x$	Add x to both sides.
$0=x^2+x-2$	Subtract 2 from both sides.

   The result is a quadratic equation.

3. Solve for x by factoring, or by using the quadratic formula.

   The quadratic appears to be factorable.

   $0=x^2+x-2$

   $0=(x-1)(x+2)$

   The solutions appear to be $x=-2$ or $x=1$.

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4. Verify the solution(s).

Substitute the potential solutions into the original equation, and check the result.

Let's start with $x = -2$.

$\sqrt{2-x} = x$	Original equation
$\sqrt{2-(-2)} = (-2)$	Substitute –2 for x.
$\sqrt{4} = -2$	Simplify.
$2 = -2$	

The result, $2 = -2$, is a false statement. This solution does not satisfy the original equation. –2 is an example of an extraneous solution, one that results from the process of our solution, but does not satisfy the original equation.

Check $x = 1$.

$\sqrt{2-x} = x$	Original equation
$\sqrt{2-(1)} = (1)$	Substitute 1 for x.
$\sqrt{1} = 1$	Simplify.
$1 = 1$	

The result is a true statement; therefore, this solution satisfies the original equation.

Since only the second solution satisfies the original equation, only that solution is valid.

The solution to the radical equation $\sqrt{2-x} = x$ is $x = 1$.



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Instruction

Example 4

Solve the radical equation $\sqrt{x-7}=3\sqrt{2}$ for x.

1. Isolate the radical expressions.

   The equation contains two distinct radical expressions. Each radical expression is alone on one side of the equation, so it should be possible to eliminate the radicals using the methods previously demonstrated.

2. Raise both sides of the equation to a power to eliminate the radicals, then simplify the result.

   Square both sides of the equation, then solve algebraically.

$\sqrt{x-7}=3\sqrt{2}$	Original equation
$\left(\sqrt{x-7}\right)^2=\left(3\sqrt{2}\right)^2$	Square both sides.
$x-7=9(2)$	Simplify.
$x-7=18$	
$x=25$	

   The solution appears to be $x = 25$.

3. Verify the solution(s).

   Substitute the potential solution into the original equation, and check the result.

$\sqrt{x-7}=3\sqrt{2}$	Original equation
$\sqrt{(25)-7}=3\sqrt{2}$	Substitute 25 for x.
$\sqrt{18}=3\sqrt{2}$	Simplify.
$\sqrt{9\bullet 2}=3\sqrt{2}$	
$3\sqrt{2}=3\sqrt{2}$	

   Since the solution satisfies the original equation, the solution is valid.

   The solution to the radical equation $\sqrt{x-7}=3\sqrt{2}$ is $x = 25$.

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Instruction

Example 5

Solve the radical equation $\sqrt{5}+\sqrt{x}=\sqrt{3x+5}$ for x.

1. Isolate the radical expressions.

   With three different radical expressions, this equation offers no simple path to isolating the radicals. It is not possible to get each radical on one side of the equation because there just aren't enough sides.

   In its current format, the most complex radical is alone on one side. The other two simpler radicals are on the other side. While this radical equation is not as ideally isolated as those in the previous examples, it is as isolated as we need it to be.

2. Raise both sides of the equation to a power to eliminate the radicals, then simplify the result.

   Square both sides of the equation, then simplify as much as possible.

$\sqrt{5}+\sqrt{x}=\sqrt{3x+5}$	Original equation
$\left(\sqrt{5}+\sqrt{x}\right)^2=\left(\sqrt{3x+5}\right)^2$	Square both sides.
$\left(\sqrt{5}+\sqrt{x}\right)\left(\sqrt{5}+\sqrt{x}\right)=3x+5$	Simplify.
$\sqrt{5}\left(\sqrt{5}+\sqrt{x}\right)+\sqrt{x}\left(\sqrt{5}+\sqrt{x}\right)=3x+5$	Distribute.
$\sqrt{5}\left(\sqrt{5}\right)+\sqrt{5}\left(\sqrt{x}\right)+\sqrt{x}\left(\sqrt{5}\right)+\sqrt{x}\left(\sqrt{x}\right)=3x+5$	Write each term as the product of the distribution.
$5+\sqrt{5x}+\sqrt{5x}+x=3x+5$	Simplify.
$2\sqrt{5x}=2x$	
$\sqrt{5x}=x$	

   While we may not have solved the equation yet, we have managed to rewrite it in a simpler format that we can solve algebraically using previously learned processes.

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Instruction

3. Eliminate the radical from the simplified equation.

   As in the previous step, square both sides of the equation, then simplify the result.

$\sqrt{5x} = x$	Simplified equation from the previous step
$\left(\sqrt{5x}\right)^2 = x^2$	Square both sides.
$5x = x^2$	Simplify.
$0 = x^2 - 5x$	Subtract $5x$ from both sides.
$0 = x(x - 5)$	Factor the resulting quadratic.

   We can determine the potential solutions by setting each factor of the equation $0 = x(x - 5)$ equal to 0:

   $x = 0$

   $x - 5 = 0$

   Our two potential solutions are $x = 0$ and $x = 5$.

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4. Verify the solution(s).

   Substitute the potential solutions into the original equation, and check the result.

   Let's start with $x = 0$.

$\sqrt{5}+\sqrt{x}=\sqrt{3x+5}$	Original equation
$\sqrt{5}+\sqrt{(0)}=\sqrt{3(0)+5}$	Substitute 0 for x.
$\sqrt{5}+0=\sqrt{0+5}$	Simplify.
$\sqrt{5}=\sqrt{5}$	

   The result, $\sqrt{5}=\sqrt{5}$, is a true statement. The solution $x = 0$ appears to be valid.

   Now check $x = 5$.

$\sqrt{5}+\sqrt{x}=\sqrt{3x+5}$	Original equation
$\sqrt{5}+\sqrt{(5)}=\sqrt{3(5)+5}$	Substitute 5 for x.
$\sqrt{5}+\sqrt{5}=\sqrt{15+5}$	Simplify.
$2\sqrt{5}=\sqrt{20}$	
$2\sqrt{5}=\sqrt{4\bullet 5}$	
$2\sqrt{5}=2\sqrt{5}$	

   The result, $2\sqrt{5}=2\sqrt{5}$, is a true statement. The solution $x = 5$ also appears to be valid.

   Both potential solutions satisfy the original equation; therefore, both are valid.

   The solutions to the radical equation $\sqrt{5}+\sqrt{x}=\sqrt{3x+5}$ are $x = 0$ and $x = 5$.



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Instruction

Example 6

Solve the radical equation $\sqrt[3]{3x+6}+10=7$ for x.

1. Isolate the radical expression.

   Not every radical represents a square root. Higher roots are represented with a radical symbol and a root on the "shelf" of the radical symbol. The solution process, however, is essentially the same.

   $\sqrt[3]{3x+6}+10=7$ Original equation

   $\sqrt[3]{3x+6}=-3$ Subtract 10 from both sides.

2. Raise both sides of the equation to a power to eliminate the radical, then simplify the result.

   Eliminating the radical this time requires raising each side to the third power (i.e., cubing each side), rather than squaring each side.

   $\left(\sqrt[3]{3x+6}\right)^3=(-3)^3$ Cube both sides.

   $3x+6=-27$ Simplify.

   $3x=-33$

   $x=-11$

   The solution appears to be $x=-11$.

3. Verify the solution(s).

   Substitute the potential solution into the original equation, and check the result.

   $\sqrt[3]{3x+6}+10=7$ Original equation

   $\sqrt[3]{3(-11)+6}+10=7$ Substitute -11 for x.

   $\sqrt[3]{-27}+10=7$ Simplify.

   $-3+10=7$

   $7=7$

   Since the solution satisfies the original equation, the solution is valid.

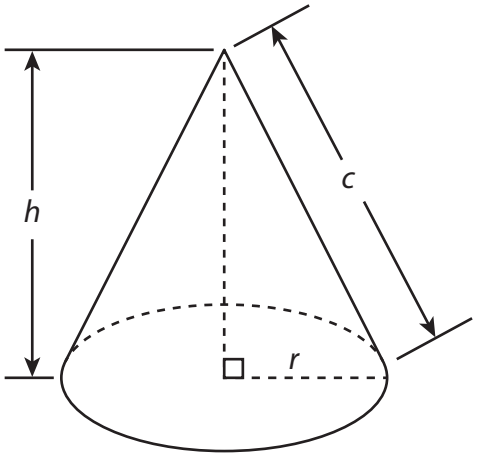
   The solution to the radical equation $\sqrt[3]{3x+6}+10=7$ is $x=-11$.

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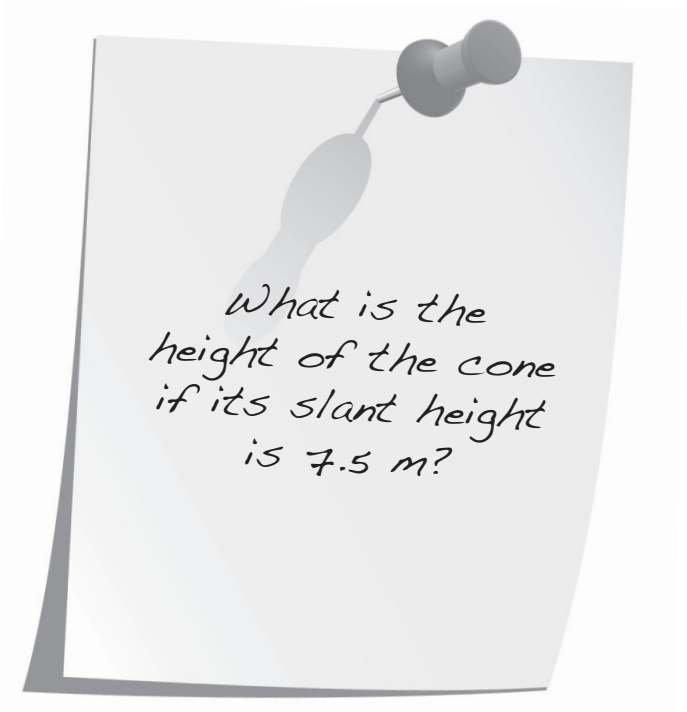
Problem-Based Task 2B.2.2: A Towering Cone

SMP	
1 ✓	2 ✓
3	4 ✓
5	6
7	8

The height, *h*, of a right circular cone is the perpendicular distance between the base and the peak. The *slant height* of such a cone, *c*, is the distance along the outer edge from the peak to the perimeter of the circular base, as shown in the diagram below. The algebraic relationship between the cone's height, its slant height, and the radius of its base is described by the equation $c = \sqrt{r^2 + h^2}$, where *c* is the slant height, *r* is the radius of the base, and *h* is the height of the cone from base to tip.



A cone-shaped top is to be added atop a tower that has a diameter of 9 m. What is the height of the cone if its slant height is 7.5 m?



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Problem-Based Task 2B.2.2: A Towering Cone

Coaching

a. What does the solution represent?

b. In what units might the solution be measured?

c. What variable is a good choice to represent the solution?

d. What other information do we need to know before we can determine the solution?

e. Review the information given in the problem statement. How can we use the given information to determine the missing information?

f. What algebraic equation describes the connection between the known information and the information needed to solve the problem?

g. Solve the equation determined in part f.

h. Remember what you set out to find. Is the potential solution you found in part g the answer to that problem?

i. In the context of the original problem, does this solution make sense?

j. What is the height of the cone described in the problem?

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Instruction

Problem-Based Task 2B.2.2: A Towering Cone

Coaching Sample Responses

a. What does the solution represent?

The problem is asking for the height of the cone; that is, the perpendicular distance from the base to the tip of the cone.

b. In what units might the solution be measured?

Since the diameter and the slant height are both given in meters, it is reasonable to expect the answer to be in meters, too.

c. What variable is a good choice to represent the solution?

Since we are looking for the height, the variable h is both simple and descriptive, and matches the relationship stated in the problem.

d. What other information do we need to know before we can determine the solution?

In order to find the height of the cone, we need to know the slant height, c, and the radius, r.

e. Review the information given in the problem statement. How can we use the given information to determine the missing information?

As stated in part d, we need to determine the slant height, c, and the radius, r. The problem directly states the length of the slant height: $c = 7.5$ m. While the problem does not tell us the radius r, it does state the diameter: 9 m. Since the radius is equal to half the diameter, $r = \frac{1}{2}d = \frac{1}{2}(9) = 4.5$. The radius is 4.5 m.

f. What algebraic equation describes the connection between the known information and the information needed to solve the problem?

The problem states that the height, slant height, and radius are related by the specific equation $c = \sqrt{r^2 + h^2}$.

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g. Solve the equation determined in part f.

Substitute the known values for c (7.5) and r (4.5) into the equation and solve for h.

$c = \sqrt{r^2 + h^2}$

$7.5 = \sqrt{4.5^2 + h^2}$

$7.5 = \sqrt{20.25 + h^2}$

$56.25 = 20.25 + h^2$

$36 = h^2$

$\sqrt{36} = \sqrt{h^2}$

$6 = h$

h. Remember what you set out to find. Is the potential solution you found in part g the answer to that problem?

Verify the potential solution found in part g by substituting it into the original equation.

$c = \sqrt{r^2 + h^2}$

$7.5 = \sqrt{4.5^2 + (6)^2}$

$7.5 = \sqrt{20.25 + 36}$

$7.5 = \sqrt{56.25}$

$7.5 = 7.5$

The result is a true statement; therefore, $h = 6$ is a solution to the original problem.

i. In the context of the original problem, does this solution make sense?

A height of 6 m makes sense. It is a positive value (a height could not be negative or 0). This solution is also less than the slant height, and should be, because it is perpendicular to the base, representing a shorter distance.

j. What is the height of the cone described in the problem?

The height of a cone with a slant height of 7.5 m and a base diameter of 9 m is 6 m.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2B.2.2: Solving Radical Equations

For problems 1–3, solve each radical equation for x. Round answers to the nearest thousandth.

1. $\sqrt{x-3}=6$

2. $\sqrt{x+6}=x$

3. $4\sqrt[3]{x+1}+5=1$

Use the given information and what you know about radical equations to solve problems 4–10. Round answers to the nearest thousandth.

4. A rectangular window is twice as wide as it is long. The area enclosed by the window is numerically equivalent to the length of the shortest side of the window. Find the area of the window.

5. A right triangle has a hypotenuse measuring 17 inches long. If one leg measures 8 inches, what is the length of the other leg?

6. When an object is dropped from a great height, gravity causes the object to speed up as it falls. When the object strikes the surface, the velocity at which the object was traveling at the moment of impact (ignoring air resistance) can be determined using the formula $v=\sqrt{2gh}$, where v represents the object's velocity (in feet per second), g is the object's acceleration due to gravity (in feet per second per second), and h is the height of the object (in feet). On Earth, an object's acceleration due to gravity is 32 ft/s^2. From what height must a stone be dropped to reach a velocity of 128 ft/s at the moment it hits the ground?

7. The time it takes an object to fall from a great height depends on the object's acceleration due to gravity, as well as the height from which the object is dropped. The formula $h=\frac{1}{2}gt^2$ can be used to describe this relationship, with t representing time (in seconds), h representing the height of the object (in feet), and g representing the object's acceleration due to gravity, which is 32 ft/s^2 on Earth. If a cliff diver leaps into the water from a height of 164 feet, how long will it take him to reach the water? Assume that air resistance is negligible and does not have an effect.

continued

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8. Accident investigators can estimate the speed of a vehicle involved in a crash by examining the skid marks left by the tires on the pavement. The algebraic relationship is shown by the formula $s = \sqrt{30fd}$, with s representing the speed in miles per hour and d representing the skid distance in feet. The value f is determined by the friction of the pavement: wet concrete is much more slippery than hot asphalt. A typical value for f, assuming a dry roadway, is 0.7. After a minor car accident on a dry day, Charlie insisted to his parents that he wasn't exceeding the 35 mph speed limit. An accident investigator found that the skid marks left by Charlie's tires were 75 feet long. Does the investigator's skid analysis indicate Charlie was telling the truth?

9. In an electrical circuit, the current is related to the power and resistance according to the rule $I = \sqrt{\frac{P}{R}}$, where I represents the current (measured in amps), P represents power (measured in watts), and R represents resistance (measured in ohms). What is the electrical power drawn by a device that has a resistance of 15 ohms with a current of 4 amps?

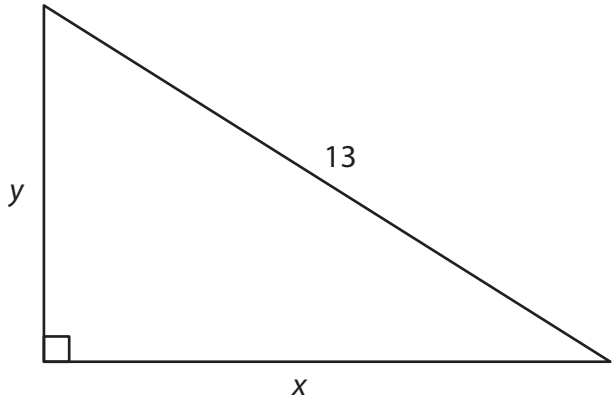
10. For a sphere with radius r, the volume, V, can be found using the formula $V = \frac{4}{3}\pi r^3$. Find the radius of a sphere with a volume of 288π.

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Lesson 2B.2.3: Solving Systems of Equations

Warm-Up 2B.2.3

Marco wants to sell his car, and plans to post signs around town. To make the most of the limited amount of poster board he has, Marco decides that each sign will be a right triangle with a hypotenuse of 13 inches. The diagram below shows the dimensions of Marco's design. Use the diagram and the given information to solve the problems that follow.



1. Use the Pythagorean Theorem to write an equation relating the length of each leg of the triangular sign to the length of the hypotenuse. Solve this equation for y.
2. The perimeter of the triangular sign can be determined by finding the sum of the lengths of the sides of the triangle. If Marco's sign has a perimeter of 30 inches, what is the equation, solved for y, that represents the perimeter of the triangle?
3. Use the equations written in problems 1 and 2 to complete the table below. Round values to the nearest hundredth.

x	1	2	3	4	5	6	7	8	9	10	11	12
Value of y based on equation from problem 1												
Value of y based on equation from problem 2												

4. At what values of x are the values of y for each equation equal? In the context of the problem, what do these x-values represent?

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Lesson 2B.2.3: Solving Systems of Equations

Common Core State Standard

A–REI.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★

Warm-Up 2B.2.3 Debrief

1. Use the Pythagorean Theorem to write an equation relating the length of each leg of the triangular sign to the length of the hypotenuse. Solve this equation for y.

   The Pythagorean Theorem states that for any right triangle, $a^2 + b^2 = c^2$, where a and b are the lengths of the legs and c is the length of the hypotenuse.

   In the diagram, the lengths of the legs are x and y and the length of the hypotenuse is 13 inches.

   Substitute these values into the Pythagorean Theorem formula, then solve the result for y.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$(x)^2 + (y)^2 = (13)^2$	Substitute x for a, y for b, and 13 for c.
$x^2 + y^2 = 169$	Simplify.
$y^2 = 169 - x^2$	Subtract x^2 from both sides.
$y = \sqrt{169 - x^2}$	Take the square root of both sides.

   The equation that represents the relationship between the lengths of the sides of a right triangle solved for y is $y = \sqrt{169 - x^2}$.

2. The perimeter of the triangular sign can be determined by finding the sum of the lengths of the sides of the triangle. If Marco's sign has a perimeter of 30 inches, what is the equation, solved for y, that represents the perimeter of the triangle?

   The lengths of the sides are x, y, and 13.

   The sum of these lengths is $x + y + 13 = 30$.

   Solve this equation for y.

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$x + y + 13 = 30$	Sum of the lengths of the triangle
$y + 13 = 30 - x$	Subtract x from both sides.
$y = 17 - x$	Subtract 13 from both sides.

The equation that represents the perimeter of the triangle solved for y is $y = 17 - x$.

3. Use the equations written in problems 1 and 2 to complete the table below. Round values to the nearest hundredth.

x	1	2	3	4	5	6	7	8	9	10	11	12
Value of *y* based on equation from problem 1	12.96	12.85	12.65	12.37	12.00	11.53	10.95	10.25	9.38	8.31	6.93	5.00
Value of *y* based on equation from problem 2	16	15	14	13	12	11	10	9	8	7	6	5

4. At what values of x are the values of y for each equation equal? In the context of the problem, what do these x-values represent?

   By looking at the table, we can see that when the value of x is equal to 5, the value of y for both equations is equal to 12.

   Also, when the value of x is equal to 12, the value of y for both equations is equal to 5.

   The x-values represent the lengths of the legs of the triangle, 5 inches and 12 inches, when the length of the hypotenuse is equal to 13 inches and the perimeter is equal to 30 inches.

Connection to the Lesson

- Students will use tables as well as technology to find the intersection(s) of two equations and will interpret the meaning of the intersection(s).
- Students will also recognize the implications of an inconsistent or dependent system of equations.

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Prerequisite Skills

This lesson requires the use of the following skills:

- completing a table of values for a given equation (F-IF.2)
- graphing multiple equations on a graphing calculator (A-REI.11)
- understanding that the graph of an equation is the set of all the solutions of the equation plotted on a coordinate plane (A-REI.10)
- graphing various forms of equations, including linear, polynomial, rational, and radical equations (F-IF.7)

Introduction

Functions can be used to model real-world situations where we may be interested in a certain event or circumstance while we are also concerned with another. Those two events may be described by two different variables or functions, each describing the very same thing. For instance, you may need to determine the number of tables to rent for a wedding reception, and your table choices include round tables that seat 8 people and square tables that seat 6 people. Or, you might want to compare two separate savings accounts that have different interest rates. Only by studying both models at the same time (that is, by solving the system) can the situation be fully understood, and the information you need extracted from it. In this lesson, we will apply technology to create tables and graphs that can reveal useful information about specific function models.

Key Concepts

- A **system of equations** is a set of equations with the same unknowns.
- The **solution set** for a system of equations is the set of ordered pairs that represent all of the solutions to an equation or a system of equations.
- The solutions to a system of equations can be identified in a table by finding points at which the *y*-value of each function is the same for a particular *x*-value. In most technology-based tables, this means the *y*-values will be in the same row.
- Solutions to a system of equations can be identified in a graph by finding the intersection of the functions.
- The **points of intersection** of a graphed system of equations are the ordered pairs where the graphed functions intersect on a coordinate plane.
- The solution of a system of $f(x)$ and $g(x)$ occurs where $f(x) = g(x)$ and the difference between these two values is 0.

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- Also recall that the solution to a system of equations is written using braces, {}.
- Recall that a system of equations may have an infinite number of solutions, a finite number of solutions, or no real solutions.
- It is possible that a system of equations does not intersect. The solution to such a system of equations is the **empty set**, written as Ø or { } (a set of braces with no coordinates inside).
- A system of equations may intersect at every point, meaning the equations overlap completely. This type of system is known as a **dependent system**.
- A graphing calculator is often used to graph the equations within a system. You can approximate where the equations intersect from a table of values or use the "trace" features of the calculator to find intersection points.
- To confirm the solution to a system, substitute the coordinates for x and y in both of the original equations. If the intersection point is approximated, the results will be nearly equal; otherwise, the results will be equal.

Common Errors/Misconceptions

- not recognizing that the values in a table are subject to rounding errors that may be imposed elsewhere, and lead to the illusion of a solution
- not realizing that on a graphing calculator, the scale of the graph may be such that the pixilated character of a function may appear to be an intersection, when it is not
- not understanding that solutions must satisfy every function in a system

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Guided Practice 2B.2.3

Example 1

Use a graph to solve the system $\begin{cases} f(x)=\sqrt{x} \\ g(x)=4-0.5x \end{cases}$. Verify that the identified coordinate pairs are solutions.

1. Create a graph of the two functions, $f(x)$ and $g(x)$.

   Graph each equation on the same coordinate plane, by hand or using a graphing calculator.

   To graph the system on your graphing calculator, use the directions appropriate to your model.

   On a TI-83/84:

   Step 1: Press the [Y=] button.

   Step 2: Type the first function into Y_1. Use [2ND][x^2] to enter radicals and the [X, T, θ, n] button for the variable x. Press [ENTER].

   Step 3: Type the second function into Y_2. Press [ENTER].

   Step 4: Press [ZOOM]. Arrow down to 6: ZStandard. Press [ENTER].

   On a TI-Nspire:

   Step 1: Press the [home] key.

   Step 2: Arrow over to the graphing icon and press [enter].

   Step 3: Type the first function next to $f1(x)$, using [ctrl][x^2] to enter radicals and the [X] button for the letter x. Press [enter].

   Step 4: To graph the second function, press [menu] and arrow down to 3: Graph Type, then arrow right to 1: Function. Press [enter].

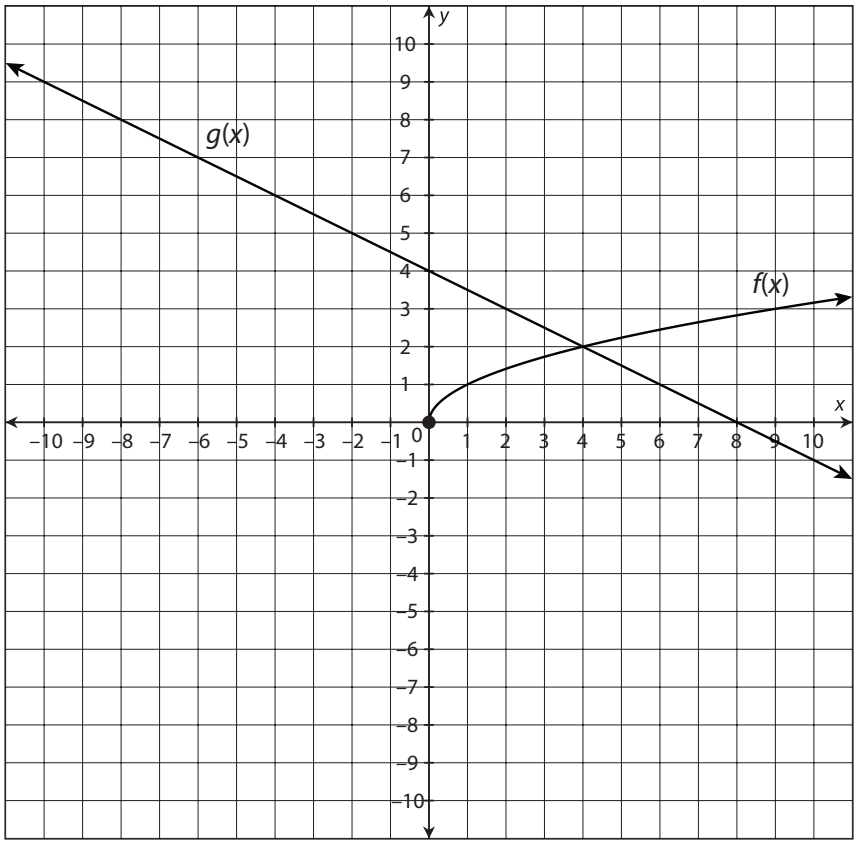
   Step 5: Type the second function next to $f2(x)$. Press [enter].

   Step 6: To change the viewing window, press [menu]. Select 4: Window/Zoom and select A: Zoom – Fit.

(*continued*)

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Either method should yield a graph resembling the following:



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2. Find the coordinates of any apparent intersections.

   Looking at the graph, estimate where the two equations intersect. It may be necessary to plot additional points.

   You may also determine the approximate point(s) of intersection on your graphing calculator:

   On a TI-83/84:

   Step 1: Press [2ND][TRACE] to call up the CALC menu. Arrow down to 5: intersect. Press [ENTER].

   Step 2: At the prompt, use the up/down arrow keys to select the Y_1 equation, and press [ENTER].

   Step 3: At the prompt, use the up/down arrow keys to select the Y_2 equation, and press [ENTER].

   Step 4: At the prompt, use the arrow keys to move the cursor close to an apparent intersection, and press [ENTER]. The coordinates of the intersection points are displayed.

   On a TI-Nspire:

   Step 1: Press [menu]. Arrow down to 6: Analyze Graph, then arrow right to 4: Intersection. Press [enter].

   Step 2: Use the pointing hand to click on each graphed line. The coordinates of the intersection points are displayed.

   The point of intersection is (4, 2).

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3. Verify that the identified coordinate pair is a solution to the original system of equations.

   In order for a coordinate to be a solution to a system, the coordinate must satisfy both equations of the system.

   Substitute the coordinate pair found in step 2 into each of the original equations to determine whether it results in a true statement.

   The potential solution is (4, 2). Let 4 = x, and let 2 = $f(x)$ and $g(x)$.

$f(x)=\sqrt{x}$	Original equation
$(2)=\sqrt{(4)}$	Substitute 4 for x and 2 for $f(x)$.
$2 = 2$	Simplify.

$g(x) = 4 - 0.5x$	Original equation
$2 = 4 - 0.5(4)$	Substitute 4 for x and 2 for $g(x)$.
$2 = 4 - 2$	Simplify.
$2 = 2$	

   The identified point (4, 2) satisfies both $f(x)$ and $g(x)$.

   The solution to the system of equations $\begin{cases} f(x)=\sqrt{x} \\ g(x)=4-0.5x \end{cases}$ is {(4, 2)}, as identified on the graph as the intersection of the functions $f(x)$ and $g(x)$, and verified algebraically.

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Example 2

Create a table to approximate the real solution(s), if any, to the system $\begin{cases} f(x)=1+\frac{1}{x} \\ g(x)=x+1 \end{cases}$.

1. Create a table of values for the two functions, $f(x)$ and $g(x)$.

   Use a graphing calculator to generate a table of values for each function.

   On a TI-83/84:

   Step 1: Press the [Y=] button.

   Step 2: Type the first function into Y_1, using parentheses for the fraction and the [X, T, θ, n] button for the variable x. Press [ENTER].

   Step 3: Type the second function into Y_2. Press [ENTER].

   Step 4: Press [2ND][WINDOW] to call up the TABLE SETUP screen. Set TblStart to –5 by typing [(–)][5], and ΔTbl to 1, then press [2ND][GRAPH] to call up the TABLE screen. A table of values for both equations will be displayed.

   On a TI-Nspire:

   Step 1: Press the [home] key.

   Step 2: Arrow over to the calculator icon and press [enter].

   Step 3: Define $f(x)$ by entering "define" and then the function using the buttons on your keypad. Press [enter].

   Step 4: Define $g(x)$ by entering "define" and then the function using the buttons on your keypad. Press [enter].

   Step 5: Press the [home] key. Arrow over to the spreadsheets page, the fourth icon over, and press [enter].

   Step 6: Press [ctrl][T] to switch to a table window. From the list of defined functions that appears, select $f(x)f1$, and press [enter]. This will display a table of values for each value of x in the function $f(x)$.

   Step 7: Press the right arrow key. From the list of defined functions that appears, select $g(x)g1$, and press [enter]. This will display a table of values for each value of x in the function $g(x)$.

(*continued*)

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The calculator will yield the table of values shown as follows. Use the up and down arrow keys as necessary to see the y-values for each x-value shown as follows.

x	–5	–4	–3	–2	–1	0	1	2	3
$f(x)$	0.8	0.75	0.67	0.5	0	undefined	2	1.5	1.33
$g(x)$	–4	–3	–2	–1	0	1	2	3	4

2. Estimate the points of intersection.

Compare the y-values for the same x-value in your completed table.

Look for places where the difference between the values for $f(x)$ and $g(x)$ is decreasing. If the difference between $f(x)$ and $g(x)$ is increasing, then any intersection occurs in the other direction. The point of intersection occurs when $f(x)$ is equal to $g(x)$ or the difference is 0.

Notice that $f(x)$ equals $g(x)$ when $x = -1$ and when $x = 1$.

The points (–1, 0) and (1, 2) appear to be the solutions to the system.

3. Verify that the identified coordinate pairs are solutions to the original system of equations.

Substitute the coordinates found in the previous step into each of the original equations to determine whether each results in a true statement.

Start with the potential solution (–1, 0). Let $-1 = x$, and let $0 = f(x)$ and $g(x)$.

$f(x) = 1 + \frac{1}{x}$ Original equation

$(0) = 1 + \frac{1}{(-1)}$ Substitute –1 for x and 0 for $f(x)$.

$0 = 1 + (-1)$ Simplify.

$0 = 0$

$g(x) = x + 1$ Original equation

$(0) = (-1) + 1$ Substitute –1 for x and 0 for $g(x)$.

$0 = 0$ Simplify.

The identified point (–1, 0) satisfies both $f(x)$ and $g(x)$, and is therefore a valid solution.

(*continued*)

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Test the second potential solution, (1, 2). Let 1 = x, and let 2 = $f(x)$ and $g(x)$.

$f(x)=1+\frac{1}{x}$	Original equation
$(2)=1+\frac{1}{(1)}$	Substitute 1 for x and 2 for $f(x)$.
$2 = 1 + (1)$	Simplify.
$2 = 2$	

$g(x) = x + 1$	Original equation
$(2) = (1) + 1$	Substitute 1 for x and 2 for $g(x)$.
$2 = 2$	Simplify.

The identified point (1, 2) also satisfies both $f(x)$ and $g(x)$.

The solution to the system of equations $\begin{cases} f(x)=1+\frac{1}{x} \\ g(x)=x+1 \end{cases}$ is {(–1, 0), (1, 2)}, as identified using a table of values showing where $f(x) = g(x)$, and verified algebraically.



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Example 3

A system contains the equations $y = 2 + \sqrt{x-3}$ and $4x - 4y = 19$. Solve the system by creating a table of values on a graphing calculator.

1. Determine whether the given equations are written in "$y =$" form and rewrite if necessary.

   Entering equations into calculators and spreadsheets generally demands that each equation be in "$y =$" form. While the first equation is already in "$y =$" form, the second is not.

   Use algebraic techniques to rewrite the equation $4x - 4y = 19$ in "$y =$" form.

$4x - 4y = 19$	Original equation
$-4y = -4x + 19$	Subtract $4x$ from both sides.
$\frac{-4}{-4}y = \frac{-4}{-4}x + \frac{19}{-4}$	Divide all terms by -4.
$y = x + \frac{19}{-4}$	Simplify.
$y = x - 4.75$	

   The equation $4x - 4y = 19$ written in "$y =$" form is $y = x - 4.75$.

2. Create a table of values for the two functions.

   Use the calculator steps shown in Example 2 to create a table of values for each equation of the defined system.

   A table of values for both equations appears below. Let "y_1" represent the equation $y = 2 + \sqrt{x-3}$ and "y_2" represent the equation $y = x - 4.75$.

x	2	3	4	5	6	7	8	9	10
$y_1 = 2 + \sqrt{x-3}$	undefined	2	3	3.41	3.73	4	4.24	4.45	4.65
$y_2 = x - 4.75$	–2.75	–1.75	–0.75	0.25	1.25	2.25	3.25	4.25	5.25

   Scrolling up and down through the list reveals that there are no x-values for which both functions have the same y-value. A closer inspection shows that the y-values for the two functions are closest in value when $x = 9$: $y_1 = 4.45$ and $y_2 = 4.25$.

   To better approximate the points of intersection, you must find additional values.

(*continued*)

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Use the following directions to determine additional values on your graphing calculator.

On a TI-83/84:

Step 1: Press [2ND][WINDOW] to call up the TABLE SETUP screen. Set TblStart to 8 and ΔTbl to 0.1, then press [2ND] [GRAPH] to call up the TABLE screen.

On a TI-Nspire:

Step 1: Press [menu]. Arrow down to 5: Table, then right and down to select 5: Edit Table Settings. Press [enter].

Step 2: Change "Table Start" to 8 and "Table Step" to 0.01. Arrow down to "OK" and press [enter].

Examine the revised table of values.

x	8.0	8.1	8.2	8.3	8.4	8.5	8.6	8.7	8.8	8.9	9.0	9.1	9.2	9.3
$y_1 = 2+\sqrt{x-3}$	4.24	4.26	4.28	4.30	4.32	4.35	4.37	4.39	4.41	4.43	4.45	4.47	4.49	4.51
$y_2 = x - 4.75$	3.25	3.35	3.45	3.55	3.65	3.75	3.85	3.95	4.05	4.15	4.25	4.35	4.45	4.55

The small changes in decimals can be hard to analyze. However, you might notice that when $x \le 9.2$, $y_2 > y_1$. That relationship switches for $x \ge 9.3$, when $y_2 < y_1$. This is an indication that something happens when $9.2 \le x \le 9.3$.

In order to look at the table more closely, change the table settings again.

Follow the directions previously outlined, but this time, start the table at 9.2, and set the interval to 0.01.

Examine the revised table again.

x	9.20	9.21	9.22	9.23	9.24	9.25	9.26	9.27	9.28	9.29	9.30
$y_1 = 2+\sqrt{x-3}$	4.49	4.492	4.494	4.496	4.498	4.5	4.502	4.504	4.506	4.508	4.51
$y_2 = x - 4.75$	4.45	4.46	4.47	4.48	4.49	4.5	4.51	4.52	4.53	4.54	4.55

When $x = 9.25$, both y_1 and $y_2 = 4.5$.

There is a potential solution to the system at (9.25, 4.5).

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Instruction

3. Continue examining the table of values for additional matching y-values.

   On your graphing calculator, set the table interval back to 0.1, or even 1, in order to verify there are no other potential solutions.

   For this system, it appears that the only potential solution is at (9.25, 4.5).

4. Verify that the identified coordinate pair is a solution to the original system of equations.

   Substitute the coordinate pair into each of the original equations to determine whether it results in a true statement.

   In order for a coordinate pair to be solution to a system, the coordinates must satisfy both equations of the system. Because our coordinate pair is estimated, it is quite possible that our results will be nearly equal, not exactly equal.

   The potential solution is (9.25, 4.5). Let 9.25 = x, and let y = 4.5.

$4x - 4y = 19$	Original equation
$4(9.25) - 4(4.5) = 19$	Substitute 9.25 for x and 4.5 for y.
$37 - 18 = 19$	Simplify.
$19 = 19$	

$y = 2 + \sqrt{x-3}$	Original equation
$(4.5) = 2 + \sqrt{(9.25)-3}$	Substitute 9.25 for x and 4.5 for y.
$4.5 = 2 + \sqrt{6.25}$	Simplify.
$4.5 = 2 + 2.5$	
$4.5 = 4.5$	

   The identified point (9.25, 4.5) satisfies both of the original equations.

   The solution to the system of equations defined by $y = 2 + \sqrt{x-3}$ and $4x - 4y = 19$ is {(9.25, 4.5)}, as identified using a table of values showing where $y_1 = y_2$, and verified algebraically.

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Instruction

Example 4

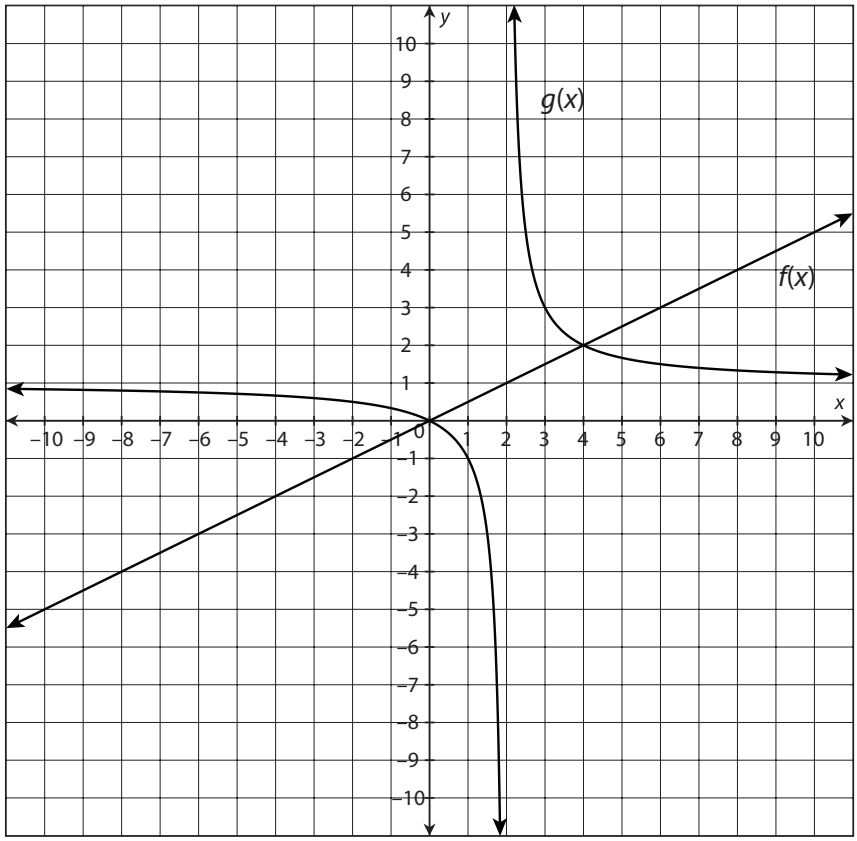
Solve the system of equations $\begin{cases} f(x)=0.5x \\ g(x)=1+\dfrac{2}{x-2} \end{cases}$ by graphing.

1. Graph both equations on the same set of axes.

   Graph each equation on the same coordinate plane, by hand or using a graphing calculator.

   To graph the system on your graphing calculator, use the directions outlined in Example 1.

   Either method should yield a graph resembling the following:



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Instruction

2. Find the coordinates of any apparent intersections.

   On your graph, estimate where the two equations intersect. It may be necessary to plot additional points or to adjust the window settings on your calculator.

   Looking at the graph indicates that these two functions may intersect twice, at approximately $x = 0$ and $x = 4$. One of those intersections appears to be the origin.

   You can more precisely determine the approximate point(s) of intersection using features on your graphing calculator.

   On a TI-83/84:

   Step 1: Press [2ND][TRACE] to call up the CALC menu. Arrow down to 5: intersect. Press [ENTER].

   Step 2: At the prompt, use the up/down arrow keys to select the Y_1 equation, and press [ENTER].

   Step 3: At the prompt, use the up/down arrow keys to select the Y_2 equation, and press [ENTER].

   Step 4: At the prompt, use the arrow keys to move the cursor close to an apparent intersection, and press [ENTER]. The coordinates of the intersection points are displayed.

   On a TI-Nspire:

   Step 1: Press [menu]. Arrow down to 6: Analyze Graph, then arrow right to 4: Intersection. Press [enter].

   Step 2: Use the pointing hand to click on each graphed line. The coordinates of the intersection points are displayed.

   The points of intersection are (0, 0) and (4, 2).

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3. Verify that the identified coordinate pairs are solutions to the original system of equations.

   Substitute each coordinate pair into the equations to determine whether each results in a true statement.

   The first potential solution is (0, 0). Let 0 = x, and let 0 = $f(x)$ and $g(x)$.

$f(x) = 0.5x$	Original equation
$(0) = 0.5(0)$	Substitute 0 for x and 0 for $f(x)$.
$0 = 0$	Simplify.

$g(x)=1+\frac{2}{x-2}$	Original equation
$(0)=1+\frac{2}{(0)-2}$	Substitute 0 for x and 0 for $g(x)$.
$0=1+\frac{2}{-2}$	Simplify.
$0 = 1 + (-1)$	
$0 = 0$	

   The identified point (0, 0) satisfies both of the original equations, and is therefore a valid solution.

(*continued*)

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Follow the same process to verify the potential solution (4, 2).

$f(x) = 0.5x$	Original equation
$(2) = 0.5(4)$	Substitute 4 for x and 2 for $f(x)$.
$2 = 2$	Simplify.

$g(x)=1+\frac{2}{x-2}$	Original equation
$(2)=1+\frac{2}{(4)-2}$	Substitute 4 for x and 2 for $g(x)$.
$2=1+\frac{2}{2}$	Simplify.
$2 = 1 + (1)$	
$2 = 2$	

The identified point (4, 2) also satisfies both of the original equations.

The solution to the system of equations $f(x) = 0.5x$ and $g(x)=1+\frac{2}{x-2}$ is {(0, 0), (4, 2)} as identified on the graph as the intersections of the functions $f(x)$ and $g(x)$, and verified algebraically.



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Instruction

Example 5

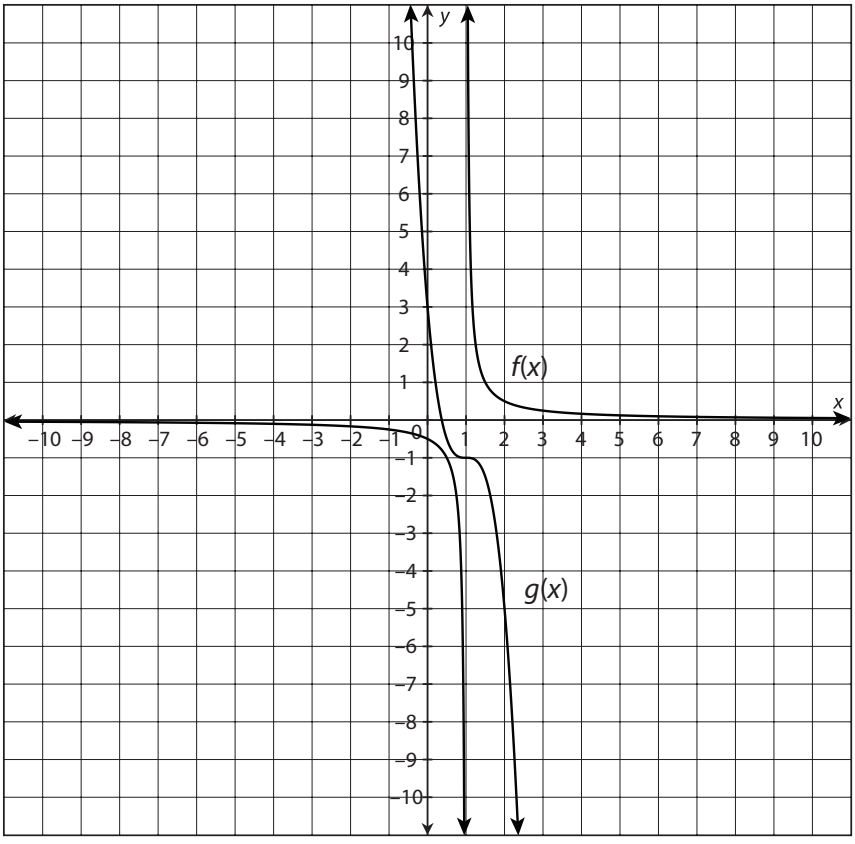
Solve the system of equations $\begin{cases} f(x)=\dfrac{1}{2(x-1)} \\ g(x)=3-12x+12x^2-4x^3 \end{cases}$ by graphing.

1. Create a graph of the two functions, $f(x)$ and $g(x)$.

   Graph each equation on the same coordinate plane, by hand or using a graphing calculator.

   To graph the system on your graphing calculator, use the directions outlined in Example 1.

   Either method should yield a graph resembling the following:



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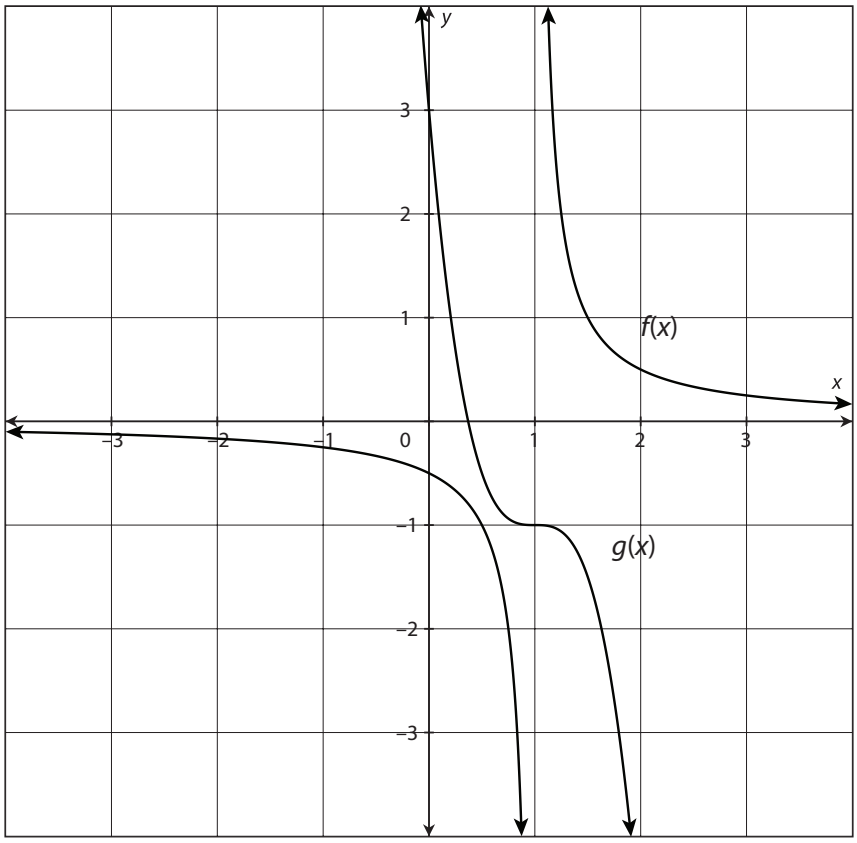
Instruction

2. Find the coordinates of any apparent intersections.

The graph indicates that these two functions never intersect.

They come close where $0 \le x \le 1$, but zooming in on that area of the graph shows that they never cross.

Follow the steps outlined in Example 1 to zoom in to the area in question. The graph should resemble the following:



There is no intersection; therefore, there is no solution to this system.

There is no real solution to the system of equations

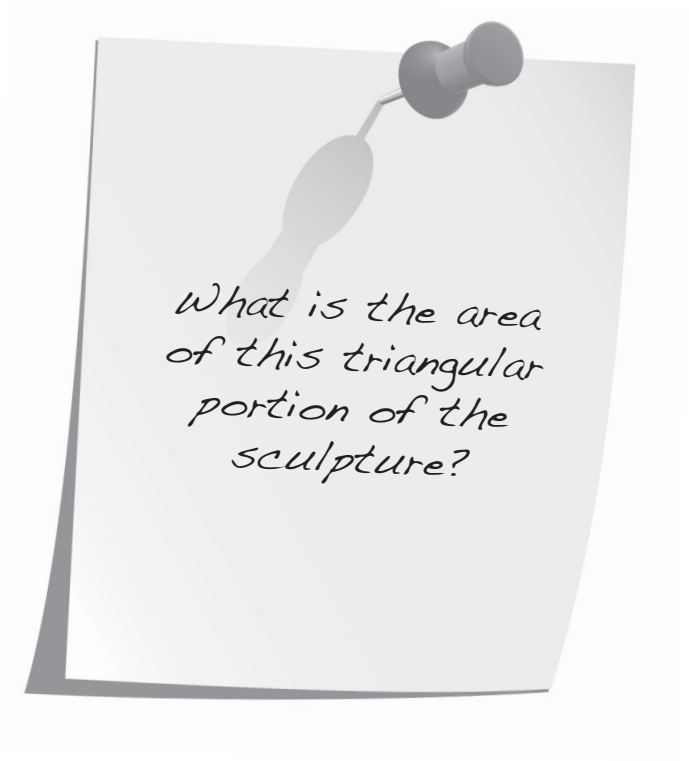
$$\begin{cases} f(x)=\dfrac{1}{2(x-1)} \\ g(x)=3-12x+12x^2-4x^3 \end{cases}.$$

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Problem-Based Task 2B.2.3: Measuring a Masterpiece

SMP	
1	2 ✓
3	4 ✓
5	6 ✓
7	8

A new sculpture was just unveiled at a local park. Part of the sculpture is a right triangle that has a hypotenuse of 29 meters. One leg is 1 meter longer than the other. What is the area of this triangular portion of the sculpture?



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Problem-Based Task 2B.2.3: Measuring a Masterpiece

Coaching

a. What will the solution represent? In what units will the solution be measured?

b. What are the variables in this problem? How can they effectively be represented?

c. How does the problem explicitly state that the variables are related? Write this relationship as an equation.

d. How does the Pythagorean Theorem mathematically describe the relationship of the hypotenuse and legs of a right triangle?

e. What system is defined by the equations determined in parts c and d? What does this system represent?

f. Solve the system determined in part e.

g. What coordinates represent the solution(s) to this system?

h. In the context of the problem, what do those coordinates represent? Justify your response.

i. What are the lengths of the sides of the triangle?

j. What is the area of the triangular portion of the sculpture?

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Instruction

Problem-Based Task 2B.2.3: Measuring a Masterpiece

Coaching Sample Responses

a. What will the solution represent? In what units will the solution be measured?

The problem is asking for the area of a triangle, so the solution will represent the area of the triangular portion of the sculpture. Area is often measured in square units, with each unit representing length. Because one side of the triangle is known, and it is measured in meters, the answer is likely to be in square meters, or m^2.

b. What are the variables in this problem? How can they effectively be represented?

The variables are the lengths of the two legs. While b and h, representing base and height, are good choices, most calculators are much easier to operate using the variables x and y. Use x to represent the base, and y to represent the height.

c. How does the problem explicitly state that the variables are related? Write this relationship as an equation.

The problem states: "The length of one leg is 1 meter more than the length of the other." We can convert this statement into the mathematical equation $y = x + 1$.

d. How does the Pythagorean Theorem mathematically describe the relationship of the hypotenuse and legs of a right triangle?

The Pythagorean Theorem relates the legs of a right triangle (usually called a and b) to the length of the hypotenuse (c) as $a^2 + b^2 = c^2$. The problem gives the length of the hypotenuse as 29 m. Using x and y to represent the legs of the triangle and 29 m as the length of the hypotenuse, we can rewrite the Pythagorean Theorem as follows: $x^2 + y^2 = 29^2$, or $x^2 + y^2 = 841$. Rewriting this equation in our more familiar "$y =$" form, we have $y = \sqrt{841 - x^2}$.

e. What system is defined by the equations from parts c and d? What does this system represent?

The system of equations defined by $y = x + 1$ and $y = \sqrt{841 - x^2}$ is $\begin{cases} y = x + 1 \\ y = \sqrt{841 - x^2} \end{cases}$. This system represents the lengths of each leg of the triangle in the problem.

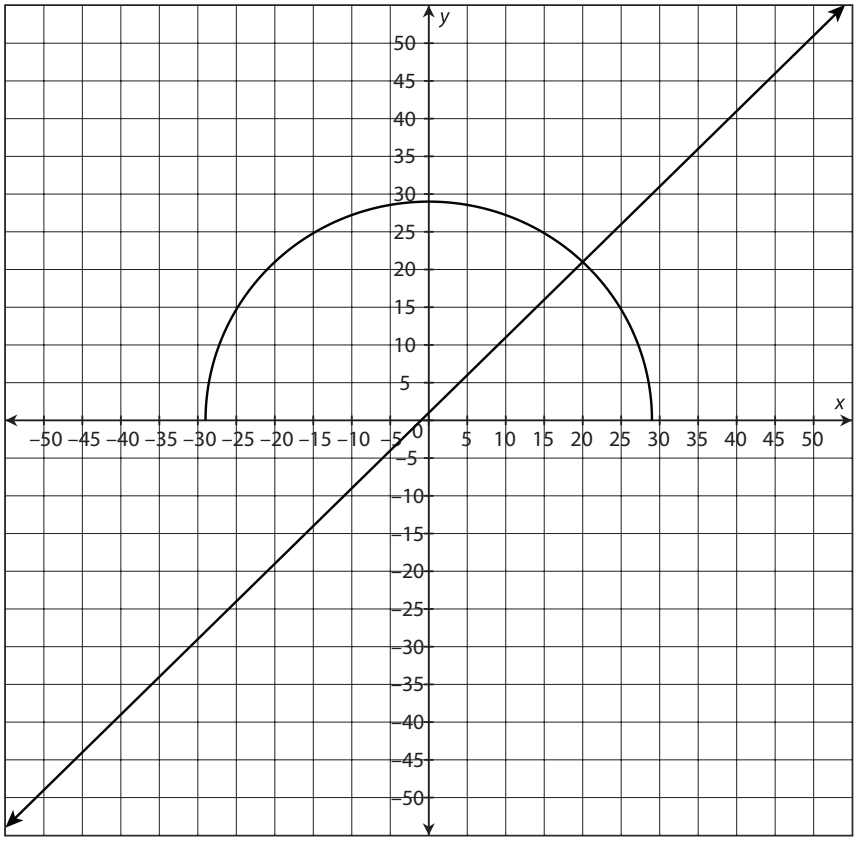
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f. Solve the system determined in part e.

Because the equations in the system are already in "$y =$" form, it makes sense to try to solve this system through graphing.

Enter the two equations, $y_1 = x + 1$ and $y_2 = \sqrt{841 - x^2}$, into a graphing calculator.

Select an appropriate graphing window.



g. What coordinates represent the solution(s) to this system?

Use the calculator's trace or analytical tools to find the point(s) of intersection of the graph.

The two equations intersect at only one point: (20, 21).

h. In the context of the problem, what do those coordinates represent? Justify your response.

The point (20, 21) corresponds to the lengths of the legs of the right triangle. The y-value is 1 unit greater than the x-value, satisfying the problem scenario requirement that "one leg is 1 meter longer than the other." Substituting these values for x and y into the Pythagorean Theorem, we find that $20^2 + 21^2 = 400 + 441 = 841 = 29^2$. This satisfies the other requirement of the problem scenario—that the triangle is a right triangle—because the squares of the lengths of the legs sum to the square of the length of the hypotenuse, 29.

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i. What are the lengths of the sides of the triangle?

The sides of the triangle are 20 m, 21 m, and 29 m long.

j. What is the area of the triangular portion of the sculpture?

With the lengths of the legs now determined, the area of this triangle can be calculated using the formula for the area of a triangle: $A = \frac{1}{2}bh$.

Let $b = 20$ and $h = 21$.

$$A = \frac{1}{2}(20)(21)$$
$$A = 210 \text{ m}^2$$

The area of the triangular portion of the sculpture is 210 m^2.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2B.2.3: Solving Systems of Equations

For problems 1–4, solve each of the following systems by graphing. If necessary, rewrite equations in the form "$y =$". Round to the nearest hundredth.

1. $\begin{cases} f(x) = 3x - 5 \\ g(x) = \frac{1}{x} + \frac{x+4}{6x} \end{cases}$

2. $3y - x = 17$ and $y = \sqrt{x} + 5$

3. $\begin{cases} f(x) = 5 - \frac{x}{4} \\ g(x) = \frac{x}{x-6} - \frac{5}{x-3} \end{cases}$

4. $\begin{cases} f(x) = x \\ g(x) = \frac{2}{x+2} + \frac{x}{3} \end{cases}$

Use the given information and what you have learned about systems of equations to solve problems 5 and 6.

5. The table below shows values for two functions, $f(x)$ and $g(x)$, at given values of x.

x	–7	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7	8
$f(x)$	136	105	78	55	36	21	10	3	0	1	6	15	28	45	66	91
$g(x)$	330	206	118	60	26	10	6	8	10	6	–10	–44	–102	–190	–314	–480

Based on the information in the table, what can you conclude about solutions to the system of equations defined by $f(x)$ and $g(x)$?

6. A system of two linear equations has no real solution. What can you conclude about the two lines?

continued

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For problems 7–10, determine the solution(s), if any, to each of the described systems. Round to the nearest hundredth.

7. Debbie says that the point (–1, 3) is the only solution to the system of equations $y=\frac{2x-1}{x}$ and $y=1+\sqrt{x-1}$. Explain why her solution is (or is not) correct.

8. Solve the system defined by the equations $y=\frac{x+6}{3}$ and $y=2-\frac{4x+4}{x^2-2x-3}$.

9. A right triangle has a hypotenuse of 37 cm. The length of one leg is 3 times the length of the other. What is the area of the triangle?

10. The period of a pendulum is given by the lowercase Greek letter *tau*, τ, and depends on the length L of the pendulum, and the pendulum's gravitational acceleration, g, according to the formula $\tau=2\pi\sqrt{\frac{L}{g}}$. For what length(s) in meters is the period of an Earth-bound pendulum numerically equal to 4 times the pendulum's length in meters? The gravitational acceleration for an object on Earth is about 9.8 m/s^2.

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Progress Assessment

Circle the letter of the best answer.

1. What is the solution to the equation $\sqrt{x}+2=9$?

   a. $x = 121$
   b. $x = 79$
   c. $x = 77$
   d. $x = 49$

2. What is the solution to the equation $\frac{x}{3}-\frac{x+2}{4}=1$?

   a. $x=-\frac{1}{2}$
   b. $x = 18$
   c. $x = -1$
   d. $x = 7$

3. What is the solution to the system of equations defined by $y=8-\sqrt{x}$ and $y=8-\frac{x}{3}$?

   a. {(9, 5)}
   b. {(0, 8), (9, 5)}
   c. {(9, 5), (0, 9)}
   d. {(0, 8)}

4. What is the solution to the equation $\frac{x}{x+1}+\frac{5}{x-1}=1$?

   a. $x = 5$
   b. $x = 2$
   c. $x=-1\frac{1}{2}$
   d. $x = -2$

5. What is the solution to the system of equations defined by $y=\frac{1}{x+3}$ and $y=1+\sqrt{x+2}$?

   a. {(1, –2)}
   b. {(–2, 1)}
   c. {(–1, 2), (3.3, 3.3)}
   d. {(3.3, 3.3)}

continued

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6. What is the solution to the equation $2+\sqrt{x-20}=\sqrt{x}$?

   a. $x = 36$
   b. $x = -36$
   c. $x = \pm 36$
   d. The equation has no real solution.

7. What is the solution to the equation $\frac{1}{x-1}+\frac{x}{x-2}=\frac{7}{x^2-3x+2}$?

   a. $x = \pm 3$
   b. $x = 3$
   c. $x = -2$
   d. $x = -5$

8. What is the solution to the system of equations defined by $y = -x - 3$ and $y=2-\frac{1}{x+5}$?

   a. {(–6, 3), (–4, 1)}
   b. {(–6, 3)}
   c. {(–4, 1)}
   d. The system has no real solutions.

9. What is the solution to the equation $\sqrt{2x-1}=x-2$?

   a. $x = -1$
   b. $x = 3$
   c. $x = 5$
   d. $x=1\frac{1}{2}$

10. Which of the following coordinates is closest to a solution for the system of equations $y=\frac{1}{x}+\frac{3}{\sqrt{x+1}}$ and $y=1-\frac{(x+4)^3}{4}$?

    a. (–0.05, –18.18)
    b. (–0.06, –14.32)
    c. (–0.07, –9.67)
    d. (–0.08, –7.03)

continued

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Assessment

Use the given information to answer the questions that follow.

11. A Foucault pendulum is a simple device used to demonstrate the Earth's rotation. Foucault pendulums consist of a weight at the bottom, a fixed-point support structure at the top, and a cable or string in between. Frank's Foucault Foundry makes large-scale precision display pendulums to sell to various science museums, research universities, and planetariums. Frank can make any size Foucault pendulum, up to 65 m in length. The larger the pendulum, the more expensive it is to build. Frank uses the model $y = 800 + 225\sqrt{2x - 5}$ to estimate the selling price for his pendulums. The cost to build each pendulum is related to the length x of the entire pendulum: $y = 750 + 80x$. This relationship can be represented by the system of equations $\begin{cases} f(x) = 800 + 225\sqrt{2x - 5} \\ g(x) = 750 + 80x \end{cases}$.

    a. What is the solution to the system of equations? Round your answer to the nearest hundredth.

    b. What does the solution to the system of equations represent in relation to the context of this problem?

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Unit 2B Assessment

Circle the letter of the best answer.

1. Which simplified rational expression is equivalent to $\frac{81x^2+27x}{9x}$?

   a. $9x^2+3x$
   c. $81x+27$
   b. $9x+3$
   d. $9x^2+27x$

2. Which simplified rational expression is equivalent to $\frac{x^2+9}{3x}+\frac{6x}{x+1}$?

   a. $\frac{x^3+19x^2+9x+9}{3x^2+3x}$
   c. $\frac{x^2+6x+9}{3x^2+3x}$
   b. $\frac{x^2+6x+9}{3x+3}$
   d. $\frac{x^3+19x^2+9x+9}{3x+3}$

3. Which simplified rational expression is equivalent to $\frac{5x+1}{3x}-\frac{x-2}{x^2}$?

   a. $\frac{4x-x-1}{-x^2+3x}$
   c. $\frac{5x^2-2x+6}{-x^2+3x}$
   b. $\frac{5x^2-2x+6}{3x^3}$
   d. $\frac{5x^2-2x+6}{3x^2}$

4. Which simplified rational expression is equivalent to $\frac{x}{x+1}\bullet\frac{5x}{2x-1}$?

   a. $\frac{5x^2}{2x-1}$
   c. $\frac{5x^2}{2x^2+x-1}$
   b. $\frac{5x}{2x+1}$
   d. $\frac{5x}{2x^2+x+2}$

continued

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5. Which simplified rational expression is equivalent to $\frac{3x^3+9x^2+6x}{2x+4}$?

   a. $\frac{(3x^2+3x)}{2}$

   b. $\frac{3x^3+9x^2+6x}{2x+4}$

   c. $3x^2+3x$

   d. $\frac{x^2+3x+4}{3x+2}$

6. Which simplified rational expression is equivalent to $\frac{x^2-9}{x+1} \div \frac{x+3}{x}$?

   a. $\frac{x^3+3x^2-9x-27}{x^2+x}$

   b. $\frac{x^3-x^2-13x-3}{-x^2+x}$

   c. $\frac{x^2-9}{4x+4}$

   d. $\frac{x^2-3x}{x+1}$

7. What is the solution to the equation $\frac{x-4}{5x}=1+\frac{1}{5x}$?

   a. $x=-\frac{6}{5}$

   b. $x=\frac{6}{5}$

   c. $x=-\frac{5}{4}$

   d. $x=\frac{5}{4}$

8. What is the solution to the equation $\frac{3}{2+x}-\frac{1}{x}=\frac{1}{5x}$?

   a. $x=-\frac{4}{3}$

   b. $x=\frac{4}{3}$

   c. $x=\frac{3}{4}$

   d. $x=-\frac{3}{4}$

9. What is the solution to the equation $4+\sqrt{x-1}=8$?

   a. $x = 17$

   b. $x = 15$

   c. $x = 143$

   d. $x = 144$

continued

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10. What is the solution to the equation $x+1=\sqrt{4x+9}$?

    a. $x = -4$
    b. $x = 2$
    c. $x = 16$
    d. $x = 4$

11. What is the solution to the system $\begin{cases} f(x)=\frac{1}{x}+4 \\ g(x)=2x+3 \end{cases}$?

    a. {(1, 5)}
    b. {(0.5, 2), (1, 5)}
    c. {(–0.5, 2), (1, 5)}
    d. {(–0.5, 2)}

12. What is the solution to the system $\begin{cases} f(x)=x-7 \\ g(x)=\sqrt{5x+1} \end{cases}$?

    a. {(16, 9)}
    b. {(9, 16)}
    c. {(–0.58, –7.58), (8.58, 1.58)}
    d. {(–2, –9)}

Use what you have learned about rational and radical relationships to complete the following problems.

13. It takes Alden 9 hours to bale all the hay in a field, but his brother, Anthony, can do the same job in 7 hours. If they bale all the hay in an entire field together, approximately how long will it take them to complete the job?

14. A rectangle measured in centimeters has a width described by the expression $\frac{2x^2+2x}{x+1}$, and a length described by the expression $\frac{2x^2+3x}{x+6}$.

    a. What simplified rational expression describes the perimeter of the rectangle?
    b. What simplified rational expression describes the area of the rectangle?

15. Paco believes that one solution to the system of equations defined by the functions $f(x)=6-\sqrt{x}$ and $g(x) = 2x + 3$ is at the point (1, 5).

    a. Verify that the point (1, 5) is indeed a solution to the defined system.
    b. Graph the two functions on the same coordinate plane.
    c. Identify any other points of intersection.
    d. Explain what the points of intersection, if any, mean in terms of the functions.

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Answer Key

Lesson 1: Operating with Rational Expressions

Pre-Assessment, p. U2B-1

1. a
2. b
3. c
4. c
5. a

Warm-Up 2B.1.1, p. U2B-5

1. There are 21 students present in Ms. Appleton's class.
2. They have eaten $\frac{5}{6}$ of the cake, so $\frac{1}{6}$ remains for the other teachers.
3. Answers will vary; three noteworthy responses are $\frac{4x^4-16x^2}{4}$, $x^2(x^2-4) = x^2(x-2)(x+2)$, and $(x^2-2x)(x^2+2x)$.

Practice 2B.1.1: Structures of Rational Expressions, p. U2B-22

1. $5, x \neq 0$
2. $\frac{3}{2}, x \neq -4$
3. $x+7, x \neq 0$
4. $6x^3-2x+3, x \neq 0$
5. $\frac{1}{x+1}, x \neq -1, x \neq -4$
6. $2x-1, x \neq 0, x \neq -7$
7. $\frac{x^3-6x^2-7x}{x^2+4x+3} = \frac{x(x^2-6x-7)}{x^2+4x+3} = \frac{x(x-7)\cancel{(x+1)}}{(x+3)\cancel{(x+1)}} = \frac{x(x-7)}{x+3} = \frac{x^2-7x}{x+3}, x \neq -3, x \neq -1$
8. $\frac{3x-8}{5}$ meters, $x > \frac{8}{3}$
9. $\frac{4}{3}$ centimeters
10. $(x-3)$ inches and $\frac{x-4}{2}$ inches, $x > 4$

Warm-Up 2B.1.2, p. U2B-23

1. They ate $\frac{7}{12}$ of the pizza.
2. $\frac{1}{4}$ of the pizza remains.
3. $\frac{1}{12}$ of a pizza, or one slice, remains for Mrs. Shepard.

Practice 2B.1.2: Adding and Subtracting Rational Expressions, p. U2B-44

1. $\frac{22}{5x}, x \neq 0$
2. $\frac{x^2-11x+11}{x-x^2}, x \neq 0, x \neq 1$
3. $\frac{x^3+3x-32}{8x}, x \neq 0$
4. $\frac{x^2+x+5}{5x}, x \neq 0$
5. $\frac{5x^2+14x}{x+1}, x \neq -1$
6. $\frac{5x^2-7x+1}{x^2-x-2}, x \neq 2, x \neq -1$
7. $\frac{33x^2+11x+13}{6x^2-7x-3}, x \neq \frac{3}{2}, x \neq \frac{-1}{3}$
8. $\frac{8x+17}{24}$ km
9. $\frac{4x^2+35x-3}{x+7}$ hectares, $x > -7$
10. $\frac{12x^2+40x+32}{5x+15}$ cm, $x > -3$

Warm-Up 2B.1.3, p. U2B-45

1. $\frac{7}{12}$ cm^2
2. $\frac{289}{16}$ cm^2 or $18\frac{1}{16}$ cm^2
3. $\frac{3x}{8}$ m^2

Practice 2B.1.3: Multiplying Rational Expressions, p. U2B-59

1. $\frac{5}{2x^2+3}, x \neq 0$
2. $\frac{2x^2+7x}{x^2-2x-3}, x \neq 3, x \neq -1$
3. $\frac{x^3+3x^2+x+3}{x^2-4x}, x \neq 4, x \neq 0$
4. $\frac{x+2}{15}, x \neq 0$
5. $\frac{x+2}{5-x}, x \neq 5$
6. $x \neq -5, x \neq -2$

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7. $\frac{5x+5}{x} \bullet \frac{x^3+3x^2}{x^2-1} \bullet \frac{x-1}{5x} = \frac{\cancel{5}\,(x+1)}{\cancel{x}} \bullet \frac{\cancel{x}\,(x+3)}{(x-1)\,\cancel{(x+1)}} \bullet$

$\frac{\cancel{(x-1)}}{\cancel{5x}} = \frac{x+3}{1} = x+3,\ x \neq -1, x \neq 1, x \neq 0$

8. $\frac{2x^2+x}{2x-10}$ inches^2, $x > 5$

9. $\frac{x^2+4x+4}{x-3}$ g • cm/s, $x > 3$

10. $\frac{7x^2+42x}{x+7}$ meters^3, $x > 5$

Warm-Up 2B.1.4, p. U2B-61

1. $\frac{25}{6}$ cm or $4\frac{1}{6}$ cm
2. $\frac{8}{9}$ m
3. Each side of the square measures $(2x + 3)$ cm.

Practice 2B.1.4: Dividing Rational Expressions, p. U2B-81

1. $\frac{2x^2}{7}, x \neq 0$
2. $\frac{3}{25}, x \neq 0$
3. $\frac{72x+27}{5x}, x \neq 0$
4. $\frac{2x^2-8}{x+x^2}, x \neq 0, x \neq -1, x \neq -2$
5. $\frac{x^2-2x-15}{2x^2-8x} \bullet \frac{32-2x^2}{x^2-13x+40} \div \frac{x+4}{8x-x^2}$

$= \frac{x^2-2x-15}{2x^2-8x} \bullet \frac{32-2x^2}{x^2-13x+40} \bullet \frac{8x-x^2}{x+4}$

$= \frac{(x-5)(x+3)}{2x(x-4)} \bullet \frac{2(16-x^2)}{(x-8)(x-5)} \bullet \frac{x(8-x)}{x+4}$

$= \frac{(x-5)(x+3)}{2x(x-4)} \bullet \frac{2(4-x)(4+x)}{(x-8)(x-5)} \bullet \frac{x(8-x)}{x+4}$

$= \frac{\cancel{(x-5)}\,(x+3)}{\cancel{2x}\,(x-4)} \bullet \frac{\cancel{2}(4-x)\,(4+x)}{(x-8)\,\cancel{(x-5)}} \bullet \frac{\cancel{x}\,(8-x)}{\cancel{x+4}}$

$= \frac{(x+3)}{\cancel{(x-4)}} \bullet \frac{-1\,\cancel{(x-4)}}{\cancel{(x-8)}} \bullet \frac{-1\,\cancel{(x-8)}}{1} = x+3,$

$x \neq -4, x \neq 0, x \neq 4, x \neq 5, x \neq 8$

6. $(2x+3)$ or $(2x+3)+\frac{0}{x-5}, x \neq 5$
7. $(x-9)+\frac{8}{x+1}, x \neq -1$
8. $\frac{2x+4}{3}$ kilometers per hour, $x > 2$
9. $\frac{15x^2+25x}{x^2-16}$ grams per meter^3, $x > 4$
10. $\frac{2x^3-8x^2+16x-10}{x^3-2x^2-5x+10}$ meters, $x > \sqrt{5}$

Progress Assessment, p. U2B-83

1. a
2. d
3. c
4. a
5. b
6. a
7. d
8. b
9. c
10. b

11. a. The rectangle has a perimeter of $\frac{8x^3+22x^2-10x-150}{x^2+5x}$ cm, where $x > 0$.

b. The rectangle has an area of $(3x^2 - 5x - 12)$ cm^2 with $x > 1$.

Lesson 2: Solving Rational and Radical Equations

Pre-Assessment, p. U2B-86

1. a
2. b
3. d
4. c
5. d

Warm-Up 2B.2.1, p. U2B-90

1. 45 minutes
2. 30 minutes

Practice 2B.2.1: Solving Rational Equations, p. U2B-109

1. $x = 9$ and $x = 7$
2. $x = -\frac{3}{7}$
3. $x = -1$
4. $\frac{t}{3} + \frac{t}{2} = 1$
5. No; it results in a denominator of 0.
6. ≈ 3.73 hours
7. 7 and –9
8. 10 hours by pipe; 15 hours by hose
9. $\frac{5}{3}$ hours
10. passenger train: 86 mph; freight train: 46 mph

Warm-Up 2B.2.2, p. U2B-110

1. 3.9 ft
2. 3.12 ft
3. ≈ 4.24 ft

Practice 2B.2.2: Solving Radical Equations, p. U2B-128

1. $x = 39$
2. $x = 3$

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3. $x = -2$
4. 0.5 units^2
5. 15 inches
6. 256 feet
7. ≈ 3.2 seconds
8. Charlie is not being truthful: if he had actually been driving the speed limit of 35 mph, the skids would have measured only about 58 feet, rather than the 75-foot marks measured at the scene.
9. 240 watts
10. 6 units

Warm-Up 2B.2.3, p. U2B-130

1. $y = \sqrt{169 - x^2}$
2. $y = 17 - x$
3.

x	1	2	3	4	5	6	7	8	9	10	11	12
Value of y based on equation from problem 1	12.96	12.85	12.65	12.37	12.00	11.53	10.95	10.25	9.38	8.31	6.93	5.00
Value of y based on equation from problem 2	16	15	14	13	12	11	10	9	8	7	6	5

4. By looking at the table, we can see that when the value of x is equal to 5, then the value of y for both equations is equal to 12. It is also true that when the value of x is equal to 12, then the value of y for both equations is equal to 5. These points represent the lengths of the triangles, 5 inches and 12 inches, when the length of the hypotenuse is equal to 13 inches and the perimeter is equal to 30 inches.

Practice 2B.2.3: Solving Systems of Equations, p. U2B-156

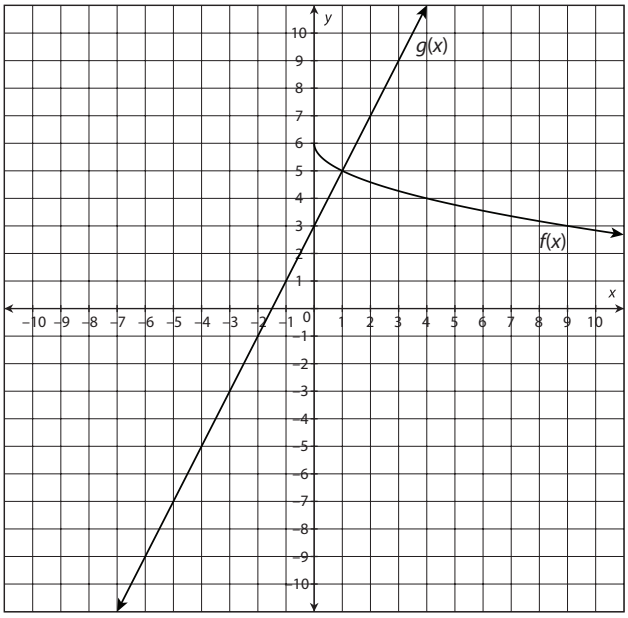
1. {(–0.28, –5.83), (2, 1)}
2. {(1, 6), (4, 7)}
3. {(8, 3), (2, 4.5)}
4. {(–3, –3), (1, 1)}
5. At the beginning of the table, $g(x) > f(x)$. However, $f(x)$ and $g(x)$ swap positions between –4 and –3, so that $g(x) < f(x)$. There may be an intersection (and therefore a solution to the system) when $-4 \le x \le -3$. $g(x) < f(x)$ until $x = 0$, suggesting another intersection when $-1 \le x \le 0$. There is another sign change when $2 \le x \le 3$. There appear to be three solutions to the system in this range.
6. If the system has no solution, there are no points of intersection. Since both equations are lines, they must be parallel.
7. Debbie is not correct. The point (–1, 3) does not satisfy both equations; it only satisfies the first equation. A detailed examination of the graph or table reveals that the functions only intersect at the point (1, 1), not at the point (–1, 3).
8. no real solution
9. The legs of the triangle measure 12 cm and 35 cm, yielding an area of 210 cm^2.
10. A length of approximately 0.252 m results in a period of approximately 1.01 seconds.

Progress Assessment, p. U2B-158

1. d
2. b
3. b
4. c
5. b
6. a
7. a
8. a
9. c
10. b
11. a. {(2.80, 974), (14.27, 1892)}
    b. The solution set represents each length of the pendulum that results in the company breaking even.

Unit Assessment p. U2B-161

1. b
2. a
3. d
4. c
5. a
6. d
7. c
8. b
9. a
10. d
11. c
12. a
13. It will take the brothers approximately 4 hours to complete the job together.
14. a. $\dfrac{8x^2 + 30x}{x + 6}$ centimeters
    b. $\dfrac{4x^3 + 6x^2}{x + 6}$ centimeters^2
15. a. $f(1) = 6 - \sqrt{(1)} = 5$; $g(1) = 2(1) + 3 = 5$
    b.



    c. There are no other points of intersection.
    d. The point of intersection located at (1, 5) represents the solution to the defined system of equations.

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Student Book Answer Key

Lesson 1: Operating with Rational Expressions

Practice 2B.1.1: Structures of Rational Expressions, p. U2B-12

1. $\frac{4x}{3}, x \neq 0$
2. $-\frac{5}{8}, x \neq 9$
3. $\frac{1}{2x^3 - 3x}, x \neq 0, x \neq \pm\sqrt{\frac{3}{2}}$
4. $\frac{2x^2}{x+3}, x \neq -3$
5. $\frac{1}{2x+1}, x \neq -\frac{1}{2}$
6. $\frac{8x^2 - 3x}{x-3}, x \neq -5, x \neq 3, x = 0$
7. $\frac{54x + 249x^2 + 27x^3}{18x^3 + 40x^2 + 8x} = \frac{3x(18 + 83x + 9x^2)}{2x(9x^2 + 20x + 4)} =$
   $\frac{3\cancel{x}(x+9)\cancel{(9x+2)}}{2\cancel{x}\cancel{(9x+2)}(x+2)} = \frac{3(x+9)}{2(x+2)} = \frac{3x+27}{2x+4}$
8. $\frac{3x}{x+1}$ meters, $x > 0$
9. $(3x - 9)$ kilometers per minute, $x > 3$
10. $12x + 12$ days; $x > -1$

Practice 2B.1.2: Adding and Subtracting Rational Expressions, p. U2B-25

1. $\frac{x}{7}$
2. $\frac{3x^3 + 2x + 1}{x^2}, x \neq 0$
3. $\frac{2x^2 + 50}{x^2 - 25}, x \neq \pm 5$
4. $\frac{8x^2 + 8x - 15}{5x + 5}, x \neq -1$
5. $\frac{2x^3 - x^2 + 5x - 1}{x^2 + 1}$
6. $\frac{x^2 - 4x - 27}{x^2 - 12x + 35}, x \neq 5, x \neq 7$
7. $\frac{9x^2 + 8x - 38}{6x^2 - 19x + 10}, x \neq \frac{5}{2}, x \neq \frac{2}{3}$
8. $\frac{x+5}{x-3}$ cookies, $x > 3$
9. $\frac{7x^3 + 7x^2 + 7x + 2}{2x^2 + x}$ centimeters, $x > 1$
10. $\frac{x^2 - 5x - 18}{x-6}$ meters, $x > 6$

Practice 2B.1.3: Multiplying Rational Expressions, p. U2B-33

1. $\frac{2x^2 + 9x + 10}{2x}, x \neq 0$
2. $\frac{3x^3 + 9x^2 + 21x}{x^2 + x - 20}, x \neq 4, x \neq -5$
3. $\frac{5}{x-1}, x \neq -8, x \neq -\frac{3}{2}, x \neq 1$
4. $\frac{x+1}{5}, x \neq 0, x \neq 9$
5. $\frac{2x-3}{x^2+1}, x \neq -4, x \neq -3, x \neq -1$
6. $x \neq \pm 4$
7. $\frac{x^2 - 2x}{9 + 3x} \bullet \frac{3x+6}{x^2-4} = \frac{x\cancel{(x-2)}}{\cancel{3}(3+x)} \bullet \frac{\cancel{3}\cancel{(x+2)}}{\cancel{(x+2)}\cancel{(x-2)}} = \frac{x}{3+x} = \frac{x}{x+3}$, $x \neq -3$
8. $\frac{9x^3 + 12x}{x+4}$ Newtons, $x \neq -4$
9. $\frac{5x^2 + 10x + 5}{x^3 - 3x^2}$ centimeters^2, $x > 3$
10. $\frac{x+1}{x^2}$ meters^3, $x \neq -5, x \neq 4, x \neq 0, x \neq 4$

Practice 2B.1.4: Dividing Rational Expressions, p. U2B-45

1. $\frac{6x - 21}{5x}, x \neq 0$
2. $\frac{9}{x-3}, x \neq 3$
3. $\frac{5x^2 - x}{2x+6}, x \neq -3, x \neq 0$
4. $\frac{x^2 - 3x}{x^2 - 7x - 18}, x \neq -2, x \neq 3, x \neq 9$

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5. $\frac{15x^2+27x-6}{x^2+x-2} \div \frac{5x-1}{3x-3} = \frac{15x^2+27x-6}{x^2+x-2} \bullet \frac{3x-3}{5x-1} =$

$\frac{3\cancel{(x+2)}\ \cancel{(5x-1)}}{\cancel{(x+2)}\ \cancel{(x-1)}} \bullet \frac{3\cancel{(x-1)}}{\cancel{(5x-1)}} = 9$

6. $(3x^2+1)$ or $(3x^2+1)+\frac{0}{x-2}, x \neq 2$

7. $(x+4)+\frac{11}{x-1}, x \neq 1$

8. $\frac{2x^2-4x}{7x-4}$ hours, $x > 3$

9. $\frac{x^2-9}{x^2-2x}$ grams per meter^3, $x > 3$

10. $\frac{x-1}{x+6}$ pies, $x > 1$

Lesson 2: Solving Rational and Radical Equations

Practice 2B.2.1: Solving Rational Equations, p. U2B-61

1. $x = 20$
2. $x = -3$ or $x = -8$
3. $x = -4$
4. $x = \frac{6}{7}$ hour or ≈ 51.4 minutes
5. 4 and 6
6. 240 minutes (4 hours)
7. $x = 3\frac{9}{13}$
8. It takes Emily 3.75 hours (3 hours 45 minutes) to climb the cliff.
9. 6 hours
10. $v = 8$ km/h

Practice 2B.2.2: Solving Radical Equations, p. U2B-73

1. $x = -6$
2. $x = 8$
3. $x = -5$
4. The other leg measures 156 mm.
5. $v = 2.4$ m/s
6. $L = 21.75$ ft
7. $g = 9.8$ m/s^2
8. Each side of the base measures 18 cm.
9. The pendulum length is about 20 m.
10. Mars is about 1.53 AU from the Sun.

Practice 2B.2.3: Solving Systems of Equations, p. U2B-94

1. {(1, 2)}
2. {(2.7, 1.35), (–3.7, –1.85)}
3. {(7, 4)}
4. {(–1, 2), (0.33, 4.67)}
5. Throughout this range of data, $g(x) < f(x)$. While the difference decreases around $x = 2$, $g(x)$ never becomes greater than $f(x)$. Since the graphs never appear to cross, there appears to be no solution in this range. It is possible that there is a pair of solutions between any two given x-values: however, this does not seem to agree with what appear to be smooth, consistent patterns.
6. The first equation is a linear equation, and the second is a radical equation. It is not possible for the graphs of a linear equation and a radical equation to have more than two intersection points. There is, therefore, no way for the system to have more than two solutions. It is possible that there is only one solution, or even none—but not more than two.
7. The answer key is incorrect. The point (1, 1) fails to satisfy the first equation. Therefore, it cannot be a solution to the system.
8. (5, 1); approximately (–2.35, 2.6) and (–0.088, –9.73)
9. The legs of the triangle measure 33 ft and 56 ft, yielding an area of 924 ft^2.
10. The resistance is approximately 379 ohms, and the power is approximately 37.9 watts.

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Instruction

Goal: To simplify, add, subtract, multiply, and divide rational expressions, and solve rational equations

Common Core State Standards

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

A–APR.6 Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

A–APR.7 (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

A–REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Student Activities Overview and Answer Key

Station 1

Students will find three sets of index cards at this station: "Original Expression" cards, "Simplified Expression" cards, and "Restrictions on the Variable(s)" cards. Students will simplify each rational expression as a group. They will then match each original expression with its correct "Simplified Expression" card and "Restrictions on the Variable(s)" card.

Answers

Original expression	Simplified expression	Restrictions on the variable(s)
1. $\dfrac{4x}{16x^2+8x}$	G. $\dfrac{1}{4x+2}$	AA. $x \neq 0, x \neq -\dfrac{1}{2}$
2. $\dfrac{x^2-y^2}{(x-y)^2}$	D. $\dfrac{x+y}{x-y}$	CC. or GG. $x \neq y$
3. $\dfrac{x^4-y^4}{3x-3y}$	J. $\dfrac{x^3+xy^2+x^2y+y^3}{3}$	CC. or GG. $x \neq y$
4. $\dfrac{x^4-1}{1-x^2}$	A. $-x^2-1$	II. $x \neq -1, x \neq 1$
5. $\dfrac{3y^2-12y+12}{3y^2+12y-36}$	H. $\dfrac{y-2}{y+6}$	EE. $y \neq -6, y \neq 2$

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Instruction

Station 2

Students will create addition and subtraction problems involving rational expressions. Students will roll a number cube to select the numerators and denominators from a table of expressions that corresponds to each number on the cube. They will then simplify the expressions and compare and discuss answers with group members.

Answers

1–6. Answers will vary according to the numbers rolled. Check the answers for each student-generated problem for accuracy. Sample answer: If the numbers rolled are 4, 6, 5, and 1, the following rational expressions result:

$$\frac{\text{numerator}\#4}{\text{denominator}\#5}+\frac{\text{numerator}\#6}{\text{denominator}\#1}=\frac{2x}{4x^2+8x+4}+\frac{2x^2}{x+1}=\frac{4x^3+4x^2+x}{2(x+1)^2}$$

Station 3

Students will find a list of simplified expressions at this station. Each student will choose one, and then make up a problem involving either multiplication or division of rational expressions that will result in the chosen expression. Group members will swap papers and simplify one another's problems. Students will then make up a new problem, swap papers, and simplify the next problem, repeating this process until all 4 problems have been created and simplified.

Answers

1–4. Student-generated problems will vary, but should result in the chosen expression. Sample answer: For an answer of $\frac{2}{3x}$, one multiplication problem that yields this expression is $\frac{2(x+1)(x+1)}{3(x+1)(x+2)}\bullet\frac{x+2}{x(x+1)}$. This problem becomes $\frac{2x^2+4x+2}{3x^2+9x+6}\bullet\frac{x+2}{x^2+x}$.

Station 4

At this station, group members will split into two teams. Teams will each choose a card with a rational equation on it, and determine the number of solutions to the equation. Teams score points equal in number to the number of solutions for the equation solved. Play continues until all equations have been solved. The team with a score closest to 0 at the end of the game wins.

Answers

1. no solutions
2. $x = -1$
3. no solutions
4. $x = 4, x = -3$
5. $x = \frac{3}{2}$
6. $x = 3, x = -4$
7. $x = \frac{5}{2}$
8. $x = 9$
9. $x = 1$
10. $x = 2, x = -8$

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Instruction

Materials List/Setup

Station 1 3 sheets of card stock in different colors, for printing out the Station 1 cards from the Card Master Sheets that follow. There are 25 cards in all, with the following expressions and restrictions:

- Original expressions:

1. $\frac{4x}{16x^2+8x}$ 2. $\frac{x^2-y^2}{(x-y)^2}$ 3. $\frac{x^4-y^4}{3x-3y}$ 4. $\frac{x^4-1}{1-x^2}$ 5. $\frac{3y^2-12y+12}{3y^2+12y-36}$

- Simplified expressions:

A. $-x^2-1$ B. $\frac{x^2+y^2}{3}$ C. $x-y$ D. $\frac{x+y}{x-y}$ E. x^2

F. $-\frac{1}{3}$ G. $\frac{1}{4x+2}$ H. $\frac{y-2}{y+6}$ I. $\frac{1}{16x^2+2}$ J. $\frac{x^3+xy^2+x^2y+y^3}{3}$

- Restrictions on the variable(s):

AA. $x\neq 0, x\neq -\frac{1}{2}$ BB. $x\neq -\frac{1}{2}$ CC. $x\neq y$ DD. $x\neq 0$ EE. $y\neq -6, y\neq 2$

FF. $y\neq 2x$ GG. $x\neq y$ HH. $y\neq -6$ II. $x\neq -1, x\neq 1$ JJ. $x\neq 1$

Station 2 six-sided number cube

Station 3 white paper

Station 4 10 index cards, with the following equations written on them. *Note*: These equations are also included in the Card Master Sheets for your convenience.

1. $\frac{3}{2x-4}=\frac{6}{x^2-4}$
2. $\frac{x}{x-3}+\frac{1}{x-4}=\frac{1}{x^2-7x+12}$
3. $\sqrt{2x}=\sqrt{x-5}$
4. $\frac{1}{3x}-\frac{1}{3}=-\frac{4}{x^2}$
5. $\sqrt{x^2+4}=x+1$
6. $x+1=\frac{12}{x}$
7. $\frac{5}{2x+9}=\frac{5}{4(x+1)}$
8. $\sqrt{x}=\sqrt{x-8}+2$
9. $\sqrt[3]{3x+5}=2$
10. $3-x=\sqrt{2x^2-7}$

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Instruction

Card Master Sheets: Station 1

There are three sets of cards for this station: Original Expressions, Simplified Expressions, and Restrictions on the Variable(s). Copy each set of cards onto a different color of card stock, then cut them out. You may wish to write the name of the type of card on the opposite side.

Original Expressions

1. $\dfrac{4x}{16x^2+8x}$

2. $\dfrac{x^2-y^2}{(x-y)^2}$

3. $\dfrac{x^4-y^4}{3x-3y}$

4. $\dfrac{x^4-1}{1-x^2}$

5. $\dfrac{3y^2-12y+12}{3y^2+12y-36}$

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Instruction

Simplified Expressions

A. $-x^2 - 1$	B. $\frac{x^2 + y^2}{3}$	C. $x - y$
D. $\frac{x + y}{x - y}$	E. x^2	F. $-\frac{1}{3}$

(*continued*)

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Instruction

Simplified Expressions

G.

$$\frac{1}{4x+2}$$

H.

$$\frac{y-2}{y+6}$$

I.

$$\frac{1}{16x^2+2}$$

J.

$$\frac{x^3+xy^2+x^2y+y^3}{3}$$

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Instruction

Restrictions on the Variable(s)

AA. $x \neq 0, x \neq -\frac{1}{2}$	BB. $x \neq -\frac{1}{2}$	CC. $x \neq y$
DD. $x \neq 0$	EE. $y \neq -6, y \neq 2$	FF. $y \neq 2x$

(*continued*)

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Instruction

Restrictions on the Variable(s)

GG. $x \neq y$	HH. $y \neq -6$
II. $x \neq -1, x \neq 1$	JJ. $x \neq 1$

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Card Master Sheets: Station 4

Prepare these 10 game cards for the teams at this station. You may wish to copy or print these equations onto card stock, or make copies that you can cut out and paste onto index cards.

1. $\frac{3}{2x-4}=\frac{6}{x^2-4}$	2. $\frac{x}{x-3}+\frac{1}{x-4}=\frac{1}{x^2-7x+12}$	3. $\sqrt{2x}=\sqrt{x-5}$
4. $\frac{1}{3x}-\frac{1}{3}=-\frac{4}{x^2}$	5. $\sqrt{x^2+4}=x+1$	6. $x+1=\frac{12}{x}$

(*continued*)

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7. $\frac{5}{2x+9}=\frac{5}{4(x+1)}$

8. $\sqrt{x}=\sqrt{x-8}+2$

9. $\sqrt[3]{3x+5}=2$

10. $3-x=\sqrt{2x^2-7}$

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Instruction

Discussion Guide

To support students in reflecting on the activities and synthesizing their ideas, and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. How are simplifying rational expressions and multiplying rational expressions alike? Explain.
2. Are there any benefits to factoring before multiplying? If so, what are they?
3. Which do you think is easier, adding rational expressions or multiplying them? Explain.
4. How do you determine the domain for a rational function?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. For both simplifying and multiplying rational expressions, you begin by factoring the numerators and denominators and crossing out common factors.
2. By factoring first, you can simplify the expressions before multiplying. If you multiply first, it will be more difficult to factor these larger expressions, so you risk having an answer that is not in simplest form.
3. Answers will vary. Sample answer: Multiplying rational numbers is easier because you do not need to first find a common denominator.
4. The domain is all real numbers except for those that will result in a denominator of 0 in the original equation.

Possible Misunderstandings/Mistakes

- Thinking that the domain is found by evaluating the simplified expression rather than the original expression
- Trying to simplify rational expressions by crossing off similar terms instead of similar factors
- Forgetting to check for extraneous solutions when solving rational equations

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Station 1

At this station, you will find three sets of cards:

- 5 "Original Expression" cards with rational expressions, numbered 1–5
- 10 "Simplified Expression" cards, lettered A–J
- 10 "Restrictions on the Variable(s)" cards, lettered AA–JJ

You will work with your group to simplify the five original expressions. First, have one group member select an "Original Expression" card and turn it face up so everyone can see it. Then, work together to simplify the expression. Determine the matching "Simplified Expression" card and the correct "Restrictions on the Variable(s)" card. Record the letters and expressions for the matching cards in the table below. Take turns selecting "Original Expression" cards until all the "Original Expression" cards have been selected. Note that there are more cards to match than you will need, and two of the restriction cards are identical.

Original expression	Simplified expression	Restrictions on the variable(s)
1. $\frac{4x}{16x^2+8x}$		
2. $\frac{x^2-y^2}{(x-y)^2}$		
3. $\frac{x^4-y^4}{3x-3y}$		
4. $\frac{x^4-1}{1-x^2}$		
5. $\frac{3y^2-12y+12}{3y^2+12y-36}$		

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Station 2

At this station, you will find a six-sided number cube. The table below contains six "numerator" expressions and six "denominator" expressions. The expressions correspond to each number on the number cube. You will use the number cube and table to create three rational addition problems that you will then simplify.

To create each addition problem, roll the number cube four times. Use the first two rolls to choose the two numerators and use the next two rolls to choose the two denominators for the expressions to be added. Record each addition problem on the next page. Make sure each group member writes the same problem. Each group member will simplify each problem individually, and then compare answers. Discuss the answers until all group members agree on the result and understand any errors; then, record your results.

Example

If you roll a 3 and a 6 to choose the numerators, and then roll a 1 and a 2 to choose the denominators, the resulting addition problem would be $\frac{x}{x+1}+\frac{2x^2}{2x+2}$.

Numerators		**Denominators**	
Number rolled	**Numerator**	**Number rolled**	**Denominator**
1	1	1	$x + 1$
2	2	2	$2x + 2$
3	x	3	$x^2 + x$
4	$2x$	4	$x^2 + 2x + 1$
5	x^2	5	$4x^2 + 8x + 4$
6	$2x^2$	6	$x^2 + 3x + 2$

continued

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Station Activities Set 1: Rational Expressions and Equations

Roll the number cube four times and record each addition problem. Once all group members have recorded these problems, work individually to simplify the expressions. Then, compare your results.

1.

2.

3.

Rewrite the problems you created in 1–3, using the operation of subtraction instead of addition. Work on your own to simplify the new problems and then compare your answers with the group. Discuss the answers until all group members agree on the result and understand any errors; then, record your results.

4.

5.

6.

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Station 3

The following are eight simplified expressions.

1. $\frac{1}{x+1}$
2. 1
3. $x+9$
4. $\frac{x-4}{x+3}$
5. $\frac{2}{3x}$
6. $\frac{2x}{x+2}$
7. $\frac{x+7}{x-2}$
8. $\frac{x+4}{9x}$

Select one expression and then write a problem involving multiplying or dividing rational expressions that will result in the chosen simplified expression. Each group member should choose a different expression and write a different problem.

Once everyone has written a problem, swap papers so each person in the group has a new paper. Simplify the problem your group member wrote, and show your work. Write your initials beside the problem number so it's clear which group member simplified that problem.

Then, choose a new expression from the list above and write a new problem on your group member's paper. Once everyone is ready, swap papers again. Check your group member's work for the previous problem, and then simplify the next problem. Repeat this process until all problems 1–4 have been written, simplified, and checked.

Example

Suppose you chose option 6: $\frac{2x}{x+2}$. A problem that results in this simplified expression is $\frac{x(x-7)}{(x+2)(x+4)} \bullet \frac{2(x+4)}{x-7}$. To make it trickier for your group member, multiply the factors and write the problem as $\frac{x^2-7x}{x^2+6x+8} \bullet \frac{2x+8}{x-7}$.

continued

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Based on the simplified expression you chose, write a problem that involves multiplying or dividing rational expressions in the appropriate space below. Swap papers with your group members, and then simplify the problem your group member wrote. Show your work and write your initials next to the number of the problem you simplified.

1. Problem:

2. Problem:

3. Problem:

4. Problem:

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Station 4

At this station, you will find a set of cards, each labeled with a different equation. Divide your group into two teams, shuffle the cards, and then follow the guidelines and rules to play this equation game.

How to Play

Place the cards face down in a pile. Each team takes turns choosing a card and solving the equation on the card. Record the problems and the number of solutions in the table provided. Your team gains 1 point for each solution an equation has; that is, an equation with 1 solution is worth 1 point and an equation with 2 solutions is worth 2 points. An equation with no solutions brings your total score back down to 0. Continue to add or subtract points from round to round based on the number of solutions. Once all the cards have been chosen and solved, the team with the score closest to 0 is the winner.

Rules

- Players may challenge their opponents by checking the other team's work.
- There is no penalty for challenging an answer that is found to be correct. If an answer is challenged and found to be incorrect, the team with the incorrect answer must have their score revised based on the correct answer.
- Extraneous solutions do not count toward your score.
- Remember: If the equation has no solutions, all previous points are canceled, and your score is set to 0.

Team 1		
Problem	**Number of solutions**	**Score**
Final score		

Team 2		
Problem	**Number of solutions**	**Score**
Final score		

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Instruction

Goal: To investigate several methods for solving non-linear systems of equations

Common Core State Standard

A–REI.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★

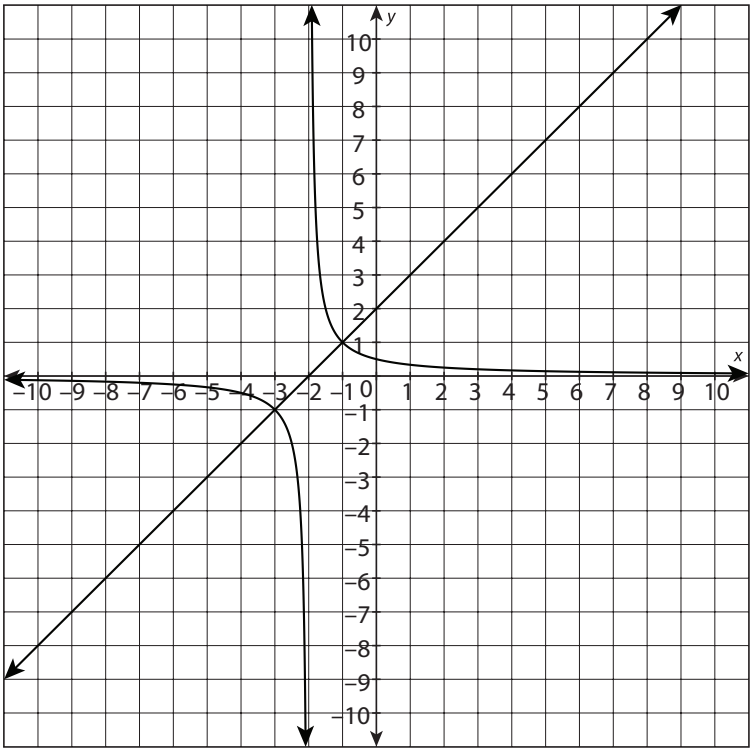
Student Activities Overview and Answer Key

Station 1

Students will form two teams. Instruct students to solve the systems of equations by hand or on a graphing calculator, and then estimate the points of intersection. Teams earn points according to how close their estimated points of intersection are to the actual points of intersection.

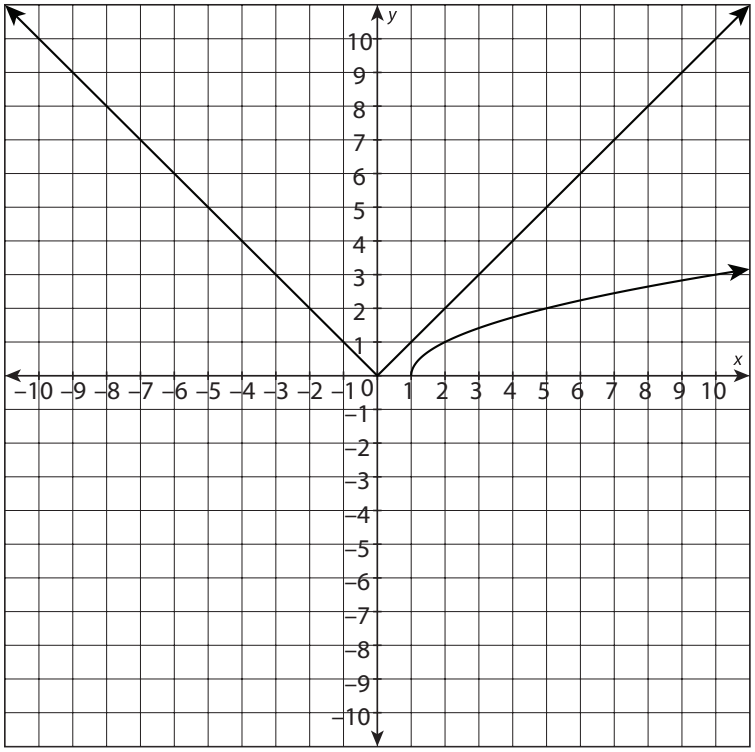
Answers

1. (–1, 1) and (–3, –1)

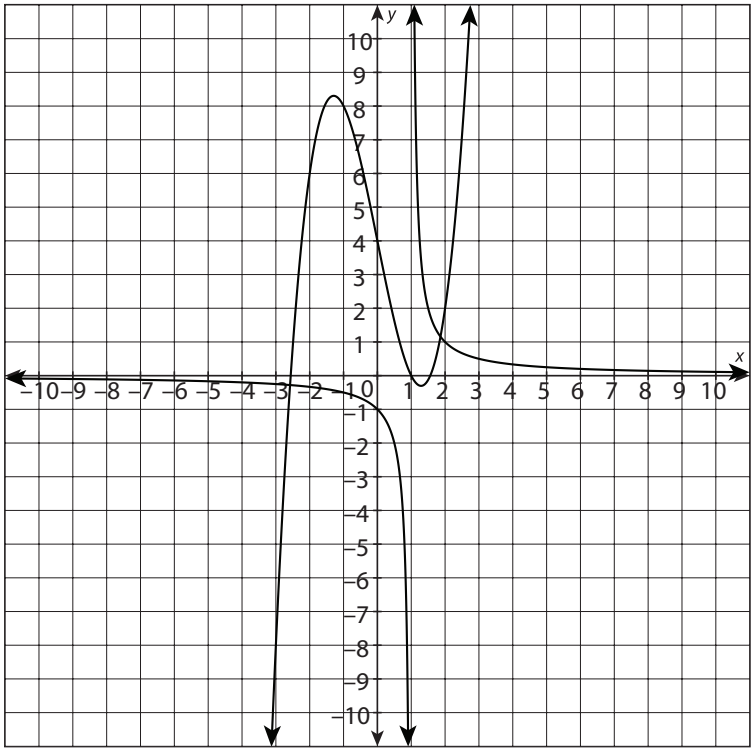


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2. no solutions

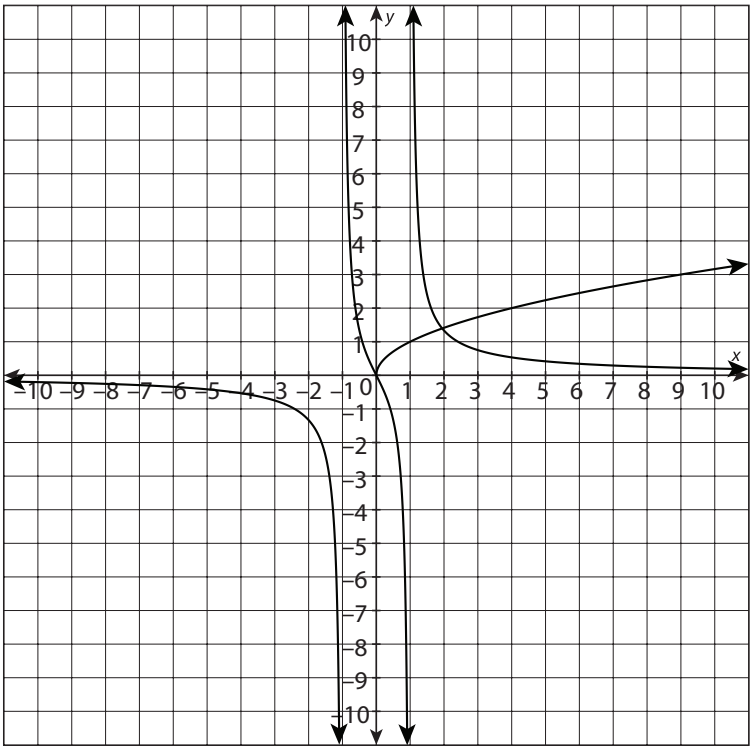


3. (–2.6, –0.3) and (1.9, 1.2)



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4. (0, 0) and (1.9, 1.4)



Station 2

Students will approximate the solutions to systems of equations by making tables of values, or by analyzing tables of values on a graphing calculator to see which *x*-value produces *y*-values that are approximately equal for both functions.

Answers

1. (1.16, 6.4)
2. (0.24, –2.96)
3. (0.75, 0.75)
4. (1, 2)

Station 3

Students will divide into two teams to solve systems of equations both algebraically and by using a graphing calculator. Six systems are provided; one team will solve the first three systems algebraically, and the other three systems by graphing. The other team will solve the first three systems by graphing, and the other three systems algebraically. Teams will then compare answers.

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Answers

1. $\left(\frac{4}{3}, \frac{5}{3}\right)$
2. (5.1, 7.2) and (0.9, –1.2)
3. no solutions
4. $\left(-\frac{1}{2}, -2\right)$
5. (1.7, 2) and (–1.7, 2)
6. (3, 0) and (4, 1)

Station 4

Students solve systems of equations by graphing the functions on graphing calculators and finding the point(s) of intersection.

Answers

1. (–1, 0) and (5.6, 2.6)
2. (–2.2, 2.1), (–1, 4), and (0.4, 3.3)
3. no solutions
4. (–4.2, –0.1), (–2.3, 0.9), and (1.5, 2.7)
5. (–30.9, 9.9)
6. By understanding the expected shape of each function, you will know if there are any "hidden curves" that don't show up in the viewing window. Be sure to adjust the viewing window so that all important features of each function are showing. For example, if you graph a quadratic function, but in the viewing window it looks like a line, you know that you need to zoom out because a quadratic function should be U-shaped.

Materials List/Setup

All Stations graphing calculators

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Instruction

Discussion Guide

To support students in reflecting on the activities and synthesizing their ideas, and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. What are some advantages and disadvantages of each of the following methods of solving systems of equations: graphing by hand, solving with tables, solving algebraically, and graphing on the calculator?
2. Which method do you prefer and why?
3. Why do you have to be sure to check all solutions found algebraically?
4. How can you be sure not to miss any solutions when solving a system of equations on your graphing calculator?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. Graphing by hand takes more time; also, some functions are difficult to graph by hand, and solutions will only be approximate, but this method can be done without a calculator. Solving with tables is time-consuming, but it leads to a good understanding of the process of successive approximations. Solving algebraically might lead to equations that are difficult to solve, and many answers found this way are extraneous solutions; however, it does give the exact solution(s). Graphing on the calculator requires setting the window size correctly in order to see some solutions, but in general this method is quick and accurate.
2. Answers will vary, but should include justification.
3. Solving systems of equations algebraically often leads to extraneous solutions, so each answer should be checked in both original equations to ensure the solution is valid.
4. Think about what the given functions should look like, and make sure you see these shapes in your window. If your window is too small, some curves of your function (and, therefore, possible solutions) could be hidden from view.

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Instruction

Possible Misunderstandings/Mistakes

- Not checking for extraneous solutions when solving a system of equations algebraically
- Setting an inappropriate window size on the graphing calculator, such that not all solutions are visible
- Incorrectly entering equations into the graphing calculator
- Making errors when solving systems of equations algebraically
- Graphing the functions incorrectly when graphing by hand, thus leading to an incorrect solution

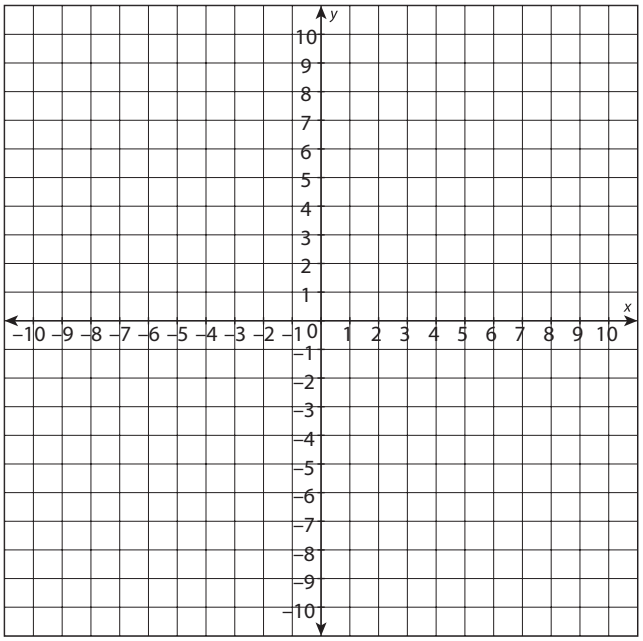
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Station 1

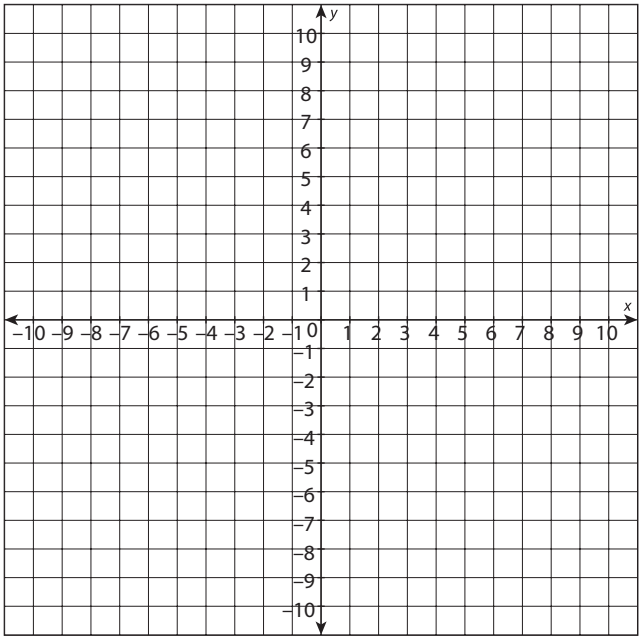
Divide your group into two teams. Each team will solve the following systems of equations by graphing, and then estimate the points of intersection. When all four systems have been solved, check the other team's work. The team whose estimated points of intersection are closest to the actual solution for each system scores a point. The team with the most points at the end wins.

Graph each system of equations that follows, by hand or using a graphing calculator. If using a graphing calculator, sketch your graph on the coordinate plane provided.

1. $\begin{cases} y = x + 2 \\ y = \dfrac{1}{x+2} \end{cases}$



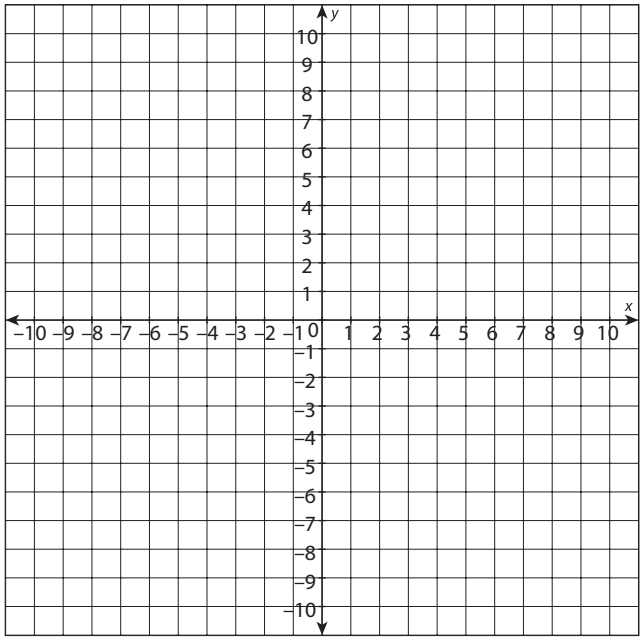
2. $\begin{cases} y = \sqrt{x-1} \\ y = |x| \end{cases}$



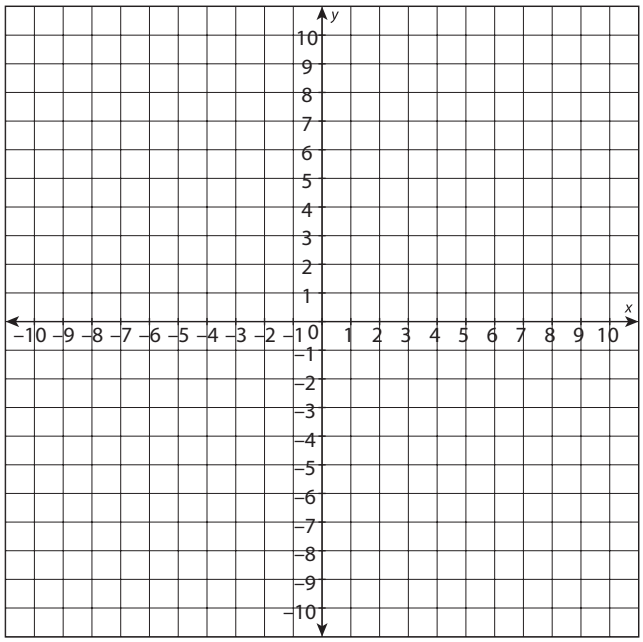
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3. $\begin{cases} y = \dfrac{1}{x-1} \\ y = x^3 - 5x + 4 \end{cases}$



4. $\begin{cases} y = \dfrac{2x}{x^2-1} \\ y = \sqrt{x} \end{cases}$



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Station 2

All of the systems of equations on the next page have a solution with an x-value between –3 and 3. Use tables to approximate this solution to the nearest hundredth. You may look at the graph on a graphing calculator to make your first guess. *Note*: The systems of equations may have other solutions as well.

Example

Given the system $\begin{cases} f(x)=\dfrac{x+1}{x^2-4} \\ g(x)=-\dfrac{1}{4}x-1 \end{cases}$, graphing these two functions on a graphing calculator shows that the point of intersection is between 1 and 2 on the x-axis.

Calculate the y-values of each function for x-values between 1 and 2. Determine when the y-values are closest together in value.

x	$f(x)$	$g(x)$
1.4	–1.18	–1.35
1.5	–1.43	–1.38
1.6	–1.81	–1.40

Of these three x-values, the y-values for $x = 1.5$ are the closest together. To get a more accurate approximation, calculate the y-values for numbers close to 1.5.

x	$f(x)$	$g(x)$
1.48	–1.37	–1.37
1.49	–1.40	–1.37
1.50	–1.43	–1.38
1.51	–1.46	–1.38
1.52	–1.49	–1.38

When x equals 1.48, both $f(x)$ and $g(x)$ are approximately equal to –1.37. The approximate solution to the system is (1.48, –1.37).

continued

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Fill in the table of values for each system of equations, then use the methods previously demonstrated to find an approximate solution that falls between –3 and 3 on the *x*-axis.

1. $\begin{cases} f(x)=\dfrac{1}{x-1} \\ g(x)=x^2-6x+12 \end{cases}$

x	*f*(*x*)	*g*(*x*)

3. $\begin{cases} f(x)=\dfrac{x^2-3x+2}{x^3} \\ g(x)=\left|x+1\right|-1 \end{cases}$

x	*f*(*x*)	*g*(*x*)

2. $\begin{cases} f(x)=x-\sqrt{x+10} \\ g(x)=-4\left|x+2\right|+6 \end{cases}$

x	*f*(*x*)	*g*(*x*)

4. $\begin{cases} f(x)=\sqrt{x}+x^2 \\ g(x)=\dfrac{x^2+1}{x} \end{cases}$

x	*f*(*x*)	*g*(*x*)

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Station 3

Below are six systems of equations. Divide your group into two teams. Team 1 will solve the first three systems algebraically. Then they will solve the next three systems by graphing on the graphing calculator and finding the point(s) of intersection. Team 2, on the other hand, will solve the first three systems by graphing on the graphing calculator and finding the point(s) of intersection. Then, they will solve the remaining three systems algebraically. Teams will then compare answers, discuss differences, and resolve any errors. Be sure to check for extraneous solutions when solving algebraically. Round the point(s) of intersection to the nearest tenth.

1. $\begin{cases} y = \sqrt{x^2 + 1} \\ y = 2x - 1 \end{cases}$

2. $\begin{cases} y = \dfrac{x+3}{x-4} \\ y = 2x - 3 \end{cases}$

3. $\begin{cases} y = \sqrt{x^2 - 7} \\ y = x - 12 \end{cases}$

4. $\begin{cases} y = \dfrac{1}{x} \\ y = \dfrac{x-2}{x^2+1} \end{cases}$

5. $\begin{cases} y = \sqrt{x^2 + 1} \\ y = -x^2 + 5 \end{cases}$

6. $\begin{cases} y = \sqrt{x - 3} \\ y = \lvert x - 3 \rvert \end{cases}$

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Station 4

Work with your group to solve the following systems of equations by graphing them on a calculator and finding the point(s) of intersection. As you graph each system, be aware of the shape of each type of function. Make sure that the viewing window on the graphing calculator is an appropriate size that will allow you to see all points of intersection. For each system, round the point(s) of intersection to the nearest tenth.

1. $\begin{cases} y=\sqrt{x+1} \\ y=\dfrac{x^2-1}{x^2-4x+3} \end{cases}$

2. $\begin{cases} y=\dfrac{x^2-2x+1}{x^2} \\ y=x^3+2x^2+3 \end{cases}$

3. $\begin{cases} y=x^3+5x^2+24 \\ y=\sqrt{x-8} \end{cases}$

4. $\begin{cases} y=\dfrac{x+3}{x^2+x-2} \\ y=\dfrac{1}{2}x+2 \end{cases}$

5. $\begin{cases} y=\sqrt{-3x+5} \\ y=\left|x+6\right|-15 \end{cases}$

6. How can you be sure that you have found all the solutions to a system of equations when you are solving with a graphing calculator?

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APPENDIX

Problem-Based Task Implementation Guides

Introduction

This addendum contains a collection of Implementation Guides (IGs) designed to support the use of the Problem-Based Tasks found throughout this unit.

Each Implementation Guide includes an overview and suggestions for introducing and facilitating the task. These contain specific recommendations for selected Standards for Mathematical Practice (SMPs) to highlight, guidance on addressing common errors and misconceptions associated with the task, and ideas for monitoring and coaching students as they work through the task. Finally, there is guidance for debriefing the task, with special attention to the SMPs and relevant Key Concepts from the instruction, as well as information on suitable extensions and alternate strategies.

Throughout the Implementation Guides, suggestions are provided in a question/prompt format to support your efforts to elicit student thinking and facilitate classroom discourse.

We developed these Implementation Guides for our Problem-Based Tasks at the request of math educators, and with advice and feedback from mathematics supervisors and teachers. Please let us know if they are helpful. Visit us at www.walch.com, follow us on Twitter (@WalchEd), or e-mail suggestions to customerservice@walch.com.

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Implementation Guide

Problem-Based Task 2B.1.1: Rewriting a Rational Expression

Common Core State Standards

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.*

Task Overview

Focus

How are rational expressions simplified? Can there be more than one way to simplify a rational expression? In this task, students will analyze the work of three fictional students, who each approached a problem differently, to see who is and who is not correct.

This activity will provide practice with:

- recognizing quadratic expressions
- identifying the difference between two squares
- factoring quadratic expressions
- simplifying rational expressions
- analyzing and critiquing the work of others

Introduction

This task should be used to explore or apply the skill of simplifying rational expressions. Students should already be familiar with factoring quadratic expressions before beginning this task. If necessary, review the process of simplifying fractions with numbers in the numerator and denominator, mirroring the process of factoring, and simplifying when algebraic expressions are in fractions.

Begin by reading the problem and clarifying what is being asked. Encourage students to work at formulating clear, solid reasons and explanations as to why they think each student's work is correct or not.

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  The focus of the task is to critique the reasoning of the three students in the scenario. Students will work on constructing viable arguments as to whether they believe each student is correct or not, and why. Have students put their ideas into writing, and encourage students who disagree with each other to discuss and explain their thinking.

- **SMP 6:** Attend to precision.

  Make sure students are using math terms and definitions clearly and accurately to explain their reasoning. Strive toward precise mathematical language, such as using the correct terms for numerator and denominator (rather than "top" and "bottom"), and simplifying by a common factor (rather than "crossing out").

- **SMP 7:** Look for and make use of structure.

  To simplify the given rational expression, it is important for students to recognize the structure of the expressions in the numerator and denominator. They should recognize that the numerator represents the difference of two squares (and in simplifying this, they encounter another difference of two squares), and they should recognize that the denominator is a quadratic expression that can be factored. Make sure that students notice the common factor that exists in the numerator and denominator.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- canceling individual terms of the numerator or denominator, rather than factors

  In the expression given in this problem, $\frac{t^4 - 16}{t^2 - t - 6}$, some students may try to simplify the 16 and the 6, or the t^4 and the t^2. Remind them that in order to simplify, there must be a factor that is identical in the numerator and denominator. The terms in the given expression are not factors.

- forgetting to check for invalid values in the denominator

  Remind students that it is important to identify any values for the variable that would make the expression undefined. In the original expression, $t \neq 3$ or $t \neq -2$. In the simplified expression, $t \neq 3$.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students have trouble beginning the task, ask them to first simplify the expression themselves before analyzing the answers given by Faith, Jia, and Kayla.
- If students are not sure how to simplify the expression on their own, ask them to draw parallels between simplifying numerical fractions and simplifying the algebraic fraction. For example, to simplify the fraction $\frac{18}{24}$, you can first factor the numerator and the denominator:

  $$\frac{18}{24} = \frac{3 \cdot 6}{4 \cdot 6}$$

  Then simplify by a factor of 6:

  $$\frac{18}{24} = \frac{3 \cdot \cancel{6}}{4 \cdot \cancel{6}} = \frac{3}{4}$$
- Explain that this process can be paralleled with algebraic fractions: Factor the numerator and denominator, then eliminate the common factors.
- If students have difficulty factoring the numerator, ask other students if they recognize the form that it is in. (**Answer:** It is the difference between two squares.)
- If students have difficulty factoring the denominator, ask if they recognize the form of this expression (it is a quadratic expression), and have them help each other to factor this.
- Students may factor the numerator as $(t^2 - 4)(t^2 + 4)$ and stop there. Prompt them to check if either of these factors can be factored more. (**Answer:** The factor $(t^2 - 4)$ is again the difference between two squares: $t^2 - 4 = (t - 2)(t + 2)$.)
- Ask students to look closely at the language in Faith's response. Are t^4 and t^2 factors, as she states? (**Answer:** Yes) Even if these were factors, would this prevent the expression from being simplified? (**Answer:** No)
- Ask students if they can think of a way to alter the rational expression so it cannot be simplified, and state the reason. (**Sample answer:** $\frac{t^4 - 15}{t^2 - t - 6}$ cannot be simplified, because the numerator cannot be factored, so there are no common factors between the numerator and denominator.)
- Make sure students understand how Jia arrived at her answer, and are able to verbally explain her process.

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- After students discover that Jia's answer is correct, they may automatically assume that Kayla's answer is incorrect, because it looks different from Jia's answer. Encourage students to see if they can find a way to transform Jia's answer into Kayla's answer.
- After multiplying the terms in the numerator of Jia's answer, students will notice that the signs are different from the signs in Kayla's answer. Some students will assume this means the two answers are not equivalent. Ask students to think of a way to change the sign of a term (multiply by –1), and to use this to alter the signs to make Jia's and Kayla's answers match.
- Ask whether one form of the answer is better than the other form. (**Answer:** No; both forms are equivalent.)
- Ask students if they can think of another way to write the answer. (**Sample answer:** Rearrange the terms and change the signs of the terms of Kayla's answer: $\frac{t^3 - 2t^2 + 4t - 8}{t - 3}$.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow them to clarify these points for each other.

Debriefing the Task

- Compare strategies and ways of justifying responses. Focus on the use of precise mathematical language and clarity.
- Ask students to present their reasoning for whether or not Faith, Jia, or Kayla is correct. Students can build on and improve each other's explanations.

Connecting to Key Concepts

- A rational expression can often be written in many equivalent forms.

  In this task, the given rational expression was written in two different forms by Jia and Kayla.
- Factoring the expressions in the numerator and denominator can reveal such forms. Rules learned for factoring polynomial expressions can be useful in finding factors within rational expressions.

  The expressions in the numerator and denominator of the rational expression were rewritten in factored form so that the expression could be simplified.
- A quadratic expression is an algebraic expression that can be written in the form $ax^2 + bx + c$.

  The expression in the denominator, $t^2 - t - 6$, is a quadratic expression.

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- Because division by 0 is undefined, a rational expression cannot include a value of the variable that leads to 0 in the denominator.

  For the given expression, t cannot equal –2 or 3, because these values would lead to 0 in the denominator. Once the expression is simplified, the restriction on the value of x is limited to 3.

  If the given rational expression defined a function, $f(t) = \frac{t^4 - 16}{t^2 - t - 6}$, then the domain would be all real numbers except $x = -2$ or $x = 3$. Even when simplified, the domain of the initial function is retained.

Extending the Task

- Provide students with a different rational expression, and ask them to come up with several equivalent expressions. For example, for the expression $\frac{x^2 + 3x - 4}{x^4 - 1}$, some equivalent expressions are $\frac{x + 4}{(x + 1)(x^2 + 1)}$ and $\frac{x + 4}{x^3 + x^2 + x + 1}$.
- Another option for extending the task is to have students write an "original expression" if the simplified expression is, for example, $\frac{x - 3}{x^2 + 4}$. (Answers will vary. As long as the same expression is multiplied by the numerator and denominator, the rational expression will remain the same other than possibly introducing new restrictions on the variable.)

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **FOR SMP 3, ASK:** "Did you construct viable arguments, and did you critique the reasoning of others?" (**Answer:** Yes. I explained why Faith's answer could not be correct, and why Jia's and Kayla's answers are correct, by providing logical arguments and explanations. I also critiqued the reasoning of other students in my class when we discussed our reasoning on the given problem.)
- **FOR SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I tried to use correct math terminology in my explanations, and to explain my reasoning in a mathematically correct and precise way.)
- **FOR SMP 7, ASK:** "How did you make use of structure when solving the problem?" (**Answer:** I noticed the structure in the expressions found in the numerator and denominator of the rational expression, and that they could both be factored.)

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Alternate Strategies or Solutions

Substitute a random value of x into the given expression and into Jia's and Kayla's expressions to see if the resulting fractions are equal. For example, if we substitute 0 in for x, we have:

- The original expression, $\frac{t^4 - 16}{t^2 - t - 6}$, gives $\frac{(0)^4 - 16}{(0)^2 - (0) - 6} = \frac{-16}{-6} = \frac{8}{3}$.
- Jia's expression, $\frac{(t - 2)(t^2 + 4)}{t - 3}$, gives $\frac{[(0) - 2][(0)^2 + 4)]}{(0) - 3} = \frac{(-2)(4)}{-3} = \frac{-8}{-3} = \frac{8}{3}$.
- Kayla's expression, $\frac{8 - 4t + 2t^2 - t^3}{3 - t}$, gives $\frac{8 - 4(0) + 2(0)^2 - (0)^3}{3 - (0)} = \frac{8}{3}$.

All three expressions are equal to $\frac{8}{3}$ when 0 is substituted for t, which indicates they are equivalent.

Technology

No technology is needed for this task.

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Problem-Based Task 2B.1.3: The Area of a Triangle

Common Core State Standards

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

A–APR.7 (+) Understand that rational expressions form a system analogous to the rational numbers closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Task Overview

Focus

How do you calculate the area of a right triangle? In this task, students calculate the area of a right triangle whose sides are expressed as rational expressions.

This activity will provide practice with:

- finding the area of a right triangle
- identifying the restrictions on the variable in a rational expression
- multiplying rational expressions
- simplifying rational expressions

Introduction

This task should be used to explore or apply multiplication of rational expressions. Students may need a refresher on how to multiply fractions and find the area of a right triangle. Allow students to clarify these skills if necessary.

Begin by reading the problem and clarifying what the legs of a right triangle are.

leg of a right triangle — either of the two sides of a right triangle that form the right angle

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 6:** Attend to precision.

  Students should strive toward precise mathematical language, such as using the correct terms for numerator and denominator (rather than "top" and "bottom") as they describe or verify their work, or critique the work of others. Make sure students attend to precision in their calculations and simplify if possible. Emphasize that a complete and precise answer will state which values of x are not allowed.

- **SMP 7:** Look for and make use of structure.

  Encourage students to equate the structure of multiplying numerical fractions to the structure of algebraic fractions, and to explain and draw on the parallels between the two processes.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  Students have previously calculated the area of a triangle where the side lengths are given as numerical values. In this problem, students will be able to apply the same reasoning and repeat the process, but use algebraic expressions as the side lengths instead of numbers. Encourage students to maintain oversight of the process while attending to the details of the algebraic calculations.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- canceling individual terms of the numerator or denominator rather than factors

  Students may try to simplify the fractions by crossing out terms that are not factors, such as the x's in the numerator and denominator of the two expressions that represent the legs of the triangle. Remind them that simplification of rational expressions only happens when there are identical factors in the numerator and denominator, and that these x's are not factors.

- forgetting to check for invalid values in the denominator

  Make sure students understand how to determine which values of x are not permitted for the given expressions. In this problem, x cannot be equal to –1 or 0, because when those values are substituted into the expression in the denominator, the result is 0.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students have trouble remembering how to find the area of a right triangle, ask a student who remembers to do a numerical example to help the others remember.
- Students may forget the factor of $\frac{1}{2}$ in the formula for the area of the triangle, and just multiply the two dimensions together. Remind them of the formula, and explain that the answer they found is the area of a rectangle with these dimensions, and that the triangle is half of this rectangle.
- If students try to find a common denominator for the two rational expressions, ask if this is necessary. (**Answer:** No; a common denominator is not needed to multiply fractions.)
- Ask students if there are common factors that can be used to simplify the expressions before multiplying. (**Answer:** Yes; there is a factor of x in both the numerator and denominator: $\frac{1}{2} \bullet \frac{3x(x-1)}{x^2(x+1)} = \frac{1}{2} \bullet \frac{3(x-1)}{x(x+1)}$.)
- Ask students if the order in which they multiply the three factors matters. (**Answer:** No.)
- If students forget to check for x-values that will produce a 0 in the denominator, remind them to do this.
- Ask if they can think of any other restrictions on the value of x. (The side lengths of the triangle must be positive numbers because they are measuring a distance, so $\frac{3x}{x+1} > 0$ and $\frac{x-1}{x^2} > 0$. Solving these two inequalities for x leads to the restriction that x must be greater than 1.)
- Encourage students to verify their solution by choosing a value for x and substituting it into the given expressions and their answers to make sure the results are the same.
- Make sure students include units with their answer. (The units are cm^2.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow them to clarify these points for each other.

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Debriefing the Task

Have students share their solutions, justify their reasoning, and critique the reasoning of others. Ask if there is more than one method to get the answer.

Connecting to Key Concepts

- The product of a collection of rational expressions is the product of the numerators divided by the product of the denominators.

  As was done in this task, the rational expressions, including the $\frac{1}{2}$, were multiplied by dividing the product of the numerators by the product of the denominators.

- Any values that make the denominator equal to 0 must be identified.

  In this problem, the values of x that lead to a 0 in the denominator are –1 and 0. Further analysis of restrictions on the variable lead to the fact that x must be greater than 1 for the side lengths to be positive.

- Similar to numerical fractions, rational expressions can often be simplified by eliminating common factors from the numerator and denominator.

  In this task, the numerator and denominator of the expression for the area of the triangle were simplified by eliminating the common factor of x.

Extending the Task

To extend the task, ask students to write a new problem that involves finding the area or volume of a different geometric figure whose sides are given as rational expressions. Make sure they state restrictions on the variable. After writing a problem and creating its answer key, students can exchange papers and solve each other's problems.

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Implementation Guide

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **FOR SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I used precise mathematical language when justifying my reasoning, I did each step carefully, and I made sure the final answer was in its simplest form.)
- **FOR SMP 7, ASK:** "How did you make use of structure when solving the problem?" (**Answer:** I saw the parallels between a calculation that I am comfortable with, multiplication of numerical fractions, and the multiplication of rational expressions that was done in this task.)
- **FOR SMP 8, ASK:** "How did you use the regularity and repeated reasoning within the problem to help you solve it?" (**Answer:** I used repeated reasoning, as I noticed that the reasoning behind finding the area of a triangle is performed the same whether the side lengths are given numerically or algebraically.)

Alternate Strategies or Solutions

- The three fractions can be multiplied in a different order than that shown in the Coaching questions.
- The three fractions can be multiplied first and then simplified by a factor of x at the end of the process, although this method leads to lengthier calculations.
- Students may try different values of x to verify their restrictions on the variable. For instance, if $x = \frac{1}{2}$, one of the legs of the triangle measures –2 cm, which is not possible.

Technology

Technology is not needed for this activity, but it may be used by students who would like to use it to investigate the problem.

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Problem-Based Task 2B.2.1: Snow Removal

Common Core State Standard

A–REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Task Overview

Focus

If a job takes a given number of minutes for two people to complete, how long does it take for each person to complete if working alone? "Work problems" in algebra are based on ratios, and the equations that model these situations use rational expressions to represent these ratios. These problems are generally much easier to solve using algebra than without algebra. This task allows students an opportunity to solve this type of real-world problem.

This activity will provide practice with:

- modeling a situation with a rational equation
- solving a rational equation
- finding a common denominator for rational expressions
- interpreting the result in the context of the problem

Introduction

This task should be used to explore or to apply the topic of solving rational equations. This type of problem, which is about the time two people take to complete a job together versus the time it would take each person to complete the job alone, can be quite difficult for students, so it is recommended that students be given plenty of time to explore and discuss this task.

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  This type of problem is often difficult for students. Encourage students to think of the portion of the shoveling job that can be done by each person.

  Encourage students to break the task into smaller pieces.

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Encourage students to use non-algebraic methods if necessary, such as trial and error.

Have students explain the problem in their own words to make sure that it is well understood.

Ask students to make initial guesses about the answer before they begin any calculations. What answer would be too big? What answer would be too small?

Encourage students to talk about what they do know and what they wish they knew.

- **SMP 2:** Reason abstractly and quantitatively.

  Students reason abstractly when they are able to translate the problem into an algebraic equation. Encourage this abstract thinking and ask questions about how they set up their equation. What does the variable stand for? What does each term in your equation represent?

  Students will be reasoning quantitatively when they are making educated guesses as to what a reasonable answer might look like, or when they test the validity of the answer found with the algebraic model.

  Ask if a table could be used to answer the question posed in the task, and encourage students to turn this quantitative method into an algebraic equation.

- **SMP 4:** Model with mathematics.

  Students who are able to represent this concrete problem with a mathematical equation or any another abstract representation are modeling with mathematics. Point this out to students as they are working. Tell them they are trying to create a symbolic model of the problem.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- multiplying only some terms of the rational equation when eliminating rational numbers

  Remind students that in order to keep the equation balanced, the same operation must be done to both sides of the equation. In this problem, they may multiply both sides of the equation by $2t$. Show that this can first be written as $2t\left(\frac{40}{t}+\frac{40}{2t}\right)=2t(1)$, so that it is clear what is being done to both sides of the equation. Next, simplify this by distributing the $2t$ to arrive at the result illustrated in the Coaching questions.

- confusing the expressions "twice as long" and "half as long" when writing algebraically

  The problem states that it takes Scot half the time that it would take Levi to do the job alone. This means that it takes Levi twice as long as Scot. If t represents Scot's time, then $2t$ represents Levi's time.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students have trouble getting started, encourage them to write down what they know and what they need to find out. Ask, "What is the value that you can represent with a variable?" (**Answer:** We can represent Scot's time with the variable. We could instead represent Levi's time with the variable, but this would mean that Scot's time would be $\frac{t}{2}$, which would be more difficult to solve due introducing another fraction into the problem.)
- To help struggling students understand the problem better, ask, "Who completes the job faster?" (**Answer:** Scot) "How much faster?" (**Answer:** twice as fast as Levi) Ask, "Will the time it takes either of them working alone be less than or greater than 40 minutes?" (**Answer:** greater than 40 minutes)
- To help students set up the equation, ask, "If both boys working together take 40 minutes to complete the job, what portion of the job does Scot complete in 40 minutes?" (**Answer:** $\frac{40}{t}$) "What portion of the job does Levi complete in 40 minutes?" (**Answer:** $\frac{40}{2t}$)
- Ask students what number or ratio can be used to represent the entire job. (**Answer:** 1, or $\frac{1}{1}$)
- To provide an opportunity for students to help or guide each other, ask if anyone can state the equation verbally. (**Answer:** Scot's portion of the job plus Levi's portion of the job equals the entire job.)
- If students correctly set up the equation but don't know how to solve it, ask what they can multiply by to eliminate the fractions. (**Answer:** the least common denominator)
- Encourage students to pursue alternate solution strategies besides the strategy described above and in the Coaching questions.
- If students would like to try to solve the problem by guessing and checking, encourage this as a valid solution strategy.
- Ask if there are any values of the variable that must be excluded from the possible solution set. (**Answer:** The variable t cannot equal 0, as this would lead to a zero in the denominator and would indicate that no work was done, and it cannot be negative, because the brothers cannot work for a negative number of minutes.)
- Once the equation is solved, make sure students return to the question being asked, and interpret their answer in the context of the problem. If students arrive at a solution of 60, ask what this 60 represents and if it answers the question being asked. (**Answer:** The 60 represents the number of minutes it would take Scot to complete the job if he was working alone. The problem is asking for the amount of time it would take Levi to complete the job alone, so 60 is not the answer to the question being asked.)

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- What is the answer to the question being asked? (**Answer:** Since it takes Levi twice as long as Scot, it must take Levi 120 minutes, or 2 hours, to complete the job if working alone.)
- Encourage students to double-check their solution to be sure that it makes sense and that it "works" when substituted in the found equation.
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow them to clarify these points for each other.

Debriefing the Task

There are several paths that lead to the solution to this problem. Encourage students to share their strategies, and compare their strategies to other strategies that were used. Ask if there is a strategy used by someone else in the class that they prefer over their own strategy, and why.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Finding the solution to a rational equation often requires multiplying both sides of the equation by a common denominator to eliminate any rational terms

  To solve the equation that models the situation, the common denominator for the rational terms is $2t$. We multiplied both sides of the equation by $2t$ in order to eliminate the rational terms.

- The process of solving rational equations can be applied to real-world examples in which a specific task that may be done alone is to be completed cooperatively.

  This task is an example of this type of problem. Point out to students that these work problems can always be modeled in a similar fashion, using rational equations.

Extending the Task

- To extend the task, you can change the parameters to create a slightly different problem. For example:

  If Levi took 3 times as long as Scot to shovel the driveway, and the entire driveway could be completed in 60 minutes if the brothers work together, how long does it take Levi to shovel the driveway if working alone? (**Answer:** 240 minutes, or 4 hours)

- Alternatively, the given information can be varied to create a new problem:

  If it takes Scot 24 minutes to shovel the driveway alone, and it takes Levi 16 minutes to shovel the same driveway alone, how long will it take them if they are working together? (**Answer:** 9.6 minutes)

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you demonstrate perseverance?" (**Answer:** I wrote down the information given in the problem, and I identified the information I was looking for. When I got stuck, I worked with other students to get ideas on how to write the equation.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** Assigning a variable and using this variable to model the situation with an equation represents abstract reasoning.) "Which of your strategies represent quantitative reasoning?" (**Answer:** When I tried guessing and checking different possible answers, and made a table to show my guesses, this represented quantitative reasoning.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I modeled the situation with a rational equation, and I represented the time that it took Scot to shovel the driveway alone with the variable t.)

Alternate Strategies or Solutions

- Along with the solution strategy presented in the Coaching questions, students may use one of the following two strategies, or they may come up with another strategy that is not listed here. Accept all strategies that lead to the correct solution, and emphasize that although there may be strategies that are quicker than others, all strategies are valid.
- Students may use guessing and checking to find the answer.

  If students are unable to come up with an equation, they may guess and check. They know that since it takes the brothers 40 minutes working together, it must take each boy more than 40 minutes working alone. Furthermore, they know that Levi takes twice as long as Scot to complete the job alone. Thus, they may make an initial guess of 50 minutes for Scot and 100 minutes for Levi. This would mean that Scot does $\frac{1}{50}$ of the job per minute, and Levi does $\frac{1}{100}$ of the job per minute. In 40 minutes of working together, they should complete 1 entire job. To test the solution, check whether or not the following is true:

  $$40\left(\frac{1}{50}\right)+40\left(\frac{1}{100}\right)=1$$

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When you simplify this equation, you find that it is not true. The result is $\frac{120}{100} \neq 1$. This shows that they must take longer than the initial guess. Try new values. This time, you might guess that it takes 60 minutes for Scot and 120 minutes for Levi. Test these values.

$$40\left(\frac{1}{60}\right) + 40\left(\frac{1}{120}\right) = 1$$

If you simplify this, you find that it is true. The result is 1 = 1. Therefore, using guessing, checking, and a bit of logic, the answer is found to be 60 minutes for Scot and 120 minutes for Levi.

- Students might write an alternate form of the algebraic equation.

  This alternate equation is based on writing what each brother accomplishes in 1 minute, and how much of the job is completed in 1 minute.

  Students may reason that if it takes Scot t minutes to complete the job alone, then he must complete $\frac{1}{t}$ of the job per minute. Likewise, if Levi takes twice as long as Scot, then he can complete the job alone in $2t$ minutes, so he does $\frac{1}{2t}$ of the job per minute. And in 1 minute, $\frac{1}{40}$ of the job is complete, since it takes 40 minutes for the entire job when the boys work together. This leads to the following equation:

  $$\frac{1}{t} + \frac{1}{2t} = \frac{1}{40}$$

  Notice that if you multiply both sides of this equation by 40, the result is the same as the equation that was presented in the Coaching questions.

  Solve this equation by multiplying both sides by the LCD, $40t$, and the result is the same as above: Scot completes the job alone in 60 minutes, so Levi completes the job in 120 minutes.

Technology

Students can use calculators to help with the calculations or with guessing and checking.

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Problem-Based Task 2B.2.3: Measuring a Masterpiece

Common Core State Standard

A–REI.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★

Task Overview

Focus

How can a system of equations be applied to solve a real-world problem? How can graphing technology aide in determining solutions of a system of equations? In this task, students will create a system of equations representing the unknown lengths of the legs of a right triangle, and they will use graphing technology to find the solution of the system of equations. They will also determine the area of the right triangle which represents the area of the triangular portion of a sculpture.

This activity will provide practice with:

- identifying variables which represent components of a problem
- writing a mathematical equation to represent the relationship between two variables
- applying the Pythagorean Theorem
- creating a system of equations
- graphing a system of equations using technology
- interpreting the solution of a system of equations in the context of a scenario
- calculating the area of a triangle

Introduction

This task should be used to explore or to apply the skill of finding the solutions to a system of equations approximately, e.g., using technology to graph the functions. Students should already be familiar with graphing multiple equations on a graphing calculator and graphing various forms of equations.

Begin by reading the problem and clarifying the meaning of the following terms:

point(s) of intersection	in a graphed system of equations, the ordered pair(s) where graphed functions intersect on a coordinate plane
Pythagorean Theorem	a theorem that relates the length of the hypotenuse of a right triangle (c) to the lengths of its legs (a and b). The theorem states that $a^2 + b^2 = c^2$.
system of equations	a set of equations with the same unknowns

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they recognize that they will first need to create a system of two equations which represent the unknown lengths of the legs of a right triangle. They will reason that the solution to this system of equations can be found by identifying the point of intersection of the two graphs of the equations. They will use graphing technology to determine the ordered pair of the point of intersection, which represents the solution. Once the solution has been determined and numerical values representing the side lengths of the triangle have been found, they will recognize that the area formula for a triangle will be used to find the area of the triangular portion of the sculpture. They will reason quantitatively as they apply the Pythagorean Theorem in order to create the system of equations by rewriting the theorem in "$y=$" form. They will also perform calculations using the area of a triangle formula.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled by a system of equations that can be graphed in order to find the solution. They will recognize that the unknown lengths of the legs of a right triangle are represented by the two equations in the system. They will interpret the point of intersection of the system of equations as the lengths of the legs of the triangle. Once these lengths have been determined, they will substitute them into the formula for the area of a triangle, as the base and the height of the triangle, in order to find the area of the triangular portion of the sculpture.

- **SMP 6:** Attend to precision.

  Students will use precision as they first carefully follow and apply the structure of the Pythagorean Theorem, which will allow them to create the system of equations which represents the unknown lengths of the legs of a triangle. They will recognize that the equations can be written in the form "$y =$", so that the format of the equations will allow for more efficient graphing with their calculators. They will use precision as they determine the location of the point of intersection of the equations using the tracing feature on the calculator. They will also use precision as they substitute the values for the base and height of the triangle into the area of a triangle formula and perform the calculations in order to find the area. Encourage them to use the correct terminology throughout the task, specifically when referring to the system of equations and the components of a right triangle, as well as the variables in the area of a triangle formula.

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Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- not understanding that solutions must satisfy every function in a system

  Remind students that the values for the x- and y-coordinates of the solution of the system must be true for both equations. Remind them to substitute $x = 20$ and $y = 21$ into both equations in the system in order to verify that they both yield true statements.

- incorrectly representing the relationship between the two variables and possibly incorrectly writing it as $y = x - 1$

  Have students reread the information in the task about the relationship between the two legs of the right triangle. Remind them that the phrase "1 meter longer than" is represented by addition; therefore the relationship is $y = x + 1$.

- forgetting to square the hypotenuse of the triangle when applying the Pythagorean Theorem

  Have students write the Pythagorean Theorem, $a^2 + b^2 = c^2$, at the top of their papers. Remind them that each of the three sides of the right triangle must be squared, so in the task, the value of the hypotenuse must be squared: $(29)^2 = 841$.

- not choosing an appropriate graphing window (scales on the x-axis and y-axis)

  Remind students that when the equations are graphed on the calculator, both equations need to be visible such that the point of intersection can be seen. Discuss possible appropriate scales for the axes with students.

- forgetting to divide the product of the base and height by 2 when finding the area of the triangle

  Have students write the formula for the area of a triangle, $A = \frac{1}{2}bh$, at the top of their papers. Remind them to substitute 20 for the base and 21 for the height, find their product, and then divide by 2 in order to get the area.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- As students begin working on the task, ask them, "Why is it practical to use the variable x to represent the base of a triangle and y to represent the height of the triangle? Could the variables be switched? Explain." (**Answer:** It is practical to think of the base of a triangle along a horizontal direction, so since the x-axis runs in a horizontal direction, it makes sense for the x-variable to represent the base. Since height is typically represented in a vertical direction, and the y-axis runs in a vertical direction, it makes sense for the y-variable to represent the height. The variables could be switched, as the triangle could be rotated 90° in either direction.)

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- As students are working on the task, ask them, "What is another way you can write the equation for the Pythagorean Theorem, instead of "$x^2 + y^2 = 29^2$", using only x as the variable in the equation? What is this solution method of a system of equations called?" (**Answer:** Since an equation for the longer of the two legs, which is represented by the variable y, can be written as $y = x + 1$, then "$x + 1$" can be substituted in for y in the equation for the Pythagorean Theorem: $x^2 + (x + 1)^2 = 29^2$. This method is called the substitution method.)
- As an extension to the previous question, ask students, "How can $x^2 + (x + 1)^2 = 29^2$ be used to solve for x? Explain the steps." (**Answer:** When "$x + 1$" is substituted in for y in the Pythagorean Theorem, the equation becomes a quadratic equation and can be solved with the following steps:

$x^2 + (x + 1)^2 = 29^2$	Original equation
$x^2 + x^2 + 2x + 1 = 841$	Expand the binomial $(x + 1)^2$.
$2x^2 + 2x + 1 = 841$	Combine like terms.
$2x^2 + 2x - 840 = 0$	Set the equation equal to 0.
$2(x^2 + x - 420) = 0$	Factor out the common factor of 2.
$2(x + 21)(x - 20) = 0$	Factor the quadratic.
$2 = 0; x + 21 = 0; x - 20 = 0$	Set each factor equal to 0.
$x = -21; x = 20$	Solve for x.

- After students have shown the steps for solving for x in a quadratic equation, ask them, "In the context of the scenario, which value for x is valid? What do you need to do in order to find the value of y?" (**Answer:** Since distance cannot be negative, the only value of the solutions to the quadratic equation that are valid in the context of the scenario is $x = 20$. In order to find the value of y, 20 can be substituted in the equation $y = x + 1$; $y = 20 + 1$; $y = 21$. The value of y is 21. Note that this matches with the solution found when the two equations are graphed.)
- As students prepare to graph the equations using their calculators or other graphing technology, ask them, "Why is it important to make sure that the scales used for the x-axis and y-axis are appropriate and similar in terms of intervals? (**Answer:** It is important that the graphs of both equations appear on the screen, especially where they intersect each other. If the scales for the axes were too disproportionate, then the point of intersection might be missed.)
- As students are working through the task and locating the point of intersection of the two equations on the graph, ask them, "Why does it make sense that the ordered pair (20, 21) is a valid solution to the system of equations, in terms of what each coordinate represents?" (**Answer:** The information in the problem states that one leg of the triangle is 1 meter longer than the other leg. Since the y-coordinate is 21, and this value is 1 more than the x-coordinate of 20, this ordered pair makes sense.)

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- Ask students, "If the two equations were graphed on the same coordinate plane, and they did not intersect at any point, what would that mean in the context of the scenario?" (**Answer:** If the two equations did not intersect, then there is no solution to this system of equations. In the context of the scenario, there would not be two valid lengths of legs of the right triangle that would work with a hypotenuse of 29 meters.)
- After students have found the point of intersection as the ordered pair (20, 21), ask them, "How can you use your solution to the system of equations in order to verify that the triangular portion of the sculpture is indeed a right triangle?" (**Answer:** To verify that the three given lengths of the triangle form a right triangle, each value can be substituted into the Pythagorean Theorem: $(20)^2 + (21)^2 = (29)^2$; 400 + 441 = 841; 841 = 841. Since the given lengths yield a true statement, then this proves that the triangle with these lengths form a right triangle.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss how the scenario could be modeled by first creating a system of equations to represent the unknown lengths of the legs of the right triangle, and then by graphing the system of equations on the same coordinate plane in order to find the intersection. Discuss why the ordered pair that represents the intersection of the graphs of the two equations is the solution to the system, and ask students to explain its meaning in the context of the scenario. Also ask students to explain how they applied the solution and found the area of the triangular portion of the sculpture. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and their steps for applying the Pythagorean Theorem in order to create an equation to represent the relationship between the hypotenuse and legs of a right triangle. Discuss other possible methods of finding the solution(s) to a system of equations, and discuss the practicality of using each method based on the equations represented in this scenario. Focus on the use of precise mathematical language and clarity, specifically when discussing how the variables represent the unknown lengths of the legs of a right triangle, as well as how the point of intersection represents the solution to the system of equations. Also focus on the variables used in the formula for the area of a triangle.

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Connecting to Key Concepts

Make explicit connections to key concepts:

- A **system of equations** is a set of equations with the same unknowns.

  In this task, students will create a system of two equations which represent the two unknown side lengths of the right triangle. They will represent these sides using the variables x and y.

- Solutions to a system of equations can be identified in a graph by finding the intersection of the functions.

  Students will use graphing technology to graph the system of equations on the same coordinate plane, and they will use their technology to locate the intersection of the two equations.

- The **points of intersection** of a graphed system of equations are the ordered pairs where the graphed functions intersect on a coordinate plane.

  Students will identify the point of intersection of the two equations, and they will represent this point of intersection as an ordered pair.

- To confirm the solution to a system, substitute the coordinates for x and y in both of the original equations. If the intersection point is approximated, the results will be nearly equal; otherwise, the results will be equal.

  Students will verify that the solution of the system of equations, (20, 21), is valid as they will substitute both coordinates into each equation in order to verify that both equations yield true statements.

Extending the Task

To extend the task, ask students to work with a partner to recreate the task, but this time with the following information: "The hypotenuse of the right triangle has a length of 30 meters. One leg of the triangle is 6 meters less than the other leg." Ask students to create a system of equations and then graph them in order to find the point of intersection, representing the unknown lengths of the legs of the triangle. Also ask them to find the area of the triangle." Ask for volunteers to share their results, specifically explaining the steps and procedures they used in order to determine the unknown leg lengths. Discuss any difficulty that students may have encountered while working through this extended task. (**Answer:** The legs of the right triangle are 24 meters and 18 meters. The area of the triangle is 216 m^2.)

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I recognized that I first needed to create a system of equations which represented the unknown lengths of the legs in the right triangle. I reasoned that the solution to this system of equations could be found by identifying the point of intersection of the two graphs of the equations. I used graphing technology to determine the ordered pair of the point of intersection, which represented the solution. Once the solution was determined and numerical values representing the side lengths of the triangle were found, I recognized that the area formula for a triangle could be used to find the area of the triangular portion of the sculpture.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I reasoned quantitatively as I applied the Pythagorean Theorem in order to create the system of equations by rewriting the theorem in "$y =$" form, and I also performed calculations using the area of a triangle formula.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by recognizing that the scenario can be modeled by a system of equations that can be graphed in order to find the solution. I recognized that the unknown lengths of the legs are represented by the two equations in the system. I interpreted the point of intersection of the system of equations as the lengths of the legs of the triangle. Once these lengths were determined, I substituted these values into the formula for the area of a triangle, as the base and the height of the triangle, in order to find the area of the triangular portion of the sculpture.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision as I first carefully followed and applied the structure of the Pythagorean Theorem, which allowed me to create the system of equations which represented the unknown lengths of the legs of a triangle. I recognized that the equations could be written in the form "$y =$", so that the format of the equations allowed for more efficient graphing with my calculator. I used precision as I determined the location of the point of intersection of the equations using the tracing feature on my calculator. I also used precision when I substituted the values for the base and height of the triangle into the area of a triangle formula and performed the calculations in order to find the area. I was careful to use correct terminology throughout the task, specifically when referring to the system of equations and the components of a right triangle, as well as the variables in the area of a triangle formula.)

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Alternate Strategies or Solutions

- Some students may choose to find the solution of the system of equations by other methods such as substitution. They may choose to use the Pythagorean Theorem and create an equation in one variable, using the value of "$x + 1$" for the variable y. Then, they may solve for x once they have created the quadratic equation. Encourage them to verify their results with the results obtained from graphing the system of equations. Some students may choose to find the solutions by the elimination method. Remind students that this method will be more time intensive because of the exponents involved in one of the equations. Encourage them to use substitution as an alternate method to check their results from graphing.
- Some students may choose to create a table of values for both equations in order to find the common ordered pair, or they may create a table of values in order to graph both equations by hand on graph paper. Remind them that it may take a great deal of time in order to plot enough points to represent each equation, and that it is possible that they do not graph the points that will represent the solution to the system of equations.

Technology

Students can use calculators or other graphing technology for this task.

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Unit 2 Comprehensive Assessment

Circle the letter of the best answer.

1. What is the result of $\left(3x^5-12x^3+2x^2\right)-\left(8x^6-5x^5+4\right)$?

   a. $-8x^6-8x^5-12x^3+2x^2+4$
   b. $11x^5+7x^3+6x^2$
   c. $-5x^5-4x^5+2x^2+4$
   d. $-8x^6+8x^5-12x^3+2x^2-4$

2. What is the result of $\left(5x^3+4x^2+6x\right)\left(3x^2-2\right)$?

   a. $5x^3+7x^2+6x-2$
   b. $15x^5+12x^4+8x^3-8x^2-12x$
   c. $15x^5+12x^4+18x^3$
   d. $15x^5+12x^4+18x^3-2$

3. Which expression represents the factored form of $\left(x^4-64\right)$?

   a. $(x^2+8)(x^2-8)$
   b. $(x+32)(x-2)$
   c. $(x+8)(x-8)$
   d. $(x^2+32)(x^2-2)$

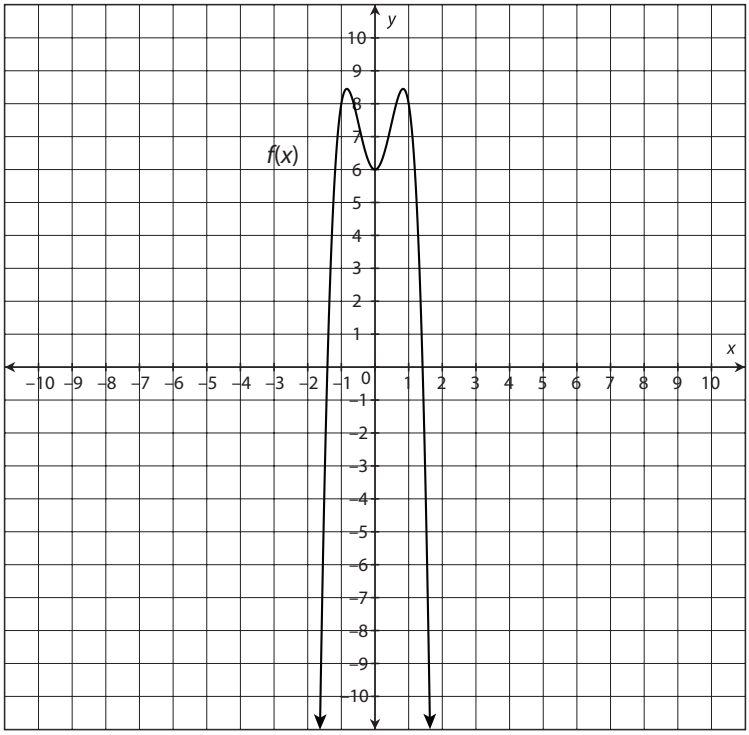
4. Which expression represents the expanded form of $(4x-y)^3$?

   a. $16x^2-8xy+y^2$
   b. $64x^3-48x^2y+12xy^2-y^3$
   c. $12xy-3y$
   d. $16x^2-8xy+4x+y^2+y$

continued

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5. What must be true about the degree and leading coefficient of the graphed polynomial?



a. The polynomial is an odd-degree polynomial and has a positive coefficient.

b. The polynomial is an odd-degree polynomial and has a negative coefficient.

c. The polynomial is an even-degree polynomial and has a positive coefficient.

d. The polynomial is an even-degree polynomial and has a negative coefficient.

6. What are all the possible rational roots for the polynomial equation $12x^3+5x^2-16=0$?

a. $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$

b. $\pm\frac{1}{16}, \pm\frac{3}{16}, \pm\frac{1}{8}, \pm\frac{3}{16} \pm\frac{3}{4} \pm1, \pm\frac{3}{2}, \pm2, \pm3, \pm4, \pm6, \pm15$

c. $\pm\frac{1}{12}, \pm\frac{1}{6}, \pm\frac{1}{4}, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm\frac{2}{3}, \pm1, \pm\frac{4}{3}, \pm2, \pm\frac{8}{3}, \pm4, \pm\frac{16}{3}, \pm8, \pm16$

d. $\pm1, \pm2, \pm4, \pm8, \pm16$

continued

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7. Graph the following system to determine the approximate real solution(s), if any solutions exist.

$$\begin{cases} f(x)=2x-2 \\ g(x)=4x^3-8x^2-4x+8 \end{cases}$$

a. {(1, 0)}

b. {(–1.16, –4.32), (1, 0), (2.16, 2.32)}

c. {(–1.16, –4.32), (–1, 0), (2.16, 2.32)}

d. There are no real solutions to this system.

8. What is the sum of the following series, if a sum exists?

$$\sum_{k=1}^{\infty} 5\left(\frac{2}{3}\right)^{k-1}$$

a. 15

b. $\frac{1}{15}$

c. $\frac{45}{2}$

d. No sum exists.

9. What is the sum of the following series, if a sum exists?

$$\sum_{k=1}^{\infty} 2\left(\frac{5}{4}\right)^{k-1}$$

a. $\frac{8}{5}$

b. $\frac{5}{8}$

c. 8

d. No sum exists.

10. Which simplified rational expression is equivalent to $\frac{7x+2}{2x}-\frac{x+3}{x^2}$?

a. $\frac{7x^2-6}{2x^2}$

b. $\frac{7x^2-6}{2x^3}$

c. $\frac{6x-1}{2x-x^2}$

d. $\frac{8x+5}{2x^2}$

continued

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11. Which simplified rational expression is equivalent to $\frac{2x}{2x+1} \bullet \frac{3x}{x+2}$?

    a. $\frac{6x^2}{2x^2+2}$

    b. $\frac{6x}{2x^2+5x+2}$

    c. $\frac{6x^2}{2x^2+x+2}$

    d. $\frac{6x^2}{2x^2+5x+2}$

12. What is the solution to the system of equations defined by the functions $f(x)=2+\sqrt{x-3}$ and $g(x)=\frac{1}{x}-4$?

    a. {(12, 5)}

    b. $\left\{\left(\frac{1}{4},0\right)\right\}$

    c. $\left\{\left(\frac{1}{4},0\right),(12,5)\right\}$

    d. There are no real solutions to this system.

Use what you have learned about algebraic relationships to complete all parts of problems 13–15.

13. Use Pascal's Triangle and the Binomial Theorem to find the terms requested for the expanded form of $(-x+2y)^6$.

    a. term 1

    b. term 2

    c. term 4

    d. term 6

continued

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14. A system of equations is defined by $f(x) = -x^3 + 2x + 4$ and $g(x) = x + 10$.

    a. Graph the two functions on the same coordinate plane.

    b. Identify any points of intersection.

    c. Explain what the points of intersection, if any, mean in terms of the functions.

15. A system of equations is defined by $f(x)=\frac{2x+3}{x}$ and $g(x)=3\sqrt{x-2}$.

    a. Graph the two functions on the same coordinate plane.

    b. Identify any points of intersection.

    c. Explain what the points of intersection, if any, mean in terms of the functions.

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Answer Key

Unit 2 Comprehensive Assessment
p. U2-CA1

1. d
2. b
3. a
4. b
5. d
6. c
7. b
8. a
9. d
10. a
11. d
12. d

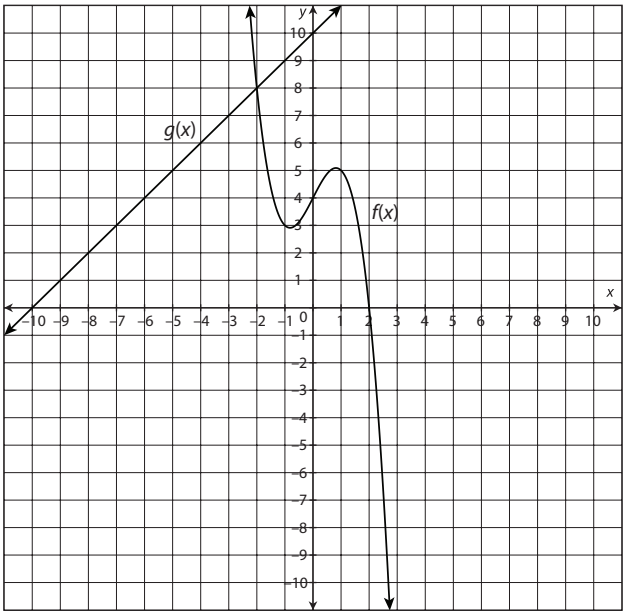
13. a. x^6

    b. $-12x^5y$

    c. $-160x^3y^3$

    d. $-192xy^5$

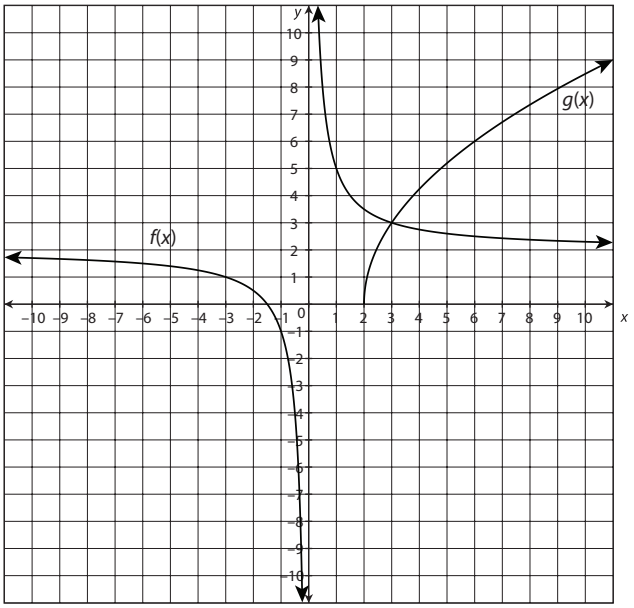
14. a.



    b. {(–2, 8)}

    c. The point of intersection located at (–2, 8) represents the solution to the defined system of equations.

15. a.



    b. {(3, 3)}

    c. The point of intersection located at (3, 3) represents the solution to the defined system of equations.

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Common Core State Standards

Mathematics III

Integrated Pathway

Teacher Resource

Unit 3

WALCH

INTEGRATED MATH

This program was developed and reviewed by experienced math educators who have both academic and professional backgrounds in mathematics. This ensures: freedom from mathematical errors, grade level appropriateness, freedom from bias, and freedom from unnecessary language complexity.

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Standards Correlations

Each lesson in this Integrated Pathway: Mathematics III program was written specifically to address the Common Core State Standards. Each lesson lists the standards covered in all the sub-lessons, and each sub-lesson lists the standards addressed in that particular section. In this section, you'll find a comprehensive list mapping the sub-lessons to the CCSS.

Guide to Common Core State Standards Annotation

As you use this program, you will come across a symbol included with the Common Core standards for some of the lessons and activities. These symbols are explained below.

Symbol: ⋆

Denotes: Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (⋆).

From http://www.walch.com/CCSS/00003

Symbol: (+)

Denotes: College and Career Readiness Standards

Advanced mathematics standards that are required in higher-level courses such as advanced statistics may also be included in lower-level courses. These additional standards are denoted by (+). According to the Common Core State Standards Initiative, "the evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6–8."

From http://www.walch.com/CCSS/00004

Connections to Future Courses

This section provides a map between topics introduced in each unit of this course and subsequent courses where each topic is revisited and built upon.

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CCSS INTEGRATED PATHWAY: MATHEMATICS III STANDARDS CORRELATIONS

Unit 3: Trigonometry of General Triangles and Trigonometric Functions				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Radians and the Unit Circle**			
	3.1.1	Radians	F–TF.1 F–TF.2	U3-5
	3.1.2	The Unit Circle	F–TF.2	U3-21
	3.1.3	Special Angles in the Unit Circle	F–TF.2	U3-45
	3.1.4	Evaluating Trigonometric Functions	F–TF.2	U3-69
Lesson 2	**Trigonometry of General Angles**			
	3.2.1	Proving the Law of Sines	G–SRT.9 (+) G–SRT.10 (+)	U3-99
	3.2.2	Proving the Law of Cosines	G–SRT.10 (+)	U3-123
	3.2.3	Applying the Laws of Sines and Cosines	G–SRT.11 (+)	U3-142
Lesson 3	**Graphs of Trigonometric Functions**			
	3.3.1	Periodic Phenomena and Amplitude, Frequency, and Midline	F–TF.5★	U3-167
	3.3.2	Using Trigonometric Functions to Model Periodic Phenomena	F–TF.5★	U3-185

TOPICS FROM FUTURE COURSES: CCSS INTEGRATED PATHWAY MATHEMATICS III

Math III	Topic introduced	Course/unit where addressed	How addressed
Unit 3	Calculating the sine, cosine, and tangent for any triangle using the laws of sines and cosines; trigonometric ratios, including angles measured in radians; the unit circle	Precalculus, Calculus, or Math IV	representing and operating with vectors using trigonometric ratios, trigonometric functions, and complex numbers; evaluating special angles in right triangles using the unit circle and concepts of periodicity

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Conceptual Activities

Use these interactive open education and/or Desmos resources to build conceptual understanding of mathematical ideas. (*Note*: Activity links will be monitored and repaired or replaced as necessary.)

- Desmos. "Burning Daylight."

  http://www.walch.com/ca/01059

  In this activity, students use sinusoids to model daylight data for two U.S. cities. They predict which city has more total daylight during a given year, and then use their model to calculate an answer to that question.

- Desmos. "Graphing the Sine Function Using Amplitude, Period, and Vertical Translation."

  http://www.walch.com/ca/01060

  Students will build a visual understanding of amplitude, period, and phase shift in this introduction to trigonometric graphing. They will use this understanding to find models for given graphs of the sine function.

- Desmos. "Marbleslides: Periodics."

  http://www.walch.com/ca/01061

  In this activity, students will transform periodic functions to send marbles through stars.

- Desmos. "Polygraph: Sinusoids."

  http://www.walch.com/ca/01062

  This activity is designed to spark vocabulary-rich conversations about sinusoids. Key vocabulary terms that may appear in student questions include *amplitude*, *periods*, *maximum*, *minimum*, and *shift*.

- Desmos. "Polygraph: Sinusoids with Vertical Transformations."

  http://www.walch.com/ca/01063

  This activity is designed to spark vocabulary-rich conversations about vertical transformations of sinusoids. Key vocabulary terms that may appear in student questions include *translation*, *dilation*, *amplitude*, *midline*, and *sinusoidal axis*.

- Desmos. "Trigonometric Graphing: Introduction to Amplitude and Vertical Shift."

  http://www.walch.com/ca/01064

  In this activity, students will informally explore range, midline, and amplitude of trigonometric functions. They'll use what they learn about the relationships to write equations of sine and cosine graphs.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Convert $\frac{\pi}{3}$ radians to degrees.
   a. 30°
   b. 60°
   c. 90°
   d. 180°

2. In which quadrant of the unit circle does the terminal side of an angle measuring $\frac{7\pi}{6}$ radians fall?
   a. Quadrant I
   b. Quadrant II
   c. Quadrant III
   d. Quadrant IV

3. At which point does the terminal side of an angle measuring $\frac{\pi}{4}$ radians intersect the unit circle?
   a. $\left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$
   b. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
   c. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
   d. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

4. Which trigonometric functions are positive for an angle measuring $\frac{4\pi}{5}$ radians?
   a. all of them
   b. cosine and secant
   c. sine and cosecant
   d. tangent and cotangent

5. What is the value of $\tan\theta$ if θ is the angle between the x-axis and the ray extending from the origin through (3, 4)?
   a. $\frac{3}{5}$
   b. $\frac{3}{4}$
   c. $\frac{4}{3}$
   d. $\frac{5}{3}$

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Lesson 1: Radians and the Unit Circle

Instruction

Common Core State Standards

SMP	
1 ✓	2 ✓
3 ✓	4
5	6 ✓
7 ✓	8 ✓

F–TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F–TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Essential Questions

1. What is a radian?
2. What is a unit circle and how is it helpful?
3. What is a reference angle and how is it found?
4. How do you find the point at which the terminal side of an angle intersects the unit circle?
5. What are the special angles and how do you find their trigonometric ratios?

WORDS TO KNOW

arc length the distance between the endpoints of an arc; written as $d(\overset{\frown}{ABC})$ or $m\overset{\frown}{AC}$

central angle an angle with its vertex at the center of a circle

cosecant the reciprocal of sine, $\csc\theta = \frac{1}{\sin\theta}$; the cosecant of $\theta =$ $\csc\theta = \frac{\text{length of hypotenuse}}{\text{length of opposite side}}$

cosine a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the side adjacent to the length of the hypotenuse; the cosine of $\theta = \cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$

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cotangent — the reciprocal of tangent, $\cot\theta = \frac{1}{\tan\theta}$; the cotangent of θ = $\cot\theta = \frac{\text{length of adjacent side}}{\text{length of opposite side}}$

coterminal angles — angles that, when drawn in standard position, share the same terminal side

initial side — the stationary ray of an angle from which the measurement of the angle starts

radian — the measure of the central angle that intercepts an arc equal in length to the radius of the circle; π radians = 180°

reference angle — the acute angle that the terminal side makes with the *x*-axis. The sine, cosine, and tangent of the reference angle are the same as that of the original angle (except for the sign, which is based on the quadrant in which the terminal side is located).

secant — the reciprocal of cosine, $\sec\theta = \frac{1}{\cos\theta}$; the secant of θ = $\sec\theta = \frac{\text{length of hypotenuse}}{\text{length of adjacent side}}$

sine — a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the hypotenuse; the sine of θ = $\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$

standard position (of an angle) — a position in which the vertex of the angle is at the origin of the coordinate plane and is the center of the unit circle. The angle's initial side is located along the positive *x*-axis and the terminal side may be in any location.

subtended arc — the section of an arc formed by a central angle that passes through the circle, thus creating the endpoints of the arc

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tangent	a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the adjacent side; the tangent of $\theta = \tan\theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$
terminal side	for an angle in standard position, the movable ray of an angle that can be in any location and which determines the measure of the angle
***theta* (θ)**	a Greek letter commonly used to refer to unknown angle measures
unit circle	a circle with a radius of 1 unit. The center of the circle is located at the origin of the coordinate plane.

Recommended Resources

- BetterExplained.com. "Intuitive Guide to Angles, Degrees, and Radians."

  http://www.walch.com/rr/00218

  This website provides a creative explanation of the radian system of angle measure.
- Khan Academy. "Unit Circle Definition of Trig Functions."

  http://www.walch.com/rr/00219

  This site includes videos and examples of how to use the unit circle to find trigonometric functions.
- MathIsFun.com. "Interactive Unit Circle."

  http://www.walch.com/rr/00220

  This site includes an interactive unit circle and the corresponding points on a coordinate plane.

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Lesson 3.1.1: Radians

Warm-Up 3.1.1

Marissa is running a race on a circular track that has a radius of 200 feet. She is competing against three other runners, and they have all taken their marks along the inside of the track. Marissa's coach stands in the center of the circle to watch as the race gets underway. After 30 seconds, the coach notes each racer's location:

- Marissa is about 125° from the starting point.
- The second racer is about 115° from the starting point.
- The third racer is about 120° from the starting point.
- The fourth racer is about 135° from the starting point.

For any circle, $s = \pi r \frac{\theta}{180}$, where s = arc length, r = radius, and θ = the measure of the central angle of the arc in degrees. Use this information to answer the questions that follow. Round your answers to the nearest foot.

1. Approximately how far has Marissa run around the track?

2. Approximately how far has the second racer run?

3. Approximately how far has the third racer run?

4. Approximately how far has the fourth racer run?

5. Approximately how far is Marissa from the leader?

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Lesson 3.1.1: Radians

Common Core State Standards

F–TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F–TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Warm-Up 3.1.1 Debrief

1. Approximately how far has Marissa run around the track?

   Use the given formula for arc length to determine how far Marissa has run. Substitute known values for the radius (200) and the angle measure (125), and then solve for s.

   $s = \pi r \frac{\theta}{180}$ Formula for arc length

   $s = \pi(200)\frac{(125)}{180}$ Substitute 200 for r and 125 for θ.

   $s \approx 436$ Use your calculator to simplify.

   Marissa has run approximately 436 feet around the track.

2. Approximately how far has the second racer run?

   Use the given formula for arc length, substituting known values for the radius (200) and the angle measure (115); then, solve for s.

   $s = \pi r \frac{\theta}{180}$ Formula for arc length

   $s = \pi(200)\frac{(115)}{180}$ Substitute 200 for r and 115 for θ.

   $s \approx 401$ Use your calculator to simplify.

   The second racer has run approximately 401 feet.

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3. Approximately how far has the third racer run?

   Again, use the given formula for arc length, substituting known values for the radius (200) and the angle measure (120); then, solve for s.

   $s = \pi r \frac{\theta}{180}$ Formula for arc length

   $s = \pi(200)\frac{(120)}{180}$ Substitute 200 for r and 120 for θ.

   $s \approx 419$ Use your calculator to simplify.

   The third racer has run approximately 419 feet.

4. Approximately has far has the fourth racer run?

   Once again, use the given formula for arc length, substituting known values for the radius (200) and the angle measure (135); then, solve for s.

   $s = \pi r \frac{\theta}{180}$ Formula for arc length

   $s = \pi(200)\frac{(135)}{180}$ Substitute 200 for r and 135 for θ.

   $s \approx 471$ Use your calculator to simplify.

   The fourth racer has run approximately 471 feet.

5. Approximately how far is Marissa from the leader?

   The leader, the fourth racer, has run approximately 471 feet.

   Marissa has run approximately 436 feet.

   Subtract to find the difference.

   $471 - 436 = 35$

   Marissa is approximately 35 feet behind the leader.

Connection to the Lesson

- Students will extend their understanding of angle measures to include radians.
- Students will discover the relationship between radians and the arc length of a circle.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- measuring angles with degrees (7.G.2)
- identifying the basic parts of a circle, e.g., radius, circumference, and arc (7.G.4)
- using conversion factors to convert units of measurement (N-Q.1)

Introduction

The most familiar unit used to measure angles is the degree, where one degree represents $\frac{1}{360}$ of a full rotation. This unit of measurement and its value originated from ancient mathematicians. Some modern theorists propose the number 360 was chosen because the ancient Babylonian calendar had 360 days in the year. Although the number of degrees in a full rotation appears to have been an arbitrary choice, there is another system of angle measurement that is not arbitrary.

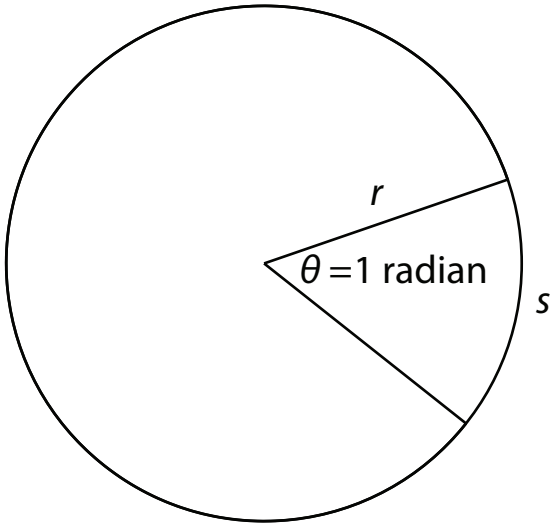
Key Concepts

- A **radian** is the measure of the central angle that intercepts an arc equal in length to the radius of the circle.
- A **central angle** is an angle with its vertex at the center of a circle.
- The radian system of measurement compares the length of the arc that the angle subtends to (intersects) the radius.
- The **subtended arc** is the section of an arc formed by a central angle that passes through the circle, thus creating the endpoints of the arc.
- The formula used to represent this relationship is $\theta = \frac{s}{r}$, in which θ is the angle measure in radians, s is the measure of the **arc length** (the distance between the endpoints of an arc), and r is the radius of the circle.
- The lowercase Greek letter ***theta*** **(θ)** is commonly used to refer to an unknown angle measure.

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Instruction

- The measure of an angle described as θ is 1 radian when the arc length equals the radius, as shown in the figure.



- This can be verified mathematically with the aforementioned formula, $\theta = \frac{s}{r}$; for any arc length s or radius r, if $s = r$, then $\frac{s}{r} = 1$.
- One radian is approximately equal to 57.3°. While this may appear to be an arbitrary value, recall that the number 360 was an arbitrarily chosen number. Radians are connected mathematically to the properties of a circle in a more easily identifiable way.
- Radians are often expressed in terms of π, which provides an exact measurement instead of a decimal approximation.
- A full rotation (360°) is equal to 2π radians. This is because the arc length of a full rotation is also the circumference of a circle, equal to $2\pi r$. Thus, when $2\pi r$ (the arc length, s) is divided by r, the result is $\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi$ radians.
- Therefore, a half rotation (180°) is equal to π radians (since $\frac{360^\circ}{2} = 180^\circ$, so by substituting 2π radians for 360°, $\frac{2\pi \text{ radians}}{2} = \pi$ radians), and a 90° rotation is equal to $\frac{\pi}{2}$ radians.

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Instruction

- To convert from radians to degrees or vice versa, use an appropriate conversion factor based on the relationship between π radians and 180°. If converting from radians to degrees, multiply by $\frac{180°}{\pi \text{ radians}}$ so that the radians cancel and degrees remain. Alternately, if converting from degrees to radians, multiply by $\frac{\pi \text{ radians}}{180°}$ so that the degrees cancel and radians remain.

Common Errors/Misconceptions

- using the incorrect conversion factor when converting; i.e., using the reciprocal
- mistakenly believing that $\pi = 180°$ instead of π radians = 180°

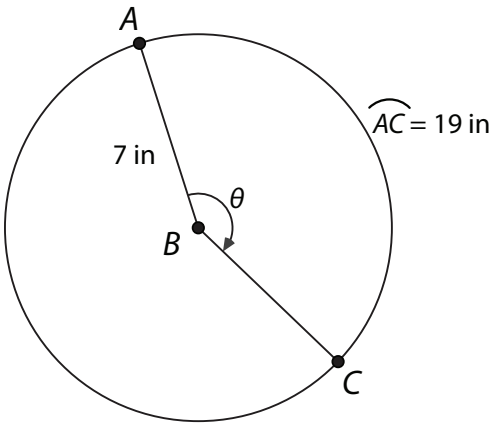
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Instruction

Guided Practice 3.1.1

Example 1

Given the diagram of $\odot B$, find the measure of θ in radians. Round your answer to the nearest ten-thousandth.



1. Identify the length of the radius and the arc length.

   The length of the radius of the circle is 7 inches and the length of the arc the angle subtends is 19 inches. Thus, $r = 7$ and $s = 19$.

2. Substitute r and s into the formula $\theta = \frac{s}{r}$ and solve for θ.

   The formula $\theta = \frac{s}{r}$ describes the relationship among an angle measure in radians, an arc length, and the radius of a circle.

   Substitute the known values of r and s into the formula, then solve for θ to determine the measure of the angle in radians.

$\theta = \frac{s}{r}$	Formula for the measure of an angle in radians
$\theta = \frac{(19)}{(7)}$	Substitute 19 for s and 7 for r.
$\theta \approx 2.7143$	Use a calculator to simplify.

   The measure of the angle is approximately 2.7143 radians.

   Notice that the angle measure in radians shows the ratio of the arc length to the length of the radius. In this instance, the arc length is approximately 2.7143 times the length of the radius.

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Instruction

Example 2

Convert 78° to radians. Give your answer as an exact answer and also as a decimal rounded to the nearest ten-thousandth.

1. Determine which conversion factor to use.

   Since degrees need to be converted to radians, multiply by $\frac{\pi \text{ radians}}{180^\circ}$ so that the degrees in the denominator will cancel out and radians will remain.

2. Multiply 78° by the conversion factor.

   Recall that π radians = 180°. Thus, $\frac{\pi \text{ radians}}{180^\circ} = 1$, since dividing one quantity by its equivalent is equal to 1. Therefore, multiply 78° by the chosen conversion factor in order to convert the degree measure to its radian equivalent.

   $78^\circ \bullet \frac{\pi \text{ radians}}{180^\circ}$ Multiply by the conversion factor, $\frac{\pi \text{ radians}}{180^\circ}$.

   $\frac{78\pi}{180}$ Multiply, and cancel out the degree symbols.

   $\frac{13\pi}{30}$ Reduce the fraction.

   Converted to radians, the exact measure of 78° is $\frac{13\pi}{30}$ radians.

3. Use your calculator to find the measure of the angle as a decimal.

   Multiply 13 by π and divide by 30.

   $\frac{13\pi}{30} \approx 1.3614$

   Converted to radians, the decimal measure of 78° is approximately 1.3614 radians.

   Recall that radians compare the value of the arc length to the value of the radius; therefore, for this 78° angle, the arc length is approximately 1.3614 times the length of the radius.

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Instruction

Example 3

Convert $\frac{2\pi}{3}$ radians to degrees.

1. Determine which conversion factor to use.

   Since radians need to be converted to degrees, multiply by $\frac{180°}{\pi \text{ radians}}$ so that the radians in the denominator will cancel out and degrees will remain.

2. Multiply $\frac{2\pi}{3}$ radians by the conversion factor.

$\frac{2\pi \text{ radians}}{3} \cdot \frac{180°}{\pi \text{ radians}}$	Multiply by the conversion factor, $\frac{180°}{\pi \text{ radians}}$.
$\frac{360\pi°}{3\pi}$	Multiply the numerators and denominators, canceling out the radians.
$120°$	Reduce the fraction, canceling out π.

Converted to degrees, the measure of $\frac{2\pi}{3}$ radians is 120°.

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Instruction

Example 4

Convert 0.5793 radian to degrees. Round your answer to the nearest tenth.

1. Determine which conversion factor to use.

   Since radians need to be converted to degrees, multiply by $\frac{180°}{\pi \text{ radians}}$.

2. Multiply 0.5793 radian by the conversion factor.

$0.5793 \text{ radian} \cdot \frac{180°}{\pi \text{ radians}}$	Multiply by the conversion factor, $\frac{180°}{\pi \text{ radians}}$.
$\frac{180°(0.5793)}{\pi}$	Cancel out the radians.
$\approx 33.2°$	Use a calculator to simplify.

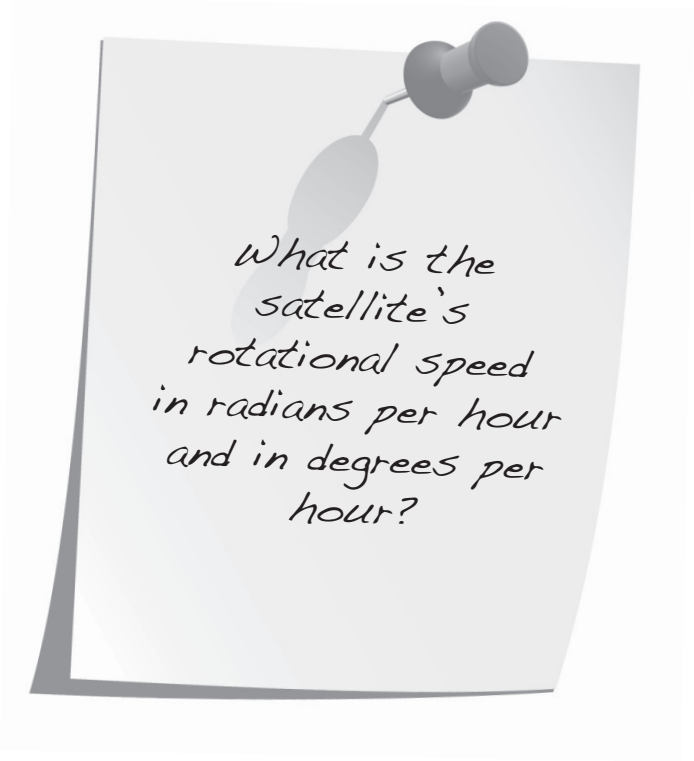
Converted to degrees, 0.5793 radian is approximately 33.2°.

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Problem-Based Task 3.1.1: A Satellite's Story

SMP	
1 ✓	2 ✓
3	4
5	6 ✓
7	8

A communications satellite located approximately 22,223 miles above the Earth's surface orbits the planet at a speed of about 7,000 miles per hour. What is the satellite's rotational speed in radians per hour and in degrees per hour? The diameter of the Earth is 7,918 miles. Round radians to the nearest thousandth and degrees to the nearest tenth.



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Problem-Based Task 3.1.1: A Satellite's Story

Coaching

a. How many miles does the satellite travel in 1 hour?

b. How can you find the radius of the satellite's orbit?

c. What is the radius of the Earth?

d. What is the radius of the satellite's orbit?

e. How many radians does the satellite travel in 1 hour?

f. What is the satellite's rotational speed in radians per hour?

g. How many degrees does the satellite travel in 1 hour?

h. What is the satellite's rotational speed in degrees per hour?

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Instruction

Problem-Based Task 3.1.1: A Satellite's Story

Coaching Sample Responses

a. How many miles does the satellite travel in 1 hour?

Since *distance* = *rate* • *time*, or $d = rt$, multiply the rate at which the satellite travels by the desired time frame. The satellite is traveling at a speed of about 7,000 miles per hour. The time is 1 hour.

$d = rt$

$d = 7000(1) = 7000$

The satellite travels approximately 7,000 miles in 1 hour.

b. How can you find the radius of the satellite's orbit?

The radius of the satellite's orbit is the total distance from the center of the Earth to the satellite's position. To find this, add the Earth's radius to the satellite's altitude.

c. What is the radius of the Earth?

The diameter of the Earth is 7,918 miles, and the radius is half the length of the diameter.

$7918 \div 2 = 3959$

The radius of the Earth is 3,959 miles.

d. What is the radius of the satellite's orbit?

Add the Earth's radius determined in part c (3,959 miles) to the satellite's altitude as given in the problem statement (approximately 22,223 miles).

$22{,}223 + 3{,}959 = 26{,}182$

The radius of the satellite's orbit is approximately 26,182 miles.

e. How many radians does the satellite travel in 1 hour?

Use the formula $\theta = \frac{s}{r}$, substituting 7,000 for the arc length (s) and 26,182 for the radius (r).

$$\theta = \frac{s}{r} = \frac{(7{,}000)}{(26{,}182)} \approx 0.267 \text{ radian}$$

The satellite travels approximately 0.267 radian in 1 hour.

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f. What is the satellite's rotational speed in radians per hour?

The satellite travels about 0.267 radian each hour. Thus, its rotational speed is approximately 0.267 radian per hour.

g. How many degrees does the satellite travel in 1 hour?

Convert 0.267 radian to degrees. Multiply by the conversion factor, $\frac{180^\circ}{\pi \text{ radians}}$.

$$0.267 \text{ radian} \bullet \frac{180^\circ}{\pi \text{ radians}} = \frac{48.06^\circ}{\pi} \approx 15.3^\circ$$

The satellite travels approximately 15.3° in 1 hour.

h. What is the satellite's rotational speed in degrees per hour?

Since the satellite travels about 15.3° each hour, its rotational speed is approximately 15.3° per hour.

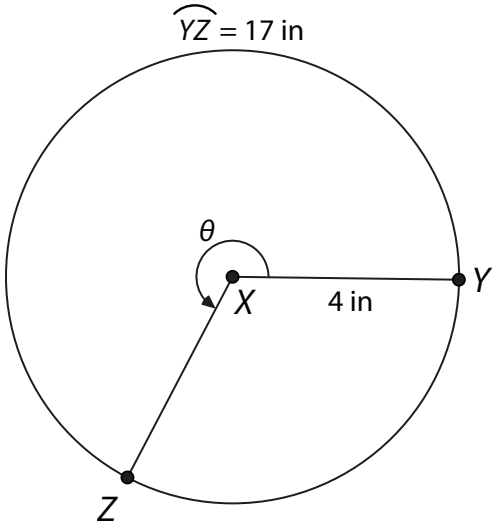
Recommended Closure Activity

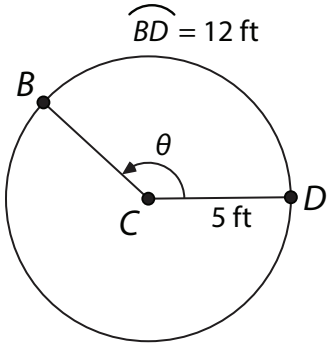
Select one or more of the essential questions for a class discussion or as a journal entry prompt.

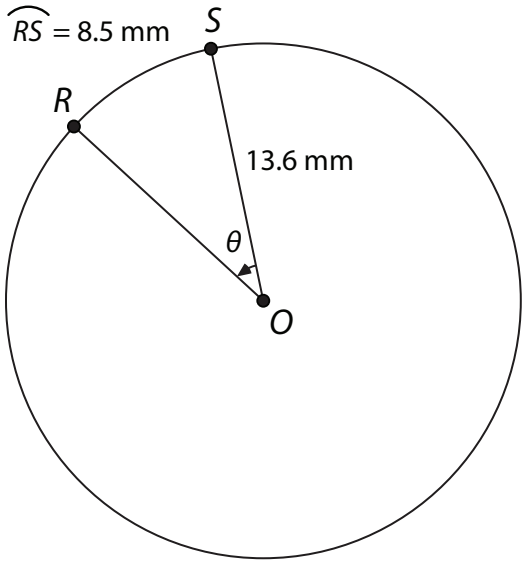
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Practice 3.1.1: Radians

For problems 1–3, use the information in the diagrams to find the angle measure of θ in radians. Round your answer to the nearest ten-thousandth, if necessary.

1. 

2. 

3. 

continued

SWB p. 13; SRB Practice 3.1.1 p. U3-11

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For problems 4–7, convert each radian measure to degrees. Round your answer to the nearest tenth, if necessary.

4. $\frac{8\pi}{9}$ radians

5. $\frac{15\pi}{11}$ radians

6. 3.0672 radians

7. 0.9431 radian

Read the following scenario, and use the information in it to complete problems 8–10.

> Mitra, Charlene, and Abbie are taking turns spinning the merry-go-round at the park. They each spin the carousel at a different speed in degrees per second. How fast is each girl's spin in radians per second? Supply an exact answer and also a decimal approximation rounded to the nearest ten-thousandth.

8. Mitra's spin speed: 215° per second

9. Charlene's spin speed: 21° per second

10. Abbie's spin speed: 104° per second

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Lesson 3.1.2: The Unit Circle

Warm-Up 3.1.2

Anna and her friend Isabella are sitting on a spinning ride at their favorite amusement park. Once the ride gets up to speed, it spins at a rate of 115° per second. Use this information to answer the questions that follow. Remember that $\theta = \frac{s}{r}$.

1. How fast does the ride spin in radians per second? Give an exact answer.

2. Anna is sitting 22 feet from the center of the ride. How far does she travel each second? Round your answer to the nearest tenth.

3. Isabella is sitting 17 feet from the center of the ride. How far does she travel each second? Round your answer to the nearest tenth.

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Instruction

Lesson 3.1.2: The Unit Circle

Common Core State Standard

F–TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Warm-Up 3.1.2 Debrief

1. How fast does the ride spin in radians per second? Give an exact answer.

   Convert 115° to radians using a conversion factor that will cancel out the degrees.

   $115° \cdot \frac{\pi \text{ radians}}{180°}$ Multiply 115° by $\frac{\pi \text{ radians}}{180°}$.

   $\frac{115\pi}{180}$ Simplify.

   $\frac{23\pi}{36}$ Reduce.

   The ride spins at a rate of $\frac{23\pi}{36}$ radians per second.

2. Anna is sitting 22 feet from the center of the ride. How far does she travel each second? Round your answer to the nearest tenth.

   The distance Anna travels is equal to *s*, the arc length, in the equation $\theta = \frac{s}{r}$.

   Since Anna is sitting 22 feet away from the ride's center, this distance represents the radius, *r*. Therefore, $r = 22$ feet.

   The angle measure, θ, can be determined by multiplying the ride's spin rate, $\frac{23\pi}{36}$ radians per second, by Anna's travel time, 1 second. Thus, $\theta = \frac{23\pi}{36}$ radians.

   Substitute this information into the formula and solve for *s* to determine the arc length or distance traveled.

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$\theta = \frac{s}{r}$	Formula for an angle measure in radians
$r \bullet \theta = s$	Multiply both sides by r.
$s = r \bullet \theta$	Apply the Symmetric Property of Equality.
$s = (22) \bullet \left(\frac{23\pi}{36} \right)$	Substitute 22 for r and $\frac{23\pi}{36}$ for θ.
$s \approx 44.2$	Simplify.

Anna travels approximately 44.2 feet each second.

3. Isabella is sitting 17 feet from the center of the ride. How far does she travel each second? Round your answer to the nearest tenth.

   Use the rearranged version of $\theta = \frac{s}{r}$: $s = r \bullet \theta$.

   Since Isabella is only 17 feet from the center of the ride, $r = 17$.

   The ride still spins at the same rate, and the time is still 1 second, so the angle measure is the same, $\theta = \frac{23\pi}{36}$ radians.

   Substitute the known information and solve for s.

$s = (17) \bullet \left(\frac{23\pi}{36} \right)$	Substitute 17 for r and $\frac{23\pi}{36}$ for θ.
$s \approx 34.1$	Simplify.

   Isabella travels approximately 34.1 feet each second.

Connection to the Lesson

- Students will continue to study radians and discover how radians relate to the unit circle.
- Students will become more familiar with using radians to measure angles.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- measuring angles in radians (F-TF.1)
- converting between degrees and radians (F-TF.1)
- finding the basic trigonometric functions of sine, cosine, and tangent (F-TF.2)

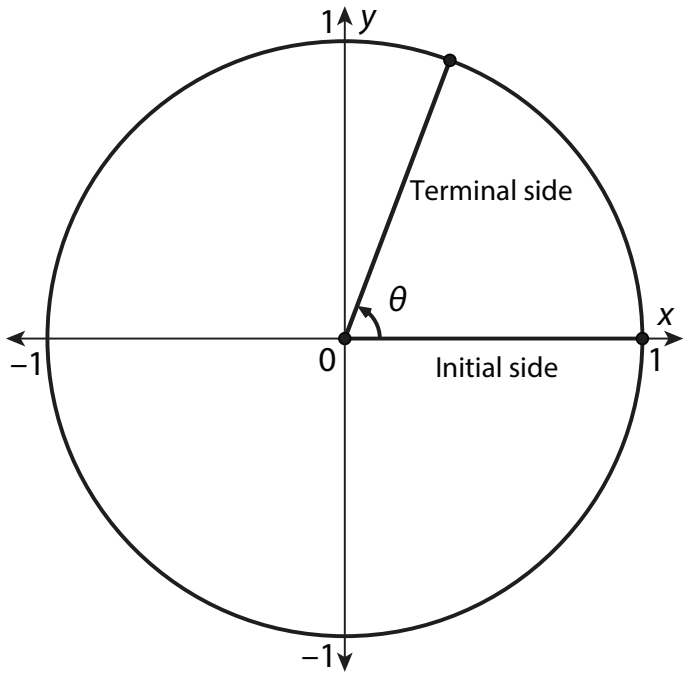
Introduction

A **unit circle** is a circle that has a radius of 1 unit, with the center of the circle located at the origin of the coordinate plane. Because $r = 1$ in the unit circle, it can be a useful tool for discussing arc lengths and angles in circles. An angle in a unit circle can be studied in radians or degrees; however, since radians directly relate an angle measure to an arc length, radian measures are more useful in calculations.

Key Concepts

- Angles are typically in **standard position** on a unit circle. This means that the center of the circle is placed at the origin of the coordinate plane, and the vertex of the angle is on the origin at the center of the circle. The **initial side** of the angle (the stationary ray from which the measurement of the angle starts) is located along the positive x-axis. The **terminal side** (the movable ray that determines the measure of the angle) may be in any location.

An angle in standard position on the unit circle



SRB p. U3-13

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Instruction

- The terminal side of the angle may be rotated counterclockwise to create a positive angle or clockwise to create a negative angle.
- To sketch an angle in radians on the unit circle, remember that halfway around the circle (180°) is equal to π radians and that a full rotation (360°) is equal to 2π radians. Then use the fraction of π to estimate the angle's location, if it falls somewhere between these measures.
- Within the unit circle, each angle has a **reference angle**. The reference angle is always the acute angle that the terminal side makes with the *x*-axis. The reference angle's sine, cosine, and tangent are the same as that of the original angle except for the sign, which is based on the quadrant in which the terminal side is located.
- Recall that a right triangle has one right angle and two acute angles (less than 90°). Sine, cosine, and tangent are trigonometric functions of an acute angle θ in a right triangle and are determined by the ratios of the lengths of the opposite side, adjacent side, and the hypotenuse of that triangle, summarized as follows.
  - The **sine** of $\theta = \sin\theta = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$.
  - The **cosine** of $\theta = \cos\theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$.
  - The **tangent** of $\theta = \tan\theta = \dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$.
- To find a reference angle, first sketch the original angle to determine which quadrant it lies in. Then, determine the measure of the angle between the terminal side and the *x*-axis. The following table shows the relationships between the reference angle and the original angle (θ) for each quadrant.

Quadrant	Reference angle (degrees)	Reference angle (radians)
I	same as θ	same as θ
II	$180° - \theta$	π radians $- \theta$
III	$\theta - 180°$	$\theta - \pi$ radians
IV	$360° - \theta$	2π radians $- \theta$

- If an angle is larger than 2π radians (360°), subtract a full rotation (2π radians or 360°) until the angle is less than 2π radians (360°). Then, find the reference angle of the resulting angle.
- The coordinates of the point at which the terminal side intersects the unit circle are always given by $(\cos\theta, \sin\theta)$, where θ is the measure of the angle.

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Instruction

Common Errors/Misconceptions

- mistakenly thinking that the reference angle in Quadrants II and IV is the angle that the terminal side makes with the *y*-axis
- reversing sin θ and cos θ when attempting to find the coordinates of the point where the terminal side intersects the unit circle

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Instruction

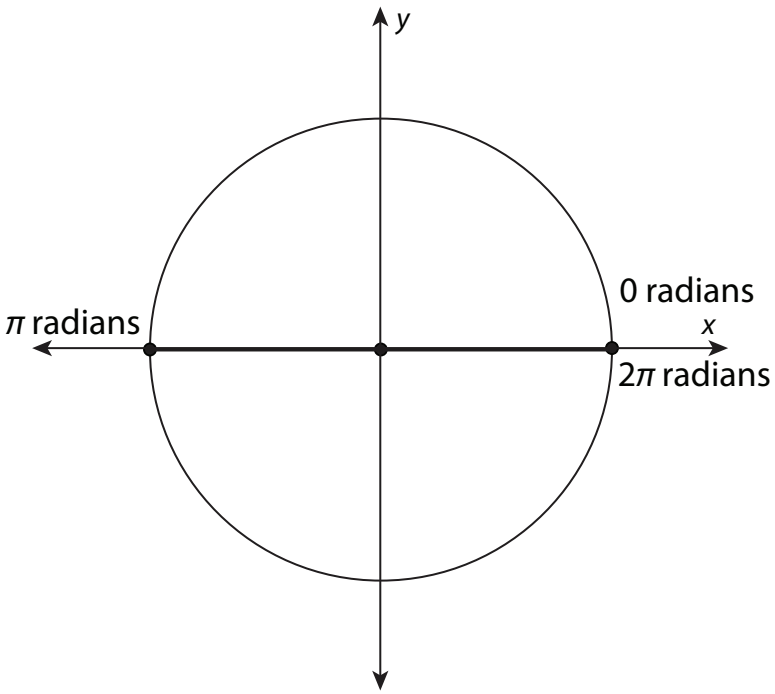
Guided Practice 3.1.2

Example 1

On a unit circle, sketch angles that measure $\frac{2\pi}{3}$ radians, $\frac{\pi}{4}$ radian, and $\frac{9\pi}{7}$ radians.

1. Sketch a unit circle, and then label π radians and 2π radians.

   A half rotation (180°) is π radians and a full rotation (360°) is 2π radians. Notice that 0 radians and 2π radians are in the same location on the unit circle, but represent different angle measures.

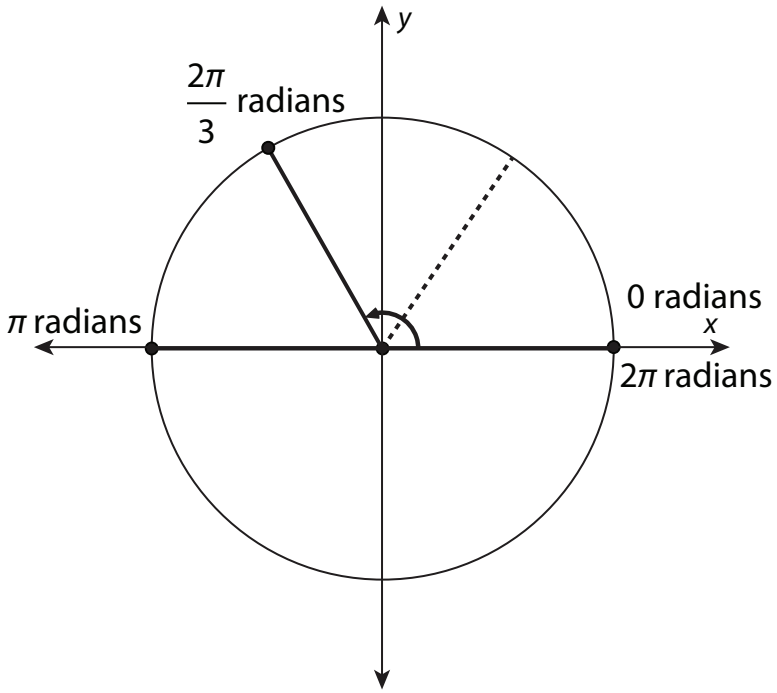


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Instruction

2. Sketch $\frac{2\pi}{3}$ radians.

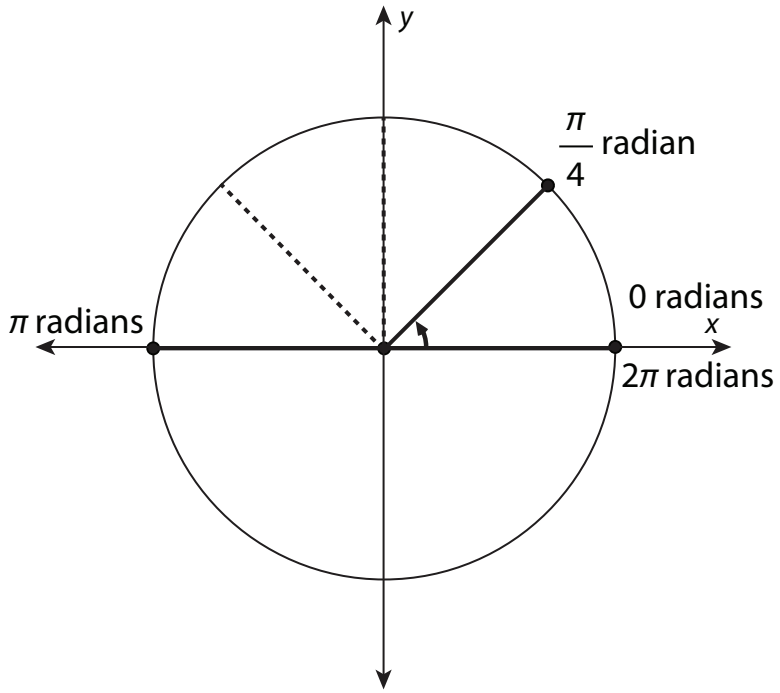
$\frac{2\pi}{3}$ is the same as $\frac{2}{3}\pi$. In other words, the terminal side is $\frac{2}{3}$ of the way between 0 and π. Thus, imagine the semicircle between 0 radians and π radians split into thirds, and then sketch the angle $\frac{2}{3}$ of the way around the semicircle.



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3. Sketch $\frac{\pi}{4}$ radian.

   $\frac{\pi}{4}$ is the same as $\frac{1}{4}\pi$. In other words, it is $\frac{1}{4}$ of the way to π. Thus, imagine the semicircle between 0 radians and π radians split into fourths, and then sketch the angle $\frac{1}{4}$ of the way around the semicircle.



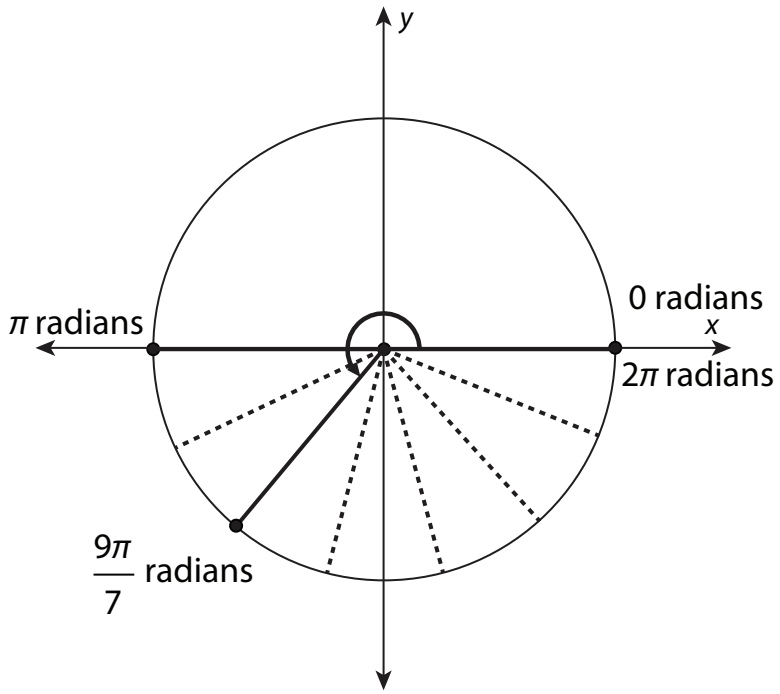
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4. Sketch $\frac{9\pi}{7}$ radians.

   $\frac{9\pi}{7}$ is the same as $\frac{9}{7}\pi$, which is equal to $1\frac{2}{7}\pi$.

   Because this value is greater than π, it goes beyond π radians. It is $\frac{2}{7}$ of the way past π.

   Imagine the semicircle between π radians and 2π radians split into sevenths, and then sketch the angle $\frac{2}{7}$ of the way around the semicircle.

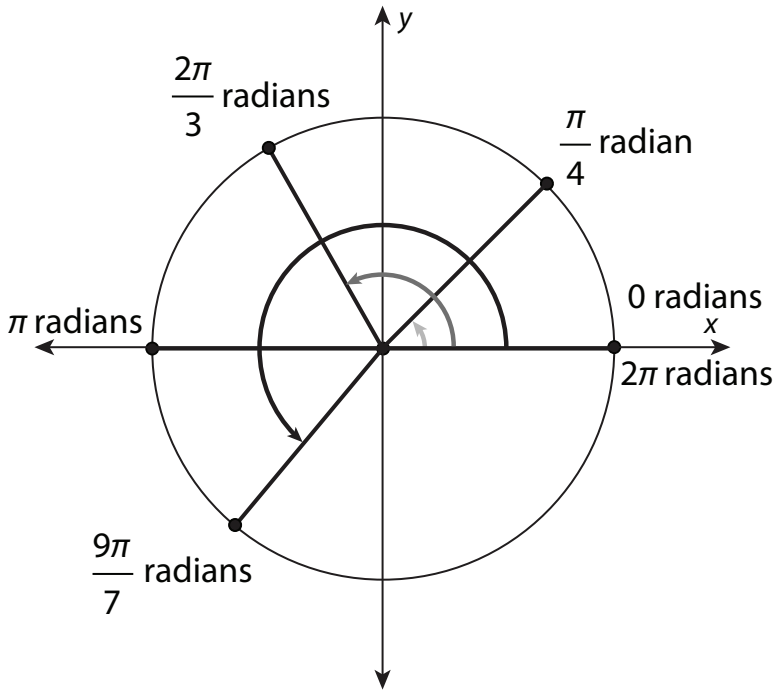


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Instruction

5. Summarize your findings.

The diagram shows the final unit circle with angles that measure $\frac{2\pi}{3}$ radians, $\frac{\pi}{4}$ radian, and $\frac{9\pi}{7}$ radians.



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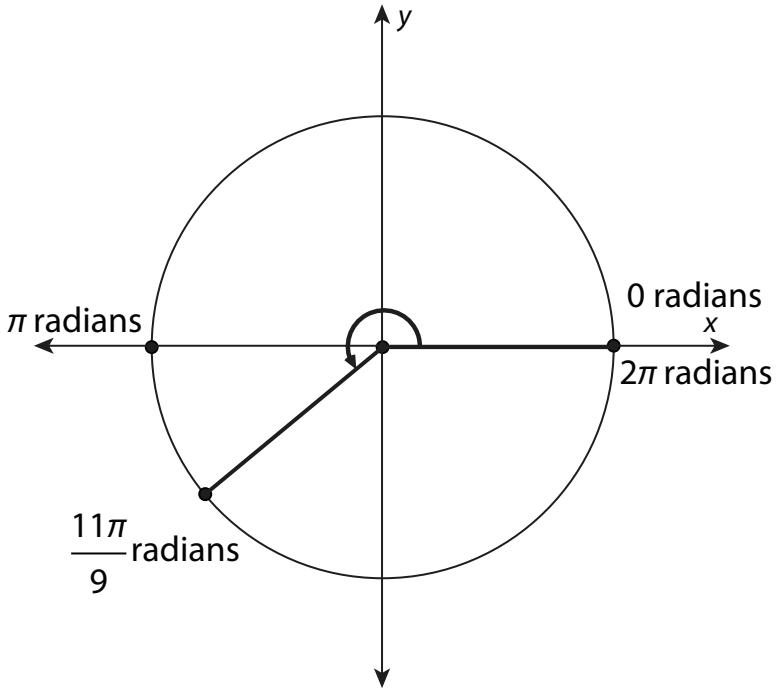
Instruction

Example 2

Find the reference angles for angles that measure $\frac{11\pi}{9}$ radians, $\frac{3\pi}{5}$ radians, and 5.895 radians.

1. Sketch an angle with a measure of $\frac{11\pi}{9}$ radians on the unit circle.

   $\frac{11\pi}{9}$ radians is the same as $1\frac{2}{9}\pi$ radians; therefore, this angle will be $\frac{2}{9}$ of the way between π radians and 2π radians.



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Instruction

2. Determine the measure of the angle between the terminal side and the *x*-axis.

   Since the terminal side falls in Quadrant III, subtract π radians from the original angle measure, $\frac{11\pi}{9}$ radians, to find the measure of the reference angle.

   $\frac{11\pi}{9} - \pi$ Subtract π from the original angle measure.

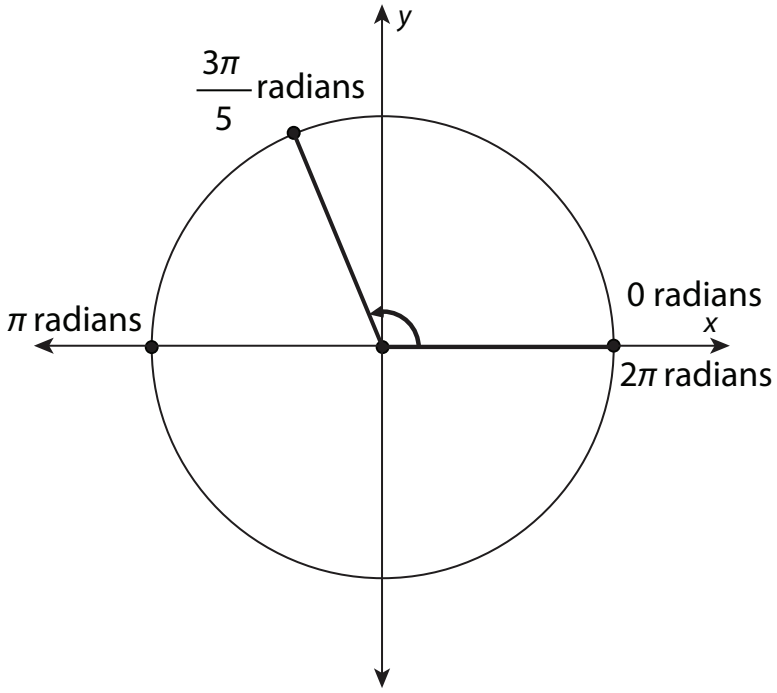
   $= \frac{11\pi}{9} - \frac{9\pi}{9}$ Rewrite π as a fraction with a common denominator.

   $= \frac{2\pi}{9}$ Subtract.

   The reference angle for $\frac{11\pi}{9}$ radians is $\frac{2\pi}{9}$ radians.

3. Sketch $\frac{3\pi}{5}$ radians.

   Sketch $\frac{3\pi}{5}$ radians $\frac{3}{5}$ of the way between 0 and π radians.



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Instruction

4. Determine the measure of the angle between the terminal side and the *x*-axis.

   Since the terminal side falls in Quadrant II, subtract $\frac{3\pi}{5}$ radians from π radians to find the measure of the reference angle.

   $\pi - \frac{3\pi}{5}$ Subtract the original angle measure from π.

   $= \frac{5\pi}{5} - \frac{3\pi}{5}$ Rewrite π as a fraction with a common denominator.

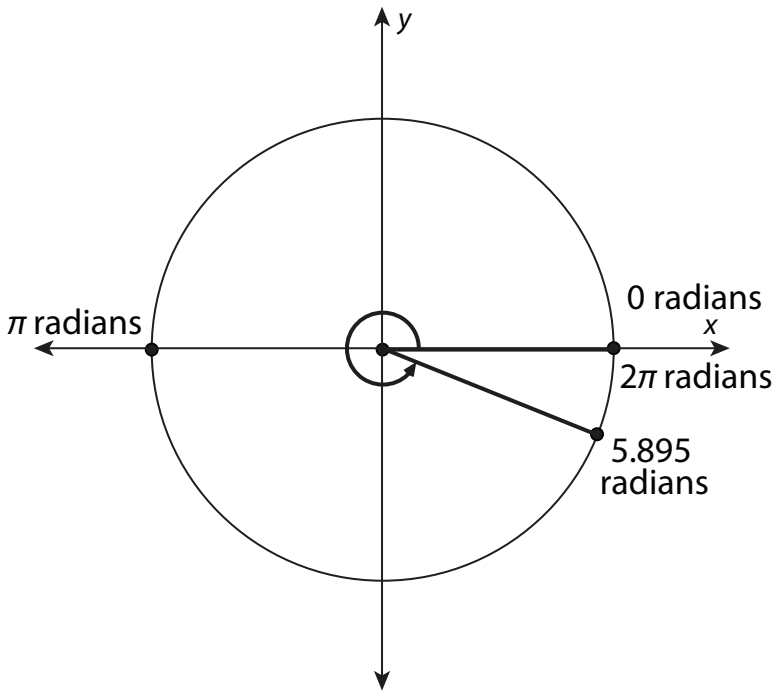
   $= \frac{2\pi}{5}$ Subtract.

   The reference angle for $\frac{3\pi}{5}$ radians is $\frac{2\pi}{5}$ radians.

5. Sketch 5.895 radians.

   Since π is not included in this measurement, we must use decimal approximations. π is approximately 3.14 and 2π is approximately 6.28.

   5.895 is fairly close to 6.28 and thus will fall in Quadrant IV.



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6. Determine the measure of the angle between the terminal side and the x-axis.

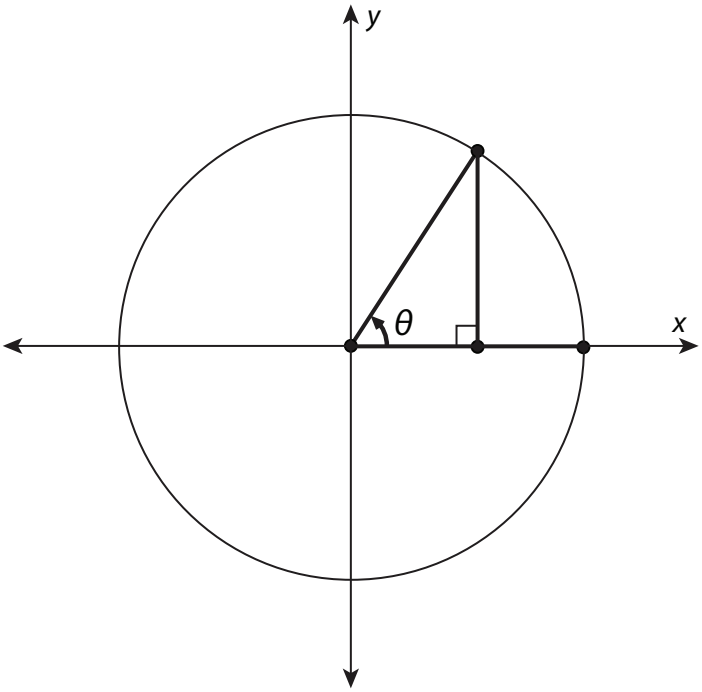
   Since the terminal side falls in Quadrant IV, subtract 5.895 radians from 2π radians to find a more precise measure of the reference angle.

   $2\pi - 5.895 \approx 0.388$

   The measure of the reference angle for 5.895 radians is approximately 0.388 radian.

Example 3

Use the following diagram of an angle in the unit circle to demonstrate why the point where the terminal side intersects the unit circle is $(\cos \theta, \sin \theta)$.



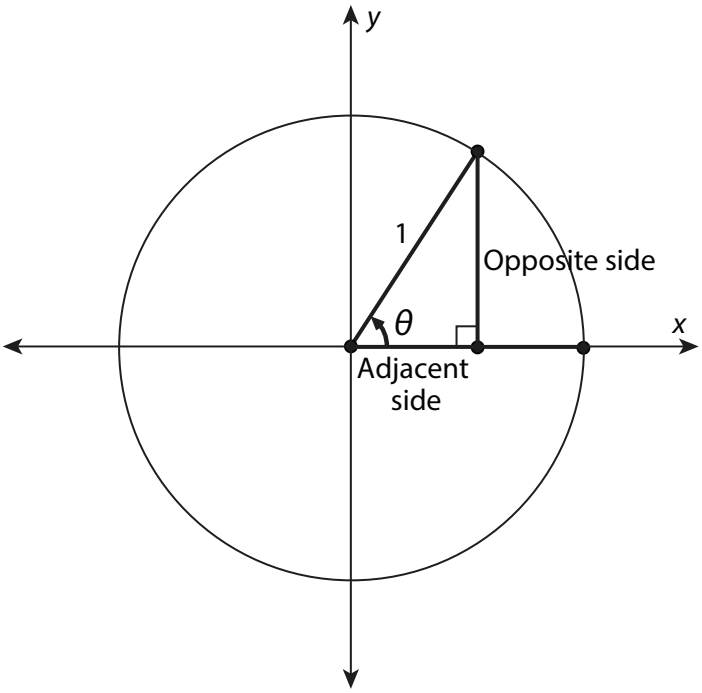
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Instruction

1. Label the three sides of the triangle.

   The hypotenuse of the triangle is also the radius of the circle. Since it is a unit circle, the radius is 1.

   Label the opposite and adjacent sides.



2. Use the cosine ratio to write a statement for the length of the adjacent side.

$$\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$$ Cosine ratio

$$\cos\theta = \frac{\text{length of adjacent side}}{(1)}$$ Substitute 1 for the length of the hypotenuse.

$$\cos\theta = \text{length of adjacent side}$$ Divide by 1.

The length of the adjacent side is equal to cos θ.

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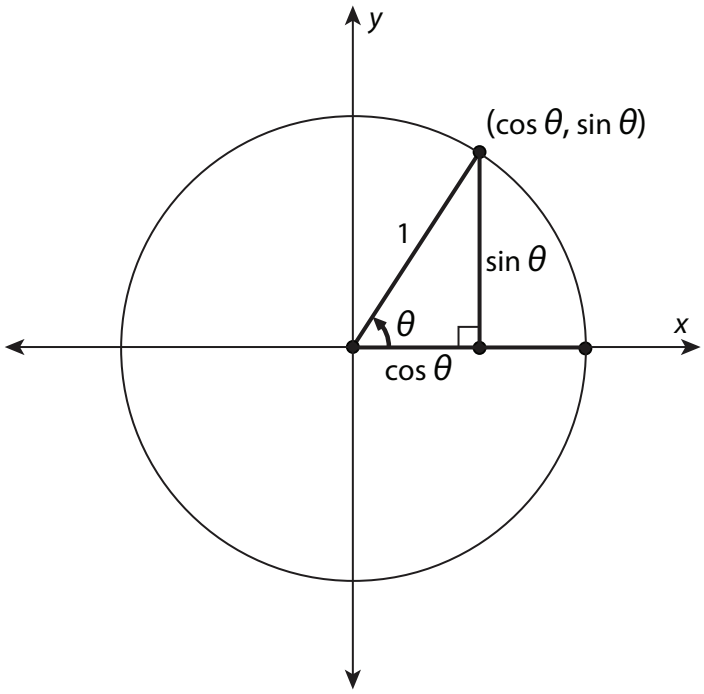
Instruction

3. Use the sine ratio to write a statement for the length of the opposite side.

$\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$	Sine ratio
$\sin\theta = \frac{\text{length of opposite side}}{(1)}$	Substitute 1 for the length of the hypotenuse.
$\sin\theta = \text{length of opposite side}$	Divide by 1.

The length of the opposite side is equal to $\sin\theta$.

4. Label the diagram to show the coordinates of the point of intersection of the terminal side and the unit circle.



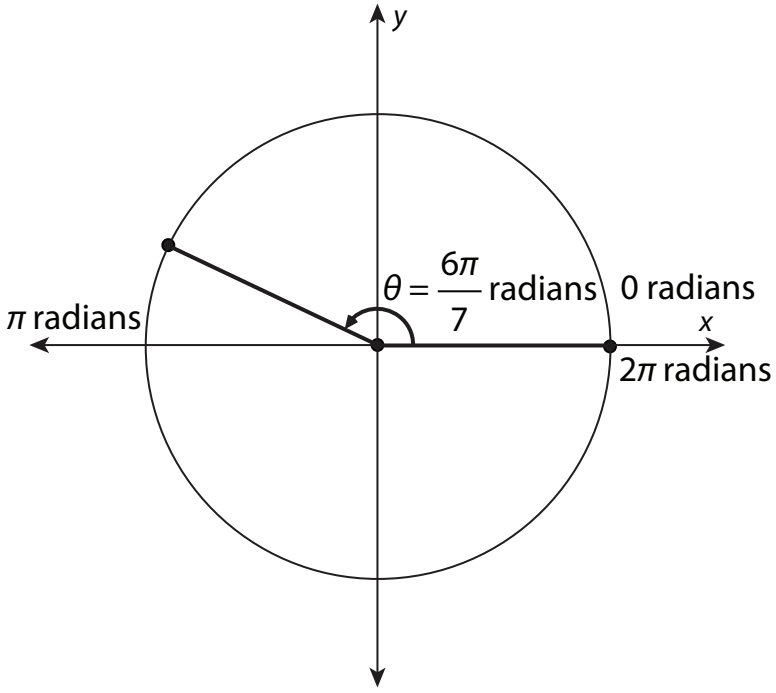
The coordinates of the point where the terminal side intersects the unit circle are $(\cos\theta, \sin\theta)$.

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Instruction

Example 4

Find the coordinates of the point where the terminal side intersects the unit circle. Round each coordinate to the nearest hundredth.



1. Find $\sin\theta$ and $\cos\theta$.

   Since the coordinates of the point where the terminal side intersects the unit circle are given by $(\cos\theta, \sin\theta)$, use this to determine the value of each coordinate.

   We are given in the diagram that $\theta = \frac{6\pi}{7}$ radians ; therefore,

   substitute $\frac{6\pi}{7}$ for θ: $\cos\theta = \cos\frac{6\pi}{7}$ and $\sin\theta = \sin\frac{6\pi}{7}$.

   Ensure your calculator is in radian mode, and then calculate $\cos\frac{6\pi}{7}$ and $\sin\frac{6\pi}{7}$.

   $\cos\frac{6\pi}{7} \approx -0.9010$

   $\sin\frac{6\pi}{7} \approx 0.4339$

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Instruction

2. Write the coordinates of the point of intersection of the terminal side and the unit circle.

   The coordinates of the point of intersection are (cos θ, sin θ).

   Substitute the rounded coordinates: –0.90 for cos θ and 0.43 for sin θ.

   The approximate coordinates of the point of intersection of the terminal side are (–0.90, 0.43).



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Problem-Based Task 3.1.2: A Sticker's Journey

SMP	
1 ✓	2 ✓
3 ✓	4
5	6
7 ✓	8

While riding home from school one day, Darius pauses to show his friend Emma his new bicycle. He explains that the wheels each have a radius of 1 foot. Emma notices that the front tire has a sticker on it, located exactly to the right of the center of the wheel. When Darius backpedals, the sticker moves counterclockwise around the wheel. Emma observes that after Darius backpedaled for 1 second, the sticker had moved 2.5 feet from its original location. At that point, how far is the sticker from the ground? Round your answer to the nearest tenth. Recall the formula $\theta = \frac{s}{r}$, where θ is the angle measure in radians, s is the arc length, and r is the radius.



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Problem-Based Task 3.1.2: A Sticker's Journey

Coaching

a. Draw a sketch of the scenario.

b. How can you determine how many radians the sticker moved?

c. How many radians did the sticker move?

d. What type of circle is the wheel?

e. How can you determine how far the sticker is from the ground?

f. What is the distance from the ground to the center of the wheel?

g. How much higher is the sticker than the center of the wheel?

h. How far is the sticker from the ground?

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Instruction

Problem-Based Task 3.1.2: A Sticker's Journey

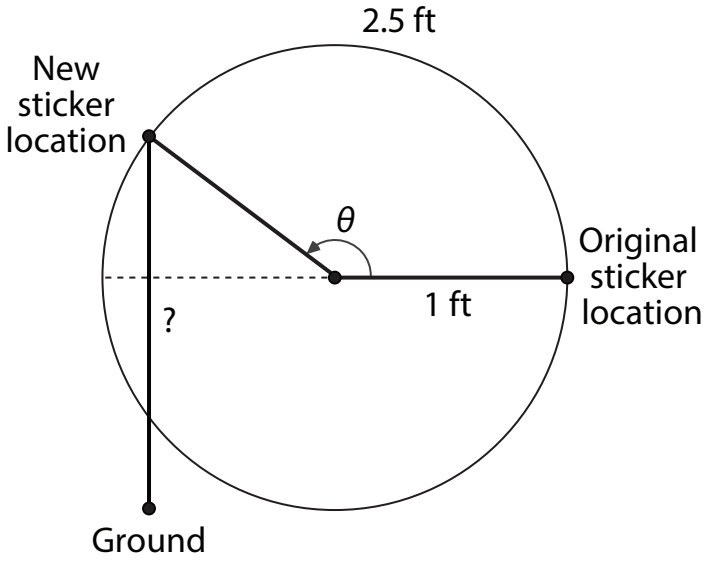
Coaching Sample Responses

a. Draw a sketch of the scenario.

The sticker moved 2.5 feet, so the arc length is 2.5 feet.

The radius is 1 foot.

The angle measure (θ) is unknown.



b. How can you determine how many radians the sticker moved?

Use the formula $\theta = \frac{s}{r}$.

c. How many radians did the sticker move?

Substitute 2.5 for s and 1 for r into the formula $\theta = \frac{s}{r}$, and then solve for θ.

$$\theta = \frac{s}{r}$$

$$\theta = \frac{(2.5)}{(1)}$$

$\theta = 2.5$ radians

The sticker moved 2.5 radians.

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d. What type of circle is the wheel?

Since the radius is 1, the wheel is a unit circle.

e. How can you determine how far the sticker is from the ground?

Find the y-coordinate of the sticker's location using the fact that the point of intersection of the terminal side and the unit circle is $(\cos \theta, \sin \theta)$. The y-coordinate will tell how much higher the sticker is than the center of the wheel.

The distance from the ground to the center of the wheel is the same as the radius.

Add these two distances together to find the height of the sticker.

f. What is the distance from the ground to the center of the wheel?

The radius of the circle is 1, so the distance from the ground to the center of the wheel is also 1.

g. How much higher is the sticker than the center of the wheel?

The y-coordinate of the point of intersection of the terminal side and the unit circle is $\sin \theta$. Substitute 2.5 for θ and calculate the result (after first making sure the calculator is in radian mode).

$\sin 2.5 \approx 0.598$

The sticker is approximately 0.598 feet higher than the center of the wheel.

h. How far is the sticker from the ground?

The distance from the ground to the center of the wheel is 1 foot. The distance from the center of the wheel to the sticker is approximately 0.598 feet. To find the total distance of the sticker from the ground, add these two distances.

$1 + 0.598 = 1.598$ feet

The sticker is approximately 1.6 feet from the ground.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.1.2: The Unit Circle

For problems 1–3, sketch each radian measure on the unit circle.

1. $\frac{2\pi}{9}$ radians
2. $\frac{8\pi}{5}$ radians
3. $\frac{5\pi}{6}$ radians

For problems 4–7, find the reference angle for each angle measure.

4. 318°
5. $\frac{4\pi}{5}$ radians
6. $\frac{\pi}{3}$ radians
7. 4.087 radians

For problems 8–10, find the coordinates of the point where the terminal side of the angle intersects the unit circle. Round each coordinate to the nearest hundredth.

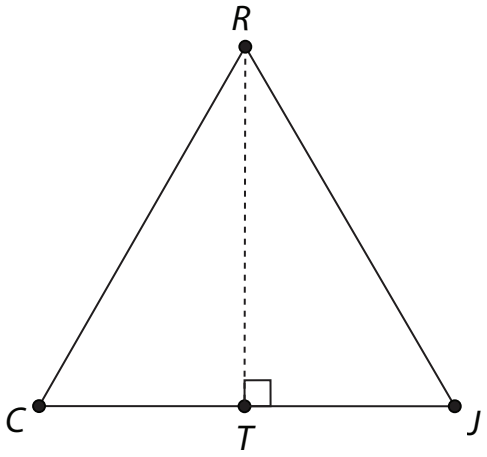
8. $\frac{5\pi}{4}$ radians
9. $\frac{\pi}{6}$ radian
10. 2.834 radians

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Lesson 3.1.3: Special Angles in the Unit Circle

Warm-Up 3.1.3

Raeann and her three friends are all in the school's marching band. For their new halftime show, one of their formations is an equilateral triangle. Raeann, Chris, and Judson stand at the three vertices of the triangle (points *R*, *C*, and *J*, respectively). Their friend Trisha stands halfway between Chris and Judson (at point *T*). Raeann and Chris are 20 feet apart. Use this information and the diagram to answer the questions that follow.



1. How far apart are Chris and Judson?

2. How far apart are Chris and Trisha?

3. How far apart are Raeann and Trisha? Give an exact answer.

4. What are the measures of the angles of $\triangle CTR$ in degrees?

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Lesson 3.1.3: Special Angles in the Unit Circle

Common Core State Standard

F–TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Warm-Up 3.1.3 Debrief

1. How far apart are Chris and Judson?

   Since Chris is at point C and Judson is at point J, the length of $\overline{CJ}$ must be found.

   Since $\triangle RCJ$ is an equilateral triangle, all three of its sides are equal.

   Since Raeann and Chris are 20 feet apart, side $\overline{RC}$ is 20 feet long.

   Since the three sides of $\triangle RCJ$ are equal, $\overline{CJ}$ is also 20 feet long.

   Therefore, Chris and Judson are 20 feet apart.

2. How far apart are Chris and Trisha?

   Trisha (point T) is halfway between Chris (point C) and Judson (point J).

   Since $\overline{CJ}$ is 20 feet long, $\overline{CT}$ (the distance between Chris and Trisha) must be half that distance.

   $20 \div 2 = 10$

   Chris and Trisha are 10 feet apart.

3. How far apart are Raeann and Trisha? Give an exact answer.

   $\triangle CTR$ is a right triangle, so the Pythagorean Theorem can be used to find the length of $\overline{RT}$ (the distance between Raeann and Trisha).

   Let $\overline{RC}$, determined to be 20 feet, be c, the hypotenuse.

   Let $\overline{CT}$, determined to be 10 feet, be a.

$c^2 = a^2 + b^2$	Pythagorean Theorem
$(20)^2 = (10)^2 + b^2$	Substitute 20 for c and 10 for a.
$400 = 100 + b^2$	Simplify the exponents.
$300 = b^2$	Subtract 100 from both sides.
$\sqrt{300} = b$	Take the square root of both sides.
$10\sqrt{3} = b$	Simplify the square root.

   Raeann and Trisha are $10\sqrt{3}$ feet apart, which is approximately 17.3 feet apart.

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4. What are the measures of the angles of $\triangle CTR$ in degrees?

   The measures of the angles can be determined as follows:

   - $m\angle CTR = 90°$ because it is a right angle.
   - $m\angle RCT = 60°$ because the measure of each angle of an equilateral triangle is 60°. (The sum of the angles of a triangle is always 180°, and since the angles are equal, 180 ÷ 3 = 60.)
   - $m\angle CRT = 30°$ because the sum of the measure of the angles of a triangle is 180° and 180 – 90 – 60 = 30.

   Therefore, the measures of the angles of $\triangle CTR$ are 90°, 60°, and 30°.

Connection to the Lesson

- Students will continue to examine special angles and the relationships between the sides of the triangles created.
- Students will study special angles as radians within the context of the unit circle.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

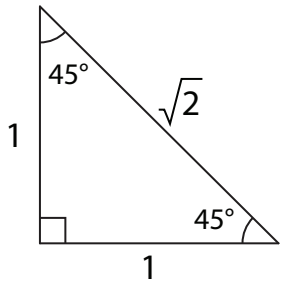
- using the patterns of the 30°–60°–90° triangle and the 45°–45°–90° triangle to find the lengths of the sides of those triangles given one side (G-SRT.6)
- finding the coordinates of the point where the terminal side of an angle intersects the unit circle (G-GPE.4)
- finding reference angles within the unit circle (F-TF.2)
- finding the sine and cosine of special angles (F-TF.2)
- converting between radian and degree measures and vice versa (F-TF.1)

Introduction

Special angles exist within the unit circle. For these special angles, it is possible to calculate the exact coordinates for the point where the terminal side intersects the unit circle. The patterns of the 30°–60°–90° triangle and the 45°–45°–90° triangle can be used to find these points.

Key Concepts

- Recall the pattern for a 45°–45°–90° triangle:

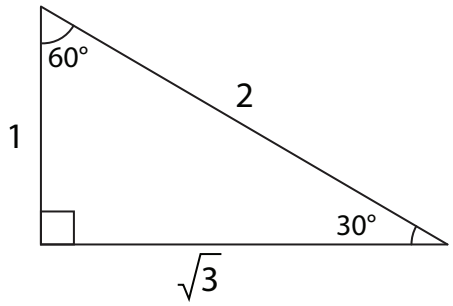


- Notice the two legs are the same length and the hypotenuse is equal to the length of a leg times $\sqrt{2}$.

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Instruction

- Recall the pattern for a 30°–60°–90° triangle:

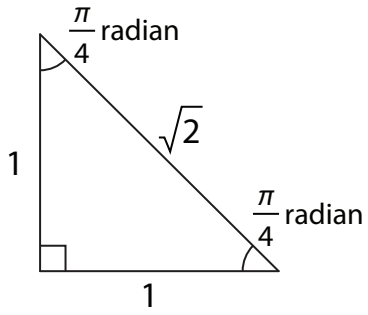


- Notice the hypotenuse is twice as long as the short leg, and the longer leg is equal to the length of the short leg times $\sqrt{3}$.

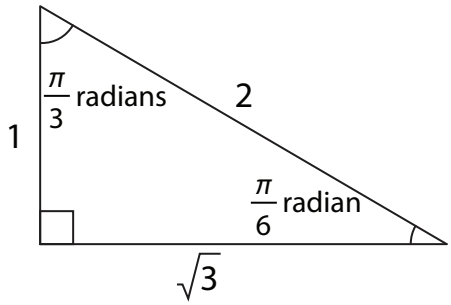
- Each special angle can be viewed in radians as well as degrees: $30° = \frac{\pi}{6}$ radian, $60° = \frac{\pi}{3}$ radians, $90° = \frac{\pi}{2}$ radians, and $45° = \frac{\pi}{4}$ radian.

- Thus, the patterns for the special triangles, noted in radians instead of degrees, are as follows:

45°–45°–90° triangle



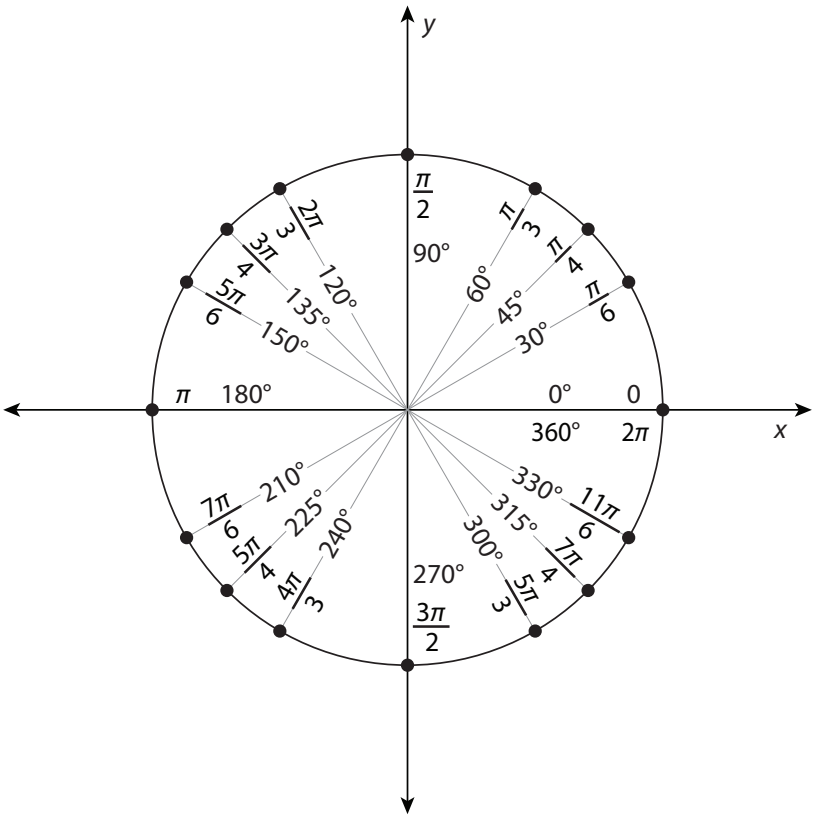
30°–60°–90° triangle



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Instruction

- The special angles continue around the unit circle and can be identified as all angles whose reference angles are special angles.

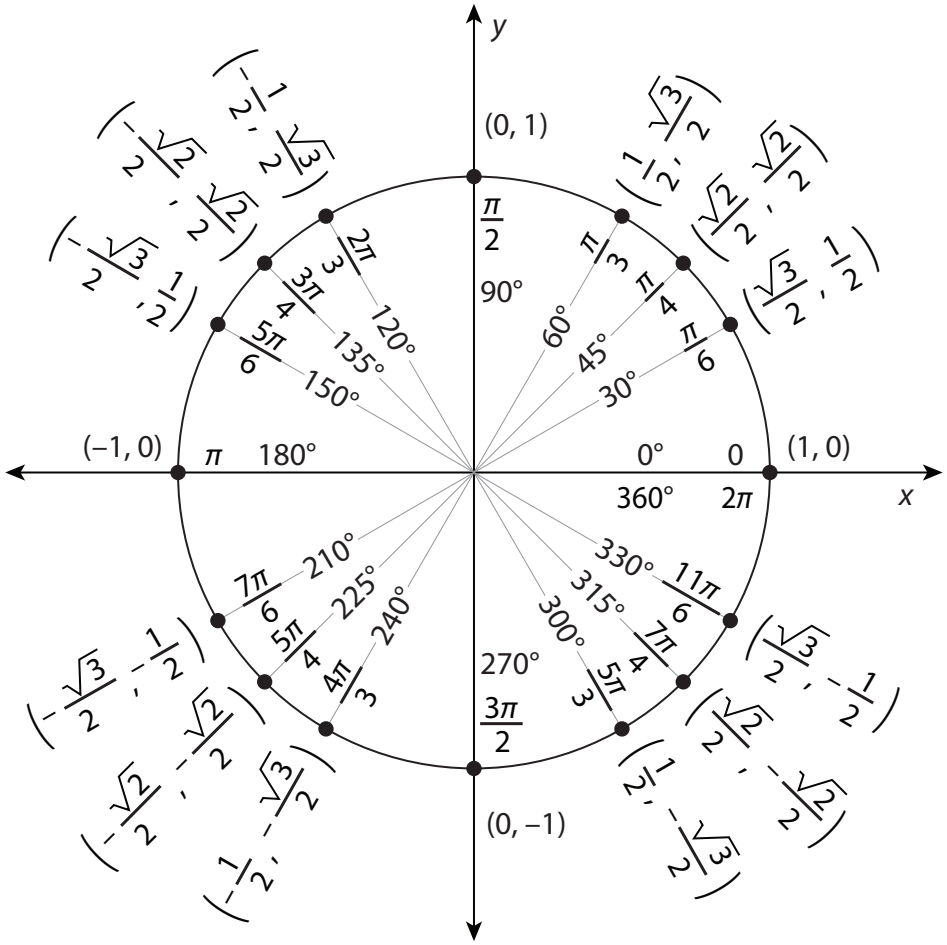


- To find the coordinates of the point where the terminal side of a special angle intersects the unit circle, first identify the reference angle. Then use the pattern to identify cos θ and sin θ. Recall that the coordinates of the point where the terminal side intersects the unit circle are always (cos θ, sin θ). However, since the reference angle was used to find cos θ and sin θ, remember to account for negative coordinates based on which quadrant the point is located in.

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Instruction

- The following illustration gives the coordinates of the points where the terminal side of each special angle intersects the unit circle. While these coordinates can be memorized, it is helpful to understand how to derive them for a given problem.



Common Errors/Misconceptions

- forgetting to account for a negative $\cos\theta$ or $\sin\theta$ when using reference angles
- thinking that, for a 30°–60°–90° triangle, the hypotenuse is $\sqrt{3}$ and the longest leg is 2
- switching the coordinates of the point where the terminal side of the angle intersects the unit circle

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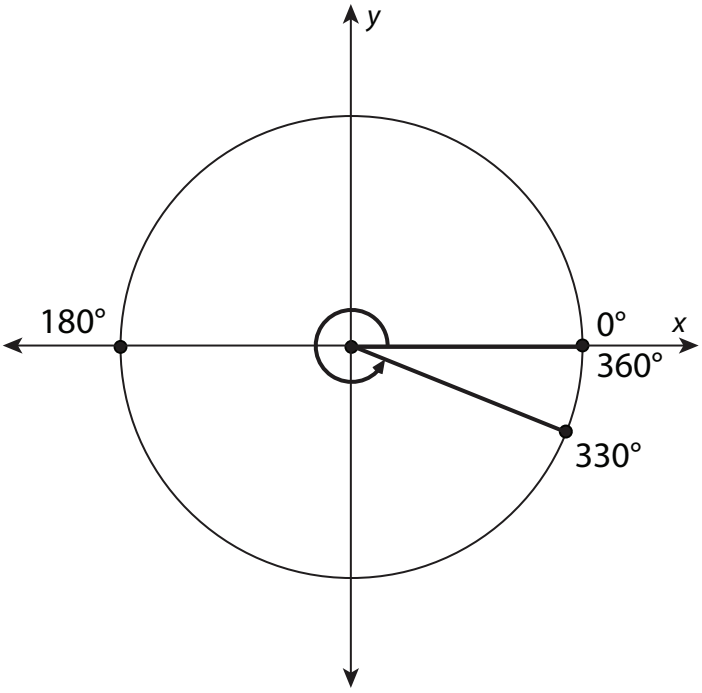
Guided Practice 3.1.3

Example 1

Find the coordinates of the point where the terminal side of a 330° angle intersects the unit circle.

1. Sketch the angle on the unit circle and identify the location of the terminal side.

   A 330° angle is close to a full rotation (360°).



   The terminal side falls in Quadrant IV.

2. Identify the reference angle.

   The reference angle is the angle that the terminal side makes with the *x*-axis. Since the terminal side is located in Quadrant IV, subtract 330° from 360° to find the reference angle.

   $360 - 330 = 30$

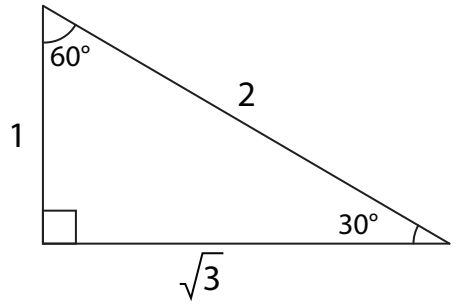
   The reference angle for 330° is 30°.

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Instruction

3. Find the cosine and sine of the reference angle.

Remember the pattern for a 30°–60°–90° triangle:



Use the ratios for sine and cosine, substituting in the values from the triangle.

$\cos\theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$ Cosine ratio

$\cos(30°) = \dfrac{(\sqrt{3})}{(2)}$ Substitute $\sqrt{3}$ for the adjacent side, 2 for the hypotenuse, and 30° for θ.

The cosine of the reference angle is $\dfrac{\sqrt{3}}{2}$.

$\sin\theta = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$ Sine ratio

$\sin(30°) = \dfrac{(1)}{(2)}$ Substitute 1 for the opposite side, 2 for the hypotenuse, and 30° for θ.

The sine of the reference angle is $\dfrac{1}{2}$.

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Instruction

4. Determine the coordinates of the point where the terminal side intersects the unit circle.

   The coordinates of the point where the terminal side intersects the unit circle are $(\cos \theta, \sin \theta)$.

   The sine and cosine of the reference angle are the same as the sine and cosine of the original angle except for the sign, which is based on the quadrant in which the terminal side is located.

   Since the terminal side is in Quadrant IV, the x-coordinate $(\cos \theta)$ must be positive and the y-coordinate $(\sin \theta)$ must be negative.

   Therefore, the coordinates of the point at which the terminal side intersects the unit circle are $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.



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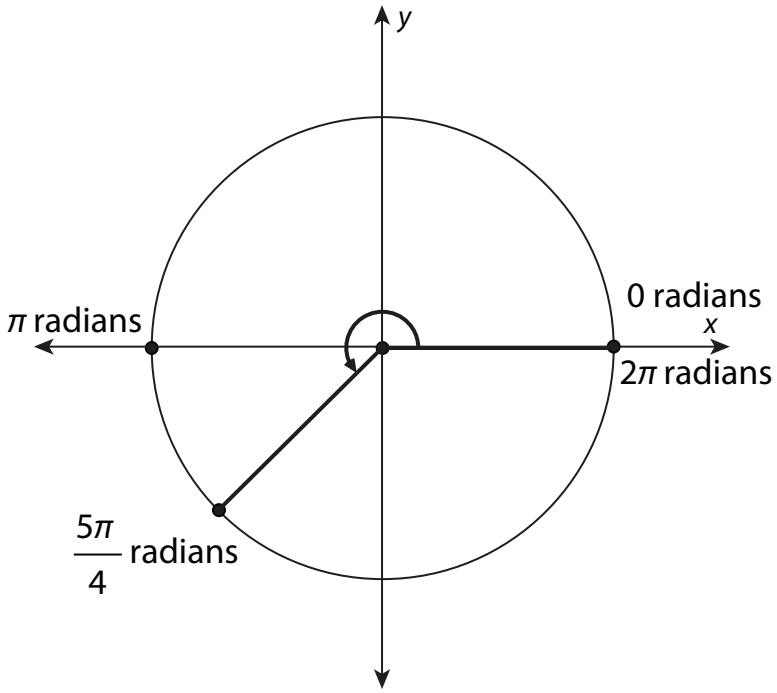
Instruction

Example 2

Find the coordinates of the point where the terminal side of an angle with a measure of $\frac{5\pi}{4}$ radians intersects the unit circle.

1. Sketch the angle on the unit circle and identify the location of the terminal side.

   $\frac{5\pi}{4}$ is the same as $1\frac{1}{4}\pi$ and thus is $\frac{1}{4}$ of the way between π radians and 2π radians.



   The terminal side falls in Quadrant III.

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Instruction

2. Identify the reference angle.

The reference angle is the angle that the terminal side makes with the x-axis. Since it is located in Quadrant III, subtract π radians from $\frac{5\pi}{4}$ radians to find the reference angle.

$\frac{5\pi}{4}-\pi$ Subtract π from the original angle measure.

$=\frac{5\pi}{4}-\frac{4\pi}{4}$ Rewrite π as a fraction over a common denominator.

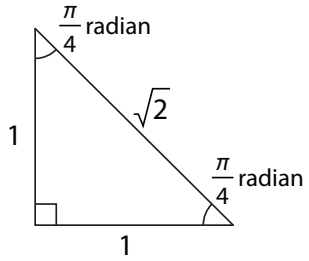
$=\frac{\pi}{4}$ Subtract.

The reference angle for $\frac{5\pi}{4}$ radians is $\frac{\pi}{4}$ radian.

3. Find the cosine and sine of the reference angle.

$\frac{\pi}{4}$ radian is the same as 45°. Recall the pattern for a 45°–45°–90° triangle measured in radians:

45°–45°–90° triangle



Substitute values from the triangle into the ratios for sine and cosine.

$\cos\theta=\frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$ Cosine ratio

$\cos\left(\frac{\pi}{4}\right)=\frac{(1)}{\left(\sqrt{2}\right)}$ Substitute 1 for the adjacent side, $\sqrt{2}$ for the hypotenuse, and $\frac{\pi}{4}$ radian for θ.

$\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$ Multiply the numerator and denominator by $\sqrt{2}$ to rationalize the denominator.

(*continued*)

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Instruction

The cosine of the reference angle is $\frac{\sqrt{2}}{2}$.

$\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$ Sine ratio

$\sin\left(\frac{\pi}{4}\right) = \frac{(1)}{\left(\sqrt{2}\right)}$ Substitute 1 for the opposite side, $\sqrt{2}$ for the hypotenuse, and $\frac{\pi}{4}$ radian for θ.

$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ Multiply the numerator and denominator by $\sqrt{2}$ to rationalize the denominator.

The sine of the reference angle is $\frac{\sqrt{2}}{2}$.

4. Determine the coordinates of the point where the terminal side intersects the unit circle.

   The coordinates of the point where the terminal side intersects the unit circle are $(\cos\theta, \sin\theta)$.

   The sine and cosine of the reference angle are the same as the sine and cosine of the original angle except for the sign, which is based on the quadrant in which the terminal side is located.

   Since the terminal side is in Quadrant III, both the x-coordinate ($\cos\theta$) and the y-coordinate ($\sin\theta$) must be negative.

   Therefore, the coordinates of the point at which the terminal side intersects the unit circle are $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

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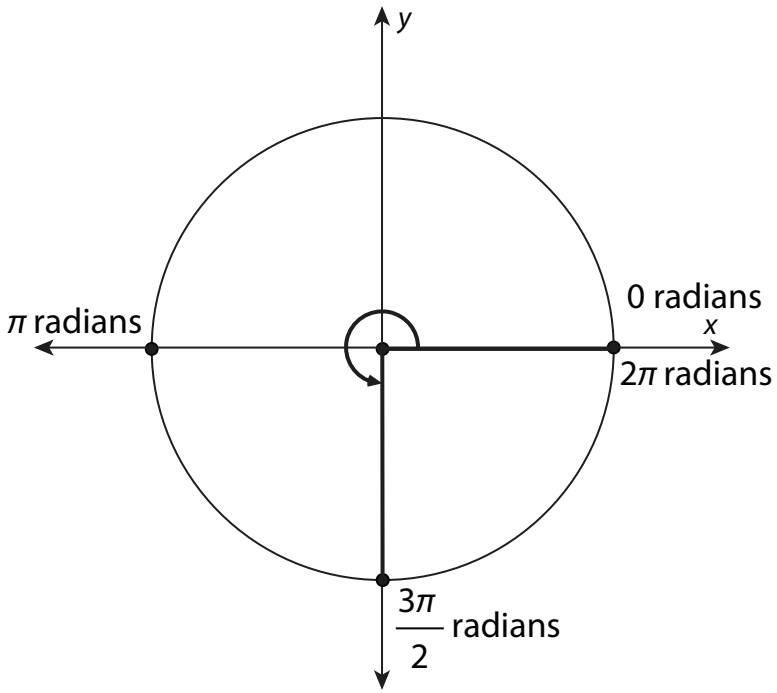
Instruction

Example 3

Find the coordinates of the point where the terminal side of an angle with a measure of $\frac{3\pi}{2}$ radians intersects the unit circle.

1. Sketch the angle on the unit circle and identify the location of the terminal side.

   $\frac{3\pi}{2}$ is the same as $1\frac{1}{2}\pi$ and thus is $\frac{1}{2}$ of the way between π radians and 2π radians.



   The terminal side is located along the *y*-axis.

2. Determine the coordinates of the point where the terminal side intersects the unit circle.

   Since the point is located on the *y*-axis, the *x*-coordinate must be 0.

   Since the radius of the unit circle is 1, the *y*-coordinate must be –1.

   The coordinates of the point where the terminal side intersects the unit circle are (0, –1).

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Instruction

Example 4

Sketch the three special angles that are located in Quadrant II. Label the coordinates of the points where their terminal sides intersect the unit circle. Use degrees.

1. Identify the special angles that are located in Quadrant II.

   The special angles of a unit circle are 30°, 45°, 60°, 90°, and their multiples.

   For the angle to fall in Quadrant II, its measure must be larger than 90° and smaller than 180°.

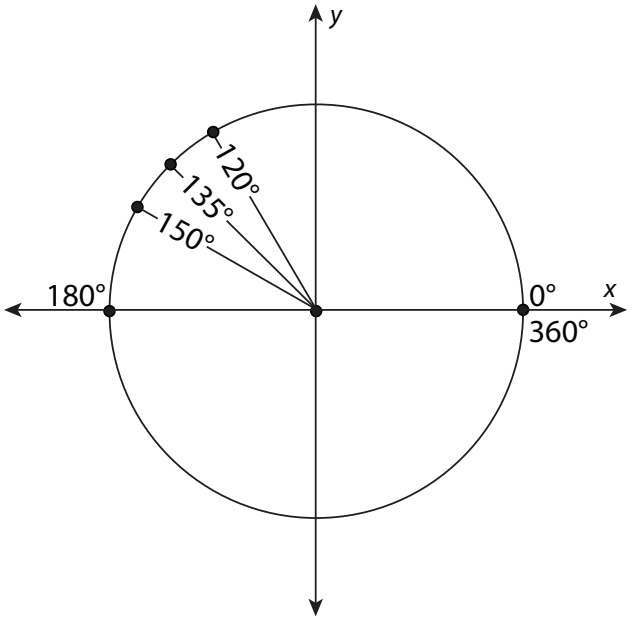
   The multiples of 30° (up to 180°) are 60°, 90°, 120°, 150°, and 180°. The only multiples of 30° that fall in Quadrant II are 120° and 150°.

   The multiples of 45° (up to 180°) are 90°, 135°, and 180°. The only one of these that falls in Quadrant II is 135°.

   The multiples of 60° and 90° are included in the multiples of 30°.

   Therefore, the special angles that are located in Quadrant II are 120°, 135°, and 150°.

2. Sketch 120°, 135°, and 150° angles on the unit circle.



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Instruction

3. Identify the reference angles for the 120°, 135°, and 150° angles.

   The reference angle is the angle that the terminal side makes with the *x*-axis. Since these angles are located in Quadrant II, subtract each original angle measure from 180° to find its reference angle.

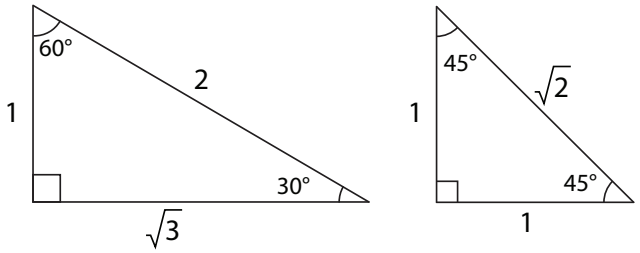
   180 – 120 = 60 The reference angle for 120° is 60°.

   180 – 135 = 45 The reference angle for 135° is 45°.

   180 – 150 = 30 The reference angle for 150° is 30°.

4. Find the cosine and sine of each reference angle.

   Remember the patterns for a 30°–60°–90° triangle and a 45°–45°–90° triangle:



   Use the ratios for cosine and sine, substituting in the values from the special right triangles for each angle measure.

   Recall that the cosine ratio is $\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$, and the sine ratio is $\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$.

   For a 60° reference angle:

   $\cos 60° = \frac{1}{2}$ $\sin 60° = \frac{\sqrt{3}}{2}$

   For a 45° reference angle:

   $\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

   For a 30° reference angle:

   $\cos 30° = \frac{\sqrt{3}}{2}$ $\sin 30° = \frac{1}{2}$

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5. Determine the coordinates of the point where each terminal side intersects the unit circle and label the coordinates on the sketch.

   The coordinates of the point where the terminal side intersects the unit circle are $(\cos\theta, \sin\theta)$.

   The sine and cosine of the reference angle are the same as the sine and cosine of the original angle except for the sign, which is based on the quadrant in which the terminal side is located.

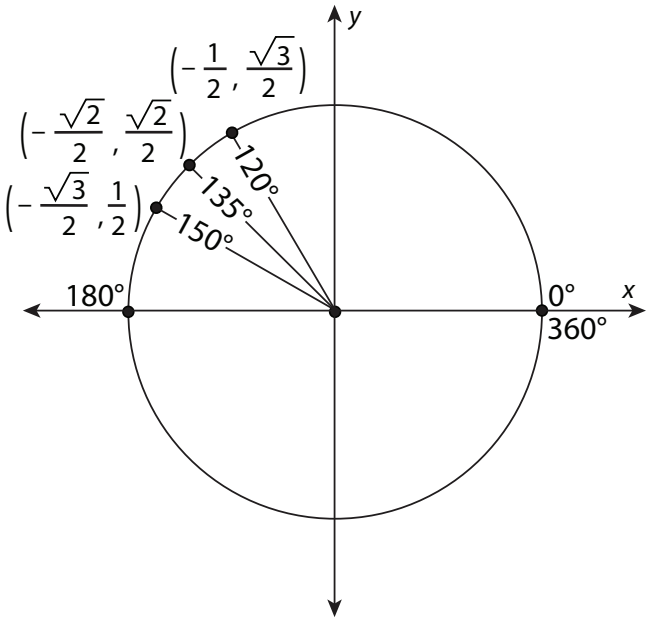
   Since the terminal sides are in Quadrant II, the x-coordinate ($\cos\theta$) must be negative and the y-coordinate ($\sin\theta$) must be positive.

   The terminal side of the 120° angle (whose reference angle is 60°) intersects the unit circle at $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

   The terminal side of the 135° angle (whose reference angle is 45°) intersects the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

   The terminal side of the 150° angle (whose reference angle is 30°) intersects the unit circle at $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

   Label these coordinates on the sketch.

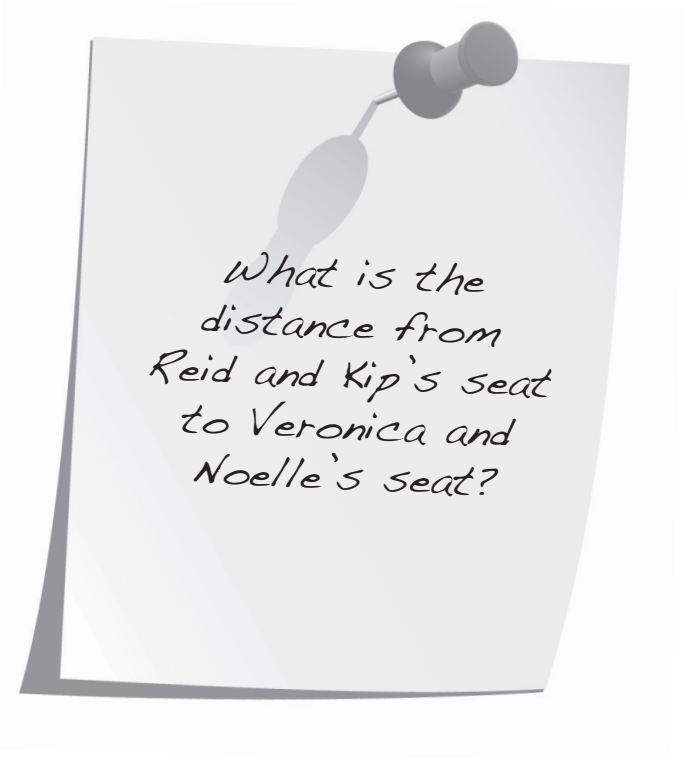


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Problem-Based Task 3.1.3: Ferris Wheel Fun

SMP	
1 ✓	2 ✓
3	4
5	6
7 ✓	8

Andre is waiting for his friends to finish riding a Ferris wheel that has a radius of 1 decameter (1 decameter = 10 meters). When Andre first looks over at the wheel, he sees that Veronica and Noelle are in a seat that is directly to the right of the center of the wheel. His other friends, Reid and Kip, are in a seat that is exactly at the top of the Ferris wheel. A minute later, he sees that the wheel has rotated exactly 120° counterclockwise as additional passengers are loaded. At this point, what is the distance from Reid and Kip's seat to Veronica and Noelle's seat? Give an exact answer.



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Problem-Based Task 3.1.3: Ferris Wheel Fun

Coaching

a. Since the radius of the Ferris wheel is 1 decameter, what type of circle is this?

b. On this circle, at what degrees are Andre's paired friends when he first looks at the wheel?

c. After the ride rotates 120° counterclockwise, at what degrees are his paired friends located?

d. How can you find the approximate height of the seats?

e. How high are Veronica and Noelle from the lowest point on the Ferris wheel?

f. How high are Reid and Kip from the lowest point on the Ferris wheel?

g. What is the distance from Reid and Kip's seat to Veronica and Noelle's seat?

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Problem-Based Task 3.1.3: Ferris Wheel Fun

Coaching Sample Responses

a. Since the radius of the Ferris wheel is 1 decameter, what type of circle is this?

Since the radius is 1, it is a unit circle.

b. On this circle, at what degrees are Andre's paired friends when he first looks at the wheel?

Veronica and Noelle are to the right of the center, which is at 0°.

Reid and Kip are at the top of the wheel, which is at 90°.

c. After the ride rotates 120° counterclockwise, at what degrees are his paired friends located?

Add 120° to each degree measure from which the friends started.

Veronica and Noelle started at 0°, so 0 + 120 = 120. Veronica and Noelle are now located at 120°.

Reid and Kip started at 90°, so 90 + 120 = 210. Reid and Kip are now located at 210°.

d. How can you find the approximate height of the seats?

In a unit circle, the coordinates of the point where the terminal side intersects the unit circle are $(\cos \theta, \sin \theta)$. The y-coordinate corresponds to the height, so we must first find $\sin \theta$ for each angle.

However, this gives the distance above or below the x-axis, or the center of the Ferris wheel. Thus, we must add or subtract the y-coordinate from the radius (1) to find the relative height of each seat.

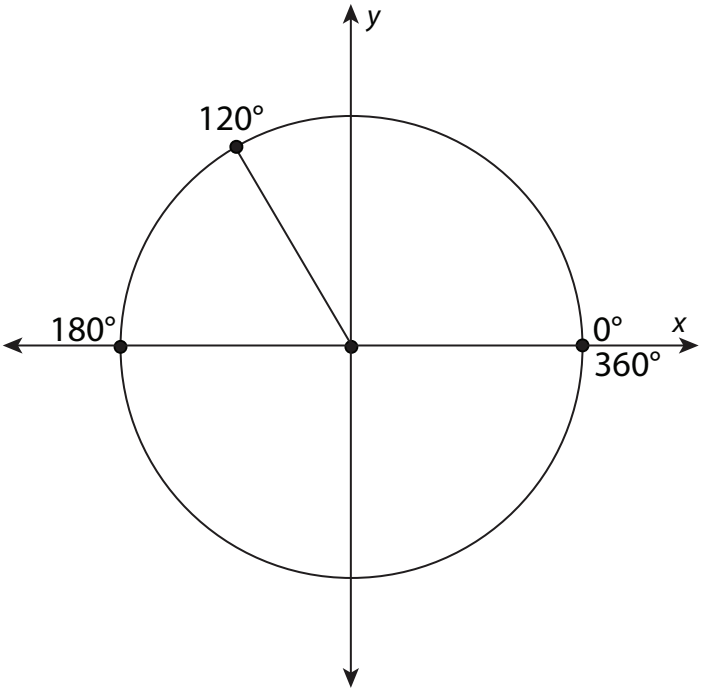
Since the Ferris wheel never touches the ground, the actual heights of the seats will be somewhat higher than the answers found, but this will not matter since the question is asking the difference between their heights. (For example, if both seats are an extra 0.1 decameter off the ground, this will not affect how far apart they are.)

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e. How high are Veronica and Noelle from the lowest point on the Ferris wheel?

Veronica and Noelle are at 120°.

To find the reference angle, sketch the angle:

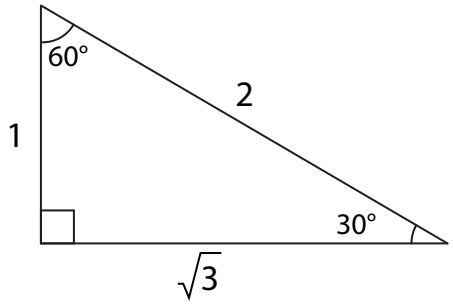


The reference angle is the angle the terminal side makes with the *x*-axis. Since the terminal side is in Quadrant II, subtract 120° from 180° to find the reference angle.

180 – 120 = 60

The reference angle is 60°.

Find sin 60° using the pattern for a 30°–60°–90° triangle:



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Use the sine ratio and the values from the special right triangle.

$$\sin 60° = \frac{\sqrt{3}}{2}$$

Veronica and Noelle are $\frac{\sqrt{3}}{2}$ decameter above the center of the Ferris wheel. The Ferris wheel's radius is 1 decameter. Add these distances to find the height of Veronica and Noelle's location.

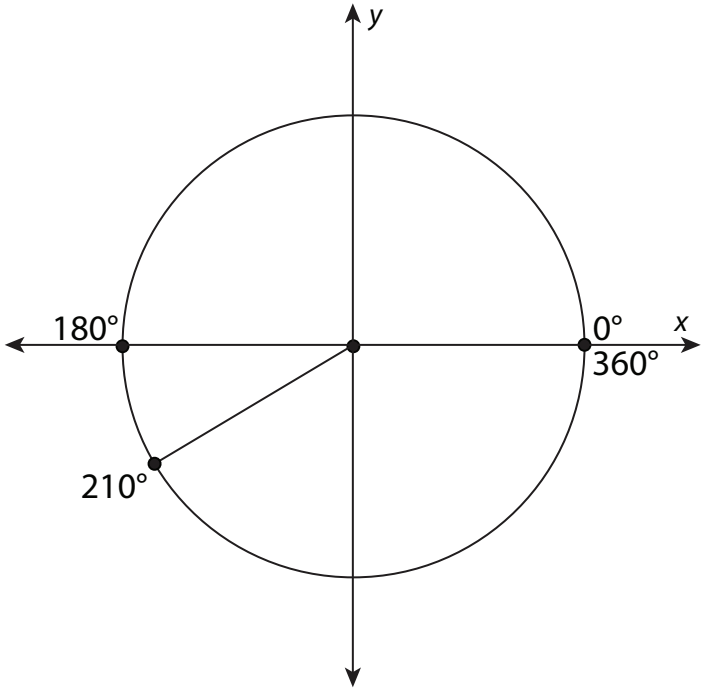
$$1 + \frac{\sqrt{3}}{2} = \frac{2}{2} + \frac{\sqrt{3}}{2} = \frac{2+\sqrt{3}}{2}$$

Veronica and Noelle are $\frac{2+\sqrt{3}}{2}$ or approximately 1.87 decameters above the bottom of the Ferris wheel.

f. How high are Reid and Kip from the lowest point on the Ferris wheel?

Reid and Kip are at 210°.

To find the reference angle, sketch the angle:



Since the terminal side is in Quadrant III, subtract 180° from 210° to find the reference angle.

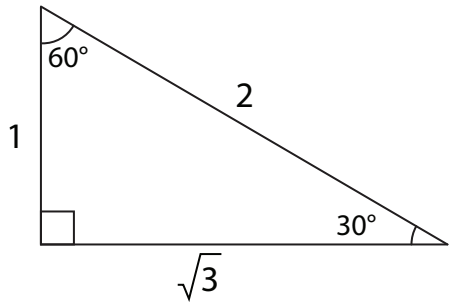
210 – 180 = 30

The reference angle is 30°.

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Instruction

Find sin 30° using the pattern for a 30°–60°–90° triangle:



Use the sine ratio and the values from the special right triangle.

$$\sin 30° = \frac{1}{2}$$

Reid and Kip are $\frac{1}{2}$ decameter below the center of the Ferris wheel (because in Quadrant III, the *y*-coordinate would be negative, indicating it is below the center). The Ferris wheel's radius is 1 decameter. Subtract $\frac{1}{2}$ from 1 to find the height of Reid and Kip's location.

$$1 - \frac{1}{2} = \frac{1}{2}$$

Reid and Kip are $\frac{1}{2}$ decameter above the lowest point of the Ferris wheel.

g. What is the distance from Reid and Kip's seat to Veronica and Noelle's seat?

Veronica and Noelle are $\frac{2+\sqrt{3}}{2}$ decameters above the center of the Ferris wheel.

Reid and Kip are $\frac{1}{2}$ decameter above the lowest point of the Ferris wheel.

Subtract to find the distance from Veronica and Noelle's seat to Reid and Kip's seat.

$$\frac{2+\sqrt{3}}{2} - \frac{1}{2} = \frac{2+\sqrt{3}-1}{2} = \frac{1+\sqrt{3}}{2}$$

Veronica and Noelle are $\frac{1+\sqrt{3}}{2}$ or approximately 1.37 decameters above Reid and Kip.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.1.3: Special Angles in the Unit Circle

For problems 1–9, find the coordinates of the point where the terminal side of the angle intersects the unit circle. Give exact answers.

1. 45°

2. 210°

3. 90°

4. 300°

5. $\frac{\pi}{3}$ radians

6. $\frac{7\pi}{4}$ radians

7. 2π radians

8. $\frac{5\pi}{6}$ radians

9. $\frac{7\pi}{6}$ radians

Use your knowledge of unit circles to complete problem 10.

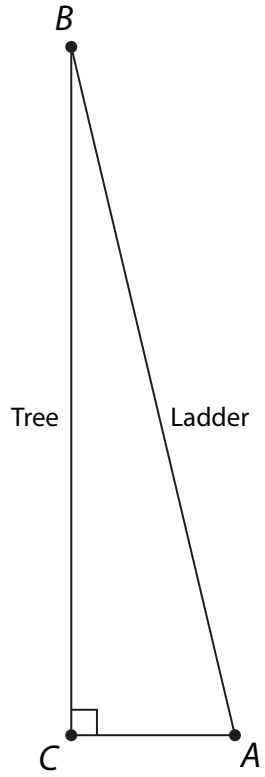
10. Create a unit circle that contains all the special angles in degrees. Label the terminal point of each angle with its coordinates.

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Lesson 3.1.4: Evaluating Trigonometric Functions

Warm-Up 3.1.4

A group of firefighters is called in to rescue a kitten stuck in a tree. The firefighters lean a 25-foot ladder up against the tree so that the base of the ladder rests 6 feet away from the base of the tree, as shown in the diagram. Answer the following questions and round each answer to the nearest tenth.



1. How high up the tree does the ladder reach?

2. What is the measure of the angle the ladder makes with the ground (angle A)?

3. As the firefighter climbs down the ladder with the kitten, he stops 3 feet from the base of the ladder and jumps off. How far off the ground was he when he jumped?

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Lesson 3.1.4: Evaluating Trigonometric Functions

Common Core State Standard

F–TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Warm-Up 3.1.4 Debrief

1. How high up the tree does the ladder reach?

   The tree, the ladder, and the ground make a right triangle in which the ladder is the hypotenuse. To determine how high up the tree the ladder reaches, find the length of side $\overline{BC}$ using the Pythagorean Theorem.

   Let the ladder's length, 25 feet, be the hypotenuse, c.

   Let the distance between the ladder and the base of the tree, 6 feet, be b.

$c^2 = a^2 + b^2$	Pythagorean Theorem
$(25)^2 = a^2 + (6)^2$	Substitute 25 for c and 6 for b.
$625 = a^2 + 36$	Simplify the exponents.
$589 = a^2$	Subtract 36 from both sides.
$\sqrt{589} = a$	Take the square root of both sides.
$24.3 \approx a$	Calculate the square root of 589.

   The ladder reaches approximately 24.3 feet high on the tree.

2. What is the measure of the angle that the ladder makes with the ground (angle A)?

   For the triangle that includes the angle that the ladder makes with the ground (angle A), 6 feet would be the length of the adjacent side and 25 feet would be the length of the hypotenuse.

   Therefore, use the cosine ratio to determine the measure of the angle.

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$\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$	Cosine ratio
$\cos(A) = \frac{(6)}{(25)}$	Substitute 6 for the adjacent side, 25 for the hypotenuse, and A for the angle.
$m\angle A = \arccos\frac{6}{25}$	Take the arccosine of both sides.
$m\angle A \approx 76.1°$	Calculate the approximate measure of angle A.

The measure of the angle that the ladder makes with the ground is approximately 76.1°.

3. As the firefighter climbs down the ladder with the kitten, he stops 3 feet from the base of the ladder and jumps off. How far off the ground was he when he jumped?

   We can solve this problem by envisioning a new, smaller triangle within the previous triangle.

   The measure of angle A is the same, approximately 76.1°, and 3 feet is the length of the new triangle's hypotenuse, since this is the length of ladder the firefighter "skipped" by jumping off early. The firefighter's height, h, from the ground is the length of the opposite side.

   Therefore, use the sine ratio to determine the length of the opposite side (the height).

$\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$	Sine ratio
$\sin(76.1°) = \frac{(h)}{(3)}$	Substitute h for the opposite side, 3 for the hypotenuse, and 76.1° for the angle measure.
$3 \bullet \sin 76.1° = h$	Multiply both sides by 3.
$2.9 \approx h$	Use your calculator to simplify.

The firefighter was approximately 2.9 feet off the ground when he jumped.

Connection to the Lesson

- Students will extend their knowledge of trigonometric functions to relate them to the unit circle.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- finding the sine, cosine, tangent, cosecant, secant, and cotangent of angles (G-SRT.6)
- using radians to measure angles (F-TF.1)
- finding the point where the terminal side of an angle intersects the unit circle (G-GPE.4)
- understanding the pattern of special angles within the unit circle (F-TF.2)
- using the Pythagorean Theorem to find the length of the third side of a right triangle (8.G.6)

Introduction

The six trigonometric functions (sine, cosine, tangent, cosecant, secant, and cotangent) can be used to find the length of the sides of a triangle or the measure of an angle if the length of two sides is given. Previously these functions could only be applied to angles up to 90°. However, by using radians and the unit circle, these functions can be applied to any angle.

Key Concepts

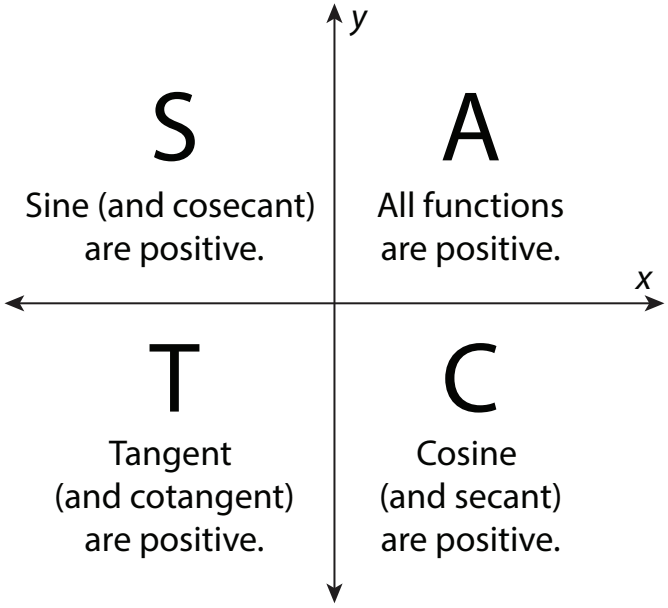
- Recall that sine is the ratio of the length of the opposite side to the length of the hypotenuse, cosine is the ratio of the length of the adjacent side to the length of the hypotenuse, and tangent is the ratio of the length of the opposite side to the length of the adjacent side. (You may have used the mnemonic device SOHCAHTOA to help remember these relationships: Sine equals the Opposite side over the Hypotenuse, Cosine equals the Adjacent side over the Hypotenuse, and Tangent equals the Opposite side over the Adjacent side.)
- Three other trigonometric functions, cosecant, secant, and cotangent, are reciprocal functions of the first three. **Cosecant** is the reciprocal of the sine function, **secant** is the reciprocal of the cosine function, and **cotangent** is the reciprocal of the tangent function.
  - The cosecant of $\theta = \csc\theta = \dfrac{\text{length of hypotenuse}}{\text{length of opposite side}}$; $\csc\theta = \dfrac{1}{\sin\theta}$
  - The secant of $\theta = \sec\theta = \dfrac{\text{length of hypotenuse}}{\text{length of adjacent side}}$; $\sec\theta = \dfrac{1}{\cos\theta}$
  - The cotangent of $\theta = \cot\theta = \dfrac{\text{length of adjacent side}}{\text{length of opposite side}}$; $\cot\theta = \dfrac{1}{\tan\theta}$

SRB p. U3-43

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Instruction

- The quadrant in which the terminal side is located determines the sign of the trigonometric functions. In Quadrant I, all the trigonometric functions are positive. In Quadrant II, the sine and its reciprocal, the cosecant, are positive and all the other functions are negative. In Quadrant III, the tangent and its reciprocal, the cotangent, are positive, and all other functions are negative. In Quadrant IV, the cosine and its reciprocal, the secant, are positive, and all other functions are negative.
- You can use a mnemonic device to remember in which quadrants the functions are positive: All Students Take Calculus (ASTC).



- However, instead of memorizing this, you can also think it through each time, considering whether the opposite and adjacent sides of the reference angle are positive or negative in each quadrant.
- To find a trigonometric function of an angle given a point on its terminal side, first visualize a triangle using the reference angle. The x-coordinate becomes the length of the adjacent side and the y-coordinate becomes the length of the opposite side. The length of the hypotenuse can be found using the Pythagorean Theorem. Determine the sign by remembering the ASTC pattern or by considering the signs of the x- and y-coordinates.
- To find the trigonometric functions of special angles, first find the reference angle and then use the pattern to determine the ratio.
- For angles larger than 2π radians (360°), subtract 2π radians (360°) to find a **coterminal angle**, an angle that shares the same terminal side, that is less than 2π radians (360°). Repeat if necessary.
- For negative angles, find the reference angle and then apply the same method.

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Instruction

Common Errors/Misconceptions

- using the incorrect trigonometric ratio
- forgetting to consider whether the trigonometric ratios are negative
- mistaking the quadrants in which each trigonometric function is positive

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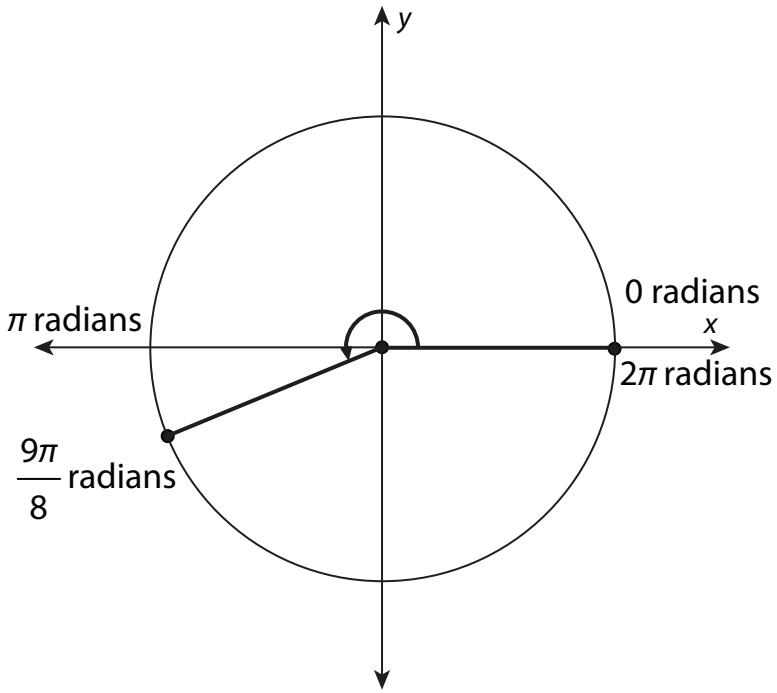
Guided Practice 3.1.4

Example 1

What is the sign of each trigonometric ratio for an angle with a measure of $\frac{9\pi}{8}$ radians?

1. Sketch the angle to determine in which quadrant it is located.

   $\frac{9\pi}{8}$ radians is the same as $1\frac{1}{8}\pi$ radians, so the terminal side falls $\frac{1}{8}$ of the way between π radians and 2π radians.

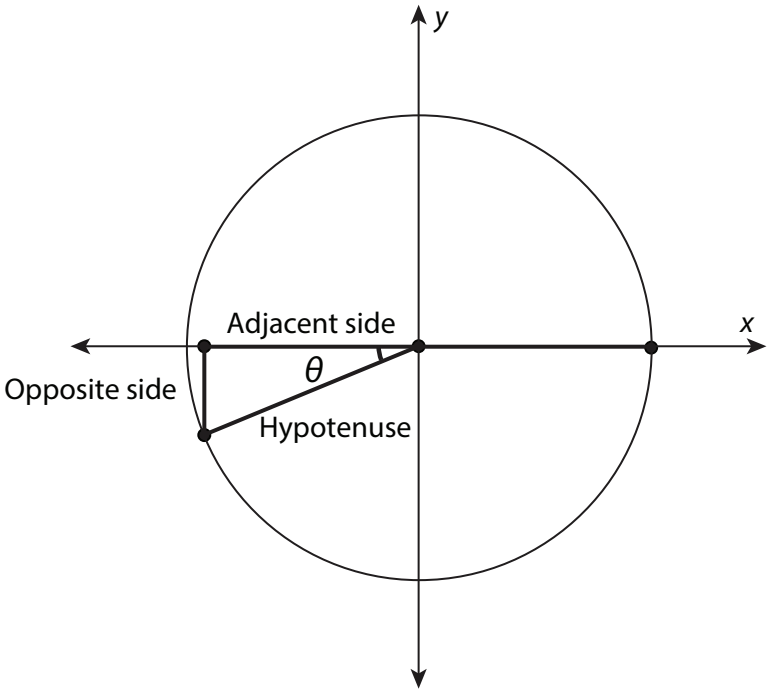


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Instruction

2. Determine the signs of the lengths of the opposite side, adjacent side, and hypotenuse for the reference angle.

   The reference angle is the angle the terminal side makes with the *x*-axis. Draw the triangle associated with this reference angle and label its sides.



   The adjacent side along the *x*-axis is negative since *x* is negative to the left of the origin. The opposite side (which corresponds to the *y*-coordinate of the terminal side) is also negative since *y* is negative below the origin. The hypotenuse is positive since it is the length of the radius.

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3. Use the definitions of the trigonometric functions to determine the sign of each and check the results by using the acronym ASTC.

   Organize the information in a table to better see the relationships among the functions.

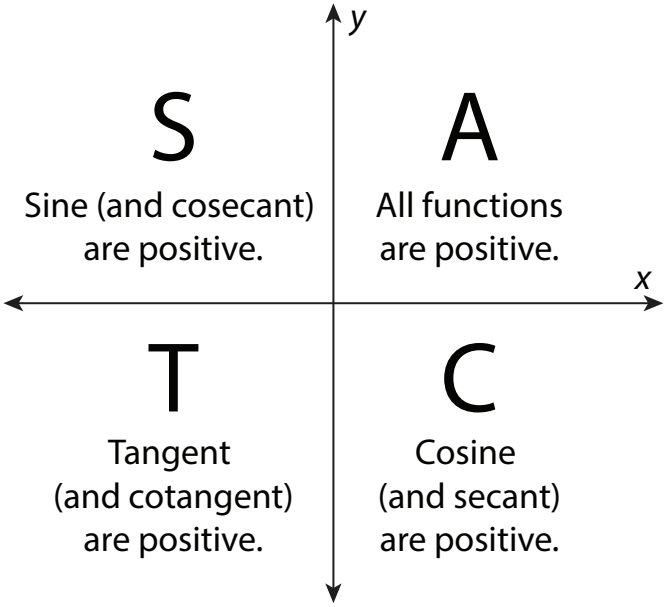
Trigonometric function	Description	Result
$\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$	For the reference angle, the sign of length of the opposite side is negative and the sign of the length of the hypotenuse is positive.	Negative
$\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$	For the reference angle, the length of the adjacent side is negative and the length of the hypotenuse is positive.	Negative
$\tan\theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$	For the reference angle, the length of the opposite side is negative and the length of the adjacent side is negative.	Positive
$\csc\theta = \frac{\text{length of hypotenuse}}{\text{length of opposite side}}$	Cosecant is the reciprocal of sine. Since the sine is negative, its reciprocal is also negative.	Negative
$\sec\theta = \frac{\text{length of hypotenuse}}{\text{length of adjacent side}}$	Secant is the reciprocal of cosine. Since the cosine is negative, its reciprocal is also negative.	Negative
$\cot\theta = \frac{\text{length of adjacent side}}{\text{length of opposite side}}$	Cotangent is the reciprocal of tangent. Since the tangent is positive, its reciprocal is also positive.	Positive

(*continued*)

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Instruction

Recall the diagram from the Key Concepts.



In Quadrant III, where the terminal side of the angle is located, only the tangent and cotangent are positive. This matches the results previously found.

Therefore, for an angle with a measure of $\frac{9\pi}{8}$ radians, the tangent and cotangent are positive, and all other trigonometric functions are negative.

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Instruction

Example 2

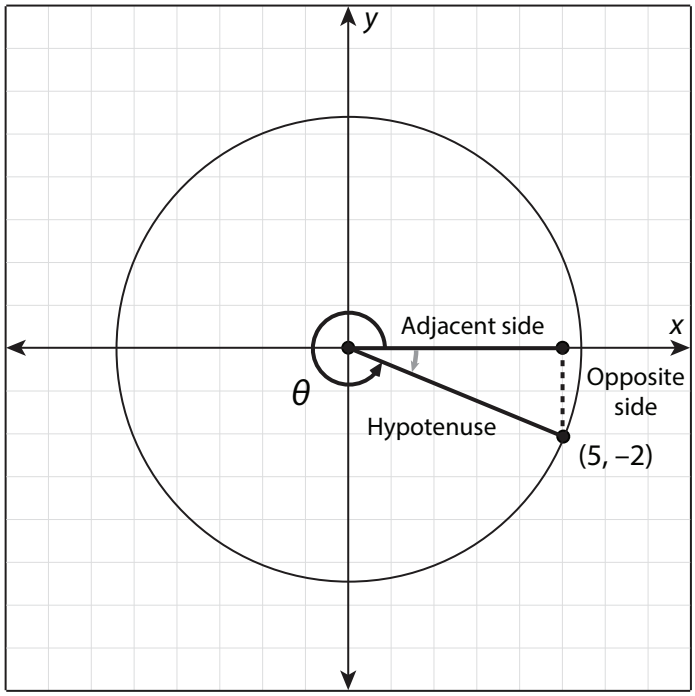
Find sin θ if θ is a positive angle in standard position with a terminal side that passes through the point (5, –2). Give an exact answer.

1. Sketch the angle and draw in the triangle associated with the reference angle.

   Recall that a positive angle is created by rotating counterclockwise around the origin of the coordinate plane.

   Plot (5, –2) on a coordinate plane and draw the terminal side extending from the origin through that point.

   The reference angle is the angle the terminal side makes with the *x*-axis.



   Notice that θ is nearly 360°, so the reference angle is in the fourth quadrant.

   The magnitude of the *x*-coordinate is the length of the adjacent side and the magnitude of the *y*-coordinate is the length of the opposite side. The hypotenuse can be found using the Pythagorean Theorem. Determine the sign of sin θ by recalling the ASTC pattern or by considering the signs of the *x*- and *y*-coordinates.

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Instruction

2. Find the length of the opposite side and the length of the hypotenuse.

Sine is the ratio of the length of the opposite side to the length of the hypotenuse; therefore, these two lengths must be determined.

The length of the opposite side is the magnitude of the y-coordinate, 2.

Since the opposite side length is known to be 2 and the adjacent side length, 5, can be determined from the sketch, the hypotenuse can be found by using the Pythagorean Theorem.

$c^2 = a^2 + b^2$	Pythagorean Theorem
$c^2 = (2)^2 + (5)^2$	Substitute 2 for a and 5 for b.
$c^2 = 4 + 25$	Simplify the exponents.
$c^2 = 29$	Add.
$c = \sqrt{29}$	Take the square root of both sides.

The length of the hypotenuse is $\sqrt{29}$ units.

3. Find $\sin\theta$.

Now that the lengths of the opposite side and the hypotenuse are known, substitute these values into the sine ratio to determine $\sin\theta$.

$\sin\theta = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$	Sine ratio
$\sin\theta = \dfrac{(2)}{\left(\sqrt{29}\right)}$	Substitute 2 for the opposite side and $\sqrt{29}$ for the hypotenuse.
$\sin\theta = \dfrac{2\sqrt{29}}{29}$	Rationalize the denominator.

According to ASTC, in Quadrant IV only the cosine and secant are positive. The sine is negative.

For a positive angle θ in standard position with a terminal side that passes through the point $(5, -2)$, $\sin\theta = -\dfrac{2\sqrt{29}}{29}$.

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Instruction

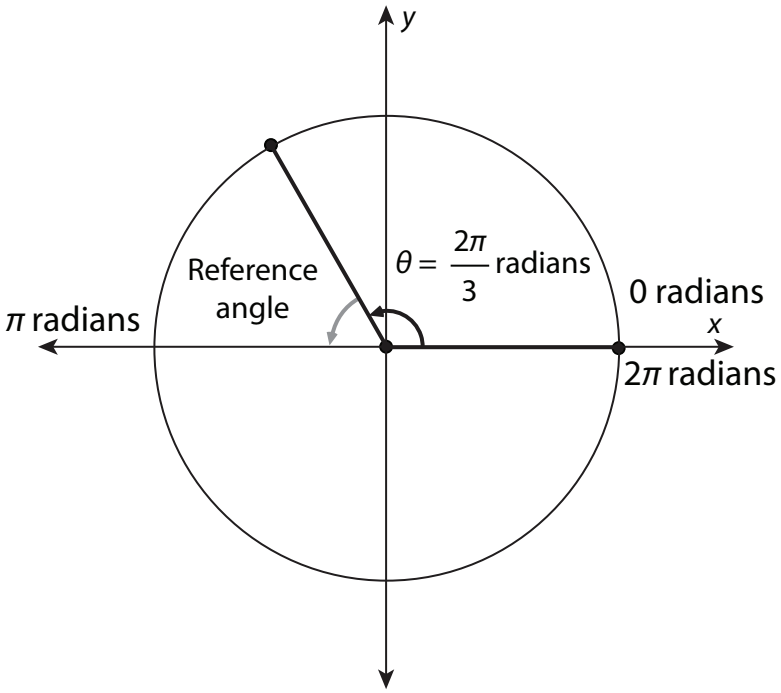
Example 3

Find $\csc\frac{2\pi}{3}$. Give an exact answer.

1. Sketch the angle and determine its reference angle.

   $\frac{2\pi}{3}$ is $\frac{2}{3}$ of the way between 0 and π radians.

   The reference angle is the angle the terminal side makes with the *x*-axis.



   The reference angle falls in Quadrant II. Therefore, subtract the original angle measure from π radians to find the reference angle.

$$\pi-\frac{2\pi}{3}=\frac{3\pi}{3}-\frac{2\pi}{3}=\frac{1\pi}{3}=\frac{\pi}{3}$$

   The reference angle for $\frac{2\pi}{3}$ radians is $\frac{\pi}{3}$ radians.

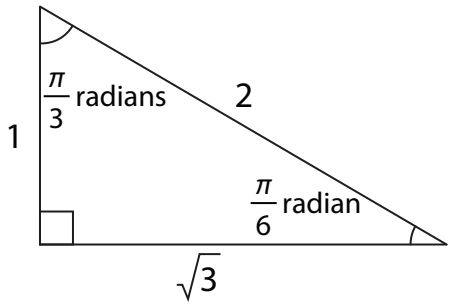
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Instruction

2. Use the pattern for the special angle to find csc θ.

Recall that $\frac{\pi}{3}$ radians is the measure of a special angle in a right triangle. Remember the pattern for the special angle:

30°–60°–90° triangle



Use the values from this right triangle to determine csc θ.

$\csc\theta = \frac{\text{length of hypotenuse}}{\text{length of opposite side}}$ Cosecant ratio

$\csc\left(\frac{\pi}{3}\right) = \frac{(2)}{\left(\sqrt{3}\right)}$ Substitute $\frac{\pi}{3}$ for θ, 2 for the hypotenuse, and $\sqrt{3}$ for the opposite side.

$\csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3}$ Rationalize the denominator.

Recall ASTC: in Quadrant II, only the sine and cosecant are positive. Thus, $\csc\frac{\pi}{3}$ must be positive. (Recall that both the opposite side and the hypotenuse are positive in Quadrant II.)

The answer confirms that the cosecant is positive; therefore, $\csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3}$.

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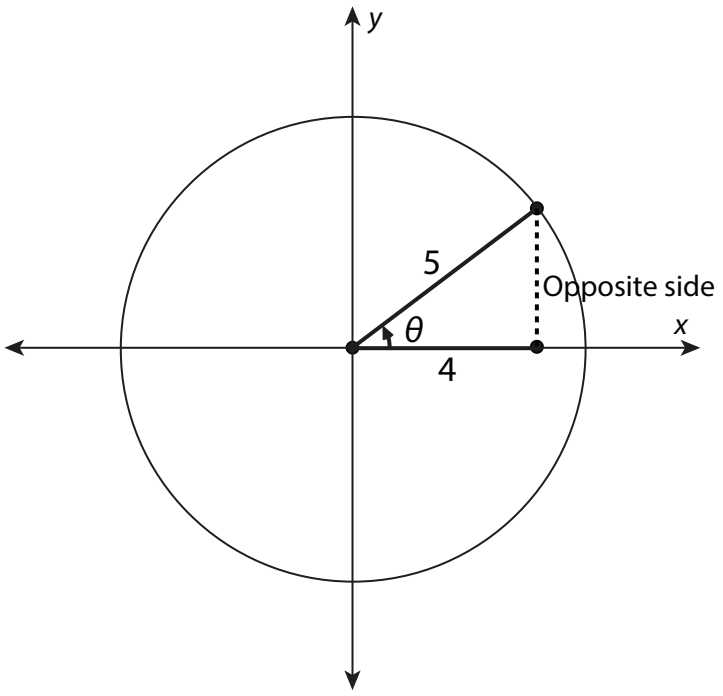
Instruction

Example 4

Given $\cos\theta = \frac{4}{5}$, if θ is in Quadrant I, find cot θ.

1. Sketch an angle in Quadrant I, draw the associated triangle, and label the sides with the given information.

   Cosine is the ratio of the length of the adjacent side to the length of the hypotenuse. Since $\cos\theta = \frac{4}{5}$, 4 is the length of the adjacent side and 5 is the length of the hypotenuse.



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Instruction

2. Use the Pythagorean Theorem to find the length of the opposite side.

Since the lengths of two sides of the triangle are given, substitute these values into the Pythagorean Theorem and solve for the missing side length.

$c^2 = a^2 + b^2$	Pythagorean Theorem
$(5)^2 = (4)^2 + b^2$	Substitute 5 for c and 4 for a.
$25 = 16 + b^2$	Simplify the exponents.
$9 = b^2$	Subtract 16 from both sides.
$3 = b$	Take the square root of both sides.

The length of the opposite side is 3 units.

3. Find the cotangent.

Use the values from the triangle to determine the cotangent.

$$\cot\theta = \frac{\text{length of adjacent side}}{\text{length of opposite side}}$$ Cotangent ratio

$$\cot\theta = \frac{(4)}{(3)}$$ Substitute 4 for the adjacent side and 3 for the opposite side.

In Quadrant I, all trigonometric ratios are positive, which coincides with the answer found.

Given $\cos\theta = \frac{4}{5}$, for an angle θ in Quadrant I, $\cot\theta = \frac{4}{3}$.

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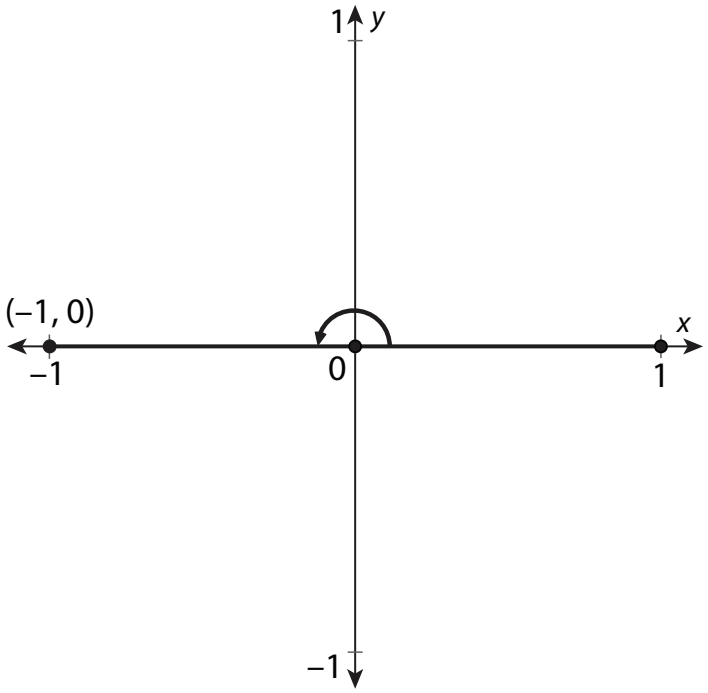
Instruction

Example 5

Find cos θ if θ is a positive angle in standard position with a terminal side that passes through the point (–1, 0). Give an exact answer.

1. Sketch the angle.

   Plot the point (–1, 0) and draw the terminal side through it.



2. Determine the reference angle.

   The reference angle is the angle the terminal side makes with the *x*-axis. In this case, the reference angle is 0.

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Instruction

3. Determine the lengths of the opposite side, adjacent side, and hypotenuse.

The length of the adjacent side always corresponds with the x-coordinate, and the length of the opposite side always corresponds with the y-coordinate. (Picture a 1° angle if needed.)

The length of the opposite side is 0 (the x-coordinate).

The length of the adjacent side is –1 (the y-coordinate).

To find the length of the hypotenuse, use the Pythagorean Theorem.

$c^2 = a^2 + b^2$	Pythagorean Theorem
$c^2 = (-1)^2 + (0)^2$	Substitute –1 for a and 0 for b.
$c^2 = 1 + 0$	Simplify the exponents.
$c^2 = 1$	Add.
$c = 1$	Take the square root of both sides.

The length of the opposite side is 0, the length of the adjacent side is –1, and the length of the hypotenuse is 1.

4. Find $\cos\theta$.

Cosine is the ratio of the length of the adjacent side to the length of the hypotenuse.

Substitute the known values into the cosine ratio and solve.

$\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$	Cosine ratio
$\cos\theta = \frac{(-1)}{(1)}$	Substitute –1 for the adjacent side and 1 for the hypotenuse.
$\cos\theta = -1$	Simplify.

In Quadrant II, sine and cosecant are positive, and all other functions are negative, which coincides with the answer found.

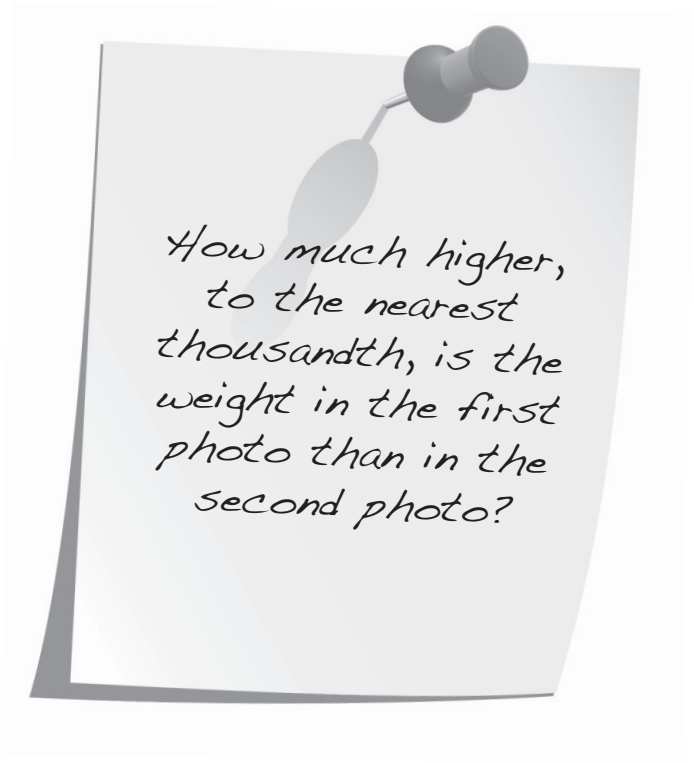
For a positive angle θ in standard position with a terminal side that passes through the point (–1, 0), $\cos\theta = -1$.

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Problem-Based Task 3.1.4: Spring into Action

SMP	
1	2
3	4
5	6 ✓
7	8 ✓

A weight hangs on the end of a coiled spring; when the weight is pulled and released, the spring begins to bounce up and down at regular intervals. The weight's displacement, or distance, from equilibrium (its resting position) is given by the formula $y(t)=\frac{1}{2}\cos 3t$, in which $y(t)$ is the displacement in feet and t is the time in seconds. Randall snaps a photo 0.1 second after the weight is released and again exactly 1 second later. How much higher, to the nearest thousandth, is the weight in the first photo than in the second photo? (Note that the formula uses radians.)



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Problem-Based Task 3.1.4: Spring into Action

Coaching

a. What is the value of t for each photo taken?

b. What is the weight's location when the first photo is taken?

c. What is the weight's location when the second photo is taken?

d. How much higher is the weight in the first photo than in the second?

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Instruction

Problem-Based Task 3.1.4: Spring into Action

Coaching Sample Responses

a. What is the value of t for each photo taken?

When the first photo is taken, 0.1 second has passed. Thus, $t = 0.1$.

The second photo is taken exactly 1 second later. Therefore, add the two times: $t = 0.1 + 1.0 = 1.1$.

b. What is the weight's location when the first photo is taken?

Substitute 0.1 for t in the formula $y(t) = \frac{1}{2}\cos 3t$.

$$y(t) = \frac{1}{2}\cos 3t$$

$$y(0.1) = \frac{1}{2}\cos 3(0.1)$$

$$y(0.1) \approx 0.4777$$

In the first photo, the weight is approximately 0.4777 feet above equilibrium.

c. What is the weight's location when the second photo is taken?

Substitute 1.1 for t in the formula $y(t) = \frac{1}{2}\cos 3t$. So, $y(1.1) \approx -0.4937$.

In the second photo, the weight is approximately 0.4937 feet below equilibrium.

d. How much higher is the weight in the first photo than in the second?

Subtract to find the difference.

$0.4777 - (-0.4937) = 0.9714$

The weight is approximately 0.971 feet higher in the first photo than in the second.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.1.4: Evaluating Trigonometric Functions

For problems 1–4, determine the specified trigonometric ratio for each angle with a terminal side that passes through the given point. Give exact answers.

1. cos θ; (–2, 7)

2. cot θ; (4, –5)

3. sec θ; (0, –1)

4. sin θ; (–12, –5)

For problems 5–7, determine the specified trigonometric ratio for each special angle. Give exact answers.

5. $\sin\frac{7\pi}{6}$ radians

6. $\tan\frac{3\pi}{4}$ radians

7. $\csc\frac{\pi}{3}$ radians

For problems 8–10, each angle is described by one of its trigonometric ratios and the quadrant in which its terminal side is located. Find the requested trigonometric ratio for the angle. Give an exact answer.

8. Find $\cos\theta$ given $\tan\theta=\frac{2}{3}$ with a terminal side in Quadrant III.

9. Find $\sec\theta$ given $\sin\theta=-\frac{3}{5}$ with a terminal side in Quadrant IV.

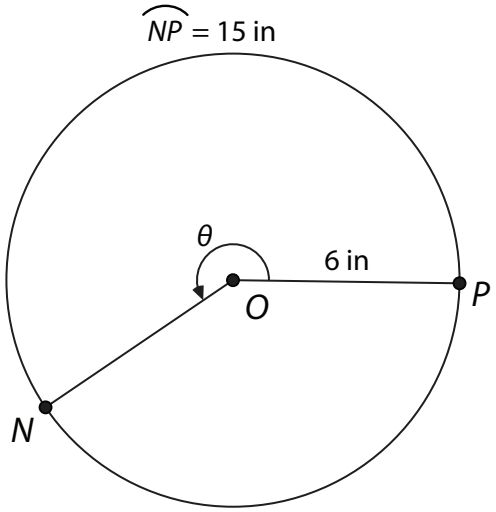
10. Find $\cot\theta$ given $\cos\theta=-\frac{1}{2}$ with a terminal side in Quadrant II.

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Progress Assessment

Circle the letter of the best answer.

1. What is the measure of the angle shown?



   a. 0.4 radian
   b. 2.5 radians
   c. 45 radians
   d. 90 radians

2. Convert 125° to radians.

   a. $\frac{25\pi}{36}$ radians
   b. $\frac{25\pi}{72}$ radians
   c. $\frac{36\pi}{25}$ radians
   d. $\frac{72\pi}{25}$ radians

3. Convert $\frac{9\pi}{5}$ radians to degrees.

   a. 100°
   b. 162°
   c. 324°
   d. 648°

4. If $\theta = \frac{13\pi}{11}$ radians, what is the measure of its reference angle?

   a. $\frac{\pi}{11}$ radians
   b. $\frac{2\pi}{11}$ radians
   c. $\frac{8\pi}{11}$ radians
   d. π radians

continued

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Assessment

5. If $\theta = \frac{5\pi}{8}$ radians, at what point does the terminal side of the angle intersect the unit circle?

   a. (–0.38, 0.92)
   b. (0.03, 0.99)
   c. (0.92, –0.38)
   d. (0.99, 0.33)

6. If $\theta = \frac{5\pi}{3}$ radians, at what point does the terminal side of the angle intersect the unit circle?

   a. $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
   b. $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
   c. $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
   d. $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

7. For an angle with a measure of $\frac{5\pi}{7}$ radians, which of its trigonometric ratios are positive?

   a. all of them
   b. cosine and secant
   c. sine and cosecant
   d. tangent and cotangent

8. What is $\cos\theta$ if θ is an angle in standard position with a terminal side that passes through the point (6, –3)?

   a. $-\frac{1}{2}$
   b. $\frac{2}{15}$
   c. $\frac{\sqrt{5}}{5}$
   d. $\frac{2\sqrt{5}}{5}$

9. What is $\csc\frac{7\pi}{6}$?

   a. $-\frac{1}{2}$
   b. $-\frac{\sqrt{3}}{2}$
   c. $-\frac{2\sqrt{3}}{3}$
   d. –2

continued

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Assessment

10. If $\sin\theta = -\frac{4}{5}$, and the terminal side of the angle is in Quadrant III, what is cot θ?

    a. $-\frac{4}{3}$

    b. $-\frac{3}{4}$

    c. $\frac{3}{4}$

    d. $\frac{4}{3}$

Use what you have learned about radians and the unit circle to complete all parts of the following problem.

11. Alison is trying to boost her endurance by running interval sprints around a circular track. After 1 minute, she has run $\frac{3}{8}$ of the way around the track.

    a. What is the measure of the angle created by her 1-minute run? State your answer in radians.

    b. At what exact point would the angle determined in part a intersect the unit circle?

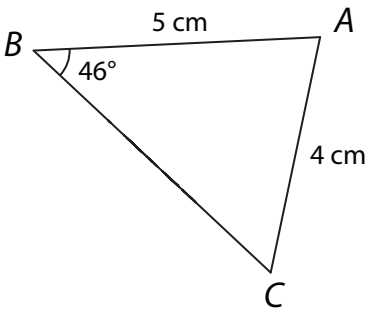
    c. What is the tangent of the angle determined in part a?

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Pre-Assessment

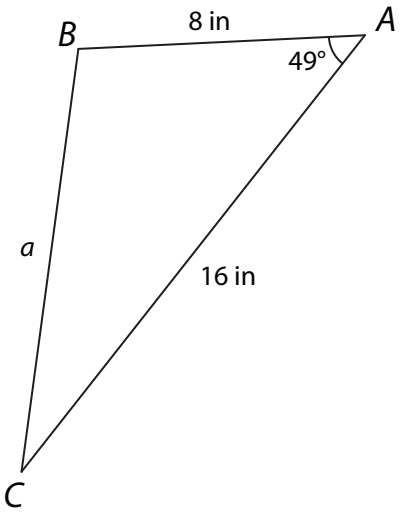
Circle the letter of the best answer.

1. What is $m\angle C$?



a. 35°
b. 41°
c. 52°
d. 64°

2. What is the length of side a?



a. 10.5 in
b. 12.3 in
c. 13.6 in
d. 15.4 in

3. In $\triangle RST$, $m\angle R = \frac{\pi}{4}$ radians, $m\angle S = \frac{\pi}{3}$ radians, and $ST = 7$ mm. Find RT.

a. 7 mm
b. 8.6 mm
c. 9.3 mm
d. 11.4 mm

continued

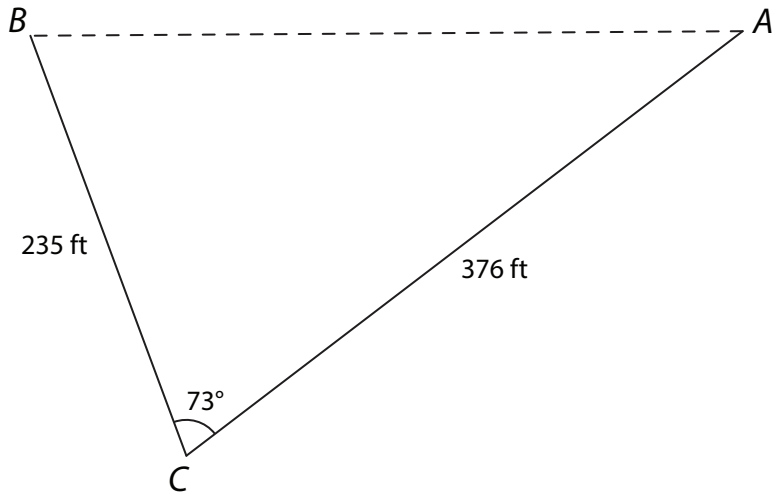
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Assessment

4. In $\triangle LMN$, $LM = 6$ feet, $MN = 5$ feet, and $LN = 8$ feet. Find $m\angle L$.

   a. 38.6°
   b. 48.5°
   c. 51.4°
   d. 92.9°

5. A surveyor wants to measure the length of a rock formation. He walks 235 feet from one end of the rock (point *B*), turns so he creates a 73° angle, and walks 376 feet to the other end of the formation. How long is the rock formation, represented by *AB*?



   a. 186 feet
   b. 381 feet
   c. 413 feet
   d. 572 feet

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Lesson 2: Trigonometry of General Angles

Instruction

Common Core State Standards

SMP
1 ✓ 2 ✓
3 ✓ 4 ✓
5 6
7 ✓ 8

G–SRT.9 (+) Derive the formula $A = 1/2\ ab\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

G–SRT.10 (+) Prove the Laws of Sines and Cosines and use them to solve problems.

G–SRT.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Essential Questions

1. How can you prove the Law of Sines?
2. How can you find the area of a triangle when given two sides and the included angle?
3. How can you prove the Law of Cosines?
4. When should you use the Law of Sines and the Law of Cosines?

WORDS TO KNOW

altitude the perpendicular line from a vertex of a figure to its opposite side; height

ambiguous case a situation wherein the Law of Sines produces two possible answers. This only occurs when the lengths of two sides and the measure of the non-included angle are given (SSA).

arccosine the inverse of the cosine function, written $\cos^{-1}\theta$ or $\arccos\theta$

arcsine the inverse of the sine function, written $\sin^{-1}\theta$ or $\arcsin\theta$

cosine a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the side adjacent to the length of the hypotenuse; the cosine of $\theta = \cos\theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$

SRB p. U3-58

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Instruction

included angle — the angle between two sides

Law of Cosines — a formula for any triangle which states $c^2 = a^2 + b^2 - 2ab\cos C$, where C is the included angle in between sides a and b, and c is the nonadjacent side across from $\angle C$

Law of Sines — a formula for any triangle which states $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, where a represents the measure of the side opposite $\angle A$, b represents the measure of the side opposite $\angle B$, and c represents the measure of the side opposite $\angle C$

oblique triangle — a triangle that does not contain a right angle

sine — a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the hypotenuse; the sine of $\theta = \sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$

tangent — a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the adjacent side; the tangent of $\theta = \tan\theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$

vertex — a point at which two or more lines meet

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Recommended Resources

- AlgebraLAB.org. "Law of Sines and Law of Cosines #1."

  http://www.walch.com/rr/00221

  Users can practice solving problems involving the Laws of Sines and Cosines and then check their answers. The multiple-choice practice problems feature "hint" buttons and embedded links to the definitions of terms within problems. Scroll to the bottom of the page and click "Show Related AlgebraLab Documents" to reveal links to additional resources, including lessons, extra practice, study aids, and word problem examples.

- Khan Academy. "Proof of the Law of Cosines."

  http://www.walch.com/rr/00222

  This video tutorial details the proof of the Law of Cosines, a proof of the Law of Sines, and examples of the Law of Cosines. Users can also access practice problems through the sidebar menu.

- MathWarehouse.com. "Law of Sines and Cosines."

  http://www.walch.com/rr/00223

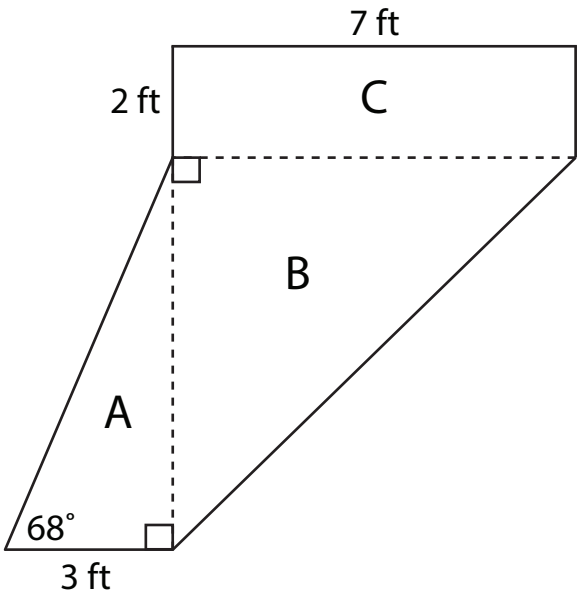
  This site provides a handy chart illustrating which law applies to different situations, as well as a video and practice problems.

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Lesson 3.2.1: Proving the Law of Sines

Warm-Up 3.2.1

Alexandra planted a vegetable garden and split it into three different plots, as shown in the diagram. Use this diagram to answer the questions that follow. Round answers to the nearest tenth.



1. Alexandra planted herbs in plot A. What is the area of this plot? *Hint*: To find the height of the triangle, use the tangent ratio, $\tan\theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$.

2. What is the area of the entire garden?

3. If Alexandra wants to put up a fence around plot A, how much fencing will she need to buy?

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Instruction

Lesson 3.2.1: Proving the Law of Sines

Common Core State Standards

G–SRT.9 (+) Derive the formula $A = 1/2\ ab\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

G–SRT.10 (+) Prove the Laws of Sines and Cosines and use them to solve problems.

Warm-Up 3.2.1 Debrief

1. Alexandra planted herbs in plot A. What is the area of this plot? *Hint*: To find the height of the triangle, use the tangent ratio, $\tan\theta = \dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$.

   First, find the height of the triangle using trigonometric ratios. Let h represent the height of the triangle. Since h is opposite of the known angle measure (68°) and the length of the adjacent side is 3 feet, use the tangent ratio, $\dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$, to find h.

$\tan\theta = \dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$	General equation for the tangent ratio
$\tan(68°) = \dfrac{(h)}{(3)}$	Substitute 68° for θ, h for the length of the opposite side, and 3 for the length of the adjacent side.
$3\tan 68° = h$	Multiply both sides of the equation by 3.
$3 \bullet (2.475) = h$	Substitute 2.475 for tan 68°.
$7.43 \approx h$	Multiply.

   The height (h) of the triangle is approximately 7.43 feet.

   Next, use the height and the base to find the area of the triangle.

$A = \dfrac{1}{2}bh$	Formula for the area of a triangle
$A = \dfrac{1}{2}(3)(7.43)$	Substitute 3 for b and 7.43 for h.
$A = 11.1$	Simplify.

   The area of plot A is 11.1 ft^2. (Recall that area is measured in square units.)

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2. What is the area of the entire garden?

   To find the area of the entire garden, first find the area of each individual plot.

   Find the area of plot B, a triangle with the same height as plot A.

   $A = \frac{1}{2}bh$ Formula for the area of a triangle

   $A = \frac{1}{2}(7)(7.43)$ Substitute 7 for b and 7.43 for h.

   $A = 26.0$ Simplify.

   The area of plot B is 26.0 ft^2.

   Find the area of plot C, a rectangle.

   $A = lw$ Formula for the area of a rectangle

   $A = (7)(2)$ Substitute 7 for l and 2 for w.

   $A = 14.0$ Simplify.

   The area of plot C is 14.0 ft^2.

   Next, add the areas of all three plots together to find the total area of the garden.

   $11.1 + 26.0 + 14.0 = 51.1$

   The area of the entire garden is 51.1 ft^2.

3. If Alexandra wants to put up a fence around plot A, how much fencing will she need to buy?

   To determine the necessary amount of fencing, find the perimeter of plot A.

   First, find the length of the hypotenuse using trigonometric ratios. Since the length of the side adjacent to the given angle is known, use the cosine ratio, $\frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$.

   $\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$ General equation for the cosine ratio

   $\cos(68°) = \frac{(3)}{(x)}$ Substitute 68° for θ, 3 for the length of the adjacent side, and x for the length of the hypotenuse.

   $x \cos 68° = 3$ Multiply both sides of the equation by x.

   $x = \frac{3}{\cos 68°}$ Divide both sides by cos 68°.

   $x \approx 8.01$ Divide.

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The hypotenuse is approximately 8.01 feet long.

Add the three sides to find the perimeter.

$3 + 7.43 + 8.01 = 18.44$

The perimeter of the garden is approximately 18.4 feet. The fence must be the same length.

Connection to the Lesson

- Students will extend their knowledge of trigonometric functions by learning how to use the Law of Sines.
- Students will use trigonometric functions to develop a new formula for the area of a triangle.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- finding the sine of an angle (G-SRT.6)
- using the sine function to find a missing side or angle of a triangle (G-SRT.8)
- finding the area of a triangle (6.G.1)
- measuring angles in radians (F-TF.1)
- finding the measure of the third angle of a triangle (in degrees or radians) (8.G.6)
- given an angle of a triangle, identifying sides that are adjacent and opposite to that angle (G-SRT.6)
- identifying the hypotenuse of a right triangle (G-SRT.6)
- finding the perimeter of a triangle (3.MD.8)

Introduction

The trigonometric functions (sine, cosine, and tangent) can be used to solve for unknown angle measures and side lengths of right triangles. However, the Law of Sines, which is derived from the definition of sine, can be used to help solve for unknown angle measures and side lengths of any triangle, even if it is not a right triangle.

The sine ratio can also be used to develop a formula for the area of a triangle when two sides and the **included angle** (the angle between two sides) are known.

Key Concepts

- The trigonometric function sine is the ratio of the length of the opposite side to the length of the hypotenuse; the sine of $\theta = \sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$.
- The cosine function is the ratio of the length of the side adjacent to the length of the hypotenuse; the cosine of $\theta = \cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$.
- The tangent function is the ratio of the length of the opposite side to the length of the adjacent side; the tangent of $\theta = \tan\theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$.
- The Law of Sines can be derived from the definition of the sine function.

SRB p. U3-61

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Instruction

- The **Law of Sines** states $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$, where a represents the measure of the side opposite angle A, b represents the measure of the side opposite angle B, and c represents the measure of the side opposite angle C.
- The Law of Sines is true for any triangle, not just a right triangle.
- To use the Law of Sines, the measures of an angle and the side opposite that angle must be known.
- To find an unknown angle or side using the Law of Sines, substitute in the known information; eliminate the fraction that is incomplete and will not help to solve the equation; then, solve the resulting equation.
- The **ambiguous case** of the Law of Sines occurs when the law is used to find an angle measure given two sides and a non-included angle (SSA). The Law of Sines may produce one answer, two answers, or no answer.
- However, in the ambiguous case, if the measure of the angle found using the Law of Sines is greater than the other angle given, there is a second valid answer. To find it, subtract the measure of the first answer from 180° (or π radians). To ensure that an answer is valid, remember that the three angles of a triangle must add up to 180° or π radians. If the sum of two of the angles is greater than 180° or π radians, then the answer is not valid.
- The area of any triangle can be found using the formula $\frac{1}{2}ab\sin C$, in which angle C is the included angle in between the sides with lengths a and b. This formula can be used when two sides and the included angle are known.
- Before finding the area of a triangle, the altitude of the triangle must first be found. Recall that the **altitude** of a triangle is a perpendicular line segment that connects a **vertex,** a point at which two or more lines meet, to a point on the line of its opposite side.
- If you know the sine of an angle and want to find the angle measure, use arcsine. **Arcsine** is the inverse of the sine function, written $\sin^{-1}\theta$ or $\arcsin\theta$.

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Instruction

- To find the sine of an angle, or if you know the sine of an angle and want to find the angle measure, use a calculator.

On a TI-83/84:

Step 1: Make sure your calculator is in the correct mode (degrees or radians).

Step 2: For sine, press [SIN], then input the angle measure. Press [ENTER]. For arcsine, press [2ND][SIN] and then input the sine ratio of an angle measure. Press [ENTER]. The calculator will calculate the angle measure.

On a TI-Nspire:

Step 1: Make sure your calculator is in the correct mode (degrees or radians).

Step 2: From the home screen, arrow down to the Calculate icon and press [enter].

Step 3: For sine, press [trig] and select "sin," then input the angle measure. Press [enter]. For arcsine, press [trig] and select "$\sin^{-1}$," and then input the sine ratio of an angle measure. Press [enter]. The calculator will calculate the angle measure.

Common Errors/Misconceptions

- neglecting to ensure the calculator is in the proper mode (degrees or radians)
- forgetting to check for the ambiguous case
- incorrectly substituting measures for sides and angles into a fraction or formula
- using the measurement of an adjacent side instead of the opposite side in a fraction or formula

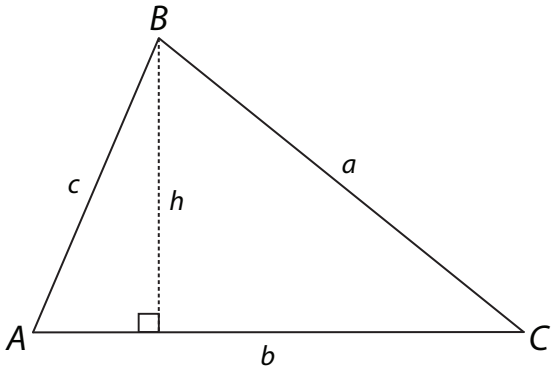
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Instruction

Guided Practice 3.2.1

Example 1

Prove the Law of Sines, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$, using the given information for the diagram of $\triangle ABC$ with altitude h and sides a, b, and c.



1. Use the definition of sine to create statements for angles A and C.

   Recall that the sine is equal to the ratio of the length of the opposite side to the length of the hypotenuse of a right triangle.

   Therefore, the ratios for $\sin A$ and $\sin C$ are as follows:

   $$\sin A=\frac{h}{c} \qquad \sin C=\frac{h}{a}$$

   Note that a, b, c, and h not only identify the sides and altitude of the triangle, they also represent the lengths of these sides and the altitude. A, B, and C identify the angles and also represent the measures of these angles.

2. Solve $\sin A$ and $\sin C$ for h.

   For $\sin A=\frac{h}{c}$, multiply both sides by c to get $c \sin A = h$.

   For $\sin C=\frac{h}{a}$, multiply both sides by a to get $a \sin C = h$.

SWB p. 51; SRB p. U3-63

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Instruction

3. Set the equations equal to each other and rearrange.

Since both expressions equal h, they must also equal each other:

$c \sin A = a \sin C$

Divide both sides by $\sin A$ and $\sin C$.

$\frac{c \sin A}{\sin A} = \frac{a \sin C}{\sin A}$ Divide both sides by $\sin A$.

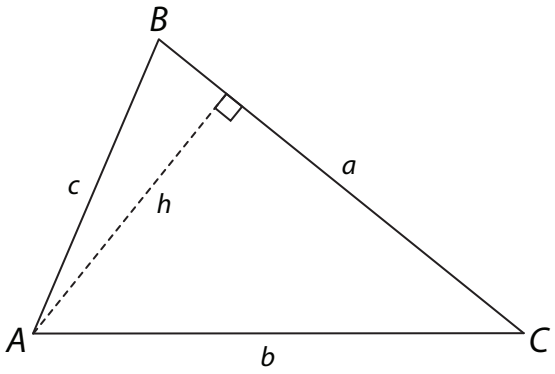
$c = \frac{a \sin C}{\sin A}$ Simplify.

$\frac{c}{\sin C} = \frac{\frac{a \sin C}{\sin A}}{\sin C}$ Divide both sides by $\sin C$.

$\frac{c}{\sin C} = \frac{a}{\sin A}$ Simplify.

4. Repeat the proof for angles B and C.

Start by drawing an altitude from angle A to side a.



5. Use the definition of sine to create statements for angles B and C.

Set up ratios for $\sin B$ and $\sin C$:

$\sin B = \frac{h}{c}$ $\sin C = \frac{h}{b}$

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Instruction

6. Solve the equations for sin B and sin C for h.

For $\sin B = \frac{h}{c}$, multiply both sides by c to get $c \sin B = h$.

For $\sin C = \frac{h}{b}$, multiply both sides by b to get $b \sin C = h$.

7. Set the equations equal to each other and rearrange.

Since both expressions equal h, they must also equal each other.

$c \sin B = b \sin C$

Divide both sides by sin B and sin C.

$\frac{c \sin B}{\sin B} = \frac{b \sin C}{\sin B}$ Divide both sides by sin B.

$c = \frac{b \sin C}{\sin B}$ Simplify.

$\frac{c}{\sin C} = \frac{\frac{b \sin C}{\sin B}}{\sin C}$ Divide both sides by sin C.

$\frac{c}{\sin C} = \frac{b}{\sin B}$ Simplify.

8. Combine the two resulting formulas from steps 3 and 7 to complete the proof.

From step 3, $\frac{c}{\sin C} = \frac{a}{\sin A}$.

From step 7, $\frac{c}{\sin C} = \frac{b}{\sin B}$.

Since both $\frac{a}{\sin A}$ and $\frac{b}{\sin B}$ are equal to $\frac{c}{\sin C}$,

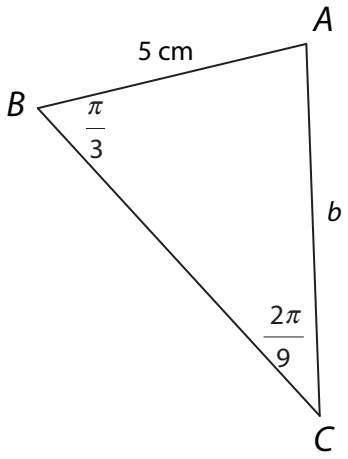
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, thus proving the Law of Sines.

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Instruction

Example 2

For $\triangle ABC$, find the length of side b if $m\angle B = \frac{\pi}{3}$ radians and $m\angle C = \frac{2\pi}{9}$ radians.



1. Substitute the known values into the Law of Sines.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ Formula for the Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin\left(\frac{\pi}{3}\right)} = \frac{(5)}{\sin\left(\frac{2\pi}{9}\right)}$$ Substitute 5 for c, $\frac{\pi}{3}$ for $m\angle B$, and $\frac{2\pi}{9}$ for $m\angle C$.

Applying the Law of Sines, the given triangle can be represented by $\frac{a}{\sin A} = \frac{b}{\sin\frac{\pi}{3}} = \frac{5}{\sin\frac{2\pi}{9}}$.

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Instruction

2. Determine which two fractions are necessary for solving the equation and then solve.

Applying the Law of Sines to this example, the following equation results: $\frac{a}{\sin A}=\frac{b}{\sin\frac{\pi}{3}}=\frac{5}{\sin\frac{2\pi}{9}}$.

Eliminate the first fraction, $\frac{a}{\sin A}$, since neither the numerator nor denominator is given.

What remains is $\frac{b}{\sin\frac{\pi}{3}}=\frac{5}{\sin\frac{2\pi}{9}}$. Solve this equation for b.

$\frac{b}{\sin\frac{\pi}{3}}=\frac{5}{\sin\frac{2\pi}{9}}$ Rewritten equation

$b=\frac{5\sin\frac{\pi}{3}}{\sin\frac{2\pi}{9}}$ Multiply both sides by $\sin\frac{\pi}{3}$ to isolate b.

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Instruction

3. Use your calculator to find the value of b.

First, use a calculator to find values for $\sin\frac{\pi}{3}$ and $\sin\frac{2\pi}{9}$.

$\sin\frac{\pi}{3} \approx 0.866$

$\sin\frac{2\pi}{9} \approx 0.643$

Substitute these values into the equation from the previous step.

$b = \frac{5\sin\frac{\pi}{3}}{\sin\frac{2\pi}{9}}$ Equation from the previous step

$b \approx \frac{5(0.866)}{(0.643)}$ Substitute 0.866 for $\sin\frac{\pi}{3}$ and 0.643 for $\sin\frac{2\pi}{9}$.

$b \approx \frac{4.33}{0.643}$ Multiply 5 • 0.866.

$b \approx 6.7$ Divide.

The length of side $b \approx 6.7$ cm.



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Instruction

Example 3

For $\triangle ABC$, find the measure of angle C if $m\angle A = 34°$, $c = 9$ meters, and $a = 7$ meters.

1. Substitute the known values into the Law of Sines.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ Formula for the Law of Sines

$$\frac{(7)}{\sin(34°)} = \frac{b}{\sin B} = \frac{(9)}{\sin C}$$ Substitute 7 for a, 34° for $m\angle A$, and 9 for c.

2. Determine which two fractions are necessary for solving the equation and then solve.

Eliminate the second fraction, $\frac{b}{\sin B}$, since neither the numerator nor the denominator is given.

What remains is $\frac{7}{\sin 34°} = \frac{9}{\sin C}$. Solve this equation for sin C.

$\frac{7}{\sin 34°} = \frac{9}{\sin C}$ Rewritten equation

$7 \sin C = 9 \sin 34°$ Eliminate the fractions by multiplying both sides of the equation by sin 34° and sin C.

$\sin C = \frac{9 \sin 34°}{7}$ Divide both sides by 7.

To solve for $m\angle C$, take the arcsine of both sides.

$$m\angle C = \arcsin\left(\frac{9 \sin 34°}{7}\right)$$

3. Use your calculator to find $m\angle C$.

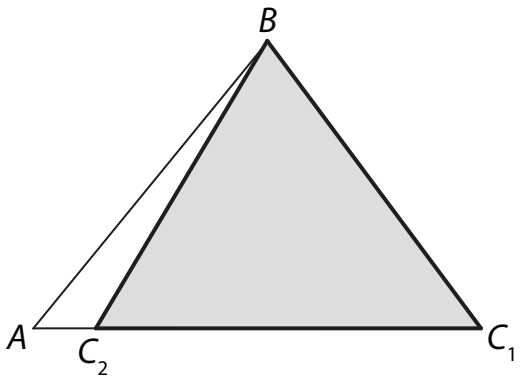
$m\angle C \approx 46.0°$

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Instruction

4. Determine whether the ambiguous case is possible to determine if there is a second valid answer.

   The ambiguous case occurs when the Law of Sines is used to find an angle measure using two sides and a non-included angle (SSA). We now know the values for two sides and an angle. Because the angle is not the included angle, there may be two possible answers. Since $m\angle C$ is greater than $m\angle A$, we know there is a second valid answer. Consider the following figure.



   Two possible triangles can result from this information. One would be acute (ABC_1) and the other obtuse (ABC_2). The Law of Sines gave us the acute angle, C_1, but another angle may exist.

   Since the measure of the angle found in step 3, $m\angle C \approx 46.0°$, is greater than that of the original angle given ($m\angle A = 34°$), there is a second valid answer.

5. Determine the second valid measure of angle C.

   Subtract the first value of angle C from 180°.

   $180 - 46 = 134$

   This answer is valid because a value of 134° for angle C, when added to the known value of 34° for angle A, still leaves 12° remaining for angle B. (Remember, the sum of the angles of a triangle must equal 180°.)

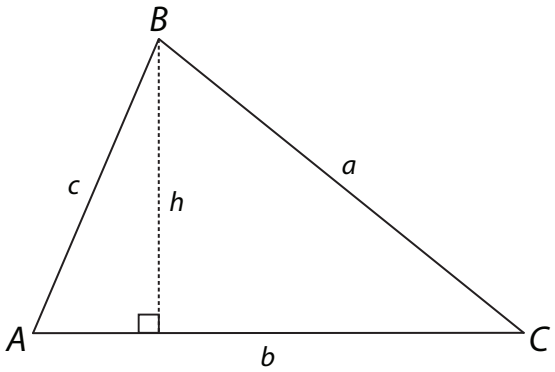
   Therefore, $m\angle C \approx 46.0°$ and $m\angle C \approx 134.0°$.

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Instruction

Example 4

Derive the formula $A = \frac{1}{2}ab\sin C$ for the area of $\triangle ABC$ with altitude h.



1. Write an expression for the altitude h using sin C.

   The sine of an angle is the ratio of the length of the opposite side to the length of the hypotenuse.

   The angle is C, the opposite side is the altitude h, and the hypotenuse of the triangle formed by angle C and h is a.

   Therefore, $\sin C = \frac{h}{a}$.

2. Solve the expression for h.

   Multiply both sides by a.

   $h = a \sin C$

3. Substitute the value for h into the basic formula for the area of a triangle.

   $A = \frac{1}{2}bh$ Formula for the area of a triangle

   $A = \frac{1}{2}b(a\sin C)$ Substitute a sin C for h.

   $A = \frac{1}{2}ab\sin C$ Apply the Commutative Property of Multiplication.

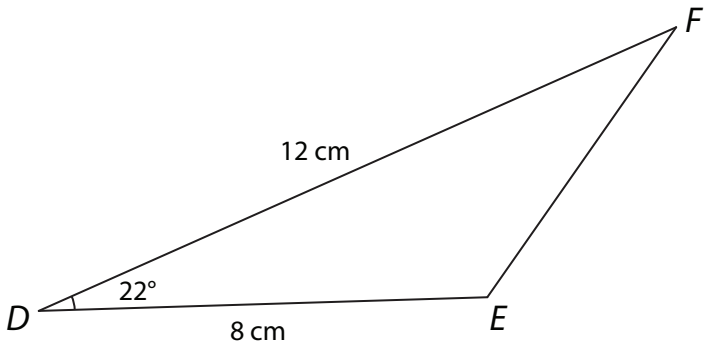
   Thus it is possible to derive the trigonometric formula $A = \frac{1}{2}ab\sin C$ for the area of $\triangle ABC$ with altitude h.

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Instruction

Example 5

Find the area of $\triangle DEF$ using the given information.



1. Substitute the values into the formula for the area of a triangle.

   Since two sides and the included angle are given, the formula $A = \frac{1}{2}ab \sin C$ can be used. The lengths of the sides are a and b, and C is the measure of the included angle.

$A = \frac{1}{2}ab \sin C$	Formula
$A = \frac{1}{2}(8)(12)\sin 22°$	Substitute 8 for a, 12 for b, and 22° for C.
$A = \frac{1}{2}(96)\sin 22°$	Multiply 8 and 12.
$A = 48 \sin 22°$	Simplify.

2. Use your calculator to find the area.

   The area of the triangle is equal to 48 sin 22°.

   Use your calculator to find the area of the triangle. Make sure your calculator is in degree mode.

   $A = 48 \sin 22° \approx 18.0$

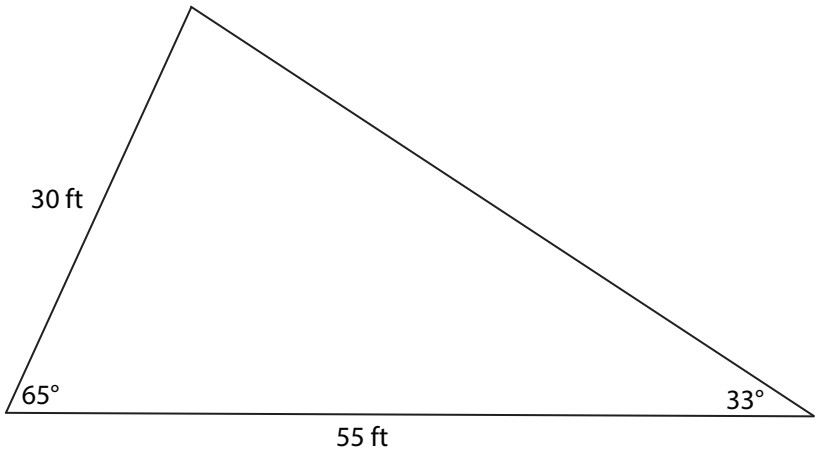
   The area of $\triangle DEF$ is approximately 18.0 cm^2.

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Problem-Based Task 3.2.1: Count the Cost

SMP	
1 ✓	2 ✓
3	4 ✓
5	6
7	8

Mr. Soliz wants to enclose a triangular plot of land to create a pen for his sheep and goats. He estimates that fencing will cost $10 per foot. He also wants to plant a certain type of grass, which he estimates will cost $0.03 per square foot. How much money, to the nearest dollar, will he need for the entire project?





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Problem-Based Task 3.2.1: Count the Cost

Coaching

a. How can you determine how much fencing will be needed?

b. What is the length of the third side of the pen?

c. How much fencing will be needed?

d. What is the estimated fencing cost?

e. What measurement must you find to determine the estimated cost of the grass?

f. Determine the measurement from part e.

g. What is the estimated cost of the grass?

h. What is the estimated total cost of the entire project?

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Instruction

Problem-Based Task 3.2.1: Count the Cost

Coaching Sample Responses

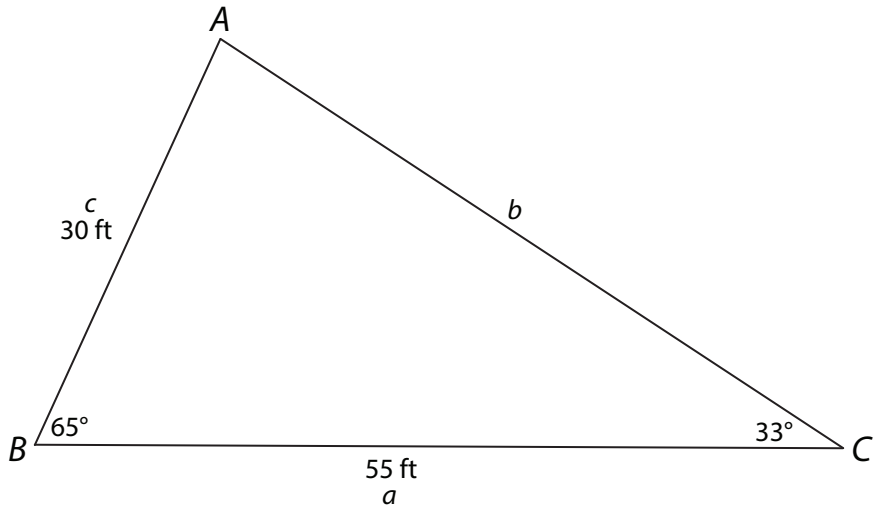
a. How can you determine how much fencing will be needed?

Fencing goes around the outside of the pen, which is the perimeter of the triangle. To find the perimeter, add the lengths of the sides.

b. What is the length of the third side of the pen?

To find the length of the third side, use the Law of Sines.

Label each angle of the figure and the opposite sides.



Substitute the known information into the formula for the Law of Sines, then solve for the missing side measure.

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\frac{55}{\sin A}=\frac{b}{\sin 65°}=\frac{30}{\sin 33°}$$

$$\frac{b}{\sin 65°}=\frac{30}{\sin 33°}$$

$$b=\frac{30\sin 65°}{\sin 33°}$$

$$b\approx 50$$

The length of side b is approximately 50 feet.

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c. How much fencing will be needed?

Add the lengths of each side of the pen to determine the total amount of fencing needed.

30 + 55 + 50 = 135

135 feet of fencing will be needed.

d. What is the estimated fencing cost?

Multiply the amount of fencing needed by the estimated price of the fencing, $10 per foot.

135 • 10 = 1350

It is estimated that the fencing will cost $1,350.

e. What measurement must you find to determine the estimated cost of the grass?

The cost of the grass depends on the square footage of the pen. So, find the area of the pen.

f. Find the measurement from part e.

Since side b (50 feet) was rounded, use the values in the original problem to arrive at a more accurate answer. Thus, use 30 feet, 55 feet, and the included angle 65°. Remember that C is the angle in between a and b, even though those values do not correspond with the sketch labels.

$$A = \frac{1}{2}ab \sin C = \frac{1}{2}(30)(55) \sin 65° \approx 748$$

The area of the pen is approximately 748 square feet.

g. What is the estimated cost of the grass?

Multiply the number of square feet by the estimated price of $0.03 per square foot.

748 • 0.03 = 22.44

The estimated cost of the grass is $22.44.

h. What is the estimated total cost of the entire project?

Add the cost of the fencing and the grass.

1350 + 22.44 = 1372.44

The estimated total cost of fencing and grass, rounded to the nearest dollar, is about $1,372.

Recommended Closure Activity

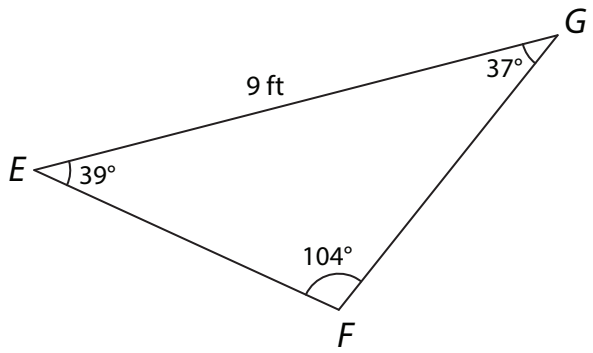
Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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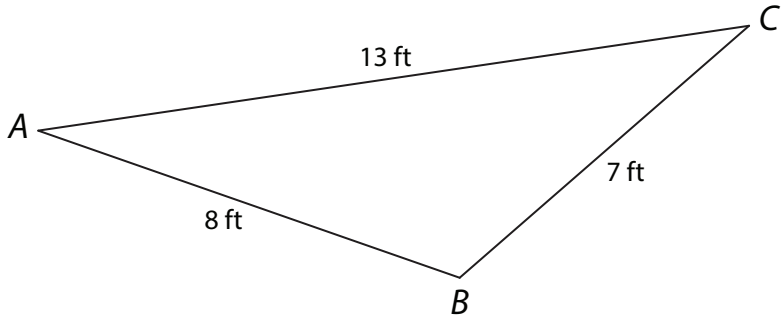
Practice 3.2.1: Proving the Law of Sines

For problems 1–5, use the Law of Sines to find each requested measurement. Round answers to the nearest hundredth.

1. What is the length of $\overline{FG}$?



2. If $m\angle B = \frac{2\pi}{3}$ radians, what is the measure of angle A?



3. For $\triangle UVW$, $UV = 6$ mm, $VW = 8$ mm, and $m\angle W = 34°$. What is the measure of angle U?

4. For $\triangle ABC$, $BC = 7$ inches, $m\angle A = 0.245$ radian, and $m\angle C = 0.784$ radian. What is the length of $\overline{AB}$?

5. For $\triangle XYZ$, $XY = 8$ feet, $XZ = 7$ feet, and $m\angle Y = \frac{\pi}{4}$ radian. What is the measure of angle Z?

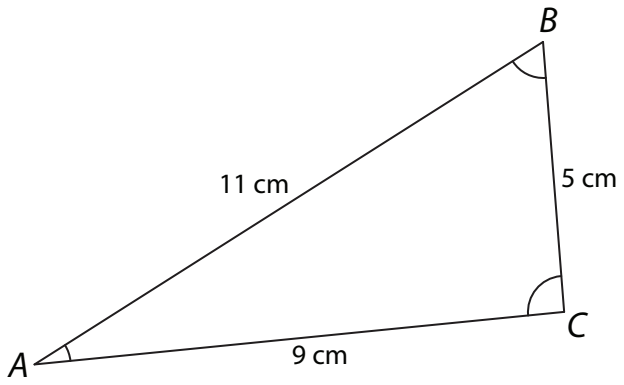
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For problems 6 and 7, find the area of each triangle. Round answers to the nearest hundredth.

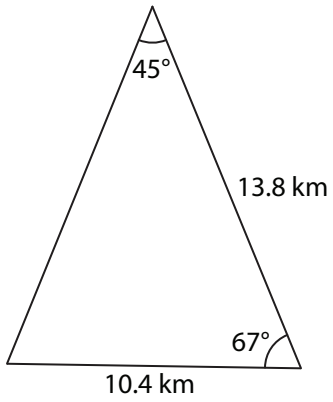
6. For $\triangle CDE$, CD = 11 meters, DE = 6 meters, EC = 8 meters, and $m\angle C = 112°$.

7. For $\triangle ABC$ as shown, $m\angle A = \frac{\pi}{7}$ radians.



For problems 8–10, find the perimeter of each triangle. Round answers to the nearest tenth.

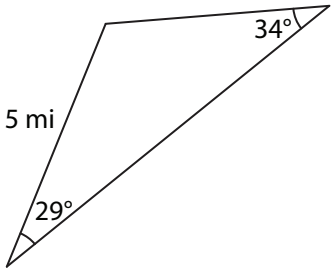
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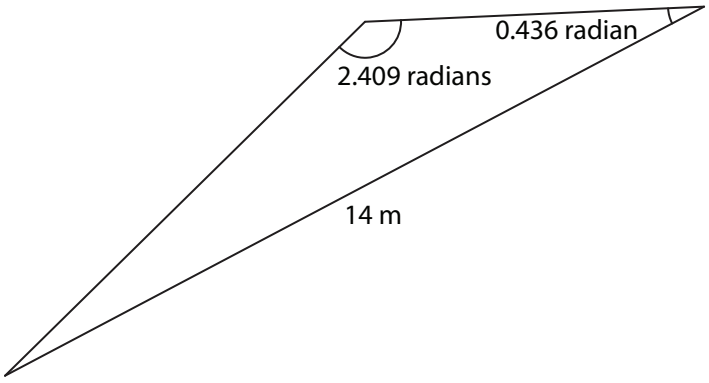
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9.



10.

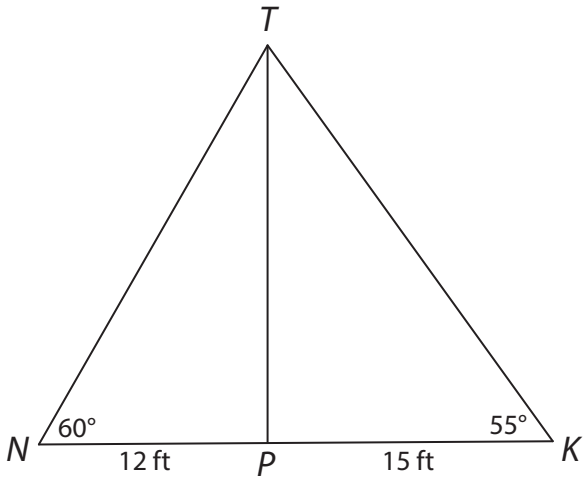


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Lesson 3.2.2: Proving the Law of Cosines

Warm-Up 3.2.2

Paul, Nate, and Kirsten want to practice shooting their paintball guns. They set up a target and stand due south of it. Nate then walks 12 feet to the west to face the target at a 60° angle. Kirsten walks 15 feet to the east to face the target at a 55° angle. The diagram shows the positions of Nate, Paul, and Kirsten (*N*, *P*, and *K*) in relation to the target, *T*. Use the diagram to answer the questions that follow. Round answers to the nearest whole foot.



1. How far is Nate from the target?

2. How far is Kirsten from the target?

3. How far is Paul from the target?

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Instruction

Lesson 3.2.2: Proving the Law of Cosines

Common Core State Standard

G–SRT.10 (+) Prove the Laws of Sines and Cosines and use them to solve problems.

Warm-Up 3.2.2 Debrief

1. How far is Nate from the target?

   $\triangle TNP$ and $\triangle TKP$ are right triangles since Paul is due south of the target and Nate and Kirsten walked due west and east, respectively.

   Use the trigonometric ratios of $\triangle TNP$ to find the length of $\overline{NT}$. Since $\overline{NT}$ is the hypotenuse of a triangle for which we know an angle (60°) and its adjacent side (12 feet), use the cosine ratio: $\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$.

$\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$	General equation for the cosine ratio
$\cos(60°) = \frac{(12)}{(x)}$	Substitute 60° for θ, 12 for the length of the adjacent side, and x for the length of the hypotenuse.
$x \cos 60° = 12$	Multiply both sides by x.
$x = \frac{12}{\cos 60°}$	Divide both sides by cos 60°.
$x = 24$	Use your calculator to simplify.

   Nate is 24 feet from the target.

2. How far is Kirsten from the target?

   Use the trigonometric ratios of $\triangle TKP$ to find the length of $\overline{KT}$. Since we know an angle and its adjacent side, use the cosine ratio.

$\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$	General equation for the cosine ratio
$\cos(55°) = \frac{(15)}{(x)}$	Substitute 55° for θ, 25 for the length of the adjacent side, and x for the length of the hypotenuse.
$x \cos 55° = 15$	Multiply both sides by x.

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$x = \frac{15}{\cos 55°}$	Divide both sides by cos 55°.
$x \approx 26.15$	Use your calculator to simplify.

Kirsten is approximately 26 feet from the target.

3. How far is Paul from the target?

Use the trigonometric ratios of either triangle to find the length of $\overline{PT}$. Since $\overline{PT}$ is the side opposite the known angle measures, use the tangent ratio, $\tan\theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$.

$\tan\theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$	General equation for the tangent ratio
$\tan(60°) = \frac{(x)}{(12)}$	Substitute 60° for θ, x for the length of the opposite side, and 12 for the length of the adjacent side.
$x = 12 \tan 60°$	Multiply both sides by 12.
$x \approx 20.78$	Use your calculator to simplify.

Paul is approximately 21 feet from the target.

Connection to the Lesson

- Students will extend their knowledge of trigonometric functions to using the Law of Cosines.
- Students will use techniques for solving right triangles to solve oblique triangles.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- finding the cosine of an angle (G-SRT.6)
- using the cosine function to find a missing side or angle of a right triangle (G-SRT.8)
- measuring angles in radians (F-TF.1)
- finding the measure of the third angle of a triangle (in degrees or radians) when the measure of two angles are known (8.G.6)
- using the Law of Sines to find an unknown angle or side of an oblique triangle (G-SRT.10)
- using the Pythagorean Theorem (G-SRT.10)
- factoring common factors from a polynomial (A-SSE.2)

Introduction

The sine, cosine, and tangent functions can be used to find unknown lengths of sides or measures of angles in right triangles, and in certain cases the Law of Sines can be used to find missing pieces of **oblique triangles** (triangles that do not contain a right angle). If two angles and a side are known (Angle-Side-Angle or ASA, or Angle-Angle-Side or AAS), or two sides and the non-included angle are known (Side-Side-Angle or SSA), the Law of Sines can be used to solve the triangle. However, sometimes there is not enough information to use the Law of Sines. In this case, one must use the Law of Cosines, which is derived from a variety of other trigonometric formulas.

Key Concepts

- Recall that the cosine function is the ratio of the length of the side adjacent to the length of the hypotenuse; the cosine of $\theta = \cos\theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$.
- The **arccosine** is the inverse of the cosine function, written $\cos^{-1}\theta$ or $\arccos\theta$.
- The Law of Cosines can be used to solve any triangle when two sides and the included angle are known (SAS) or when three sides are known (SSS).
- The **Law of Cosines** states $c^2 = a^2 + b^2 - 2ab\cos C$, where C is the included angle in between sides a and b, and c is the (nonadjacent) side across from angle C.
- The Law of Cosines can be derived from the Pythagorean Theorem using the definitions of sine and cosine, along with the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$.

SRB p. U3-76

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Instruction

- The Law of Cosines is true for any triangle, not just a right triangle.
- To find the cosine or arccosine of a given angle measure, use a graphing calculator.

On a TI-83/84:

Step 1: Make sure your calculator is in the correct mode (degrees or radians).

Step 2: For cosine, press [COS], then input the angle measure. Press [ENTER]. For arccosine, press [2ND][COS] and then input the angle measure. Press [ENTER].

On a TI-Nspire:

Step 1: Make sure your calculator is in the correct mode (degrees or radians).

Step 2: For cosine, press [trig] and select "cos," then input the angle measure. Press [enter]. For arccosine, press [trig] and select "$\cos^{-1}$," and then input the angle measure. Press [enter].

Common Errors/Misconceptions

- neglecting to ensure the calculator is in the proper mode (degrees or radians)
- forgetting to square the sides or take the square root
- using the Law of Sines instead of the Law of Cosines

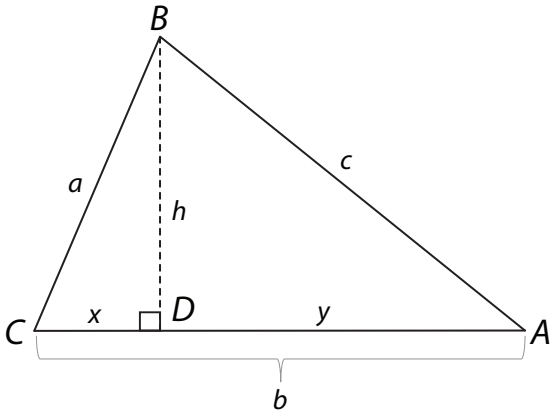
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Instruction

Guided Practice 3.2.2

Example 1

Given the diagram of $\triangle ABC$ with altitude h, prove the Law of Cosines.



1. Create statements for h, x, and y using the sine and cosine of angle C.

   Recall that in a right triangle, the sine is equal to the ratio of the length of the opposite side to the length of the hypotenuse, and the cosine is the ratio of the length of the adjacent side to the length of the hypotenuse.

$\sin C = \frac{h}{a}$	Set sin C equal to the ratio of sides h and a.
$h = a \sin C$	Multiply both sides by a and apply the Symmetric Property of Equality.
$\cos C = \frac{x}{a}$	Set cos C equal to the ratio of sides x and a.
$x = a \cos C$	Multiply both sides by a and apply the Symmetric Property of Equality.
$x + y = b$	Side b equals the sum of segments x and y.
$y = b - x$	Subtract x from both sides.
$y = b - (a \cos C)$	Substitute $a \cos C$ for x.

   The expressions for h, x, and y are $h = a \sin C$, $x = a \cos C$, and $y = b - a \cos C$.

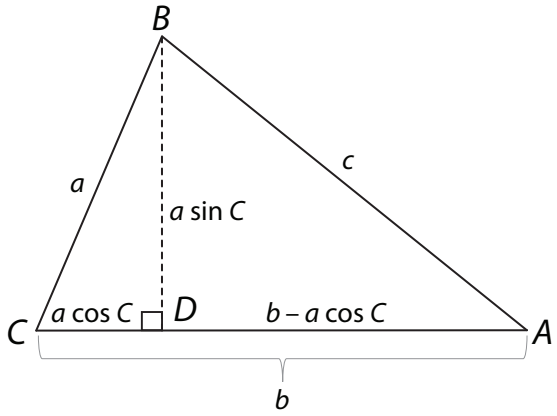
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SWB p. 65; SRB p. U3-78

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Instruction

The new values for h, x, and y can now be substituted into the diagram, as shown.



2. Use the Pythagorean Theorem and the known expressions to write an equation for $\triangle ADB$.

   From the diagram, we know that $\triangle ADB$ is a right triangle.

   Recall that the Pythagorean Theorem states that for a right triangle, $a^2 + b^2 = c^2$, where a and b are the legs and c is the hypotenuse.

$c^2 = a^2 + b^2$	Pythagorean Theorem
$c^2 = (a \sin C)^2 + (b - a \cos C)^2$	Substitute $a \sin C$ for a and $b - a \cos C$ for b.

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Instruction

3. Expand and rearrange the statement to develop the Law of Cosines.

$c^2 = (a \sin C)^2 + (b - a \cos C)^2$	Statement from the previous step
$c^2 = a^2 \sin^2 C + b^2 - 2ab \cos C + a^2 \cos^2 C$	Expand each squared term.
$c^2 = a^2 \sin^2 C + a^2 \cos^2 C + b^2 - 2ab \cos C$	Rearrange the terms.
$c^2 = a^2(\sin^2 C + \cos^2 C) + b^2 - 2ab \cos C$	Factor out the common a^2 from the first two terms on the right side of the equation.
$c^2 = a^2(1) + b^2 - 2ab \cos C$	Substitute 1 for $\sin^2 C + \cos^2 C$ (from the trigonometric identity $\sin^2 \theta + \cos^2 \theta = 1$).
$c^2 = a^2 + b^2 - 2ab \cos C$	Multiply.

The result, $c^2 = a^2 + b^2 - 2ab \cos C$, is the Law of Cosines.

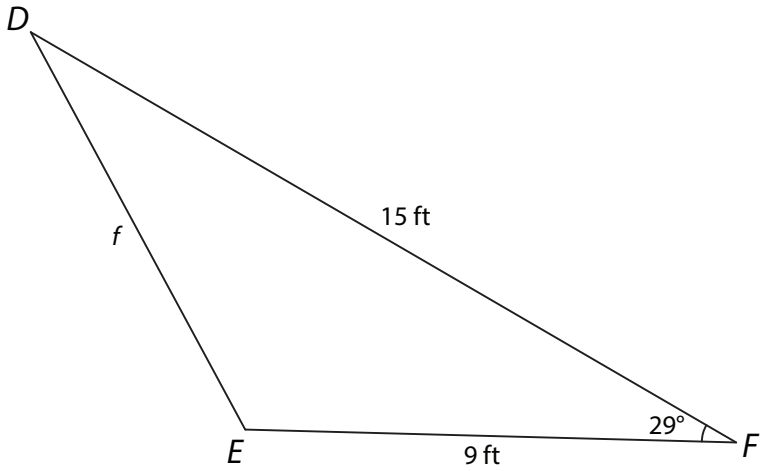


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Instruction

Example 2

Given the diagram of $\triangle DEF$, find the length of side f. Round the length to the nearest tenth.



1. Identify which values are equivalent to a, b, and c in the Law of Cosines formula.

   Recall that the Law of Cosines states $c^2 = a^2 + b^2 - 2ab \cos C$, where C is the included angle in between sides a and b, and c is the (nonadjacent) side across from angle C.

   Examine the diagram labels to determine which sides are equivalent to a, b, and c.

   Since sides are named after their opposite angles, the side labeled "9 ft" is side d and the side labeled "15 ft" is side e. Side f is given.

   In this triangle, we can match the angle names and sides: $a = d$, $b = e$, and $c = f$. Therefore, $\angle F$ is the included angle of sides d and e.

   $\angle C$ is the included angle in between sides a and b. Thus $a = 9$, $b = 15$, and $C = 29°$.

   c is the side across from angle C, so $c = f$.

2. Substitute the known values into the Law of Cosines formula.

   Since two sides and the included angle are known (SAS), the Law of Cosines can be used to find the length of the third side.

$c^2 = a^2 + b^2 - 2ab \cos C$	Formula for the Law of Cosines
$(f)^2 = (9)^2 + (15)^2 - 2(9)(15) \cos (29°)$	Substitute 9 for a, 15 for b, f for c, and 29° for C.

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Instruction

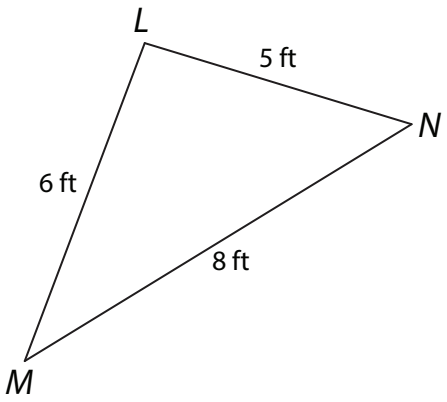
3. Solve the resulting equation for f.

$f^2 = 9^2 + 15^2 - 2(9)(15)\cos 29°$	Equation from the previous step
$f^2 = 81 + 225 - 270\cos 29°$	Simplify the squares and multiply.
$f^2 = 306 - 270\cos 29°$	Add.
$f = \sqrt{306 - 270\cos 29°}$	Take the square root of both sides.
$f \approx 8.4$	Use a calculator to find the answer.

The length of side f is approximately 8.4 feet.

Example 3

Given the diagram of $\triangle LMN$, find the measure of angle L in radians.



1. Identify which values are equivalent to a, b, and c in the Law of Cosines formula.

   Side c is the side across from angle C. Since we are looking for angle L, let angle L be equivalent to angle C. Thus, $c = 8$.

   a and b are the sides that make up angle C. The sides that make up angle L are 5 feet and 6 feet, so $a = 5$ and $b = 6$.

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Instruction

2. Substitute the known values into the Law of Cosines formula.

Since three sides are known (SSS), the Law of Cosines can be used to find the measure of an angle.

$c^2 = a^2 + b^2 - 2ab \cos C$	Formula for the Law of Cosines
$(8)^2 = (5)^2 + (6)^2 - 2(5)(6) \cos (L)$	Substitute 8 for c, 5 for a, 6 for b, and L for C.
$64 = 25 + 36 - 60 \cos L$	Simplify the squares and multiply.
$64 = 61 - 60 \cos L$	Add.

3. Solve the resulting equation for angle L.

$64 = 61 - 60 \cos L$	Equation from the previous step
$3 = -60 \cos L$	Subtract 61 from both sides.
$\frac{3}{-60} = \cos L$	Divide both sides by –60.
$m\angle L = \arccos\left(-\frac{3}{60}\right)$	Take the arccosine of both sides.
$m\angle L \approx 1.621$ radians	Use your calculator in radian mode to find $\arccos\left(-\frac{3}{60}\right)$.

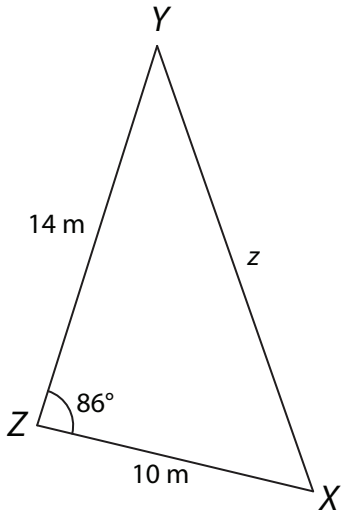
The measure of angle L is approximately 1.621 radians.

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Instruction

Example 4

Given the diagram of $\triangle XYZ$, solve the triangle by finding the missing sides and angles.



1. Use the Law of Cosines formula to find the length of side z.

   To solve a triangle, you must find the length of all the sides and all the angles.

   Since two sides and the included angle are known (SAS), the Law of Cosines can be used to find the third side.

$c^2 = a^2 + b^2 - 2ab \cos C$	Formula for the Law of Cosines
$(z)^2 = (14)^2 + (10)^2 - 2(14)(10) \cos (86°)$	Substitute z for c, 14 for a, 10 for b, and 86° for C.
$z^2 = 196 + 100 - 280 \cos 86°$	Simplify the squares and multiply.
$z^2 = 296 - 280 \cos 86°$	Add.
$z = \sqrt{296 - 280 \cos 86°}$	Take the square root of both sides.
$z \approx 16.6$	Use your calculator to find the answer.

   The length of side z is approximately 16.6 meters.

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Instruction

2. Find the measure of angle X.

Now that you know the length of side z, you can use either the Law of Sines or the Law of Cosines to find angle X.

$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$ Formula for the Law of Sines

$\frac{(z)}{\sin(Z)}=\frac{(x)}{\sin(X)}$ Substitute z for a, x for b, Z for A, and X for B. The third ratio is unnecessary.

$\frac{(16.6)}{\sin(86°)}=\frac{(14)}{\sin X}$ Substitute 16.6 for z, 14 for x, and 86° for Z.

$16.6 \sin X = 14 \sin 86°$ Multiply both sides by sin 86° and sin X.

$\sin X=\frac{14\sin 86°}{16.6}$ Divide both sides by 16.6.

$m\angle X=\arcsin\left[\frac{14\sin 86°}{16.6}\right]$ Take the arcsine of both sides.

$m\angle X \approx 57.3°$

The measure of angle X is approximately 57.3°.

3. Find the measure of angle Y.

The sum of the angles of a triangle is always 180°. Subtract the known values for angles X and Z from 180° to find $m\angle Y$.

180 – 86 – 57.3 = 36.7

The measure of angle Y is approximately 36.7°.

4. Summarize your findings.

The missing angle and side measures for $\triangle XYZ$ are $z \approx 16.6$ meters, $m\angle X \approx 57.3°$, and $m\angle Y \approx 36.7°$.

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Problem-Based Task 3.2.2: Travel Trouble

SMP	
1 ✓	2 ✓
3	4
5	6
7 ✓	8

An airplane took off from an airport (point *A*) and flew due west for 1 hour, 50 minutes at an average ground speed of 410 miles per hour (mph) until it experienced technical problems and had to divert to a nearby airport to land. The plane turned 45° to the north and traveled for 15 minutes at an average speed of 285 mph before landing. When the plane landed (at point *C*), how far was it from its original departure airport? Use the equation for distance, $d = rt$, where d is distance, r is speed, and t is time. Round answers to the nearest mile. Assume all distances and speeds include the effects of the wind.





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Problem-Based Task 3.2.2: Travel Trouble

Coaching

a. How far did the plane travel before changing course?

b. How far did the plane travel after changing course?

c. What is the measure of $\angle ABC$?

d. How can you find the distance from the departure airport (point A) to the plane's landing point (point C)?

e. What is the distance from the departure airport (point A) to the plane's landing point (point C)? Round your answer to the nearest mile.

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Instruction

Problem-Based Task 3.2.2: Travel Trouble

Coaching Sample Responses

a. How far did the plane travel before changing course?

The plane traveled for 1 hour, 50 minutes at an average speed of 410 mph.

The units must match; therefore, 1 hour, 50 minutes must be converted to hours.

Convert 50 minutes to hours.

$$50\text{ min} \bullet \frac{1\text{ hr}}{60\text{ min}} = \frac{50\text{ min}}{1} \bullet \frac{1\text{ hr}}{60\text{ min}} = \frac{50}{60}\text{ hr} = \frac{5}{6}\text{ hr}$$

Therefore, 1 hour, 50 minutes = $1\frac{5}{6}$ hours.

Substitute this value and the average speed, 410 mph, into the formula for distance, $d = rt$.

$d = rt$

$d = (410)\left(1\frac{5}{6}\right)$

$d \approx 751.67$ miles

The plane traveled approximately 751.67 miles before changing course.

b. How far did the plane travel after changing course?

After changing course, the plane traveled for 15 minutes at a speed of 285 mph.

Convert 15 minutes to hours.

$$15\text{ min} \bullet \frac{1\text{ hr}}{60\text{ min}} = \frac{15}{60}\text{ hr} = \frac{1}{4}\text{ hr} = 0.25\text{ hr}$$

Then, use this information and the known speed to find the distance traveled using the formula $d = rt$.

$d = rt$

$d = (285)(0.25)$

$d = 71.25$

The plane traveled 71.25 miles after changing course.

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c. What is the measure of $\angle ABC$?

$\angle ABC$ and the 45° angle are supplementary because they make a straight line.

Supplementary angles add up to 180°. Therefore, to find $m\angle ABC$, subtract 45° from 180°.

$180 - 45 = 135$

Therefore, $m\angle ABC = 135°$.

d. How can you find the distance from the departure airport (point *A*) to the plane's landing point (point *C*)?

Draw a line from point *A* to point *C* to create $\triangle ABC$.

The lengths of two sides ($\overline{AB}$ and $\overline{BC}$) and the included angle ($\angle ABC$) are known, so this is an example of SAS. Therefore, the Law of Cosines can be used to find the length of the third side, $\overline{AC}$.

e. What is the distance from the departure airport (point *A*) to the plane's landing point (point *C*)? Round your answer to the nearest mile.

Use the Law of Cosines and the known values to determine the distance, $\overline{AC}$.

$c^2 = a^2 + b^2 - 2ab \cos C$

$c^2 = (751.67)^2 + (71.25)^2 - 2(751.67)(71.25) \cos (135°)$

$c = \sqrt{751.67^2 + 71.25^2 - 2(751.67)(71.25)\cos 135°}$

$c \approx 804$ miles

The distance from the departure airport to the plane's landing point is approximately 804 miles.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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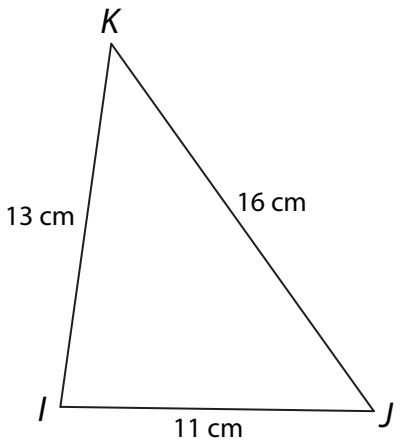
Practice 3.2.2: Proving the Law of Cosines

For problems 1–4, use the Law of Cosines to find the measure of each requested angle. Round degrees to the nearest tenth and radians to the nearest thousandth.

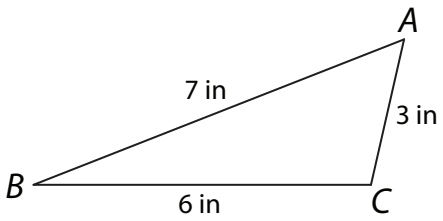
1. For $\triangle VWX$, find the measure of angle V in degrees if $VW = 15$ meters, $WX = 10$ meters, and $VX = 9$ meters.

2. For $\triangle NMO$, find the measure of angle O in radians if $NM = 4$ inches, $NO = 5$ inches, and $MO = 8$ inches.

3. Find the measure of angle I in radians.



4. Find the measure of angle B in degrees.



continued

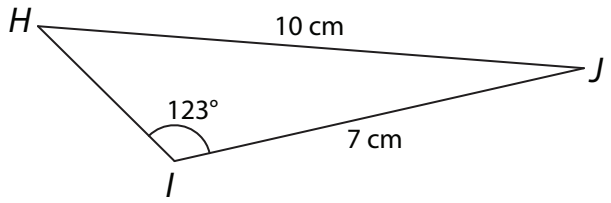
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For problems 5–7, solve each triangle by finding the measure of all the unknown angles and sides. Use degrees for angle measurements and round all answers to the nearest whole number.

5. For $\triangle RST$, $RS = 8$ in, $RT = 5$ in, and $m\angle R = 86°$.

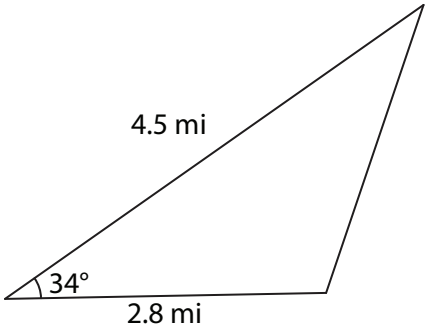
6. For $\triangle ABC$, $AB = 9$ meters, $BC = 4$ meters, and $AC = 6$ meters.

7. For $\triangle HIJ$, refer to the diagram.

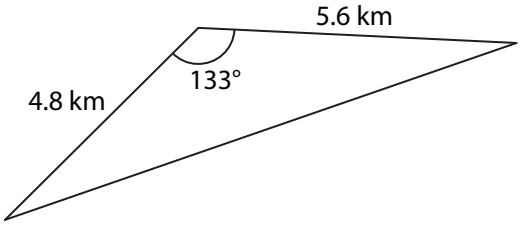


For problems 8–10, determine the amount of fencing that would be needed to completely enclose each triangular plot of land. Round answers to the nearest tenth.

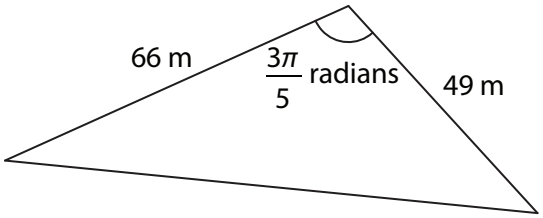
8.



9.



10.



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Lesson 3.2.3: Applying the Laws of Sines and Cosines

Warm-Up 3.2.3

Mr. Lawson bought boards to build a triangular-shaped sandpit for his sons. He cut two of the boards to be 4 feet long and the third to be 6 feet long. Answer the following questions about the sandpit. Use degrees for angle measures and round answers to the nearest tenth.

1. What will be the measure of the angle formed by the 4-foot boards?

2. Determine the measures of the other two angles.

3. What will be the area of the sandpit?

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Lesson 3.2.3: Applying the Laws of Sines and Cosines

Common Core State Standard

G–SRT.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Warm-Up 3.2.3 Debrief

1. What will be the measure of the angle formed by the 4-foot boards?

   Use the Law of Cosines to find the measure of the angle.

$c^2 = a^2 + b^2 - 2ab\cos C$	Formula for the Law of Cosines
$(6)^2 = (4)^2 + (4)^2 - 2(4)(4)\cos C$	Substitute 6 for c, 4 for a, and 4 for b.
$36 = 16 + 16 - 32\cos C$	Simplify.
$36 = 32 - 32\cos C$	Add.
$4 = -32\cos C$	Subtract 32 from both sides.
$\frac{4}{-32} = \cos C$	Divide both sides by –32.
$m\angle C = \arccos\left(-\frac{1}{8}\right)$	Simplify and take the arccosine of both sides.
$m\angle C \approx 97.2°$	Use your calculator to simplify.

   The measure of the angle formed by the 4-foot boards is approximately 97.2°.

2. Determine the measures of the other two angles.

   Since there are two lengths with the same measure, the triangle is isosceles, and the other two angles will have the same measure. Since the angles of a triangle add up to 180, subtract $m\angle C$ from 180°.

   $180 - 97.2 = 82.8$

   The other two angles must add up to 82.8°.

   Since both angles are equal, divide 82.8 by 2.

   $82.8 \div 2 = 41.4$

   The other two angles each measure 41.4°.

   (*Note*: This answer could also be found using the Law of Sines or the Law of Cosines.)

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3. What will be the area of the sandpit?

   Use the formula $A = \frac{1}{2}ab\sin C$ to find the area.

$A = \frac{1}{2}ab\sin C$	Formula for the area of a triangle
$A = \frac{1}{2}(4)(4)\sin(97.2°)$	Substitute 4 for a, 4 for b, and 97.2° for C.
$A = 8\sin 97.2°$	Simplify.
$A \approx 7.9$	Use your calculator to simplify.

   The area of the sandpit will be approximately 7.9 ft^2.

Connection to the Lesson

- Students will apply their knowledge of the Law of Sines and Law of Cosines to solve real-life problems.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- measuring angles in radians (F-TF.1)
- finding the measure of the third angle of a triangle (in degrees or radians) (8.G.6)
- using the Law of Sines to find an unknown angle or side measure of an oblique triangle (G-SRT.10)
- using the Law of Cosines to find an unknown angle or side measure of an oblique triangle (G-SRT.10)
- solving oblique or right triangles using the Law of Sines, the Law of Cosines, and the basic trigonometric functions of sine, cosine, and tangent (G-SRT.10)

Introduction

Because the Law of Sines and the Law of Cosines can be used to solve oblique triangles, they have many real-world applications. Two of the most common applications are in surveying and navigation.

Key Concepts

- In problems describing a real-world situation, if no picture is given, draw and label a diagram.
- Recall that the Law of Sines should be used when two angles and a side are known (ASA or AAS) or two sides and the non-included angle are known (SSA). The Law of Cosines must be used when two sides and the included angle are known (SAS) or when three sides are known (SSS).
- To find the area of a triangle when the height is not known, use the formula $A = \frac{1}{2}ab\sin C$.
- The calculated answer to a real-world problem should always be examined to determine if the answer is reasonable given the context.
- When appropriate, answers should include a unit label.

Common Errors/Misconceptions

- mistakenly using the Law of Sines instead of the Law of Cosines
- incorrectly drawing or labeling diagrams
- forgetting to correctly label answers on a diagram
- neglecting to check for ambiguous angle measures

SRB p. U3-89

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Instruction

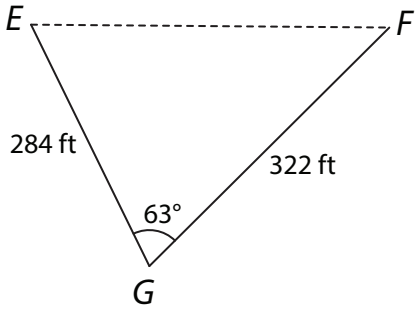
Guided Practice 3.2.3

Example 1

A surveyor needs to measure the width of a building. He identifies two points E and F at ground level on opposite sides of the building. He then walks to a point a distance away from the building: point G. He measures the distance from E to G as 284 feet, the distance from F to G as 322 feet, and the measure of $\angle G$ as 63°. How wide is the building? Round your answer to the nearest foot, then determine whether it is reasonable given the context of the problem.

1. Draw and label a sketch of the situation.

   This will help to organize the known information.



2. Find EF using the Law of Cosines.

   Since two sides and the included angle are known (SAS), the Law of Cosines must be used to find the length of the third side.

$c^2 = a^2 + b^2 - 2ab \cos C$	Formula for the Law of Cosines
$g^2 = (284)^2 + (322)^2 - 2(284)(322) \cos (63°)$	Substitute g for c, 284 for a, 322 for b, and 63° for C.
$g = \sqrt{284^2 + 322^2 - 2(284)(322)\cos 63°}$	Take the square root of both sides.
$g \approx 318$	Solve using a calculator.

   The width of the building is approximately 318 feet.

3. Determine if the answer is reasonable.

   Based on the other side lengths, yes, this answer is a reasonable estimate for the width of the building.

SWB p. 77; SRB p. U3-90

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Instruction

Example 2

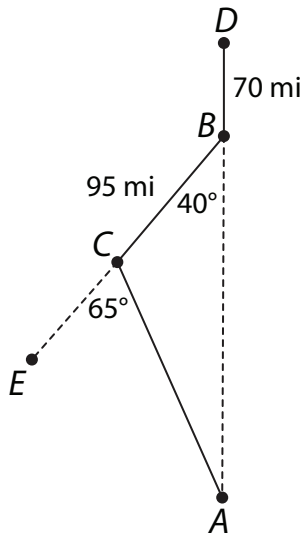
Hideko is a pilot with a small two-seater plane. She took off from the airport and headed south toward her hometown. After flying for 70 miles, Hideko turned the plane 40° to the west to avoid a large storm. She remained on this course for 95 miles before turning back 65° to the east and continuing on to land near her hometown. How much farther did Hideko fly than she would have if she could have remained on her original course? Determine whether your answer is reasonable given the context of the problem.

1. Draw and label a sketch of the situation.

   Draw a sketch that includes all the known information.

   Each change in the course is a change from the direction in which the plane was previously heading.

   Label the points with letters.



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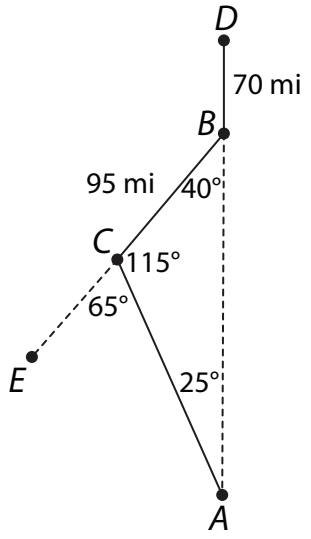
Instruction

2. Find the measure of $\angle BCA$ and $\angle CAB$.

Since $\angle BCA$ and $\angle ECA$ form a straight line, they are supplementary angles. Supplementary angles sum to 180, so subtract $m\angle ECA$ (65°) from 180° to find $m\angle BCA$: 180 – 65 = 115.

Since the sum of the angle measures of a triangle is 180°, subtract $m\angle BCA$ (115°) and $m\angle CBA$ (40°) from 180° to find the remaining angle of the triangle, $m\angle CAB$: 180 – 115 – 40 = 25.

Update the sketch with this new information.



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Instruction

3. Find AC using the Law of Sines.

The Law of Sines must be used since two angles (A and BCA) and one side (BC) are known (AAS).

$\frac{a}{\sin A}=\frac{b}{\sin B}$ — Use the portion of the Law of Sines formula in which a and b are the lengths of the sides across from angles A and B, respectively.

$\frac{(95)}{\sin(25°)}=\frac{b}{\sin(40°)}$ — Substitute 95 for a, 25° for A, and 40° for B.

$b=\frac{95\sin 40°}{\sin 25°}$ — Multiply both sides by sin 40°.

$b \approx 144.5$ — Use your calculator to find the answer. Round to the nearest tenth.

AC is approximately 144.5 miles. Hideko flew approximately 144.5 miles after turning back toward her hometown.

4. Find AB using the Law of Sines.

The Law of Sines is more appropriate for this situation than the Law of Cosines, because the original information can be substituted into the Law of Sines formula, thus reducing the chance for error. (The problem can also be solved using the Law of Cosines, but if a mistake were made in finding AC, AB would also be incorrect.)

$\frac{a}{\sin A}=\frac{c}{\sin C}$ — Use the portion of the Law of Sines formula in which a and c are the measures of the sides across from angles A and C, respectively.

$\frac{(95)}{\sin(25°)}=\frac{c}{\sin(115°)}$ — Substitute 95 for a, 25° for A, and 115° for C.

$c=\frac{95\sin 115°}{\sin 25°}$ — Multiply both sides by sin 115°.

$c \approx 203.7$ — Use your calculator to find the answer. Round to the nearest tenth.

AB is approximately 203.7 miles.

Hideko was 203.7 miles away from her hometown when she had to change course.

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Instruction

5. Determine how much farther Hideko flew as a result of changing course.

   The 70 miles Hideko flew before initially changing course are irrelevant, because the distance flown before the detour has no effect on how much distance was added to the trip. Therefore, this figure can be ignored and the difference can be calculated starting at the point at which Hideko turned west away from the storm.

   Add BC and AC to find the total distance Hideko flew between the point at which she turned west and when she landed near her home.

   $95 + 144.5 = 239.5$

   Hideko flew a total of 239.5 miles from the detour point to her hometown.

   If she had stayed on course, she would have flown only 203.7 miles from the detour point to her hometown.

   Subtract to find the difference.

   $239.5 - 203.7 = 35.8$

   Hideko flew an additional 35.8 miles as a result of the detour.

6. Consider whether or not the answer is reasonable.

   Yes, 35.8 miles is a reasonable answer. If a much larger answer such as 443 miles had been found, this answer would not make sense because that is a greater distance than the entire trip.

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Instruction

Example 3

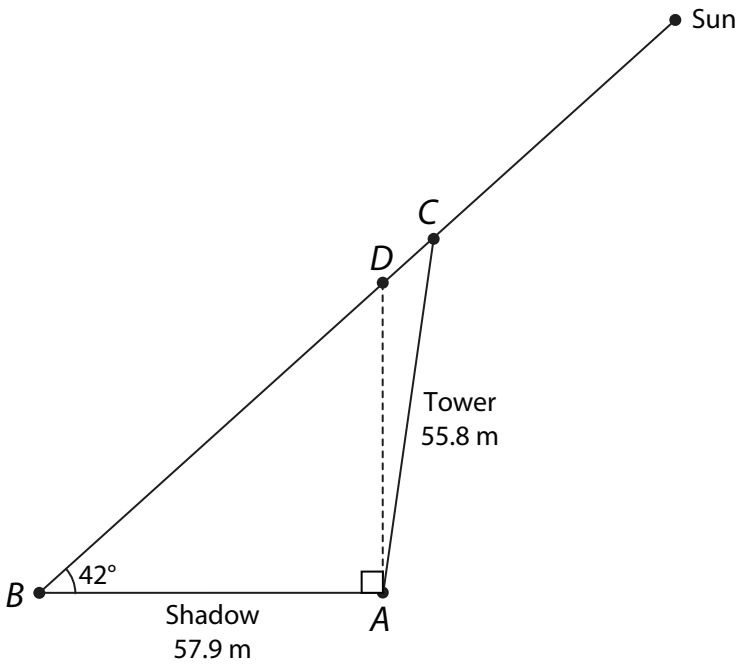
The Leaning Tower of Pisa, which is 55.8 meters tall, currently leans to the southeast. When the angle of elevation of the sun is 42° and the tower is leaning toward the sun, the tower's shadow is 57.9 meters long. How many degrees does the tower lean? Round your answer to the nearest degree, then determine whether your answer is reasonable given the context of the problem.

1. Draw and label a sketch of the situation.

   Sketch the situation and label the known information.

   Draw a dotted vertical line to represent where the tower would be if it were perfectly vertical.

   Label the points with letters.



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Instruction

2. Find the measure of $\angle BCA$.

Since there is not enough information yet to find $m\angle BAC$, $m\angle BCA$ must be found first.

Two sides and an angle are known (SSA), so use the Law of Sines.

$\frac{b}{\sin B} = \frac{c}{\sin C}$ — Use the portion of the Law of Sines formula in which b and c are the measures of the sides opposite angles B and C, respectively.

$\frac{(55.8)}{\sin(42°)} = \frac{(57.9)}{\sin C}$ — Substitute 55.8 for b, 57.9 for c, and 42° for B.

$55.8 \sin C = 57.9 \sin 42°$ — Cross multiply.

$\sin C = \frac{57.9 \sin 42°}{55.8}$ — Divide both sides by 55.8.

$m\angle BCA = \arcsin\left[\frac{57.9 \sin 42°}{55.8}\right]$ — Take the arcsine of both sides.

$m\angle BCA \approx 44.0°$ — Use your calculator to find the answer. Round to the nearest tenth.

The measure of $\angle BCA$ is approximately 44°.

3. Find the measure of $\angle BAC$.

The angle measures of a triangle sum to 180°. Subtract $m\angle B$ and $m\angle BCA$ from 180 to find $m\angle BAC$.

$180 - m\angle B - m\angle BCA = m\angle BAC$

$180 - 42 - 44 = 94$

The measure of $\angle BAC$ is approximately 94°.

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Instruction

4. Determine how many degrees the tower leans.

$\angle DAC$ represents the angle of the tower's lean.

$\angle DAB$ is a right angle since we drew AD to be a perfectly vertical tower, which would be perpendicular to the ground.

$m\angle DAC = m\angle BAC - m\angle DAB$	Angle Subtraction Property
$m\angle DAC = (94°) - (90°)$	Substitute 94° for $m\angle BAC$ and 90° for $m\angle DAB$.
$m\angle DAC = 4°$	Simplify.

The tower leans approximately 4°.

5. Consider whether or not the answer is reasonable.

Yes, this is a reasonable answer given the height of the tower. A slight lean of such a tall tower would be noticeable without putting the tower in danger of toppling; a larger degree could cause the tower to collapse.

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Problem-Based Task 3.2.3: Safe or Out?

SMP	
1 ✓	2
3 ✓	4
5	6
7 ✓	8

A baseball diamond is a square with 90 feet in between each base. The pitcher's mound is located on a direct line between home plate and second base and is 60.5 feet from home plate. During a pivotal play, the pitcher throws the ball to his teammate standing at first base at a speed of 132 feet per second in an attempt to get a runner out. The runner speeds back to the base in 0.6 second. Is the runner safe or out? Explain your reasoning. Use the equation for distance, $d = rt$, where d is distance, r is speed, and t is time. (*Note*: A runner is safe if he gets back to the base before the ball arrives. He is out if the ball is caught at the base before he returns.)



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Problem-Based Task 3.2.3: Safe or Out?

Coaching

a. Draw a diagram to represent this scenario.

b. What is the measure of the angle between the line from home plate to first base and the line from home plate to the pitcher's mound?

c. How can you determine the distance from the pitcher's mound to first base?

d. What is the distance from the pitcher's mound to first base?

e. How long did it take the pitcher's throw to reach first base?

f. Is the runner safe or out? Explain.

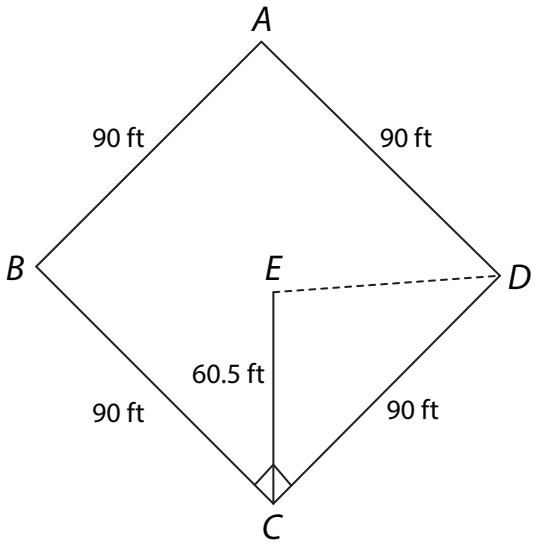
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Problem-Based Task 3.2.3: Safe or Out?

Coaching Sample Responses

a. Draw a diagram to represent this scenario.

The dashed line represents the unknown distance between the pitcher's mound and first base.



b. What is the measure of the angle between the line from home plate to first base and the line from home plate to the pitcher's mound?

In the diagram, this is $\angle DCE$.

The baseball diamond is a square, so $\angle BCD$ is a right angle.

Since the pitcher's mound is on a direct line from home plate to second base (a diagonal of the square), $\overline{CE}$ bisects $\angle BCD$.

When an angle is bisected, each resulting angle is half the measure of the original angle. Therefore, divide the measure of $\angle BCD$ by 2 to find $m\angle DCE$.

$90 \div 2 = 45$

Therefore, $m\angle DCE = 45°$.

c. How can you determine the distance from the pitcher's mound to first base?

For $\triangle CED$, two sides and the included angle are known (SAS). The Law of Cosines can be used to find the length of the third side (the distance from the pitcher's mound to first base).

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d. What is the distance from the pitcher's mound to first base?

Use the Law of Cosines and the known values to determine the distance.

$c^2 = a^2 + b^2 - 2ab \cos C$

$c^2 = 60.5^2 + 90^2 - 2(60.5)(90) \cos (45°)$

$c = \sqrt{60.5^2 + 90^2 - 2(60.5)(90)\cos 45°}$

$c \approx 63.717$

The distance from the pitcher's mound to first base is approximately 63.717 feet.

e. How long did it take the pitcher's throw to reach first base?

Use the equation for finding distance, $d = rt$, where d is distance, r is speed, and t is time.

$d = rt$

$(63.717) = (132)t$

$t = \frac{63.717}{132}$

$t \approx 0.483$ second

It took approximately 0.483 second for the pitcher's throw to reach first base.

f. Is the runner safe or out? Explain.

It took the ball 0.48 second to arrive at first base, and it took the runner 0.6 second to get back to the base. Therefore, the runner is out because the ball reached the base before he did.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.2.3: Applying the Laws of Sines and Cosines

Answer the following questions using the Law of Sines or the Law of Cosines. Round answers to the nearest tenth.

1. Adam and Karla are standing by a silo. Karla heads due north of the silo and Adam heads 28 degrees west of north. When they stop, Adam is 72 yards away from the silo and 86 yards from Karla. How far is Karla from the silo?

2. A plane flew out of Las Vegas on a 1,515-mile flight to Chicago. When the plane was 723 miles away from Chicago, the plane experienced a technical problem and the pilot had to divert to the closest available airport, which was 42 miles away from the plane's current location. When the plane landed, it was 701 miles away from Chicago. How many degrees did the pilot turn to divert to the closest airport?

3. A surveyor is trying to measure the width of a rock formation. He walks to a point far away from the formation and sets up his instruments. He measures the distance from the left side of the formation to his location as 89 feet and the distance from the right side of the formation to his location as 94 feet. He then measures the angle between these two lines of sight as 74°. How wide is the rock formation?

4. The sun is currently at a 67° angle of elevation. A tree is leaning away the sun at a 5° angle and casts an 18-foot shadow. How tall is the tree?

5. A ship sailed out of Baltimore on its way to Bermuda, which is 820 miles away. After 425 miles of smooth sailing, the captain decided to veer 35° to the left to avoid a storm. After sailing on that bearing for 167 miles, the captain turned the ship back on a direct line to Bermuda. How far must the ship travel to reach Bermuda?

continued

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6. The sun is currently at an 82° angle of elevation. A 30-foot-tall telephone pole is leaning toward the sun at a 6° angle. How long is the pole's shadow?

7. A surveyor identifies two points J and K on opposite sides of a valley. She then walks to a point a distance away from these points: point L. She measures the distance from J to L as 50 meters, the distance from K to L as 212 meters, and the measure of $\angle L$ as 57°. What is the distance across the valley?

8. In a parallelogram, opposite angles and opposite sides have the same measure. The sides of a parallelogram are 6 cm and 4 cm long, and the angles are 135° and 45°. What are the lengths of the diagonals?

9. A flagpole that is 25 feet tall is leaning at a 2° angle away from the sun. When the flagpole's shadow is 12 feet long, what is the angle of elevation of the sun?

10. Erika and Brad are both on the same street headed directly toward the Empire State Building, which is 1,454 feet tall. Erika, who is closer to the building, measures its angle of elevation from her location as 74°. Brad measures an angle of elevation. How far away is Erika from Brad?

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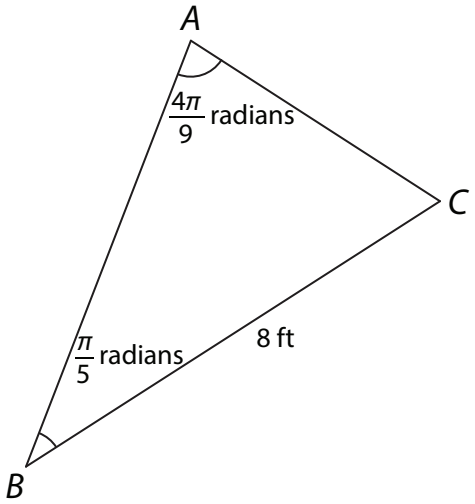
Progress Assessment

Circle the letter of the best answer.

1. In $\triangle IJK$, $IJ = 12$ in, $IK = 7$ in, and $m\angle I = \frac{\pi}{3}$ radians. What is the area of this triangle?

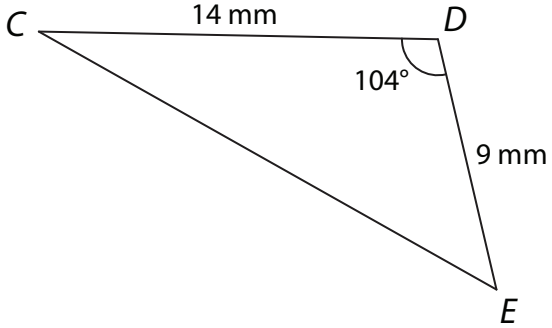
   a. $0.8\,\text{in}^2$
   b. $24.5\,\text{in}^2$
   c. $36.4\,\text{in}^2$
   d. $72.7\,\text{in}^2$

2. What is the length of $\overline{AC}$?



   a. 3.6 feet
   b. 4.8 feet
   c. 13.4 feet
   d. 37.3 feet

3. What is the length of $\overline{CE}$?



   a. 12.6 mm
   b. 14.4 mm
   c. 17.5 mm
   d. 18.4 mm

continued

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Assessment

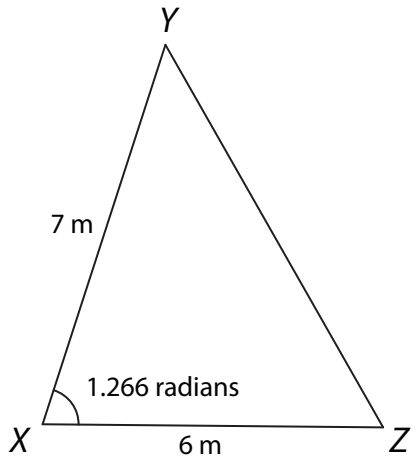
4. For $\triangle RST$, $RS = 4$ meters, $m\angle S = 86°$, and $m\angle R = 47°$. What is the length of $\overline{ST}$?

   a. 2.9 m
   b. 4 m
   c. 5 m
   d. 5.5 m

5. For $\triangle EFG$, $EF = 8$ in, $FG = 5$ in, and $m\angle E = 35°$. What is $m\angle G$?

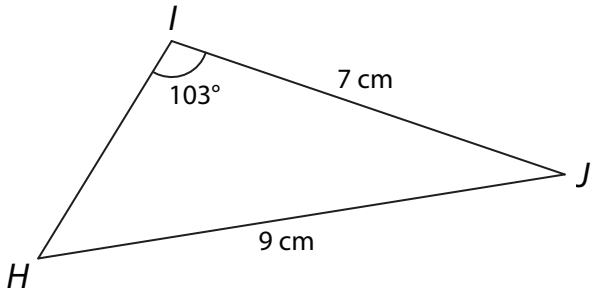
   a. 21°
   b. 67°
   c. 21° and 159°
   d. 67° and 113°

6. What is the measure of angle *Y*?



   a. 0.833 radian
   b. 0.957 radian
   c. 0.982 radian
   d. 1.042 radians

7. What is the area of $\triangle HIJ$?



   a. 14.7 cm^2
   b. 16.4 cm^2
   c. 23.9 cm^2
   d. 30.7 cm^2

continued

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8. For $\triangle LMN$, $LM = 18$ in, $MN = 12$ in, and $LN = 15$ in. What is the measure of angle L?

   a. 0.723 radian
   b. 0.848 radian
   c. 0.973 radian
   d. 1.445 radians

9. Cedric is trying to determine the distance across a river, so he identifies two trees on opposite banks of the river. He then walks to a distant point F and measures the distance from point F to each of the trees. From point F to the tree on the east bank of the river is 42 feet, and from point F to the tree on the west bank is 59 feet. If $m\angle F = 64°$, how wide is the river?

   a. 28.1 feet
   b. 55.4 feet
   c. 64.5 feet
   d. 71.7 feet

10. A 36-foot-tall tree that is leaning toward the sun casts a 40-foot-long shadow when the sun is at a 41° angle of elevation. Approximately how many degrees does the tree lean?

    a. 1°
    b. 2°
    c. 4°
    d. 7°

Use your knowledge of the Law of Sines and the Law of Cosines to answer the following questions. Round answers to the nearest whole number.

11. An airplane flew north out of Miami on a flight to Baltimore. After 887 miles of flight, the pilot turned 25° to the east to avoid the no-fly zone over Washington, D.C. He flew on that bearing for 28 miles before turning back to the west and flying into Baltimore. The entire flight covered 956 miles.

    a. How far did the plane fly after it turned back toward Baltimore?

    b. How many miles would the flight have been if the pilot had stayed on course, without detouring to avoid the no-fly zone?

    c. How many degrees did the pilot turn to the west to get back on course for Baltimore?

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. What is the period of the function $f(x) = 3 \sin (4x) + 2$?

   a. $\frac{\pi}{4}$

   b. $\frac{\pi}{2}$

   c. 2π

   d. 4π

2. What is the amplitude of the function $f(x) = 4 \cos [3(x + 1)] + 5$?

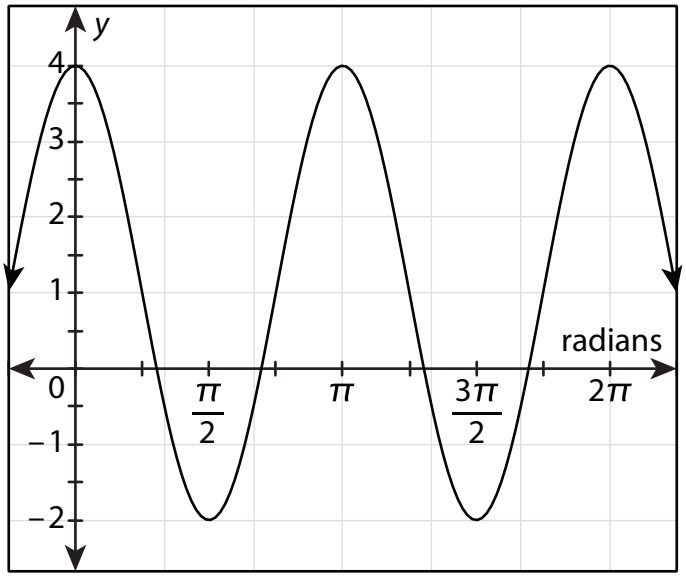
   a. 1

   b. 3

   c. 4

   d. 5

3. What is the frequency of the function shown in the graph?



   a. $\frac{1}{\pi}$

   b. $\frac{1}{2\pi}$

   c. π

   d. 2π

continued

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Assessment

4. What is the equation of a sine function that has no horizontal displacement, rises 2 units above its midline, is at $y = 3$, and has a period of $\frac{\pi}{4}$?

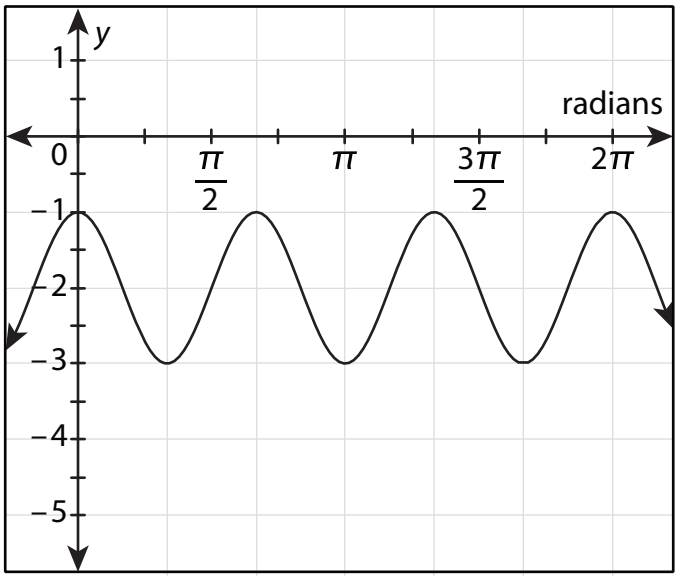
   a. $f(x)=2\sin\left(\frac{\pi}{4}x\right)+3$

   b. $f(x)=2\sin\left(\frac{\pi}{4}x+3\right)$

   c. $f(x) = 2\sin(8x) + 3$

   d. $f(x) = 2\sin(8x + 3)$

5. Which equation describes the graphed function?



   a. $f(x) = -2\sin(3x) + 1$

   b. $f(x) = -2\cos(3x) + 1$

   c. $f(x) = \sin(3x) - 2$

   d. $f(x) = \cos(3x) - 2$

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Lesson 3: Graphs of Trigonometric Functions

Instruction

SMP
1 2 ✓
3 4
5 6 ✓
7 ✓ 8

Common Core State Standard

F–TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

Essential Questions

1. What is the period of a trigonometric function and how can it be found?
2. What is the amplitude of a trigonometric function and how can it be found?
3. How is the equation of a trigonometric function determined?

WORDS TO KNOW

amplitude	the coefficient a of the sine or cosine term in a function of the form $f(x) = a \sin bx$ or $g(x) = a \cos bx$; on a graph of the cosine or sine function, the vertical distance from the y-coordinate of the maximum point on the graph to the midline of the cosine or sine curve
cosine function	a trigonometric function of the form $f(x) = a \cos [b(x - c)] + d$, in which a, b, c, and d are constants and x is a variable defined in radians over the domain $(-\infty,\infty)$
cycle	the smallest representation of a cosine or sine function graph as defined over a restricted domain that is equal to the period of the function
frequency of a periodic function	the reciprocal of the period for a periodic function; indicates how often the function repeats
maximum	the greatest value or highest point of a function
midline	in a cosine function or sine function of the form $f(x) = \sin x + d$ or $g(x) = \cos x + d$, a horizontal line of the form $y = d$ that bisects the vertical distance on a graph between the minimum and maximum function values
minimum	the least value or lowest point of a function

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period	in a cosine or sine function graph, the horizontal distance from a maximum to a maximum or from a minimum to a minimum; one repetition of the period of a function is called a *cycle*
periodic function	a function whose values repeat at regular intervals
periodic phenomena	real-life situations that repeat at regular intervals and can be represented by a periodic function
sine curve	a curve with a constant amplitude and period, which are given by a sine or cosine function; also called a *sine wave* or *sinusoid*
sine function	a trigonometric function of the form $f(x) = a \sin [b(x - c)] + d$, in which a, b, c, and d are constants and x is a variable defined in radians over the domain $(-\infty,\infty)$
sine wave	a curve with a constant amplitude and period given by a sine or cosine function; also called a *sine curve* or *sinusoid*
sinusoid	a curve with a constant amplitude and period given by a sine or cosine function; also called a *sine curve* or *sine wave*

Recommended Resources

- LearnZillion. "Graph Sinusoidal Functions by Plotting Points."

  http://www.walch.com/rr/00224

  This video series explores topics such as how to graph a sine curve by plotting points and how the terminal ray of the unit circle creates the sine curve.

- The Math Page. "Trigonometry."

  http://www.walch.com/rr/00225

  This website provides detailed instructions for how to graph the various trigonometric functions.

- MathIsFun.com. "Graphs of Sine, Cosine, and Tangent."

  http://www.walch.com/rr/00226

  This website shows the various trigonometric functions and how they relate to one another.

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Lesson 3.3.1: Periodic Phenomena and Amplitude, Frequency, and Midline

Warm-Up 3.3.1

Spencer and Evan hang a cluster of bananas at the end of a spring, so that the bananas oscillate up and down once the spring is disturbed. They want to calculate how far the cluster will move from its resting position over time. This distance, known as the displacement from equilibrium, is given by the formula $f(t) = \cos 2t$, in which $f(t)$ is the displacement in feet and t is the time in seconds. Use this information to answer the questions that follow. Round all answers to the nearest hundredth. (*Note*: The formula uses radians.)

1. How far from equilibrium will the bananas be 1 second after the spring is disturbed?

2. How far from equilibrium will the bananas be 2 seconds after the spring is disturbed?

3. How far from equilibrium will the bananas be 3 seconds after the spring is disturbed?

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Instruction

Lesson 3.3.1: Periodic Phenomena and Amplitude, Frequency, and Midline

Common Core State Standard

F–TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

Warm-Up 3.3.1 Debrief

1. How far from equilibrium will the bananas be 1 second after the spring is disturbed?

   Substitute 1 for the time, t, in the given formula for displacement, and then solve. Make sure that your calculator is in radian mode before calculating the final value.

$f(t) = \cos 2t$	Formula for the displacement
$f(1) = \cos 2(1)$	Substitute 1 for t.
$f(1) = \cos 2$	Multiply.
$f(1) \approx -0.42$	Use your calculator to simplify.

   Since the amount of displacement is a negative value, this indicates that the bananas are below equilibrium—that is, their position is lower than the original resting position.

   After 1 second, the bananas will be 0.42 feet below equilibrium.

2. How far from equilibrium will the bananas be 2 seconds after the spring is disturbed?

   Substitute 2 for the time, t, in the given formula for displacement, and then solve.

$f(t) = \cos 2t$	Formula for the displacement
$f(2) = \cos 2(2)$	Substitute 2 for t.
$f(2) = \cos 4$	Multiply.
$f(2) \approx -0.65$	Use your calculator to simplify.

   After 2 seconds, the bananas will be 0.65 feet below equilibrium.

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3. How far from equilibrium will the bananas be 3 seconds after the spring is disturbed?

   Substitute 3 for the time, t, in the given formula for displacement, and then solve.

$f(t) = \cos 2t$	Formula for the displacement
$f(3) = \cos 2(3)$	Substitute 3 for t.
$f(3) = \cos 6$	Multiply.
$f(3) \approx 0.96$	Use your calculator to simplify.

   Since the displacement is positive, the bananas will be above equilibrium.

   After 3 seconds, the bananas will be 0.96 feet above equilibrium.

Connection to the Lesson

- Students will extend their knowledge of trigonometric functions to examine their graphs and properties.
- Students will apply the properties of trigonometric functions to real-world scenarios.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

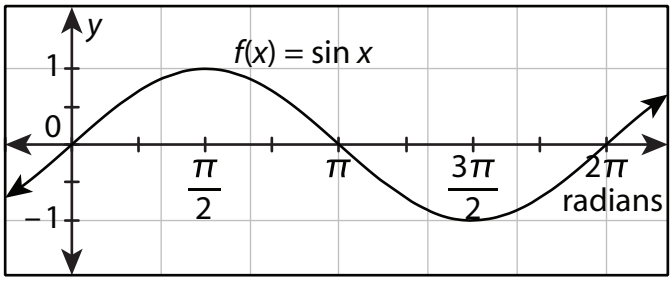
- using radians to describe angle measure (F-TF.1)
- finding the sine, cosine, and tangent of an angle (G-SRT.6)

Introduction

The sine, cosine, and tangent functions are **periodic functions** since their values repeat at regular intervals. While the unit circle can be used to study their properties, graphing these functions on a coordinate plane is also useful, since a coordinate plane allows the functions' periodic properties to be examined more closely.

Key Concepts

- The basic sine function, $f(x) = \sin x$, can be graphed using a table of x- and y-values. The graph reveals the basic pattern of the sine curve, as shown.

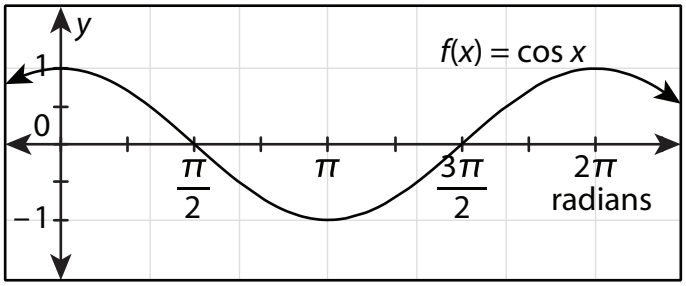


- A **sine curve** has a constant amplitude and period given by a sine or cosine function. It is also called a **sine wave** or **sinusoid**.
- The **amplitude** is the vertical distance from the y-coordinate of the maximum point on the graph to the midline of the curve. The **period** is the horizontal distance from a maximum to a maximum or from a minimum to a minimum. One repetition of the period of a function is called a **cycle**, which is the smallest representation of a cosine or sine function graph as defined over a restricted domain that is equal to the period of the function.
- The basic cosine function, $f(x) = \cos x$, can also be graphed using a table of x- and y-values, which results in the following graph. Notice that the graph of the basic cosine function is also a sine curve, but it has been shifted $\frac{\pi}{2}$ radians to the left.

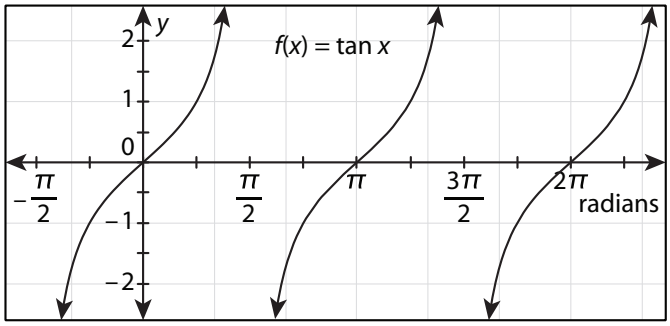
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- Likewise, the basic tangent function, $f(x) = \tan x$, can be graphed using a table of x- and y-values. However, its graph is quite different from the graphs of the sine and cosine functions. The tangent function repeats twice as often and approaches both positive and negative infinity.



- The **maximum** of a function is the function's greatest value or highest point. The **minimum** of a function is the least value of the function, or its lowest point. A function can have several maxima and minima.
- The period of a sine or cosine function can be determined by finding the horizontal distance between two maxima or between two minima. The period also equals the length of one repetition of the function.
- Note that, in the past, a relative maximum was defined as the greatest value of a function for a particular interval of the function and a relative minimum was defined as the least value of a function for a particular interval of the function. With sine and cosine waves, however, the functions repeat, and the maximum and minimum are the greatest and least values, not for a particular interval, but for the entire function.
- The **midline** of a periodic function is the horizontal line located halfway between a function's minimum and maximum.
- The amplitude of a sine or cosine function can be found by determining how far the function rises above its midline.

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Instruction

- The **frequency of a periodic function** is the reciprocal of its period and indicates how often the function repeats. The higher the frequency, the smaller each wave, thus resulting in more waves appearing in a given portion of the graph.
- The general form of the **sine function** is $f(x) = a \sin [b(x - c)] + d$, and the general form of the **cosine function** is $f(x) = a \cos [b(x - c)] + d$. For both functions, $|a|$ is the amplitude, $\frac{2\pi}{b}$ is the period, c is the horizontal shift (also called the phase shift), and d is the vertical shift.
- When calculating the number and locations of these features of a function, viewing the functions on a graphing calculator can be helpful in confirming your answers.

Common Errors/Misconceptions

- thinking that b is the value of the period
- believing that the amplitude is the entire vertical distance from the maximum point to the minimum point
- confusing the period with the frequency, and vice versa

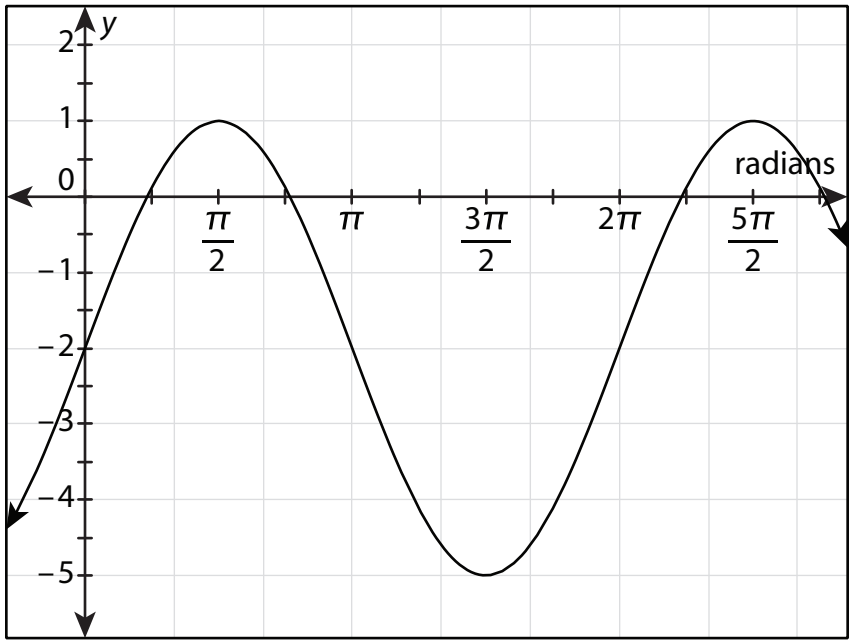
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Instruction

Guided Practice 3.3.1

Example 1

Determine the period, frequency, midline, and amplitude of the graphed function.



1. Determine the period.

   The period is the length of one cycle of the curve, or the horizontal distance the curve travels before it repeats. It can be found by determining the horizontal distance between two maximum points.

   There are two clearly identifiable maximum points: one at $x=\frac{\pi}{2}$ and another at $x=\frac{5\pi}{2}$. Subtract these values to find the distance between them.

   $$\frac{5\pi}{2}-\frac{\pi}{2}=\frac{4\pi}{2}=2\pi$$

   The period of this function is 2π.

   This means the length of one repetition of the curve is 2π.

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Instruction

2. Determine the frequency.

The frequency tells how often a function repeats. It is found by taking the reciprocal of the period.

The period is 2π, so the frequency is $\frac{1}{2\pi}$.

This means that the curve repeats $\frac{1}{2\pi}$ of a time from the interval 0 to 1. The larger the frequency, the more waves appear in a given portion of the graph.

3. Determine the midline.

The midline of a periodic function is the horizontal line located halfway between the function's minimum and maximum points.

By inspecting the graph, we can see that the maximum point occurs at $y = 1$, and the minimum occurs at $y = -5$.

Find the average of 1 and –5.

$$\frac{1+(-5)}{2}=\frac{-4}{2}=-2$$

The average is –2; therefore, the midline occurs at $y = -2$.

We can verify this by inspecting the graph and confirming that the horizontal line $y = -2$ runs through the middle of the curve.

4. Determine the amplitude.

The amplitude of a sine or cosine function is the height of the curve. It is found by determining how far the function rises above its midline.

The midline is $y = -2$, and the highest the curve rises is $y = 1$.

Subtract to find the distance between them. Or, count the vertical distance from the midline to the maximum point on the graph.

$1 - (-2) = 3$

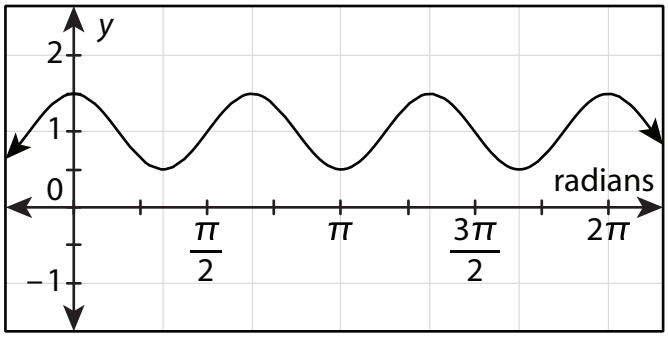
The amplitude of the function is 3.

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Instruction

Example 2

Determine the period, frequency, midline, and amplitude of the graphed function.



1. Determine the period.

   The period can be found by determining the horizontal distance between two maximum points, but in this case, the exact x-value of the maximum points is unclear. Instead, find the period by determining the length of one repetition of the curve.

   The curve repeats 3 times from 0 to 2π. Thus, one repetition is $\frac{1}{3}$ of 2π or $\frac{2\pi}{3}$.

   The period of this function is $\frac{2\pi}{3}$.

2. Determine the frequency.

   The frequency can be found by taking the reciprocal of the period.

   The period, as determined in step 1, is $\frac{2\pi}{3}$. The reciprocal of this is $\frac{3}{2\pi}$; therefore, the frequency is $\frac{3}{2\pi}$.

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Instruction

3. Determine the midline.

Recall that the midline is the horizontal line located halfway between the function's minimum and maximum points.

By inspecting the graph, we can see that the maximum point occurs at $y = 1.5$, and the minimum occurs at $y = 0.5$.

Find the average of 1.5 and 0.5.

$$\frac{1.5+0.5}{2}=\frac{2}{2}=1$$

The average is 1; therefore, the midline occurs at $y = 1$.

We can verify this by inspecting the graph and confirming that the horizontal line $y = 1$ runs through the middle of the curve.

4. Determine the amplitude.

The amplitude is found by determining how far the function rises above its midline.

The midline is $y = 1$, and the highest the curve rises is $y = 1.5$.

Subtract to find the distance between them. (Or, count the vertical distance from the midline to the maximum point on the graph.)

$1.5 - 1 = 0.5$

The amplitude of the function is 0.5.

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Instruction

Example 3

Determine the period, frequency, midline, and amplitude of the function.

$$f(x)=2\cos\left(\frac{1}{3}x\right)-4$$

1. Determine the period.

   In the general form of the cosine function, $f(x) = a\cos[b(x-c)] + d$, the period is $\frac{2\pi}{b}$.

   In $f(x)=2\cos\left(\frac{1}{3}x\right)-4$, $b=\frac{1}{3}$ since $\frac{1}{3}$ is the coefficient of x.

   Substitute $\frac{1}{3}$ for b in the formula for the period and then solve.

$\frac{2\pi}{b}$	Formula for the period
$=2\pi\div\left(\frac{1}{3}\right)$	Rewrite the fraction as division and substitute $\frac{1}{3}$ for b.
$=2\pi\bullet\frac{3}{1}$	To divide, multiply by the reciprocal of the second fraction.
$=6\pi$	Simplify.

   The period of the function is 6π.

   This means the length of one repetition of the curve is 6π.

2. Determine the frequency.

   The frequency is the reciprocal of the period.

   The reciprocal of 6π is $\frac{1}{6\pi}$. Therefore, the frequency of the function is $\frac{1}{6\pi}$.

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Instruction

3. Determine the midline.

   In the general form of the cosine function, $f(x) = a \cos [b(x - c)] + d$, d is the horizontal shift. In other words, d indicates how many units the midline has moved from the origin.

   In the function $f(x) = 2 \cos\left(\frac{1}{3}x\right) - 4$, $d = -4$. This means that the entire function has shifted down 4 units and that the midline is at $y = -4$.

4. Find the amplitude of the function.

   In the general form of the cosine function, $f(x) = a \cos [b(x - c)] + d$, $|a|$ is the amplitude.

   In the function $f(x) = 2 \cos\left(\frac{1}{3}x\right) - 4$, $a = 2$.

   Therefore, the amplitude of the function is 2.

   This means that the curve rises 2 units above the midline.

Example 4

Describe the difference between the function $f(x) = 3 \sin (x - 1)$ and the function $g(x) = 3 \sin [2(x - 1)]$.

1. Identify the difference between the two functions and determine the property to which it relates.

   The functions are nearly identical, but for one crucial difference. By inspection, we can see that the second function has a coefficient for the term $x - 1$, whereas the first function does not.

   The general form of the sine function is $f(x) = a \sin [b(x - c)] + d$.

   The difference in the two functions is the b value, the value of the period.

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Instruction

2. Determine the period of each function.

First, find the period of $f(x) = 3 \sin (x - 1)$.

Use the formula for the period, and substitute 1 for b, since the x has no coefficient and the b value is understood to be 1.

$\frac{2\pi}{b}$ Formula for the period

$= \frac{2\pi}{(1)}$ Substitute 1 for b.

$= 2\pi$ Divide.

The period of $f(x) = 3 \sin (x - 1)$ is 2π.

Next, find the period of $g(x) = 3 \sin [2(x - 1)]$.

Use the formula for the period, and substitute 2, the coefficient of the $x - 1$ term, for b.

$\frac{2\pi}{b}$ Formula for the period

$= \frac{2\pi}{(2)}$ Substitute 2 for b.

$= \pi$ Divide.

The period of $g(x) = 3 \sin [2(x - 1)]$ is π.

3. Describe the difference between the functions.

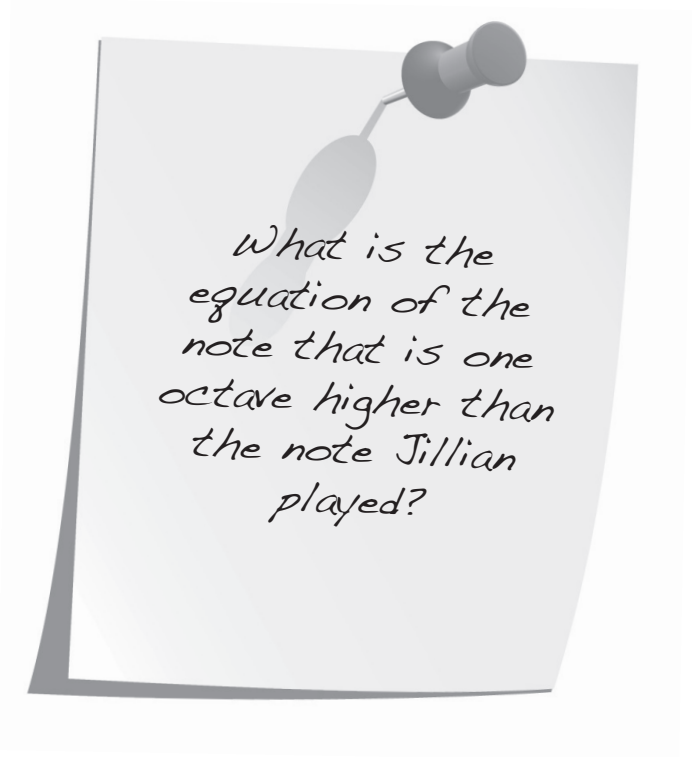
The period of $f(x) = 3 \sin (x - 1)$ is 2π and the period of $g(x) = 3 \sin [2(x - 1)]$ is π. The period of $f(x)$ is twice as long as that of $g(x)$. Therefore, one repetition of the function $f(x)$ will be twice as long as one repetition of the function $g(x)$.

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Problem-Based Task 3.3.1: Perfect Pitch

SMP	
1	2 ✓
3	4
5	6 ✓
7 ✓	8

In music, an octave is an interval between the first note and the eighth note on a musical scale. The first and eighth notes are the same, though one is higher than the other. Since sound travels in waves, a single note can be described by a sine curve, and the pitch of the note is determined by its frequency in hertz (Hz). If two notes are one octave apart, the higher note will have twice the frequency of the lower note. Jillian plays a note on the piano that can be described by the function $f(x) = \sin(880\pi t)$, where t is time in seconds. What is the equation of the note that is one octave higher than the note Jillian played?



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Problem-Based Task 3.3.1: Perfect Pitch

Coaching

a. What is the period of the note Jillian played?

b. What is the frequency of the note Jillian played?

c. What is the frequency of the note that is one octave higher than the first note played?

d. What is the period of the note that is one octave higher than the first note played?

e. How can you find the value of b for the equation of the note that is one octave higher?

f. What is the value of b for the equation of the note that is one octave higher?

g. What is the equation of the note that is one octave higher than the first note played?

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Instruction

Problem-Based Task 3.3.1: Perfect Pitch

Coaching Sample Responses

a. What is the period of the note Jillian played?

In the general form of the sine function, $f(x) = a \sin [b(x - c)] + d$, the period is $\frac{2\pi}{b}$.

In the function $f(x) = \sin (880\pi t)$, $b = 880\pi$ since 880π is the coefficient of the variable. Substitute this value for b in the formula for the period and solve.

$$\frac{2\pi}{b} = \frac{2\pi}{(880\pi)} = \frac{1}{440}$$

The period of the function is $\frac{1}{440}$.

b. What is the frequency of the note Jillian played?

The frequency is the reciprocal of the period.

The reciprocal of $\frac{1}{440}$ is 440.

The unit for the frequency of a note is typically given in hertz (Hz).

Thus, the frequency of the note Jillian played is 440 Hz.

c. What is the frequency of the note that is one octave higher than the first note played?

A note that is one octave higher will have double the frequency of the first note.

$440 \bullet 2 = 880$

The frequency of the note that is one octave higher is 880 Hz.

d. What is the period of the note that is one octave higher than the first note played?

The frequency is the reciprocal of the period.

The reciprocal of 880 is $\frac{1}{880}$.

Therefore, the period of the note that is one octave higher is $\frac{1}{880}$.

e. How can you find the value of b for the equation of the note that is one octave higher?

The period equals $\frac{2\pi}{b}$. Substitute the value of the period and solve for b.

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f. What is the value of b for the equation of the note that is one octave higher?

Use the formula for the period, substitute the known values, and solve for b.

$$\text{period} = \frac{2\pi}{b}$$

$$\left(\frac{1}{880}\right) = \frac{2\pi}{b}$$

$$1 \cdot b = 880 \cdot 2\pi$$

$$b = 1760\pi$$

The value of b in the new equation will be 1760π.

g. What is the equation of the note that is one octave higher than the first note played?

In the original equation, $f(x) = \sin(880\pi t)$, only the b value will change because only the frequency and the period of the note have changed.

The new b value is 1760π.

Replace 880π with 1760π to determine the new equation, $f(x) = \sin(1760\pi t)$.

The equation of the note that is one octave higher than the first note played is $f(x) = \sin(1760\pi t)$.

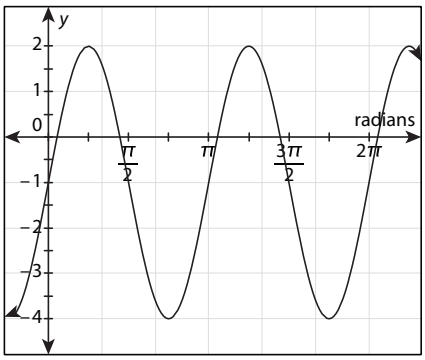
Recommended Closure Activity

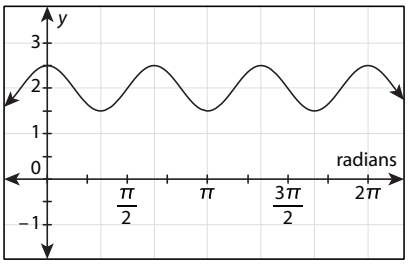
Select one or more of the essential questions for a class discussion or as a journal entry prompt.

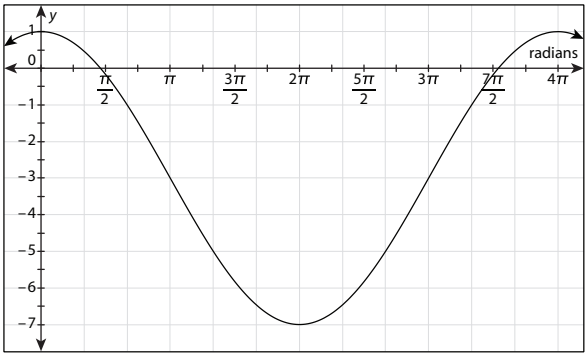
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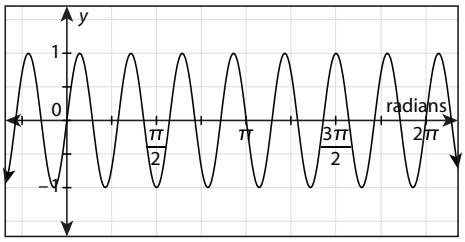
Practice 3.3.1: Periodic Phenomena and Amplitude, Frequency, and Midline

For problems 1–7, determine the period, frequency, midline, and amplitude of each trigonometric function.

1. 

3. 

2. 

4. 

5. $f(x) = \cos(x + 5)$

6. $f(x) = \frac{1}{2}\sin(2x) - 8$

7. $f(x) = 3\sin\left[\frac{2}{5}(x-1)\right] + 4$

For problems 8–10, the function for the periodic motion of a spring is given. Determine the frequency of the spring, or how many times it oscillates per second.

8. $f(x) = 4\sin(6\pi t)$

9. $f(x) = 3\sin(\pi t)$

10. $f(x) = \sin\left(\frac{3}{2}\pi t\right)$

SWB p. 91; SRB Practice 3.3.1 p. U3-113

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Lesson 3.3.2: Using Trigonometric Functions to Model Periodic Phenomena

Warm-Up 3.3.2

Sound travels in waves, and a single note can be described by a sine curve. The larger the amplitude, the louder the note will be. Pitch refers to how high or low a note is. Since the pitch of a note is determined by its frequency, the greater the frequency, the higher the note will be.

Grant sat down at the piano and played a series of notes that can be described by the following functions. Use the given information about sound waves and the note functions to answer the questions.

- Note 1: $f(x) = \sin(880\pi t)$
- Note 2: $g(x) = 3\sin(784\pi t)$
- Note 3: $h(x) = 2.5\sin(1397\pi t)$
- Note 4: $j(x) = 2\sin(1568\pi t)$
- Note 5: $k(x) = 0.75\sin(392\pi t)$

1. Which note was the loudest?

2. Which note was the quietest?

3. Which note was the highest in pitch?

4. Which note was the lowest in pitch?

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Instruction

Lesson 3.3.2: Using Trigonometric Functions to Model Periodic Phenomena

Common Core State Standard

F–TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

Warm-Up 3.3.2 Debrief

1. Which note was the loudest?

   The larger the amplitude, the louder the note will be.

   In the general form of the sine function, $f(x) = a \sin [b(x - c)] + d$, $|a|$ is the amplitude. Comparing the a values of the given functions for the notes shows that the note with the highest amplitude is note 2, which has an amplitude of 3.

   Note 2 was the loudest.

2. Which note was the quietest?

   If the note with the largest amplitude is the loudest, then the note with the smallest amplitude must be the quietest.

   Therefore, compare the amplitudes of the functions (the a values) to determine the note with the smallest amplitude.

   Recall that a function with no a value represents a coefficient of 1; thus, the amplitude of note 1, described by the equation $f(x) = \sin (880\pi t)$, is 1.

   Therefore, note 5, with an amplitude of 0.75, was the quietest.

3. Which note was the highest in pitch?

   The greater the frequency, the higher the note's pitch will be.

   Recall that the frequency is the reciprocal of the period.

   In the general form of the sine function, $f(x) = a \sin [b(x - c)] + d$, the period is $\frac{2\pi}{b}$. Thus, taking the reciprocal of this formula, the frequency equals $\frac{b}{2\pi}$.

   The greater the b value, the higher the frequency will be, so determine which note has the greatest b value.

   Compare the b values of the functions.

   Note 4 has the greatest b value (1568π) and thus has the highest frequency; therefore, note 4 was the highest in pitch.

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4. Which note was the lowest in pitch?

   If the note with the highest frequency has the highest pitch, then the note with the lowest frequency must have the lowest pitch.

   Compare the frequencies of the functions (the b values) to determine which note has the lowest frequency.

   Note 5 has the smallest b value (392π) and thus has the lowest frequency; therefore, note 5 was the lowest in pitch.

Connection to the Lesson

- Students will use trigonometric functions to model periodic phenomena with specified amplitudes and frequencies.
- Students will determine the equation for a sine or cosine function given specific properties of the function.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

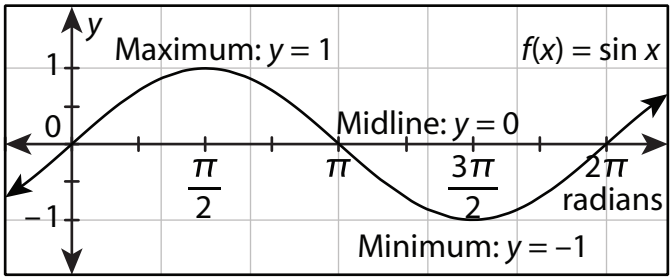
- finding the amplitude, period, frequency, and midline of a trigonometric function when given an equation or a graph of the function (F-TF.5)
- determining reciprocals (5.NF.7)

Introduction

Each sine or cosine function can be distinguished by its period, frequency, midline, and amplitude. Being able to recognize these aspects of a sine or cosine function can be useful in modeling real-life situations that repeat at regular intervals, or **periodic phenomena**. Once the period, frequency, midline, and amplitude are known, the equation of the function can be determined.

Key Concepts

- Recall that the general form of the sine function is $f(x) = a \sin [b(x - c)] + d$ and the general form of the cosine function is $f(x) = a \cos [b(x - c)] + d$. For both functions, $|a|$ is the amplitude, $\frac{2\pi}{b}$ is the period, c is the horizontal shift (also known as the phase shift), and d is the vertical shift.
- To find the equation of a function, determine the *a, b, c,* and *d* values, and then substitute them into the general form.
- Since the only difference between the sine and cosine curve is the horizontal shift, these functions can be described both in terms of a sine function and in terms of a cosine function.
- A basic sine function starts at the midline and travels up to its maximum point. Upon reaching the maximum, the function then curves down past the midline to its minimum point, and then returns to the midline.

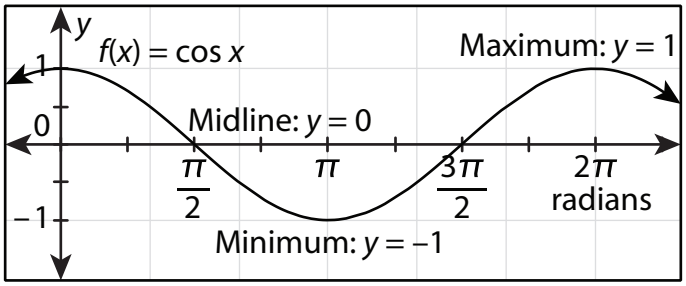


SRB p. U3-114

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Instruction

- A basic cosine function, on the other hand, starts at the maximum point and then travels down past the midline to the minimum point before returning up to the maximum point.



Common Errors/Misconceptions

- thinking that b is the period
- misinterpreting the sign and direction of the horizontal shift, c
- switching c and d, the horizontal and vertical shifts
- incorrectly determining where the sine or cosine curve begins (when shifted horizontally)

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Guided Practice 3.3.2

Example 1

Find the equation of a sine function with no horizontal shift whose frequency is 2. The function rises 3 units above its midline, which is $y = -1$.

1. Determine the value of a.

   a represents the amplitude of the function, or how far the function rises above its midline.

   Since the function rises 3 units above its midline, its amplitude is 3.

   The value of a, the amplitude, is 3.

2. Determine the value of b.

   The period (the length of one cycle of the curve) is equal to $\frac{2\pi}{b}$, and the frequency is the reciprocal of the period.

   The reciprocal of the given frequency, 2, is $\frac{1}{2}$. Thus, $\frac{1}{2}$ is the period.

   Substitute this value into the formula for the period and solve for b.

$\text{period} = \frac{2\pi}{b}$	Formula for the period
$\left(\frac{1}{2}\right) = \frac{2\pi}{b}$	Substitute $\frac{1}{2}$ for the period.
$1(b) = 2(2\pi)$	Cross multiply to eliminate the fractions.
$b = 4\pi$	Multiply.

   The value of b is 4π.

SWB p. 97; SRB p. U3-116

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3. Determine the value of c.

   c is the horizontal shift. Since the function has no horizontal shift, $c = 0$.

4. Determine the value of d.

   d is the vertical shift.

   Since the function's midline is $y = -1$, this means that the entire function has been shifted down one unit (–1), so $d = -1$.

5. Substitute a, b, c, and d into the general form of the sine function.

   Substitute the known values into the general form of the sine function and simplify.

$f(x) = a \sin [b(x - c)] + d$	General form of the sine function
$f(x) = (3) \sin \{(4\pi)[x - (0)]\} + (-1)$	Substitute 3 for a, 4π for b, 0 for c, and –1 for d.
$f(x) = 3 \sin (4\pi x) - 1$	Simplify.

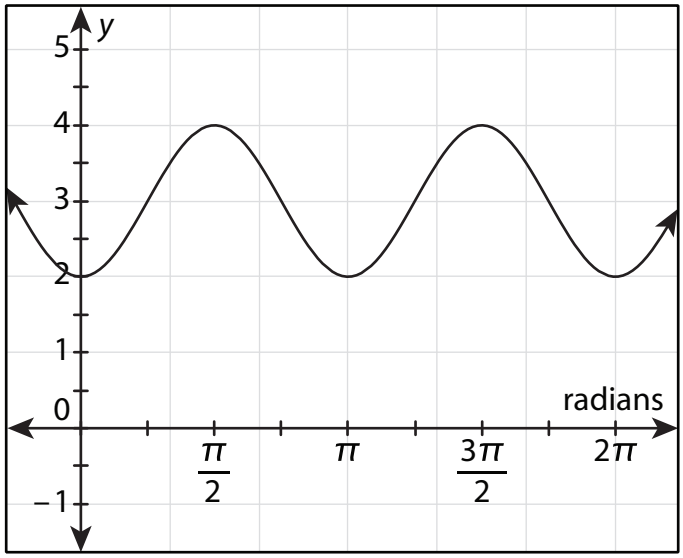
   The equation of the function is $f(x) = 3 \sin (4\pi x) - 1$.

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Instruction

Example 2

Write an equation to describe the graphed function.



1. Choose to use either the sine or cosine form.

   This periodic function can be described by a sine function or a cosine function. Either function will work; however, since the cosine curve begins at the maximum point, it may allow for more efficient calculations of the features. At the origin, the curve is not at the midline or maximum point, so there is a horizontal shift for both the sine and cosine function.

   There is an identifiable maximum at $x = \frac{\pi}{2}$. Thus, this function can be described as a cosine function that has been shifted $\frac{\pi}{2}$ units to the right.

   (Note that the curve can also be described as a sine function that has been shifted $\frac{\pi}{4}$ units to the right, but this example will focus on finding the equation by using the cosine function.)

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Instruction

2. Determine the value of a.

a is the amplitude, or how far the function rises above its midline.

The midline is $y = 3$, and the function rises 1 unit above its midline.

Therefore, the value of a, the amplitude, is 1.

3. Determine the value of b.

The period (the length of one cycle of the curve) is equal to $\frac{2\pi}{b}$.

One cycle of this cosine curve starts at $x = \frac{\pi}{2}$ and ends at $x = \frac{3\pi}{2}$.

Subtract to find the length of one cycle of the curve.

$$\frac{3\pi}{2} - \frac{\pi}{2} = \frac{2\pi}{2} = \pi$$

The period of the function is π.

Substitute this value into the formula for the period of the function and solve for b.

$\text{period} = \frac{2\pi}{b}$	Formula for the period
$(\pi) = \frac{2\pi}{b}$	Substitute π for the period.
$\pi b = 2\pi$	Multiply both sides of the equation by b to eliminate the fraction.
$b = \frac{2\pi}{\pi}$	Divide both sides of the equation by π.
$b = 2$	Simplify.

The value of b is 2.

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Instruction

4. Determine the value of c.

   c is the horizontal shift.

   As determined in step 1, the cosine function has been shifted $\frac{\pi}{2}$ units to the right. Shifts to the right are designated as positive; shifts to the left are negative.

   The value of c, the horizontal shift, is $\frac{\pi}{2}$.

5. Determine the value of d.

   d is the vertical shift.

   The midline is $y = 3$; thus, the function has been shifted up 3 units (+3).

   The value of d, the vertical shift, is 3.

6. Substitute a, b, c, and d into the general form of the cosine function.

   Substitute the known values into the general form of the cosine function and simplify.

   $f(x) = a\cos[b(x-c)] + d$ — General form of the cosine function

   $f(x) = (1)\cos\left\{(2)\left[x-\left(\frac{\pi}{2}\right)\right]\right\}+(3)$ — Substitute 1 for a, 2 for b, $\frac{\pi}{2}$ for c, and 3 for d.

   $f(x) = \cos\left[2\left(x-\frac{\pi}{2}\right)\right]+3$ — Simplify.

   $f(x) = \cos(2x-\pi)+3$ — Distribute to simplify further, if desired.

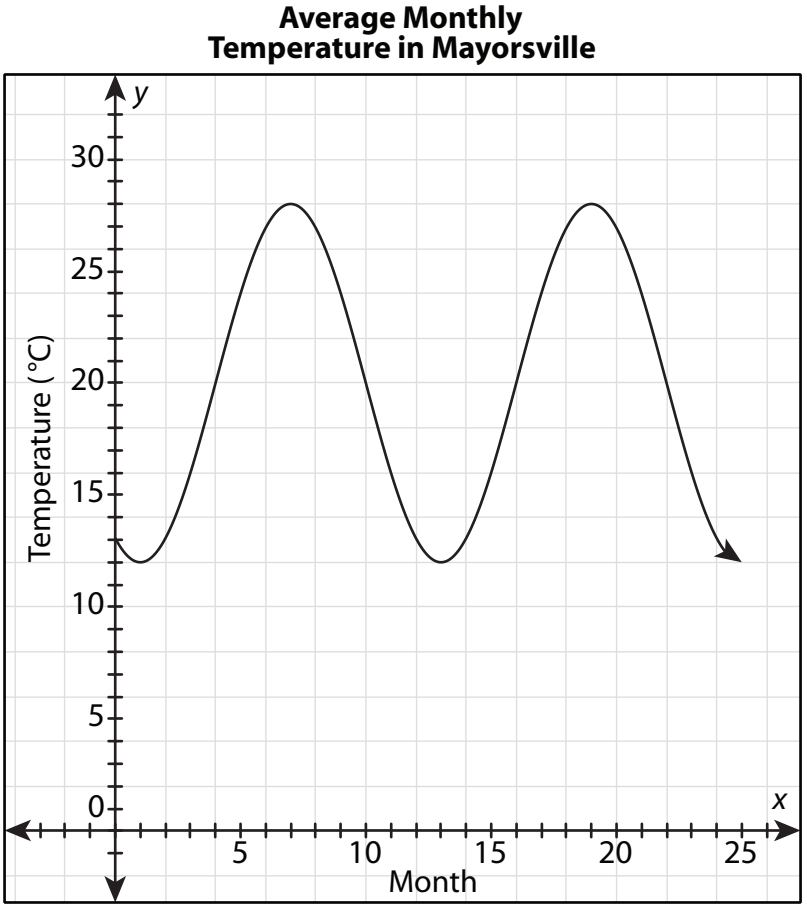
   The equation of the function is $f(x) = \cos\left[2\left(x-\frac{\pi}{2}\right)\right]+3$ or $f(x) = \cos(2x-\pi)+3$.

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Instruction

Example 3

The following graph shows historical average monthly temperatures for the town of Mayorsville starting in January 2000. Write an equation for the graphed function.



1. Choose to use either the sine or cosine form.

   This periodic function can be described by either a sine function or a cosine function. However, since the cosine curve begins at the maximum point, it may allow for more efficient calculations of the features. At the origin, the curve is not at the midpoint or maximum point, so there is a horizontal shift for both the sine and cosine function.

   There is an identifiable maximum at $x = 7$. Thus, this function can be described as a cosine function that has been shifted 7 units to the right.

   (Similar to the curve in Example 2, this curve can also be described as a sine function that has been shifted 4 units to the right, but this example will focus on finding the equation by using the cosine function.)

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Instruction

2. Determine the value of a.

a is the amplitude, or how far the function rises above its midline.

The function's maximum point is at $y = 28$ and its minimum point is at $y = 12$.

Determine the average of these two values.

$$\frac{28+12}{2}=\frac{40}{2}=20$$

The average of 28 and 12 is 20, so the midline is $y = 20$.

Subtract to determine how far the function rises above its midline.

$28 - 20 = 8$

The value of a, the amplitude, is 8.

3. Determine the value of b.

The period (the length of one cycle of the curve) is equal to $\frac{2\pi}{b}$.

One cycle of this cosine curve starts at $x = 7$ and ends at $x = 19$.

Subtract to find the length of one cycle of the curve.

$19 - 7 = 12$

The period of the function is 12.

Substitute this value into the formula for the period and solve for b.

$\text{period}=\frac{2\pi}{b}$	Formula for the period
$(12)=\frac{2\pi}{b}$	Substitute 12 for the period.
$12b = 2\pi$	Multiply both sides of the equation by b to eliminate the fraction.
$b=\frac{2\pi}{12}$	Divide both sides of the equation by 12.
$b=\frac{\pi}{6}$	Simplify.

The value of b is $\frac{\pi}{6}$.

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Instruction

4. Determine the value of c.

   c is the horizontal shift.

   As determined in step 1, the cosine function has been shifted 7 units to the right. Shifts to the right are designated as positive; shifts to the left are negative.

   The value of c, the horizontal shift, is 7.

5. Determine the value of d.

   d is the vertical shift.

   As determined in step 2, the midline is $y = 20$. Thus, the function has been shifted up 20 units (+20).

   The value of d, the vertical shift, is 20.

6. Substitute a, b, c, and d into the general form of the cosine function.

   Substitute the known values into the general form of the cosine function and simplify.

   $f(x) = a\cos[b(x-c)] + d$ General form of the cosine function

   $f(x)=(8)\cos\left\{\left(\frac{\pi}{6}\right)[x-(7)]\right\}+(20)$ Substitute 8 for a, $\frac{\pi}{6}$ for b, 7 for c, and 20 for d.

   $f(x)=8\cos\left[\frac{\pi}{6}(x-7)\right]+20$ Simplify.

   The equation of the function is $f(x)=8\cos\left[\frac{\pi}{6}(x-7)\right]+20$.

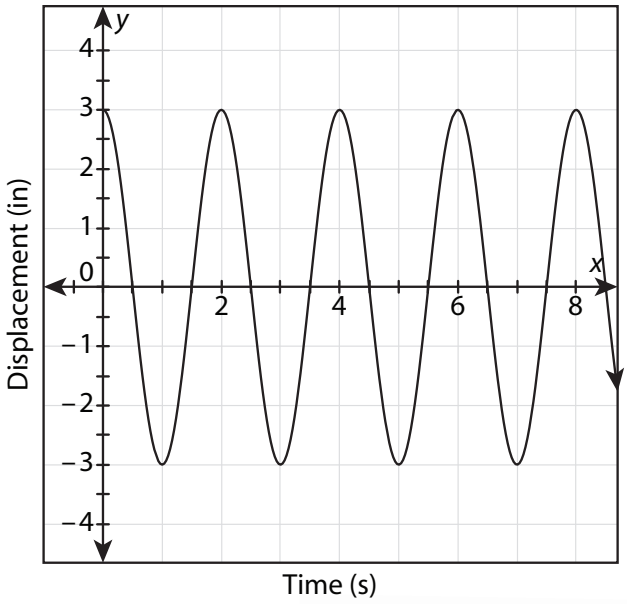
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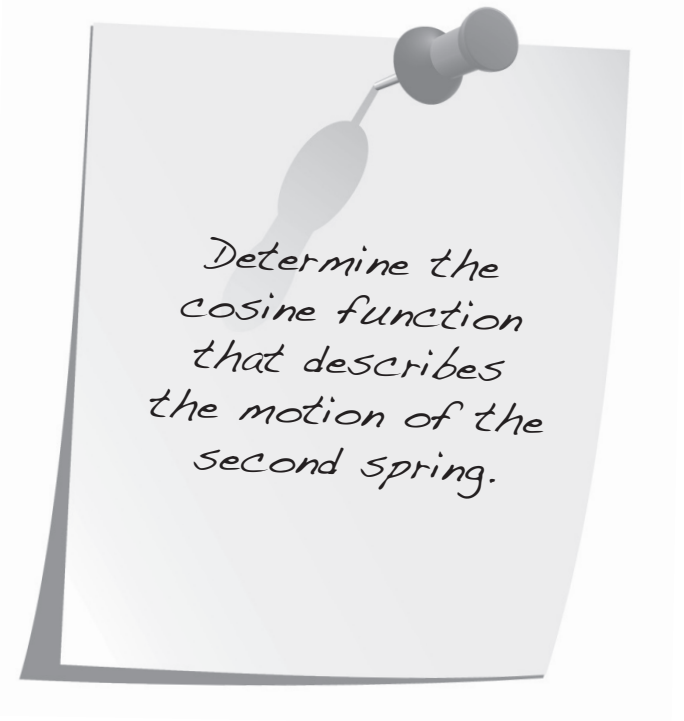
Problem-Based Task 3.3.2: Spring Displacement

SMP	
1	2 ✓
3	4
5	6
7 ✓	8

Ramzi observed the motion of a spring and plotted its displacement on the graph shown. He then observed a second spring that behaved in the same way except that its amplitude was twice as large as the first and its period was half as long. Determine the cosine function that describes the motion of the second spring.

Spring Displacement over Time in Seconds





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Problem-Based Task 3.3.2: Spring Displacement

Coaching

a. What is the amplitude of the first spring?

b. What is the amplitude of the second spring?

c. What is the period of the first spring?

d. What is the period of the second spring?

e. What is the value of b for the second spring?

f. What are the values of a, c, and d for the second spring?

g. What is the equation that describes the cosine function for the motion of the second spring?

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Instruction

Problem-Based Task 3.3.2: Spring Displacement

Coaching Sample Responses

a. What is the amplitude of the first spring?

The amplitude is the distance the function rises above its midline.

The midline of this function is $y = 0$, and the function's highest point is $y = 3$, which is 3 units above the midline.

The amplitude of the first spring is 3.

b. What is the amplitude of the second spring?

The amplitude of the second spring is twice the amplitude of the first spring, so multiply the amplitude of the first spring by 2.

$3 \cdot 2 = 6$

The amplitude of the second spring is 6.

c. What is the period of the first spring?

The period is the length of one cycle of the curve and can be found by determining the distance between two maximum points.

One cycle of the curve starts at a maximum of $x = 0$ and ends at another maximum of $x = 2$.

The length of one cycle of the curve is 2.

The period of the first spring is 2.

d. What is the period of the second spring?

The period of the second spring is half as long as the period of the first spring, so divide the period of the first spring by 2.

$2 \div 2 = 1$

The period of the second spring is 1.

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e. What is the value of b for the second spring?

Substitute 1, the value determined for the period, into the formula for the period and solve for b.

$$\text{period} = \frac{2\pi}{b}$$
$$(1) = \frac{2\pi}{b}$$
$$b = 2\pi$$

For the second spring, $b = 2\pi$.

f. What are the values of a, c, and d for the second spring?

a is the amplitude, which was determined in part b: $a = 6$.

c is the horizontal displacement, which is 0: $c = 0$.

d is the vertical displacement, and since the midline is at $y = 0$, there is no vertical displacement: $d = 0$.

Thus, for the second spring, $a = 6$, $c = 0$, and $d = 0$.

g. What is the equation that describes the cosine function for the motion of the second spring?

Substitute the values for a, b, c, and d into the general form of the cosine function and simplify.

$f(x) = a \cos [b(x - c)] + d$

$f(x) = (6) \cos \{(2\pi)[x - (0)]\} + (0)$

$f(x) = 6 \cos (2\pi x)$

The function which describes the motion of the second spring is $f(x) = 6 \cos (2\pi x)$.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.3.2: Using Trigonometric Functions to Model Periodic Phenomena

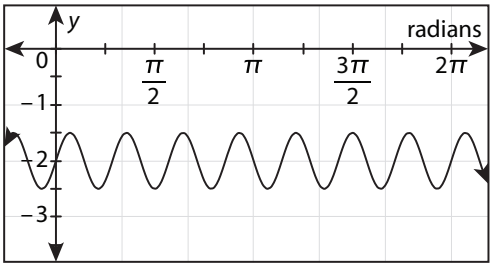
For problems 1–7, write a trigonometric equation to describe each function. *Note*: Some problems may have more than one correct answer, but only one answer is needed.

1. A sine curve that is shifted $\frac{\pi}{3}$ units to the left has a midline at $y = 2$ and rises 4 units above the midline. The length of one cycle of its curve is 2π.

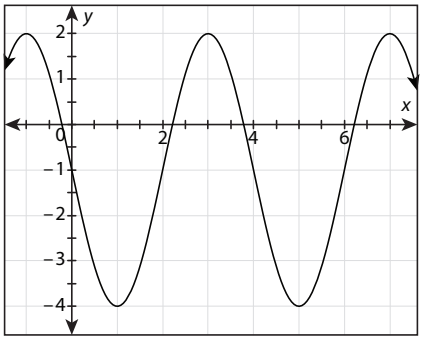
2. A sine curve with no horizontal shift has a frequency of $\frac{2}{\pi}$ and rises 3 units above its midline, which is at $y = 1$.

3. A cosine curve has two consecutive maximum points at $\left(\frac{\pi}{4}, 4\right)$ and $\left(\frac{3\pi}{4}, 4\right)$, and a minimum point at $\left(\frac{\pi}{2}, -6\right)$.

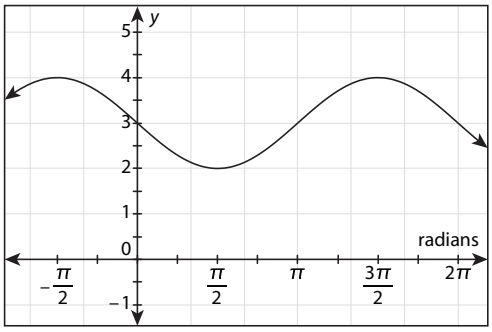
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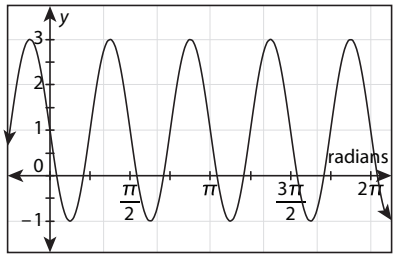
6.



5.



7.



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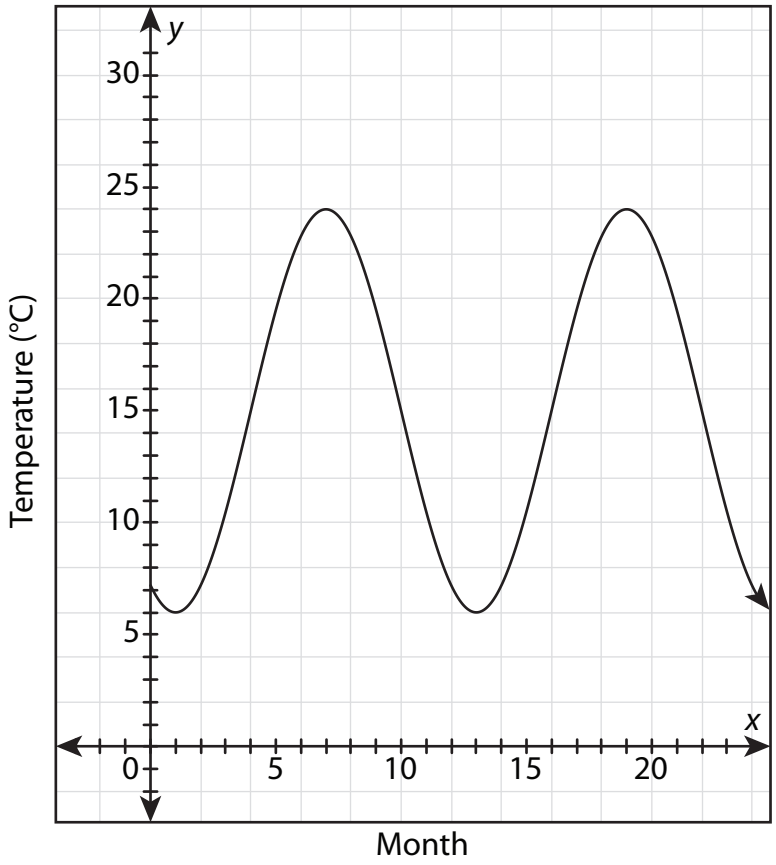
For problems 8–10, write the trigonometric equation of the function that models each periodic phenomenon.

8. When Eve released a spring, it completed one oscillation (cycle) per second and traveled up to 4 inches from equilibrium (its midline). She knows the oscillation can be described by a cosine function with no horizontal or vertical displacement. Write an equation to describe this function.

9. Brandon played a note on his clarinet that had an amplitude of 2 and a frequency of 740 Hz. He knows this note can be described by a sine function with no horizontal or vertical displacement. Write an equation to describe this function.

10. Since January 2005, the average monthly temperatures of Swansburg have fluctuated in accordance with the function shown in the following graph. Write an equation to describe this function.

Average Monthly Temperature in Swansburg



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Progress Assessment

Circle the letter of the best answer.

1. What is the frequency of the function $f(x) = 3 \sin (5x) - 1$?

   a. $\frac{1}{5}$

   b. 5

   c. $\frac{5}{2\pi}$

   d. $\frac{2\pi}{5}$

2. What is the midline of the function $f(x) = 2 \cos [4(x - 3)] - 5$?

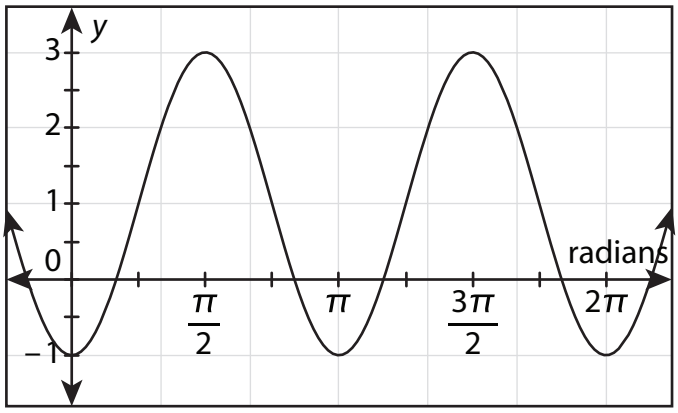
   a. $y = -5$

   b. $y = -3$

   c. $y = 2$

   d. $y = 4$

3. What is the period of the graphed function?



   a. $\frac{1}{\pi}$

   b. $\frac{2}{3\pi}$

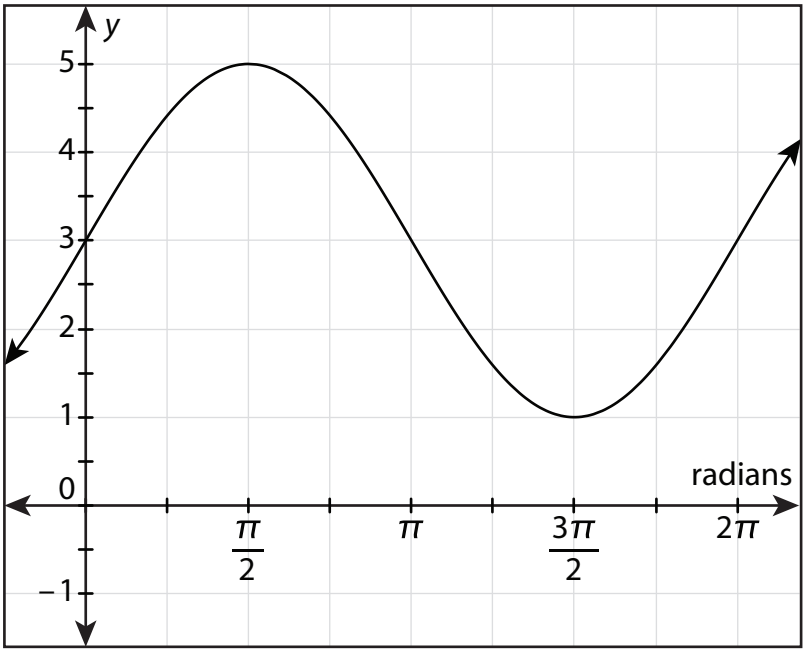
   c. $\frac{3\pi}{2}$

   d. π

continued

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4. What is the amplitude of the graphed function?



a. 1
b. 2
c. 3
d. 4

5. Which function has a period of $\frac{\pi}{2}$?

a. $f(x) = \cos(2x)$
b. $f(x) = 3\sin(4x) - 6$
c. $f(x)=\frac{1}{2}\cos\left(\frac{2}{\pi}x\right)+4$
d. $f(x)=2\sin\left[\frac{\pi}{2}(x+3)\right]$

6. Which function has an amplitude of 1?

a. $f(x)=\frac{1}{2}\sin\left(\frac{\pi}{4}x\right)+1$
b. $f(x)=\sin\left[4\left(x-\frac{\pi}{7}\right)\right]$
c. $f(x) = 2\cos(3x + 1)$
d. $f(x) = 6\cos(x) - 2$

continued

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7. A cosine curve has a maximum point at (2, –1) and a period of 5. Its amplitude is 3. Which equation describes this function?

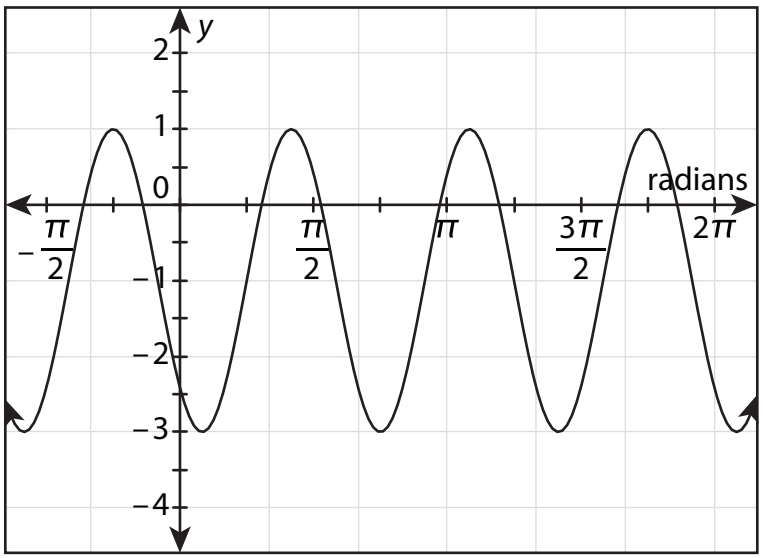
   a. $f(x)=3\cos\left[\frac{2\pi}{5}(x-2)\right]-4$

   b. $f(x)=3\cos\left[\frac{2\pi}{5}(x+2)\right]-1$

   c. $f(x) = 3\cos[5(x-2)]-4$

   d. $f(x) = 3\cos[5(x+2)]-1$

8. Which equation describes the graphed function?



   a. $f(x)=2\cos\left[\frac{2\pi}{3}\left(x+\frac{\pi}{4}\right)\right]-1$

   b. $f(x)=2\cos\left[3\left(x+\frac{\pi}{4}\right)\right]-1$

   c. $f(x)=2\sin\left[\frac{2\pi}{3}\left(x+\frac{\pi}{4}\right)\right]-1$

   d. $f(x)=2\sin\left[3\left(x+\frac{\pi}{4}\right)\right]-1$

9. Avery played a note on her piccolo that had an amplitude of 3 and a frequency of 2,637 Hz. Which function could be used to describe this note?

   a. $f(x) = 3\sin(2637x)$

   b. $f(x) = 3\sin(2637\pi x)$

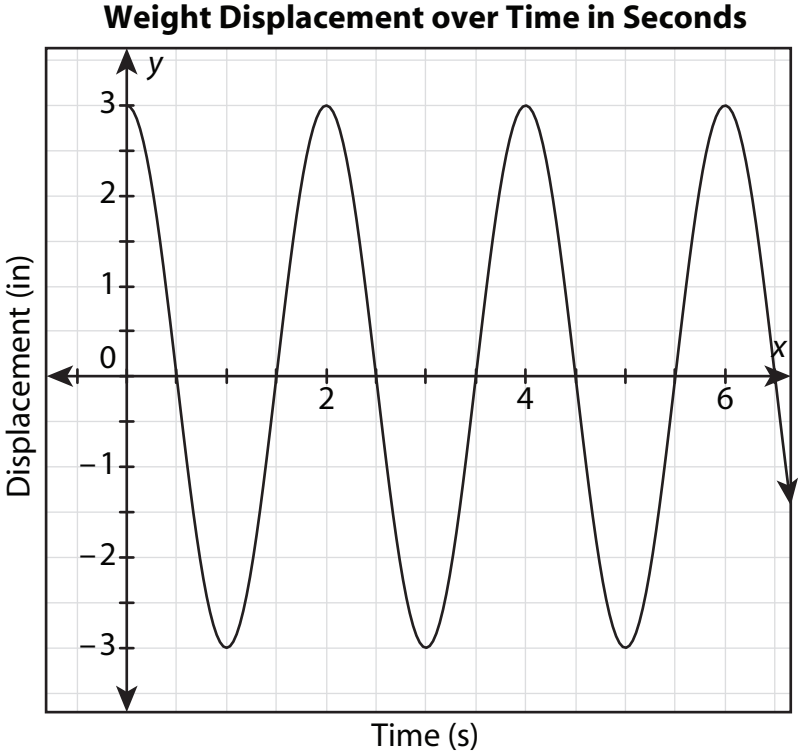
   c. $f(x) = 3\sin(5274x)$

   d. $f(x) = 3\sin(5274\pi x)$

continued

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10. The following graph shows the displacement of the weight at the end of an oscillating spring. Which equation describes the graphed function?



a. $f(x) = 3\cos(2x)$

b. $f(x) = 3\cos(\pi x)$

c. $f(x) = 3\sin(2x)$

d. $f(x) = 3\sin(\pi x)$

Use your knowledge of trigonometric functions and periodic phenomena to answer the questions about the function described.

11. A trigonometric function has two consecutive maximum points at $\left(-\frac{\pi}{4}, 3\right)$ and $\left(\frac{7\pi}{4}, 3\right)$, and a minimum point at $\left(\frac{3\pi}{4}, 1\right)$.

    a. What is the period of the function?

    b. What is the midline of the function?

    c. What is the amplitude of the function?

    d. What is the equation of the function?

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Unit 3 Assessment

Circle the letter of the best answer.

1. Convert 320° to radians.

   a. $\frac{9\pi}{16}$ radians

   b. $\frac{64\pi}{73}$ radians

   c. $\frac{73\pi}{64}$ radians

   d. $\frac{16\pi}{9}$ radians

2. If $m\angle\theta = \frac{17\pi}{9}$ radians, what is the measure of its reference angle?

   a. $\frac{\pi}{9}$ radian

   b. $\frac{9\pi}{17}$ radians

   c. $\frac{8\pi}{9}$ radians

   d. 2π radians

3. If $m\angle\theta = \frac{5\pi}{6}$ radians, at what point does the terminal side of the angle intersect the unit circle?

   a. $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

   b. $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

   c. $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

   d. $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

4. What is $\tan\theta$ if θ is an angle in standard position whose terminal side passes through the point (–5, 2)?

   a. $-\frac{2}{5}$

   b. $-\frac{5}{2}$

   c. $\frac{2\sqrt{29}}{29}$

   d. $-\frac{5\sqrt{29}}{29}$

continued

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5. What is $\sec \frac{5\pi}{4}$?

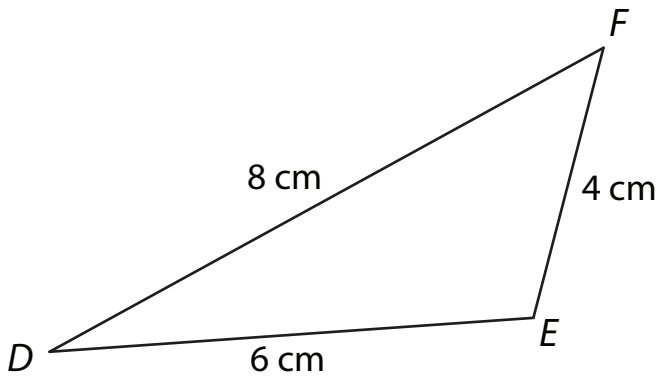
   a. $-\sqrt{2}$

   b. $\sqrt{2}$

   c. $-\frac{\sqrt{2}}{2}$

   d. $\frac{\sqrt{2}}{2}$

6. What is the area of $\triangle DEF$ if $m\angle D = \frac{5\pi}{36}$ radians?



   a. 10.1 cm^2

   b. 13.5 cm^2

   c. 20.3 cm^2

   d. 21.8 cm^2

7. For $\triangle LMN$, $LM = 5$ m, $MN = 4$ m, and $m\angle L = 38^\circ$. What are $m\angle N$ and $m\angle M$?

   a. $m\angle N = 150.5^\circ$ and $m\angle M = 29.5^\circ$

   b. $m\angle N = 91.7^\circ$ and $m\angle M = 50.3^\circ$

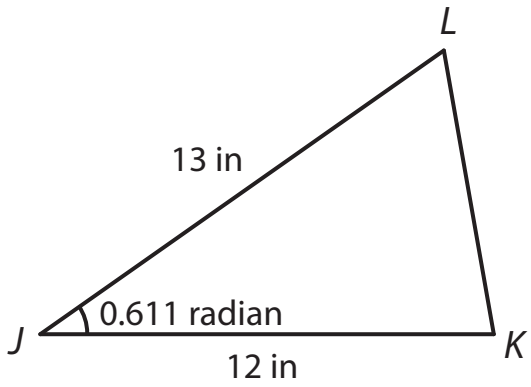
   c. $m\angle N = 29.5^\circ$ and $m\angle M = 150.5^\circ$

   d. $m\angle N = 50.3^\circ$ and $m\angle M = 91.7^\circ$

continued

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8. What is $m\angle L$?



a. 0.558 radian
b. 0.671 radian
c. 0.908 radian
d. 1.139 radians

9. A surveyor is trying to measure the width of a lake. He walks to a point far away from the lake and sets up his instruments. He measures the distance from the left side of the lake to his location as 385 feet and the distance from the right side of the lake to his location as 412 feet. He then measures the angle between these two lines of sight as 32°. How wide is the lake?

a. approximately 221 feet
b. approximately 231 feet
c. approximately 387 feet
d. approximately 428 feet

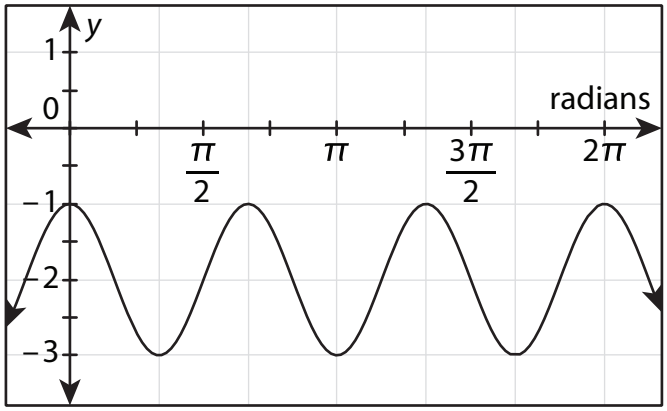
10. The sun is currently at a 68° angle of elevation. A tree is leaning toward the sun at a 5° angle and casts a shadow that is 9 feet long. How tall is the tree?

a. approximately 3 feet
b. approximately 8 feet
c. approximately 18 feet
d. approximately 29 feet

continued

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11. What is the frequency of the function?



a. $\frac{1}{3}$

b. $\frac{2\pi}{3}$

c. $\frac{3}{2\pi}$

d. 3

12. Anderson played a note on his bass guitar that had an amplitude of 2 and a frequency of 55 Hz. Which of the following functions could be used to describe this note?

a. $f(x) = 2 \sin (55x)$

b. $f(x) = 2 \sin (55\pi x)$

c. $f(x) = 2 \sin (110x)$

d. $f(x) = 2 \sin (110\pi x)$

Use what you have learned about the trigonometry of general triangles and trigonometric functions to complete all parts of the following problems.

13. The terminal side of angle θ is in Quadrant IV and $\sin \theta = -\frac{3}{5}$.

a. What is $\tan \theta$?

b. What is $\cos \theta$?

c. What is $\csc \theta$?

continued

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14. A flock of migratory birds flew from Baltimore headed for the Ivory Coast, 4,875 miles away. After 1,876 miles of smooth flying, the lead bird veered 34° to the right to avoid a storm. Then, after flying on that path for 97 miles, the lead bird turned the flock back on a direct line to the Ivory Coast.

    a. To the nearest mile, approximately how many more miles must the flock travel to reach the Ivory Coast?

    b. To the nearest mile, approximately how many extra miles did the detour add to the trip?

15. A trigonometric function has two consecutive maximum points at $\left(\frac{\pi}{4}, 1\right)$ and $\left(\frac{3\pi}{4}, 1\right)$, and a minimum point at $\left(\frac{\pi}{2}, -3\right)$.

    a. What is the period of the function?

    b. What is the midline of the function?

    c. What is the amplitude of the function?

    d. What is the equation of the function?

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Answer Key

Lesson 1: Radians and the Unit Circle

Pre-Assessment, p. U3-1

1. b
2. c
3. c
4. c
5. c

Warm-Up 3.1.1, p. U3-5

1. approximately 436 feet
2. approximately 401 feet
3. approximately 419 feet
4. approximately 471 feet
5. approximately 35 feet

Practice 3.1.1: Radians, p. U3-19

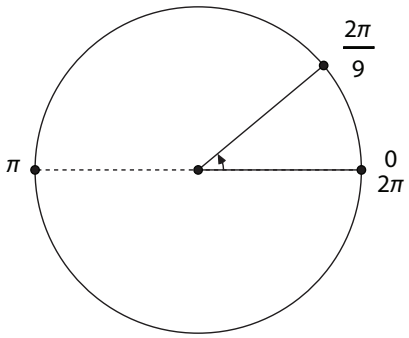
1. 4.25 radians
2. 2.4 radians
3. 0.625 radian
4. 160°
5. 245.5°
6. 175.7°
7. 54.0°
8. Mitra's spin speed: $\frac{43\pi}{36}$ radians per second; 3.7525 radians per second
9. Charlene's spin speed: $\frac{7\pi}{60}$ radian per second; 0.3665 radian per second
10. Abbie's spin speed: $\frac{26\pi}{45}$ radians per second; 1.8151 radians per second

Warm-Up 3.1.2, p. U3-21

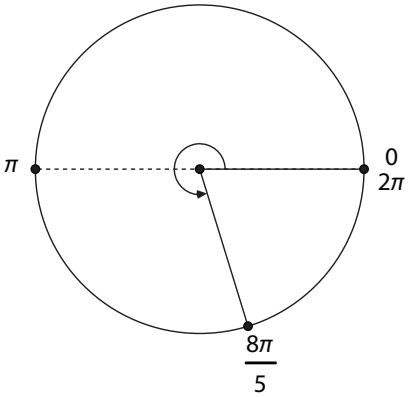
1. $\frac{23\pi}{36}$ radians per second
2. approximately 44.2 feet each second
3. approximately 34.1 feet each second

Practice 3.1.2: The Unit Circle, p. U3-44

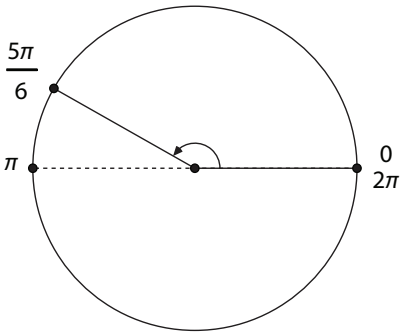
1.



2.



3.



4. 42°
5. $\frac{\pi}{5}$ radian
6. $\frac{\pi}{3}$ radians
7. 0.945 radian
8. (–0.71, –0.71)
9. (0.87, 0.5)
10. (–0.95, 0.30)

Warm-Up 3.1.3, p. U3-45

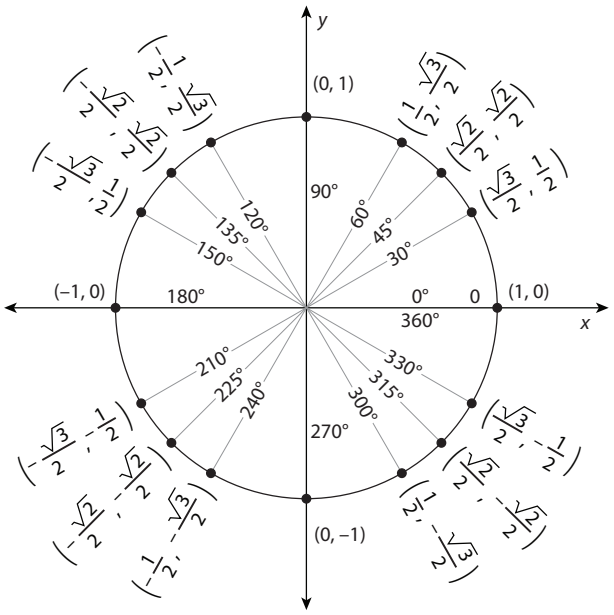
1. 20 feet
2. 10 feet
3. $10\sqrt{3}$ feet, or approximately 17.3 feet
4. $m\angle CTR = 90°$; $m\angle RCT = 60°$; $m\angle CRT = 30°$

Practice 3.1.3: Special Angles in the Unit Circle, p. U3-68

1. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
2. $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
3. (0, 1)
4. $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

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5. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
6. $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
7. (1, 0)
8. $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
9. $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
10.



Warm-Up 3.1.4, p. U3-69

1. approximately 24.3 feet
2. approximately 76.1°
3. approximately 2.9 feet

Practice 3.1.4: Evaluating Trigonometric Functions, p. U3-90

1. $-\frac{2\sqrt{53}}{53}$
2. $-\frac{4}{5}$
3. $\frac{1}{0}$ or undefined
4. $-\frac{5}{13}$
5. $-\frac{1}{2}$
6. $-\frac{1}{1} = -1$
7. $\frac{2\sqrt{3}}{3}$
8. $-\frac{3\sqrt{13}}{13}$
9. $\frac{5}{4}$
10. $-\frac{\sqrt{3}}{3}$

Progress Assessment, p. U3-91

1. b
2. a
3. c
4. b
5. a
6. d
7. c
8. d
9. d
10. c
11. a. $\frac{3\pi}{4}$ radians
    b. $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
    c. –1

Lesson 2: Trigonometry of General Angles

Pre-Assessment, p. U3-94

1. d
2. b
3. b
4. a
5. b

Warm-Up 3.2.1, p. U3-99

1. 11.1 ft²
2. 51.1 ft²
3. 18.4 ft

Practice 3.2.1: Proving the Law of Sines, p. U3-120

1. 5.84 ft
2. 0.49 radian
3. 48.21° and 131.79°
4. 20.38 in
5. 0.94 radian and 2.20 radians
6. 40.80 m²
7. 21.48 cm²
8. 37.7 km
9. 17.3 mi
10. 28.9 m

Warm-Up 3.2.2, p. U3-123

1. 24 ft
2. approximately 26 ft
3. approximately 21 ft

Practice 3.2.2: Proving the Law of Cosines, p. U3-140

1. 40.3°
2. 0.421 radian
3. 1.452 radians
4. 25.2°
5. $ST = 9$ in, $m\angle S = 33°$, $m\angle T = 61°$
6. $m\angle A = 21°$, $m\angle B = 32°$, $m\angle C = 127°$
7. $m\angle H = 36°$, $m\angle J = 21°$, $HI = 4$ cm
8. 10.0 mi
9. 19.9 km
10. 208.6 m

Warm-Up 3.2.3, p. U3-142

1. 97.2°
2. 41.4°
3. 7.9 ft²

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Practice 3.2.3: Applying the Laws of Sines and Cosines, p. U3-158

1. 142.7 yd
2. 57.0°
3. 110.2 ft
4. 35.3 ft
5. 275.4 mi
6. 1.1 ft
7. 189.5 m
8. 9.3 cm and 4.2 cm
9. 66.0°
10. 356.2 ft

Progress Assessment, p. U3-160

1. c
2. b
3. d
4. b
5. d
6. a
7. a
8. a
9. b
10. b
11. a. 41 mi
    b. 952 mi
    c. 42°

Lesson 3: Graphs of Trigonometric Functions

Pre-Assessment, p. U3-163

1. b
2. c
3. a
4. c
5. d

Warm-Up 3.3.1, p. U3-167

1. 0.42 ft below equilibrium
2. 0.65 ft below equilibrium
3. 0.96 ft above equilibrium

Practice 3.3.1: Periodic Phenomena and Amplitude, Frequency, and Midline, p. U3-184

1. period: π; frequency: $\frac{1}{\pi}$; midline: $y = -1$; amplitude: 3
2. period: 4π; frequency: $\frac{1}{4\pi}$; midline: $y = -3$; amplitude: 4
3. period: $\frac{2\pi}{3}$; frequency: $\frac{3}{2\pi}$; midline: $y = 2$; amplitude: $\frac{1}{2}$
4. period: $\frac{2\pi}{7}$; frequency: $\frac{7}{2\pi}$; midline: $y = 0$; amplitude: 1
5. period: 2π; frequency: $\frac{1}{2\pi}$; midline: $y = 0$; amplitude: 1
6. period: π; frequency: $\frac{1}{\pi}$; midline: $y = -8$; amplitude: $\frac{1}{2}$
7. period: 5π; frequency: $\frac{1}{5\pi}$; midline: $y = 4$; amplitude: 3
8. 3
9. $\frac{1}{2}$ or 0.5
10. $\frac{3}{4}$ or 0.75

Warm-Up 3.3.2, p. U3-185

1. Note 2
2. Note 5
3. Note 4
4. Note 5

Practice 3.3.2: Using Trigonometric Functions to Model Periodic Phenomena, p. U3-202

1. $f(x) = 4\sin\left(x + \frac{\pi}{3}\right) + 2$
2. $f(x) = 3\sin(4x) + 1$
3. $f(x) = 5\cos\left[4\left(x - \frac{\pi}{4}\right)\right] - 1$
4. $f(x) = \frac{1}{2}\sin(7x) - 2$
5. $f(x) = \cos\left(x + \frac{\pi}{2}\right) + 3$
6. $f(x) = 3\cos\left[\frac{\pi}{2}(x+1)\right] - 1$ or $f(x) = 3\cos\left[\frac{\pi}{2}(x-3)\right] - 1$ or $f(x) = 3\sin\left[\frac{\pi}{2}(x-2)\right] - 1$
7. $f(x) = 2\sin\left[4\left(x - \frac{\pi}{4}\right)\right] + 1$
8. $f(x) = 4\cos(2\pi x)$
9. $f(x) = 2\sin(1480\pi x)$
10. $f(x) = 9\cos\left[\frac{\pi}{6}(x-7)\right] + 15$ or $f(x) = 9\sin\left[\frac{\pi}{6}(x-4)\right] + 15$

Progress Assessment, p. U3-204

1. c
2. a
3. d
4. b
5. b
6. b
7. a
8. b
9. d
10. b
11. a. 2π
    b. $y = 2$
    c. 1
    d. $f(x) = \cos\left(x + \frac{\pi}{4}\right) + 2$

Unit Assessment p. U3-208

1. d
2. a
3. b
4. a
5. a
6. a
7. d
8. d
9. a
10. d
11. c
12. d
13. a. $-\frac{3}{4}$
    b. $\frac{4}{5}$
    c. $-\frac{5}{3}$
14. a. 2,919 miles
    b. 17 miles
15. a. $\frac{\pi}{2}$
    b. $y = -1$
    c. 2
    d. $f(x) = 2\cos\left[4\left(x - \frac{\pi}{4}\right)\right] - 1$

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Student Book Answer Key

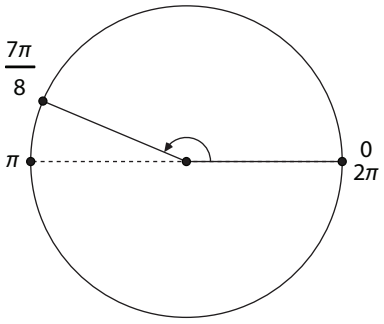
Lesson 1: Radians and the Unit Circle

Practice 3.1.1: Radians, p. U3-11

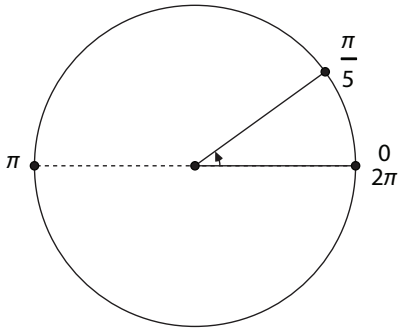
1. 4.5 radians
2. 2.7143 radians
3. 0.6923 radian
4. 260°
5. 128.6°
6. 298.0°
7. 42.3°
8. Stephen's spin speed: $\frac{37\pi}{36}$ radians per second; 3.2289 radians per second
9. Ali's spin speed: $\frac{31\pi}{60}$ radians per second; 1.6232 radians per second
10. Easton's spin speed: $\frac{61\pi}{90}$ radians per second; 2.1293 radians per second

Practice 3.1.2: The Unit Circle, p. U3-28

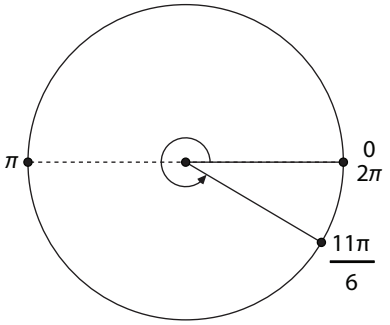
1.



2.



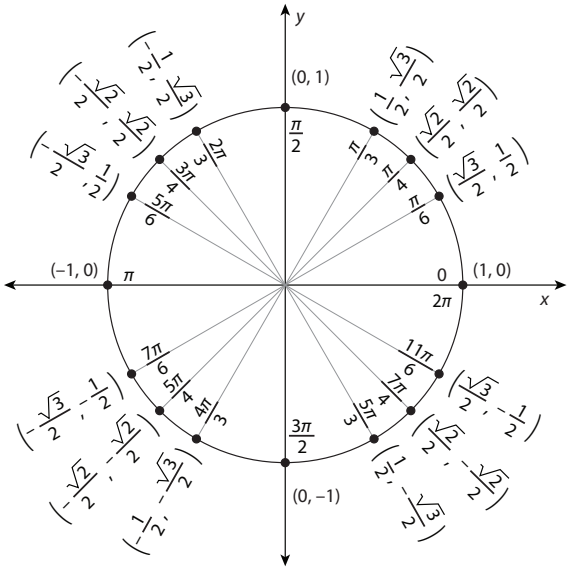
3.



4. 38°
5. $\frac{\pi}{4}$ radian
6. $\frac{2\pi}{7}$ radians
7. 0.271 radian
8. (0.90, 0.43)
9. (–0.5, –0.87)
10. (0.93, –0.38)

Practice 3.1.3: Special Angles in the Unit Circle, p. U3-42

1. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
2. $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
3. (–1, 0)
4. $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
5. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
6. $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
7. (0, –1)
8. $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
9. $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
10.



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Practice 3.1.4: Evaluating Trigonometric Functions, p. U3-57

1. $\frac{3}{5}$
2. $-\sqrt{5}$
3. $\frac{1}{0}$ or undefined
4. $-\frac{2\sqrt{5}}{5}$
5. $\frac{\sqrt{3}}{2}$
6. $\frac{1}{1}=1$
7. $\frac{1}{-\frac{1}{2}}=-2$
8. $-\frac{\sqrt{3}}{3}$
9. $\frac{4}{3}$
10. $-\frac{\sqrt{85}}{2}$

Lesson 2: Trigonometry of General Angles

Practice 3.2.1: Proving the Law of Sines, p. U3-73

1. 5.38 ft
2. 1.03 radians
3. 40.27° and 139.73°
4. 2.57 ft
5. 0.83 radian and 2.32 radians
6. 10.49 in^2
7. 18.76 cm^2
8. 40.6 km
9. 30.2 mi
10. 19.4 yd

Practice 3.2.2: Proving the Law of Cosines, p. U3-86

1. 106.6°
2. 0.776 radian
3. 1.796 radians
4. 34.8°
5. $UW = 9$ mm, $m\angle U = 26°$, $m\angle W = 80°$
6. $m\angle B = 24°$, $m\angle C = 101°$, $m\angle D = 55°$
7. $HF = 5$ cm, $m\angle G = 30°$, $m\angle H = 38°$
8. 18.9 mi
9. 18.7 km
10. 108.3 yd

Practice 3.2.3: Applying the Laws of Sines and Cosines, p. U3-98

1. 109.1 m
2. 50.7°
3. 131.7 ft
4. 33.6 ft
5. 480.5 mi
6. 34.9 ft
7. 179.2 m
8. 10.0 cm and 6.9 cm
9. 76.8°
10. 945.1 ft

Lesson 3: Graphs of Trigonometric Functions

Practice 3.3.1: Periodic Phenomena and Amplitude, Frequency, and Midline, p. U3-113

1. period: 2π; frequency: $\frac{1}{2\pi}$; midline: $y = 1$; amplitude: 1
2. period: $\frac{\pi}{2}$; frequency: $\frac{2}{\pi}$; midline: $y = -2$; amplitude: 3
3. period: $\frac{2\pi}{3}$; frequency: $\frac{3}{2\pi}$; midline: $y = 5$; amplitude: $\frac{1}{2}$
4. period: 3π; frequency: $\frac{1}{3\pi}$; midline: $y = -1$; amplitude: 1
5. period: $\frac{\pi}{2}$; frequency: $\frac{2}{\pi}$; midline: $y = 0$; amplitude: 5
6. period: π; frequency: $\frac{1}{\pi}$; midline: $y = -3$; amplitude: $\frac{1}{4}$
7. period: 7π; frequency: $\frac{1}{7\pi}$; midline: $y = 5$; amplitude: 1
8. 4
9. $\frac{3}{2}$ or 1.5
10. $\frac{5}{4}$ or 1.25

Practice 3.3.2: Using Trigonometric Functions to Model Periodic Phenomena, p. U3-124

1. $f(x)=2\sin\left[2\left(x+\frac{\pi}{5}\right)\right]+1$
2. $f(x) = 2\cos(6x) - 4$
3. $f(x)=3\cos\left[2\left(x-\frac{\pi}{2}\right)\right]-1$
4. $f(x) = 2\sin(3x) + 5$
5. $f(x)=2\cos\left(x+\frac{\pi}{4}\right)$
6. $f(x)=2\cos\left[\frac{\pi}{4}(x-1)\right]-3$ or $f(x)=2\sin\left[\frac{\pi}{4}(x+1)\right]-3$
7. $f(x)=\sin\left[4\left(x+\frac{\pi}{4}\right)\right]-1$ or $f(x)=\cos\left[4\left(x+\frac{\pi}{8}\right)\right]-1$ or $f(x)=\cos\left[4\left(x-\frac{3\pi}{8}\right)\right]-1$
8. $f(x) = 3\cos(\pi x)$
9. $f(x) = 3\sin(2960\pi x)$
10. $f(x)=7\cos\left[\frac{\pi}{6}(x-6)\right]+18$ or $f(x)=7\sin\left[\frac{\pi}{6}(x-3)\right]+18$

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Instruction

Goal: To provide opportunities for students to explore basic relationships in radian and degree measure, and in the relationship between the six trigonometric functions and the unit circle

Common Core State Standard

F–TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Student Activities Overview and Answer Key

Station 1

Students explore radians as fractions of circular rotations. They relate degree measures on the unit circle to corresponding measures in radians, then explain how to convert angle measures from radians to degrees, and from degrees to radians.

Answers

1. The circumference formula is $C = d\pi$, which equals $C = 2\pi r$. When the radius r equals 1 unit, $C = 2\pi \bullet 1$, so $C = 2\pi$.
2. Counterclockwise starting at (1, 0): $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, $2\pi/3$, $3\pi/4$, $5\pi/6$, π, $7\pi/6$, $5\pi/4$, $4\pi/3$, $3\pi/2$, $5\pi/3$, $7\pi/4$, $11\pi/6$, 2π
3. Counterclockwise starting at (1, 0): 30°, 45°, 60°, 90°, 120°, 135°, 150°, 180°, 210°, 225°, 240°, 270°, 300°, 315°, 330°, 360°
4. Clockwise starting at (1, 0): radians: $-\pi/6$, $-\pi/4$, $-\pi/3$, $-\pi/2$, $-2\pi/3$, $-3\pi/4$, $-5\pi/6$, $-\pi$, $-7\pi/6$, $-5\pi/4$, $-4\pi/3$, $-3\pi/2$, $-5\pi/3$, $-7\pi/4$, $-11\pi/6$, -2π; degrees: –30°, –45°, –60°, –90°, –120°, –135°, –150°, –180°, –210°, –225°, –240°, –270°, –300°, –315°, –330°, –360°
5.

Degrees	0°	30°	45°	60°	90°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

6. Multiply by π, then divide by 180°.
7. Divide by π, then multiply by 180°.

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Instruction

Station 2

Students explore relationships between angles and their reference angles. They work with angles in all four quadrants, using both degree and radian measures.

Answers

1. Subtract 180° from the angle measure.
2. Subtract the angle measure from 2π.
3. Quadrant II
4. Take the absolute value of the angle measure.
5. Sample answer: 100°, 80°
6. Sample answer: 200°, 40°
7. Sample answer: 300°, 60°
8. Sample answer: $5\pi/6$, $\pi/6$
9. Sample answer: $4\pi/3$, $\pi/3$
10. Sample answer: $7\pi/4$, $\pi/4$
11. Their trigonometric functions have the same absolute value, differing only in the signs. The angles at 90° and 270° have the same relationship.

Station 3

Students apply the Pythagorean Theorem to special cases of the 45°–45°–90° and 30°–60°–90° triangles. The lengths of the hypotenuses are 1 unit, so results can later be applied to the unit circle. Students find the sine and cosine values for angles measuring 0°, 30°, 45°, 60°, and 90°, and for multiples of these angles.

Answers

1. triangle with 45° acute angle: legs are each $\frac{\sqrt{2}}{2}$; triangles with 30° and 60° acute angles: legs are $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$
2. The sine equals the ratio of the opposite side to the hypotenuse. The cosine equals the ratio of the adjacent side to the hypotenuse.

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3.

Angle	0°	30°	45°	60°	90°
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Angle	0°	30°	45°	60°	90°
Sine	0	0.50	0.71	0.87	1
Cosine	1	0.87	0.71	0.50	0

4. Sample answers: The sine of x equals the cosine of $(90° - x)$. For example, the sine of 30° equals the cosine of 60°. The sine and cosine of a 45° angle are equal.

5.

Angle	30°	90°	150°	210°	330°
Sine	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
Cosine	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$

Angle	45°	90°	135°	225°	315°
Sine	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
Cosine	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$

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Angle	60°	120°	240°	300°
Sine	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
Cosine	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

6.

Angle	–330°	–210°	–150°	–30°
Sine	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
Cosine	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$

Angle	–315°	–225°	–135°	–45°
Sine	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
Cosine	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$

Angle	–300°	–240°	–120°	–60°
Sine	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
Cosine	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

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Instruction

Station 4

Students use the sine and cosine functions of special angles to label ordered pairs on unit circles.

Answers

1. Sample answer: In the diagram, $\cos\theta = \frac{x}{r}$, where r is the radius of the circle and x is the first coordinate of point P. Since $r = 1$, $\cos\theta = x$. The second coordinate of point P is y. Since $\sin\theta = \frac{y}{r}$ and $r = 1$, $\sin\theta = y$.
2. Clockwise starting at (1, 0): $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
3. Clockwise starting at (1, 0): $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
4.

Angle	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Location on the unit circle	(1, 0)	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	(0, 1)

Materials List/Setup

Station 1 none

Station 2 calculator

Station 3 scientific calculator

Station 4 none

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. Describe an angle in standard position.
2. How can the measure of an angle be interpreted as a rotation?
3. Explain how to find the reference angle.
4. What is the difference between an exact value and an approximate value for a trigonometric function?
5. Why do you think trigonometric functions are often called "circular functions"?
6. An angle θ has its vertex at the origin. The terminal side intersects the unit circle at a point P. How are the coordinates of P related to the angle θ?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. The angle is drawn on a coordinate system. The initial side is on the positive x-axis. The terminal side for a nonzero angle is in one of the four quadrants.
2. Put the vertex at the origin. One ray of the angle starts at (0, 0) and extends to the right along the x-axis. The other ray of the angle rotates in a counterclockwise direction. The number of degrees in the rotation is the measure of the angle.
3. If an angle is negative or has a measure greater than 360°, first find the coterminal angle. Identify the quadrant. In Quadrant I, the angle equals its reference angle. In Quadrant II, subtract from 180°. In Quadrant III, subtract 180°. In Quadrant IV, subtract from 360°.
4. The exact values may have square roots in them. The approximate values are decimal approximations of these square roots.
5. They can be defined by using arc lengths on a unit circle.
6. The x-coordinate of P equals the cosine of angle θ. The y-coordinate of P equals the sine of angle θ.

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Instruction

Possible Misunderstandings/Mistakes

- Incorrectly applying the Pythagorean Theorem when finding side lengths of the 45°–45°–90° and the 30°–60°–90° triangles
- Multiplying by $\frac{\pi}{180}$ instead of $\frac{180}{\pi}$ when converting radians to degrees
- Multiplying by $\frac{180}{\pi}$ instead of $\frac{\pi}{180}$ when converting degrees to radians
- Making computation errors when computing multiples of 30, 45, and 60
- Switching the x- and y-coordinates in ordered pairs
- Using the inverse trig keys on a calculator when finding reciprocal trig functions
- Incorrectly computing the reference angle
- Not rationalizing denominators
- Confusing the sine function with the cosine function, or the cosecant with the secant
- Making sign errors for values in Quadrants II, III, and IV

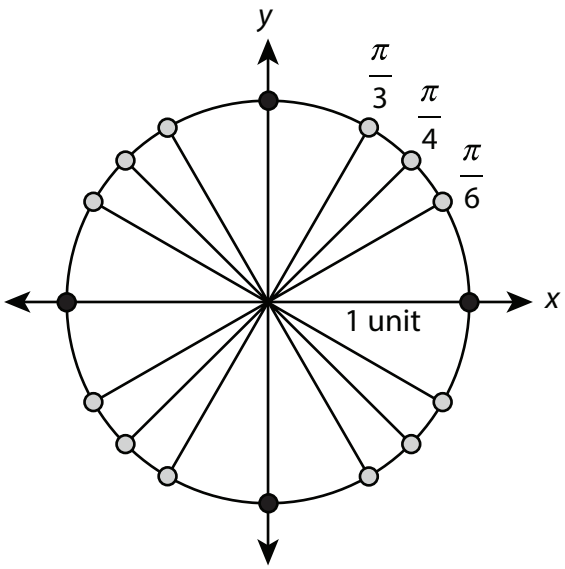
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Station 1

Work with your group to complete the problems.

1. A unit circle has a radius of 1 unit. Explain why one complete rotation around a unit circle is a distance of 2π.

2. Complete the diagram that follows to show fractional rotations around a unit circle in radians. Remember, every rotation starts at (1, 0) and goes in a counterclockwise direction.

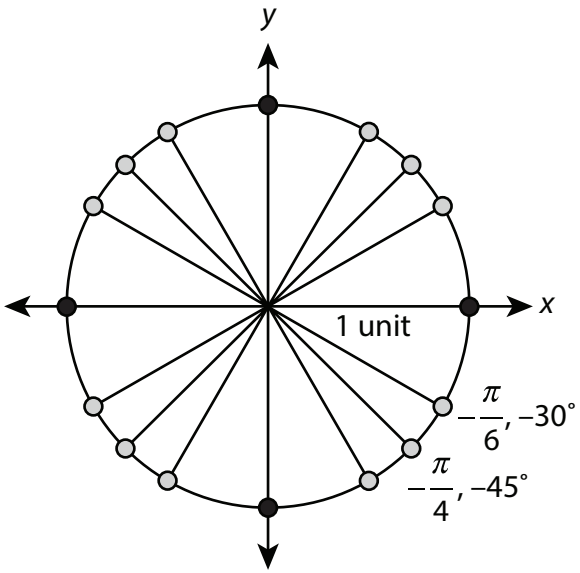


3. Angles can be measured in radians as well as in degrees. A complete rotation of a circle is 360°. A complete rotation also equals 2π radians. On the diagram for problem 2, write the degree measure next to each radian measure.

continued

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4. Repeat problems 2 and 3 for angles measured in a clockwise direction. Use the figure that follows. Label each location on the unit circle with two measures. The first two locations have been labeled for you.



5. The table that follows shows the special angles in Quadrant I. Complete the table, including both degree and radian angle measures.

Exact Trigonometric Values for Special Angles in Quadrant I

Degrees	0°	30°	45°	60°	90°
Radians					
Sine					
Cosine					

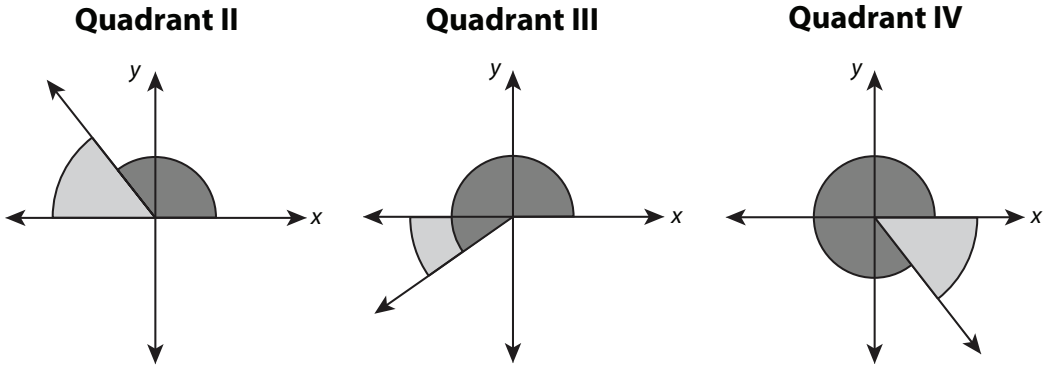
6. Explain how to change a degree measure to a radian measure.

7. Explain how to change a radian measure to a degree measure.

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Station 2

Work with your group and refer to the following figures to complete the problems. In each figure, a lightly shaded reference angle in the stated quadrant is shown for a given (darkly shaded) angle.



1. The degree measure of an angle in the third quadrant is given. Explain how to find the reference angle.

2. An angle in the fourth quadrant is measured in radians. Explain how to find the reference angle.

3. A student subtracted an angle measure from 180° to find the reference angle. In which quadrant was this angle?

4. An angle that has completed less than a full negative rotation with a negative measure lies in Quadrant IV. Explain how to find the reference angle.

5. Choose three angles in Quadrant II. Write the measures in degrees. Find the reference angle for each one.

continued

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6. Choose three angles in Quadrant III. Write the measures in degrees. Find the reference angle for each one.

7. Choose three angles in Quadrant IV. Write the measures in degrees. Find the reference angle for each one.

8. Choose three angles in Quadrant II. Write the measures in radians. Find the reference angle for each one.

9. Choose three angles in Quadrant III. Write the measures in radians. Find the reference angle for each one.

10. Choose three angles in Quadrant IV. Write the measures in radians. Find the reference angle for each one.

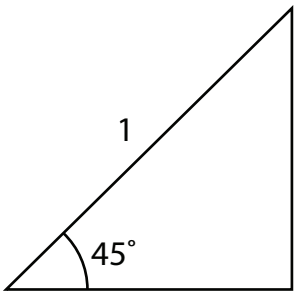
11. Explain why 0° can be considered the reference angle for 180°. Which other pair of degree measures have this same relationship?

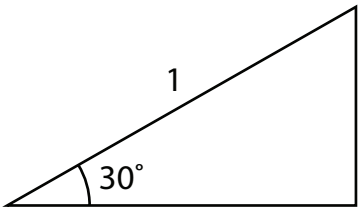
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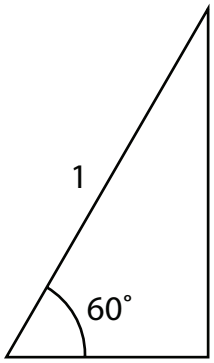
Station 3

Work with your group to complete the problems.

1. Study the triangles that follow. The hypotenuse of each triangle is 1 unit long. Use the Pythagorean Theorem to find the exact values of all side lengths. (*Hint*: An *exact* value may have a square root sign in it.)







2. How do you use the side lengths of a right triangle to find values for the sine and cosine functions?

continued

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3. Complete the following tables to show the exact and approximate values for the sine and cosine of the given angles.

Exact Trigonometric Values for Special Angles

Angle	0°	30°	45°	60°	90°
Sine	0				
Cosine	1	$\frac{\sqrt{3}}{2}$			

Approximate Trigonometric Values for Special Angles

Angle	0°	30°	45°	60°	90°
Sine			0.71		
Cosine				0.50	

4. What patterns are found in the tables? How can these patterns help you memorize these key values?

continued

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5. The tables that follow show the multiples of 30°, 45°, and 60° angles. Complete the tables with exact trigonometric values for the sine and cosine.

Exact Trigonometric Values for Multiples of 30° Angles

Angle	30°	90°	150°	210°	330°
Sine			$\frac{1}{2}$	$-\frac{1}{2}$	
Cosine			$-\frac{\sqrt{3}}{2}$		$\frac{\sqrt{3}}{2}$

Exact Trigonometric Values for Multiples of 45° Angles

Angle	45°	90°	135°	225°	315°
Sine					
Cosine					

Exact Trigonometric Values for Multiples of 60° Angles

Angle	60°	120°	240°	300°
Sine				
Cosine				

continued

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6. The tables that follow show the multiples of –30°, –45°, and –60° angles. Complete the tables with exact trigonometric values for the sine and cosine.

Exact Trigonometric Values for Multiples of –30° Angles

Angle	–330°	–210°	–150°	–30°
Sine				
Cosine				

Exact Trigonometric Values for Multiples of –45° Angles

Angle	–315°	–225°	–135°	–45°
Sine				
Cosine				

Exact Trigonometric Values for Multiples of –60° Angles

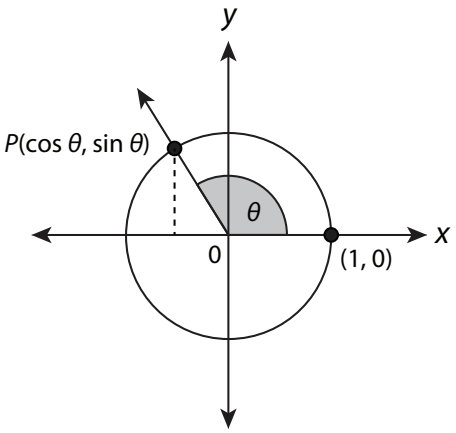
Angle	–300°	–240°	–120°	–60°
Sine				
Cosine				

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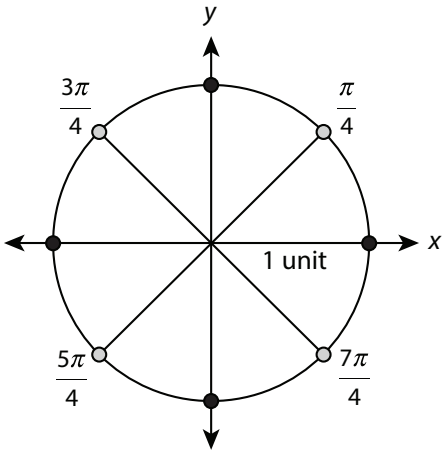
Station 4

Work with your group to complete the problems.

1. Explain how the sine and cosine functions can be used to write ordered pairs for points on the unit circle. Use the diagram that follows in your explanation.



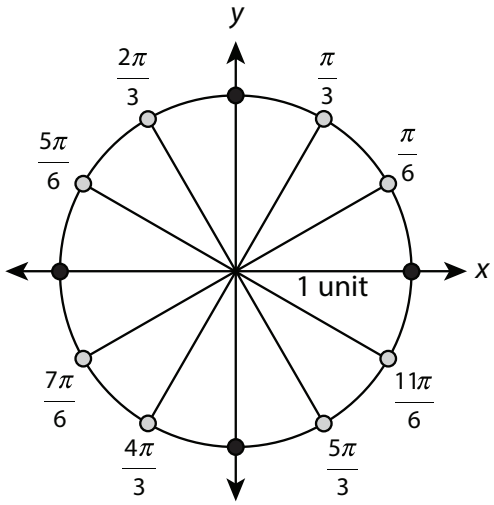
2. The bullets on the following unit circle show some multiples of 45° angles. The angles are marked in radians. Use exact values for the sine and cosine functions to write the coordinates of each point.



continued

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3. The following unit circle shows some multiples of 30° and 60° angles, measured in radians. Use exact values for the sine and cosine functions to write the coordinates of each point.



4. The chart that follows shows the special angles in the first quadrant of the unit circle, with angle measures in radians. Write ordered pairs to show the location of the intersection points on the unit circle for rays that create these angles.

Angle	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Location on the unit circle					

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Instruction

Goal: To use the Law of Sines and the Law of Cosines to solve problems

Common Core State Standards

G–SRT.9 (+) Derive the formula $A = 1/2\ ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

G–SRT.10 (+) Prove the Laws of Sines and Cosines and use them to solve problems.

G–SRT.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

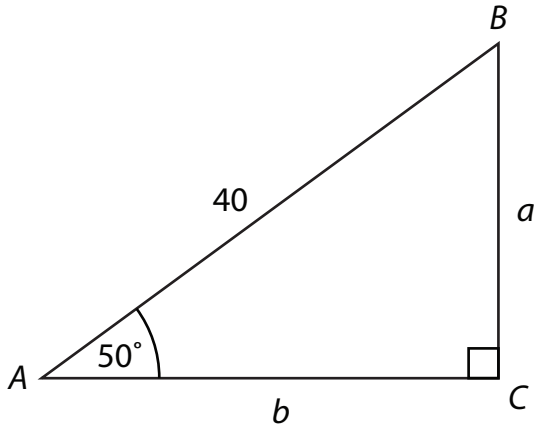
Student Activities Overview and Answer Key

Station 1

Students will review trigonometric ratios and use them to find the missing side lengths of given triangles.

Answers

1. $a = 56.8$ m
2. $a = 30.2$ in
3. Liam's answer is correct because the measure of the side given is opposite the measure of the angle given, and the unknown side length is the hypotenuse of the triangle; therefore, sine is the appropriate trigonometric ratio to use.
4. The base of the ladder is 10.4 ft from the tree.
5. a.



b. $AC = 25.7$ cm; $BC = 30.6$ cm

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Instruction

Station 2

Students will derive the formula $A = \frac{1}{2}ab\sin C$. Students will construct triangles using straws or licorice sticks. Then, students will use the derived formula and their constructions to find the area of each triangle.

Answers

1. Answers should reflect the following derivation:
   Set up a ratio, $\sin C = \frac{h}{b}$, and solve for h: $b \sin C = h$. Substitute h for a in the formula for the area of a triangle, $A = \frac{1}{2}bh$, to arrive at $A = \frac{1}{2}ab\sin C$.
2. 56.4 cm^2
3. 7.7 in^2
4. 984.8 cm^2
5. Verify accuracy of sketches; 43.3 cm^2
6. Verify accuracy of sketches; 51.4 cm^2

Station 3

Given paper and straws or licorice sticks, students will construct triangles and use the Law of Sines and the Law of Cosines to find missing side lengths or angle measures for given triangles.

Answers

1. Law of Cosines
2. Law of Sines
3. Law of Cosines
4. Law of Sines
5. Verify accuracy of sketches.
   a. 12.7 cm
   b. 13 cm. Sample answer: The length found using the Law of Cosines is similar to that found in part a.

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6. Verify accuracy of sketches.
   a. 3.5 in
   b. 3.9 in. Sample answer: The length found using the Law of Cosines is similar to that found in part a.
7. Verify accuracy of sketches; 46.0 cm^2
8. Verify accuracy of sketches; 34.3°

Station 4

Students will sketch triangles using the given information, and then apply the Law of Sines and the Law of Cosines to complete the problems.

Answers

1. Verify accuracy of sketches; 46°
2. Verify accuracy of sketches; 1,850 ft
3. Verify accuracy of sketches; 57.0 ft
4. Verify accuracy of sketches; 1,571 miles
5. Verify accuracy of sketches; 86.2°

Materials List/Setup

Station 1 scientific or graphing calculator

Station 2 scientific or graphing calculator; white paper; straws or licorice sticks

Station 3 scientific or graphing calculator; white paper; straws or licorice sticks

Station 4 scientific or graphing calculator

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Instruction

Discussion Guide

To support students in reflecting on the activities and synthesizing their ideas, and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. How can you use the Pythagorean Theorem to find the values of the six basic trigonometric functions for a given triangle?
2. If you wanted to use trigonometry to model a real-world problem with a non-right triangle, what could you do?
3. Give an example of a word problem that uses trigonometric functions in a real-world context, such as flying and landing an airplane.
4. How can you use a calculator to find the measure of an angle to the nearest degree given a sine, cosine, or tangent value?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. If you know the lengths of two sides of a right triangle, you can use the Pythagorean Theorem to find the length of the third side. From there, you can determine the values of the six trigonometric functions. For example, in a right triangle $\triangle ABC$, if side a is 3 units and the hypotenuse c is 5 units, then by using the Pythagorean Theorem, we can determine the length of side b is 4 units. Substitute these values for a, b, and c into the trigonometric functions: $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\csc\theta = \frac{5}{3}$, $\sec\theta = \frac{5}{4}$, and $\cot\theta = \frac{4}{3}$.
2. You could draw a perpendicular line from a vertex of the triangle to the opposite side of the triangle, resulting in two smaller right triangles within the original triangle.
3. Answers may vary. Sample answer: A pilot must approach an airport at an angle of 12° (the angle of depression) toward the runway. The plane is flying at an altitude of 4,200 ft. Use a trigonometric function to determine the distance at which the pilot should start the plane's descent.

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Instruction

4. You can use the inverse trigonometric functions (arcsine, arccosine, and arctangent) on a calculator to find the measure of an angle with a given sine, cosine, or tangent value. For instance, if the sine value is 0.875, use the arcsine to determine the angle measure: $\sin^{-1} 0.875 = 61.04$. Rounded to the nearest whole degree, the measure of the angle is 61°.

Possible Misunderstandings/Mistakes

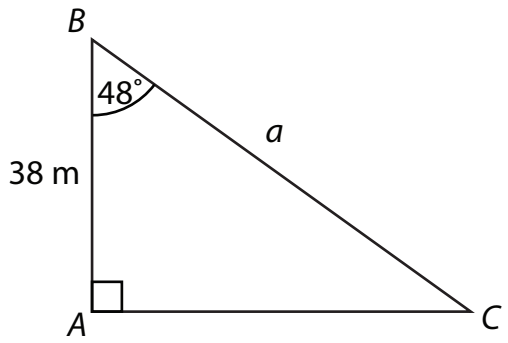
- Not using the correct trigonometric ratio; for example, stating $\sin\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ instead of $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$
- Inputting information into calculators incorrectly; for example, solving for a trigonometric function when the inverse trigonometric function is required, and vice versa
- Not setting calculators to degree mode

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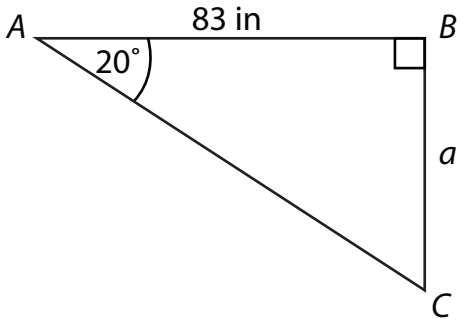
Station 1

For problems 1 and 2, use the trigonometric ratios of sine, cosine, and tangent to find the missing side length for each triangle shown. Round answers to the nearest tenth.

1. Determine the length of side *a*.



2. Determine the length of side *a*.

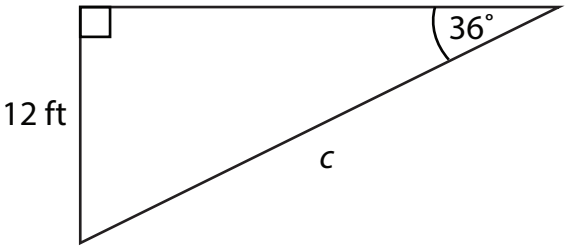


continued

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For problems 3–5, use the information given in each scenario to complete the problems.

3. Ana and Liam each found a different length for side c in the triangle shown in the following diagram. Compare Ana's work with Liam's work to determine whose answer is correct. Justify your response.



Ana's work	Liam's work
$\cos 36° = \frac{12}{c}$	$\sin 36° = \frac{12}{c}$
$c \cos 36° = 12$	$c \sin 36° = 12$
$c = \frac{12}{\cos 36°}$	$c = \frac{12}{\sin 36°}$
$c \approx 14.83$	$c \approx 20.42$

4. A 40-foot ladder leans against a tree. The ladder forms a 75° angle with the ground. How far from the tree is the base of the ladder?

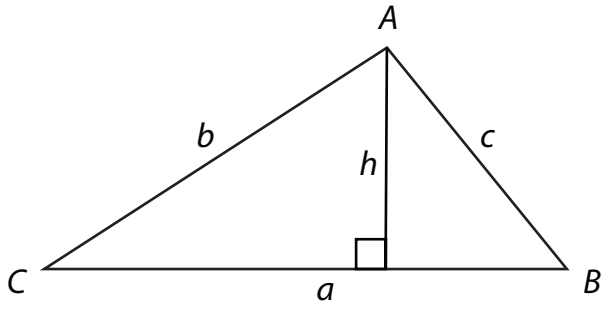
5. $\triangle ABC$ is a right triangle. $\angle C$ is a right angle, $m\angle A = 50°$, and $AB = 40$ cm.
   a. Sketch and label the triangle.
   b. Determine the lengths of $\overline{AC}$ and $\overline{BC}$.

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Station 2

Use trigonometric ratios and the provided materials, as needed, to complete the problems.

1. Derive the formula for the area of a triangle using trigonometry and the figure shown in the diagram. As a starting point, use one of the trigonometric ratios (sine, cosine, or tangent) to identify the height, h.



For problems 2–4, use the paper and straws or licorice sticks provided to construct each triangle described. Then, find the area of the triangle to the nearest tenth of a unit.

2. Construct a triangle with side lengths of 10 cm and 12 cm; the included angle measures 70°.

3. Construct a triangle with side lengths of 4 in and 5 in; the included angle measures 50°.

4. $\triangle ABC$ has adjacent side lengths of 40 cm and 50 cm; the included angle measures 100°.

continued

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For problems 5 and 6, sketch each triangle described, then use the formula found in problem 1 to determine the area of the triangle. As you sketch and construct the triangles using the straws and licorice sticks, remember to label the side opposite $\angle A$ with a, the side opposite $\angle B$ with b, and the side opposite $\angle C$ with c.

5. $\triangle ABC$ has two sides that each measure 10 cm and a non-included angle of 60°.

6. $\triangle ABC$ has adjacent sides with lengths of 10 cm and 16 cm and an included angle of 40°.

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Station 3

At this station, you will find paper and straws or licorice sticks. Use these materials, as needed, as well as the Law of Sines and the Law of Cosines to complete each problem.

For problems 1–4, state whether the Law of Sines or the Law of Cosines should be used to solve each problem.

1. Given the lengths of three sides of a triangle, find the measure of an angle.

2. Given the lengths of two sides of a triangle and the measure of an angle opposite one of those sides, find the measures of the two unknown angles.

3. Given the lengths of two sides of a triangle and the measure of an included angle, find the length of the side opposite the known angle measure.

4. Given the measure of two angles of a triangle and the length of one side, find the lengths of the missing sides.

For problems 5 and 6, use the paper and straws or licorice sticks provided to construct the triangle described, then use the construction to answer the questions.

5. Construct a triangle with side lengths of 10 cm and 12 cm; the included angle measures 70°.
   a. Use trigonometric ratios to determine the length of the third side of the triangle (the side opposite the 70° angle).

   b. Verify the length of the third side using the Law of Sines or the Law of Cosines. What do you notice about your answer?

continued

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6. Construct a triangle with side lengths of 4 in and 5 in; the included angle measures 50°.

   a. Use trigonometric ratios to determine the length of the third side of the triangle (the side opposite the 50° angle).

   b. Verify the length of the third side using the Law of Sines or the Law of Cosines. What do you notice about your answer?

For problems 7 and 8, create a sketch of each triangle described, then use your sketch to answer the question.

7. For $\triangle ABC$, $AC = 12$ cm, $CB = 10$ cm, and $m\angle C = 50^\circ$. What is the area of $\triangle ABC$?

8. For $\triangle ABC$, $AC = 14$ cm, $CB = 22$ cm, and the area is 87 cm^2. What is the measure of $\angle C$?

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Station 4

Sketch each triangle described and then apply the Law of Sines or the Law of Cosines to complete each problem. Unless otherwise indicated, round answers to the nearest tenth.

1. Clevon forms a triangle with three straws measuring 20, 25, and 18 inches. To the nearest whole degree, find the measure of the angle enclosed by the 20- and 25-inch straws.

2. Olivia flies a kite on a 1,850-foot string at an angle of elevation of 80° with the ground. An observer notes that the angle formed between the kite and Olivia is 100°. How far is the observer from the kite?

3. Two stabilizing wires extend from the top of a pole to the ground, forming right triangles on either side of the pole. One wire forms an angle of 38° with the ground and the other wire forms an angle of 67° with the ground. The distance between where each of the wires attaches to the ground is 30 meters. How tall is the pole?

continued

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4. A plane took off from an airport and flew in a straight path for 2 hours 30 minutes. The pilot then headed 9° to the west and flew 2 hours in the new direction before landing the plane. If the pilot maintained an average speed of 350 mph, how far away from the starting point did he land?

5. Two planes took off in opposite directions from one airport at the same time. One plane flew at an average speed of 250 mph and the other flew at an average speed of 450 mph. After 2 hours, the planes were 1,000 miles apart. What is the measure of the angle between their flight paths at that time?

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APPENDIX

Problem-Based Task Implementation Guides

Introduction

This addendum contains a collection of Implementation Guides (IGs) designed to support the use of the Problem-Based Tasks found throughout this unit.

Each Implementation Guide includes an overview and suggestions for introducing and facilitating the task. These contain specific recommendations for selected Standards for Mathematical Practice (SMPs) to highlight, guidance on addressing common errors and misconceptions associated with the task, and ideas for monitoring and coaching students as they work through the task. Finally, there is guidance for debriefing the task, with special attention to the SMPs and relevant Key Concepts from the instruction, as well as information on suitable extensions and alternate strategies.

Throughout the Implementation Guides, suggestions are provided in a question/prompt format to support your efforts to elicit student thinking and facilitate classroom discourse.

We developed these Implementation Guides for our Problem-Based Tasks at the request of math educators, and with advice and feedback from mathematics supervisors and teachers. Please let us know if they are helpful. Visit us at www.walch.com, follow us on Twitter (@WalchEd), or e-mail suggestions to customerservice@walch.com.

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Implementation Guide

Problem-Based Task 3.1.1: A Satellite's Story

Common Core State Standards

F–TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F–TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Task Overview

Focus

How can radians be used as a measure of distance around a circle? How can measurements be converted from radians to degrees, and vice versa? In this task, students will determine the rotational speed in radians per hour and also degrees per hours of a satellite that is orbiting the Earth.

This activity will provide practice with:

- determining a radius of a circle given the diameter
- applying the formula for finding an angle measure in radians based on the arc length and the radius of the circle
- calculating the number of radians and degrees an object travels
- converting from radians to degrees and vice versa
- interpreting radian measure in the context of a real-world scenario

Introduction

This task should be used to explore or to apply the skill of understanding radian measure of an angle as the length of the arc on the unit circle subtended by the angle. Students should already be familiar with measuring angles with degrees, identifying the basic parts of a circle, e.g. radius, circumference, and arc, and using conversion factors to convert units of measurement.

Begin by reading the problem and clarifying the meaning of *arc length* and *radian*.

arc length distance between the endpoints of an arc

radian measure of the central angle that intercepts an arc equal in length to the radius of the circle; π radians = 180°

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine the satellite's rotational speed in radians per hour and also in degrees per hour. To accomplish this task, they will recognize that first it will be necessary to find the radius of the satellite's orbit so that this value can be used in the formula to determine the number of radians the satellite travels per hour. Once they have determined the number of radians, which they will recognize is also the rotational speed of the satellite, then they will convert this value to degrees using the appropriate conversion factor. This conversion will allow them to determine the number of degrees that the satellite travels per hour, which is also the rotational speed of the satellite in degrees per hour.

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they interpret the written information given in the task and identify the steps necessary to solve the problem. They will recognize that they will first need to find the radius of the Earth, which is the result of finding half of the diameter of the Earth. Then they will recognize that they will need to find the radius of the satellite's orbit by finding the sum of the Earth's radius and the distance between the Earth's surface and the location of the satellite. They will reason that the radius of the satellite's orbit will be used in the formula that is used to find the number of radians that the satellite travels per hour. They will reason quantitatively as they apply the formula to find the number of radians, as well as when they use the conversion factor for radians to degrees, in order to find the number of degrees that the satellite travels per hour.

- **SMP 6:** Attend to precision.

  Students will use precision as they first determine the value of the Earth's radius by dividing the given diameter by two. After they have added the Earth's radius to the distance that the satellite is above the Earth's surface to find the radius of the satellite's orbit, they will carefully substitute in the values for the arc length and the radius to find the number of radians that the satellite travels per hour. They will recognize that the resulting number of radians will need to be represented by at least three decimal places, so they will round the value to the nearest thousandth. They will also use precision as they use the conversion factor of radians to degrees, and they will round the number of degrees to the nearest tenth of a degree. Encourage them to use the correct terminology throughout the task, specifically when referring to radius and arc length, as well as when discussing the components of the conversion factor of radians to degrees.

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Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- forgetting to add the Earth's radius onto the distance above the surface that the satellite is orbiting when determining the radius of the satellite's orbit

  Encourage students to draw a diagram to represent the Earth and the satellite's position above the Earth's surface. Remind them that the radius of the satellite's orbit is the total distance from the center of the Earth to the satellite's position, so the Earth's radius must be added to the distance the satellite is about the Earth's surface.

- using the incorrect conversion factor when converting; i.e., using the reciprocal

  Have students write the conversion factors for radians to degrees, and degrees to radians, at the top on their papers. Remind them that the goal is to use the conversion factor that allows the desired unit to cancel out, leaving only the unit that is being converted to.

- mistakenly believing that $\pi = 180°$ instead of π radians = 180°

  Remind students that the value of π is approximately 3.14, so it cannot be equal to 180°. Remind them that the value of π radians on a circle is 180°.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin working on the task, ask them, "What formula can be used to find the distance the satellite travels in any given number of hours? What do the components of the formula represent?" (**Answer:** The formula to find the distance the satellite travels is $d = rt$, where d represents the total distance the satellite travels, r represents the rate of speed that the satellite travels, and t represents the amount of time the satellite travels.)

- As an extension to the previous question, ask students, "How can the distance formula be used to find the number of hours it would take for the satellite to travel 50,000 miles?"

  (**Answer:** The formula $d = rt$ can be used to find the number of hours it would take for the satellite to travel 50,000 miles by substituting 50,000 miles in for the value of d; substituting 7,000 miles in for the value of r, since it is given that the satellite travels 7,000 miles per hour; and solving for the unknown value of t: $d = rt$: $50{,}000 = 7000t$; $\frac{50{,}000}{7{,}000} = t$; $t \approx 7.1$ hours. It would take the satellite approximately 7.1 hours to travel a distance of 50,000 miles at a rate of 7,000 miles per hour.)

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Implementation Guide

- As students are working through the task, ask them, "Why is it necessary to add the radius of the Earth to the distance of the satellite above the Earth's surface in order to find the radius of the satellite's orbit?" (**Answer:** The radius of the satellite has two components which comprise the total radius – the distance from the center of the Earth to the Earth's surface, and the distance from the Earth's surface to where the satellite is located above the Earth's surface.)
- Ensure students' understanding of the formula $\theta = \frac{s}{r}$. Ask them, "In the formula $\theta = \frac{s}{r}$, what do the variables represent in the context of the scenario given in the task?" (**Answer:** The variable θ represents the number of radians that the satellite travels per hour. The variable s represents the arc length, or the distance the satellite travels per hour, and r represents the radius of the satellite's orbit.)
- Ask students, "Why do you think the value of the number of radians was rounded to the nearest thousandth? Do you think the value should be rounded to fewer or more decimal places?" (**Answer:** The value of the number of radians the satellite travels per hour is given as approximately 0.267 of a radian. If the value had been rounded to the nearest whole radian, then the result would be 0, which is not accurate or precise enough in the context of the problem. The value could be rounded to more decimal places, but if it were rounded to fewer, then the number of degrees would not be as accurate and precise.)
- As students are converting the number of radians per hour to degrees per hour, ask them, "Why do you use the conversion factor $\frac{180^\circ}{\pi \text{ radians}}$ instead of $\frac{\pi \text{ radians}}{180^\circ}$ when convertingfrom radians to degrees?" (**Answer:** The conversion factor $\frac{180^\circ}{\pi \text{ radians}}$ is used because the goal is to cancel the units of radians so that only the units of degrees are left. Since the value 0.267 radians is used in the numerator, then when the conversion factor is applied, the units of radians cancel out, leaving only units of degrees.)
- After students have determined the satellite's rotational speed in radians and degrees, ask them, "Approximately how many hours would it take the satellite to make one whole orbit around the Earth? Explain how you would figure this out." (**Answer:** The satellite travels 15.3° in one hour. There are 360° in one full orbit of the Earth, since there are 360° in a circle. To determine the number of hours it would take the satellite to travel 360°, divide 360 by 15.3: $360 \div 15.3 \approx 23.5$. It would take the satellite approximately 23.5 hours to make one whole orbit around the Earth.)

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- As an extension to the previous question, ask students, "If the number 23.5 hours was rounded to the nearest whole number, the number of hours it would take the satellite to make one whole orbit around the Earth would be 24. What other scientific fact do you know of that involves the Earth and the value of 24 hours?" (**Answer:** It takes the Earth approximately 24 hours to make one full rotation on its axis.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss why it was necessary to find the radius of the Earth, although the diameter was the given value in the problem. Ask students to explain the components of the formula to find the number of radians that the satellite travels, specifically which values represent the arc length and the radius. Also discuss why it was necessary to add the Earth's radius to the distance that the satellite was located above the Earth's surface in order to find the total radius of the satellite's orbit. Ask students to explain their choice of conversion factors in order to find the number of degrees the satellite traveled per hour, and ask them to explain how they applied the conversion factor. Focus on the units involved in the conversion factor and how the appropriate units were canceled out. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to explain how the speed that the satellite travels per hour relates to the value of the arc length of a circle. Also ask them to explain how the radius of the satellite's orbit relates to the value used in the formula to find the number of radians that the satellite travels. Discuss possible errors in applying the conversion factor of radians to degrees. Also discuss how the results can be interpreted in the context of the scenario. Focus on the use of precise mathematical language and clarity, specifically when discussing the relationships between arc length of a circle and the distance and speed the satellite travels, as well as how the values of s and r are represented in the formula used to find the number of radians the satellite travels per hour.

Connecting to Key Concepts

Make explicit connections to key concepts:

- A **radian** is the measure of the central angle that intercepts an arc equal in length to the radius of the circle.

  In this task, students will recognize a radian as a unit of measurement, and they will determine how many radians the satellite is traveling per hour.

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- Radians are often expressed in terms of π, which provides an exact measurement instead of a decimal approximation.

  Students will recognize that π is used when referring to radians in the conversion factor for radians to degrees.

- The formula used to represent the relationship between the arc length and the radius of a circle is $\theta = \frac{s}{r}$, in which θ is the angle measure in radians, s is the measure of the **arc length** (the distance between the endpoints of an arc), and r is the radius of the circle.

  Students will apply the formula for the relationship between arc length and the radius of a circle in order to determine the number of radians that the satellite travels in one hour.

- To convert from radians to degrees or vice versa, use an appropriate conversion factor based on the relationship between π radians and 180°. If converting from radians to degrees, multiply by $\frac{180°}{\pi \text{ radians}}$ so that the radians cancel and degrees remain. Alternately, if converting from degrees to radians, multiply by $\frac{\pi \text{ radians}}{180°}$ so that the degrees cancel and radians remain.

  Students will convert from radians to degrees using the conversion factor $\frac{180°}{\pi \text{ radians}}$ in order to determine the satellite's rotational speed in degrees per hour.

Extending the Task

- To extend the task, ask students to determine the approximate number of miles that the satellite travels in one full orbit of the Earth. Also ask them to write down their thoughts about why in the context of the scenario, the word "approximate" is used. Ask for volunteers to share their results and explain each step in their calculations. Discuss the factors that would affect the satellite maintaining a constant speed as it is orbiting the Earth.
- Another option for extending the task is to ask students to work through the task again, but this time with a different value for the distance that the satellite is located about the Earth's surface. Also provide a different number of miles that the satellite travels per hour. Note to students why the one value that cannot be altered in the new problem is the diameter of the Earth. Ask for volunteers to share the results of their new problem, and ask them to explain their steps and calculations.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of this problem by first recognizing that the goal of the task is to determine the satellite's rotational speed in radians per hour and also in degrees per hour. To accomplish this task, I recognized that first it was necessary to find the radius of the satellite's orbit so I could use this value in the formula to determine the number of radians the satellite travels per hour. Once I determined the number of radians, which I recognized was also the rotational speed of the satellite, then I converted this value to degrees using the appropriate conversion factor. This conversion allowed me to determine the number of degrees that the satellite travels per hour, which was also the rotational speed of the satellite in degrees per hour.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I recognized that I first needed to interpret the written information given in the task and identify the steps necessary to solve the problem. I recognized my first step was to find the radius of the Earth, which was the result of finding half of the given diameter of the Earth. Next, I recognized that I needed to find the radius of the satellite's orbit by finding the sum of the Earth's radius and the distance between the Earth's surface and the location of the satellite. I reasoned that the radius of the satellite's orbit would be used in the formula that is used to find the number of radians that the satellite travels per hour.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I reasoned quantitatively as I applied the formula to find the number of radians, as well as when I used the conversion factor for radians to degrees, in order to find the number of degrees that the satellite travels per hour.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision as I first determined the value of the Earth's radius by dividing the given diameter by two. After I added the Earth's radius to the distance that the satellite is above the Earth's surface, to find the radius of the satellite's orbit, I carefully substituted in the values for the arc length and the radius to find the number of radians that the satellite travels per hour. I recognized that the resulting number of radians needed to be represented by at least three decimal places, so I round the value to the nearest thousandth. I also used precision as I used the conversion factor of radians to degrees, and I rounded the number of degrees to the nearest tenth of a degree. I was careful to use the correct terminology throughout the task, specifically I referred to radius and arc length, as well as when I discussed the components of the conversion factor of radians to degrees.)

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Alternate Strategies or Solutions

Some students may choose to draw a diagram representing the scenario presented in the task, as this may help them to better understand the goal of the task. Encourage them to carefully label the components, specifically the diameter and radius of the Earth, as well as the distance representing the location of the satellite above the Earth's surface.

Technology

Students can use scientific calculators in this task.

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Problem-Based Task 3.1.3: Ferris Wheel Fun

Common Core State Standard

F–TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Task Overview

Focus

How can the patterns of special right triangles be used in a unit circle to find points where the terminal side of an angle intersects the unit circle? What is the relationship of the radius of a unit circle to its location on the circle? In this task, students will apply reference angles and special patterns of right triangles in order to determine the distance between two seats located on a Ferris wheel.

This activity will provide practice with:

- identifying angle measures on a unit circle
- determining reference angles on the unit circle given an original angle measurement
- sketching reference angles on a unit circle
- applying patterns of special right triangles to determine coordinates on the unit circle
- simplifying rational expressions containing radicals
- calculating the difference of the distance between two angles on the unit circle in the context of a real-world scenario

Introduction

This task should be used to explore or to apply the skill of explaining how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle. Students should already be familiar with finding the coordinates of the point where the terminal side of an angle intersects the unit circle, finding reference angles within the unit circle, and finding the sine and cosine of special angles.

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Implementation Guide

Begin by reading the problem and clarifying the meanings of *cosine*, *reference angle*, *sine*, *terminal side*, and *unit circle*.

cosine	trigonometric function of an acute angle in a right triangle that is the ratio of the length of the side adjacent to the length of the hypotenuse
reference angle	acute angle that the terminal side makes with the *x*-axis. The sine, cosine, and tangent of the reference angle are the same as that of the original angle (except for the sign, which is based on the quadrant in which the terminal side is located).
sine	trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the hypotenuse
terminal side	for an angle in standard position, the movable ray of an angle that can be in any location and which determines the measure of the angle
unit circle	circle with a radius of 1 unit. The center of the circle is located at the origin of the coordinate plane.

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine the distance between Reid and Kip's seat and Veronica and Noelle's seat on the Ferris wheel. To accomplish this task, they will first recognize that the Ferris wheel can be represented by a unit circle, because the radius of the Ferris wheel is a unit of one. Then they will determine the locations of each pair of friends' seats by identifying the angle measurements representing the locations on the Ferris wheel. The seat locations will be determined by the degree measurement after the ride has rotated 120° counterclockwise. They will then use reference angles and the pattern for a 30°–60°–90° triangle in order to find the coordinate which represents the height of each pair's seat in reference to the center of the Ferris wheel. Finally, they will subtract the smaller height from the larger height to find the distance from Veronica and Noelle's seat to Reid and Kip's seat.

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they recognize that the scenario of the Ferris wheel can be represented by a unit circle. They will interpret the locations of both Reid and Kip's seat and Veronica and Noelle's seat on the Ferris wheel as angle measurements on the unit circle. They will also reason that reference angles can be found for the angles representing these locations, and this will allow them to use the patterns of special right triangles, specifically

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a 30°–60°–90° triangle, in order to determine the height of each seat in reference to the center of the Ferris wheel. They will reason quantitatively as they determine the height of each seat's location by finding the sum of the distances of each seat above the center of the Ferris wheel and the radius of the Ferris wheel. They will also perform calculations involving rational expressions with radicals in order to find the distance from Reid and Kip's seat to Veronica and Noelle's seat.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure of the unit circle as well as apply the pattern of a special right triangle in order to accomplish the goal of the task. They will recognize that the Ferris wheel can be modeled with a unit circle, since its radius has a unit measurement of one, and the locations of the seats of the pairs of friends can be represented by angle measurements on the unit circle. They will use the pattern for a 30°–60°–90° triangle to find the heights of the seats in reference to the center of the Ferris wheel and the height from the bottom point of the Ferris wheel.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- forgetting to account for a negative $\cos\theta$ or $\sin\theta$ when using reference angles

  Encourage students to have a copy of the unit circle in front of them when working through the task. Remind them of the sign designations for each quadrant in the unit circle.

- thinking that, for a 30°–60°–90° triangle, the hypotenuse is $\sqrt{3}$ and the longest leg is 2

  Have students draw the diagram of the side lengths for a 30°–60°–90° triangle on the top of their papers. Remind them that the hypotenuse of any right triangle must be longer than either of the legs, and 2 is larger than the value of $\sqrt{3}$. Therefore, it is not possible for $\sqrt{3}$ to be the length of the hypotenuse.

- switching the coordinates of the point where the terminal side of the angle intersects the unit circle

  Remind students that the coordinates of the point where the terminal side intersects the unit circle are always $(\cos\theta, \sin\theta)$. Encourage them to remember that the word "cosine" comes before the word "sine" alphabetically, so "cos" comes before "sin" in the listing of the coordinates.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, review the meaning of reference angles and how they are applied in a unit circle. Ask them, "What is a reference angle, and how do you find the reference angle for an original angle for each quadrant?" (**Answer:** A reference angle is an acute angle that the terminal side makes with the *x*-axis. For Quadrant I, the reference angle is the same as the original angle. For Quadrant II, to find the reference angle, subtract the original angle from 180°. For Quadrant III, to find the reference angle, subtract 180° from the original angle. For Quadrant IV, to find the reference angle, subtract the original angle from 360°.)
- Before students begin the task, ask them, "What other words do you know that begin with the prefix 'deca', and what do these words mean?" (**Answer:** Answers may vary. Some common words are: decade, which means ten years; decagon, which is a polygon with ten sides and ten interior angles, and decathlon, which is a competition with events.)
- As students are working through the task, ask them, "Why is it necessary to find the sine of θ, instead of the cosine of θ, when finding the approximate height of the seats above or below the *x*-axis?" (**Answer:** The coordinates of the points that represent the locations of the seats have the format (cos θ, sin θ). The sine of θ represents the *y*-coordinate, and the *y*-coordinate represents the height of the seats since it is the vertical distance on the coordinate plane.)
- As students are working through the task, ask them, "Based on the angles which represent the locations of the two seats on the Ferris wheel after they have been rotated 120° counterclockwise, how do you know that you will use the pattern for the 30°–60°–90° triangle, and not the pattern for the 45°–45°–90° triangle?" (**Answer:** When the two seats on the Ferris wheel are rotated 120° counterclockwise, the angles representing the new locations of the seats are 120° and 210°. Since 120° is located in Quadrant II, then the reference angle is found by subtracting 120° from 180°, which is 60°. The other angle is 210°, which is located in Quadrant III. This reference angle is found by subtracting 180° from 210°, which is 30°. Therefore, these angles representing the new locations of the seats will follow the pattern of a 30°–60°–90° triangle.)
- As students are determining the coordinates representing the height of each seat above or below the *x*-axis, ask them, "Why is it necessary to add the value of the radius of the Ferris wheel to the value above the *x*-axis for Veronica and Noelle's seat?" (**Answer:** The question asks for the approximate height of the seats, which implies the height above the ground. The value found for the sine of θ for a 120° angle, $\frac{\sqrt{3}}{2}$, is the height of the seat above the *x*-axis. The radius of the Ferris wheel, which is 1, represents the distance from the *x*-axis to the ground. Therefore, to find the total height of the seat in reference to the ground, it is necessary to add both distances; $1 + \frac{\sqrt{3}}{2} = \frac{2+\sqrt{3}}{2}$ decameters.)

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- As an extension to the previous question, ask students, "Why is it necessary to subtract the value of the coordinate of sin θ from the radius of the Ferris wheel for the location of Reid and Kip's seat?" (**Answer:** The value found for the sine of the angle representing Reid and Kip's seat, which is 210°, is $\frac{1}{2}$. Since this angle is found in Quadrant III, it is below the *x*-axis. Therefore, to find the total height of the seat in reference to the ground, $\frac{1}{2}$ is subtracted from the radius value of 1: $1-\frac{1}{2}=\frac{1}{2}$ decameter.)
- Ask students, "Why are the values representing the heights of each pair of friends' seats only approximate heights? What information could have been given that would allow for the exact heights to be found?" (**Answer:** The values representing the heights of each pair of friends' seats are only approximate heights because the Ferris wheel is not positioned directly on the ground. Therefore, there is some amount of measurable height from the ground to the bottom of the Ferris wheel. If this height had been given in the problem, then this value could have been added to the heights of the seats in order to get an exact value for the heights of the seats above the ground.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss why the Ferris wheel can be represented with a unit circle. Ask students to explain how they determined the reference angles for each original angle representing the locations of each pair of friends' seats on the Ferris wheel. Discuss why the 30°–60°–90° triangle pattern could be used, and also discuss why the only coordinate that was relevant was the *y*-coordinate, representing the sine value of the angle. Ask students to explain why it was necessary to add or subtract the radius of the Ferris wheel when determining the heights of each seat. Also ask students to explain how they simplified their rational expressions with the radical values. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to explain how the values of the coordinates on the unit circle were used to help them determine the height of each seat on the Ferris wheel. Discuss why the values found for the heights were approximate heights and not exact heights, based on the context of the scenario of the position of the Ferris wheel in relation to the ground. Focus on the use of precise mathematical language and clarity, specifically when discussing the components of the unit circle, such as the quadrants, the reference angles, and how the coordinates of the point where the terminal side intersects the unit circle are always (cos θ, sin θ). Remind students to include the unit of measurement (decameters) when stating their answers.

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Connecting to Key Concepts

Make explicit connections to key concepts:

- Recall the pattern for a 30°–60°–90° triangle states that the length of the hypotenuse is twice as long as the short leg, and the longer leg is equal to the length of the short leg times $\sqrt{3}$.

  In this task, students will use this special pattern in order to find the coordinates of specific angles on the unit circle that represent the location of seats on a Ferris wheel.

- The special angles continue around the unit circle and can be identified as all angles whose reference angles are special angles.

  Students will identify specific special angles of 120° and 210°, and they will use these special angles in order to find the lengths of the coordinates which represent the distances the Ferris wheel seats are above the center of the Ferris wheel, as well as the distances that two specific seats are from each other.

- To find the coordinates of the point where the terminal side of a special angle intersects the unit circle, first identify the reference angle.

  Students will determine the reference angles for the special angles 120° and 210°, and they will find the coordinates of these angles.

Extending the Task

To extend the task, ask students to work through the following problem:

> "Andre is waiting for his friends to finish riding a Ferris wheel that has a radius of 1 decameter (1 decameter = 10 meters). When Andre first looks over at the wheel, he sees that Veronica and Noelle are in a seat that is directly to the right of the center of the wheel. His other friends, Reid and Kip, are in a seat that is directly to the left of the center of the Ferris wheel. A minute later, he sees that the wheel has rotated exactly 135° counterclockwise as additional passengers are loaded. At this point, what is the distance from Reid and Kip's seat to Veronica and Noelle's seat? Give an exact answer."

Ask for volunteers to share their results, specifically the steps and procedures used throughout the problem. Discuss the calculations involved in determining the reference angles in Quadrants II and IV, and discuss finding the value of the sine of θ for a 45°–45°–90° triangle. Also discuss how the negative value of the sine value in Quadrant IV affects the outcome of the problem. Encourage students to discuss any difficulties they had when working through the task again with different values.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the task is to determine the distance between Reid and Kip's seat and Veronica and Noelle's seat on the Ferris wheel. To accomplish this task, I first recognized that the Ferris wheel could be represented by a unit circle, because the radius of the Ferris wheel is a unit of one. Then I determined the locations of each pair of friends' seats by identifying the angle measurements representing the locations on the Ferris wheel. The seat locations were determined by the degree measurement after the ride was rotated 120° counterclockwise. I then used reference angles and the pattern for a 30°–60°–90° triangle in order to find the coordinate which represented the height of each pair's seat in reference to the center of the Ferris wheel. Finally, I subtracted the smaller height from the larger height to find the distance from Veronica and Noelle's seat to Reid and Kip's seat.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first recognized that the scenario of the Ferris wheel could be represented by a unit circle. I interpreted the locations of both Reid and Kip's seat and Veronica and Noelle's seat on the Ferris wheel as angle measurements on the unit circle. I also reasoned that reference angles could be found for the angles representing these locations, and this allowed me to use the patterns of special right triangles, specifically a 30°–60°–90° triangle, in order to determine the height of each seat in reference to the center of the Ferris wheel.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I reasoned quantitatively as I determined the height of each seat's location by finding the sum of the distances of each seat above the center of the Ferris wheel and the radius of the Ferris wheel. I also performed calculations involving rational expressions with radicals in order to find the distance from Reid and Kip's seat to Veronica and Noelle's seat.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I looked for and made use of structure when solving this problem by applying the structure of the unit circle as well as applying the pattern of a special right triangle in order to accomplish the goal of the task. I recognized that the Ferris wheel could be modeled with a unit circle, since its radius has a unit measurement of one, and the locations of the seats of the pairs of friends could be represented by angle measurements on the unit circle. I used the pattern for a 30°–60°–90° triangle to find the heights of the seats in reference to the center of the Ferris wheel and the height from the bottom point of the Ferris wheel.)

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Alternate Strategies or Solutions

- Some students may have memorized the cosine and sine values for the special angles on a unit circle and may not need to sketch a right triangle in order to determine the requested values. Remind students to be mindful of the signs of the values in each quadrant, as well as being careful not to switch the values for the cosine and sine.
- Some students may choose to sketch a diagram of the scenario in order to better understand the locations of each pair of friends' seats before and after the Ferris wheel rotated. Encourage students to carefully label the angles on the diagram, as well as the unknown parts on the diagram.

Technology

Students can use a scientific calculator for this task.

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Problem-Based Task 3.2.1: Count the Cost

Common Core State Standards

G–SRT.9 (+) Derive the formula $A = 1/2\ ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

G–SRT.10 (+) Prove the Laws of Sines and Cosines and use them to solve problems.

Task Overview

Focus

How can trigonometry be applied to calculate the perimeter and area of a triangle? How can the Law of Sines be applied to determine missing side lengths of a triangle? In this task, students will apply the formula for the perimeter and area of a triangle and the Law of Sines in order to calculate the total amount of money needed to put fencing around and plant grass inside of a triangular plot of land.

This activity will provide practice with:

- applying the Law of Sines to determine a missing side length
- sketching and labeling the sides and angles of a triangle on a diagram
- evaluating trigonometric functions with a calculator
- calculating the perimeter and area of a triangle
- estimating costs of items
- interpreting calculation results in the context of a real-world scenario

Introduction

This task should be used to explore or to apply the skill of using the Law of Sines to solve problems. Students should already be familiar with finding the sine of an angle, using the sine function to find a missing side or angle of a triangle, and finding the perimeter and area of a triangle.

Begin by reading the problem and clarifying the meaning of *Law of Sines* and *sine.*

Law of Sines a formula for any triangle which states $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, where a represents the measure of the side opposite $\angle A$, b represents the measure of the side opposite $\angle B$, and c represents the measure of the side opposite $\angle C$

sine a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the hypotenuse

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine Mr. Soliz's total cost of building a pen around a triangular-shaped plot of land in order to enclose his sheep and goats, as well as planting grass inside the pen. To accomplish this task, they will recognize that first it will be necessary to find the perimeter and area of the triangular-shaped pen. These parameters will be found by applying the Law of Sines in order to find the missing sides length of the triangle. Once this third side length has been determined, this value will be used to calculate the perimeter and the area of the triangle. Finally, the cost of the fencing will be found by multiplying the perimeter by the cost per foot of fencing, and the cost of the grass will be found by multiplying the area by the cost per square foot of grass. The sum of these two costs is the total cost for Mr. Soliz's project.

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they interpret the triangular diagram given in the task and recognize the necessity of determining the value of the third side of the triangle. They will identify that the given triangle is not a right triangle, and they will recognize that the Law of Sines can be applied to find the missing side length. They will also reason abstractly as they conclude that the area and the perimeter must be found in order to calculate the total cost of building the pen for the sheep and goats. They will reason quantitatively as they substitute values of side lengths and angles into the formula for the Law of Sines in order to determine the missing side length. They will also reason quantitatively as they calculate the cost for the fencing, as well as the cost for the grass, and finally the total cost for the entire project of building a triangular-shaped pen for the sheep and goats.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled by using geometric concepts such as area and perimeter, as well as by using trigonometric functions and formulas. They will recognize that the side lengths of the triangle represent the three side lengths of the fencing that will enclose a plot of land used as a pen for sheep and goats. They will also recognize that the area inside the triangular-shaped plot of land represents the space for the grass that will be grown in the pen. In order to accomplish the goal of the task, they will apply trigonometry and evaluate trigonometric functions for angles in the Law of Sines formula, and they will solve an equation from this formula in order to determine the missing side length of the triangular-shaped pen.

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Implementation Guide

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- neglecting to ensure the calculator is in the proper mode (degrees or radians)

  Remind students that to evaluate the sine function for the given angles in the problem, the calculator mode must be set to degrees. Encourage them to check the display on their calculators before evaluating the sine of an angle.

- incorrectly substituting measures for sides and angles into a fraction or formula

  Have students write the Law of Sines and the formula for the area of a triangle at the top of their papers. Remind them to carefully label their diagram and to label the corresponding values with the three sides lengths; a, b, and c.

- forgetting to calculate the cost of the fencing and the cost of the grass after finding the perimeter and the area of the triangle

  Remind students that the goal is to find the total cost of building fencing around and planting grass inside a triangular-shaped plot of land, so the answer must be in monetary units. Remind them that the values for the perimeter and area of the triangle must be multiplied by their respective costs per foot and per square foot to determine the cost of the project.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them to classify the triangle which represents the pen. Ask them, "What type of triangle does the sheep and goats pen represent – an acute, an obtuse, or a right triangle? Explain your answer." (**Answer:** The two given angles are 65° and 33°. The third angle is calculated by first finding the sum of the two given angles and subtracting the sum from 180°: 65 + 33 = 98; 180 – 98 = 82. The third angle of the triangle measures 82°. Since all three angles are less than 90°, the triangle is an acute triangle.)

- Before students begin the task, ask, "If $\angle C$ had been labeled as having a measure of 25° instead of the given value of 33°, and the value of side c was changed accordingly, what alternate method could you use to solve for the missing side length? Explain." (**Answer:** If $\angle C$ had a measure of 25° instead of 33°, then the measure of $\angle A$ would be 90°; therefore the triangle would be a right triangle, and the Pythagorean Theorem could be used to solve for the missing side length.)

- As students begin working on the task, ensure their conceptual understanding of the Law of Sines. Ask them, "In terms of ratios, what does the Law of Sines formula, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, represent in terms of the relationships of the sides and angles of a triangle?" (**Answer:** The Law of Sines states that the ratio of the length of side a to the value of the sine of $\angle A$ is equal to the ratio of the length of side b to the value of the sine of $\angle B$, which is also equal to the ratio of the length of side c to the value of the sine of $\angle C$.)

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- Before students apply the Law of Sines to find the value of b, ask them, "What values must be known in order to apply the Law of Sines to determine a missing side length or angle of a triangle? What values are known in this task?" (**Answer:** Since a pair of ratios contains a total of 4 components, or values, three out of the four values must be known in order to solve for one unknown side length or angle measure. In this task, two of the angle measures are known, $\angle B$ and $\angle C$, and the length of side c is known. Therefore, the Law of Sines can be used to find the length of side b.)
- As students are working through the task, ask them to consider a different pair of ratios to use in the Law of Sines. Ask them, "Instead of using the ratios of $\frac{b}{\sin B} = \frac{c}{\sin C}$, which other pair of ratios could be used to solve for the missing side length? Explain." (**Answer:** The other pair of ratios that could be used to solve for the missing side length is $\frac{a}{\sin A} = \frac{b}{\sin B}$. Although the measure of $\angle A$ is not given in the original diagram, it can be calculated by finding the sum of angles B and C and subtracting the sum from 180°. Then, both the values of the length of side a and $\angle A$ could be substituted into the Law of Sines formula.)
- As students are applying the Law of Sines and determining the missing side length, ask them, "Why do you think it makes sense to round the value of b to the nearest whole number, in terms of the values of the other two side lengths?" (**Answer:** Although the actual value of side b is given as 49.92156615 in the calculator, the other two given side lengths have already been rounded to the nearest whole number, so the third side length should be applied with the same place value as the other two lengths.)
- As an extension to the previous question, ask students, "If the value of b was given as 49.25, why would you round up to 50 instead of rounding down to 49?" (**Answer:** In the context of the scenario, if the value of 49.25 was rounded down to 49, then there would not be enough fencing material to represent this side length. By rounding up to 50, there will be more than enough, which is a better scenario than not having enough material to enclose the triangular-shaped pen.)
- As students are finding the estimated costs of the fencing and grass needed for the pen, ask them, "Why is it necessary to use the words "estimated" and "approximately" when discussing the costs of the fencing and the grass?" (**Answer:** Since the value for the missing side length, b, was rounded, it does not represent the exact value of side b. As a result when the rounded value of b is used in the calculations to find the area of the triangular-shaped pen the cost of the fencing and the grass will not be an exact amount.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask for volunteers to discuss how to accomplish the task. Discuss why it was necessary to find both the area and the perimeter of the pen to determine the final cost of the project. Ask students how they concluded that the given triangle was not a right triangle, and therefore applying the Law of Sines was necessary in order to find the missing side length of the triangle. Also ask students to explain the components of the Law of Sines formula, as well as why the ratios involving sides b and c were used in this scenario. Discuss how the values for the perimeter and area were used to estimate the costs of the fencing and grass needed, and also how the total cost of the project was determined. Encourage students to discuss any difficulties or confusion they experienced when working through the task.
- Compare students' strategies and ways of justifying responses. Ask students to explain what values of a non-right triangle must be known in order to correctly apply the Law of Sines. Also ask them to explain how they algebraically manipulated the equation in order to isolate the missing side length, b. Discuss possible errors when finding the length of b, as well as possible errors when calculating the cost of the fencing or the grass needed for the pen. Focus on the use of precise mathematical language and clarity, specifically when discussing the relationships between the sides of the triangle and their corresponding opposite angle measures. Also discuss the how the terms "per foot" and "per square foot" relate to the perimeter and area in terms of finding the total cost of the pen.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The **Law of Sines** states $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, where a represents the measure of the side opposite angle A, b represents the measure of the side opposite angle B, and c represents the measure of the side opposite angle C.

  In this task, students will apply the Law of Sines in order to accomplish the goal of the task, as they represent the scenario with a diagram of three sides and angles labeled.
- The Law of Sines is true for any triangle, not just a right triangle.

  Students will evaluate the angle measures of the given triangle representing the plot of land used, and they will conclude that the triangle is not a right triangle. They will apply the Law of Sines to solve the problem.
- To find an unknown angle or side using the Law of Sines, substitute in the known information; eliminate the fraction that is incomplete and will not help to solve the equation; then, solve the resulting equation.

  Students will first substitute in the known values of 65° and 33° for the angles, and b and 30, respectively, for the sides of the triangle. Then they will perform the steps necessary to eliminate the fraction formed by the ratio and solve the equation in order to find the missing side, b, of the triangle.

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- The area of any triangle can be found using the formula $\frac{1}{2}ab\sin C$, in which angle C is the included angle in between the sides with lengths a and b. This formula can be used when two sides and the included angle are known.

  In order to find the area of the triangular-shaped pen, students will apply the area formula by substituting in the known values for the two side lengths, 30 and 55, and also 65° for the value of the included angle, C.

Extending the Task

To extend the task, ask students to recreate the given task by first drawing an obtuse triangle on their paper, using a straightedge. Ask them to draw a large enough triangle such that the side lengths are measured in centimeters. Ask them to only measure and label two sides of the triangle. Then, ask them to use a protractor and measure and label two of the three angles. Once each student has created a triangle with the requested dimensions labeled, ask them to switch their triangles with another student. Ask each student to use the Law of Sines in order to calculate the missing side length of the triangle, and then calculate the perimeter and area of the triangle. Ask for volunteers to share their triangles and discuss the procedures and calculations used to determine the perimeter and area of the triangle. Discuss any errors that occurred, and provide the opportunity for students to help with the corrections.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of this problem by first recognizing that the goal of the task is to determine Mr. Soliz's cost of building a pen to keep his sheep and goats, as well as planting grass inside the pen. To accomplish this task, I recognized that first it was be necessary to find the perimeter and area of his triangular-shaped pen. I found these parameters by applying the Law of Sines in order to find the missing side length of the triangle. Once I found this side length, I used this value to calculate the perimeter and the area of the triangle. I found the cost of the fencing by multiplying the perimeter by the cost per foot, and I found the cost of the grass by multiplying the area by the cost per square foot of grass. Then I summed these two costs, which is the total cost for Mr. Soliz's project.)

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- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I recognized the necessity of determining the value of the third side of the triangle. I identified that the given triangle is not a right triangle, and I recognized that the Law of Sines could be applied to find the missing side length. I also reasoned abstractly as I concluded that the area and the perimeter must be found in order to calculate the total cost of building the triangular-shaped pen for the sheep and goats.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I reasoned quantitatively as I substituted values of side lengths and angles into the formula for the Law of Sines in order to determine the missing side length. I also reasoned quantitatively as I calculated the cost for the fencing, as well as the cost for the grass, and finally the total cost for the entire project of building a triangular-shaped pen for the sheep and goats.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by recognizing that the scenario can be modeled by using geometric concepts such as area and perimeter, as well as by using trigonometric functions and formulas. I recognized that the side lengths of the triangle represented the three side lengths of the fencing that enclose a plot of land used as a pen for sheep and goats. I also recognized that the area inside the triangular-shaped plot of land represented the grass that will grow in the pen. In order to accomplish the goal of the task, I applied trigonometry and evaluated trigonometric functions for angles in the Law of Sines formula, and I solved an equation from this formula in order to determine the missing side length of the triangular-shaped pen.)

Alternate Strategies or Solutions

Some students may choose to calculate the measure of $\angle A$ at the beginning of the task. As a result, they may use the values of the side length of a and the measure of $\angle A$ in the Law of Sines formula, in conjunction with the unknown side length of b and the known measure of $\angle B$, instead of the values for side c and $\angle C$, as used in the task. Remind them that as long as three out of four values are known, then it does not make a difference if the value for side a or the value of side c is used.

Technology

Students can use scientific calculators for this task.

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Problem-Based Task 3.2.3: Safe or Out?

Common Core State Standard

G–SRT.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Task Overview

Focus

How can trigonometry be applied to solve for unknown lengths of oblique triangles in real-world applications? How can the Law of Cosines be applied to determine missing side lengths of a triangle? In this task, students will apply the Law of Cosines in order to calculate the distance from the pitcher's mound to first base and use this value to determine if a runner is safe or out at first base.

This activity will provide practice with:

- drawing a diagram to represent a real-world scenario
- applying the Law of Cosines to determine a missing side length
- determining a new angle measure after bisecting the original angle
- evaluating trigonometric functions with a calculator
- calculating the value of an unknown variable in the formula $d = rt$
- interpreting calculation results in the context of a real-world scenario

Introduction

This task should be used to explore or to apply the skill of understanding and applying the Law of Cosines to find unknown measurements in right and non-right triangles. Students should already be familiar with finding the cosine of an angle, using the cosine function to find a missing angle or side of a right triangle, and applying the distance, rate, and time formula.

Begin by reading the problem and clarifying the meaning of *bisected angle* and *Law of Cosines.*

bisected angle	an angle that has been divided into two congruent angles, where each angle is half the measure of the original angle
Law of Cosines	a formula for any triangle which states $c^2 = a^2 + b^2 - 2ab\cos C$, where C is the included angle in between sides a and b, and c is the nonadjacent side across from $\angle C$

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine if a baseball player is safe or out at first base, which depends on if he made it back to first base before the ball that the pitcher threw to first base. To accomplish this task, they will recognize that first it will be necessary to find the distance from the pitcher's mound to first base. They will recognize that an oblique triangle is formed with the home plate, the pitcher's mound, and first base representing the vertices of the triangle. They will conclude that since the known values of the triangle are two sides and the included angle, they can apply the Law of Cosines in order to find the unknown distance from the pitcher's mound to first base. Then, they will use the formula for speed ($d = rt$) and compare the time it takes the runner to get back to first base to the time it takes the pitcher to throw back to first base, in order to determine if the runner is safe or out.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for determining whether or not the runner is safe or out at first base. They will base their arguments on the comparison of the time it takes the runner to run back to first base and the time it takes the pitcher to throw the ball to first base. To find the missing time value, they will first apply the Law of Cosines to find the distance from the pitcher's mound to first base, and then they will use the distance = rate • time formula in order to find the value of the time it takes the pitcher's throw to reach first base, since the time it takes the runner to make it back to first base is given in the original problem. They will present their arguments to others and explain their conclusion based on the steps and procedures they followed in order to determine the necessary values.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure and procedures involved in using the Law of Cosines and the distance = rate • time formula in order to accomplish the goal of the task. They will recognize that the structure of the oblique triangle formed within the baseball diamond contains two known side lengths and an included angle. They will use the structure of the Law of Cosines formula to find the unknown value of the distance from the pitcher's mound to first base. This value will be used in the distance = rate • time formula so that the time that it took the pitcher to throw the ball back to first base can be found.

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Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- mistakenly using the Law of Sines instead of the Law of Cosines

  Have students write down the conditions for the use of both the Law of Sines and the Law of Cosines. Remind them that the Law of Cosines is used when two sides of a triangle are known, as well as the angle in between the sides (the included angle). Point out on the diagram how the 45° angle is located in between the two known distances in the triangle.

- incorrectly drawing or labeling diagrams

  Remind students to carefully draw the diagram in parts. Have them draw a diamond first, and then encourage them to note each base of the baseball diamond at the corners of the diamond, including home plate. Also encourage them to draw any type of picture, or object that will represent the bases and pitcher's mound. Have them compare their diagrams with other students to ensure their values are labeled correctly on the diagram.

- neglecting to ensure the calculator is in the proper mode (degrees or radians)

  Remind students that in order to evaluate the cosine function for the given angle in the problem, the calculator mode must be set to degrees. Encourage them to check the display on their calculators before evaluating the cosine of the angle.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, review the properties of a square with them. Ask them, "The task states that a baseball diamond is a square. What are the properties of a square and its diagonals that could be helpful when working through this task?" (**Answer:** A square has four congruent sides and four 90° angles. The diagonals of a square bisect each other, and they bisect each of the four angles of a square, which creates two 45° angles at each corner.)

- Before students begin the task, ensure their understanding of the distinction between the Law of Sines and the Law of Cosines. Ask them, "In what scenarios is the Law of Sines applicable in a problem, and in what scenarios is the Law of Cosines applicable, in terms of known values?" (**Answer:** The Law of Sines should be used when two angles and a side are known (ASA or AAS) or two sides and the non-included angle are known (SSA). The Law of Cosines must be used when two sides and the included angle are known (SAS) or when three sides are known (SSS).)

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- After students have drawn the diagram of the baseball diamond and have labeled the components, ask them, "Can you assume that the pitcher's mound is located directly in the center of the baseball diamond? What other distances must be known in order to prove if the pitcher's mound is in the center of the baseball diamond or not?" (**Answer:** It cannot be assumed that the pitcher's mound is located directly in the center of the baseball diamond, because the diagram does not give enough information. In order to determine this, the distances from the pitcher's mound to first base, second base, and third base must be known. If these distances are all the same, then it can be concluded that the pitcher's mound is located directly in the center of the baseball diamond. This point would be the intersection point of the diagonals, and the diagonals of a square bisect each other.)
- As students begin the task, ask them, "What information is provided that indicates that the Law of Cosines will be applied to find the missing distance from the pitcher's mound to first base?" (**Answer:** The Law of Cosines is applied when either two sides and the included angle are known, or when all three sides are known. In the given scenario, the distance from home plate to first base is known, as well as the distance from home plate to the pitcher's mound. The angle between the line from home plate to first base and the line from home plate to the pitcher's mound is the included angle, because it is the angle between the two known side lengths of the triangle.)
- After students have calculated the distance between the pitcher's mound and first base, ask them, "You have found the distance between the pitcher's mound and first base to be approximately 63.717 feet. Why do you think this number has not been rounded to the nearest whole number?" (**Answer:** The value for the distance between the pitcher's mound and first base is used in the next component of the task, in the distance = rate • time formula. Since the time is represented in terms of seconds, the more precise the distance is (in other words, the more digits in the value that are used), the more accurate and precise the value for the time will be.)
- Ask students to recall their knowledge about right triangles and the Pythagorean Theorem. Ask them, "What distance can the Pythagorean Theorem be used to find in the baseball diagram? Explain." (**Answer:** Since the baseball diamond is a square, each corner measures 90°. The distance between home plate and second base, segment *AC*, can be found using the Pythagorean Theorem. The two triangles formed by this distance are triangles *ABC* and *ADC*. Angles *B* and *D* are 90°, so therefore the triangles are right triangles, with congruent side lengths of 90 feet. Segment *AC* represents the hypotenuse of triangles *ABC* and *ADC*.)
- Ask students, "How can the Law of Sines be applied to find either the measure of $\angle DEC$ or $\angle CDE$, and why can this be applied?" (**Answer:** Once the value for the distance between the pitcher's mound and first base has been found, then the Law of Sines can be applied using the known angle of 45°. Now two sides and one non-included angle are known, which is a condition of using the Law of Sines.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss how the diagram was drawn to accurately represent the scenario of the baseball diamond and the bases, as well as how an oblique triangle was formed within the diamond to further represent the scenario. Ask students to explain why the Law of Cosines was used to solve for the missing side length representing the distance from the pitcher's mound to first base, and not the Law of Sines. Also ask them to explain how they used the value for this distance to find the amount of time it took the pitcher to throw the ball to first base, using the formula $d = rt$. Discuss how the final conclusion was reached about whether or not the runner was safe or out at first base. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to explain the difference between when the Law of Sines is applicable in solving a real-world application problem with oblique triangles, and when the Law of Cosines is the correct choice instead. Discuss how the values for the time were compared in order to conclude if the runner was safe or out at first base. Focus on the use of precise mathematical language and clarity, specifically when discussing the relationships between the sides of the oblique triangle and the included angle. Also discuss how the variables in the Law of Cosines formula correspond to the parts of the triangle. Ensure the use of proper units when working through the task, specifically units of feet to represent distance, and units of seconds to represent time.

Connecting to Key Concepts

Make explicit connections to key concepts:

- In problems describing a real-world situation, if no picture is given, draw and label a diagram.

  In this task, students will interpret verbal information and translate it into a diagram to represent the scenario. They will carefully sketch and label all known values given in the problem.
- The Law of Cosines must be used when two sides and the included angle are known (SAS) or when three sides are known (SSS).

  Students will identify the necessity of using the Law of Cosines in this task because they are given the values of two side lengths of the triangle, as well as the included angle.
- The calculated answer to a real-world problem should always be examined to determine if the answer is reasonable given the context.

  Students will interpret and analyze the value of the distance from the pitcher's mound to first base, as well as the time it takes the ball to travel to first base. They will assess the reasonableness of these values in the context of the scenario.
- When appropriate, answers should include a unit label.

  Students will recognize that the side lengths of the triangle are represented in units of feet, and the time values are represented in units of seconds.

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Extending the Task

To extend the task, ask students to work with a partner to find the following missing lengths and angle measures on the baseball diamond diagram: $\overline{CA}$, $\overline{EA}$, $\overline{BE}$, $\angle AED$, and $\angle ADE$. Ask them to carefully note the steps and procedures used to find each piece of missing information. Also ask them to write down every calculation in detail and label each component with which formula was used. Ask for volunteers to share their results and discuss why the Law of Sines was used and/or the Law of Cosines was used for determining the missing side lengths or angle measures. Also discuss how the Pythagorean Theorem was used to find the distance from home plate to second base (or segment *CA*). Discuss any errors that occurred in the calculations, and provide the opportunity for students to help with the corrections.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the problem is to determine if a baseball player is safe or out at first base, which depends on if he made it back to first base before the ball that the pitcher threw to first base. To accomplish this task, I recognized that first it was necessary to find the distance from the pitcher's mound to first base. I recognized that an oblique triangle is formed with the home plate, the pitcher's mound, and first base representing the vertices of the triangle. I concluded that since the known values of the triangle were two sides and the included angle, I could apply the Law of Cosines in order to find the unknown distance from the pitcher's mound to first base. Then, I used the formula for speed ($d = rt$) and compared the time it took the runner to get back to first base to the time it took the pitcher to throw back to first base, in order to determine if the runner was safe or out.)
- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining my decision about whether or not the runner was safe or out at first base. I based my arguments on the comparison of the time it took the runner to run back to first base and the time it took the pitcher to throw the ball to first base. To find the missing time value, I first applied the Law of Cosines to find the distance from the pitcher's mound to first base, and then I used the distance = rate • time formula in order to find the value of the time it took the pitcher's throw to reach first base, since the time it took the runner to make it back to first base was given in the original problem. I presented my arguments to others and explained my conclusion based on the steps and procedures I followed to determine the necessary values.)

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- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I looked for and made use of structure when solving this problem by applying the structures and procedures involved in using the Law of Cosines and the distance = rate • time formula in order to accomplish the goal of the task. I recognized that the structure of the oblique triangle formed within the baseball diamond contained two known side lengths and an included angle. I used the structure of the Law of Cosines formula to find the unknown value of the distance from the pitcher's mound to first base. This value was used in the distance = rate • time formula so that the time that it took the pitcher to throw the ball back to first base could be found.)

Alternate Strategies or Solutions

Some students may choose to solve for the other missing lengths in the baseball diamond, such as the distance from the pitcher's mound to second base and third base. They may also find the distance between home plate and second base. Students may choose to identify triangle *BEC* as a congruent triangle to *DEC*. Remind them that if they choose to perform their calculations using triangle *BEC*, then they must use corresponding parts in order to find the distance from the pitcher's mound to first base, since that is the component of the task that is applicable for solving the problem.

Technology

Students can use scientific calculators for this task.

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Problem-Based Task 3.3.1: Perfect Pitch

Common Core State Standard

F–TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

Task Overview

Focus

How can the properties of trigonometric functions be applied to real-world scenarios? How can information about a specific sine function be obtained from the equation representing its function? In this task, students will analyze a sine function representing a note played on a piano, and they will determine a new equation for the function representing the note that is one octave higher than the original note.

This activity will provide practice with:

- interpreting and analyzing written information about periodic functions in the context of a real-world scenario
- substituting values into an expression
- determining the period and frequency of a sine function
- solving an equation for the period of a function
- creating an equation for a sine function given new values

Introduction

This task should be used to explore or to apply the skill of applying trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. Students should already be familiar with using radians to describe angle measure, and finding the sine, cosine, or tangent of an angle.

Begin by reading the problem and clarifying the meaning of *frequency*, *octave*, *period*, *pitch*, and *sine curve*.

frequency	the reciprocal of the period for a periodic function; indicates how often the function repeats
octave	an interval between the first note and the eighth note on a musical scale
period	in a cosine or sine function graph, the horizontal distance from a maximum to a maximum or from a minimum to a minimum
pitch	in music, how low or high a note is; measured in a unit called Hertz
sine curve	a curve with a constant amplitude and period, which are given by a sine or cosine function

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they interpret the written information given in the task and identify the steps necessary to solve the problem. They will recognize that they will first need to determine the period and frequency of the original note that Jillian played on the piano. They will reason that the general form of a sine function can be applied to find these values, given the equation representing the function of the original note played. They will also reason that the frequency and period of the note that is one octave higher will need to be found in order to create a new equation representing the function of this new note. They will reason quantitatively as they substitute values into the formula for the period and solve for the value of the variable, b. They will also reason quantitatively as they use this new value of b and create the equation representing the note that is one octave higher than the first note that Jillian played.

- **SMP 6:** Attend to precision.

  Students will use precision as they first determine the period and frequency of the first note that Jillian played on the piano. They will recognize that the frequency of the note that is one octave higher than the original note will have double the frequency of the first note. After they have determined the period and frequency of the note that is one octave higher, they will substitute the known values into the formula for the period and solve for the unknown value, b, which they will use to create the equation for the note that is one octave higher than the original note. Encourage students to use the correct terminology throughout the task, specifically when referring to the period and frequency of the original equation, as well as when creating the new equation.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure and procedures involved in using the general formula for the sine function, $f(x) = a \sin [b(x - c)] + d$ in order to accomplish the goal of the task. They will recognize that the variable b in the general formula represents the period of the function, and they will use this value in order to determine the frequency of the first note played. Then they will use the structure of the formula to determine the period and frequency of the note that is one octave higher. When they are creating the new equation that represents the note that is one octave higher than the first note played, they will follow the structure of the equation of the first note.

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Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- thinking that b is the value of the period

  Remind students that the value for the variable b is placed into the expression which represents the period, $\frac{2\pi}{b}$. Remind them to substitute the value of the coefficient of the variable in the equation for b in the formula for the period.

- confusing the period with the frequency, and vice versa

  Have students write the general form for the sine function at the top of their papers, labeling each of the components. Remind them that the period is represented by $\frac{2\pi}{b}$, and then once this value is found, then the frequency is the reciprocal of the value for the period.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "Besides sound waves, what other real-life applications do you know that use sine waves?" (**Answer:** Answers may vary. Some examples include: seismic waves representing earthquake activity, the periodic rotations of a crankshaft in an engine, the movement of a pendulum, a person's biorhythms, etc.)
- Before students begin the task, review the meanings of period and frequency with them. Ask them, "How would you describe what the period and frequency of a function mean?" (**Answer:** The period of a function is the distance required to complete one full cycle. The frequency of a function is how often the function repeats in a given amount of time.)
- As an extension to the previous question, ask students, "In general, what would it mean if a function has a high frequency?" (**Answer:** If a function has a high frequency, then the function repeats its wave cycle many times within a given time, or a given portion on a graph.)
- As students are working on the task, ask them to analyze the given equation representing the first note played on the piano and compare the equation to the general form of the sine function. Ask them, "The general form of a sine function is given as $f(x) = a \sin [b(x - c)] + d$. What components of this general form are missing from the given equation of $f(x) = \sin (880\pi t)$?" (**Answer:** The amplitude of a sine function is represented by the coefficient in front of the variable a. Since there is no coefficient given in the equation representing the first note, the amplitude is an understood value of 1. Since there is no value for the variables c or d, then there is no horizontal or vertical phase shift for the equation representing the first note played.)

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- As students are working through the task and finding the frequency for the first note played and the note played with a higher octave, discuss the units of measurement used for frequency. Ask them, "What are some other applications where the units of "Hertz" is used to measure something?" (**Answer:** Answers may vary. Some examples include: measuring radio waves, computer and other electronics, electric current, and water current.)
- As students are working to find the equation to represent the new note, ask them, "What does it mean that the frequency of the note that is one octave higher than the first note played is double the frequency of this first note?" (**Answer:** Doubling the frequency means that twice as many waves occur in a given period of time. When a new note is played that is an octave higher than the original note, the number of waves that were present with the first note within a given time period doubles for the new note.)
- Ask students to recall their prior knowledge about reciprocals. Ask them, "The period and frequency of a function are reciprocals of each other. In terms of the different types of slopes of lines, what types of lines have relationships with each other that are based on reciprocals?" (**Answer:** Two lines are perpendicular to each other if their slopes are negative reciprocals of each other.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss the components of the original function used to describe the first note that Jillian played on the piano. Ask students to explain the values given in the equation representing the function, including what properties are not shown with specific values. Discuss how the coefficient of the variable for t is substituted for b into the formula for the period, as well as the subsequent calculations used to find the period. Also discuss the relationship between the period and the frequency of the function. Ask students to explain how the frequency of the note that is one octave higher than the first note played is related to the original frequency of the first note played. Discuss how cross-multiplication is used to solve for the value of b in the new equation, as well as why the value of b is the only value that changed from the original equation to the new equation. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to explain the difference between the period and frequency of a function. Also ask them to explain how period and frequency relate to notes on a musical scale. Discuss how the values for the period and frequency were determined in both the original equation for the first note played, and

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also for the new equation representing the note an octave higher. Focus on the use of precise mathematical language and clarity, specifically when discussing the relationships between the period and the frequency of a function. Also discuss how the variable b is related to the expression used to describe the period, $\frac{2\pi}{b}$. Ensure the use of proper units when working through the task, specifically units of Hertz when referring to the frequency.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The **period** of a periodic function is the horizontal distance from a maximum to a maximum or from a minimum to a minimum.

  In this task, students will examine the period of a sine function in an equation representing a note played on a piano. They will also find the value of the period for a note that is one octave higher than the first note played.

- The **frequency** of a periodic function is the reciprocal of its period and indicates how often the function repeats.

  Students will examine the frequency of the function representing a note played on a piano. They will determine the value of the frequency by finding the reciprocal of the value of the period of the function.

- The general form of the **sine function** is $f(x) = a \sin [b(x - c)] + d$, where a is the amplitude, $\frac{2\pi}{b}$ is the period, c is the horizontal shift (also called the phase shift), and d is the vertical shift.

  Students will recognize that the format of the equation given to represent the first note played in the general form of the sine function. They will use this general form in order to determine the period and frequency of the first note played.

Extending the Task

- To extend the task, ask students to work through the task again, but this time ask them to create a new equation for a note played that is two octaves higher than the first note played. Then, ask them to create a new equation for a note played that is three octaves higher than the first note played. Ask for volunteers to share their new equations for each scenario, and ask them to explain how they determined the new period and frequency for the new notes. Discuss any errors that occurred in determining the new period and frequency or when creating the new equations, and provide the opportunity for students to help with the corrections.

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- Another option for extending the task is to ask students to create an example of a periodic sine function that features the amplitude, period, horizontal shift, and vertical shift. Once each student has created an equation representing the function, ask him/her to switch their equation with that of another classmate. Ask each student to identify the components of the given equation, and ask them to sketch a graph of the function. Ask for volunteers to share their answers and discuss the components of the equation, as well as the steps and procedures followed when graphing the equation. Discuss any errors that occurred in the identification of the components or the graphing, and provide the opportunity for students to help with the corrections.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I recognized that I first needed to determine the period and frequency of the original note that Jillian played on the piano. I reasoned that the general form of a sine function could be applied to find these values, given the equation representing the function of the original note played. I also reasoned that the frequency and period of the note that is one octave higher needed to be found in order to create a new equation representing the function of this new note.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I reasoned quantitatively as I substituted values into the formula for the period and solved for the value of the variable, b. I also reasoned quantitatively as I used this new value of b and created the equation representing the note that is one octave higher than the first note that Jillian played.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision as I first determined the period and frequency of the first note that Jillian played on the piano. I recognized that the frequency of the note that is one octave higher than the original note has double the frequency of the first note. After I determined the period and frequency of the note that is one octave higher, I substituted the known values into the formula for the period and solved for the unknown value, b, which I used to create the equation for the note that is one octave higher than the original note. I was careful to use the correct terminology throughout the task, specifically when I referred to the period and frequency of the original equation, as well as when I created the new equation.)

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- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I looked for and made use of structure when solving this problem by applying the structures and procedures involved in using the general formula for the sine function, $f(x) = a \sin [b(x - c)] + d$ in order to accomplish the goal of the task. I recognized that the variable b in the general formula represented the period of the function, and I used this value in order to determine the frequency of the first note played. Then I used the structure of the formula to determine the period and frequency of the note that is one octave higher. When I created the new equation that represented the note that is one octave higher than the first note played, I followed the structure of the equation of the first note.)

Alternate Strategies or Solutions

Some students may analyze the information given in the task and assume that the coefficient for the new equation will automatically be doubled because the frequency of the note played that is an octave higher than the first note played is doubled. Encourage students to follow the general format of the sine function and find the value of b using $\frac{2\pi}{b}$.

Technology

No technology is needed for this task.

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Common Core State Standards

Mathematics III

Integrated Pathway

Teacher Resource

Unit 4A

WALCH

INTEGRATED MATH

This program was developed and reviewed by experienced math educators who have both academic and professional backgrounds in mathematics. This ensures: freedom from mathematical errors, grade level appropriateness, freedom from bias, and freedom from unnecessary language complexity.

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Standards Correlations

Each lesson in this Integrated Pathway: Mathematics III program was written specifically to address the Common Core State Standards. Each lesson lists the standards covered in all the sub-lessons, and each sub-lesson lists the standards addressed in that particular section. In this section, you'll find a comprehensive list mapping the sub-lessons to the CCSS.

Guide to Common Core State Standards Annotation

As you use this program, you will come across a symbol included with the Common Core standards for some of the lessons and activities. These symbols are explained below.

Symbol: ⋆

Denotes: Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (⋆).

From http://www.walch.com/CCSS/00003

Symbol: (+)

Denotes: College and Career Readiness Standards

Advanced mathematics standards that are required in higher-level courses such as advanced statistics may also be included in lower-level courses. These additional standards are denoted by (+). According to the Common Core State Standards Initiative, "the evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6–8."

From http://www.walch.com/CCSS/00004

Connections to Future Courses

This section provides a map between topics introduced in each unit of this course and subsequent courses where each topic is revisited and built upon.

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CCSS INTEGRATED PATHWAY: MATHEMATICS III STANDARDS CORRELATIONS

Unit 4A: Mathematical Modeling of Inverse, Logarithmic, and Trigonometric Functions				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Inverses of Functions**			
	4A.1.1	Determining Inverses of Quadratic Functions	F–BF.4a	U4A-5
	4A.1.2	Determining Inverses of Other Functions	F–BF.4a	U4A-36
Lesson 2	**Modeling Logarithmic Functions**			
	4A.2.1	Logarithmic Functions as Inverses	F–BF.4a F–LE.4★	U4A-74
	4A.2.2	Common Logarithms	F–IF.8 F–LE.4★	U4A-96
	4A.2.3	Natural Logarithms	F–IF.8 F–LE.4★	U4A-114
	4A.2.4	Graphing Logarithmic Functions	F–IF.7e★	U4A-136
	4A.2.5	Interpreting Logarithmic Models	F–IF.4★ F–IF.5★ F–IF.6★	U4A-161
Lesson 3	**Modeling Trigonometric Functions**			
	4A.3.1	Graphing the Sine Function	F–IF.7e★	U4A-187
	4A.3.2	Graphing the Cosine Function	F–IF.7e★	U4A-230

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Standards Correlations

TOPICS FROM FUTURE COURSES: CCSS INTEGRATED PATHWAY MATHEMATICS III

Math III	Topic introduced	Course/unit where addressed	How addressed
Unit 4A	Finding the inverses of quadratic and other functions, including rational and trigonometric functions; making the connection between exponential functions and their inverses, logarithmic functions, as the method for solving exponential equations	Precalculus, Calculus, or Math IV	extended to complex logarithms and verifying inverses using compositions of functions
	Exploring the intervals where trigonometric functions are increasing and decreasing	Precalculus, Calculus, or Math IV	creating inverses of trigonometric functions

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Conceptual Activities

Use this interactive Desmos resource to build conceptual understanding of mathematical ideas. (*Note*: The link will be monitored and repaired or replaced as necessary.)

- Desmos. "Polygraph: Exponential & Logarithmic Functions."

  http://www.walch.com/ca/01045

  This activity is designed to spark vocabulary-rich conversations about exponential and logarithmic functions. Key vocabulary terms that may appear in student questions include *exponential*, *asymptote*, *logarithmic*, and *quadrant*.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Which equation is a quadratic function?

   a. $f(x)=\dfrac{(1+x)^3}{x+1}$

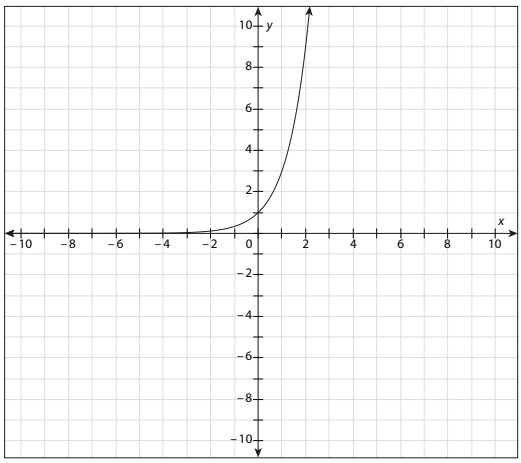
   b. $f(x) = (-3x^2)^0$

   c. $f(x) = 3x^{-2}$

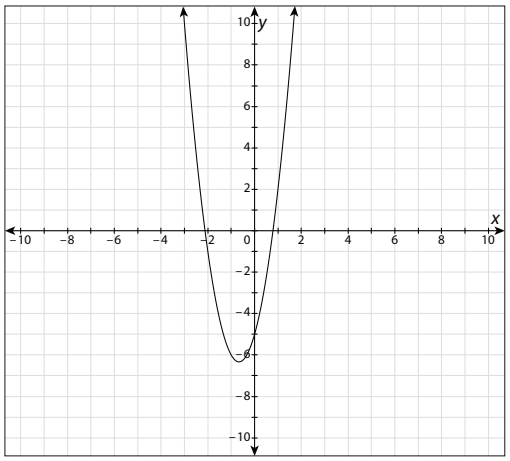
   d. $f(x)=5-\dfrac{4x}{3}$

2. Which graph shows a quadratic function?

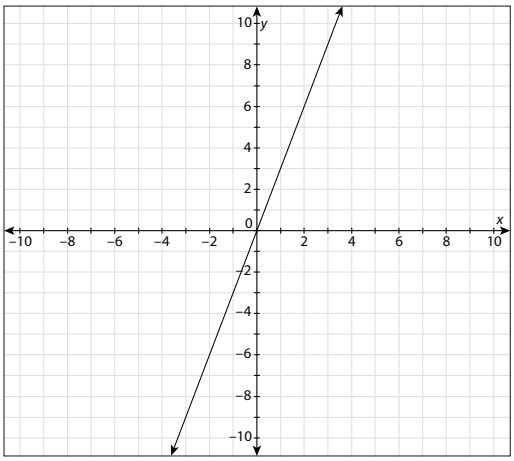
   a.



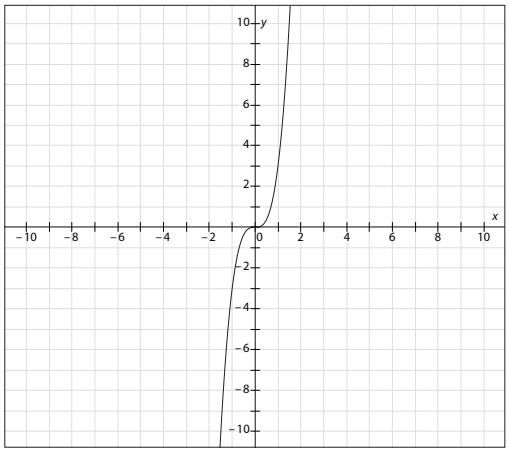
   c.



   b.



   d.



continued

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Assessment

3. Which table row represents data points on the graph of a quadratic function?

x	–2	–1	0	1	2
W	5	0	–3	–4	–3
X	0.04	0.2	1	5	25
Y	–32	–1	0	1	32
Z	–10	–5	0	5	10

a. W

b. X

c. Y

d. Z

4. What is the axis of symmetry of $f(x) = 9 - 4x - x^2$?

a. $x = 9$

b. $x = -\frac{1}{2}$

c. $x = -\frac{2}{9}$

d. $x = -2$

5. What is the domain of the quadratic function $g(x) = 9 - x^2$?

a. $[-3, 3]$

b. $(-\infty, +\infty)$

c. $[3, +\infty)$

d. $[9, -\infty)$

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Lesson 1: Inverses of Functions

Instruction

Common Core State Standard

F–BF.4 Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *or* $f(x) = (x + 1)/(x – 1)$ *for* $x \neq 1$.

SMP
1 ✓ 2 ✓
3 ✓ 4 ✓
5 6
7 ✓ 8 ✓

Essential Questions

1. How can you determine if the functions $f(x)$ and $g(x)$ are inverse functions of each other if $f(a) = b$ and $g(b) = a$?
2. How do the graphs of a function $f(x)$ and its inverse $f^{-1}(x)$ compare?
3. How do the domains and ranges of a function and its inverse compare?
4. How can you identify a function that does not have an inverse?

WORDS TO KNOW

domain	the set of all input values (x-values) that satisfy the given function without restriction
function	a relation in which every element of the domain is paired with exactly one element of the range; that is, for every value of x, there is exactly one value of y
inverse function	the function that may result from switching the x- and y-variables in a given function; the inverse of $f(x)$ is written as $f^{-1}(x)$
inverse relation	a relation $g(x)$ such that $g(f(x)) = x$ and $f(g(y)) = y$ where $f(x)$ is a function
one-to-one correspondence	the feature of a function whereby each value in the domain corresponds to a unique function value; that is, if $x = a$ and $x = b$, the two points would be $(a, f(a))$ and $(b, f(b))$, and if $a \neq b$, then $f(a) \neq f(b)$ for a function to exhibit one-to-one correspondence

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quadratic function	a function defined by a second-degree expression of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants. The graph of any quadratic function is a parabola.
range	the set of all outputs of a function; the set of y-values that are valid for the function
relation	a relationship between two variables in which at least one value of the domain or independent variable, x, is matched with one or more values of the dependent or range variable, y

Recommended Resources

- Analyzemath.com. "Inverse of Quadratic Functions."

  http://www.walch.com/rr/00227

  This site provides easy-to-follow, simple examples of how to find the inverse of a quadratic function with a restricted domain. *Note*: The shorthand "sqrt(x)" is used for $\sqrt{x}$.

- ChiliMath. "Finding the Inverse Function of a Quadratic Function."

  http://www.walch.com/rr/00228

  This tutorial covers how to quickly determine whether a quadratic function has an inverse, and if so, how to find the inverse. Example problems work through the solutions step-by-step.

- MathIsFun.com. "Inverse Functions."

  http://www.walch.com/rr/00229

  This site uses a step-by-step approach to explain and review inverse functions.

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Lesson 4A.1.1: Determining Inverses of Quadratic Functions

Warm-Up 4A.1.1

A flooring company engineer was asked to install a rectangular dance floor with a total area given by the quadratic function $A(x) = x(x - 8)$, where x represents the length of the floor in feet and $x - 8$ is the width. Pre-fabricated flooring panels can be ordered to accommodate different areas, so the engineer would like to know what the total areas could be for specific values of x.

1. Solve the quadratic function for x in terms of A, the total area of the dance floor. (*Hint*: Use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.)

2. What are the restrictions on the domain of x in the function $A(x) = x(x - 8)$? State the domain.

3. What are the restrictions on the domain of A in the function in problem 1?

4. What is a reasonable value for the length (x) of the dance floor, and what is the total area of a dance floor with that value of x?

5. What is a reasonable value for the area (A) of the dance floor, and what are the dimensions of that dance floor if the length is x and the width is $x - 8$?

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Instruction

Lesson 4A.1.1: Determining Inverses of Quadratic Functions

Common Core State Standard

F–BF.4 Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example, $f(x) = 2x^3$ or $f(x) = (x + 1)/(x – 1)$ for $x \neq 1$.*

Warm-Up 4A.1.1 Debrief

1. Solve the quadratic function for x in terms of A, the total area of the dance floor. (*Hint*: Use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.)

   Solve the quadratic function $A(x) = x(x – 8)$ for x using the quadratic formula by thinking of A, the area, as a constant: $A = x(x – 8)$. Recall that the standard form of a quadratic function is $f(x) = ax^2 + bx + c$, where $a \neq 0$.

   First, rearrange the function to write it in standard form:

$A = x(x – 8)$	Function with A in place of $A(x)$
$0 = x(x – 8) – A$	Subtract A from both sides.
$0 = x^2 – 8x – A$	Distribute.

   From this, we can determine the values to use for a, b, and c in the quadratic formula: $a = 1$, $b = –8$, and $c = –A$. Substitute these values into the formula.

   $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Quadratic formula

   $x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-A)}}{2(1)}$ Substitute 1 for a, –8 for b, and $–A$ for c.

   $x = \frac{8 \pm \sqrt{64 + 4A}}{2}$ Simplify.

   The solution for x written in terms of A is $x = \frac{8 \pm \sqrt{64 + 4A}}{2}$.

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Instruction

2. What are the restrictions on the domain of x in the function $A(x) = x(x - 8)$? State the domain.

   This is a real-world situation, so the dimensions in feet have to be positive. $A(x)$ represents the area of the floor, which must be greater than 0. Therefore, x must be greater than 8. The domain of x is given by $(8, +\infty)$.

   Note: Remember that parentheses around the domain values mean that the values go up to but do not include the values. Brackets mean that the values include the domain values. For example, the domain [1, 6] means that 1 and 6 are included in the domain.

3. What are the restrictions on the domain of A in the function in problem 1?

   This is a real-world situation, so the area has to be positive. The domain for A is given by $(0, +\infty)$.

4. What is a reasonable value for the length (x) of the dance floor, and what is the total area of a dance floor with that value of x?

   Different answers are possible, but all answers should have values of x greater than 8 feet, since the domain of x was found to be greater than 8 feet in problem 2. For example, if x is 40 feet, $x - 8$ would be 32 feet, and the area would be 1,280 square feet. A dance floor with a length of 40 feet and a width of 32 feet could reasonably be found in a large dance studio or hotel ballroom.

5. What is a reasonable value for the area (A) of the dance floor, and what are the dimensions of that dance floor if the length is x and the width is $x - 8$?

   Choose a reasonable value for the area; look back at the function for x in terms of A given by the quadratic formula in problem 1. Since this is a real-world situation, we will focus on the positive part of the $\pm$ symbol in front of the radical, as the negative part will result in negative values. That is, the function for x is given by $x(A) = \frac{8 + \sqrt{64 + 4A}}{2}$.

   Let's choose a value of 200 square feet for the area ($A = 200$). Substitute this value into the function found in problem 1 to find a reasonable value for x.

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Instruction

$x(A)=\dfrac{8+\sqrt{64+4A}}{2}$ Function for x in terms of A

$x(A)=4+\dfrac{1}{2}\sqrt{64+4A}$ Divide.

$x(200)=4+\dfrac{1}{2}\sqrt{64+4(200)}$ Substitute 200 for A.

$x(200)=4+\dfrac{1}{2}\sqrt{864}$ Simplify the operations under the radicand.

$x(200)=4+\dfrac{1}{2}(29.39)$ Simplify $\sqrt{864}$; round to the nearest hundredth.

$x(200)\approx 18.695\approx 18.7$ Simplify.

Solving this, we find that x is approximately equal to 18.7 feet, and $x - 8$ is approximately equal to 10.7 feet. The area would be $A \approx 18.7(10.7)$, which would equal 200.09 square feet, a value that is very close to our choice of 200 square feet as a reasonable value for A.

Connection to the Lesson

- Students will derive an inverse from a quadratic function.
- Students will compare the domains of a quadratic function and its inverse.
- Students will identify the real-world limitations on the domains and ranges of a quadratic function and its inverse.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- solving a quadratic function for its domain variable as one way to find the inverse of the quadratic function (A–CED.4)
- identifying the domain variables of a quadratic function (F–IF.2)
- identifying the domain of a quadratic function (F–IF.5)

Introduction

Several kinds of relationships exist between variables. For example, a **relation** is a relationship between two variables in which at least one value of the domain or independent variable, x, is matched with one or more values of the dependent or range variable, y. The **domain** refers to the set of all input values (x-values) that satisfy the given function without restriction and the **range** refers to the values of the dependent variable of the function. Sometimes, a relation is said to be a **function**, a relation in which every element of the domain is paired with exactly one element of the range; that is, for every value of x, there is exactly one value of y. Both relations and functions have inverses. In this section, we will explore how to determine the inverse of a quadratic function, as well as how to verify where the inverse exists. A **quadratic function** is a function defined by a second-degree expression of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants. The graph of any quadratic function is a parabola.

Key Concepts

- Determining the inverse of a function often requires examining the original function's domain and range.
- An **inverse relation** is a relation $g(x)$ such that $g(f(x)) = x$ and $f(g(y)) = y$ where $f(x)$ is a function.
- If the inverse relation $g(x)$ is also a function, it is said to be an **inverse function**. It is the function that results from switching the x- and y-variables in the given function, $f(x)$. The inverse of $f(x)$ is also written as $f^{-1}(x)$.
- Note that this $f^{-1}(x)$ notation does *not* mean to take the –1 power of $f(x)$. Both notations, $f^{-1}(x)$ and $g(x)$, are used to represent an inverse.
- A function $g(x)$ is the inverse of a function $f(x)$ if $g(f(x)) = x$ and $f(g(y) = y$; or, $g(x)$ is the inverse of $f(x)$ if $f(a) = b$ and $g(b) = a$.

SRB p. U4A-3

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Instruction

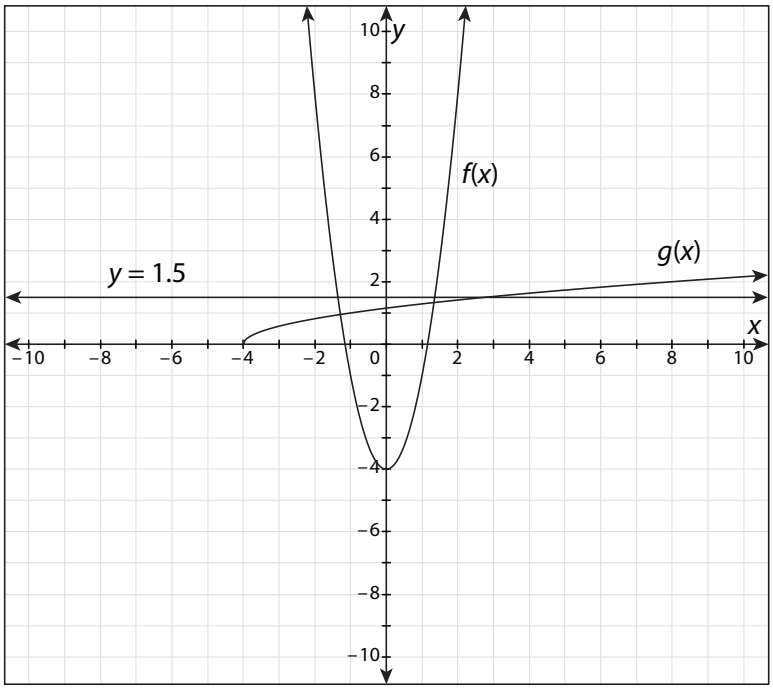
- For example, if $f(x) = x + 1$, then we can switch the domain and range variables (x and $f(x)$) and rename $f(x)$ as $g(x)$ to arrive at its inverse: the inverse of $f(x) = x + 1$ is $x = g(x) + 1$. Or, solved for $g(x)$, $g(x) = x - 1$. Notice that $f(x)$ becomes its inverse $g(x)$ once the variables are switched.
- For a function to have an inverse function, it has to satisfy the condition of **one-to-one correspondence**. This means that each value in the domain corresponds to a unique function value; that is, if $x = a$ and $x = b$, the two points would be $(a, f(a))$ and $(b, f(b))$; if $a \neq b$, then $f(a) \neq f(b)$ for a function to exhibit one-to-one correspondence. In other words, for a function to have an inverse, it cannot have the same range/output value for two different domain values. It is particularly important to check for one-to-one correspondence when finding the inverse of a quadratic function.
- Recall that a quadratic function is a second-degree function of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants.
- Since quadratic functions are parabolic, this means there is likely more than one output value for each input value, or more than one input value for each output, depending on the equation. For example, if $f(x) = x^2$, then values of $x = 2$ and $x = -2$ both result in $f(x) = 4$. The resulting points, (2, 4) and (–2, 4), do not have unique output values—they share 4 as the output. Therefore, determining the inverse of a quadratic function requires some additional steps.
- To construct an inverse relation or a possible inverse function for a quadratic function, switch the dependent and independent variables, x and y, and solve for the new function, y.
- For example, take the function $f(x) = 3x^2 - 4$. Switching the variables and renaming $f(x)$ as $g(x)$ results in $x = 3[g(x)]^2 - 4$. Simplifying and solving this new equation for $g(x)$ results in $g(x) = \pm\frac{\sqrt{3(x+4)}}{3}$.
- From the graph that follows, notice that the domain of the original function, $f(x) = 3x^2 - 4$, is $(-\infty, \infty)$, whereas the domain of the new inverse relation, $g(x) = \pm\frac{\sqrt{3(x+4)}}{3}$, is $[-4, \infty)$. (Also notice that $g(x)$ is called an inverse relation since we haven't determined whether it is a function.)

  The differing domains suggest that there is no inverse that exists over the entire domain of the original function, but there *is* an inverse over part of the domain: the interval $[-4, \infty)$. Thus for the function $f(x) = 3x^2 - 4$, the inverse function $g(x) = \pm\frac{\sqrt{3(x+4)}}{3}$ exists over the restricted domain $[-4, \infty)$. Recall the vertical line test to determine if a relation is a function. If the entire parabola for $g(x)$ were included, the relation $g(x)$ would not be a function.

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Instruction

- Another way to determine whether the inverse of a quadratic function exists is to graph the function and its possible inverse and determine whether the functions show one-to-one correspondence.
- You may recall using the vertical line test to determine if a relation is a function. Once a function passes the *vertical* line test, the *horizontal* line test can then be used to determine if the function is one-to-one; if the horizontal line passes through the function at more than one point, the function fails the test and is not one-to-one.
- To visualize this, look at the graph of $f(x) = 3x^2 - 4$ and its positive inverse, $g(x)=\frac{\sqrt{3(x+4)}}{3}$. (Note that the negative inverse, $g(x)=-\frac{\sqrt{3(x+4)}}{3}$, has been excluded from the graph.) A horizontal line has been drawn through the graph of $f(x)$ at $y = 1.5$.



- Notice that the horizontal line $y = 1.5$ intersects the original function at two points, which means that $f(x) = 3x^2 - 4$ does not have an inverse over its entire domain of $(-\infty, \infty)$. In other words, there is not one-to-one correspondence for each point on $f(x)$: the function values of $f(x)$ have *two* possible domain values over some parts of the domain.
- However, if the domain is restricted to the intervals $[0, \infty)$ or $[-4, 0]$, then $f(x)$ does have an inverse $g(x)$ because the domain of $g(x)$, $[-4, \infty)$, includes the restricted domains. Thus, it is possible for a quadratic function to have an inverse, given a restricted domain.

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Instruction

- A graphing calculator can be used to explore a quadratic function and the domain(s) over which its inverse exists.
- Note that a graphing calculator will not provide rigorous proof of the existence of an inverse and its domain. Furthermore, the horizontal line test is not as accurate a method of testing a function and its inverse for one-to-one correspondence. Therefore, when using a graphing calculator, it is necessary to check the results algebraically after plotting the functions.

On a TI-83/84:

Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

Step 2: At Y_1, enter the original function using your keypad. Press [ENTER].

Step 3: Enter the possible inverse function(s) using your keypad. *Note*:

When both positive and negative inverses are possible, each function needs to be entered separately to be graphed. For example, to graph $g(x)=\pm\frac{\sqrt{3(x+4)}}{3}$, enter $\frac{\sqrt{3(x+4)}}{3}$ at Y_2 and $-\frac{\sqrt{3(x+4)}}{3}$ at Y_3. Press [ENTER].

Step 4: To view a graph of $f(x)$ and its inverse $g(x)$, press [GRAPH].

Step 5: To view the table of values, press [2ND][GRAPH]. Take note of any undefined values for the functions.

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Instruction

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon, the second icon from the left, and press [enter].

Step 3: At *f1*(*x*), enter the original function using your keypad. Press [enter].

Step 4: To enter the possible inverse function(s), press [tab]. *Note*: When both positive and negative inverses are possible, each function needs to be entered separately to be graphed. For example, to graph $g(x)=\pm\frac{\sqrt{3(x+4)}}{3}$, enter $\frac{\sqrt{3(x+4)}}{3}$ at *f2*(*x*) and $-\frac{\sqrt{3(x+4)}}{3}$ at *f3*(*x*). Press [enter].

Step 5: To view the graphs of *f*(*x*) and its inverse *g*(*x*), press [tab]. Arrow up to display the equation you wish to graph, and then press [enter]. Repeat for each equation you wish to graph on the same screen.

Step 6: To view the table of values, press [ctrl][T]. Take note of any undefined values for the functions.

Common Errors/Misconceptions

- leaving the variables unchanged when rewriting a function as its inverse
- identifying $f^{-1}(x)$ as the inverse of a quadratic function $f(x)$ over a domain for which the quadratic function does not exhibit one-to-one correspondence
- identifying the domain in a real-world problem involving a quadratic function $f(x)$ and its inverse $f^{-1}(x)$ without taking into account the real-world restrictions on domain values
- mistakenly identifying the symbol $f^{-1}(x)$ as the reciprocal of a function instead of as the function's inverse

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Instruction

Guided Practice 4A.1.1

Example 1

Find the inverse, $g(x)$, of the function $f(x) = 4x^2$ and determine the domain value(s) over which the inverse exists.

1. Switch the domain and function variables, and then rename $f(x)$ as $g(x)$.

   Since the inverse function results from switching the input and output variables of the original function, identify which variable represents the input and which represents the output. For the function $f(x) = 4x^2$, x is the input (independent) variable, and $f(x)$ is the output (dependent) variable. Therefore, swap $f(x)$ and x.

$f(x) = 4x^2$	Original function
$x = 4[f(x)]^2$	Switch $f(x)$ and x.

   Next, rename $f(x)$ as $g(x)$, since this new relation represents the inverse of the original function.

$x = 4[f(x)]^2$	Equation with variables switched
$x = 4[g(x)]^2$	Rename $f(x)$ as $g(x)$.

   The resulting inverse relation, $x = 4[g(x)]^2$, is the *possible* inverse function of the original function. The original function, $f(x) = 4x^2$, must be evaluated to determine whether it has one-to-one correspondence.

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Instruction

2. Solve the possible inverse for $g(x)$.

$x = 4[g(x)]^2$	Possible inverse
$\frac{x}{4} = [g(x)]^2$	Divide both sides by 4.
$g(x)^2 = \frac{x}{4}$	Rearrange using the Symmetric Property of Equality.
$g(x) = \pm\sqrt{\frac{x}{4}}$	Take the square root of both sides.
$g(x) = \pm\frac{\sqrt{x}}{\sqrt{4}}$	Apply the square root to both the numerator and the denominator.
$g(x) = \pm\frac{\sqrt{x}}{2}$	Simplify.
$g(x) = \pm\frac{1}{2}\sqrt{x}$	Eliminate the denominator by rewriting the division as multiplication by a fraction.

3. Determine the domain of $g(x)$.

We need to know where $g(x)$ is defined in order to know its domain, which we want to compare to the range of $f(x)$.

$g(x)$ is a radical function; specifically, it is a square root function. Recall that the value of the expression inside the radical must be positive in order for the function to be defined. The expression under the square root is x; therefore, the values of x are limited to non-negative numbers.

Thus, the domain of $g(x)$ is $[0, +\infty)$ because x is raised to the $\frac{1}{2}$ power (recall that $\sqrt{x} = x^{\frac{1}{2}}$).

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Instruction

4. Determine the range of $g(x)$.

   We need to compare the range of $g(x)$ to the domain of $f(x)$.

   There are no limitations on the output values; therefore, the range of $g(x)$ is $(-\infty, +\infty)$.

5. Determine the domain of $f(x)$.

   The domain of $f(x)$ is $(-\infty, +\infty)$. The domain will need to be restricted to those values over which $f(x)$ shows one-to-one correspondence.

6. Determine whether the function $f(x)$ exhibits one-to-one correspondence.

   The function $f(x)$ can only have an inverse over the part of its domain for which it exhibits one-to-one correspondence.

   We can determine that the function $f(x) = 4x^2$ does not show one-to-one correspondence for all range values because $x = a$ and $x = -a$ yield equal values for $f(x)$: $f(a) = 4a^2$ and $f(-a) = 4(-a)^2 = 4a^2$. You can apply the horizontal line test and exclude portions of $f(x)$ where this test fails; that is, where the horizontal line crosses $f(x)$ at more than one point.

7. Determine the parts of its domain over which $f(x)$ exhibits one-to-one correspondence.

   The restricted domains of $f(x)$ are $(-\infty, 0]$ and $[0, +\infty)$. Only the point (0, 0) is not restricted, because all of the other points do not have one-to-one correspondence. The horizontal line crosses the graph at two points for all other points on the graph.

8. Determine the range of $f(x)$.

   Because $f(x)$ is a quadratic function and x^2 can never be negative, the range of $f(x)$ is $[0, +\infty)$.

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Instruction

9. Match domains to ranges to find the inverse(s) of $f(x)$.

In order for a function to have an inverse, it must be one-to-one. Therefore, match the restricted domains of $f(x)$ to the range of $g(x)$.

Restricted domain of $f(x)$:	Inverse function with matching range:
$[0, +\infty)$	$g(x)=\frac{1}{2}\sqrt{x}$
$(-\infty, 0]$	$g(x)=-\frac{1}{2}\sqrt{x}$

Check that the range of $f(x)$ matches the domain of $g(x)$.

The range of $f(x)$ is $[0, +\infty)$. The domain of $g(x)$ is $[0, +\infty)$.

The inverse of $f(x) = 4x^2$ on the restricted domain $0 \le x < +\infty$ is $g(x)=\frac{1}{2}\sqrt{x}$.

The inverse of $f(x) = 4x^2$ on the restricted domain $-\infty < x \le 0$ is $g(x)=-\frac{1}{2}\sqrt{x}$.



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Instruction

Example 2

Find the inverse, $f^{-1}(x)$, of the function $f(x) = x(x - 1)$ and determine the domain and range value(s) over which the inverse exists.

1. Determine whether an inverse exists for this function.

   When $f(x)$ is simplified, it will contain x^2, so $f(x)$ is a quadratic function and its graph is a parabola. A parabola does not have an inverse because it does not have one-to-one correspondence. For any y-value in the range of $f(x) = x(x - 1)$, except at the vertex, there are two corresponding x-values.

   In order for an inverse function to exist, we must restrict the domain of the given function so that it represents half of the parabola, as determined by the axis of symmetry.

   The axis of symmetry is the line $x = -\frac{b}{2a}$.

   Simplify the given equation in order to determine the values of a and b.

   $f(x) = x(x - 1)$

   $f(x) = x^2 - x$

   a is the coefficient of the x^2 term and b is the coefficient of the x term, so $a = 1$ and $b = -1$. Substitute these values to find the axis of symmetry.

   $$x = -\frac{b}{2a}$$

   $$x = -\frac{(-1)}{2(1)} = \frac{1}{2}$$

   The axis of symmetry is $x = \frac{1}{2}$. As long as we restrict the domain of $f(x)$ such that either $x \le \frac{1}{2}$ or $x \ge \frac{1}{2}$, the function will have an inverse.

   Let's use the restricted domain $\left[\frac{1}{2}, +\infty\right)$.

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Instruction

2. Determine the range of $f(x)$.

Because $f(x)$ is a parabolic function, its range is limited.

Since the coefficient of the quadratic term is positive, the parabola opens up, and the minimum of the range is the y-coordinate of the minimum of the parabola.

The x-coordinate of the minimum is known; it is the axis of symmetry, $x=\frac{1}{2}$. To determine the y-coordinate, substitute $x=\frac{1}{2}$ into the original function, $f(x) = x(x - 1)$. Then, solve for $f(x)$.

$f(x) = x(x - 1)$	Original function
$f\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)\left[\left(\frac{1}{2}\right)-1\right]$	Substitute $\frac{1}{2}$ for x.
$f\left(\frac{1}{2}\right)=-\frac{1}{4}$	Simplify.

The minimum of the function is located at $\left(\frac{1}{2},-\frac{1}{4}\right)$. Therefore, the range of $f(x)$ is $\left[-\frac{1}{4},+\infty\right)$.

The restricted domain and corresponding range for $f(x)$ are now known. With this restricted domain, the function now has an inverse.

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Instruction

3. Calculate the inverse by switching the domain and function variables, and then renaming $f(x)$ as $f^{-1}(x)$.

Swap $f(x)$ and x.

$f(x) = x(x - 1)$	Original function
$x = f(x)[f(x) - 1]$	Switch $f(x)$ and x.

Next, rename $f(x)$ as $f^{-1}(x)$, since this new relation represents the inverse rather than the original function.

$x = f(x)[f(x) - 1]$	Equation with variables switched
$x = f^{-1}(x)[f^{-1}(x) - 1]$	Rename $f(x)$ as $f^{-1}(x)$.

The inverse for the original function on the restricted domain is $x = f^{-1}(x)[f^{-1}(x) - 1]$.

4. Rewrite the inverse in a form that can be solved for $f^{-1}(x)$.

In this example, $f^{-1}(x)$ can be thought of as the dependent variable in a quadratic function, and x can be thought of as an independent variable. Here, $f^{-1}(x)$ is a single variable. Along with x, it can be manipulated according to the usual rules of algebra.

$x = f^{-1}(x)[f^{-1}(x) - 1]$	Inverse
$0 = f^{-1}(x)[f^{-1}(x) - 1] - x$	Subtract x from both sides.
$0 = [f^{-1}(x)]^2 - f^{-1}(x) - x$	Distribute $f^{-1}(x)$ over $[f^{-1}(x) - 1]$.

Thus, $x = f^{-1}(x)[f^{-1}(x) - 1]$ becomes $0 = [f^{-1}(x)]^2 - f^{-1}(x) - x$.

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5. Solve for $f^{-1}(x)$ using the quadratic formula.

The quadratic formula is used for this equation since this function cannot be factored using whole numbers. In this equation, the coefficients in the quadratic formula are $a = 1$, $b = -1$, and $c = -x$. Substitute these values into the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic formula

$$\left[f^{-1}(x)\right] = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-x)}}{2(1)}$$ Substitute $f^{-1}(x)$ for x, 1 for a, -1 for b, and $-x$ for c.

$$f^{-1}(x) = \frac{1 \pm \sqrt{1 + 4x}}{2}$$ Simplify.

$$f^{-1}(x) = \frac{1}{2} \pm \frac{\sqrt{1 + 4x}}{2}$$ Rewrite as separate terms, with the terms in the numerator each written over the common denominator.

The inverse function is either $f^{-1}(x) = \frac{1}{2} + \frac{\sqrt{1 + 4x}}{2}$ or $f^{-1}(x) = \frac{1}{2} - \frac{\sqrt{1 + 4x}}{2}$. These equations represent the lower and upper halves of a "sideways" parabola. To determine the correct inverse function, look at the domain and range of each. Recall that the domain of the original function becomes the range of the inverse function, and the range of the original function becomes the domain of the inverse function.

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Instruction

6. Determine the domain of $f^{-1}(x)$.

The domain of $f^{-1}(x)$ should be the same as the range of the original function, $f(x)$, which is $\left[-\frac{1}{4}, +\infty\right)$. Verify this by determining where $f^{-1}(x)$ is defined. The domain of $f^{-1}(x)=\frac{1}{2}\pm\frac{\sqrt{1+4x}}{2}$ can be found by looking at the binomial under the radical sign, $1 + 4x$. Since the radical represents a real number value, $1 + 4x$ must be non-negative; therefore, $1+4x\geq 0$. Solve this inequality for x.

$1+4x\geq 0$	Inequality
$4x\geq -1$	Subtract 1 from both sides of the inequality.
$x\geq -\frac{1}{4}$	Divide both sides of the inequality by 4.

Since x is greater than *or equal to* $-\frac{1}{4}$, this value is included in the domain of $f^{-1}(x)$. Therefore, the domain of $f^{-1}(x)$ is $\left[-\frac{1}{4}, +\infty\right)$. This is the same as the range of $f(x)$.

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Instruction

7. Determine the range of $f^{-1}(x)$.

The range of $f^{-1}(x)$ should be the same as the restricted domain of the original function, $f(x)$, which is $\left[\frac{1}{2}, +\infty\right)$. Remember that we have two options for $f^{-1}(x)$, either $f^{-1}(x)=\frac{1}{2}+\frac{\sqrt{1+4x}}{2}$ or $f^{-1}(x)=\frac{1}{2}-\frac{\sqrt{1+4x}}{2}$. Determine which of these is the correct inverse for $f(x)$ on its restricted domain by finding the range of $f^{-1}(x)$ and comparing it to the domain of $f(x)$.

For $f^{-1}(x)$, the lower boundary of the domain is $-\frac{1}{4}$. Substitute this x-value into either option for $f^{-1}(x)$ and solve.

$f^{-1}(x)=\frac{1}{2}+\frac{\sqrt{1+4x}}{2}$ Upper part of $f^{-1}(x)$

$f^{-1}\left(-\frac{1}{4}\right)=\frac{1}{2}+\frac{\sqrt{1+4\left(-\frac{1}{4}\right)}}{2}$ Substitute $-\frac{1}{4}$ for x.

$f^{-1}\left(-\frac{1}{4}\right)=\frac{1}{2}+\frac{\sqrt{1-1}}{2}$ Multiply.

$f^{-1}\left(-\frac{1}{4}\right)=\frac{1}{2}+\frac{0}{2}$ Subtract.

$f^{-1}\left(-\frac{1}{4}\right)=\frac{1}{2}$ Add.

(*continued*)

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Instruction

When $x=-\frac{1}{4}$, $f^{-1}(x)=\frac{1}{2}$. Recall that the axis of symmetry for the original parabola was $x=\frac{1}{2}$. Thus, the axis of symmetry for the "sideways" parabola is $f^{-1}(x)=\frac{1}{2}$. Therefore, the range is either $\left(-\infty,\frac{1}{2}\right]$ or $\left[\frac{1}{2},+\infty\right)$. Since we know that the range of $f^{-1}(x)$ must match the domain of $f(x)$, it must be $\left[\frac{1}{2},+\infty\right)$, which is the range for the upper part of the inverse function, $f^{-1}(x)=\frac{1}{2}+\frac{\sqrt{1+4x}}{2}$.

8. Summarize your conclusions.

The inverse of $f(x) = x(x - 1)$ on the restricted domain $\left[\frac{1}{2},+\infty\right)$ is $f^{-1}(x)=\frac{1}{2}+\frac{\sqrt{1+4x}}{2}$. The domain of $f^{-1}(x)$ is $\left[-\frac{1}{4},+\infty\right)$ and the range of $f^{-1}(x)$ is $\left[\frac{1}{2},+\infty\right)$.

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Instruction

Example 3

Use a graphing calculator to graph the quadratic function $f(x) = x^2 - 3x - 1$ and its inverse, $g(x)$. Write an equation for the possible inverse, $g(x)$, using algebraic methods. Then, verify your written equation using points from the graphs of $f(x)$ and $g(x)$.

1. Graph $f(x)$ and its inverse, $g(x)$, using a graphing calculator.

   A graphing calculator does not provide the solid proof that the algebraic methods do for the existence of an inverse and its domain. Therefore, it is necessary to check the results algebraically after the functions are plotted. For this reason, after graphing $f(x)$, you will also need to calculate the table of values for the graph.

   Follow the instructions particular to your calculator model to graph the original function, $f(x) = x^2 - 3x - 1$, and its inverse, $g(x)$.

   On a TI-83/84:

   Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

   Step 2: Enter the original function on the Y_1 line.

   Step 3: Press [GRAPH] to see a graph of $f(x)$.

   Step 4: Press [2ND][GRAPH] to display a table of x- and y-values.

   Step 5: To see a graph of the inverse of $f(x)$, press [2ND][PRGM] to display the DRAW menu. Select 8: DrawInv.

   Step 6: After "DrawInv" and the cursor appear on the screen, press [VARS]. Arrow over to Y-VARS and select 1: Function.

   Step 7: Highlight 1: Y_1 and press [ENTER]. This will display "DrawInv Y_1." Press [ENTER] to graph the inverse of the function entered as Y_1.

   On a TI-Nspire:

   Step 1: Press the [home] key.

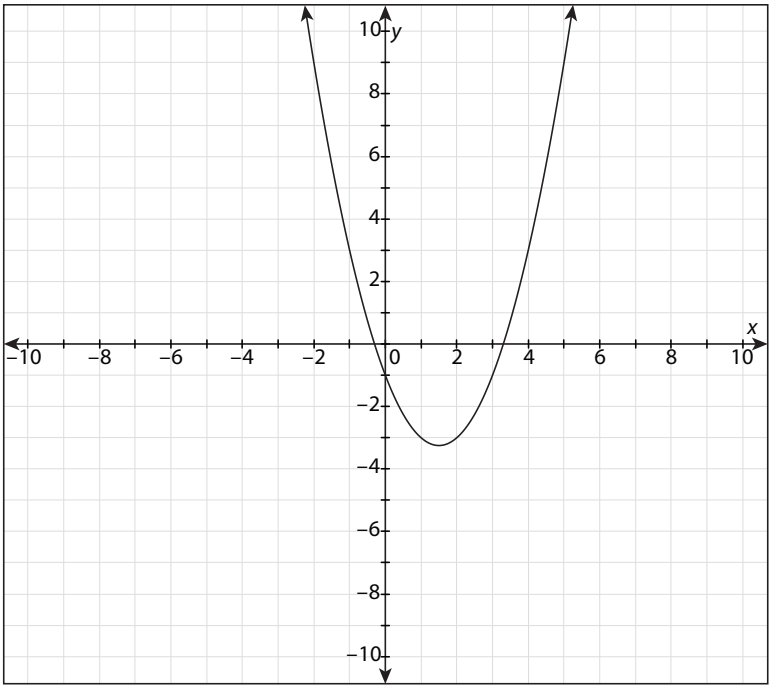
   Step 2: Arrow over to the graphing icon, the second icon from the left, and press [enter].

   Step 3: At $f1(x)$, enter the original function. Press [enter].

(*continued*)

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Instruction



Step 4: To view the table of values, press [ctrl][T]. The function table will be displayed to the right of the graph.

Step 5: To hide the table, press [ctrl][k] to select it (signified by a blinking table border), then press [clear]. *Note*: The table can be displayed again later by pressing [ctrl][T]. After graphing the inverse function, return to this table so that you can observe how the graph of the inverse results from exchanging the x- and y-values found in the table of values for $f(x)$.

Step 6: Constructing the inverse of a function requires that the graph of $f(x)$ be reflected across a constructed line of the form $y = x$. First, add a grid to the screen by pressing the [menu] key and selecting 2: View. Select 5: Show Grid.

Step 7: To construct the function $y = x$, press [menu] and select 7: Points & Lines. Then select 4: Line. A pencil cursor will be displayed on the graph.

Step 8: Use the NavPad to move the cursor to a point of the form (a, a) and press the click button. Repeat by moving the cursor to a different point of the form (b, b) and then pressing the click button. A line of the form $y = x$ will be displayed.

(*continued*)

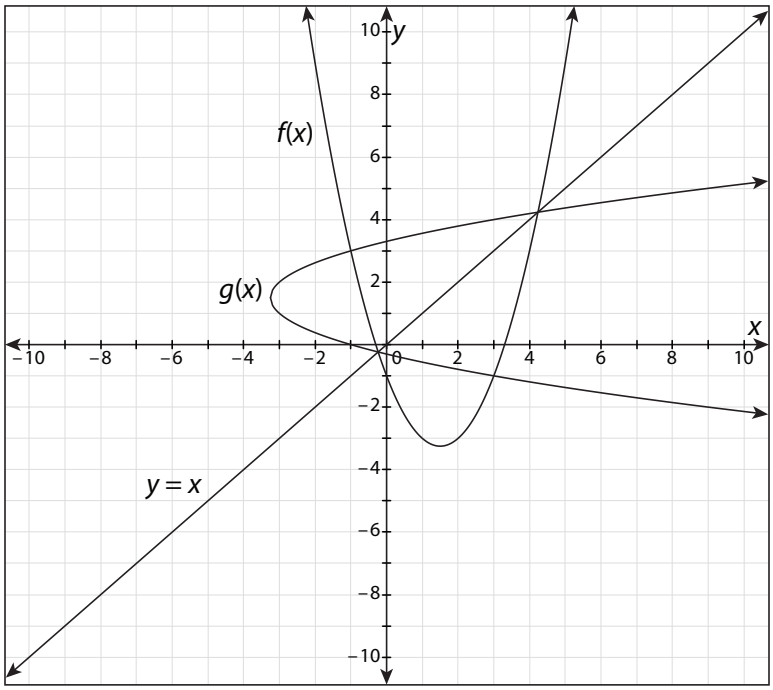
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Note: A function reflected across the line $y = x$ will produce the inverse because the values of x and y will be exchanged; i.e., point A (a, b) will become point B (b, a).

Step 9: To start the reflection process, press [menu] and select B: Transformation. Then select 2: Reflection. Move the cursor to the function and press the click button to name a point on the function. Then, move the cursor to the line $y = x$ and press the click button. The image of the point on the function will appear. Press [enter] to save the points shown.

Step 10: To see the graph of the inverse, press [menu] and select A: Construction. Then select 6: Locus. Click on the image point and press [enter], then click on the point on the original function to see the inverse relation that goes with the function.

Either calculator will create a graph resembling the one that follows. Observe that this graph of the inverse results from exchanging the x- and y-values found in the table of values created for $f(x)$.



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Instruction

2. Write a possible inverse, $g(x)$, for the function $f(x) = x^2 - 3x - 1$.

Recall that the first step in writing the inverse of a function is to switch the domain and function variables.

$f(x) = x^2 - 3x - 1$	Original function
$x = [f(x)]^2 - 3[f(x)] - 1$	Switch $f(x)$ and x.

Then, rename $f(x)$ as $g(x)$, since this new relation represents the inverse.

$x = [f(x)]^2 - 3[f(x)] - 1$	Equation with variables switched
$x = [g(x)]^2 - 3[g(x)] - 1$	Rename $f(x)$ as $g(x)$.

Rewrite the inverse in standard form.

$x = [g(x)]^2 - 3[g(x)] - 1$	Possible inverse
$0 = [g(x)]^2 - 3[g(x)] - 1 - x$	Subtract x from both sides.
$0 = [g(x)]^2 - 3[g(x)] - (x + 1)$	Distribute -1 over $1 - x$.

Solve this equation for $g(x)$ by using the quadratic formula. Let $a = 1$, $b = -3$, and $c = -(x + 1)$.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Quadratic formula
$[g(x)] = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)[-(x+1)]}}{2(1)}$	Substitute 1 for a, -3 for b, $-(x + 1)$ for c, and $g(x)$ for x.
$g(x) = \frac{3 \pm \sqrt{9 + 4x + 4}}{2}$	Simplify.
$g(x) = \frac{3}{2} \pm \frac{\sqrt{4x+13}}{2}$	Rewrite as separate terms, with the terms in the numerator each written over the common denominator.

The equation $g(x) = \frac{3}{2} \pm \frac{\sqrt{4x+13}}{2}$ is a possible inverse for the function $f(x) = x^2 - 3x - 1$.

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Instruction

3. Use points from the graph to verify the possible inverse.

Refer to the table of values you created in step 1 for the graph of $f(x)$. Substitute an ordered pair from the table of values into the possible inverse equation determined in step 2. This will verify that the inverse is produced by exchanging the domain and function values for the points of the inverse relation graph.

Let's use the ordered pair (0, –1) from the table of values for $f(x)$.

Reverse the coordinates: (–1, 0).

Substitute these reversed coordinates into the equation for $g(x)$, then compare the resulting coordinates to the graph of $g(x)$.

$g(x)=\frac{3}{2}\pm\frac{\sqrt{4x+13}}{2}$ Possible inverse of $f(x)$

$g(-1)=\frac{3}{2}\pm\frac{\sqrt{4(-1)+13}}{2}$ Substitute –1 for x.

$g(-1)=\frac{3}{2}\pm\frac{\sqrt{9}}{2}$ Simplify.

Solve each of these equations for $g(-1)$.

$g(-1)=\frac{3}{2}+\frac{\sqrt{9}}{2}$ $\quad$ $g(-1)=\frac{3}{2}-\frac{\sqrt{9}}{2}$

$g(-1)=\frac{3}{2}+\frac{3}{2}$ $\quad$ $g(-1)=\frac{3}{2}-\frac{3}{2}$

$g(-1) = 3$ $\quad$ $g(-1) = 0$

$g(-1) = 0$ or 3; the corresponding ordered pairs are (–1, 0) and (–1, 3). Compare these points to the graph of $f(x)$.

The ordered pair (–1, 3) is also a point on the graph of $f(x)$, so the possible inverse relation has been verified.

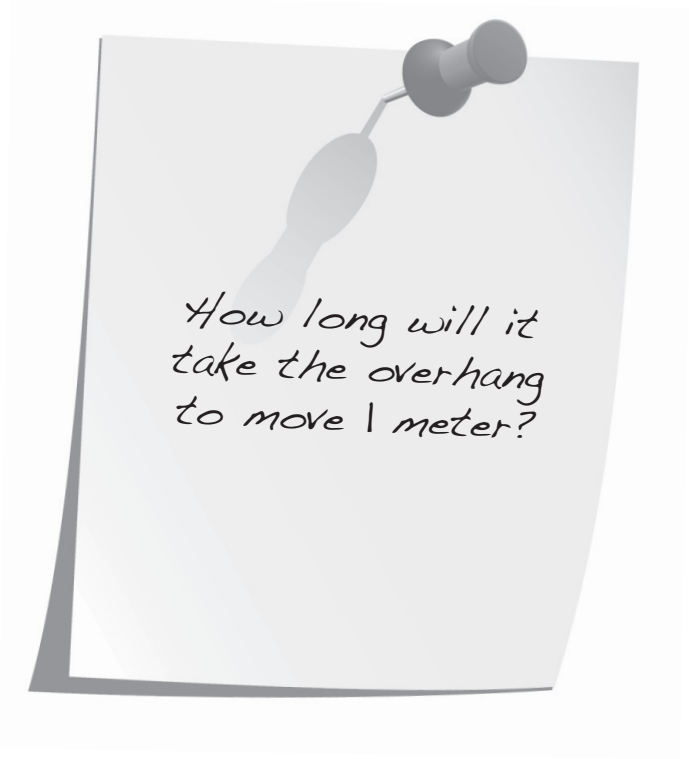


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Problem-Based Task 4A.1.1: Sliding Rocks

SMP	
1	2 ✓
3	4
5	6
7 ✓	8 ✓

A geologist for the Department of Natural Resources has been studying a rock formation popular with mountain climbers. The rocks form an overhang that is actually one large metamorphic rock block that is sliding past another rock along what is called a slip fault. The geologist has determined that the overhang is disappearing as the rock formation slides past the other rock at its base, at a rate of 10 centimeters per decade. The overhang's movement is also accelerating at a rate of 2 centimeters per second squared. The overhang's motion can be approximated by the quadratic function $f(x) = 10x + x^2$, where $f(x)$ is the distance in centimeters the overhang slides and x is the time in decades. Write an inverse function for $f(x)$ and determine the domain values over which the inverse exists in this real-world situation. How long will it take the overhang to move 1 meter?



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Problem-Based Task 4A.1.1: Sliding Rocks

Coaching

a. What is the domain of $f(x)$ and how does this relate to the existence of its inverse function?

b. What is the first step in writing an inverse function $f^{-1}(x)$ for $f(x)$?

c. What is the domain of the inverse function $f^{-1}(x)$ and how does this compare to the domain of $f(x)$?

d. Find the inverse function $f^{-1}(x)$.

e. What do the two possible solutions for the inverse function mean about the direction of motion of the overhang?

f. What did x and $f(x)$ represent in the original function? What do x and $f^{-1}(x)$ represent in the inverse function?

g. When would you use the original function $f(x)$ and when would you use the inverse function $f^{-1}(x)$?

h. Use the inverse function to determine how many decades it will take for the overhang to move 1 meter.

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Instruction

Problem-Based Task 4A.1.1: Sliding Rocks

Coaching Sample Responses

a. What is the domain of $f(x)$ and how does this relate to the existence of its inverse function?

The function $f(x) = 10x + x^2$ is a quadratic, so its domain is all real numbers, $(-\infty,+\infty)$. The inverse is *not* defined for all real numbers because there are unequal values of x for which the function values are equal. For example, $f(0) = 0$ and $f(-10) = 0$. However, if the domain of $f(x)$ is restricted to $[0,+\infty)$, the inverse function does exist since there is now one-to-one correspondence between the domain and function values. In addition, since x represents time, we know that $x \geq 0$.

b. What is the first step in writing an inverse function $f^{-1}(x)$ for $f(x)$?

Switch $f(x)$ and x in $f(x) = 10x + x^2$, and then change $f(x)$ to $f^{-1}(x)$.

$f(x) = 10x + x^2$

$x = 10[f^{-1}(x)] + [f^{-1}(x)]^2$

c. What is the domain of the inverse function $f^{-1}(x)$ and how does this compare to the domain of $f(x)$?

The domain of the inverse function, $f^{-1}(x)$, defined in part b, is the same as the range of the original function. The original function, in the context of this problem, has a domain of $[0,+\infty)$. When $x = 0$, $f(x)$ is also equal to 0. Since the original function is a parabola that opens upward, when $x > 0$, then $f(x) > 0$ as well. Thus, the range of the original function in the context of this problem is also $[0,+\infty)$. Therefore, the domain of the inverse function, $f^{-1}(x)$, is $[0,+\infty)$.

d. Find the inverse function $f^{-1}(x)$.

First, rewrite the function from part b in standard form so that it can be solved using the quadratic formula.

$x = 10[f^{-1}(x)] + [f^{-1}(x)]^2$

$0 = 10[f^{-1}(x)] + [f^{-1}(x)]^2 - x$

$0 = [f^{-1}(x)]^2 + 10[f^{-1}(x)] - x$

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Instruction

In the quadratic formula, let $a = 1$, $b = 10$, $c = -x$, and $x = f^{-1}(x)$. Substitute and simplify.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\left[f^{-1}(x)\right] = \frac{-(10) \pm \sqrt{(10)^2 - 4(1)(-x)}}{2(1)}$$

$$f^{-1}(x) = \frac{-10 \pm \sqrt{100 + 4x}}{2}$$

$$f^{-1}(x) = -5 \pm \frac{\sqrt{4(25 + x)}}{2}$$

$$f^{-1}(x) = -5 \pm \frac{2\sqrt{25 + x}}{2}$$

$$f^{-1}(x) = -5 \pm \sqrt{25 + x}$$

Note that without considering the context of the problem, the domain of $f^{-1}(x)$ is $25 + x \geq 0$, or $x \geq -25$. But in the context of the problem, x must be positive.

e. What do the two possible solutions for the inverse function mean about the direction of motion of the overhang?

The overhang is described as "disappearing," which could be taken to mean that the length of the overhang is decreasing. If so, the positive acceleration and velocity apply to this "downward" movement. Therefore, the inverse function solution $-5 + \sqrt{25 + x}$ applies to this net downward movement of the overhang. The inverse function solution $-5 - \sqrt{25 + x}$ applies to an upward movement of the overhang, which was not observed by the geologist in this situation. Which direction is taken as negative and which is taken as positive is inconsequential, but once chosen, the sign of the directional movement has to be consistently applied to acceleration, distance, and velocity.

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f. What did x and $f(x)$ represent in the original function? What do x and $f^{-1}(x)$ represent in the inverse function?

In the original function, x represented time in decades, and $f(x)$ represented the distance in centimeters that the overhang slides.

In the inverse function, x represents the distance in centimeters that the overhang slides, and $f^{-1}(x)$ represents the time in decades.

g. When would you use the original function $f(x)$ and when would you use the inverse function $f^{-1}(x)$?

In $f(x)$, the independent variable x represents time in decades, and $f(x)$ represents distance. You would use this equation if you knew the time and wanted to calculate the distance the rock moved.

In $f^{-1}(x)$, the independent variable x represents distance and $f^{-1}(x)$ represents time in decades. You would use this equation if you knew the distance the rock moved and wanted to calculate the time.

h. Use the inverse function to determine how many decades it will take for the overhang to move 1 meter.

Recall from part e that the inverse function $f^{-1}(x) = -5 + \sqrt{25 + x}$ applies to the net downward movement of the overhang.

First convert 1 meter to 100 centimeters, because the formula is in terms of centimeters. Substitute 100 for x in $f^{-1}(x)$ and simplify.

$$f^{-1}(x) = -5 + \sqrt{25 + x}$$

$$f^{-1}(100) = -5 + \sqrt{25 + (100)}$$

$$f^{-1}(100) \approx 6.2$$

It will take approximately 6.2 decades, or 62 years, for the overhang to move 1 meter.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4A.1.1: Determining Inverses of Quadratic Functions

For problems 1–7, write the inverse for each quadratic function and the restricted domain over which the quadratic function can have an inverse.

1. $f(x) = 3x^2 - 5$
2. $g(x) = 7x - x^2$
3. $h(x) = (x + 1)(x - 2)$
4. $a(x) = 8x^2 - 3$
5. $b(x) = x^2 - 5x$
6. $c(x) = 3x^2 - 2x - 1$
7. $d(x) = 2x^2 + 1$

For problems 8–9, use what you know about determining inverses of quadratic functions to answer the questions.

8. The area of a rectangular carpet is given by the quadratic function $A(x) = x(x + 30)$, where $A(x)$ is the area and x is the width of the carpet in feet. What are the restrictions on the domain of the function and its inverse?

9. The kinetic energy of a motorcycle is given by the function $K(v) = \frac{1}{2}mv^2$, where m is the mass of the motorcycle and v is the velocity with which the motorcycle is moving. If the domain of v is [–90, 120], over what domain(s) does the inverse exist?

Use the given information to complete all parts of problem 10.

10. A social media site has been operating for 8 years. The growth in the number of site users is approximated by the quadratic function $N(y) = 25y^2 - 85y + 30$, where $N(y)$ is the number of users in millions and y is the number of years since the site was launched.
    a. Write an inverse for the function.
    b. State the domain of the inverse.
    c. State the domain of the inverse based on the real-world characteristics of the problem.

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Lesson 4A.1.2: Determining Inverses of Other Functions

Warm-Up 4A.1.2

The courthouse clock lost power during an electrical storm, but the decorative pendulum bob that normally swings once every 3 seconds continued to swing until friction brought it to a stop. At the instant the clock lost power, the period of the pendulum T could be written as the radical function $T(L)=2\pi\sqrt{\frac{L}{g}}$, where T is the amount of time in seconds needed for one complete swing of the pendulum to and from its beginning location. The constant g is the magnitude of the acceleration due to gravity (approximately 9.8 meters per second squared, or 9.8 m/s^2), and L is the length of the pendulum in meters. The period of the pendulum is proportional to the square root of its length. Over what real-world domain does this function have an inverse?

1. What is the domain of the function?

2. Does the function have an inverse? Explain.

3. Write the inverse of the function.

4. What is the domain of the inverse?

5. What is the shared domain for the function and its inverse? Give a numerical answer using interval notation.

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Instruction

Lesson 4A.1.2: Determining Inverses of Other Functions

Common Core State Standard

F–BF.4 Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example, $f(x) = 2x^3$ or $f(x) = (x + 1)/(x – 1)$ for $x \neq 1$.*

Warm-Up 4A.1.2 Debrief

1. What is the domain of the function?

   Since the variable L is under a square root sign and since it represents the length of a real-world object, the lower bound of the domain of L is any value greater than 0. The upper bound of the domain at the instant the clock stopped can be found by evaluating the function at $T = 3$ seconds.

2. Does the function have an inverse? Explain.

   This question can be answered by asking another question: Does the function show one-to-one correspondence of its domain and function values?

   Looking at the function $T(L)$, it can be seen that the variables L and g are both under a square root. All of the values for $T(L)$ must be positive. This means that for each domain value of L in the function, there is a unique function value of $T(L)$. Therefore, the function shows one-to-one correspondence.

3. Write the inverse of the function.

   Switch the domain and function variables $T(L)$ and L, then rewrite the result so that it is in inverse function notation.

   $T(L) = 2\pi\sqrt{\frac{L}{g}}$ Original function

   $L = 2\pi\sqrt{\frac{T(L)}{g}}$ Switch $T(L)$ and L.

   $L = 2\pi\sqrt{\frac{T^{-1}(L)}{g}}$ Rewrite the function using $T^{-1}(L)$ in place of $T(L)$.

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Next, solve the equation for $T^{-1}(L)$.

$L^2 = \frac{4\pi^2 T^{-1}(L)}{g}$	Square both sides of the equation and simplify.
$gL^2 = 4\pi^2 T^{-1}(L)$	Multiply both sides of the equation by g.
$\frac{gL^2}{4\pi^2} = T^{-1}(L)$	Divide both sides of the equation by $4\pi^2$.
$T^{-1}(L) = \frac{gL^2}{4\pi^2}$	Rewrite using the Symmetric Property of Equality.

The inverse of the function is $T^{-1}(L) = \frac{gL^2}{4\pi^2}$.

4. What is the domain of the inverse?

   The mathematical domain of L for the inverse is $(-\infty, +\infty)$, but as described earlier, the real-world restrictions on L range from greater than 0 to the length of a pendulum that has a period of 3 seconds. Therefore, the restricted domain is (0, 3].

5. What is the shared domain for the function and its inverse? Give a numerical answer using interval notation.

   Both the function and its inverse share the domain that ranges from greater than 0 to the length of a pendulum that has a period of 3 seconds. To find L for 3 seconds, substitute the given values for $T(L)$ and g into the original function and solve for L: $T(L) = 3$ seconds; $g \approx 9.8$ m/s^2.

$T(L) = 2\pi\sqrt{\frac{L}{g}}$	Original function
$(3) = 2\pi\sqrt{\frac{L}{(9.8)}}$	Substitute 3 for $T(L)$ and 9.8 for g.
$9 = \frac{4\pi^2 L}{9.8}$	Square both sides of the equation.
$88.2 = 4\pi^2 L$	Multiply both sides of the equation by 9.8.

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$\frac{88.2}{4\pi^2} = L$	Divide both sides of the equation by $4\pi^2$.
$L = \frac{88.2}{4\pi^2}$	Rewrite using the Symmetric Property of Equality.
$L \approx 2.23$	Simplify.

The pendulum is approximately 2.23 meters long. Therefore, the real-world domain of the pendulum at the instant the clock lost power was approximately (0, 2.23].

Connection to the Lesson

- Students will recognize that a quadratic relation can be the inverse of a non-quadratic function.
- Students will determine if a function has an inverse.
- Students will compare the domains of functions and their inverses.
- Students will recognize the real-world limitations on the domain and range of a function and its inverse.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- determining if a function has an inverse by checking one-to-one correspondence (F–IF.1)
- solving a function for its domain variable as one way to find the inverse of the function (F–BF.4a)
- identifying the domain variables of a function and its inverse (F–IF.2)
- identifying the domain of a function and the domain over which the inverse of the function is defined (F–IF. 5)

Introduction

Finding the inverse of functions that are not quadratic—such as radical, rational, and cubic functions—is similar to the process for finding the inverse of a quadratic function. As with quadratic functions, the first step in writing the expression for the possible inverse of a function is to switch the dependent and independent values of the original function, then solve for the new function. Then, to verify that the possible inverse actually exists, you must determine whether the function and its possible inverse exhibit one-to-one correspondence. That is, a function $f(x)$ only has an inverse if each value in the domain corresponds to a unique function value. In this lesson, we will determine and verify the inverses for different types of functions.

Key Concepts

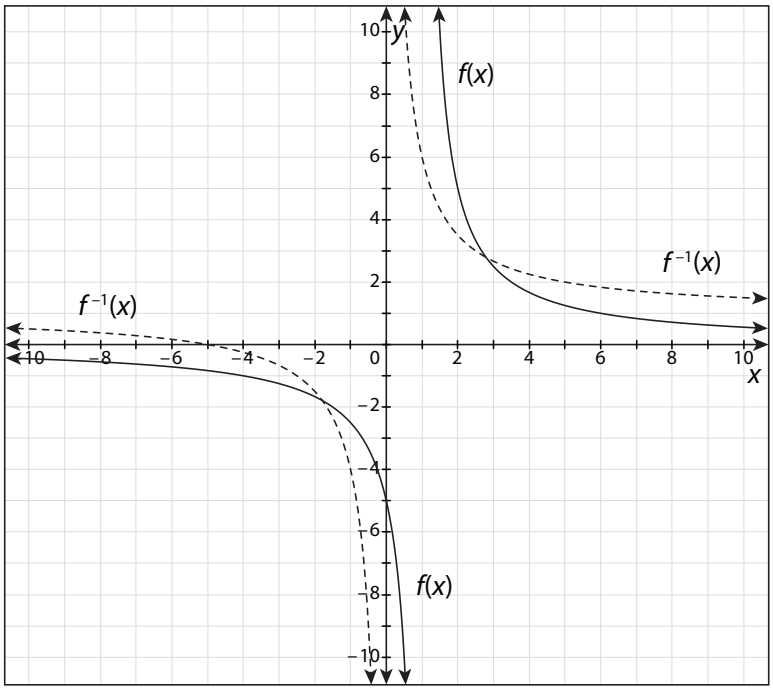
- Recall that a function is a relationship between two variables in which each value of the domain or independent variable, x, is matched with one unique value of the dependent or range variable, y.
- An inverse relation, $g(x)$ or $f^{-1}(x)$, is the relation that results from switching the x- and y-variables in the given function, $f(x)$. This relation may be a function or it may be made a function by applying restrictions.
- For example, take the rational function $f(x)=\frac{5}{x-1}$. Switching the variables and rewriting with $f^{-1}(x)$ in place of $f(x)$ yields the equation $x=\frac{5}{f^{-1}(x)-1}$.
- Remember: $f^{-1}(x)$ notation does *not* mean to take the –1 power of $f(x)$.

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Instruction

- As with quadratic functions, this equation needs to be simplified and solved algebraically for $f^{-1}(x)$. The resulting equation, $f^{-1}(x)=\frac{x+5}{x}$, is the possible inverse of the original function.
- Does the possible inverse of the function exist? To visualize this, compare the graphs of the two functions.



- Notice that the domain of the original function is $(-\infty,1)\cup(1,+\infty)$, whereas the domain of the inverse is $(-\infty,0)\cup(0,+\infty)$. From the graph of the function, it can be seen that the *x*-values of the domain of the original function extend from negative infinity to 1 and from 1 to positive infinity. This excludes 1 from the domain of the original function. The domain of the inverse extends from negative infinity to 0 and from 0 to positive infinity. The domain of the inverse excludes 0.
- Examine the graph of the function and its inverse and use the horizontal line test. It can be seen that for the function and its inverse, each value in the domain corresponds to a unique function value. Therefore, the function and its inverse exhibit one-to-one correspondence. The original function approaches the horizontal asymptote $y = 0$ as x approaches $\pm\infty$, and the inverse approaches the horizontal asymptote $y = 1$ as x approaches $\pm\infty$.

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- To find the domain over which *both* a function and its inverse exist, look at the individual domains of each and where they are defined. In this example, the combined domain is $(-\infty,0)\cup(0,1)\cup(1,+\infty)$ because the function and the inverse have different domain values at which they are undefined.
- As with quadratic functions and their inverses, a graphing calculator can be used to explore a function and the domain(s) over which its inverse exists. Refer to the instructions provided in the previous sub-lesson.

Common Errors/Misconceptions

- leaving the variables unchanged when rewriting a function as its inverse
- identifying $f^{-1}(x)$ as the inverse of a function $f(x)$ over a domain for which the function does not exhibit one-to-one correspondence
- identifying the domain in a real-world problem involving a function and its inverse without taking into account the real-world restrictions on domain values
- mistakenly identifying the symbol $f^{-1}(x)$ as the reciprocal of a function instead of as the function's inverse

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Guided Practice 4A.1.2

Example 1

Find the inverse of the function $f(x)=\frac{x}{x-1}$ if it exists, and determine the domain value(s) shared by the inverse and the function.

1. Switch the domain and function variables, and then rewrite using inverse function notation.

   Switch $f(x)$ and x; then, replace $f(x)$ with $f^{-1}(x)$.

   $f(x)=\frac{x}{x-1}$ Original function

   $x=\frac{f(x)}{[f(x)]-1}$ Switch the variables.

   $x=\frac{f^{-1}(x)}{[f^{-1}(x)]-1}$ Rewrite using $f^{-1}(x)$ notation.

   The resulting relation, $x=\frac{f^{-1}(x)}{[f^{-1}(x)]-1}$, is the possible inverse of the original function. It must be evaluated to determine whether it has one-to-one correspondence with the original function.

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Instruction

2. Solve the possible inverse for $f^{-1}(x)$.

$x=\dfrac{f^{-1}(x)}{\left[f^{-1}(x)\right]-1}$	Possible inverse
$x\bullet\left\{\left[f^{-1}(x)\right]-1\right\}=f^{-1}(x)$	Multiply both sides of the equation by $\left[f^{-1}(x)\right]-1$.
$x\left[f^{-1}(x)\right]-x=f^{-1}(x)$	Distribute.
$x\left[f^{-1}(x)\right]=f^{-1}(x)+x$	Add x to both sides of the equation.
$x\left[f^{-1}(x)\right]-f^{-1}(x)=x$	Subtract $f^{-1}(x)$ from both sides of the equation.
$\left[f^{-1}(x)\right](x-1)=x$	Factor out $f^{-1}(x)$ from the left side of the equation.
$f^{-1}(x)=\dfrac{x}{x-1}$	Divide both sides of the equation by $x-1$.

3. Determine the domain of $f^{-1}(x)$.

The domain will be needed in order to define where $f^{-1}(x)$ exists in comparison to $f(x)$. The domain of $f^{-1}(x)$ is $(-\infty,1)\cup(1,+\infty)$ because $f^{-1}(x)$ is undefined at $x = 1$. Therefore, the domain is all real numbers except $x = 1$.

4. Determine the range of $f^{-1}(x)$.

The range of $f^{-1}(x)$ is also $(-\infty,1)\cup(1,+\infty)$ since $f^{-1}(x)$ has a horizontal asymptote at $y = 1$. Therefore, the range is all real numbers except $y = 1$.

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Instruction

5. Determine whether the function $f(x)$ exhibits one-to-one correspondence.

   The function can only have an inverse over the part of its domain for which it exhibits one-to-one correspondence.

   To determine this, graph the function and the possible inverse relation. From the graph, visually determine if the possible inverse relation is a function by applying the vertical line test. If both the function and its possible inverse relation pass the horizontal line test, then they have one-to-one correspondence.

   Follow the steps appropriate to your calculator model to view the function and its inverse.

   On a TI-83/84:

   Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

   Step 2: Enter the original function on the Y_1 line.

   Step 3: Press [GRAPH] to see a graph of $f(x)$. For better viewing, adjust the domain and function variables by pressing [WINDOW] and changing the values.

   Step 4: To view a table of domain and function values, press [2ND] [GRAPH]. This shows that this function exhibits one-to-one correspondence.

   Step 5: To see a graph of the inverse of $f(x)$, press [2ND][PRGM] to display the DRAW menu. Select 8: DrawInv.

   Step 6: After "DrawInv" and the cursor appear on the screen, press [VARS]. Arrow over to Y-VARS and select 1: Function.

   Step 7: Highlight 1: Y_1 and press [ENTER]. This will display "DrawInv Y_1." Press [ENTER] to graph the inverse of the function entered as Y_1.

(*continued*)

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Instruction

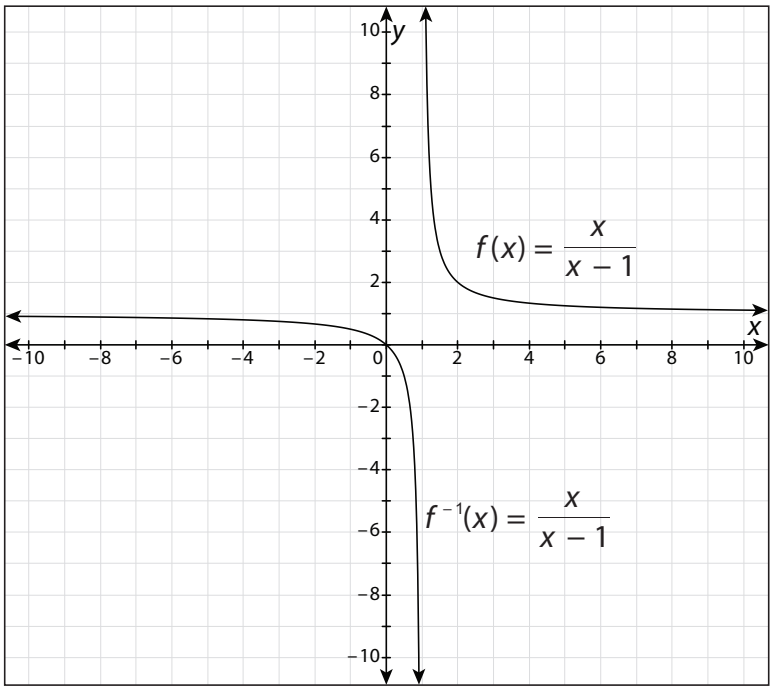
On a TI-Nspire:

Step 1: Press [home].

Step 2: Arrow over to the graphing icon, the second icon from the left, and press [enter].

Step 3: At *f1*(*x*), enter the original function. Press [enter].

Step 4: Press [tab], and then enter the inverse function at *f2*(*x*). Press [enter].



Either calculator will show that the graphs of $f(x)$ and $f^{-1}(x)$ are the same graph.

6. Compare the domains of the function and its inverse to determine where $f(x)$ has an inverse, $f^{-1}(x)$.

   Both the function and its inverse have a domain of $(-\infty,1)\cup(1,+\infty)$. (*Note*: This is a situation in which a function and its inverse have identical equations.)

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Instruction

Example 2

Find the inverse of the radical function $f(x)=\sqrt[3]{x+2}$ if it exists, and determine the domain over which the function and its inverse exist.

1. Switch the domain and function variables, and then rewrite using inverse function notation.

   This is the first step in determining if the function has an inverse.

   Switch $f(x)$ and x; then, replace $f(x)$ with $f^{-1}(x)$.

$f(x)=\sqrt[3]{x+2}$	Original function
$x=\sqrt[3]{[f(x)]+2}$	Switch the variables.
$x=\sqrt[3]{[f^{-1}(x)]+2}$	Rewrite using $f^{-1}(x)$ notation.

   Next, solve for $f^{-1}(x)$.

$x=\sqrt[3]{[f^{-1}(x)]+2}$	Possible inverse
$x^3=[f^{-1}(x)]+2$	Cube both sides of the equation.
$x^3-2=f^{-1}(x)$	Subtract 2 from both sides of the equation.
$f^{-1}(x)=x^3-2$	Apply the Symmetric Property of Equality.

   The possible inverse of $f(x)=\sqrt[3]{x+2}$ is $f^{-1}(x)=x^3-2$.

2. Determine the domain of the function and the domain of the inverse.

   The domain of both the function and its inverse is $(-\infty,+\infty)$ since nonzero cube roots can be positive or negative.

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Instruction

3. Determine whether the function $f(x)$ exhibits one-to-one correspondence with its inverse.

   To do this, graph the function and the possible inverse relation. From the graph, visually determine if the possible inverse relation is a function by applying the vertical line test. If both the function and its possible inverse relation pass the horizontal line test, then they have one-to-one correspondence.

   Follow the directions appropriate to your calculator model.

   On a TI-83/84:

   Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

   Step 2: Enter the original function on the Y_1 line.

   Step 3: Press [GRAPH] to see a graph of $f(x)$. For better viewing, adjust the domain and function variables by pressing [WINDOW] and changing the values.

   Step 4: To view a table of domain and function values, press [2ND] [GRAPH]. This shows that the function exhibits one-to-one correspondence.

   Step 5: To see a graph of the inverse of $f(x)$, press [2ND][PRGM] to display the DRAW menu. Select 8: DrawInv.

   Step 6: After "DrawInv" and the cursor appear on the screen, press [VARS]. Arrow over to Y-VARS and select 1: Function.

   Step 7: Highlight 1: Y_1 and press [ENTER]. This will display "DrawInv Y_1." Press [ENTER] to graph the inverse of the function entered as Y_1.

   On a TI-Nspire:

   Step 1: Press [home].

   Step 2: Arrow over to the graphing icon, the second icon from the left, and press [enter].

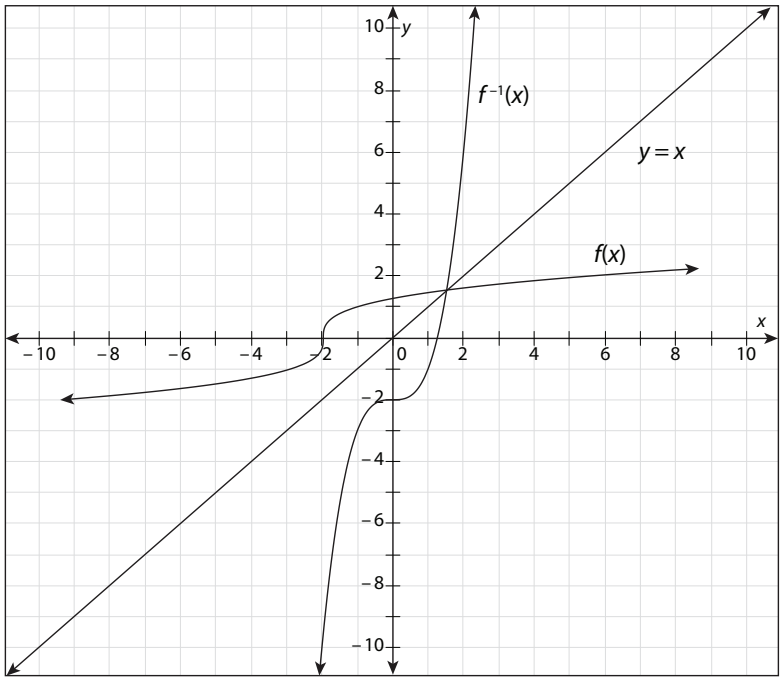
   Step 3: At $f1(x)$, enter the original function. Press [enter].

(*continued*)

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Instruction

Step 4: Press [tab], and then enter the inverse function at *f2*(*x*). Press [enter].



As both graphing calculators show, the function and its inverse are one-to-one over the domain $(-\infty, +\infty)$.

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Instruction

Example 3

Find the inverse of the absolute-value function $f(x)=|x+2|$ if it exists, and determine the domain and range of the inverse.

1. Write the inverse of the function and simplify it.

   Switch $f(x)$ and x; then, replace $f(x)$ with $f^{-1}(x)$.

   $f(x)=|x+2|$ Original function

   $x=|[f(x)]+2|$ Switch the variables.

   $x=|f^{-1}(x)+2|$ Rewrite using $f^{-1}(x)$ notation.

   The absolute-value notation implies that $x = f^{-1}(x) + 2$ or $x = -[f^{-1}(x) + 2]$. Simplify and solve each of these equations for $f^{-1}(x)$.

   $x = f^{-1}(x) + 2$ $\qquad$ $x = -[f^{-1}(x) + 2]$

   $x - 2 = f^{-1}(x)$ $\qquad$ $x = -f^{-1}(x) - 2$

   $f^{-1}(x) = x - 2$ $\qquad$ $f^{-1}(x) = -x - 2$

   The possible inverse can be written as $f^{-1}(x) = x - 2$ or $-x - 2$.

2. Graph the function and its inverse on a graphing calculator.

   On a TI-83/84:

   Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

   Step 2: Enter the function on the Y_1 line. To enter the absolute value function, select [2ND][CATALOG] and choose "abs(". Press [ENTER].

   Step 3: Enter the possible inverse functions at Y_2 and Y_3. Press [ENTER].

   Step 4: To see the graphs of the functions, press [GRAPH].

(*continued*)

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Instruction

On a TI-Nspire:

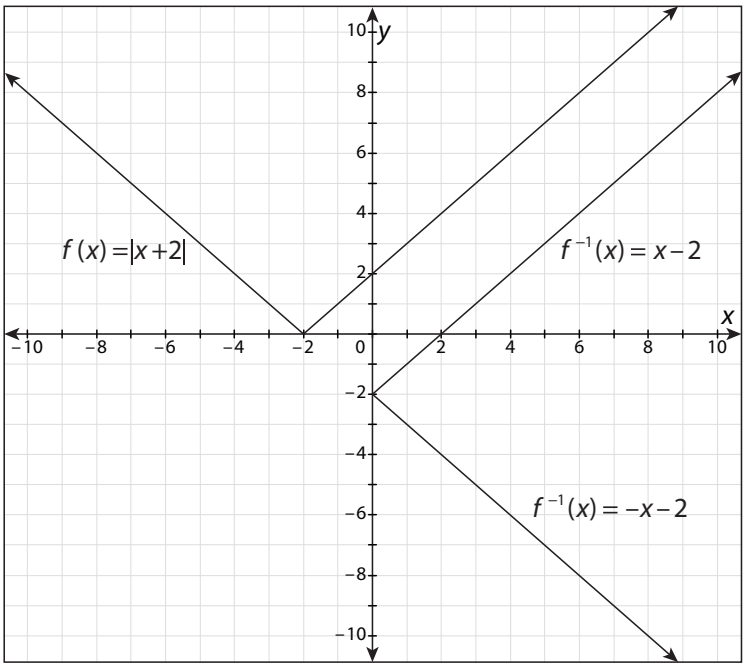
Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon, the second icon from the left, and press [enter].

Step 3: At *f1*(*x*), enter the original function. To enter the absolute value symbol, press [ctrl][×] to open the math expressions palette, then select the absolute value symbol. Press [enter].

Step 4: Press [tab] to enter the possible inverse function at *f2*(*x*) and *f3*(*x*). Press [enter]. The graphs will be displayed.

The resulting graph should resemble the following.



3. Determine the domain and range of the function.

The minimum point on the graph of the function occurs when $|x+2|=0$, which can be represented by the two equalities $x + 2 = 0$ and $-(x + 2) = 0$. The solution to these equations is $x = -2$. Therefore, the domain of the function is $(-\infty, +\infty)$. The range of the function is $[0, +\infty)$ since its minimum value is 0, which occurs at $x = -2$.

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Instruction

4. Identify the restricted domain(s) over which the function can have an inverse.

   The graph shows that the function is not one-to-one over its domain, but it is one-to-one over the restricted domain of $(-\infty, -2]$ or $[-2, +\infty)$. Note that the function can be divided at the point (–2, 0). Therefore, the restricted values of the domain could be $(-\infty, -2]$ or $[-2, +\infty)$.

5. Find the domain and range of the inverse.

   The inverse can be represented by $f^{-1}(x) = x - 2$ and $f^{-1}(x) = -x - 2$.

   The domain of the inverse is the range of the function. Therefore, the domain of the inverse is $[0, +\infty)$.

   The range of the inverse is the domain of the function. Therefore, the range of the inverse is $(-\infty, -2]$ for $f^{-1}(x) = -x - 2$ and $[-2, +\infty)$ for $f^{-1}(x) = x - 2$.

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Instruction

Example 4

Graph the cubic function $f(x) = x^3 - 4x^2 + 3x$ and determine the restricted domain(s) over which its inverse exists.

1. Graph the function using a graphing calculator.

   This is the quickest way to see the function's domain and to specify the restricted domain(s) over which it has an inverse.

   Follow the directions appropriate to your calculator model.

   On a TI-83/84:

   Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

   Step 2: Enter the function on the Y_1 line. Press [GRAPH].

   On a TI-Nspire:

   Step 1: Press the [home] key.

   Step 2: Arrow over to the graphing icon, the second icon from the left, and press [enter].

   Step 3: At *f1*(*x*), enter the original function. Press [enter].

   Keep this view on the calculator for reference in the next example step.

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2. Identify the restricted domain(s) over which the function is not one-to-one.

   This can be approximated visually, but the graphing calculator's trace features will give a more precise result.

   A visual approximation suggests that the restricted domain over which the function is not one-to-one is from $x \approx 0.5$ to $x \approx 3$. This is the portion of the graph that does not pass the horizontal line test.

   Confirm this using the method appropriate to your calculator model.

   On a TI-83/84:

   Step 3: Press [TRACE].

   Step 4: Use the left and right arrow keys to move the cursor to the left local maximum at about $x = 0.5$. Notice that the function has a maximum value at the point (0.42553192, 0.62934032).

   Step 5: Move the cursor to the right to the point at which the function has approximately the same value as in the previous step, which is (3.0851064, 0.54747021).

   On a TI-Nspire:

   Step 4: Press [menu]. Select 5: Trace, and then select 1: Graph Trace.

   Step 5: Move the cursor to the left local maximum at about $x = 0.5$. Notice that at exactly the maximum function value, "maximum" appears. This point should be (0.45, 0.63).

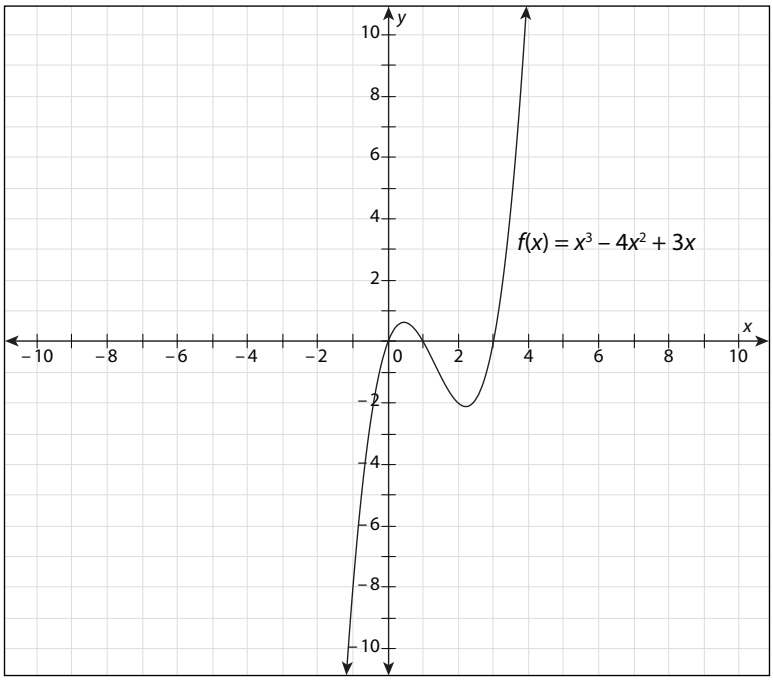
   Step 6: Move the cursor right to the point on the curve that is at approximately the same function value as the maximum found in the previous step. (This may require changing the trace settings: press [menu], select 5: Trace, and then select 3: Trace Step.... Set the trace step to 0.1 and press [enter] to return to the graph.) The coordinates that are close to the function value of 0.63113 found in step 5 are (3.1, 0.65).

(*continued*)

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Instruction

Either calculator will create a graph resembling the following. Keep this view on the calculator for reference in the next example step.



3. Graph the inverse of each function and give the domain value(s) over which the inverse is not one-to-one.

   The graphing calculator can be used to display the inverse, which will give an approximation of the restricted domain(s) over which it is one-to-one and how this compares to the restricted domains of the original function.

   On a TI-83/84:

   Step 6: To see a graph of the inverse of $f(x)$, press [2ND][PRGM] to display the DRAW menu. Select 8: DrawInv.

   Step 7: After "DrawInv" and the cursor appear on the screen, press [VARS]. Arrow over to Y-VARS and select 1: Function.

(*continued*)

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Step 8: Highlight 1: Y_1 and press [ENTER]. This will display "DrawInv Y_1." Press [ENTER] to graph the inverse of the function entered as Y_1. *Note*: The "DrawInv" function on the TI-83/84 is not interactive, so only an approximation of the restricted domain value(s) is possible by viewing the graph.

Step 9: The inverse does not appear to be one-to-one over the restricted domain from approximately $x = -2$ to approximately $x = 0.5$. To see a more exact value of the locality of $x = 0.5$, press [WINDOW] and reset the end points of the minimum for each axis to –1 and the maximum of each axis to 1. To see the inverse, repeat steps 6–8. By tracing the original function, we can see that the right most point of the inverse is relatively close to the point (0.65957447, 0.52550976).

On a TI-Nspire:

Step 7: Constructing the inverse of a function requires that the graph of $f(x)$ be reflected across a constructed line of the form $y = x$. First, add a grid to the screen by pressing [menu] and selecting 2: View. Select 5: Show Grid.

Step 8: To construct the function $y = x$, press [menu] and select 7: Points & Lines. Then select 4: Line. A pencil cursor will be displayed on the graph.

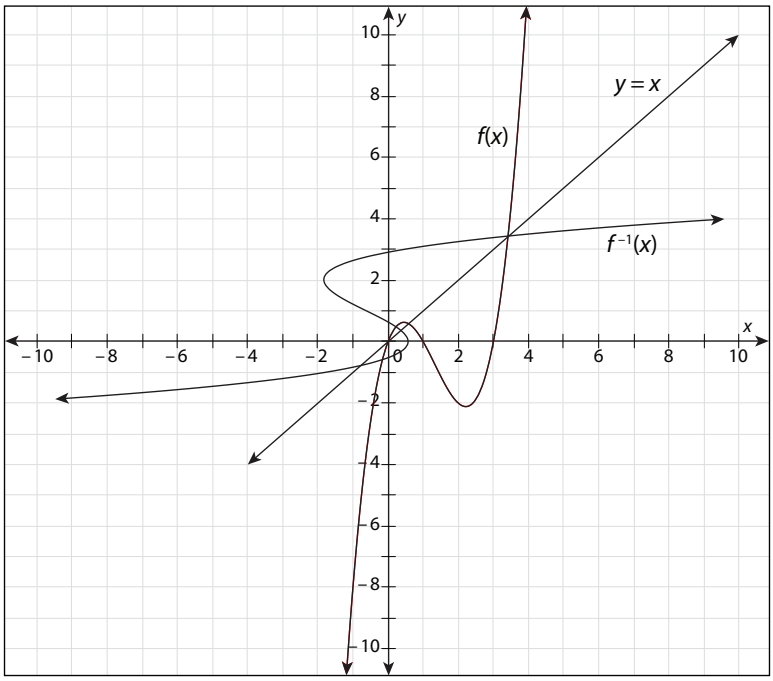
Step 9: Use the NavPad to move the cursor to a point of the form (a, a) and press the click button. Repeat by moving the cursor to a different point of the form (b, b) and then pressing the click button. A line of the form $y = x$ will be displayed. *Note*: A function reflected across the line $y = x$ will produce the inverse because the values of x and y will be exchanged; i.e., point A (a, b) will become point B (b, a).

Step 10: To start the reflection process, press [menu] and select B: Transformation. Then select 2: Reflection. Move the cursor to the function and press the click button to name a point on the function. Then, move the cursor to the line $y = x$ and press the click button. The image of the point on the function will appear. Press [enter] to save the points shown.

(*continued*)

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Step 11: To see the graph of the inverse, press [menu] and select A: Construction. Then select 6: Locus. Click on the image point and press [enter], then click on the point on the original function to see the inverse relation that goes with the function.



Notice that the function and its inverse are reflected across the line $g(x) = x$.

The turning points of $f^{-1}(x)$ appear to be at about $x = -2$ and $x = 0.6$.

4. Compare the domain(s) over which the inverse is one-to-one with the domain(s) over which the function is one-to-one.

   This will determine where the function and its inverse are defined, and can be seen on the graph. The function and its inverse exist over the restricted domain intervals $(-\infty,-2)$ and $(0.6,+\infty)$. The finite domain values are approximations.

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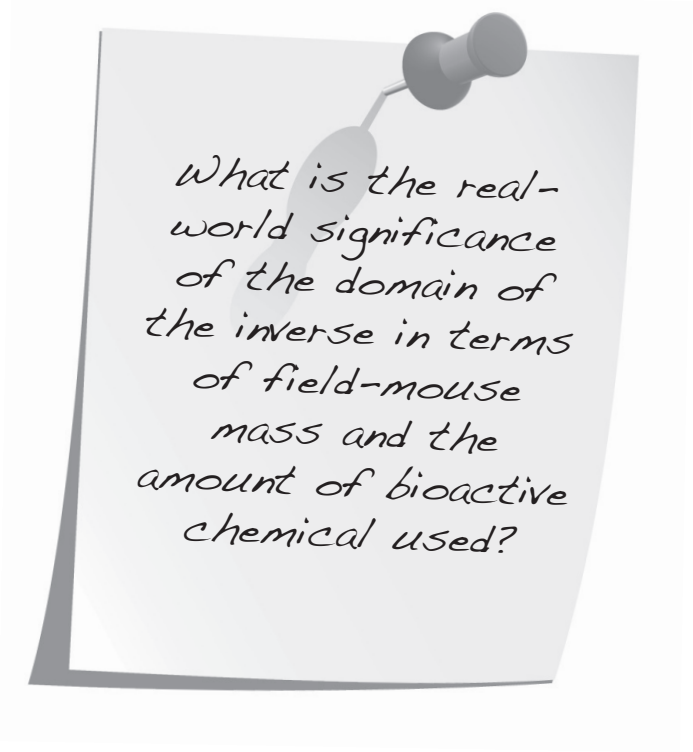
Problem-Based Task 4A.1.2: The Rat Snake's Many-Mouse Diet

SMP	
1 ✓	2
3 ✓	4 ✓
5	6
7	8

In a multi-county region of southeast Georgia, the rat snake population is growing faster than the area can support. If the rat snakes' usual prey populations become scarce, the snakes may resort to eating other small mammals, which could harm the balance of the existing ecosystem.

A herpetologist with the Georgia Department of Agriculture received a grant to introduce a bioactive chemical into the diet of rat snakes. This chemical would control the rat snake population so that beneficial small-mammal populations would not be endangered. The herpetologist plans to introduce the bioactive chemical into the diet of field mice, one of the rat snake's prey species, by spraying the chemical on the field mice's feeding habitats.

The amount of the bioactive chemical that can be tolerated by the field mice is given by the function $f(x)=\sqrt[3]{x-3}+4$, where x is the mass of an individual field mouse in grams, and $f(x)$ is the amount of bioactive chemical solution in grams per 100 liters sprayed on the feeding habitats. Determine the domain of the function if the mice in the habitat each have a mass of 25 grams or less. How does the domain of the function compare to the domain of its inverse? What is the real-world significance of the domain of the inverse in terms of field-mouse mass and the amount of bioactive chemical used?



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Problem-Based Task 4A.1.2: The Rat Snake's Many-Mouse Diet

Coaching

a. What is the mathematical domain of this function without reference to a real-world setting?

b. What is the restricted domain of the real-world function?

c. Graph the function and use the graph to verify your answers to parts a and b.

d. What concentration of the chemical solution can be tolerated by a 25-gram mouse?

e. What is the range of this function in the context of the problem?

f. Find the inverse of the function and describe where it exists.

g. What is the range of the inverse function?

h. When would it be helpful to use the original function? When would it be helpful to use the inverse function?

i. What does the domain of the inverse function mean for the concentration of the bioactive chemical tolerated by any field-mouse mass?

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Problem-Based Task 4A.1.2: The Rat Snake's Many-Mouse Diet

Coaching Sample Responses

a. What is the mathematical domain of this function without reference to a real-world setting?

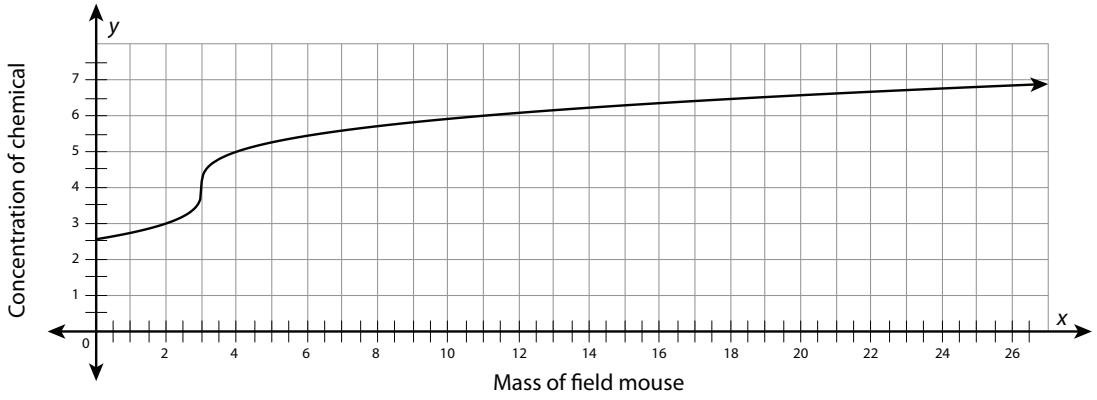
Notice that all values of x are valid for a cube root function, from negative to positive infinity, including 0. (This can be seen in the graph in part c, too.) Therefore, the complete domain for the function $f(x)=\sqrt[3]{x-3}+4$ is $(-\infty,+\infty)$.

b. What is the restricted domain of the real-world function?

Real mice do not have negative mass, and the problem restricts the definition of a field mouse's mass to 25 grams or less. Therefore, the real-world restricted domain is (0, 25].

c. Graph the function and use the graph to verify your answers to parts a and b.

The graph of the function indicates the values of x for which the function is defined.



The function graph starts at $f(0)=\sqrt[3]{(0)-3}+4\approx 2.56$, so the smallest useful concentration of the bioactive chemical is about 2.56 grams per 100 liters. As the mouse mass increases, the tolerable chemical concentration increases sharply for mice between 2 and 4 grams, and then it begins to level off between 6 and 7 grams per 100 liters.

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d. What concentration of the chemical solution can be tolerated by a 25-gram mouse?

Substitute $x = 25$ grams into the original function. For $x = 25$, the concentration is $f(25) = \sqrt[3]{(25)-3}+4 \approx 6.80$ grams per 100 liters of solution.

e. What is the range of this function in the context of the problem?

When $x = 0$, $f(x) \approx 2.56$. When $x = 25$, $f(x) \approx 6.80$. Therefore, the range is [2.56, 6.80]. The concentration of the chemical that is used should be between these two values.

f. Find the inverse of the function and describe where it exists.

To find the inverse function, switch $f(x)$ and x in the function $f(x) = \sqrt[3]{x-3}+4$, replace $f(x)$ with $f^{-1}(x)$, and solve for $f^{-1}(x)$.

$$f(x) = \sqrt[3]{x-3}+4$$
$$x = \sqrt[3]{f^{-1}(x)-3}+4$$
$$x-4 = \sqrt[3]{f^{-1}(x)-3}$$
$$(x-4)^3 = \left(\sqrt[3]{f^{-1}(x)-3}\right)^3$$
$$x^3-12x^2+48x-64 = f^{-1}(x)-3$$
$$x^3-12x^2+48x-61 = f^{-1}(x)$$
$$f^{-1}(x) = x^3-12x^2+48x-61$$

The inverse function is $f^{-1}(x) = x^3-12x^2+48x-61$. Recall that the domain of the inverse is the same as the range of the original function. The domain of this function is all real numbers, but in the context of this problem, the domain is [2.56, 6.80]. This indicates the possible allowable concentrations of the bioactive chemical.

Notice that in the original function, x is the mass of the mouse and $f(x)$ is the concentration of the chemical. In the inverse function, x is the concentration of the chemical and $f^{-1}(x)$ is the mass of the mouse.

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g. What is the range of the inverse function?

The range of an inverse function is the same as the domain of the original function. Thus, the range of the inverse function is (0, 25]. This means that concentrations of bioactive chemical between 2.56 grams and 6.80 grams per 100 liters of solution are appropriate for mice ranging from 0 to 25 grams.

h. When would it be helpful to use the original function? When would it be helpful to use the inverse function?

The original function is useful if you know the weight of a mouse (x) and you want to calculate the concentration of chemical ($f(x)$) that the mouse can tolerate.

The inverse function is useful if you have a certain concentration of chemical (x) and you want to know the weight of a mouse ($f^{-1}(x)$) that can tolerate this concentration.

i. What does the domain of the inverse function mean for the concentration of the bioactive chemical tolerated by any field-mouse mass?

Theoretically, the concentration of bioactive chemical needed for field mice is between 2.56 and 6.80 grams per 100 liters of solution. However, the amount that a mouse can tolerate varies depending on the mouse's weight. The higher concentration of the chemical is more likely to bring the snake population under control, but smaller mice may be unable to survive the toxicity of the dosage. Using the lower concentration would assure that the mouse population is not harmed, but this concentration may not be sufficient to control the snake population.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4A.1.2: Determining Inverses of Other Functions

For problems 1–4, write a function for each description, and then determine the inverse of the function.

1. a rational function that has a zero at $x = 2$ and is undefined at $x = -3$

2. a polynomial function that has solutions at $x = 0$ and $x = 1$

3. a radical function that is the square root of the sum of 1 more than 3 times a number

4. an absolute-value function that is half of the absolute value of the quantity 5 less 2 times a number

For problems 5–7, determine the shared restricted domain(s) over which each function and its inverse are defined.

5. $f(x)=\dfrac{2}{1-x}$

6. $g(x)=\sqrt[4]{3x^2}$

7. $h(x)=|4-2x^2|$

continued

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For problems 8–10, read the scenarios and use the information given in each to complete the problems.

8. At an automobile manufacturing plant, it takes one model of a robotic welder t seconds to weld a car chassis frame. The time it takes a second model to weld the frame is $t - 1$ seconds. The function that describes how long it takes the two welders working together to weld the frame is $\frac{1}{W(t)} = \frac{1}{t} + \frac{1}{t-1}$. Find the inverse of the function and describe the restricted domain over which it is defined.

9. A privately funded orbiting spacecraft has to decelerate in order to change the altitude of its orbital path around Earth. The velocity of the orbiter is given by the function $v(x) = \sqrt{v_0{}^2 + 2ax}$, where v_0 is the orbiter's velocity at the time its deceleration starts, a is the deceleration in kilometers per minute squared, and x is the distance in kilometers the orbiter travels during the deceleration burn. If $v_0 = 20$ kilometers per minute and $a = -5$ kilometers per minute squared, write the inverse of the velocity function and state the restricted domain of x for the inverse.

10. The amount of variation in the pressure that a natural gas pipeline can withstand without triggering an alarm at the closest monitoring station is given by the absolute function $f(x) = |x - 25|$, where $x \geq 0$. The alarm is triggered if the pressure variation function rises above 40 units. Write the inverse of this function and describe the restricted domain of the inverse and the function itself.

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Progress Assessment

Circle the letter of the best answer.

1. Which points lie on the graph of the inverse of a function with a graph that contains the points (0, 4), (3, 9), and (6, 15)?

   a. (0, –4), (3, –9), and (6, –15)

   b. (0, 4), (–3, 9), and (–6, 15)

   c. (4, 0), (9, 3), and (15, 6)

   d. (4, 0), (9, –3), and (15, –6)

2. If g is the inverse of f and $f(3) = 4$, what is $f(3) + g(4)$?

   a. 6

   b. 7

   c. 8

   d. 12

3. Over which of the following domain(s) is the function $f(x) = 5x^2$ one-to-one?

   a. $(-\infty, +\infty)$

   b. $(-\infty,0] \cap [0,+\infty)$

   c. $(-\infty,0] \cup (0,+\infty)$

   d. $(-\infty,0) \cup (0,+\infty)$

4. What is the inverse of the function $f(x) = 36 - 4x^2$ that exists over a restricted domain?

   a. $f^{-1}(x) = 4 - 36x^2$

   b. $f^{-1}(x) = 36x^2 - 4$

   c. $f^{-1}(x) = \pm\frac{1}{4}\sqrt{36-x}$

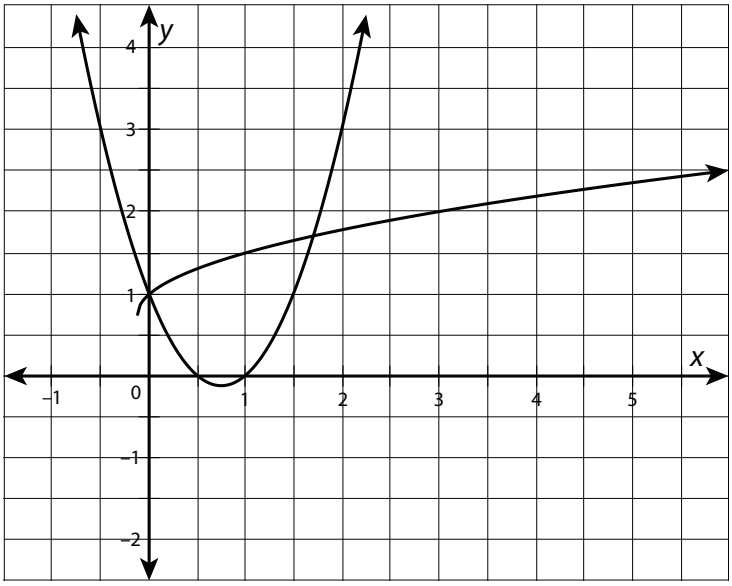
   d. $f^{-1}(x) = \pm\frac{\sqrt{36-x}}{2}$

continued

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Assessment

5. The graph shows a function $f(x)$ and one part of its inverse, $f^{-1}(x)$.



What is a restricted domain over which the function *and* its inverse exist?

a. $(-\infty, +\infty)$

b. $(-\infty, 1]$

c. $\left[\frac{3}{4}, +\infty\right]$

d. $[1, +\infty)$

6. Over what domain are the function $f(x)=\frac{x-4}{x+4}$ and its inverse, $f^{-1}(x)=-4\left(\frac{x+1}{x-1}\right)$, one-to-one?

a. $(-\infty,-4)\cup(-4,1)\cup(1,+\infty)$

b. $(-\infty,-4)\cup(-4,4)\cup(4,+\infty)$

c. $(-\infty,-1)\cup(-1,1)\cup(1,+\infty)$

d. $(-\infty,1)\cup(1,4)\cup(4,+\infty)$

7. What type of function is the inverse of the function $f(x)=\sqrt[4]{x^2-5}$?

a. a fourth-degree polynomial function

b. a fourth-root radical function

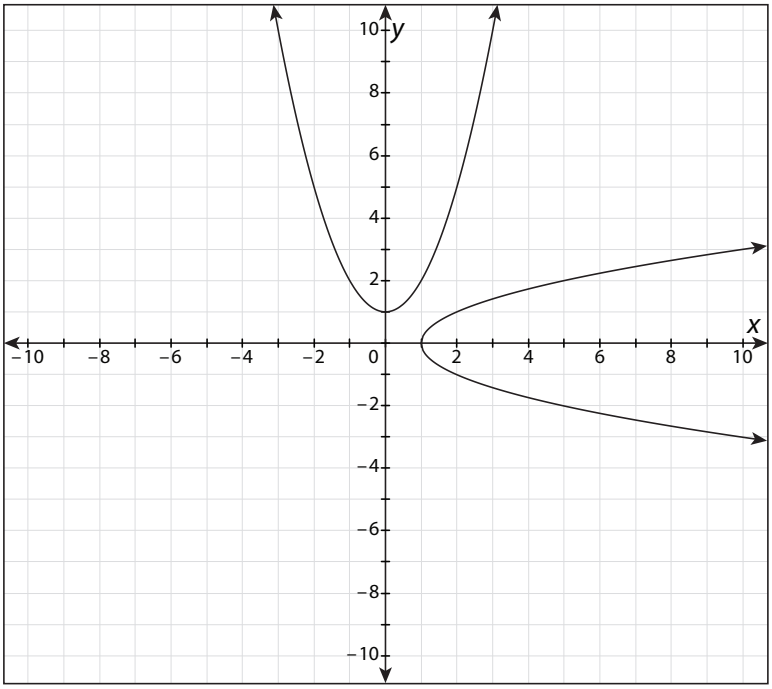
c. a quadratic function

d. a square root function

continued

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8. The function $f(x) = x^2 + 1$ is shown on the graph along with its inverse.



What is the restricted domain over which the function and its inverse are one-to-one and defined?

a. $(-\infty,-1)\cup(1,+\infty)$

b. $(-1,0)\cup(0,1)$

c. $[-1,0]\cup[0,1]$

d. $[1,+\infty)$

9. The movement of a glacier is described by the function $\frac{1}{C(t)}=\frac{1}{3t}+\frac{1}{2t+3}$, where t is the time in days and $C(t)$ is the distance in meters. What is the real-world domain of the inverse of the function?

a. $t>0$

b. $t>\frac{3}{5}$

c. $t>\frac{3}{2}$

d. $t>3$

continued

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Assessment

10. A minnow swimming at a rate of 5 centimeters per second accelerates at a rate of 20 centimeters per second squared to escape a predator. The function describing the distance the minnow swims during the burst of acceleration is $d(t) = 5t + 10t^2$. The inverse of this function is derived from the equation $t = 5[d^{-1}(t)] + 10[d^{-1}(t)]^2$. When solving this equation for the conditions in this problem, what is true of the resulting radical function?

    a. Only the negative value of the radical is used.

    b. Only the positive value of the radical is used.

    c. Both the positive and negative values of the radical are used.

    d. Neither the positive or negative values of the radical are used.

Use your knowledge of functions and inverses to complete the following problem.

11. A university veterinarian injects a cow with a new bovine vaccine. The concentration $f(x)$ of the vaccine in the animal's system over time x is described by the function $f(x) = \frac{5x}{0.01x^2 + 5}$, where $f(x)$ is the concentration of vaccine in grams per liter of solution and x is the time in minutes since the cow was injected. Determine the value of the function after 1 hour. Write the equation of the inverse function, $f^{-1}(x)$. Then, explain why $f^{-1}(x)$ is undefined for $x < 11.2$ minutes.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Which logarithmic expression corresponds to the exponential function $f(x) = 10^x$?

   a. $\log_x f(x) = 10$

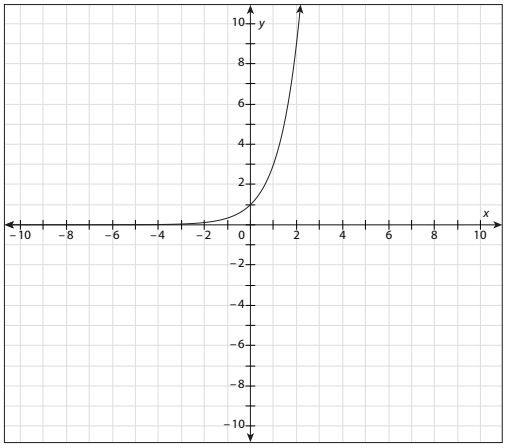
   b. $\log_{10} f(x) = x$

   c. $\log_{10} f(x) = 10$

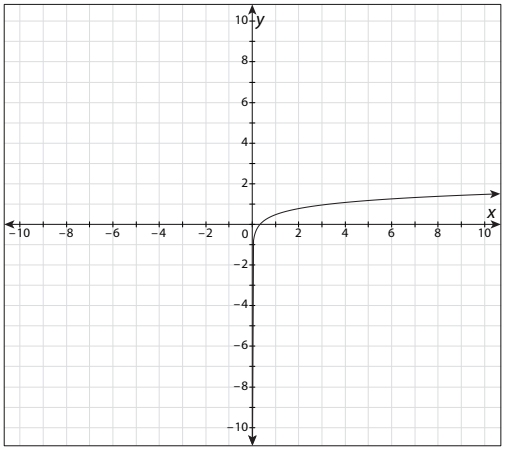
   d. $\log_x f(x) = x$

2. Which is the graph of a logarithmic function?

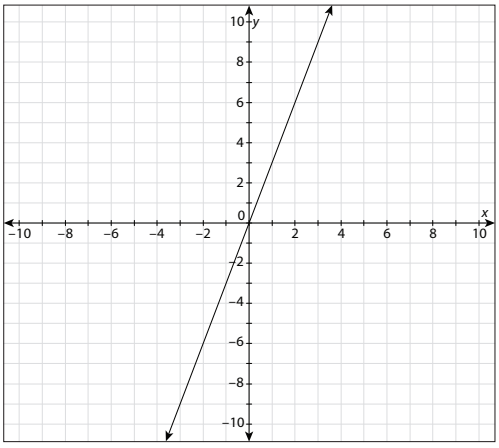
   a.



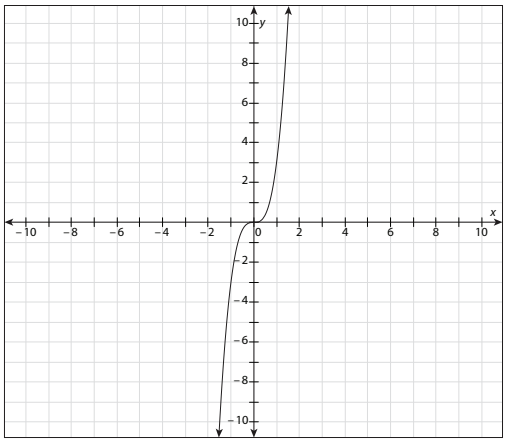
   c.



   b.



   d.



continued

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Assessment

3. Which table row represents data points on the graph of a logarithmic function?

x	–2	–1	0	1	2
Q	—	—	—	0.70	1
R	0.08	0.4	2	10	50
S	–18	–0.5	0	0.5	18
T	–2	–1	0	1	2

a. Q

b. R

c. S

d. T

4. Which exponential function corresponds to the logarithmic expression $\log_4 g(x) = x + 3$?

a. $g(x) = 3 \bullet 4^x$

b. $g(x) = 4 \bullet x^3$

c. $g(x) = 4^{3x}$

d. $g(x) = 4^{x+3}$

5. What is the base of the natural logarithm?

a. 2

b. e

c. π

d. 10

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Lesson 2: Modeling Logarithmic Functions

Instruction

Common Core State Standards

SMP
1 ✓ 2 ✓
3 ✓ 4 ✓
5 ✓ 6 ✓
7 8

F–IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F–IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F–IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F–IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

F–BF.4 Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *or* $f(x) = (x + 1)/(x – 1)$ *for* $x \neq 1$.

F–LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$, where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.★

SRB p. U4A-45

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Instruction

Essential Questions

1. If an exponential function is $f(x) = a^x$, what is its inverse function?
2. What exponential function has as its inverse the logarithmic function $\log_a g(x) = x$?
3. What are the domain and range of the logarithmic function that is the inverse of the exponential function $f(x) = a^x$?
4. What function is the inverse of the exponential function $f(x) = ab^{cx}$?
5. How do the values of an exponential function and its inverse logarithmic function vary over a domain and what are the restrictions, if any, on the domain values?

WORDS TO KNOW

argument	the result of raising the base of a logarithm to the power of the logarithm, so that b is the argument of the logarithm $\log_a b = c$
base	the quantity that is being raised to an exponent in an exponential expression; in a^x, a is the base; or, the quantity that is raised to an exponent which is the value of the logarithm, such as 2 in the equation $\log_2 g(x) = 3 - x$
common logarithm	a base-10 logarithm which is usually written without the number 10, such as $\log x = \log_{10} x$
e	an irrational number with an approximate value of 2.71828; e is the base of the natural logarithm ($\ln x$ or $\log_e x$)
exponential function	a function that has a variable in the exponent, such as $f(x) = 5^x$
logarithmic function	the inverse of an exponential function; for the exponential function $f(x) = 5^x$, the inverse logarithmic function is $g(x) = \log_a x$
natural logarithm	a logarithm whose base is the irrational number e; usually written in the form "ln," which means "$\log_e$"
power	the result of raising a base to an exponent; 32 is a power of 2 since $2^5 = 32$

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Recommended Resources

- The Math Page. "Logarithmic and Exponential Functions."

  http://www.walch.com/rr/00230

  This tutorial offers a brief, step-by-step introduction to functions and inverses. The site offers an abundant and diverse selection of examples and problems as well.

- Patrick JMT: Just Math Tutorials. "Properties of Logarithms."

  http://www.walch.com/rr/00231

  This 20-minute instructional video reviews graphing logarithmic functions and the rules of logarithms, and provides detailed instruction on applying these rules.

- S.O.S. Math. "Logarithms and Their Inverses."

  http://www.walch.com/rr/00232

  This comprehensive site gives detailed examples and explanations of how to find the inverses of logarithmic functions.

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Lesson 4A.2.1: Logarithmic Functions as Inverses

Warm-Up 4A.2.1

In the metric system, sound intensity is measured in watts per square meter. For example, the average sound intensity at a rock concert is 0.1 watt per square meter. The threshold of human hearing is about 10^{-12} watt per square meter.

1. What is 0.1 watt per square meter written as an expression using a base of 10 and an exponent?

2. How much greater is the sound intensity at an average rock concert than the threshold of human hearing?

3. Write an exponential function with a base of 10 and a power of x to represent sound intensity, $I(x)$.

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Instruction

Lesson 4A.2.1: Logarithmic Functions as Inverses

Common Core State Standards

F–BF.4 Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *or* $f(x) = (x + 1)/(x – 1)$ *for* $x \neq 1$.

F–LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$, where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.★

Warm-Up 4A.2.1 Debrief

1. What is 0.1 watt per square meter written as an expression using a base of 10 and an exponent?

   Recall that in order to rewrite 0.1 using a base of 10 and an exponent, first move the decimal place to the right to create the whole number 1. Then determine the exponent by counting how many places the decimal was moved. To move from 0.1 to 1.0, the decimal was moved 1 unit to the right, creating an exponent of –1. Therefore, 0.1 written using base 10 with an exponent is 10^{-1}.

2. How much greater is the sound intensity at an average rock concert than the threshold of human hearing?

   To determine this, divide the intensity at the rock concert by the threshold of human hearing.

   $$\frac{10^{-1}}{10^{-12}} = 10^{(-1)-(-12)} = 10^{-1+12} = 10^{11}$$

   The sound intensity of the average rock concert is 100 billion times greater than the threshold of human hearing.

3. Write an exponential function with a base of 10 and a power of x to represent sound intensity, $I(x)$.

   The simplest exponential function is given by $I(x) = 10^x$.

Connection to the Lesson

- Students will write exponential functions based on data, and write the inverses of exponential functions as logarithmic functions.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- determining the dependent and independent variables in an exponential function based on data from a graph or in a problem statement (F–IF.2)
- recognizing that a logarithmic function is the inverse of the related exponential function by comparing their graphs on the same axes (F–IF.5)

Introduction

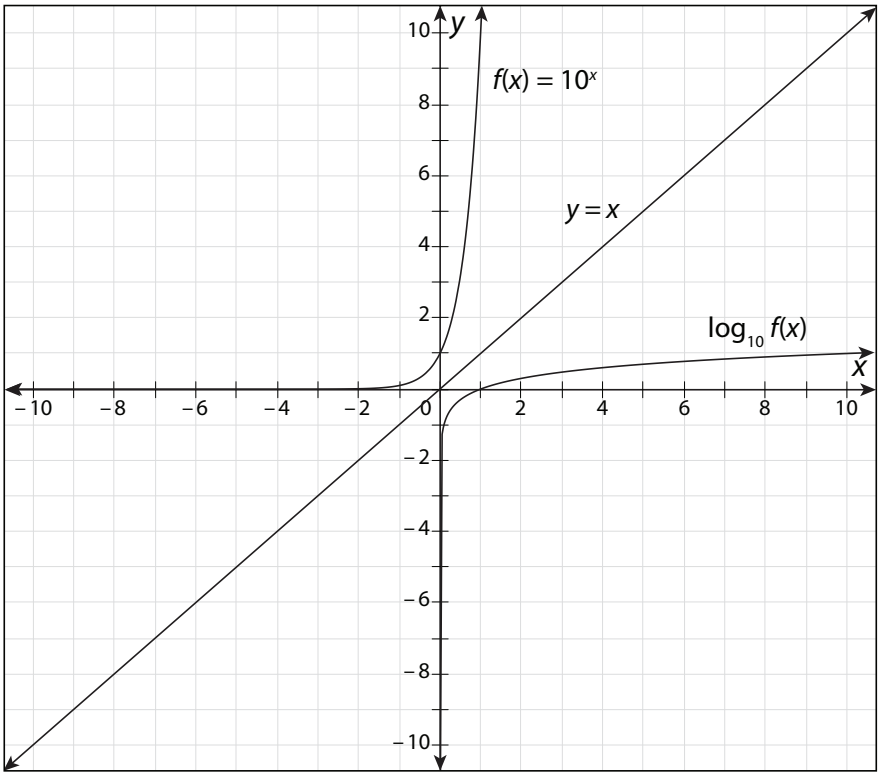
In this course, you have studied a variety of functions, such as trigonometric functions, quadratic functions, and the inverses of functions. You have worked with exponents in the past and probably realize that exponents are not always whole numbers. You may also recall that sometimes exponents contain variables.

An **exponential function** is a function that has a variable in the exponent, such as $f(x) = 5^x$. The **power** is the result of raising a base to an exponent; 32 is a power of 2 since $2^5 = 32$. The exponent is the value of the function's corresponding logarithm, such as x in the logarithmic function $x = \log_5 f(x)$ and its corresponding exponential function, $f(x) = 5^x$.

Like other functions, exponential functions have inverses, which are called logarithmic functions. A **logarithmic function** is the inverse of an exponential function. For example, for the exponential function $f(x) = 5^x$, the inverse logarithmic function is $x = \log_5 f(x)$. If the exponential function is of the form $f(x) = a^x$, then the logarithmic function is of the form $\log_a f(x) = x$. This confirms the relationship between a function $f(x) = y$ and its inverse, $g(y) = x$. This relationship can also be seen from the following graph of an exponential function, $f(x) = 10^x$, and its inverse logarithmic function, $\log_{10} f(x)$.

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Instruction



Notice that the exponential function and its inverse logarithmic function are reflected across the line $f(x) = x$ (often written as $y = x$). For example, this means that for the value $x = 3$, the exponential function is given by $f(3) = 10^3$ and its inverse logarithmic function is $\log_{10} f(3) = \log_{10} (10^3) = 3$. In real-world problems, such as the sound-intensity example in the Warm-Up, there will be situations in which the inverse function is more effectively used than the function from which the inverse is derived. Knowledge of the real-world domain of the function can help make the decision about whether the function or its inverse has more meaning. Another factor in deciding which function to work with is how simplified the expressions and numbers are for each function.

Key Concepts

- As the graph in the Introduction shows, the exponential function and its inverse are one-to-one over their domains. The domain of the exponential function is $(-\infty,+\infty)$. However, the domain of the logarithmic function is $(0,+\infty)$.
- The range of the exponential function is $(0,+\infty)$. The range of the logarithmic function is $(-\infty,+\infty)$. This information provides more evidence that the logarithmic function is the inverse of the exponential function.

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- In the graphed example, the value of the exponential function is 1 at $x = 0$ because $f(0) = 10^0 = 1$. Correspondingly, the value of the inverse logarithmic function is 0 at $x = 1$ because $\log_{10}(1) = 0$.
- Exponential functions with more constants can be explored using the properties of exponents or by looking at data tables generated by a graphing calculator.
- Use a graphing calculator to explore the domain, range, and other key points of the function $4 \cdot 3^{2x}$ and its inverse logarithmic function by looking at data tables of domain and function values. Follow the directions appropriate to your calculator model.

On a TI-83/84:

Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

Step 2: At Y1, use your keypad to enter values for the function. Use [X, T, θ, n] for x and [x^2] for any exponents.

Step 3: Press [GRAPH]. Press [WINDOW] to adjust the graph's axes.

Step 4: Press [2ND][GRAPH] to display a table of values. Look at the domain values around $x = 0$.

On a TI-Nspire:

Step 1: Press [home] to display the Home screen.

Step 2: Arrow down to the graphing icon, the second icon from the left, and press [enter].

Step 3: Enter the function to the right of "$f1(x) =$" and press [enter].

Step 4: To adjust the x- and y-axis scales on the window, press [menu] and select 4: Window and then 1: Window Settings. Enter each setting as needed. Tab to "OK" and press [enter].

Step 5: To see a table of values, press [menu] and scroll down to 2: View, then 5: Show Table.

- Either calculator will show exponential function values that approach 0 as x becomes negative and that increase as x becomes positive.

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- To show the corresponding function values for the inverse logarithmic function, switch the x- and y-values, as shown in the following table.

Exponential function	x	–2	–1	0	1	2
	y	0.05	0.44	4	36	324
Logarithmic function	x	0.05	0.44	4	36	324
	y	–2	–1	0	1	2

- Notice that the logarithmic function does not exist for negative domain values.
- The logarithmic function values can be verified with the data table.
- For example, $f(x) = 4 \bullet 3^{2x}$, so $x = \frac{1}{2}\log_3\left(\frac{f(x)}{4}\right)$. For $x = 0$, $\log_3\left(\frac{f(0)}{4}\right) = 0 \rightarrow \log_3\left(\frac{4}{4}\right) = \log_3(1) = 0$.
- Notice that the coefficient of 4 in the function changes the value of the function to 4 at $x = 0$, and it changes the value of x to 4 when the value of the inverse function is 0.
- Finally, the basic definitions and rules of exponents and logarithms will be needed in order to manipulate and calculate exponential and logarithmic functions, summarized as follows.

Terms and Rules for Logarithms

- In a logarithmic equation, $\log_a b = c$, a is the base, b is the argument, and c is the logarithm of b to the base a.
- The **base** is the quantity that is being raised to an exponent in an exponential expression, such as a in the expression a^x, or the quantity that is raised to an exponent which is the value of the logarithm, such as 2 in the function $\log_2 g(x) = 3 - x$.
- The **argument** is the result of raising the base of a logarithm to the power that is the value of the logarithm, so that b is the argument of the logarithm $\log_a b = c$.

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- You may recall the rules for working with exponents; for example, according to the Product of Powers Property, when multiplying two exponents with the same base, keep the base and add the powers: $a^x \bullet a^y = a^{x+y}$. The rules for various operations with logarithms are derived from the rules for exponents. The following table lists some exponent rules, followed by the equation and name of the related logarithmic rule.

Exponent rule	Related logarithm rule	Logarithm rule name
$a^x \bullet a^y = a^{x+y}$	$\log_a (x \bullet y) = \log_a x + \log_a y$	Product rule
$\frac{a^x}{a^y} = a^{x-y}$	$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$	Quotient rule
$(a^x)^y = a^{x \bullet y}$	$\log_a x^y = y \bullet \log_a x$	Power rule

- Another rule, the base change rule, allows for computing with logarithms other than base 10; one form of the equation for this rule is $\log_b a = \frac{\log_{10} a}{\log_{10} b}$. (Other forms will be discussed later.)
- This rule is particularly useful when working with calculators that only calculate with logarithms with bases of e (natural logarithms) and 10 (common logarithms).
- The irrational number ***e*** has a value of approximately 2.71828. A **natural logarithm** is a logarithm with a base of e. Natural logarithms are usually written in the form "ln," which means "$\log_e$." For example, $f(x) = \ln (1 - x)$ is understood to be the inverse of the function for the exponential function $g(x) = 1 - e^x$.
- A **common logarithm**, on the other hand, is a logarithm with a base of 10. When writing a common logarithm, the 10 is usually omitted, such that $\log x = \log_{10} x$. For example, the logarithmic function $f(x) = \log x$ is understood to be the inverse function for the exponential function $g(x) = 10^x$.

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Common Errors/Misconceptions

- incorrectly identifying the domain and range variables in an exponential function and in its inverse logarithmic function
- confusing the base with the power in expressing an exponential function as a logarithmic function, or vice versa
- misidentifying the domains of exponential functions and their inverse logarithmic functions
- misinterpreting the coefficients of a base and of a variable in a power in an exponential function when writing the inverse logarithmic function
- misapplying the rules of exponents and logarithms in rewriting exponential and logarithmic functions

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Guided Practice 4A.2.1

Example 1

Write the inverse of the exponential function $f(x) = 0.1 \cdot 2^{0.3x}$.

1. Isolate the exponential term.

   This is necessary in order to use the logarithmic function definition $\log_a f(x) = x$.

   Divide both sides of the equation by 0.1; this is equal to multiplication by 10.

   $f(x) = 0.1 \cdot 2^{0.3x}$ becomes $10 \cdot f(x) = 2^{0.3x}$.

2. Rewrite the result as a logarithm.

   $10 \cdot f(x) = 2^{0.3x}$ becomes $\log_2 [10 \cdot f(x)] = 0.3x$.

3. Isolate the exponent variable, x.

   $\log_2[10 \cdot f(x)] = 0.3x$ becomes $x = \frac{10}{3} \cdot \log_2[10 \cdot f(x)]$.

4. Use the rules of logarithms to rewrite the result so that the simplest expression possible can be used to evaluate the function numerically.

   Use the product rule for logarithms to rewrite the expression as the sum of two logarithms:

   $$\frac{10}{3} \cdot \log_2[10 \cdot f(x)] = \frac{10}{3}[\log_2 10] + \frac{10}{3}[\log_2 f(x)] = x$$

   Use the power rule to rewrite the separate logarithms as exponentials:

   $$\frac{10}{3}[\log_2 10] + \frac{10}{3}[\log_2 f(x)] = \log_2 10^{\frac{10}{3}} + \log_2 f(x)^{\frac{10}{3}} = x$$

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5. Switch the domain and function variables to write the logarithmic inverse as a logarithmic function.

   This step is necessary if the logarithmic function is considered independent of the exponential function from which it was derived.

   $$\log_2 10^{\frac{10}{3}} + \log_2 x^{\frac{10}{3}} = f^{-1}(x)$$

   The inverse of the exponential function $f(x) = 0.1 \cdot 2^{0.3x}$ is $f^{-1}(x) = \log_2 10^{\frac{10}{3}} + \log_2 x^{\frac{10}{3}}$.

Example 2

Find the exponential function that is the inverse of the logarithmic function $g(x) = \log_6 x^2 - \log_6 25$.

1. Switch the domain and function variables.

   $g(x) = \log_6 x^2 - \log_6 25$ becomes $x = \log_6 [g(x)]^2 - \log_6 25$.

2. Use the rules of logarithms to rewrite the result so that the simplest expression possible can be found for the logarithmic function before it is converted to exponential form.

$x = \log_6 [g(x)]^2 - \log_6 25$	Logarithmic function from step 1
$x = \log_6 [g(x)]^2 - \log_6 5^2$	Rewrite 25 as 5^2.
$x = 2 \cdot \log_6 g(x) - 2 \cdot \log_6 5$	Apply the power rule to both logarithmic terms.
$x = 2 \cdot [\log_6 g(x) - \log_6 5]$	Factor 2 out from both terms.
$x = 2 \cdot \log_6 \left[\frac{g(x)}{5}\right]$	Use the quotient rule to rewrite the logarithmic terms.

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3. Solve for the logarithmic term.

   The equation may be easier to work with by applying the Symmetric Property of Equality so that x is on the right side of the equation.

   $$2 \bullet \log_6 \left[\frac{g(x)}{5}\right] = x$$

   $$\log_6 \left[\frac{g(x)}{5}\right] = \frac{x}{2}$$

   $$\log_6 [0.2 \bullet g(x)] = 0.5x$$

4. Write the exponential function from the simplified logarithmic function by using the definition of an exponential function and its inverse.

$\log_6 [0.2 \bullet g(x)] = 0.5x$	Logarithmic function
$0.2 \bullet g(x) = 6^{0.5x}$	Rewrite as an exponential function.
$g(x) = 5 \bullet 6^{0.5x}$	Divide both sides by 0.2.

   The inverse of the logarithmic function $g(x) = \log_6 x^2 - \log_6 25$ is the exponential function $g(x) = 5 \bullet 6^{0.5x}$.

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Example 3

Use a logarithmic function to solve the exponential equation $4^{\left(\frac{x-3}{x}\right)}=5$.

1. Rewrite the exponential equation as its logarithmic form.

   This is an alternative method to using the properties of exponents to solve the equation.

   $4^{\left(\frac{x-3}{x}\right)}=5$ becomes $\log_4 5=\frac{x-3}{x}$.

2. Simplify the result algebraically.

$\log_4 5=\frac{x-3}{x}$	Inverse logarithmic function
$x \bullet \log_4 5 = x - 3$	Multiply both sides by x.
$3 = x - x \bullet \log_4 5$	Add and subtract from both sides.
$3 = x(1 - \log_4 5)$	Factor out x.
$x=\frac{3}{1-\log_4 5}$	Divide both sides by $1 - \log_4 5$.

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Instruction

3. Solve the original exponential equation using the rules of exponents.

This will serve as a check on the logarithmic approach used in steps 1 and 2.

$4^{\left(\frac{x-3}{x}\right)}=5$	Original equation
$4^{x-3}=5^{x}$	Raise both sides to the power of x.
$\frac{4^{x}}{4^{3}}=5^{x}$	Rewrite subtracted exponents as a fraction.
$\left(\frac{4}{5}\right)^{x}=4^{3}$	Cross multiply and simplify.
$\frac{4}{5}=4^{\frac{3}{x}}$	Raise both sides to the $\frac{1}{x}$ power.

Simplify using the definition of logarithm and the quotient rule.

$\frac{4}{5}=4^{\frac{3}{x}}$	Simplified exponential equation
$\log_4\left(\frac{4}{5}\right)=\frac{3}{x}$	Rewrite as a logarithm.
$\log_4 4-\log_4 5=\frac{3}{x}$	Apply the quotient rule.

The resulting expression can be rearranged to equal x, yielding the result found with the logarithmic function in step 2:

$\log_4 4-\log_4 5=\frac{3}{x}$ becomes $x=\frac{3}{1-\log_4 5}$.



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Example 4

Write the domain and function value of the exponential function $f(x) = 1.23 \cdot 2^{0.7x}$ and its inverse at a domain value of $x = 1.05$. Use a graphing or scientific calculator and the rules of exponents and logarithms to verify your results.

1. Substitute $x = 1.05$ into the function $f(x) = 1.23 \cdot 2^{0.7x}$.

   Evaluate the function using either a graphing calculator or a scientific calculator with exponentiation functionality.

   $$f(1.05) = 1.23 \cdot 2^{0.7(1.05)} \approx 2.05$$

2. Write the domain and function value of the inverse of the exponential function at $x = 1.05$.

   Switch the domain and function values of the exponential function to determine the domain and function values of the inverse logarithmic function at $x = 1.05$.

   The corresponding values for the inverse logarithmic function are given by the ordered pair (2.05, 1.05).

3. Verify that the ordered pair determined in step 2 satisfies the inverse logarithmic function.

   To verify that (2.05, 1.05) satisfies the inverse logarithmic function, first determine the inverse logarithmic function from the original function using the definition of logarithm and the quotient and power rules.

$f(x) = 1.23 \cdot 2^{0.7x}$	Original function
$0.7x = \log_2\left[\dfrac{f(x)}{1.23}\right]$	Divide by 1.23 and rewrite as a logarithm.
$x = \dfrac{10}{7}\log_2 f(x) - \dfrac{10}{7}\log_2 1.23$	Divide by 0.7 and apply the quotient rule.
$x = \log_2 f(x)^{\frac{10}{7}} - \log_2 1.23^{\frac{10}{7}}$	Apply the power rule.

(*continued*)

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Switch x and $f(x)$ to write the inverse logarithmic function.

$$f(x)=\log_2 x^{\frac{10}{7}}-\log_2 1.23^{\frac{10}{7}}$$

Substitute the value of x found in step 2.

$$f(2.05)=\log_2 (2.05)^{\frac{10}{7}}-\log_2 1.23^{\frac{10}{7}} \quad \text{Substitute 2.05 for } x.$$

$$f(2.05)=\log_2\left(\frac{2.05}{1.23}\right)^{\frac{10}{7}} \quad \text{Apply the quotient rule.}$$

$$f(2.05)\approx\frac{10}{7}\log_2 1.67 \quad \text{Apply the power rule.}$$

4. Use a graphing calculator to estimate the value of the inverse logarithmic function at $x = 2.05$.

   The process for estimating the function value will depend on the calculator used. Follow the directions specific to your calculator model.

   On a TI-83/84:

   Note: The TI-83/84 can only calculate natural (base-e) and common (base-10) logarithms, so the base change rule $\log_b a=\frac{\log_{10} a}{\log_{10} b}$ must be used to rewrite the expression found in step 3 before proceeding:

   $$\frac{10}{7}\log_2 1.67=\frac{10}{7}\left(\frac{\log_{10} 1.67}{\log_{10} 2}\right)$$

   Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

   Step 2: Use the keypad and [LOG] key to enter the values for the expression. Press [ENTER].

(*continued*)

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On a TI-Nspire:

Note: The TI-Nspire can calculate the logarithm directly without first converting it to a base-10 logarithm.

Step 1: Press [home].

Step 2: Arrow down to the calculator icon, the first icon from the left, and press [enter]. Press [ctrl][clear] to create a new document or to clear any previous calculations on the current document.

Step 3: Press [ctrl][10^x] to bring up the $\log_{\square}\square$ field.

Step 4: Enter the argument of the logarithm into the blank subscript field. Tab to the next blank field to enter the base. Press [enter].

Step 5: To multiply the result by $\frac{10}{7}$, press [ctrl][(–)]. Then, enter the numbers and the operations using your keypad.

Press [enter].

Either calculator will return a result of approximately 1.057.

5. Compare the results of steps 2 and 4.

Both procedures result in an ordered pair of approximately (2.05, 1.06) for the inverse logarithmic function.



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Problem-Based Task 4A.2.1: Healing the Waters

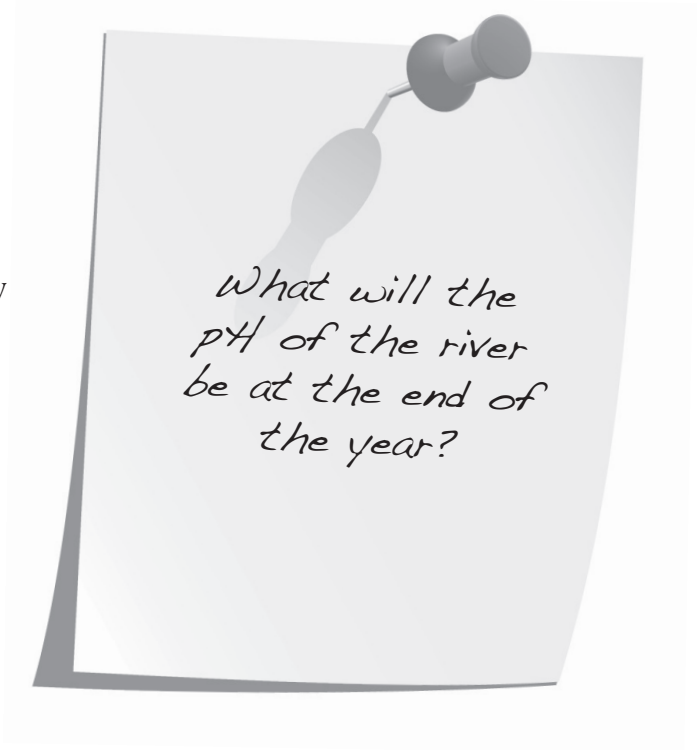
SMP	
1 ✓	2
3	4 ✓
5	6 ✓
7	8

The Chattamulgee River Keepers have embarked on a year-long river cleanup. They aim to work with businesses and towns along the river to reduce the acidity of the water so that a greater variety of plant and animal life can live in the river.

The river's acidity can be measured in two ways:

- by determining the concentration of the chemical responsible for the river's acidity. This concentration is measured in milligrams per liter and is usually a number with a magnitude of 10^{-6}. This is a measure of the mass of a single hydronium ion in a solution measured in liters by the pH of the solution, defined as $\text{pH} = \log_{10}\left(\frac{1}{C}\right) = -\log_{10} C$, where C is the concentration of the chemical responsible for the level of acidity in the solution.
- by determining the pH level of the river. The pH level is the preferred measurement of river acidity because it is a number between 0 and 14 instead of a very small number represented by scientific notation. The pH of "neutral" water (non-acidic) is 7. Other common pH measures are bleach, which is very alkaline (the "opposite" of acidic) with a pH of 13, and gastric or stomach acid, which has a pH of 1.

At the beginning of the year, the pH of the river is 5.2. After one month of education and cleanup activities, the river pH is 5.4. By what amount did the concentration of the acid-producing chemical in the river change after one month? What will the pH of the river be at the end of the year? (Assume that the pH changes by the same amount each month during the year.)



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Problem-Based Task 4A.2.1: Healing the Waters

Coaching

a. What is the concentration of the acid-producing chemical at the beginning of the project?

b. What is the concentration of the acid-producing chemical after one month?

c. Describe the change over the first month in terms of the pH and in terms of the acid-producing chemical concentration.

d. Was the river more or less acidic at the beginning of the project than after one month of cleanup? Explain.

e. Will the river be more or less acidic at the end of the project year? Explain.

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Instruction

Problem-Based Task 4A.2.1: Healing the Waters

Coaching Sample Responses

a. What is the concentration of the acid-producing chemical at the beginning of the project?

The concentration of the acid-producing chemical can be found by rewriting the pH formula.

$$\text{pH} = \log_{10}\left(\frac{1}{C}\right)$$

$$\text{pH} = -\log_{10} C$$

$$C = 10^{-\text{pH}}$$

In other words, the pH is the power of 10 of the concentration. At the beginning of the project, the pH is 5.2, so the concentration is $10^{-5.2}$, which is a number that can be difficult to evaluate without the use of a calculator.

Evaluated using a calculator, $10^{-5.2} \approx 0.000006$.

The concentration of the acid-producing chemical is about 6 millionths of a milligram per liter.

b. What is the concentration of the acid-producing chemical after one month?

Substitute the pH of the river after one month, 5.4, into the rewritten expression for the concentration C from part a. Then, evaluate using a calculator.

$$C = 10^{-\text{pH}}$$

$$C = 10^{-(5.4)}$$

$$C \approx 0.000004$$

After one month, the concentration of the acid-producing chemical is about 4 millionths of a milligram per liter.

c. Describe the change over the first month in terms of the pH and in terms of the acid-producing chemical concentration.

The concentration of the acid-producing chemical was reduced from 6 millionths of a milligram per liter to 4 millionths of a milligram per liter over one month, a reduction factor of $\frac{1}{3}$.

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d. Was the river more or less acidic at the beginning of the project than after one month of cleanup? Explain.

According to the data in the problem, at the beginning of the project the river's pH was 5.2, which is less neutral than "pure" water, which has a pH of 7. After one month, the pH had increased to 5.4. The pH represents a negative exponent, so a smaller negative exponent means a greater concentration of the acid-producing chemical. Therefore, the river was more acidic at the beginning of the project.

e. Will the river be more or less acidic at the end of the project year? Explain.

If the pH changes continue at the same rate for 12 months, the change in the pH can be given by the expression $12 \cdot (5.4 - 5.2) = 2.4$. Therefore, the pH of the river after one year will be $5.2 + 2.4$ or 7.6. This pH level is almost equal to the neutral pH of 7. Any solution with a pH greater than 7 is described as being an alkaline solution, as opposed to an acidic solution for pH levels less than 7. The river will be less acidic at the end of the project year, but what made it slightly alkaline will have to be monitored, too.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4A.2.1: Logarithmic Functions as Inverses

For problems 1–3, write the inverse of each logarithmic function.

1. $f(x) = 2 \bullet \log_{10}(x + 1) - 3$

2. $g(x) = \dfrac{2 \bullet \log_3 x + 4}{\log_3 x - 5}$

3. $h(x) = 0.25 \bullet \log_5 x^{-0.25}$

For problems 4–7, state the domain and range of the logarithmic function.

4. $a(x) = 4 - \log_6(x - 6)$

5. $b(x) = \dfrac{9}{\log_6 3x}$

6. $c(x) = \log_7(2 - x) + 7$

7. $d(x) = \dfrac{1}{4} \bullet \log_3(2x - 1)$

continued

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Use the information given in each scenario to complete problems 8–10.

8. The intensity of a particular sound is 10^{-8} watt per meter squared. The intensity of a second type of sound is 10^{-4} watt per meter squared. The decibel loudness of a sound is given by the function $D(i) = \log_{10}\left(\frac{i}{i_0}\right)$, in which $i_0 = 10^{-12}$ watt per meter squared, the threshold of human hearing. How much greater is the *loudness* (in decibels) of the second sound than the first? How much greater is the *intensity* of the second sound than the first?

9. In order to change their electrical properties, semiconductors are heated in a process called furnace annealing. The pH of an acid bath used to treat a semiconductor after it emerges from an annealing furnace is supposed to be 3.5. However, the pH can cover a range of 3.45 to 3.55 without the semiconductor being rejected. The pH is given by the formula $\text{pH} = -\log_{10} C$, where C is the concentration of the acid bath. Find the range of concentrations of the acid bath within which the semiconductor treatment is acceptable.

10. The chance of rolling a 6 on a fair number cube with faces numbered 1–6 is 1 out of 6, and the chance of rolling a 6 twice in a row is 1 out of 36. What is the chance, $C(n)$, of rolling a 6 ten times in a row? Write an exponential function for the chance $C(n)$ of rolling a 6 n times in a row. Write the inverse $C^{-1}(n)$ using a logarithm.

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Lesson 4A.2.2: Common Logarithms

Warm-Up 4A.2.2

According to the special theory of relativity, the speed of light is the same throughout the universe. It has a value of approximately 300,000,000 meters per second. One of the consequences of the relativity theory is that the length L of an object will shrink the faster it moves. The function that describes this is $L(v) = L_0 \bullet \sqrt{1 - \frac{v^2}{c^2}}$, where v is the object's velocity, c is the speed of light, L_0 is the starting length of the object, and $L(v)$ is the object's length at that velocity.

1. Write the function as a logarithm if L_0 is 1 meter.

2. Calculate the length of the object at a velocity of 1 kilometer per second.

3. Explain how the length of the object at a velocity of 1 kilometer per second changes.

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Instruction

Lesson 4A.2.2: Common Logarithms

Common Core State Standards

F–IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

F–LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$, where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.★

Warm-Up 4A.2.2 Debrief

1. Write the function as a logarithm if L_0 is 1 meter.

   We are given that $L_0 = 1$ and $c \approx 300{,}000{,}000$, which equals 3×10^8.

   Substitute the values of L_0 and c in the function and simplify it.

$$L(v) = L_0 \bullet \sqrt{1 - \frac{v^2}{c^2}}$$

$$L(v) = 1 \bullet \sqrt{1 - \frac{v^2}{\left(3 \times 10^8\right)^2}}$$

$$L(v) = \sqrt{\frac{9 \times 10^{16} - v^2}{9 \times 10^{16}}}$$

   Take the common or base-10 logarithm of both sides of the function and simplify using algebra and the rules of logarithms.

$$\log_{10} L(v) = \log_{10} \sqrt{\frac{9 \times 10^{16} - v^2}{9 \times 10^{16}}}$$

$$\log_{10} L(v) = \frac{1}{2}\left[\log_{10}\left(9 \bullet 10^{16} - v^2\right) - \log_{10} 9 - \log_{10} 10^{16}\right]$$

   Simplify as many base-10 logarithms as possible.

$$\log_{10} L(v) = \frac{1}{2}\log_{10}\left(9 \bullet 10^{16} - v^2\right) - \frac{1}{2}\log_{10} 9 - 8$$

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Use a calculator to compute any logarithms that cannot be calculated exactly.

$$\log_{10} L(v) = \frac{1}{2}\log_{10}\left(9 \bullet 10^{16} - v^2\right) - \frac{1}{2}\log_{10} 9 - 8$$

$$\log_{10} L(v) \approx \frac{1}{2}\log_{10}\left(9 \bullet 10^{16} - v^2\right) - 8.5$$

The logarithmic expression for $L(v)$ is $\log_{10} L(v) \approx \frac{1}{2}\log_{10}\left(9 \bullet 10^{16} - v^2\right) - 8.5$.

2. Calculate the length of the object at a velocity of 1 kilometer per second.

The given velocity, v, of 1 kilometer per second is equivalent to 1,000 meters per second. Rewritten as an exponent, 1000 meters per second = 10^3 meters per second.

Calculate the length of the object at a velocity v of 10^3 meters per second using the logarithmic expression for $L(v)$.

$$\log_{10} L(v) \approx \frac{1}{2}\log_{10}\left(9 \bullet 10^{16} - v^2\right) - 8.5$$

$$\log_{10} L\left(10^3\right) \approx \frac{1}{2}\log_{10}\left[9 \bullet 10^{16} - (10^3)^2\right] - 8.5$$

$$\approx \frac{1}{2}\log_{10}\left[9 \bullet 10^{16} - 10^6\right] - 8.5$$

$$\approx \frac{1}{2}\log_{10}\left(8.9999999999 \bullet 10^{16}\right) - 8.5$$

$$\approx \frac{1}{2}\log_{10}(8.9999999999) + \frac{1}{2}\log_{10}\left(10^{16}\right) - 8.5$$

$$\approx 0.5 + 8 - 8.5 \approx 0$$

This means that $\log_{10} L(v) \approx 0$ can be written as $L(v) \approx 10^0 \approx 1$.

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3. Explain how the length of the object at a velocity of 1 kilometer per second changes.

   At a speed of 1,000 meters per second, there is no change in length of this object for all practical purposes, because the change is so small that it is considered insignificant.

   To check this result, solve the original function for v = 1000 meters per second:

$$L(1000)=\sqrt{\frac{9\times10^{16}-\left(10^{3}\right)^{2}}{9\times10^{16}}}$$

$$L(1000)=\sqrt{\frac{8.9999999999\times10^{16}}{9\times10^{16}}}$$

$$L(1000)=\sqrt{0.99999999999}\approx1 \text{ meter}$$

   The special theory of relativity becomes a factor in calculating the length of an object when its velocity approaches the speed of light. The velocity in this problem is 300,000 times "slower" than the speed of light.

Connection to the Lesson

- Students will rewrite problems that involve very large and very small numbers as base-10 or common logarithms.
- Students will solve problems involving base-10 or common logarithms using calculators.
- Students will use rewritten problems with base-10 or common logarithms to verify results of problems using other functions, such as exponential or radical functions.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- using base-10 or common logarithms to solve or check problems written with other functions (F–LE.4)
- using the rules of exponents and logarithms to explain and/or simplify problem solving (F–LE.4)
- using graphing or scientific calculators to calculate and graph exponential and logarithmic functions (F–LE.4)

Introduction

Recall that a base-10 logarithm is called a common logarithm. It is usually written without the base number 10. For example, $f^{-1}(x) = 3 \bullet \log 2x$ is understood to be the inverse of the exponential function $f(x)=\frac{1}{2}\bullet 10^{\frac{x}{3}}$. The newest graphing and scientific calculators can evaluate logarithms with any base, but earlier models may only be capable of evaluating one or two logarithmic function bases. Base 10 is one of those bases. As a result, it is important to know that a logarithm of any base can be evaluated by a quotient of base-10 bases as shown in the Key Concepts that follow. Base-10 or common logarithms are also the basis for a variety of mathematical models used in the sciences and humanities. For example, pH levels and decibel measurements are based on common logarithms.

Key Concepts

- A base-10 or common logarithm is one in which $b = 10$ in the logarithmic function inverse of the exponential function. So, for $f(t) = ab^{ct}$:

$$f^{-1}(t)=\frac{1}{c}\log t-\frac{1}{c}\log a=\log\left(\frac{t}{a}\right)^{\frac{1}{c}}$$

- When written without a base number, the term "log" is understood to be a common logarithm with a base of 10. The base change rule allows a logarithm of any base to be calculated with common logarithms; one form of the base change rule is $\log_a b=\frac{\log b}{\log a}$.
- This conversion to common logarithms may not result in a simpler answer, but it will make a much wider variety of logarithmic-calculation tools available because of the widespread use of common logarithms.

SRB p. U4A-62

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Common Errors/Misconceptions

- forgetting that the base of a common logarithm is 10 and using another number or variable from a problem statement as the base
- switching the terms in a logarithmic function $a = \log b$ by, for example, mistaking a for the base and b for the exponent of the base
- multiplying the terms in a logarithmic function by 10 instead of using 10 as the base; for example, thinking that $a = \log b$ means $b = 10a$ or $a = 10b$ instead of the correct equation, $b = 10^a$

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Instruction

Guided Practice 4A.2.2

Example 1

Write the inverse of the exponential function $f(x) = 3 \cdot 10^{2x}$.

1. Identify the base of the exponential function.

   The base is 10 because 10 is raised to the $2x$ power.

2. Switch the variables and use $f^{-1}(x)$ as the inverse function variable.

   $f(x) = 3 \cdot 10^{2x}$ becomes $x = 3 \cdot \left[10^{2f^{-1}(x)}\right]$.

3. Isolate the exponential term.

   This will simplify writing the logarithmic function.

   $$x = 3 \cdot \left[10^{2f^{-1}(x)}\right]$$

   $$\frac{x}{3} = 10^{2f^{-1}(x)}$$

4. Write the inverse logarithmic function.

   Use the inverse logarithmic function definition to rewrite the equation.

   $$\frac{x}{3} = 10^{2f^{-1}(x)}$$

   $$\log\left(\frac{x}{3}\right) = 2 \cdot f^{-1}(x)$$

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5. Simplify the result.

   Use algebraic methods and the power rule of logarithms.

   $\log\left(\frac{x}{3}\right) = 2 \bullet f^{-1}(x)$ Inverse logarithmic function

   $\frac{1}{2}\log\left(\frac{x}{3}\right) = f^{-1}(x)$ Divide both sides by 2.

   $\log\left(\frac{x}{3}\right)^{\frac{1}{2}} = f^{-1}(x)$ Apply the power rule.

   The inverse of the exponential function $f(x) = 3 \bullet 10^{2x}$ is $f^{-1}(x) = \log\left(\frac{x}{3}\right)^{\frac{1}{2}}$.



Example 2

Write the exponential function that has the inverse $g(x) = 10 - 2 \bullet \log x$.

1. Switch the variables and replace $g(x)$ with $g^{-1}(x)$.

   This can also be done later in the process, but it can be helpful to complete this step first.

   $g(x) = 10 - 2 \bullet \log x$ becomes $x = 10 - 2 \bullet \log [g^{-1}(x)]$.

2. Isolate the logarithmic term.

   The logarithmic term has to be isolated to identify the power of the exponential function.

   $x = 10 - 2 \bullet \log [g^{-1}(x)]$ Equation from step 1

   $x - 10 = -2 \bullet \log [g^{-1}(x)]$ Subtract 10 from both sides.

   $\frac{1}{2}(10 - x) = \log\left[g^{-1}(x)\right]$ Divide both sides by –2.

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3. Write the exponential function from the logarithmic function.

   Use the definition of logarithm.

   $\frac{1}{2}(10-x)=\log\left[g^{-1}(x)\right]$ becomes $g^{-1}(x)=10^{\frac{1}{2}(10-x)}$.

4. Simplify the exponential function, if needed.

   In this case, the function is already simplified.

   The exponential function that has the inverse $g(x) = 10 - 2 \bullet \log x$ is $g^{-1}(x)=10^{\frac{1}{2}(10-x)}$.

Example 3

Rewrite the logarithmic function $f(x) = 3 \bullet \log_4 (x - 1) - 2$ using common logarithms.

1. Write the base change rule for changing a logarithm of the form $\log_a b$ to a common logarithm.

   $\log_a b=\frac{\log b}{\log a}$

2. Identify values for a and b in the function $f(x) = 3 \bullet \log_4 (x - 1) - 2$.

   Notice that in the rule $\log_a b=\frac{\log b}{\log a}$, the argument of $\log_a b$ is in the numerator and the base is in the denominator.

   Look at the logarithmic term of the function $f(x) = 3 \bullet \log_4 (x - 1) - 2$: $\log_4 (x - 1)$. The base is 4 and the argument is $x - 1$.

   In $\log_4 (x - 1)$, a is 4 and b is $x - 1$.

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3. Write the common logarithmic term for the logarithmic term.

   This can be done separately before substituting in the function in order to avoid possible confusion.

$$\log_4(x-1)=\frac{\log(x-1)}{\log 4}$$

4. Substitute the common logarithmic term into the original function.

$$f(x) = 3 \bullet \log_4(x-1) - 2 = 3 \bullet \left[\frac{\log(x-1)}{\log 4}\right] - 2$$

5. Simplify as needed.

   Combine the terms on the right side of the equation.

$$f(x)=3\bullet\left[\frac{\log(x-1)}{\log 4}\right]-2$$

$$f(x)=\frac{3\bullet\log(x-1)-2\bullet\log 4}{\log 4}$$

6. Replace any constant terms by evaluating the common logarithms.

   This may be necessary for some real-world problems. If this example came from a real-world problem, the logarithm of 4 would be evaluated as shown.

$$f(x)=\frac{3\bullet\log(x-1)-2\bullet\log 4}{\log 4}$$ Function from step 5

$$f(x)\approx\frac{3\bullet\log(x-1)-1.2}{0.6}$$ Substitute the calculated values of the logarithmic terms.

$f(x) = 5 \bullet \log(x-1) - 2$ Simplify.

The logarithmic function $f(x) = 3 \bullet \log_4(x-1) - 2$, rewritten as a common logarithm, is $f(x) = 5 \bullet \log(x-1) - 2$.

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Example 4

Write the inverse of the exponential function $g(x) = 2 \bullet 3^{x-4}$ using common logarithms. Evaluate any constants in the logarithmic function.

1. Isolate the exponential term in the function.

   $g(x) = 2 \bullet 3^{x-4}$ becomes $\frac{g(x)}{2} = 3^{x-4}$.

2. Write the logarithmic function.

   $\frac{g(x)}{2} = 3^{x-4}$ becomes $x - 4 = \log_3\left[\frac{g(x)}{2}\right]$.

3. Simplify the logarithmic function.

   Isolate x. Use the quotient rule on the logarithm's argument.

   $x - 4 = \log_3\left[\frac{g(x)}{2}\right]$ becomes $x = \log_3 g(x) - \log_3 2 + 4$.

4. Switch the variables and use $g^{-1}(x)$ for $g(x)$.

   $x = \log_3 g(x) - \log_3 2 + 4$ becomes $g^{-1}(x) = \log_3 x - \log_3 2 + 4$.

5. Rewrite the logarithmic terms for $g^{-1}(x)$ as common logarithmic fractions using the base change rule.

   Recall that the base change rule is $\log_a b = \frac{\log b}{\log a}$.

   The logarithmic terms for $g^{-1}(x) = \log_3 x - \log_3 2 + 4$ are $\log_3 x$ and $\log_3 2$.

   $\log_3 x = \frac{\log x}{\log 3}$ and $\log_3 2 = \frac{\log 2}{\log 3}$

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6. Substitute the common logarithmic fractions into the equation found in step 4 and simplify.

$g^{-1}(x) = \log_3 x - \log_3 2 + 4$

$$g^{-1}(x) = \left(\frac{\log x}{\log 3}\right) - \left(\frac{\log 2}{\log 3}\right) + 4$$

$$g^{-1}(x) = \frac{\log x - \log 2 + 4 \bullet \log 3}{\log 3}$$

7. Evaluate the constant terms.

Use a calculator to evaluate log 2 and log 3, and then simplify.

$$g^{-1}(x) = \frac{\log x - \log 2 + 4 \bullet \log 3}{\log 3}$$

$$g^{-1}(x) \approx \frac{\log x - 0.3 + 1.9}{0.5}$$

$g^{-1}(x) \approx 2 \bullet \log x + 3.2$

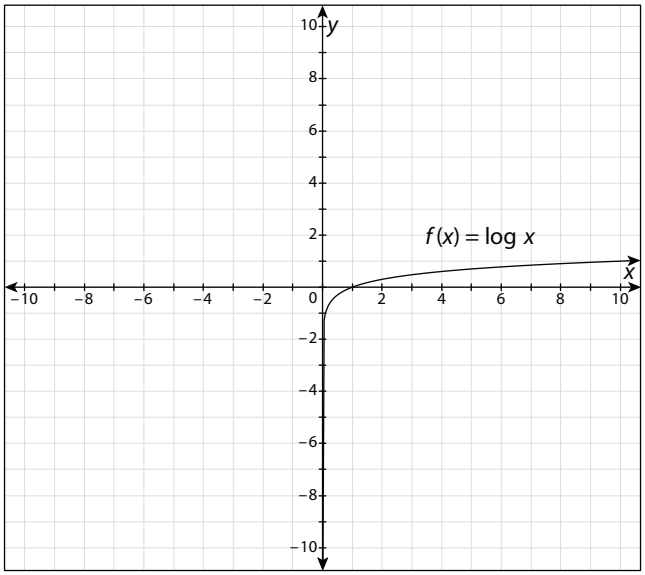
The inverse of the exponential function $g(x) = 2 \bullet 3^{x-4}$, rewritten using common logarithms, is $g^{-1}(x) \approx 2 \bullet \log x + 3.2$.

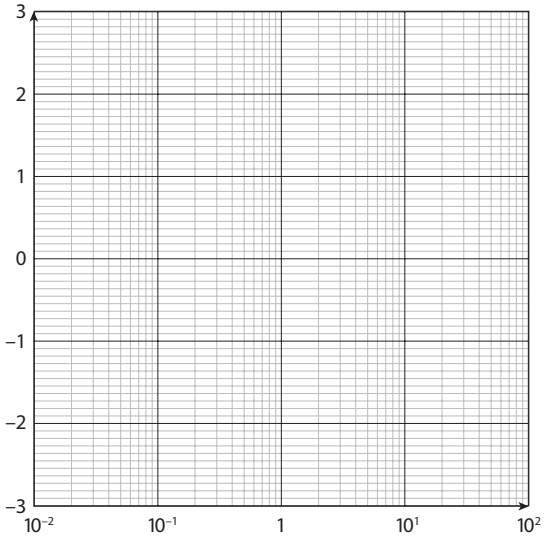
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Problem-Based Task 4A.2.2: Power Plotting and Semi-Logarithms

SMP
1 ✓ 2
3 ✓ 4
5 ✓ 6 ✓
7 8

The graph of the common logarithmic function $f(x) = \log x$ is shown in the graph on the left, beside a semi-logarithmic coordinate plane. A semi-logarithmic coordinate plane is a coordinate plane on which one axis is labeled in powers of 10 and the other axis is a standard linear axis. Either scale can be changed to produce a coordinate plane with four quadrants. Plot the points (log 0.01, 0.01), (log 0.1, 0.1), (log 1, 1), (log 10, 10), and (log 10^2, 10^2) on the provided semi-logarithmic coordinate plane. Describe the resulting graph.





Describe the resulting graph.

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Problem-Based Task 4A.2.2: Power Plotting and Semi-Logarithms

Coaching

a. What do each of the divisions between 1 and 10^1 on the *x*-axis mean?

b. What do each of the divisions between 1 and 10^{-1} on the *x*-axis mean?

c. What do each of the divisions between 10^{-1} and 10^{-2} on the *x*-axis mean?

d. What do each of the divisions between 10^1 and 10^2 on the *x*-axis mean?

e. What is the value of log 0.01 in the ordered pair (log 0.01, 0.01)?

f. Rewrite the five ordered pairs without the "log" notation.

g. Plot the ordered pairs on the semi-logarithmic coordinate plane and draw a line through them.

h. Describe the graph of the points on the semi-logarithmic coordinate plane.

i. What is the equation of the common logarithmic function of the straight line graphed on a semi-logarithmic coordinate plane?

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Instruction

Problem-Based Task 4A.2.2: Power Plotting and Semi-Logarithms

Coaching Sample Responses

a. What do each of the divisions between 1 and 10^1 on the *x*-axis mean?

Notice that there are 8 tick marks between 1 and 10^1. Including the marks for 1 and 10^1, there are 10 tick marks. The exponent is 1. Therefore, each division represents 1 unit, such as 1, 2, 3, and so on.

b. What do each of the divisions between 1 and 10^{-1} on the *x*-axis mean?

Including the marks for 1 and 10^{-1}, there are 10 tick marks. The exponent is –1. Therefore, each division represents one-tenth of a unit, such as 0.1, 0.2, 0.3, and so on.

c. What do each of the divisions between 10^{-1} and 10^{-2} on the *x*-axis mean?

Including the marks for 10^{-1} and 10^{-2}, there are 10 tick marks. The exponents increase in value from –2 to –1. Therefore, starting with 10^{-2}, each division represents one hundredth of a unit, such as 0.01, 0.02, 0.03, and so on.

d. What do each of the divisions between 10^1 and 10^2 on the *x*-axis mean?

Including the marks for 10^1 and 10^2, there are 10 tick marks. The exponents increase in value from 1 to 2. Therefore, each division represents 10 units, such as 10, 20, 30, and so on.

e. What is the value of log 0.01 in the ordered pair (log 0.01, 0.01)?

Write the definition of the logarithm: log 0.01 = –2 because $10^{-2} = 0.01$.

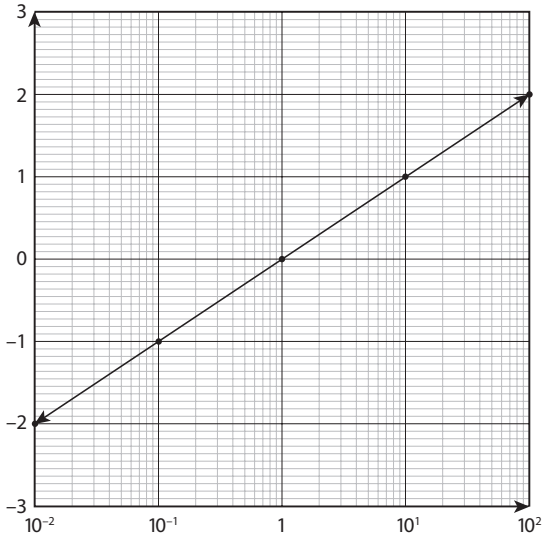
f. Rewrite the five ordered pairs without the "log" notation.

Evaluate each logarithm.

(–2, 0.01), (–1, 0.1), (0, 1), (1, 10), and (2, 100)

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g. Plot the ordered pairs on the semi-logarithmic coordinate plane and draw a line through them.



h. Describe the graph of the points on the semi-logarithmic coordinate plane.

Answers may vary, but should indicate the graph is a straight line or a linear function.

i. What is the equation of the common logarithmic function of the straight line graphed on a semi-logarithmic coordinate plane?

Recall that the vertical axis is the scale for the logarithm of x and that the horizontal axis is the scale for the domain values of x. Therefore, the resulting function is $f(x) = \log x$.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4A.2.2: Common Logarithms

For problems 1–3, write the inverse of the exponential function using common logarithms.

1. $f(x) = 3 \bullet 2^x$

2. $g(x) = 4 \bullet 3^{2x}$

3. $h(x) = 0.5 \bullet 5^{-x}$

For problems 4–7, rewrite each logarithmic function as a logarithm with a base other than 10.

4. $a(x) = \dfrac{2 \bullet \log x}{\log 4}$

5. $b(x) = \dfrac{\log 5}{5 \bullet \log x}$

6. $\dfrac{\log x^2}{\log 3} - 4$

7. $d(x) = \dfrac{\log 3 \bullet \log x}{\log 4 \bullet \log(x-1)}$

continued

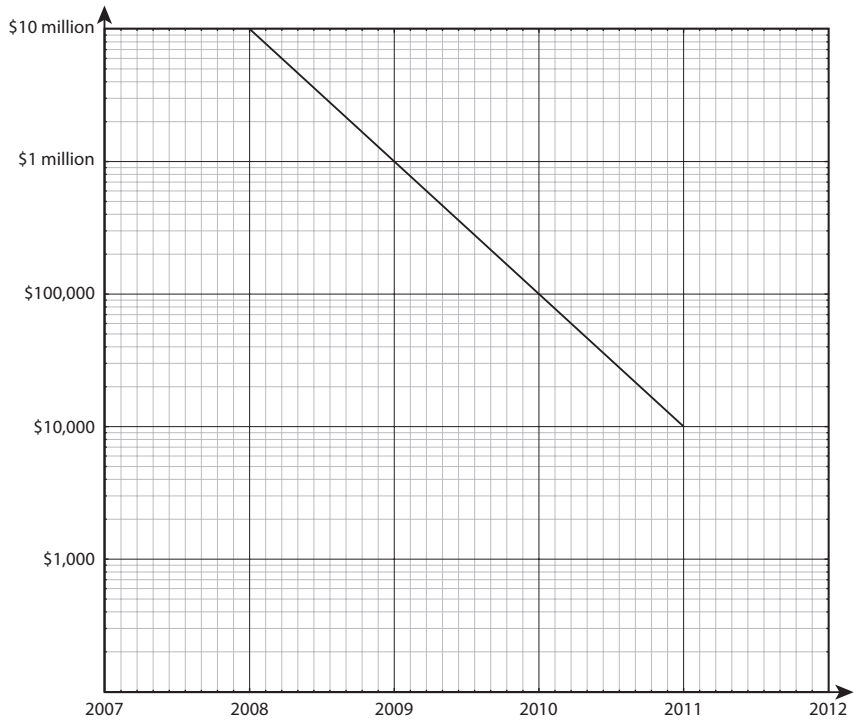
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Use the information given in each scenario to complete problems 8–10.

8. The function $R(y) = 3 \cdot 10^9 \cdot 2^y$ describes the revenue $R(y)$ in dollars from public computer "cloud" services from 2008 through 2013, where $y = 0$ for 2008. Write the inverse of this function as a common logarithmic function and determine the domain of the exponential function.

9. From 2001 to 2010, the production in megawatts of solar cells in China went from about 20 to about 11,000. The function describing this growth is $P(n) = 4 \cdot 2.15^n$, where $P(n)$ is the produced capacity in megawatts and $n = 1$ for the year 2001. Write the inverse of this function using common logarithms.

10. A semi-logarithmic coordinate plane is a coordinate plane on which one axis is labeled in powers of 10 and the other axis is a standard linear axis. The following semi-logarithmic graph shows the cost of sequencing a human genome from 2007 to 2011. Use the graph to write a common logarithmic function for the inverse of the cost function $C(y)$, where y represents years. Let $y = 1$ for 2008.



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Lesson 4A.2.3: Natural Logarithms

Warm-Up 4A.2.3

The growth of algae on a retention pond during a summer drought can be modeled by the exponential function $r(t) = 2 \cdot a^t - 1$, where a is a constant, t is the number of months, and $r(t)$ is the rate of growth in quarter acres of pond surface area per month. After one month, the growth rate r is approximately 4.437.

1. What is the value of the base of the exponential function, a?

2. The value of e is approximately equal to 2.71828. How does the calculated value for a compare to the value of the base of a natural logarithm e?

3. Why do you think a and e are not equal?

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Lesson 4A.2.3: Natural Logarithms

Common Core State Standards

F–IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

F–LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$, where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.★

Warm-Up 4A.2.3 Debrief

1. What is the value of the base of the exponential function, a?

   To find a, start by isolating the term that contains the variable t.

   $r(t) = 2 \bullet a^t - 1$

   $r(t) + 1 = 2 \bullet a^t$

   $$\frac{r(t)+1}{2} = a^t$$

   Substitute 4.437 for $r(t)$ and 1 for t, then solve for a.

   $$a^t = \frac{r(t)+1}{2}$$

   $$a^{(1)} = \frac{(4.437)+1}{2} \approx 2.7185$$

2. The value of e is approximately equal to 2.71828. How does the calculated value for a compare to the value of the base of a natural logarithm e?

   Subtract a from e:

   $2.71828 - 2.7185 = -0.00022$

   The value of a differs from e by about 0.00022.

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3. Why do you think a and e are not equal?

   The value for $r(1)$ might be an approximation, since e is an irrational number that is rounded by calculator algorithms. This would introduce error into the calculation, which uses a value of a that is given to four decimal places.

Connection to the Lesson

- Students will apply what they know about exponential and logarithmic functions to problems involving the natural logarithm base e.
- Students will make inferences about the accuracy of a calculation involving an irrational number that is rounded to a specified number of decimal places, and the effect that rounding has on the precision of an answer.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- applying the definitions and rules of logarithms (F–LE.4)
- using a graphing or scientific calculator to solve problems and graph functions expressed as logarithms (F–LE.4)

Introduction

Recall that a logarithm with a base of e is called a natural logarithm. A natural logarithm is usually written in the form "ln" to distinguish it from logarithms of other bases. For example, $f^{-1}(x) = \log_e x$ is usually written $f^{-1}(x) = \ln x$ and is the inverse of the exponential function $f(x) = e^x$.

Exponential functions with base e and natural logarithms have been found useful in describing a number of important real-world phenomena, including population growth and extinction, as well as radioactive decay of the isotopes of elements. The latter application has been useful in accurately dating the age of antiquities and fossils of now-extinct living things. The number e, like π, has been found to be useful in describing a wide variety of such real-world phenomena.

Key Concepts

- A natural logarithm is a logarithm with a base of e. The irrational number e is approximately equal to 2.71828. A natural logarithm can be written as $\log_e x$ or, more commonly, $\ln x$.
- Recall the base change rule for logarithms, which allows a logarithm of any base to be rewritten with common logarithms: $\log_a b = \dfrac{\log b}{\log a}$.
- The base change rule can be generalized so that $\log_a b$ can be written as the quotient of two logarithms to base c. This form of the base change rule is written $\log_a b = \dfrac{\log_c b}{\log_c a}$.
- The base change rule also allows any logarithm to be written as the quotient of two natural logarithms: $\log_a b = \dfrac{\ln b}{\ln a}$.
- A graphing calculator can be used to perform calculations and display graphs of natural logarithm functions and the exponential functions to which they are related.

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Instruction

On a TI-83/84:

Step 1: Use the keypad and [LN] to enter the values for the natural logarithm. Press [ENTER].

Step 2: Press [GRAPH] to view the function graph or press [2ND] [GRAPH] to see a table of values for the function.

Step 3: To calculate a specific value of the function for the domain value of $x = 1$, press [2ND][MODE] to return to the calculation screen.

Step 4: Use the keypad and [LN] to enter the values for the natural logarithm, substituting 1 for x. Press [ENTER].

Step 5: Press [2ND][GRAPH] to check the result against a table of values. The result from step 4 should be listed as the result for the function value, Y_1.

On a TI-Nspire:

Step 1: Press [home]. Arrow down to the graphing icon, the second icon from the left, and press [enter].

Step 2: Enter the function to the right of "$f1(x) =$", using [ctrl][e^x] for the natural logarithm. Press [enter].

Step 3: To calculate a specific value of this function for the domain value of $x = 1$, press [home] and select the calculator icon. Press [enter].

Step 4: Enter the function to the right of "$f1(x) =$", using [ctrl][e^x] for the natural logarithm but substituting 1 for x. Press [enter].

Step 5: To check the result against a table of values, press the [ctrl] and arrow keys to return to the screen showing the graph of the original natural logarithm function.

Step 6: Press [menu], then scroll down to 2: View, then 9: Show Table. Press [enter]. The result for $x = 1$ in the "$f1(x)$:=" column should match the result from step 4.

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Instruction

Common Errors/Misconceptions

- incorrectly calculating common and natural logarithms by confusing the bases of each type when writing their inverse exponential functions
- thinking that exponential functions always have a base of e
- applying the rules of logarithms differently for natural logarithms than for logarithms with other bases

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Guided Practice 4A.2.3

Example 1

Write the simplified logarithmic function that is the inverse of the exponential function $f(x) = 2 \cdot e^{3x-1}$.

1. Isolate the exponential term.

   In order to write the logarithmic function, the exponential term should be isolated with no coefficients or terms added to it.

   $f(x) = 2 \cdot e^{3x-1}$ can be rewritten as $\frac{1}{2} \cdot f(x) = e^{3x-1}$.

2. Rewrite the exponential term to eliminate any other constants that can be removed.

   In this case, e^1 can be removed as shown:

   $$\frac{1}{2} \cdot f(x) = e^{3x-1}$$

   $$\frac{1}{2} \cdot f(x) = \frac{e^{3x}}{e^1}$$

   $$\frac{e}{2} \cdot f(x) = e^{3x}$$

3. Write the inverse logarithmic function.

   To write the inverse, take the natural log of $\frac{e}{2} \cdot f(x) = e^{3x}$. This is one of two ways to find the logarithmic function (the other is to take the cube root of both sides of the equation). Simplify algebraically.

   $$\frac{e}{2} \cdot f(x) = e^{3x}$$

   $$\ln\left[\frac{e}{2} \cdot f(x)\right] = 3x$$

   $$x = \frac{1}{3}\ln\left[\frac{e}{2} \cdot f(x)\right]$$

   The inverse function of $\frac{e}{2} \cdot f(x) = e^{3x}$ is $x = \frac{1}{3}\ln\left[\frac{e}{2} \cdot f(x)\right]$.

SWB p. 49; SRB p. U4A-73

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4. Switch x and $f(x)$ and replace $f(x)$ with $f^{-1}(x)$.

$$f^{-1}(x)=\frac{1}{3}\ln\left[\frac{e}{2}\bullet x\right]$$

5. If possible, simplify the function using any applicable rules of logarithms.

The result in step 4 is complete, but the power rule of logarithms can be used to simplify it further.

$$f^{-1}(x)=\frac{1}{3}\ln\left[\frac{e}{2}\bullet x\right]$$

$$f^{-1}(x)=\ln\left(\frac{e}{2}\bullet x\right)^{\frac{1}{3}}$$

The simplified logarithmic function $f^{-1}(x)=\ln\left(\frac{e}{2}\bullet x\right)^{\frac{1}{3}}$ is the inverse of the exponential function $f(x) = 2 \bullet e^{3x-1}$.

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Instruction

Example 2

Write an exponential function that is the inverse of the natural logarithm function $g(x) = \ln (4 - 5x)^{0.3}$.

1. Use the rules of logarithms to simplify the function.

   This is not always needed, but it can make calculations simpler.

   $g(x) = \ln (4 - 5x)^{0.3}$ Function

   $g(x) = 0.3 \bullet \ln (4 - 5x)$ Apply the power rule.

   $\frac{10}{3} \bullet g(x) = \ln(4 - 5x)$ Divide both sides by 0.3.

2. Switch x and $g(x)$.

   This step can also be done later in the process.

   $\frac{10}{3} \bullet g(x) = \ln(4 - 5x)$ becomes $\frac{10x}{3} = \ln\{4 - 5[g(x)]\}$.

3. Rewrite the resulting expression in exponential form.

   $\frac{10x}{3} = \ln\{4 - 5[g(x)]\}$ becomes $4 - 5[g(x)] = e^{\frac{10x}{3}}$.

4. Simplify the exponential term, if possible.

   This is needed to isolate the term with the domain variable, x.

   $4 - 5[g(x)] = e^{\frac{10x}{3}}$ is already in simplest form.

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Instruction

5. Isolate $g(x)$.

$4-5[g(x)]=e^{\frac{10x}{3}}$	Equation from the previous step
$-5[g(x)]=e^{\frac{10x}{3}}-4$	Subtract 4 from both sides.
$5[g(x)]=4-e^{\frac{10x}{3}}$	Divide both sides by –1 and rearrange the right side.
$g(x)=\frac{1}{5}\left(4-e^{\frac{10x}{3}}\right)$	Divide both sides by 5.

6. Use the rules of logarithms to simplify the exponential term.

The function in step 5 is complete, but it can be simplified further.

$g(x)=\frac{1}{5}\left(4-e^{\frac{10x}{3}}\right)$	Equation from the previous step
$g(x)=\frac{4}{5}-\frac{1}{5}\left(\sqrt[3]{e^{10x}}\right)$	Distribute and apply the power rule.

The exponential function $g(x)=\frac{4}{5}-\frac{1}{5}\left(\sqrt[3]{e^{10x}}\right)$ is the inverse of the natural logarithm function $g(x) = \ln (4 - 5x)^{0.3}$.

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Example 3

Solve the logarithmic equation $10 + 2 \bullet \ln(x - 4) = 5$ for x. Express the result to three decimal places.

1. Isolate the term containing x.

   This is the same procedure used with solving any equation. In this case, the term to be isolated is $\ln(x - 4)$.

$10 + 2 \bullet \ln(x - 4) = 5$	Logarithmic equation
$2 \bullet \ln(x - 4) = -5$	Subtract 10 from both sides.
$\ln(x - 4) = -2.5$	Divide both sides by 2.

2. Rewrite the result as an exponential equation.

$\ln(x - 4) = -2.5$	Equation from the previous step
$x - 4 = e^{-2.5}$	Rewrite with a base of e.
$x = 4 + e^{-2.5}$	Add 4 to both sides.

3. Use a calculator to determine an approximate value for x to three decimal places.

   Follow the procedures appropriate to your calculator model.

   On a TI-83/84:

   Step 1: Press [Y=]. Press the [CLEAR] key to delete any other functions stored on the screen.

   Step 2: Use the keypad to enter the values for the expression, using [2ND][LN] for the natural log. Press [ENTER].

   On a TI-Nspire:

   Step 1: Press [home]. Arrow down to the calculator icon, the first icon from the left, and press [enter].

   Step 2: Use the keypad to enter the values for the expression, using [ctrl][e^x] for the natural log. Press [enter].

   Either calculator will return a rounded result of $x \approx 4.082$.



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Instruction

Example 4

For the following statement, write the natural logarithm function and find the exponential function that is the inverse of this function:

> The function that is the opposite of four tenths of the natural log of the quantity eight reduced by two times a number.

1. Translate the written information into mathematical expressions.

   Start with one phrase from the statement, and write its mathematical equivalent. Then, build on each phrase and continue writing mathematical equivalents until the entire statement is represented mathematically.

Phrase	Translation
four tenths	0.4
eight reduced by two times a number	$8 - 2x$
the opposite of four tenths of the natural log	$-0.4 \cdot \ln$
the natural log of the quantity eight reduced by two times a number	$\ln (8 - 2x)$
the opposite of four tenths of the natural log of the quantity eight reduced by two times a number	$-0.4 \cdot \ln (8 - 2x)$

2. Write the natural logarithm function using the expressions from step 1.

   $f(x) = -0.4 \cdot \ln (8 - 2x)$

3. Switch x and $f(x)$.

   $f(x) = -0.4 \cdot \ln (8 - 2x)$ becomes $x = -0.4 \cdot \ln \{8 - 2[f(x)]\}$.

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4. Isolate the natural logarithm.

$x = -0.4 \bullet \ln\{8 - 2[f(x)]\}$	Equation from the previous step
$-\frac{x}{0.4} = \ln\{8 - 2[f(x)]\}$	Divide both sides by –0.4.
$-\frac{10x}{4} = \ln\{8 - 2[f(x)]\}$	Rewrite 0.4 as a fraction.

5. Write an exponential function from the logarithmic function in step 4.

$-\frac{10x}{4} = \ln\{8 - 2[f(x)]\}$ becomes $8 - 2 \bullet f(x) = e^{-\frac{10x}{4}}$.

6. Simplify the exponent.

$8 - 2 \bullet f(x) = e^{-\frac{10x}{4}}$ becomes $8 - 2 \bullet f(x) = e^{-\frac{5x}{2}}$.

7. Isolate the term containing $f(x)$ and simplify the result as much as possible without calculating any constants.

$8 - 2 \bullet f(x) = e^{-\frac{5x}{2}}$	Equation from the previous step
$-2 \bullet f(x) = e^{-\frac{5x}{2}} - 8$	Subtract 8 from both sides.
$2 \bullet f(x) = 8 - e^{-\frac{5x}{2}}$	Divide both sides by –1 and then rearrange the right side.
$f(x) = 4 - \frac{1}{2}e^{-\frac{5x}{2}}$	Divide both sides by 2 and simplify.

8. Summarize your results.

For the given statement, the natural logarithm function is $f(x) = -0.4 \bullet \ln(8 - 2x)$ and its inverse exponential function is $f^{-1}(x) = 4 - \frac{1}{2}e^{-\frac{5x}{2}}$.

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Instruction

Example 5

Write the logarithmic function $f(x) = 5 + 2 \bullet \log_5 (x - 2)$ as a natural logarithm function.

1. Rewrite the logarithmic term of the function as the quotient of two natural logarithms.

   Determine and substitute the specific values for a, b, and c from the logarithmic term $f(x) = 5 + 2 \bullet \log_5 (x - 2)$ into the base change rule, $\log_a b = \frac{\log_c b}{\log_c a}$. For $\log_5 (x - 2)$, $a = 5$, $b = x - 2$, and $c = e$.

   Recall that the base change rule allows any logarithm to be written as the quotient of two natural logarithms: $\log_a b = \frac{\ln b}{\ln a}$. Therefore, $\log_5 (x - 2)$ rewritten as the quotient of two natural logarithms is $\log_5 (x-2) = \frac{\ln(x-2)}{\ln 5}$.

2. Substitute the result for the logarithmic term in the original function.

   By substitution, the function $f(x) = 5 + 2 \bullet \log_5 (x - 2)$ becomes $f(x) = 5 + 2 \bullet \left[\frac{\ln(x-2)}{\ln 5}\right]$.

3. Simplify the result.

   The function found in step 2 is simplified, but the result can also be written as shown.

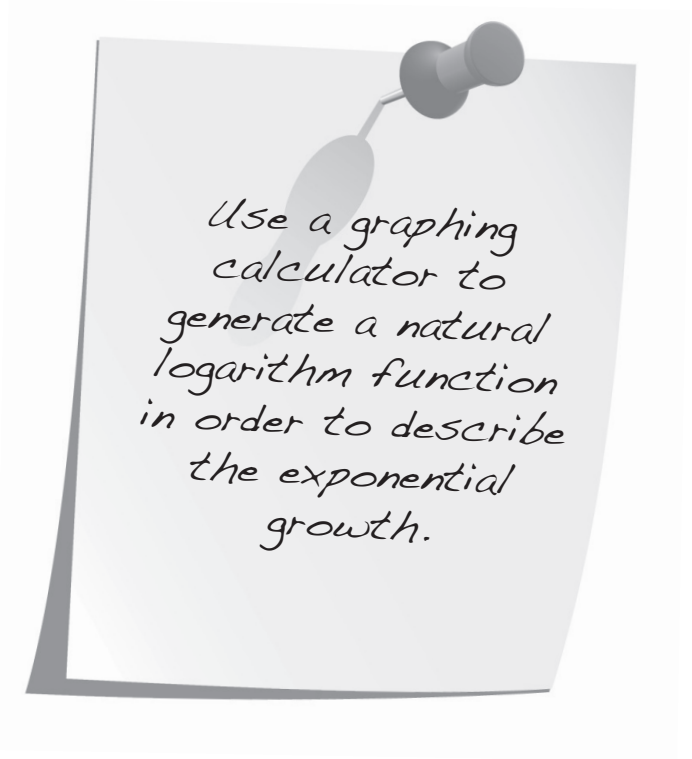
   $$f(x) = 5 + 2 \bullet \left[\frac{\ln(x-2)}{\ln 5}\right]$$
   $$f(x) = \frac{5 \bullet \ln 5 + 2 \bullet \ln(x-2)}{\ln 5}$$

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Problem-Based Task 4A.2.3: Watch IT Grow!

SMP	
1	2 ✓
3	4 ✓
5 ✓	6
7	8

According to a 2011 survey by a manufacturer of Internet networking hardware, global consumer Internet traffic grew from 2,280 petabytes ($2.28 \cdot 10^{18}$ bytes) per month in 2006 to 12,684 petabytes per month in 2010. Usage was projected to grow to 42,070 petabytes per month in 2014. It was estimated that the largest portion of Internet traffic in 2014 (about 46 percent) was from users streaming videos. Use a graphing calculator to generate a natural logarithm function in order to describe the exponential growth. Then, write the corresponding exponential function as a check for the natural logarithm function. Let the year 2006 have a value of $x = 1$, with the following years numbered sequentially; e.g., 1, 2, 3. Round the constants in the functions to two significant figures.



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Problem-Based Task 4A.2.3: Watch IT Grow!

Coaching

a. What will be the domain of the natural logarithm function if the year 2006 corresponds to the number 1 and the function runs to the year 2014?

b. What is the range of the logarithmic function representing monthly Internet traffic?

c. How can a graphing calculator be used to generate a natural logarithm function from the data in the problem?

d. What is the natural logarithm function? What is its value at $x = 2280$? What does this mean in the context of the problem?

e. What is the exponential function? What is its value at $x = 1$? What does this mean in the context of the problem?

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Instruction

Problem-Based Task 4A.2.3: Watch IT Grow!

Coaching Sample Responses

a. What will be the domain of the natural logarithm function if the year 2006 corresponds to $x = 1$ and the function runs to the year 2014?

Recall that the logarithmic function is the inverse of an exponential function. That means the domain and range of the logarithmic function are switched from the exponential function. If 2006 is $x = 1$ for the exponential function, then the domain value for the natural logarithm function is the Internet traffic in 2006, which is 2,280 petabytes per month. Using the information from the original problem, we know that in 2010 the traffic was 12,684 petabytes per month, and in 2014 the monthly traffic was projected to grow to 42,070 petabytes. The domain starts at 2,280 and goes to 12,684.

Therefore, the domain of the natural logarithm function is [2,280, 12,684].

b. What is the range of the function representing the monthly Internet traffic?

The function values of the logarithmic function are the domain values of the exponential function. If 2006 is the year represented by 1, then the other two function values are 5 for 2010 and 9 for 2014. Therefore, the range is given by [1, 9] for the natural logarithm function.

c. How can a graphing calculator be used to generate a natural logarithm function from the data in the problem?

Think about what happens when you have two or more points on a coordinate plane and you want to find out what type of graph is produced by the points. Depending on the number of points and their positions, a line of best fit can be derived. For straight lines, the point-slope or slope-intercept method can be used to generate an equation. Using a graphing calculator to generate a natural logarithm function produces the same equation.

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Instruction

On a TI-83/84:

Note: On the TI-83/84, the data entered in the table is for the exponential function. L1 represents the domain variable for the exponential function, and L2 represents the range variable. For the natural logarithm function, the data listed for L2 represents the domain variable and the data listed for L1 represents the range variable.

Step 1: Press [MODE]. Arrow down to Func and press [ENTER] to select function mode.

Step 2: Press [STAT]. Select 1: Edit to open a table. Arrow up to L1, then press [CLEAR][ENTER] to clear the list. Repeat as needed. Press [ENTER].

Step 3: From L1, press the down arrow to move your cursor into the list. Enter the known function values for each year, pressing [ENTER] after each number to navigate to the next blank spot in the list.

Step 4: Arrow over to the L2 column. Enter the corresponding values representing the number of petabytes used for each year.

Step 5: Press [STAT]. Arrow over to the CALC menu. Arrow down to 9: LnReg, "natural logarithm regression." Press [ENTER].

Step 6: To calculate the natural logarithm function, first press [2ND][2] to use the data in the table under the L2 heading as the domain variable for the logarithmic function.

Step 7: Press [,][2ND][1] to enter the data that appear under L1.

Step 8: Press [ENTER] to generate a natural logarithm function of the form $y = a + b \cdot \ln x$.

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On a TI-Nspire:

Step 1: Press [home]. Arrow over to the spreadsheet icon, the fourth icon from the left, and press [enter].

Step 2: Arrow over to column B. Enter the known function values for each year, pressing [enter] to navigate to the next blank cell.

Step 3: Arrow over to column A. Enter the corresponding values representing the number of petabytes used for each year.

Step 4: To generate a natural logarithm function with the data, press [menu]. Use the arrow key to select 4: Statistics, then 1: Stat Calculations, then B: Logarithmic Regression. Press [enter].

Step 5: Use your keypad and [tab] to enter A in the X List field and B in the Y List field. Tab to OK and press [enter]. This will generate a natural logarithm function of the form $y = a + b \bullet \ln x$. The coefficients appear on the right side of the screen in columns C and D.

Either calculator will return approximate values of $a = -20.0$ and $b = 2.7$.

d. What is the natural logarithm function? What is its value at $x = 2280$? What does this mean in the context of the problem?

The calculator activity produced a natural logarithmic equation in the form $y = a + b \bullet \ln x$. Substitute the values of a and b into this form, and change y to $f(x)$ to write the equation.

$y = a + b \bullet \ln x$

$f(x) = -20 + 2.7 \bullet \ln x$

Substitute 2,280 for x and 1 for y.

$f(1) = -20 + 2.7 \bullet \ln (2280) \approx 0.88$

Recall that 1 represents the year 2006; since the result of 0.88 is a little less than 1, this means that, according to this function, Internet traffic reached 2,280 petabytes per month late in 2005 (i.e., a little before 2006).

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e. What is the exponential function? What is its value at $x = 1$? What does this mean in the context of the problem?

Rewrite the natural logarithm function $f(x) = -20 + 2.7 \cdot \ln x$. First, switch the variables x and $f(x)$ to result in $x = -20 + 2.7 \cdot \ln f(x)$. Isolate the natural logarithmic term: $x + 20 = 2.7 \cdot \ln [f(x)]$. Then, divide both sides by 2.7.

$$\frac{x+20}{2.7} = \ln[f(x)]$$

Simplify the fraction using a calculator.

$$\frac{1}{2.7} \approx 0.37 \text{ and } \frac{20}{2.7} \approx 7.4$$

Write the corresponding exponential function: $f(x) = e^{0.37x + 7.4}$.

Let $x = 1$, which results in $f(1) = e^{7.8} \approx 2440$, representing the traffic for 2006.

This number is higher than the reported traffic for 2006, but this could be due to the rounding of the original calculated values for a and b.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4A.2.3: Natural Logarithms

For problems 1–4, use the rule $\log_a b = \frac{\ln b}{\ln a}$ to write each function as a natural logarithm function.

1. $f(x) = 2 \bullet \log_3 (4x)$

2. $g(x) = 1 - \log (x - 2)$

3. $f(x) = 5 \bullet \log_6 (4 - x) - 3 \bullet \log_6 (x - 2)$

4. $g(x) = [\log_2 (x + 1)]^3$

For problems 5–7, compare the values of the functions $f(x)$ and $g(x)$ at $x = 2$ using >, <, or = signs. Do not use a calculator.

5. $f(x) = 2 \bullet \log x$
   $g(x) = \log x$

6. $f(x) = 4 \bullet \ln (x - 1)$
   $g(x) = 2 \bullet \ln x$

7. $f(x) = \log_3 (x - 1)$
   $g(x) = \ln (x - 1)$

continued

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Use the information given in each scenario to complete problems 8–10.

8. The exponential function $N(x) = N_0(x) \bullet e^{-(\ln 2)x}$ gives the expected number of fair coins that land heads-up each time a group of coins is flipped, if after each flip the coins that land heads-up are removed. Calculate the number of coins remaining after 3 flips if there were 30 coins to start.

9. A researcher sprays a concentrated solution of bleach on a mildewed surface. The exponential function $N(t) = N_0(t) \bullet e^{-t}$ describes the amount of mildew that remains on the surface t seconds after the solution is sprayed. After how many seconds will half the original amount of mildew remain?

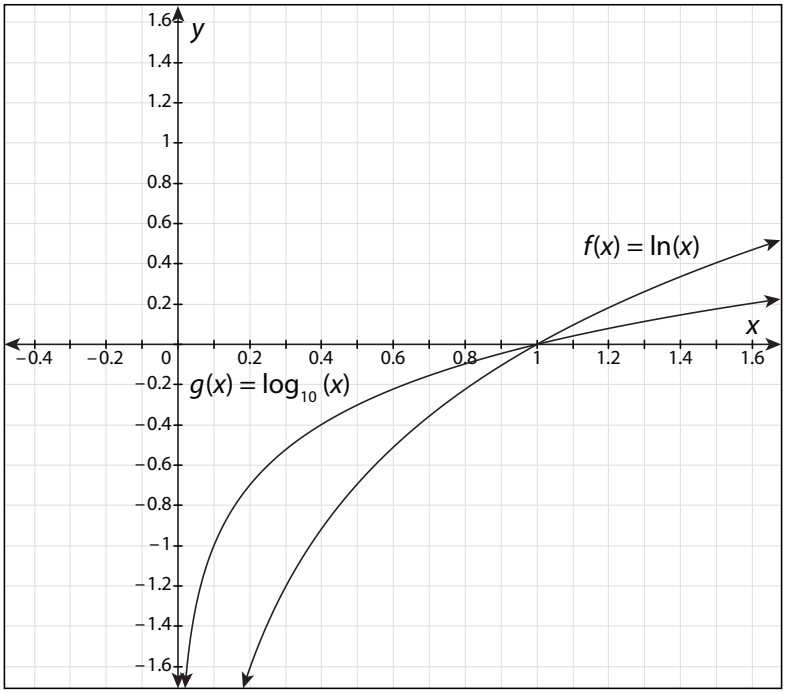
10. Carbon-14 is a radioactive isotope that decays at a steady rate over time. Because of this predictable decay rate, archaeologists can measure the amount of carbon-14 that remains in a fossil in order to determine the fossil's age. The exponential function $R(t) = R_0(t) \bullet e^{-ct}$ models the amount of radioactive carbon-14 that remains in a fossil after the time t has elapsed. The constant c is $3.83 \bullet 10^{-12}$ s^{-1} since t is in seconds (s) and the exponential term must not have any units. Write the exponential function as a natural logarithm function and find t in years if $R(t) = 0.5 \bullet R_0(t)$.

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Lesson 4A.2.4: Graphing Logarithmic Functions

Warm-Up 4A.2.4

In preparation for the upcoming Mathlete Tournament, Mrs. Blake graphed two functions as shown. She asked her students to use this graph to compare the domain and range of each function.



1. Over which restricted domain are the values of $f(x) < g(x)$?

2. Over which restricted domain are the values of $f(x) > g(x)$?

3. At what domain value is $f(x) = g(x)$?

4. What are the ranges of $f(x)$ and $g(x)$?

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Instruction

Lesson 4A.2.4: Graphing Logarithmic Functions

Common Core State Standard

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Warm-Up 4A.2.4 Debrief

1. Over which restricted domain are the values of $f(x) < g(x)$?

   As the graph shows, the restricted domain over which $f(x) < g(x)$ is $(0, 1)$.

2. Over which restricted domain are the values of $f(x) > g(x)$?

   As the graph shows, the restricted domain over which $f(x) > g(x)$ is $(1,+\infty)$.

3. At what domain value is $f(x) = g(x)$?

   As the graph shows, $f(x) = g(x)$ at $x = 1$.

4. What are the ranges of $f(x)$ and $g(x)$?

   Both functions have a range of $(-\infty,+\infty)$.

Connection to the Lesson

- Students will identify the domain and restricted domain(s) of a logarithmic function.
- Students will compare the graphs of logarithmic functions and identify the restricted domain(s) over which the functions can be compared.
- Students will identify the range of a logarithmic function.
- Students will describe how the value of a logarithmic function changes as the domain values approach the upper and lower bounds of the domain.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- calculating the intercept(s) and zero(s) of a logarithmic function (F–IF.4)
- determining the value of a function as the domain values approach the upper and lower bounds of the domain (F–IF.5)
- comparing logarithmic functions by describing what happens to the function values as the domain values vary in a specific way (F–IF.7e)

Introduction

The functions $f(x) = a \bullet \log_b c$, $g(x) = a + b \bullet \log_c d$, and $h(x) = a \bullet \log_d c$ are "families" of logarithmic functions. The graphs of logarithmic functions exhibit patterns that can help in identifying these functions when the algebraic function is not present.

Key Concepts

- Logarithmic functions or the logarithmic terms within logarithmic functions are equal to 0 when their arguments equal 1. Recall that the argument is the result of raising the base of a logarithm to the power of the logarithm, so that b is the argument of the logarithm $\log_a b = c$.
- For example, let's look at $f(x) = 30 \bullet \ln x$ at $x = 1$: $f(1) = 30 \bullet \ln 1 = 30 \bullet 0 = 0$.
- For another example, let's look at the function $g(x) = -250 + 6 \bullet \log_4 (x + 4)$. Evaluate when $x = -3$:

  $g(-3) = -250 + 6 \bullet \log_4 [(-3) + 4]$

  $g(-3) = -250 + 6 \bullet \log_4 (1)$

  $g(-3) = -250 + 0 = -250$
- Notice that the argument for each example was equal to 1.
- The values of logarithmic functions or their logarithmic terms approach positive or negative infinity as the argument of the logarithmic term approaches 0. For example, in the function $f(x) = -\ln x$, $f(0.1)$ is about 2.3, but $f(0.001)$ is almost 7. If $x = 10^{-12}$, $f(x)$ is almost 28. The function value increases continuously as the value of x decreases.
- A graphing calculator will also show this function behavior. By looking at a variety of function values of interest (such as very small values of x), it is possible to see trends in the changing values of the function. Examine very small values of x by adjusting the axis scales or using the calculator's trace feature.

SRB p. U4A-83

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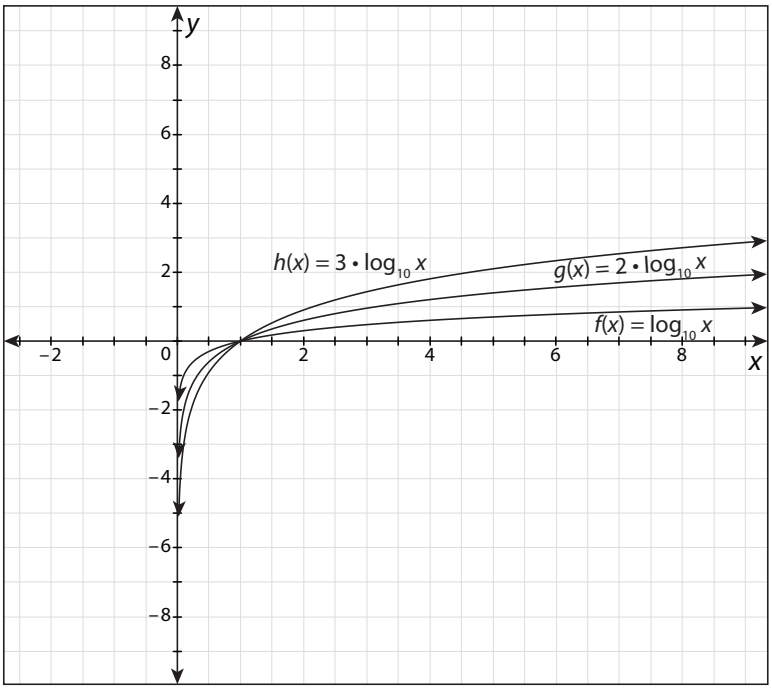
- Similarly, the value of logarithmic functions or their logarithmic terms as x becomes very large, positively or negatively, can be seen by substituting values for x or by using a calculator to calculate the values.
- For example, for the function $g(x) = \log x$, the domain is $(0,+\infty)$. As x becomes very large, the value of $g(x)$ increases, too, but at a much slower rate. The value of $g(10^3) = 3$, but the value of $g(10^{200}) = 200$. If you suspect the function value of $g(x)$ has an upper bound, try to find the highest value of the function.
- Follow these basic rules to compare logarithmic functions. However, use caution when defining domains: positive, negative, or zero domain values can result in an undefined function, or change the ordering in a comparison of function values.

Powers, Products, Quotients, Roots, and Sums of Logarithmic Functions

- Families of logarithmic functions are grouped according to the operations shown in the equations of the functions. In real-world problems, you can calculate such combined operations with logarithmic functions best by approximation techniques or with calculators. Each example that follows shows how different operations affect the graph of a logarithmic function.

***f*(*x*) = *a* • $\log_b c$**

- Compare the graphs of three functions of the form $f(x) = a \cdot \log_b c$.



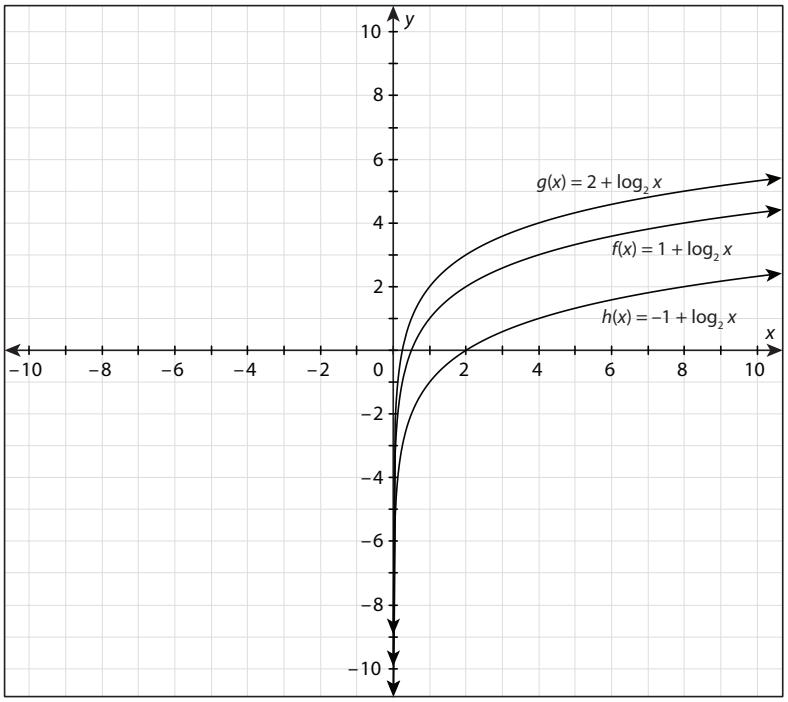
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- All three graphs pass through the point (1, 0) because any number raised to the 0 power (the *y*-value) is equal to 1 (the *x*-value). The coefficient in front of each logarithm multiplies the logarithmic value at that value of *x* by the magnitude of the coefficient. This means that $h(x) = 3 \bullet f(x) = 3 \bullet \log x$, and $g(x) = 2 \bullet f(x) = 2 \bullet \log x$.

$f(x) = a + b \bullet \log_c d$

- Observe the graphs of three functions of the form $f(x) = a + b \bullet \log_c d$.



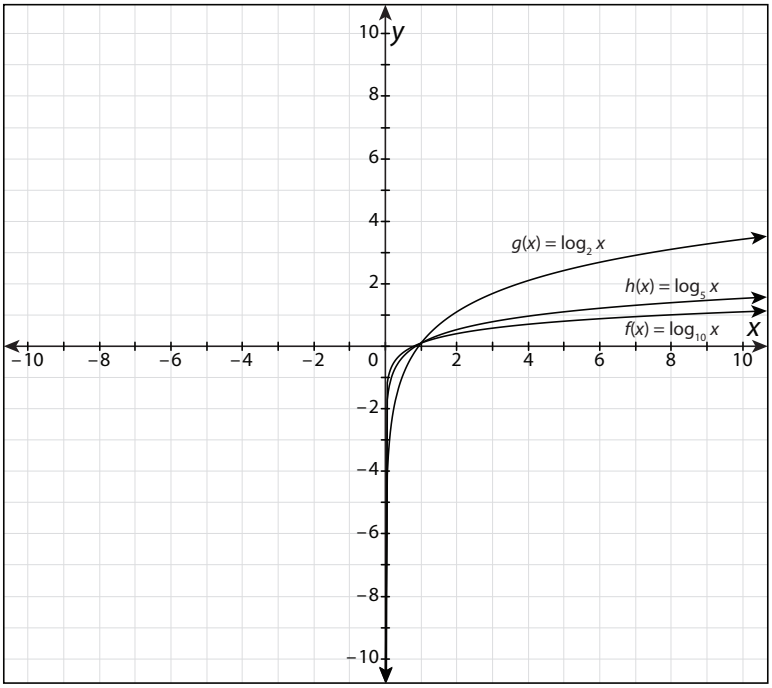
- All three graphs are continuously increasing across the domain $(0,+\infty)$. For any value of *x* in the domain, the *y*-values are related by the inequality $g(x) > f(x) > h(x)$. The *x*-intercepts are determined by the constant added to $\log_2 x$.

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$f(x) = a \bullet \log_b c$ and $g(x) = a \bullet \log_d c$

- Compare the graphs of three functions with different bases.



- All three functions contain the point (1, 0), since the logarithm of any base to the power of 0 is equal to 1. As the graphs show, the functions are related by the inequality $g(x) > h(x) > f(x)$ when $x > 1$. Comparing the bases of the three functions reveals that they are ordered in the opposite "direction" from the functions: $2 < 5 < 10$ for $g(x) > h(x) > f(x)$.

Common Errors/Misconceptions

- confusing the domain and range in a logarithmic function problem
- using an argument in a logarithmic term that is less than or equal to 0
- failing to check all of the domain and extreme-value options in graphing one or more logarithmic functions; e.g., intercepts and function values for upper and lower bounds of domains

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Guided Practice 4A.2.4

Example 1

Sketch the graphs of the functions $f(x) = 2 \bullet \log_2 x$ and $g(x) = 2 - \log_2 x$ on a coordinate plane. Describe the end behavior of each function.

1. Determine function values for $f(1)$ and $g(1)$ and write the results as ordered pairs.

   Substitute $x = 1$ into each function and solve.

   $f(1) = 2 \bullet \log_2 (1) = 2 \bullet 0 = 0$

   $g(1) = 2 - \log_2 (1) = 2 - 0 = 2$

   Write each point as an ordered pair.

   For $f(x)$: (1, 0)

   For $g(x)$: (1, 2)

2. Find the value(s) of x at which $f(x) = g(x)$.

   This will reveal any point(s) of intersection, or solutions, of the system of equations.

$2 \bullet \log_2 x = 2 - \log_2 x$	Set the functions equal to each other.
$3 \bullet \log_2 x = 2$	Add $\log_2 x$ to both sides.
$\log_2 x = \frac{2}{3}$	Divide both sides by 3.
$2^{\frac{2}{3}} = x$	Simplify.

   Using a calculator, the approximate value of x is 1.6.

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3. Find the function value at which the functions are equal.

   Substitute the approximate value of x from step 2 into each function.

   $f(1.6) \approx g(1.6) \approx \frac{4}{3}$

4. Write the approximate point at which the functions intersect.

   Since the value of x is rounded, the functions intersect at approximately $\left(1.6, \frac{4}{3}\right)$ or $\left(1.6, 1.\overline{3}\right)$.

5. Determine additional points for sketching the graph of each function.

   Choose values of x that can be evaluated easily.

   Let $x = 0.25, 0.5, 2,$ and 4.

   Solve $f(x)$ for each given value.

   $f(0.25) = 2 \bullet \log_2 (0.25) = 2 \bullet -2 = -4$

   $f(0.5) = 2 \bullet \log_2 (0.5) = 2 \bullet -1 = -2$

   $f(2) = 2 \bullet \log_2 (2) = 2 \bullet 1 = 2$

   $f(4) = 2 \bullet \log_2 (4) = 2 \bullet 2 = 4$

   Solve $g(x)$ for each given value.

   $g(0.25) = 2 - \log_2 (0.25) = 2 - (-2) = 4$

   $g(0.5) = 2 - \log_2 (0.5) = 2 - (-1) = 3$

   $g(2) = 2 - \log_2 (2) = 2 - 1 = 1$

   $g(4) = 2 - \log_2 (4) = 2 - 2 = 0$

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6. Write the additional points.

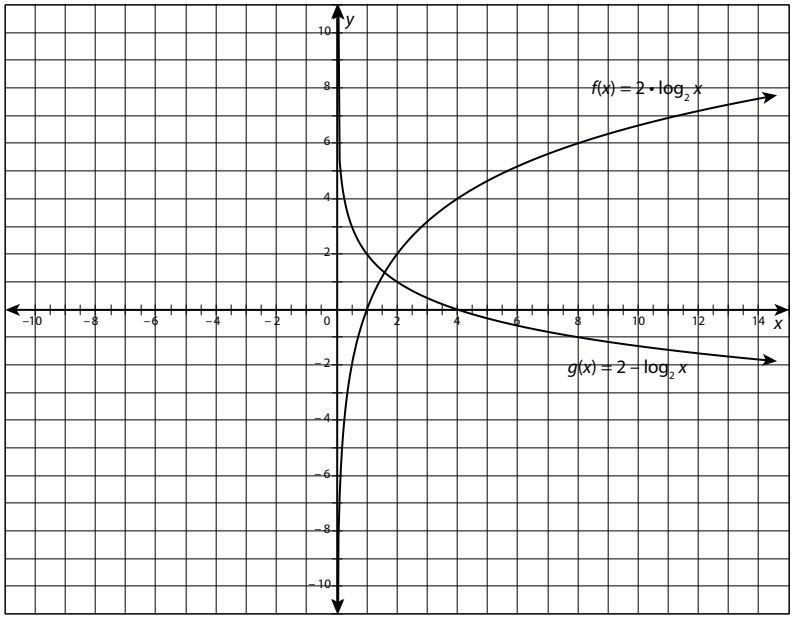
   Make sure to keep the points with the correct function.

   For $f(x)$: (0.25, –4), (0.5, –2), (2, 2), and (4, 4)

   For $g(x)$: (0.25, 4), (0.5, 3), (2, 1), and (4, 0)

7. Plot and sketch a curve to connect the points for each function.

   The completed curves might suggest that additional points should be plotted to better determine the actual shape of each function's graph.



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8. Describe what happens to the function values as x becomes very large and as x approaches 0.

   It is useful to know the limits of function values as the values go beyond the finite values used for a sketch of a graph.

   For $f(x)$: As x approaches positive infinity, $f(x)$ approaches positive infinity. As x approaches 0, $f(x)$ approaches negative infinity.

   For $g(x)$: As x approaches positive infinity, $g(x)$ approaches negative infinity. As x approaches 0, $g(x)$ approaches positive infinity.



Example 2

Sketch the graphs of $f(x) = x + \log x$ and $g(x) = \log x$ on a coordinate plane, using a graphing calculator if needed. Write $f(x)$ in terms of $g(x)$ to simplify sketching. Then, describe the end behavior of the functions.

1. Determine function values for $f(1)$ and $g(1)$ and write the results as ordered pairs.

   Substitute $x = 1$ into each function and solve.

   $f(1) = 1 + \log(1) = 1 + 0 = 1$

   $g(1) = \log(1) = 0$

   Rewrite the points as ordered pairs.

   For $f(x)$: (1, 1)

   For $g(x)$: (1, 0)



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2. Find the value(s) of x at which $f(x) = g(x)$.

   This will reveal any point(s) of intersection, or solutions, of the system of equations.

   $f(x) = g(x)$

   $x + \log x = \log x$

   $x = 0$

   The value $x = 0$ is not in the domain of these functions. Therefore, there is no point of intersection or solution for $f(x)$ and $g(x)$.

3. Determine additional points for sketching the graph of each function.

   Choose values of x for each function, and then solve the resulting equations.

   Let $x = 0.1$, 1, 2, 3, and 4.

   The x-values of 2, 3, and 4 will result in common logarithms that cannot be found easily without a calculator that has the capability of computing common logarithms.

   Solve $f(x)$ for each given value.

   $f(0.1) = (0.1) + \log (0.1) = 0.1 - 1 = -0.9$

   $f(1) = (1) + \log (1) = 1$

   $f(2) = (2) + \log (2) \approx 2.3$

   $f(3) = (3) + \log (3) \approx 3.5$

   $f(4) = (4) + \log (4) \approx 4.6$

   Solve $g(x)$ for each given value.

   $g(0.1) = \log (0.1) = -1$

   $g(1) = \log (1) = 0$

   $g(2) = \log (2) \approx 0.3$

   $g(3) = \log (3) \approx 0.5$

   $g(4) = \log (4) \approx 0.6$

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4. Write and plot the additional points.

   Make sure to keep the points with the correct function.

   For $f(x)$: (0.1, –0.9), (1, 1), approximately (2, 2.3), approximately (3, 3.5), and approximately (4, 4.6)

   For $g(x)$: (0.1, –1), (1, 0), approximately (2, 0.3), approximately (3, 0.5), and approximately (4, 0.6)

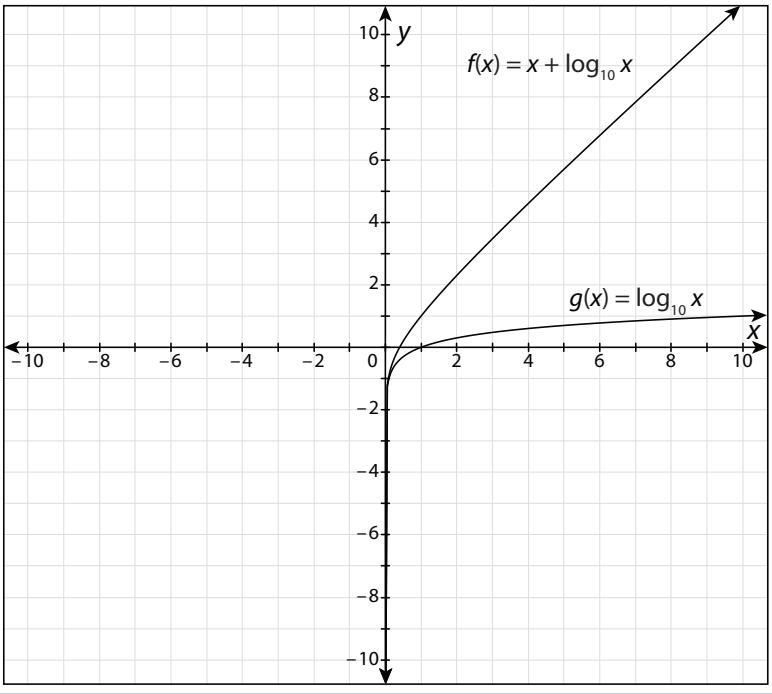
5. Write $f(x)$ in terms of $g(x)$.

   In some problems involving two or more functions graphed on the same coordinate plane, the functions are related, which can simplify graph sketching.

   $f(x) = x + \log x = x + g(x)$ or $f(x) - g(x) = x$, which means the difference between the function values at any value of x is x.

6. Plot and sketch a curve to connect the points for each function.

   The completed curves might suggest that additional points should be plotted to better determine the actual shape of each function's graph.



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7. Describe what happens to the function values for domain values less than 1.

   The graph indicates that the function values diverge.

   For $f(x)$: The function values approach negative infinity as x approaches 0. As x approaches 1, the function values converge on 1.

   For $g(x)$: The function values approach negative infinity as x approaches 0. As x approaches 1, the function values converge on 0.

8. Describe what happens to the function values for domain values greater than 1.

   The graph indicates that the function values increase.

   For $f(x)$: The function values approach positive infinity as x increases from $x = 1$ to positive infinity. As x approaches 1 from the right, the function values converge on 1.

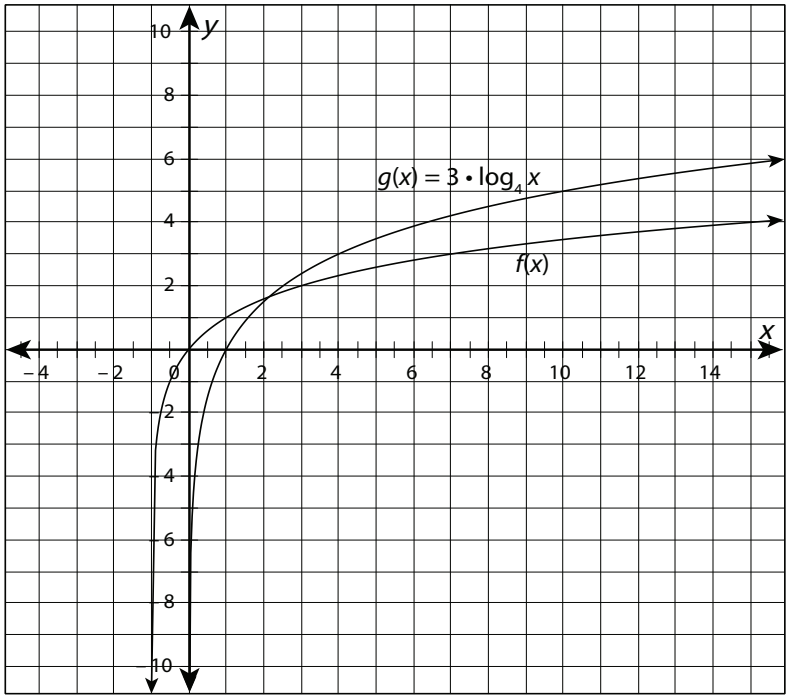
   For $g(x)$: The function values approach positive infinity as x increases from $x = 1$ to positive infinity. As x approaches 1 from the right, the function values converge on 0.

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Example 3

Use the graph of the function $f(x)$ to write the algebraic form of $f(x)$. Assume that $f(x)$ is of the form $a \cdot \log_4 (x + b)$ and includes the point (15, 4). Then, describe how to find the common solution of $f(x)$ and the function $g(x) = 3 \cdot \log_4 x$.



1. Determine the x-intercepts and their coordinates for each function.

   Inspect the graph to determine this information.

   The x-intercept of $f(x)$ is 0, so the point is (0, 0). The x-intercept of $g(x)$ is 1, so the point is (1, 0).

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2. Substitute the coordinates of the x-intercept of $f(x)$ into the form $f(x) = a \bullet \log_4 (x + b)$.

   This is the first step in writing $f(x)$.

   For $x = 0$, $f(x) = 0$. Substitute and solve.

   $f(x) = a \bullet \log_4 (x + b)$

   $f(0) = a \bullet \log_4 [(0) + b] = 0$

   Divide the logarithmic term by a.

   $\log_4 b = 0$

   $b = 1$ since $4^0 = 1$. Substitute $b = 1$ in $f(x) = a \bullet \log_4 (x + b)$.

   $f(x) = a \bullet \log_4 (x + 1)$

3. Use the other given point on the graph of $f(x)$ to find a.

   Substitute the given point (15, 4) into the $f(x)$ function and then solve.

$f(x) = a \bullet \log_4 (x + 1)$	Function for $f(x)$ found in the previous step
$(4) = a \bullet \log_4 [(15) + 1]$	Substitute 15 for x and 4 for $f(x)$.
$4 = a \bullet \log_4 (16)$	Add.
$4 = a \bullet 2$	Simplify.
$2 = a$	Divide both sides by 2.

4. Substitute the value of a into the equation for $f(x)$ found in step 2.

   This is the last step in writing $f(x)$.

   $f(x) = 2 \bullet \log_4 (x + 1)$

5. Set $f(x) = g(x)$.

   This is the first step in finding the common solution of the two functions.

   If $f(x) = 2 \bullet \log_4 (x + 1)$ and $g(x) = 3 \bullet \log_4 x$, then $2 \bullet \log_4 (x + 1) = 3 \bullet \log_4 x$ for the value of x at which a common solution exists.

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6. Simplify the resulting equation.

   Use the rules of logarithms.

   $2 \cdot \log_4 (x + 1) = 3 \cdot \log_4 x$ can be rewritten as $\log_4 (x + 1)^2 = \log_4 x^3$.

7. Rewrite the equation without logarithms.

   When logarithms of the same base are equal, the arguments of those logarithms are equal. Each logarithm has a base of 4; therefore, set the arguments equal to each other.

   $(x + 1)^2 = x^3$

8. Explain how to find x in the resulting equation.

   Since this is a cubic equation, expand the binomial and collect all terms on one side of the equation.

   $(x + 1)^2 = x^3$

   $x^2 + 2x + 1 = x^3$

   $x^3 - x^2 - 2x - 1 = 0$

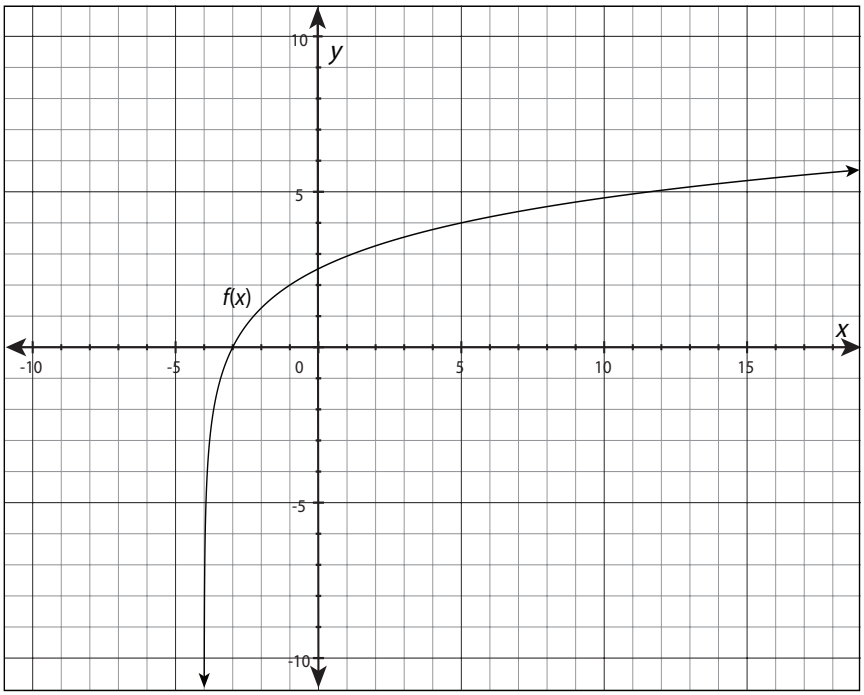
   There are no small integer solutions to this cubic equation, so x can be found by determining the root of this cubic equation on a graphing calculator. Alternatively, the point of intersection can be found by calculating it with the "calculate intersect" feature on a graphing calculator.

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Example 4

The graph shows a logarithmic function of the form $f(x) = a \bullet \log_b (x + c)$, where (–1, 2) is a point on the graph. The exact x-intercept can be determined directly from the graph. Write the x-intercept and estimate the value of the y-intercept from the graph. Then, calculate the actual y-intercept and compare this value with the estimate.



1. Identify the x-intercept from the graph and write its coordinates.

   The x-intercept can be used to help identify the constants in the general form of the function and find the y-intercept.

   The x-intercept is located at the point where the graph crosses the x-axis, or where $y = 0$.

   The x-intercept is –3, so the point containing it is (–3, 0).

2. Estimate the y-intercept from the graph.

   The y-intercept is located at the point where the graph crosses the y-axis, or where $x = 0$. From the graph, the y-intercept appears to be at $y = 2.5$, but this should be verified algebraically.

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3. Determine the equation for the y-intercept.

The y-intercept can be calculated by evaluating the function at $x = 0$. To do this, first determine the equation for the y-intercept. The function is in the form $f(x) = a \cdot \log_b (x + c)$, so substitute 0 for x and simplify.

$f(0) = a \cdot \log_b (0 + c) = a \cdot \log_b c$

The equation for the y-intercept can be written as $f(0) = a \cdot \log_b c$.

4. Use the x-intercept and the given point (–1, 2) to find the constants in the function.

To find one or more of the constants, substitute the coordinates of the x-intercept found in step 1 into the form $f(x) = a \cdot \log_b (x + c)$.

The x-intercept is located at (–3, 0).

$f(x) = a \cdot \log_b (x + c)$	Given function form
$(0) = a \cdot \log_b [(-3) + c]$	Substitute –3 for x and 0 for $f(x)$.
$0 = \log_b (c - 3)$	Divide both sides by a and simplify.
$b^0 = c - 3$	Apply the power rule of logarithms.
$1 = c - 3$	A quantity raised to a 0 power is equal to 1.
$4 = c$	Add 3 to both sides.

Therefore, if $c = 4$, then $f(x) = a \cdot \log_b (x + 4)$.

Now substitute (–1, 2) into $f(x) = a \cdot \log_b (x + 4)$ to find a.

$f(x) = a \cdot \log_b (x + 4)$	Function from the previous step
$2 = a \cdot \log_b [(-1) + 4]$	Substitute –1 for x and 2 for $f(x)$.
$2 = a \cdot \log_b 3$	Add.
$\frac{2}{\log_b 3} = a$	Divide both sides by $\log_b 3$.

The known constants are $a = \frac{2}{\log_b 3}$ and $c = 4$.

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5. Substitute the value of a into the equation for the y-intercept.

Use the equation for the y-intercept determined in step 3.

$f(0) = a \bullet \log_b c$	Equation for the y-intercept
$f(0) = \left(\frac{2}{\log_b 3}\right) \bullet \log_b (4)$	Substitute $\frac{2}{\log_b 3}$ for a and 4 for c.
$f(0) = 2 \bullet \left(\frac{\log_b 4}{\log_b 3}\right)$	Rewrite the multiplication.
$f(0) = (\log_3 4)^2$	Simplify.

6. Use a calculator to evaluate $(\log_3 4)^2$.

Rewrite $(\log_3 4)^2$ as $2 \bullet \left(\frac{\log 4}{\log 3}\right)$ using the power and quotient rules.

$$f(0) = 2 \bullet \left(\frac{\log 4}{\log 3}\right) \approx 2.52$$

The y-intercept is approximately 2.52.

7. Compare the estimated y-intercept from the graph with the calculated y-intercept.

Though the graph suggests that the y-intercept appears to be at (0, 2.5), the y-intercept is actually closer to approximately (0, 2.52), a difference of approximately 0.02.

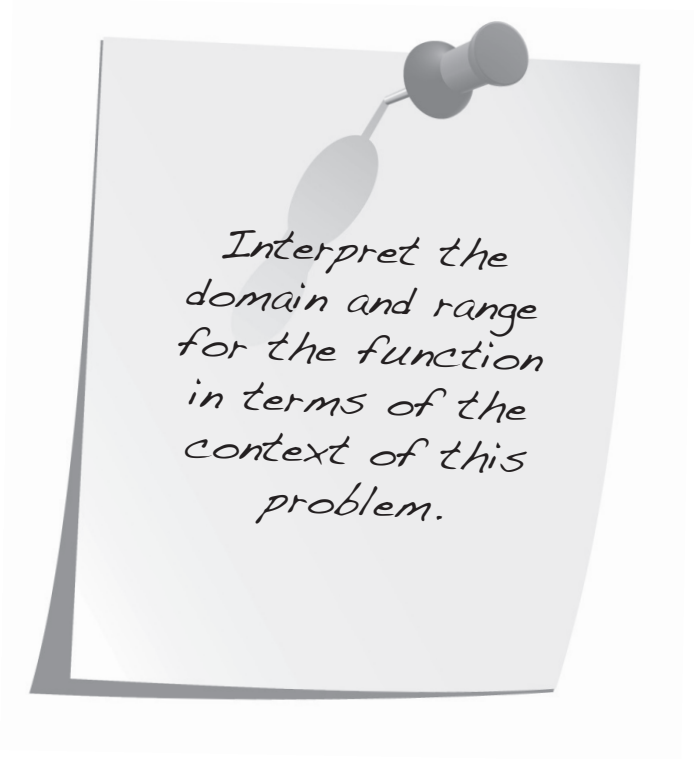


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Problem-Based Task 4A.2.4: To the Rescue!

SMP	
1 ✓	2 ✓
3	4 ✓
5 ✓	6 ✓
7	8

An emergency management agency wants to provide boaters in the state's coastal waters with updated information on the average length of rescue attempts. The natural logarithm function $N(t) = 12 - 6 \cdot \ln t$ is based on 28 recent rescue attempts. The time t, in hours, has a domain of [1, 6]. The range of the number of rescue attempts is [1, 12] and is restricted to whole numbers. For example, the greatest number of rescue attempts (12) occurred within the first hour after a distress call ($t = 1$). Use a graphing calculator to help sketch a graph of $N(t)$. Interpret the domain and range for the function in terms of the context of this problem. Also, interpret the values of $N(t)$ as its graph approaches the horizontal and vertical axes and at any intercepts on the graph.



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Problem-Based Task 4A.2.4: To the Rescue!

Coaching

a. How should the table and window settings of your graphing calculator be set for this situation?

b. What does the domain [1, 6] indicate about the time needed for rescue attempts?

c. Why is 0 not included in the domain?

d. What does the range [1, 12] mean?

e. Why is the range limited to whole numbers?

f. Why does the graphing calculator table show decimal numbers for the range?

g. Determine the value of any intercept(s) of the graph.

h. Use your graphing calculator's table of values to determine the values of $N(t)$ for $t = 7$ and $t = 8$.

i. How would you describe or interpret the value of $N(t)$ at $t = 7$ hours and $t = 8$ hours?

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Problem-Based Task 4A.2.4: To the Rescue!

Coaching Sample Responses

a. How should the table and window settings of your graphing calculator be set for this situation?

Think about the domain and range of the function and the type of numbers (e.g., integers, rational numbers) that should be displayed in the table. The results will need to be accurate in order to answer the rest of the coaching questions correctly.

Suggested window settings for the *x*-minimum, *x*-maximum, *y*-minimum, and *y*-maximum are 0, 10, 0, and 12, respectively.

Tables should start at the smallest domain value (1). The change/steps should be in increments that reflect the nature of the numbers in the domain; increments of 1 are suitable here.

b. What does the domain [1, 6] indicate about the time needed for rescue attempts?

The domain is measured in hours, so the minimum amount of time needed for a rescue attempt is 1 hour. The maximum time for a rescue attempt *in this data set* is 6 hours. Future rescue attempts could redefine the domain if the rescue times are less than 1 hour or greater than 6 hours. There is also no restriction on what kind of number *t* is. It could be an integer, but it is more likely to be a decimal fraction or an improper fraction, such as 1 hour and 20 minutes.

c. Why is 0 not included in the domain?

This would mean a time of 0 hours, which is not meaningful in a real-world situation. Technically, it is the time at which a rescue attempt started.

d. What does the range [1, 12] mean?

The number of rescue attempts that can be performed in 1 to 6 hours is anywhere from 1 to 12.

e. Why is the range limited to whole numbers?

The function is defined as the number of rescue attempts; only whole numbers make sense, as there cannot be "half" of a rescue.

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f. Why does the graphing calculator table show decimal numbers for the range?

The natural logarithm function for $N(t)$ is an approximation of a function that is based on the actual data points of (12, 1), (8, 2), (4, 3), (2, 1), and (1, 1). The "best fit" function for this data is the function $f(x) = 11.95 - 6.65 \cdot \ln x$. The data shown in the graphing calculator table is based on using the whole-number domain [1, 6] with the function $N(t) = 12 - 6 \cdot \ln x$, which produces decimal numbers for $N(t)$.

g. Determine the value of any intercept(s) of the graph.

Algebraically, the x-intercept occurs at the function value $N(t) = 0$, which gives the equation $0 = 12 - 6 \cdot \ln t$ or $\ln t = 2$, or $t \approx 7.4$ hours. Therefore, the intercept point is approximately (7.4, 0). Using the graphing calculator trace function indicates that the function value is 0 between the domain values of $t \approx 7.34$ hours and $t \approx 7.45$ hours.

h. Use your graphing calculator's table of values to determine the values of $N(t)$ for $t = 7$ and $t = 8$.

$N(7) \approx 0.3$ and $N(8) \approx -0.5$

i. How would you describe or interpret the value of $N(t)$ at $t = 7$ hours and $t = 8$ hours?

The function values are not whole numbers, so they do not represent the data that matches the real-world conditions placed on the function values. The function and its values predict that no rescue attempts will take longer than 7 hours. However, new data could modify the function and its domain and function values.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4A.2.4: Graphing Logarithmic Functions

For problems 1–4, sketch $f(x)$ and $g(x)$. Then, calculate the solution to the system of the two functions.

1. $f(x) = 1 + 2 \log_3 x$
   $g(x) = 2 - \log_3 x$

2. $f(x) = \log (x + 1)$
   $g(x) = \log (2 - x)$

3. $f(x) = 3 \bullet \ln (x + 1)$
   $g(x) = 2 \bullet \ln (x - 1)$

4. $f(x) = \log_4 x^2$
   $g(x) = \log_4 (3x + 4)$

For problems 5–7, compare the domains and ranges of the three functions in each problem. Then, state the domain(s) over which all three functions are defined.

5. $f(x) = 1 - \ln x$
   $g(x) = 1 - 2 \bullet \ln x$
   $h(x) = 2 - \ln x$

6. $f(x) = 2 \bullet \log (x - 1)$
   $g(x) = \log (x - 2)$
   $h(x) = \log (x + 1)$

7. $f(x) = 1 + 2 \bullet \log_3 (x + 4)$
   $g(x) = 2 + 4 \bullet \log_3 (x + 1)$
   $h(x) = 4 + \log_3 (x + 2)$

continued

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For problems 8–10, use the information in each problem to sketch a graph of the given function on a coordinate plane. Be sure to label the axes so that all the real-world parts of the domain and range are evident. Then, use your graph to solve the problem.

8. The longevity of soy-based ink-jet cartridges at a public transportation ticket kiosk varies according to the logarithmic function $N(t) = 19 - 6 \cdot \ln t$, where t is the time in weeks and $N(t)$ is the number of cartridges used up during the time t. If the domain of the function is [2, 20], what is the range of the function? What kind of number is $N(t)$—a fraction, an integer, negative, positive?

9. The number of mechanical failures of a NASCAR stock car over the first 40 minutes of a race is modeled by the logarithmic function $F(t) = -1.5 + 0.9 \cdot \ln t$, where t is time in minutes. What is the minimum time elapsed after which a failure is predicted to occur according to this model?

10. In a youth hockey league, the team score keeper found that the time $T(s)$ it takes a tied game to end with a winning score within 17 minutes of the overtime period is modeled by the function $T(s) = 17 - 11 \cdot \ln s$, in which s is the number of shots on goal taken by the winning team. What are the domain and range of the function?

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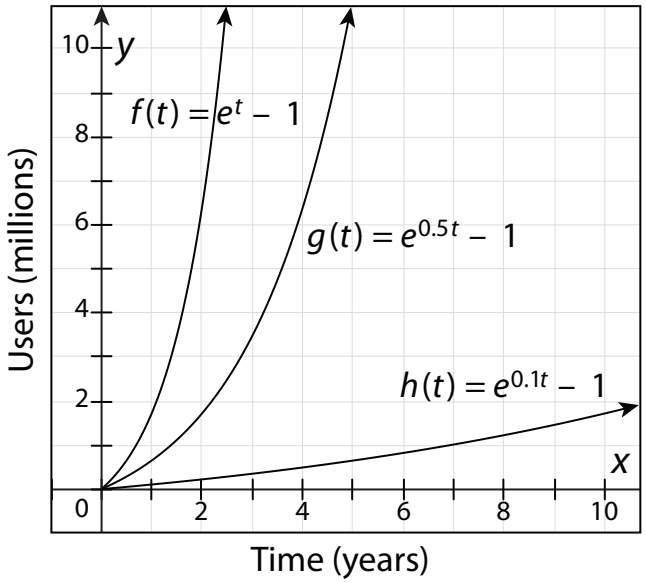
Lesson 4A.2.5: Interpreting Logarithmic Models

Warm-Up 4A.2.5

The number of people using social media, smartphones, and tablet computers has grown exponentially over the past 5 years. The following exponential functions model the growth in the number of users, in millions, for three examples of these technologies, where t represents time in years:

- a particular social media website: $f(t) = e^t - 1$
- a popular smartphone device: $g(t) = e^{0.5t} - 1$
- a certain model of tablet computer: $h(t) = e^{0.1t} - 1$

Use the following graph of the exponential functions to complete the problems. Assume that each technology was launched at the same time and thus the numbers of users are equal when $t = 0$.



1. How does the growth in the number of users for each technology correlate with the coefficient of the power in the exponential term of each function?
2. Write the inverse logarithmic function of each given exponential function.
3. How do the terms in the logarithmic functions compare in relation to the exponential growth of each technology?
4. How can the logarithmic functions describe the growth of each technology?

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Lesson 4A.2.5: Interpreting Logarithmic Models

Common Core State Standards

F–IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F–IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F–IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

Warm-Up 4A.2.5 Debrief

1. How does the growth in the number of users for each technology correlate with the coefficient of the power in the exponential term of each function?

   As the coefficient of the power of e decreases, the growth of the number of users decreases for any given year t. For example, the social media website users outnumber the smartphone users after 3 years, because $f(3) = e^3 - 1$ and $g(3) = e^{0.5(3)} - 1 = e^{1.5} - 1$. $f(3) > g(3)$ because $e^3 > e^{1.5}$.

2. Write the inverse logarithmic function of each given exponential function.

   For $f(t)$, $f(t) = e^t - 1$, so $f(t) + 1 = e^t$, which means that $\ln [f(t) + 1] = t$. Switch the variables and change f to f^{-1} and t to n, where n is the number of users measured in millions:

   $\ln (n + 1) = f^{-1}(n)$

   For $g(t)$, $g(t) = e^{0.5t} - 1$, so $g(t) + 1 = e^{0.5t}$, which means that $\ln [g(t) + 1] = 0.5 \bullet t$. Switch the variables and change g to g^{-1} and t to n.

   $\ln (n + 1) = 0.5 \bullet g^{-1}(n)$

   Then, simplify.

   $2 \bullet \ln (n + 1) = g^{-1}(n)$ Multiply both sides by 2.

   $\ln (n + 1)^2 = g^{-1}(n)$ Apply the power rule.

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For $h(t)$, $h(t) = e^{0.1t} - 1$, so $h(t) + 1 = e^{0.1t}$, which means that $\ln [h(t) + 1] = 0.1 \bullet t$. Switch the variables and change h to h^{-1} and t to n.

$\ln (n + 1) = 0.1 \bullet h^{-1}(n)$

Then, simplify.

$10 \bullet \ln (n + 1) = h^{-1}(n)$ Multiply both sides by 10.

$\ln (n + 1)^{10} = h^{-1}(n)$ Apply the power rule.

3. How do the terms in the logarithmic functions compare in relation to the exponential growth of each technology?

   The slower the growth of users of a technology, the larger the argument of the logarithmic function becomes. For example, for the social media website, the argument of the natural logarithm is $t + 1$, but for the tablet, the argument is $(t + 1)^{10}$.

4. How can the logarithmic functions describe the growth of each technology?

   The graphs depict the changing number of users per year for each technology. The inverse function is the number of years needed to see an increase in growth of a fixed number of users. Therefore, the logarithmic model implies that it will take more years as indicated by the inverse functions to see the same amount of user growth for a technology.

   For example, if the number of years elapsed is $n = 2$, then $\ln (3) = f^{-1}(2)$ for the social media website and $\ln (3)^2 = g^{-1}(2)$ for the smartphone. $g^{-1}(2) > f^{-1}(2)$ because $\ln 9 > \ln 3$, which implies that it will take the smartphone about twice as many years as the social media website to realize the same increase in the number of users.

Connection to the Lesson

- Students will identify different views of solutions to problems involving logarithmic functions compared to exponential functions.
- Students will investigate the effects of addends and coefficients of logarithmic terms in problems.
- Students will interpret the graphs of functions and apply the information from the graphs to solve problems.

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Prerequisite Skills

This lesson requires the use of the following skills:

- writing a logarithmic function from an exponential function and vice versa (F–LE.4)
- translating what a function represents in a real-world problem (F–BF.1a)
- relating different types of functions to one another in a problem setting (F–BF.1a)
- using the rules of exponentials and logarithms to simplify a function (F–LE.4)
- identifying the domain and range values of a real-world problem (F–IF.5)
- relating different types of functions to one another in a real-world scenario (F–BF.1a)

Introduction

Expressing or solving logarithmic functions in terms of exponential function models is one technique for solving real-world problems. One factor in determining which type of function is best in a given situation is how the solution to a problem affects a particular audience. For example, environmental scientists may need to present a study of the acidity or alkalinity of a freshwater pond to citizens at a town hall meeting. The citizens might best understand the results if the logarithm-based pH factor is used to describe the chemical condition of the pond rather than the actual concentration of hydronium or hydroxide ions in a sample of the pond water.

Such initial conditions as the upper and lower bound of a domain are essential to the viability of such models. For example, time is nearly always considered to be a positive quantity that moves in an ever-increasing direction. (There are exceptions to this in some of the leading-edge fields of physics, such as cosmology, but such discussions are generally beyond the scope of a mathematics course at this level.)

The ability to move accurately between a function and its inverse is often important in solving real-world problems that employ logarithms. Also, a thorough mastery of the basic rules of exponents and logarithms is essential for such problems.

Finally, be aware of the potential for a graphical or tabular presentation of a problem to aid in the application of logarithmic functions. In fact, visual models based on real data often provide a more accurate picture of a problem than an algebraic model that does not reveal the restricted domain or range of the problem in the way that a graph or table does.

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Instruction

Key Concepts

- Logarithmic functions have a wide range of applications in real-world problems, such as in the fields of biology, chemistry, ecology, and engineering. Logarithmic functions often provide an alternative approach to the use of exponential functions, which might help to increase the understanding of a problem or its solution.
- The logarithmic function is the inverse of an exponential function and vice versa. Recall two basic steps in writing one as the other:
  - A function's inverse switches the values of the domain and range values. For example, if an ordered pair of a function is (3, 5), then the ordered pair (5, 3) solves the inverse of the function.
  - To find the inverse of a function, replace the domain variable (often x) with the range variable ($f(x)$), and change the range variable to $f^{-1}(x)$.
- Apply the basic rules of exponentials and logarithms, described earlier in this lesson.
- The ability to identify the restricted domain and/or range over which a logarithmic function is defined can mean the difference between finding a solution to a problem and misinterpreting the application of that function to the particulars of the problem setting. This is especially true if a function and its inverse have roles to play in formulating and/or solving the problem.
- A graphing calculator can help with the identification of the restraints on real-world problems. The tables that are created when functions are graphed offer approximations of solutions that cannot be easily found using algebraic methods. Adjust the table and window settings as necessary to reflect the particulars of the problem.

Common Errors/Misconceptions

- neglecting to ensure that the real-world domain and/or range of a problem is compatible with the logarithmic function that models the problem
- incorrectly interpreting the representation of a logarithmic function in a problem
- when working with two or more functions, choosing the less-appropriate function to model a real-world problem and its solution
- misinterpreting a graph or table of a logarithmic function model of a real-world problem

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Guided Practice 4A.2.5

Example 1

The number of electric vehicles (E) sold in the United States within 3 years of their introduction to the market can be modeled by the logarithmic function $E(y) = 1.67 + 5.74 \cdot \ln y$, where $E(y)$ is the number of vehicles sold in thousands and y is the number of years after introduction to the market. The number of hybrid vehicles (H) sold within 3 years of their introduction to the market can be modeled by the logarithmic function $H(y) = 0.78 + 2.4 \cdot \ln y$. At what value of y was an equal number of each vehicle sold, and which type of vehicle had greater sales over the 3 introductory years? Explain your reasoning with references to the terms in the function and how they compare.

1. Set $E(y) = H(y)$.

   The two functions are equal at a value of y that satisfies the equation $E(y) = H(y)$.

   We are given that $E(y) = 1.67 + 5.74 \cdot \ln y$ and $H(y) = 0.78 + 2.4 \cdot \ln y$.

   Therefore, $1.67 + 5.74 \cdot \ln y = 0.78 + 2.4 \cdot \ln y$.

2. Solve the resulting equation for $\ln y$.

   Isolate the logarithmic terms on one side of the equation and the constants on the other.

$1.67 + 5.74 \cdot \ln y = 0.78 + 2.4 \cdot \ln y$	Set $E(y)$ equal to $H(y)$.
$5.74 \cdot \ln y = 0.78 + 2.4 \cdot \ln y - 1.67$	Subtract 1.67 from both sides.
$5.74 \cdot \ln y - 2.4 \cdot \ln y = 0.78 - 1.67$	Subtract $2.4 \cdot \ln y$ from both sides.
$\ln y\,(5.74 - 2.4) = -0.89$	Factor out $\ln y$ and simplify.
$3.34 \cdot \ln y = -0.89$	Continue to simplify.
$\ln y \approx -0.2665$	Calculate the result.

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3. Rewrite the resulting equation using an inverse function.

   The exponential function with base e is the inverse of the natural logarithm function in step 2.

   If $\ln y \approx -0.2665$, then $y \approx e^{-0.2665}$, which simplifies to $y \approx 0.766$.

4. Interpret the resulting value of y in terms of the conditions of the original problem.

   The result also has to be checked for reasonableness as a member of the domain of the real-world problem.

   Since y represents time in years, 0.766 would be the equivalent of approximately three-quarters of a year. This is the amount of time after vehicle introduction that the same number of electric vehicles and hybrid vehicles were sold. However, the sales data in the problem are based on 3 full years of sales, which implies a domain of [1, 3]. Parts of a year (e.g., 1.5 years, or a year and a half) are defined within this domain, but not for a period of time that falls outside of the domain, such as 0.75 year.

5. Check the value of y found in step 3 by substituting it back into the functions $E(y)$ and $H(y)$.

   Values should always be checked to verify that the result is accurate and satisfies the condition(s) of the problem.

   Recall that the functions represent the number of vehicles sold in thousands.

   $E(0.766) \approx 1.67 + 5.74 \bullet \ln 0.766 \approx 0.140$, or about 140 vehicles

   $H(0.766) \approx 0.78 + 2.4 \bullet \ln 0.766 \approx 0.140$, or about 140 vehicles

   The results for each function are equal, so the calculated value of y checks out.

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6. Show which vehicle type had greater sales over the 3-year period by comparing the characteristics of each function.

   Look at the terms in each function and use the values of the domain to support your claim(s).

   The constant term of $H(t)$ is less than the constant term of $E(t)$, which means that $E(t)_{\text{constant term}} > H(t)_{\text{constant term}}$. The coefficient of the logarithmic term of $E(t)$ is also greater than the coefficient of the logarithmic term of $H(t)$, which means that $E(t)_{\text{logarithmic term}} > H(t)_{\text{logarithmic term}}$. Furthermore, the logarithmic terms themselves are equal and are only defined for values of y in the range $1 \leq y \leq 3$, which also indicates they are positive. In summary:

   $$E(t)_{\text{constant term}} + E(t)_{\text{logarithmic term}} \bullet \ln y > H(t)_{\text{constant term}} + H(t)_{\text{logarithmic term}} \bullet \ln y$$

   As the result of step 5 shows, the two vehicle types had the same number of sales during a period of time of about 0.766 of a year that was not included in the functions' domain, namely, before the first sales year was over.



Example 2

A pyramid-shaped token for a board game consists of 4 congruent equilateral faces. Each face is a different color: blue, green, red, or white. Each face is equally likely to end up on the bottom if the token is rolled on the game board. What is the probability of the green face landing on the bottom? The probability of the same event a occurring n times is given by the function $f(n) = a^n$. Write a function for the green face landing on the bottom n times in a row. Then, write the inverse of the function, and explain what the inverse function describes in the context of this problem.

1. State the number of faces on the token.

   This is the first step in defining the probability event.

   There are 4 faces on the token.

2. State the probability of any of the 4 faces landing on the bottom.

   Each face has a 1 in 4, or 0.25, chance of landing on the bottom. Thus, the probability of the green face landing on the bottom is 0.25.

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3. Check your answer to step 2 by calculating the chance that *any* of the 4 faces lands on the bottom.

   This answer should add up to 1.

   The probability of a blue, green, red, *or* white face landing on the bottom is given by 0.25 + 0.25 + 0.25 + 0.25, which equals 1.

4. Write the function for the green face landing on the bottom n times in a row.

   This generalizes the result of step 2 for any number of rolls of the token.

   Let $a = \frac{1}{4}$ in the function $f(n) = a^n$ and simplify.

   $f(n) = a^n$

   $f(n) = \left(\frac{1}{4}\right)^n$

   $f(n) = (4^{-1})^n$

   $f(n) = 4^{-n}$

5. Write the inverse of the function determined in step 4.

   This will give the logarithm requested in the problem.

   The inverse of $f(n) = 4^{-n}$ is $\log_4 f(n) = -n$, which simplifies to $n = -\log_4 f(n)$.

   Switch the variables and change $f(n)$ to $f^{-1}(n)$:

   $f^{-1}(n) = -\log_4 n = \log_4 n^{-1}$

   Therefore, the inverse function is $f^{-1}(n) = \log_4 n^{-1}$.

6. Explain what the inverse function describes in the context of this problem.

   Suppose $n = 4$. The value of the function at $n = 4$ is given by the ordered pair $(4, 4^{-4})$, so the corresponding point for the inverse is $(4^{-4}, 4)$. The inverse implies that a probability of 4^{-4} corresponds to the token being rolled 4 times with the same outcome (4^{-1}) each time.

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Example 3

The owners of a West Virginia pine forest and grasslands preserve introduced a breeding pair of quail to a specific part of the property. The population of quail can be modeled by an exponential function, but a statistics teacher at the local college came up with the logarithmic function $M(n) = 8.33 - \log(50 - n)^4$, which models how the number of quail n relates to the number of months M since the introduction of the first pair to the preserve. What is the maximum number of quail (n_{max}) that can be estimated using this model? Explain your answer, and state the domain for the function. What does the constant 8.33 mean in terms of this function model? Write the inverse of the logarithmic function.

1. Name the type of number that has to be used for the variable n.

   The answer to this will also help in describing the domain.

   The variable n refers to individual quail, so n has to be a positive whole number that is equal to at least 2, since the problem specifies a breeding *pair*.

2. State the value of n for which the logarithmic term of the function is undefined.

   This has to be considered since a logarithm is involved.

   The argument of the logarithm, $50 - n$, cannot be less than or equal to 0.

3. Use the results of steps 1 and 2 to list the domain.

   The variable n is a positive whole number that is at least 2 but less than 50, so the domain is [2, 49].

4. Name the maximum number of quail that can be estimated using the logarithmic model.

   The maximum number of quail is 49, since that is the largest number for n in the domain.

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5. Determine the range of the function $M(n)$.

   M is the number of months, so $M(n)$ should be computed using the end points of the domain.

   Evaluate $M(n)$ at the beginning of the domain values, 2.

   $M(2) = 8.33 - \log (50 - 2)^4$

   $M(2) = 8.33 - \log(48)^4$

   $M(2) \approx 1.6$ months

   Evaluate $M(n)$ at the end of the domain values, 49.

   $M(49) = 8.33 - \log (50 - 49)$

   $M(49) = 8.33 - \log 1$

   $M(49) = 8.33$ months

   The time of 8.33 is the maximum time it will take for the quail population to "max out" at 49 birds.

6. Write the inverse of the logarithmic function.

   Simplify the logarithmic function using algebraic methods and the power rule of logarithms in order to write the inverse.

   $M(n) = 8.33 - \log (50 - n)^4$

   $M(n) - 8.33 = -4 \bullet \log (50 - n)$

   $8.33 - M(n) = 4 \bullet \log (50 - n)$

   $$\frac{8.33 - M(n)}{4} = \log(50 - n)$$

   Switch the variables, and change n to m and $M(n)$ to $N(m)$ to indicate that the inverse of the logarithmic function will use the number of months m as the independent variable and the variable $N(m)$ for the number of quail.

   The inverse is $\frac{8.33 - m}{4} = \log[50 - N(m)]$, which can be written as $50 - N(m) = 10^{\left(\frac{8.33 - m}{4}\right)}$ or $N(m) = 50 - 10^{2.0825 - 0.25m}$.

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Problem-Based Task 4A.2.5: When Will It Beep?

SMP	
1 ✓	2
3	4 ✓
5	6 ✓
7	8

Many models of household smoke detectors use a radioactive element as a component in the smoke-detection system. In one model, the radioactive element has a half-life of 450 years. This model will stop working (even if you change the battery annually) if the radioactivity drops to 0.2% of its original radioactivity level. The amount of radioactivity $R(t)$ left after t years of radioactive decay is related to the original amount of radioactivity R_0 by the exponential function $R(t) = R_0 \bullet e^{-ct}$, where c is a constant that is unique to the radioactive element. How long can this smoke detector be expected to function properly? Based on this result, explain why it is recommended that the battery in a smoke detector be changed annually.



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Problem-Based Task 4A.2.5: When Will It Beep?

Coaching

a. How is the half-life of the radioactive element related to the original amount of radioactivity of the element?

b. How can you use the half-life data to find the value of the constant c in the exponent of the exponential term?

c. How can the inverse of the exponential function be used to simplify finding the answer to part b? Find the value of the constant c.

d. How can you use the result of part c and the information about when radioactive decay will cause the smoke detector to stop working to find how long this will take?

e. Write an equation using the information in parts c and d, then use it to determine how long it will take for the smoke detector to stop working.

f. How does the answer to part e compare to how often the battery should be changed in a smoke detector?

g. What other factor(s) might explain why the smoke detector's battery needs to be changed annually?

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Instruction

Problem-Based Task 4A.2.5: When Will It Beep?

Coaching Sample Responses

a. How is the half-life of the radioactive element related to the original amount of radioactivity of the element?

The half-life of the radioactive element is the amount of radioactivity, $R(t)$, that is equal to half of the original amount of radioactivity, R_0. Mathematically, this can be written as $R(t) = 0.5 \bullet R_0$.

b. How can you use the half-life data to find the value of the constant c in the exponent of the exponential term?

The results from part a can be used with the given function $R(t) = R_0 \bullet e^{-ct}$ to eliminate one of the variables.

If $R(t) = R_0 \bullet e^{-ct}$ and $R(t) = 0.5 \bullet R_0$, the function can be written as $0.5 \bullet R_0 = R_0 \bullet e^{-ct}$, which simplifies to $0.5 = e^{-ct}$. This can be rewritten as $2 = e^{ct}$ by taking the reciprocal of each side of the equation, which eliminates the negative sign. We are given that $t = 450$ years for this element, so the function becomes $2 = e^{450c}$.

c. How can the inverse of the exponential function be used to simplify finding the answer to part b? Find the value of the constant c.

Additional information will be needed to find the value of c. The inverse provides the means to do that. A calculator that can compute common and natural logarithms will be needed in order to find the value of c.

The equation $2 = e^{450c}$ can be rewritten as $450 \bullet c = \ln 2$.
Therefore, $c = \dfrac{\ln 2}{450} \approx 0.0015$.
The units of c are "per year" or "y^{-1}" since the exponent of e has to be dimensionless.

d. How can you use the result of part c and the information about when radioactive decay will cause the smoke detector to stop working to find how long this will take?

The information regarding the time when the smoke detector will stop working is related to how the half-life was expressed in terms of the original amount of radioactivity.

In this case, the smoke detector will stop working when the radioactivity level, $R(t)$, is down to 0.2% of the original amount, R_0. However, 0.2% will need to be converted to a decimal: 0.2% = 0.002.

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e. Write an equation using the information in parts c and d, then use it to determine how long it will take for the smoke detector to stop working.

In order to write this equation, recall that the information to solve for t is already known. There is a relationship between $R(t)$ and R_0, and the value of c is known.

First, write the function for when $R(t)$ is down to 0.2%.

$R(t) = 0.002 \bullet R_0$

Substitute this for $R(t)$ in the original function.

$0.002 \bullet R_0 = R_0 \bullet e^{-ct}$

Substitute the value of c.

$0.002 \bullet R_0 = R_0 \bullet e^{-ct}$

$0.002 = e^{-0.0015t}$

Write the inverse natural logarithm function and simplify.

$-0.0015t = \ln(0.002)$

$$t = \frac{\ln(0.002)}{-0.0015} \approx 4{,}143 \text{ years}$$

f. How does the answer to part e compare to how often the battery should be changed in a smoke detector?

An annual change of the smoke-detector battery, and the number of years it will take the radioactive element of the smoke detector to decay to the point that the smoke detector stops working, are more than 4,000 years apart.

g. What other factor(s) might explain why the smoke detector's battery needs to be changed annually?

One possible response is that the battery is not related to the decay of the radioactivity element. The battery is used to keep other components of the smoke detector working if there is a household power failure (e.g., the alarm, signal lights).

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4A.2.5: Interpreting Logarithmic Models

Use the following information to complete problems 1–3.

> An aerial photograph of a section of undeveloped land documents the growth of a rapidly growing ground plant called kudzu over a 10-year period. A statistician estimates that the spread of the plant over the 10-year period can be modeled by the logarithmic function $A(t) = -18.5 + 30.5 \cdot \ln t$, where A is the acreage covered by kudzu and t is the number of years.

1. What is the value of $A(1)$, and what meaning does it have in the context of this problem?

2. What is the domain of the function?

3. What is the range of the function?

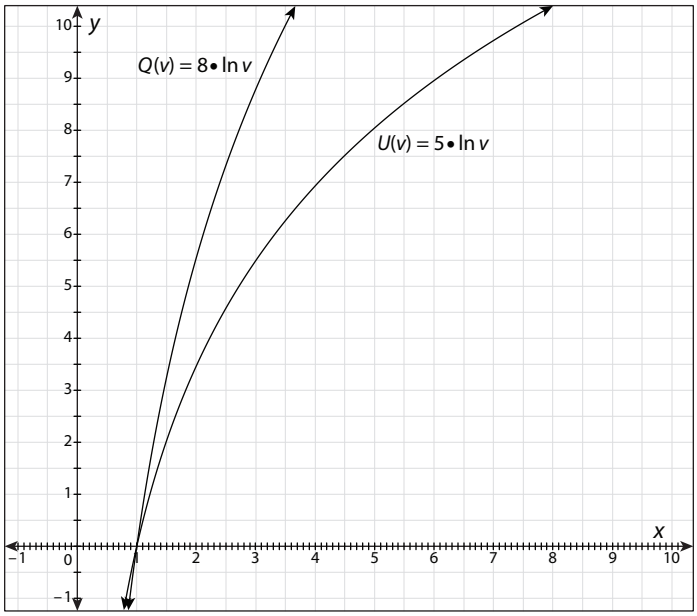
Use the given information to complete problem 4.

4. The manufacturer of a brand of orange juice would like to decrease the acidity of the original product and market the low-acid version in special health food markets. If the original juice has a pH of 3.25, and the manufacturer wants to cut the acid-producing substances in the juice in half, what will the pH of the resulting product be? (*Note*: $pH = -\log c$, where c is the concentration of hydronium or "acid" ions.)

continued

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The graph shows the work done by a mechanic's pneumatic car lift at two different temperatures. Use the graph to complete problems 5 and 6.



5. Draw a vertical line on the graph that represents the difference between the work function at the lower temperature, $U(v) = 5 \bullet \ln v$, and the work function at the higher temperature, $Q(v) = 8 \bullet \ln v$, at a domain value of $v = 2$ where v is the volume of the liquid-compression chamber of the car lift.

6. Without using a calculator, write and simplify a mathematical expression that represents this difference.

continued

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Use the following information to complete problems 7–9.

> A beagle can detect a biological trace element in the air in average concentrations of one part per ten million. Humans can detect similar substances in the air in average concentrations of one part per thousand.

7. Write fractions representing the sense of smell in beagles and humans.

8. Write common logarithms for each fraction determined in problem 7.

9. Compare the answers to problems 7 and 8, and describe a way of comparing the sense of smell of beagles and humans using powers of 10 and common logarithms.

Use the given information to complete problem 10.

10. The trumpet section of the school band has 6 trumpets that can produce a sound with an intensity level of 90 decibels (dB). Three more trumpets will be added to the trumpet section for the next school term. The band director says that this will make the band "50 percent louder." The math club president says the band will not be that loud. Who is correct, the band director or the math club president? Explain your answer using the sound-intensity function $D(I) = 10 \bullet \log\left(\frac{I}{I_0}\right)$, in which I_0 is the threshold of human hearing of 10^{-12} watt per square meter.

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Progress Assessment

Circle the letter of the best answer.

1. What is the inverse function of the exponential function $f(x) = 1 - 3^x$?

   a. $\log_3 (x - 1) = f^{-1}(x)$
   b. $\log_3 x = f^{-1}(x)$
   c. $\log_3 f^{-1}(x) = x$
   d. $\log_3 (1 - x) = f^{-1}(x)$

2. What is the inverse function of the exponential function $g(x) = 4 \cdot 5^{6-x}$?

   a. $g^{-1}(x) = \frac{1}{4}(\log_5 x - 6)$
   b. $g^{-1}(x) = 4(6 - \log_5 x)$
   c. $g^{-1}(x) = 6 + \log_5 4 - \log_5 x$
   d. $g^{-1}(x) = \log_5 x - 6 - \log_5 4$

3. What is the exponential function for which the common logarithmic function $h(x) = -\log (3x + 4)$ is the inverse?

   a. $h^{-1}(x) = \frac{1}{10^x}$
   b. $h^{-1}(x) = \frac{1}{3 \cdot 10^x} - \frac{4}{3}$
   c. $h^{-1}(x) = \frac{1}{3}(10^x - 4)$
   d. $h^{-1}(x) = \frac{1}{3 \cdot 10^x} - 4$

4. What is the common logarithmic function that is equivalent to $a(x) = -7 \cdot \log_4 x$?

   a. $a(x) = -\frac{7 \cdot \log x}{\log 4}$
   b. $a(x) = -7 \cdot \log 4x$
   c. $a(x) = -\frac{7 \cdot \log 5}{\log x}$
   d. $a(x) = -\frac{\log x}{7 \cdot \log 4}$

5. What is the value of the natural logarithm function that is equivalent to the function $b(1) = 4 \cdot e^3$?

   a. $\ln b(1) = 4 - \ln 3$
   b. $\ln b(1) = 4 + \ln 3$
   c. $\ln b(1) = 3 + \ln 4$
   d. $\ln b(1) = 3 - \ln 4$

continued

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Assessment

6. What is the domain of the natural logarithm function $c(x) = 10 \bullet \ln(x - e)$?

   a. $(-e,+\infty)$

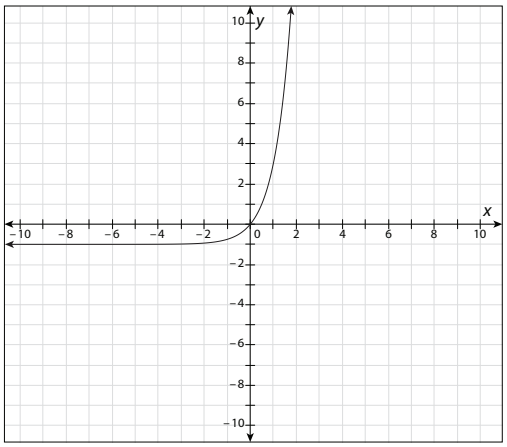
   b. $(-e,-\infty)$

   c. $(e,+\infty)$

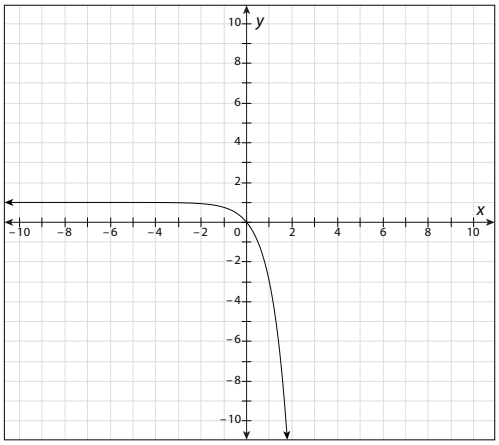
   d. $(0,+\infty)$

7. Which is the graph of $d(x) = 1 - \log_4 x$?

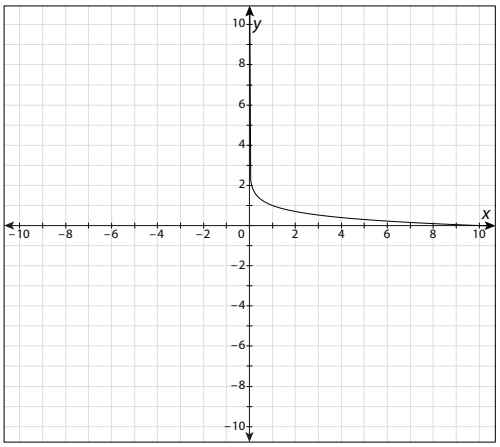
   a.



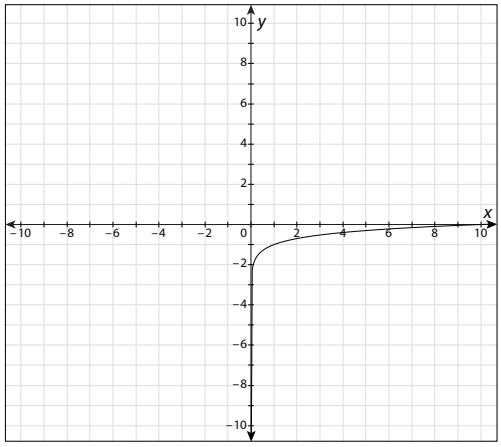
   c.



   b.



   d.

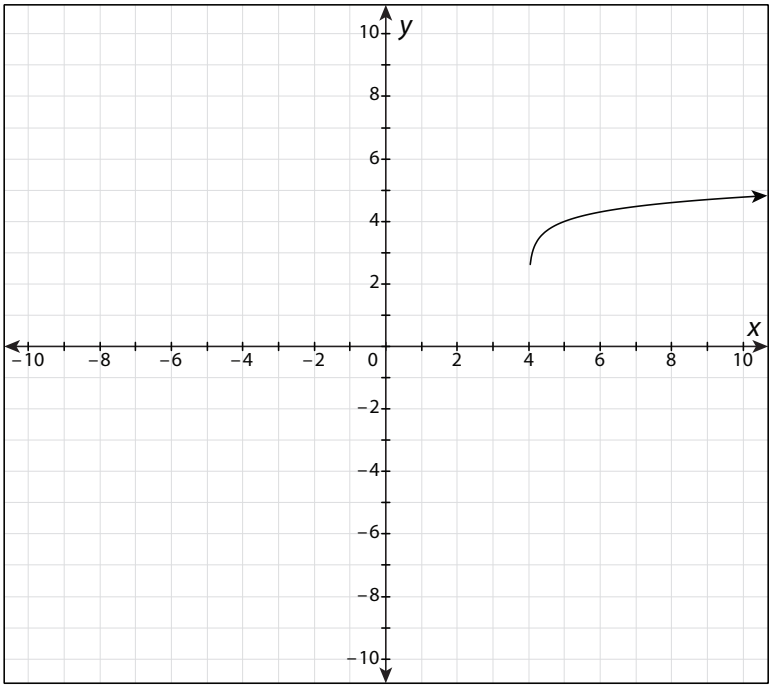


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Assessment

8. What function could be represented by the graph shown?



a. $f(x) = 4 + \log(x + 4)$

b. $f(x) = 4 - \log(x - 4)$

c. $f(x) = 4 + \log(x - 4)$

d. $f(x) = 4 - \log(x + 4)$

Use the information that follows to complete problems 9 and 10.

The natural logarithm function $\Delta G = RT \bullet \ln \frac{C_{\text{inside}}}{C_{\text{outside}}}$ is used to describe the change in energy ΔG in a biological system (such as a cell) in which R and T are constants, C_{inside} is the ion concentration of a specific chemical inside the system, and C_{outside} is the ion concentration of that chemical outside the system.

9. What is the relationship of C_{inside} to C_{outside} for any energy change in the system?

a. $C_{\text{inside}} > C_{\text{outside}}$

b. $C_{\text{inside}} + C_{\text{outside}} = 0$

c. $\frac{C_{\text{inside}}}{C_{\text{outside}}} > 0$

d. $C_{\text{inside}} = C_{\text{outside}}$

continued

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Assessment

10. What is the relationship of C_{inside} to $C_{outside}$ if $\Delta G < 0$?

    a. $C_{inside} < C_{outside}$

    b. $C_{inside} > C_{outside}$

    c. $C_{inside} = C_{outside}$

    d. $C_{inside} \bullet C_{outside} = 0$

Use the given information to complete all parts of problem 11.

11. A math instructor tells his students that the acceleration of the head of a copperhead snake as it strikes can be modeled by the natural logarithm function $\ln A(t) = t + \ln 15$, in which $A(t)$ is the acceleration of its head in meters per second squared and t is the strike time in milliseconds. The instructor says that the model is based on an exponential function and is accurate over the domain (0, 5 milliseconds). A team of students used an accelerometer probe to collect data on a copperhead in the reptile house of the city zoo, and analyzed the data using software. The students found that the acceleration of the snake's head starts from being at rest. Just before the head strikes the target, the acceleration of the head is almost 15 meters per second squared. In writing the logarithmic inverse function, the students made an error in one or more of the steps.

    For each step a through d:

    - Determine whether the step shown is correct.
    - If it is correct, state the correct operation or rule applied.
    - If it is incorrect, state the incorrect operation or rule applied.

    a. $\ln\left[\frac{A(t)}{15}\right] = \ln A(t) - \ln 15$

    b. $\ln\left[1 - \frac{A(t)}{15}\right] = \ln 1 - \ln A(t) + \ln(15)$

    c. $1 - \frac{A(t)}{15} = e^{-t} \rightarrow -t = \ln\left[1 - \frac{A(t)}{15}\right]$

    d. $15 - A(t) = 15e^{-t} \rightarrow 1 - \frac{A(t)}{15} = e^{-t}$

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. What is the degree measure of an angle that is $\frac{\pi}{6}$ radian?

   a. 0.5°
   b. 15°
   c. 30°
   d. 60°

2. What is the period of the function $g(x) = 3 \sin 4x$?

   a. $\frac{\pi}{2}$
   b. 2π
   c. 6π
   d. 8π

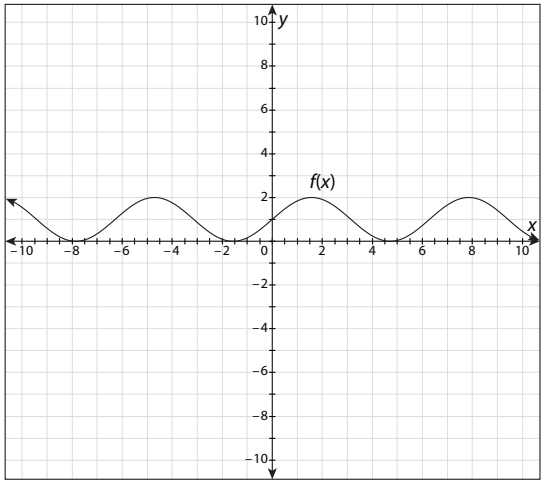
3. What are the domain and range of $f(x) = 4 \sin 2x$?

   a. $[-\infty,\infty]$ and $(-4, 4)$
   b. $(-\infty,\infty]$ and $(-4, 4]$
   c. $[-\infty,\infty)$ and $(-4, 4]$
   d. $(-\infty,\infty)$ and $[-4, 4]$

4. What are the domain and range of $g(x) = -0.5 \cos(-5x)$?

   a. $(-\infty,\infty)$ and $[-0.5, 0.5]$
   b. $(-\infty,\infty)$ and $(-0.5, 0.5]$
   c. $(-\infty,\infty)$ and $[-0.5, 0.5)$
   d. $(-\infty,\infty)$ and $(-0.5, 0.5)$

5. Which function does the graph represent?



   a. $f(x) = -\sin x - 1$
   b. $f(x) = \sin x - 1$
   c. $f(x) = \sin x + 1$
   d. $f(x) = -\sin x + 1$

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Lesson 3: Modeling Trigonometric Functions

Instruction

Common Core State Standard

SMP
1 ✓ 2 ✓
3 4 ✓
5 6 ✓
7 ✓ 8 ✓

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Essential Questions

1. What is the amplitude of a sine function given by $f(x) = a \sin bx$?
2. What is the period of the sine function given by $f(x) = a \sin bx$?
3. What is the amplitude of a cosine function given by $g(x) = a \cos bx$?
4. What is the period of the cosine function given by $g(x) = a \cos bx$?
5. How do you find the amplitude, domain, period, and range of a trigonometric function?

WORDS TO KNOW

amplitude the coefficient a of the sine or cosine term in a function of the form $f(x) = a \sin bx$ or $g(x) = a \cos bx$; on a graph of the cosine or sine function, the vertical distance from the y-coordinate of the maximum point on the graph to the midline of the cosine or sine curve

argument the term $[b(x - c)]$ in a cosine or sine function of the form $f(x) = a \sin [b(x - c)] + d$ or $g(x) = a \cos [b(x - c)] + d$

cosine a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the side adjacent to the length of the hypotenuse; the cosine of $\theta = \cos \theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$

cosine function a trigonometric function of the form $f(x) = a \cos bx$, in which a and b are constants and x is a variable defined in radians over the domain $(-\infty, \infty)$

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Instruction

cycle — the smallest representation of a cosine or sine function graph as defined over a restricted domain; equal to one repetition of the period of a function

midline — in a cosine function or sine function of the form $f(x) = \sin x + d$ or $g(x) = \cos x + d$, a horizontal line of the form $y = d$ that bisects the vertical distance on a graph between the minimum and maximum function values

parent function — a function with a simple algebraic rule that represents a family of functions. The graphs of the functions in the family have the same general shape as the parent function. For cosine and sine, the parent functions are $f(x) = \cos x$ and $g(x) = \sin x$, respectively.

period — in a cosine or sine function graph, the horizontal distance from a maximum to a maximum or from a minimum to a minimum; one repetition of the period of a function is called a *cycle*

phase shift — on a cosine or sine function graph, the horizontal distance by which the curve of a parent function is shifted by the addition of a constant or other expression in the argument of the function

radian — the measure of the central angle that intercepts an arc equal in length to the radius of the circle; π radians = 180°

sine — a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the hypotenuse;

$$\text{the sine of } \theta = \sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$$

sine function — a trigonometric function of the form $f(x) = a \sin bx$, in which a and b are constants and x is a variable defined in radians over the domain $(-\infty, \infty)$

unit circle — a circle with a radius of 1 unit. The center of the circle is located at the origin of the coordinate plane.

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Recommended Resources

- Brightstorm, Inc. "Graphs of the Sine and Cosine Functions."

  http://www.walch.com/rr/00233

  This site provides short video tutorials with accompanying transcripts on graphing sine and cosine functions.

- Illuminations. "Trigonometric Graphing."

  http://www.walch.com/rr/00234

  This graphing utility allows users to explore amplitudes, periods, and phase shifts of trigonometric functions.

- Purplemath.com. "Graphing Trigonometric Functions."

  http://www.walch.com/rr/00235

  This website explains how to graph trigonometric functions and includes abundant examples of sine and cosine graphs.

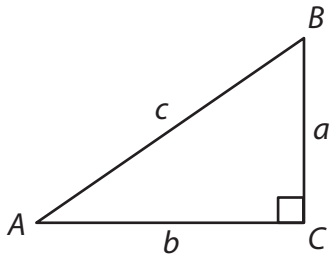
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Lesson 4A.3.1: Graphing the Sine Function

Warm-Up 4A.3.1

Sofia is preparing a presentation to the city council on the need to lengthen several ramps at the skate park. Her presentation must include mathematical justifications for the new ramp lengths in order to prove her point to the council. To make her case, Sofia decided to create a table and a diagram to help the council members understand how she used sines to find the ramp lengths based on given angles. The diagram is finished, but the table is not.

In Sofia's diagram, the sine ratio for angle A in right triangle ABC is defined as the ratio of the length of side a to the length of the hypotenuse, c.



Mathematically, the sine ratio of angle A can be written as $\sin A = \frac{a}{c}$. Similarly, the sine ratio of angle B can be written as $\sin B = \frac{b}{c}$. Use the diagram and the following questions to complete the table for Sofia's presentation.

$m\angle A$	$m\angle B$	$m\angle C$	a	b	c	$\sin A$	$\sin B$
30°			1				
45°			1				
60°			1				

1. What is the relationship between the sum of $m\angle A$ and $m\angle B$ in each row? Use this information to fill in the missing angle measures for angles B and C.
2. What is the relationship between the lengths of the three sides of right triangle ABC? Use this information to fill in the missing lengths for b and c. Simplify the resulting values for the rows for 30°, 45°, and 60° as much as possible without using a calculator.
3. Use the definition of the sine ratio to compute the values of sin A and sin B in the table. What do you notice about the sine ratios for complementary angles? (*Hint*: Complementary angles have measures that add up to the measure of a right angle.)

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Instruction

Lesson 4A.3.1: Graphing the Sine Function

Common Core State Standard

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Warm-Up 4A.3.1 Debrief

1. What is the relationship between the sum of $m\angle A$ and $m\angle B$ in each row? Use this information to fill in the missing angle measures for angles *B* and *C*.

   In each row, $m\angle A + m\angle B = 90$; that is, the angles opposite the legs of the right triangle sum to 90°.

   To solve for $m\angle B$:

Let $m\angle A = 30$.	Let $m\angle A = 45$.	Let $m\angle A = 60$.
$m\angle A + m\angle B = 90$	$m\angle A + m\angle B = 90$	$m\angle A + m\angle B = 90$
$(30) + m\angle B = 90$	$(45) + m\angle B = 90$	$(60) + m\angle B = 90$
$m\angle B = 60$	$m\angle B = 45$	$m\angle B = 30$

   As shown in the diagram, $\angle C$ is a right angle and measures 90° in each case.

   Write these measures for angles *B* and *C* in the appropriate table cells. (Refer to the completed table in the response to problem 3.)

2. What is the relationship between the lengths of the three sides of right triangle *ABC*? Use this information to fill in the missing lengths for *b* and *c*. Simplify the resulting values for the rows for 30°, 45°, and 60° as much as possible without using a calculator.

   Sides *a*, *b*, and *c* are related to one another by the Pythagorean Theorem, $a^2 + b^2 = c^2$.

   The table includes rows for three different right triangles: 30°–60°–90°, 45°–45°–90°, and 60°–30°–90°.

   In a 30°–60°–90° triangle, the side opposite the 30° angle is half the length of the hypotenuse. As stated in the table, $a = 1$, which gives the relationship $1^2 + b^2 = 2^2$. Solving for *b* results in $b = \sqrt{3}$.

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In a 45°–45°–90° triangle, sides a and b are equal in length, so the relationship among the sides is given by $1^2 + 1^2 = c^2$. Solving for c results in $c = \sqrt{2}$.

A 60°–30°–90° triangle is actually the same triangle as the 30°–60°–90° triangle. However, in this example, $a = 1$, and we do not know the values of b and c, but we do know that b is half the length of c. Therefore, we can substitute x for the unknown length of b and $2x$ for c to determine that the relationship among the three sides is $1^2 + x^2 = (2x)^2$. Solve this equation for x to determine the missing lengths of b and c.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$(1)^2 + (x)^2 = (2x)^2$	Substitute 1 for a, x for b, and $2x$ for c.
$1 + x^2 = 4x^2$	Apply the exponent.
$1 = 3x^2$	Subtract x^2 from both sides.
$\frac{1}{3} = x^2$	Divide by 3.
$\sqrt{\frac{1}{3}} = x$	Take the square root.
$\frac{1}{\sqrt{3}} = x$	Simplify the numerator.
$\frac{1}{\sqrt{3}} \bullet \frac{\sqrt{3}}{\sqrt{3}} = x$	Rationalize the denominator.
$\frac{1\sqrt{3}}{\sqrt{9}} = x$	Multiply.
$\frac{\sqrt{3}}{3} = x$	Simplify.

Since $x = b$, this means that the length of b is $\frac{\sqrt{3}}{3}$ and $c = 2 \bullet b = \frac{2\sqrt{3}}{3}$.

Write these measures for sides b and c in the appropriate table cells. (Refer to the completed table in the response to problem 3.)

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3. Use the definition of the sine ratio to compute the values of sin *A* and sin *B* in the table. What do you notice about the sine ratios for complementary angles? (*Hint*: Complementary angles have measures that add up to the measure of a right angle.)

   As given in the problem, the sine ratios are defined as $\sin A = \frac{a}{c}$ and $\sin B = \frac{b}{c}$.

   - For the 30°–60°–90° triangle:

   $$\sin 30° = \frac{a}{c} = \frac{1}{2} \qquad \sin 60° = \frac{b}{c} = \frac{\sqrt{3}}{2}$$

   - For the 45°–45°–90° triangle:

   $$\sin 45° = \frac{a}{c} = \frac{b}{c} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

   This applies to both angles *A* and *B* since each has a measure of 45°.

   - For the 60°–30°–90° triangle:

   $$\sin 60° = \frac{a}{c} \qquad \sin 30° = \frac{b}{c}$$

   $$= \frac{1}{\frac{2\sqrt{3}}{3}} = 1 \div \frac{2\sqrt{3}}{3} = 1 \bullet \frac{3}{2\sqrt{3}} \qquad = \frac{\frac{\sqrt{3}}{3}}{\frac{2\sqrt{3}}{3}} = \frac{\sqrt{3}}{3} \div \frac{2\sqrt{3}}{3} = \frac{\sqrt{3}}{3} \bullet \frac{3}{2\sqrt{3}}$$

   $$= \frac{3}{2\sqrt{3}} = \frac{3}{2\sqrt{3}} \bullet \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2} \qquad = \frac{3\sqrt{3}}{6\sqrt{3}} = \frac{3\sqrt{3}}{6\sqrt{3}} \bullet \frac{\sqrt{3}}{\sqrt{3}} = \frac{1}{2}$$

   In the 30°–60°–90° and 60°–30°–90 right triangles, the sine ratios for the 30° angles are the same and the sine ratios for the 60° angles are the same, even though the lengths of the corresponding sides differ in the two triangles.

   Write these measures for sin *A* and sin *B* in the appropriate table cells.

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Completed table:

$m\angle A$	$m\angle B$	$m\angle C$	a	b	c	$\sin A$	$\sin B$
30°	60°	90°	1	$\sqrt{3}$	2	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
45°	45°	90°	1	1	$\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	30°	90°	1	$\frac{\sqrt{3}}{3}$	$\frac{2\sqrt{3}}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

Connection to the Lesson

- Students will identify the relationships between the sine values of complementary angles.
- Students will discover that the values of the sine term (sin x) in the functions $f(x) = \sin x$ and $g(x) = a \sin x$ are equal, as shown by the different right-triangle calculations in the Warm-Up.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- converting degrees to radians and vice versa (F–TF.1)
- finding the missing measures of a right triangle using the Pythagorean Theorem (G–SRT.8)
- using radical operations to add and subtract radicals (N–RN.2)
- using radical operations to simplify radical expressions, including expressions that involve radicals raised to the second power (N–RN.2)
- identifying the sine function values in an application involving triangles or from the ordered pairs of points on a unit circle graph on a coordinate plane (F–TF.2)

Introduction

In previous lessons, you have worked with sine ratios. Recall that **sine** is a trigonometric function of an acute angle in a right triangle. It is the ratio of the length of the opposite side to the length of the hypotenuse, or $\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$. In this lesson, you will explore the graphs of sine functions.

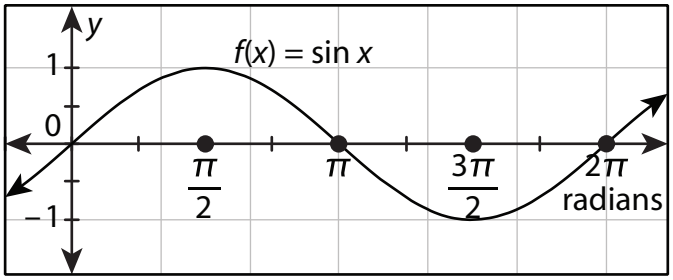
Key Concepts

- A **sine function** is a trigonometric function of the form $f(x) = a \sin bx$, in which a and b are constants and x is a variable defined in radians over the domain $(-\infty, \infty)$.
- Recall that a **radian** is the measure of the central angle that intercepts an arc equal in length to the radius of the circle; π radians = 180°.
- In order to sketch the graph of a sine function, it is necessary to know the amplitude and period of the parent function. A **parent function** is a function with a simple algebraic rule that represents a family of functions. The graphs of the functions in the family have the same general shape as the parent function. For sine, the parent function is $f(x) = \sin x$, as shown in the following graph.

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Instruction

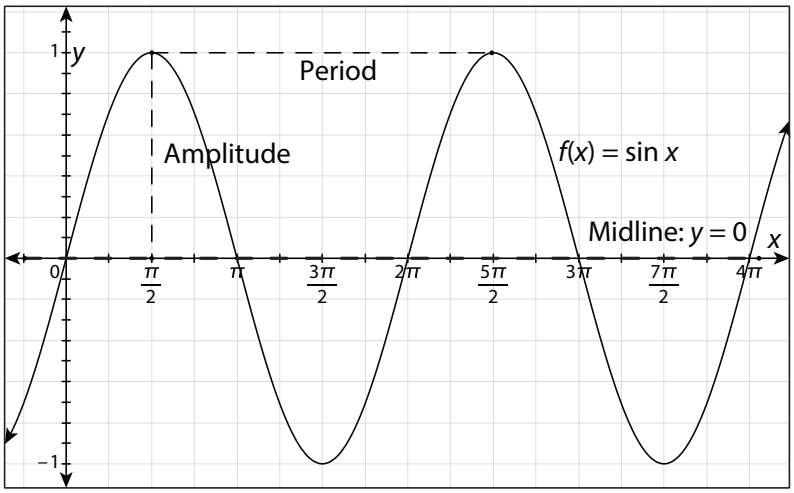
Sine Parent Function, $f(x) = \sin x$



- On this graph, the points shown on the x-axis to the right of the origin correspond to the points $\left(\frac{\pi}{2}, 0\right)$, $(\pi, 0)$, $\left(\frac{3\pi}{2}, 0\right)$, and $(2\pi, 0)$, which are multiples of π in radians. For example, $\frac{\pi}{2} \approx 1.57$ and $2\pi \approx 6.28$. (*Note*: Most graphing calculators display the multiples of π as decimal approximations.)
- The **amplitude** of a function is defined as the vertical distance from the y-coordinate of the maximum point on the graph to the midline of the curve. In the graph of the parent function $f(x) = \sin x$, the amplitude is 1, the distance from the midline (x-axis) to the highest point on the curve.
- The **midline** in a function of the form $f(x) = \sin x + d$ is a horizontal line of the form $y = d$ that bisects the vertical distance on a graph between the minimum and maximum function values. In this example, $f(x) = \sin x$ can be written as $f(x) = \sin x + 0$; therefore, the midline is at $y = 0$.
- The **period** of a graphed sine function is the horizontal distance from a maximum to a maximum or from a minimum to a minimum. In other words, it is the beginning of one complete cycle of the graph to the point at which its behavior or shape repeats. A **cycle** is the smallest representation of a sine function graph as defined over a restricted domain that is equal to the period of the function, as shown in the following graph.

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Instruction



- In this example, the period is 2π or about 6.28, which can also be expressed as 360°. (Recall that a **unit circle** has a radius of 1, which means its circumference is 2π and which creates an angle of 360°.)
- In the general form of the sine function, $f(x) = a \sin bx$, the amplitude is $|a|$ and the period is $\frac{2\pi}{b}$. In other words, any change in the value of a has a direct effect on the amplitude of the graph, and any change in the value of b has an inverse effect on the period.
- In an extended form of the general sine function, $g(x) = a \sin [b(x - c)] + d$, the constant d shifts the graph vertically, whereas the constant c shifts the graph horizontally.
- The term $[b(x - c)]$ in a sine function of the form $g(x) = a \sin [b(x - c)] + d$ is the argument of the function.
- The amount of the **phase shift**, or the horizontal distance by which the curve of a parent function is shifted, is found by setting the argument equal to 0 and solving for x. If the function is in the form $f(x) = a \sin [b(x - c)]$, the phase shift is c. If the function is in the form $f(x) = a \sin (bx + c)$, the phase shift is $-\frac{c}{b}$.

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Instruction

Expressing the Domain of a Sine Function

- The domain of a sine function is often expressed in radians. However, in some applications and real-world problems, it is useful to express the domain in degrees.
- The conversion between degrees and radians is given by the relationship 2π radians = 360 degrees. Use the following equations to convert between the two units of measurement.

$$1 \text{ radian} = \frac{180}{\pi} \approx 57.3^\circ \text{ or } 1 \text{ degree} = \frac{\pi}{180} \approx 0.02 \text{ radian}$$

- When sketching the graph of a sine function, knowing the values of the function at several critical values of x can be useful.

$$\sin 30^\circ = \sin\left(\frac{\pi}{6} \text{ radian}\right) = 0.5$$

$$\sin 45^\circ = \sin\left(\frac{\pi}{4} \text{ radian}\right) = \frac{\sqrt{2}}{2} \approx 0.71$$

$$\sin 60^\circ = \sin\left(\frac{\pi}{3} \text{ radians}\right) = \frac{\sqrt{3}}{2} \approx 0.87$$

General Characteristics of a Sine Function

- The amplitude of the general sine function $f(x) = a \sin [b(x - c)] + d$ is $|a|$. Any change in a has a *direct* effect on the amplitude of the sine graph. For example, if a increases, the value of the term $a \sin [b(x - c)]$ also increases if the other quantities stay the same.
- The period of the general sine function $f(x) = a \sin [b(x - c)] + d$ is $\frac{2\pi}{b}$. Any change in b has an *inverse* effect on the period of the sine graph. For example, if b increases, the value of the period of the function *decreases* if the other quantities stay the same.
- A useful formula for finding the period of any sine function of the form $f(x) = a \sin [b(x - c)] + d$ is to set the argument of the sine function, $[b(x - c)]$, equal to 2π and solve for x. The result is the period of the function.
- In summary, the sine function $f(x) = a \sin [b(x - c)] + d$ has the following characteristics:

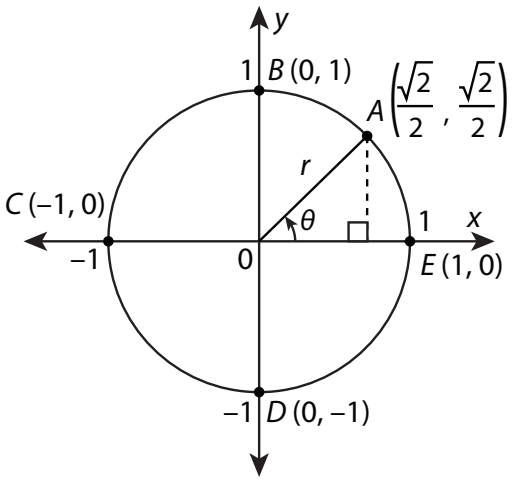
Amplitude: a Period: $\frac{2\pi}{b}$ Phase shift: c Vertical shift: d

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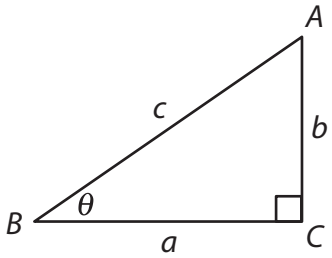
The Unit Circle

- The unit circle is a visual means of viewing sine and cosine values. The unit circle has a radius r of 1 unit and is drawn on a coordinate plane, with the center at (0, 0).



- The coordinates of point A, $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, represent the lengths of the sides of a right triangle formed by dropping an altitude to the x-axis. Those x- and y-coordinates also represent the cosine and sine ratios of a 45° (or a $\frac{\pi}{4}$-radian) angle θ, respectively, of an isosceles right triangle with a hypotenuse that is 1 unit in length. This originates from the plane geometry definition of the cosine and sine ratios.

- The sine and cosine ratios of any right triangle can be found as follows.



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Instruction

- The cosine of angle θ (abbreviated cos θ) is defined as the ratio of the length of side a (the side that is adjacent to angle θ) to the length of the hypotenuse, side c.

$$\cos\theta = \frac{a}{c} \text{ or } a = c\cos\theta$$

- The sine of angle θ (abbreviated sin θ) is defined as the ratio of the length of side b (the side that is opposite angle θ) to the length of the hypotenuse, side c.

$$\sin\theta = \frac{b}{c} \text{ or } b = c\sin\theta$$

- On the unit circle diagram, the horizontal leg of the right triangle is a and the vertical leg is b. Therefore, the coordinates of a point that lies on the unit circle are (a, b) or (c cos θ, c sin θ). For a unit circle, $c = 1$, so the coordinates of the point are (cos θ, sin θ).

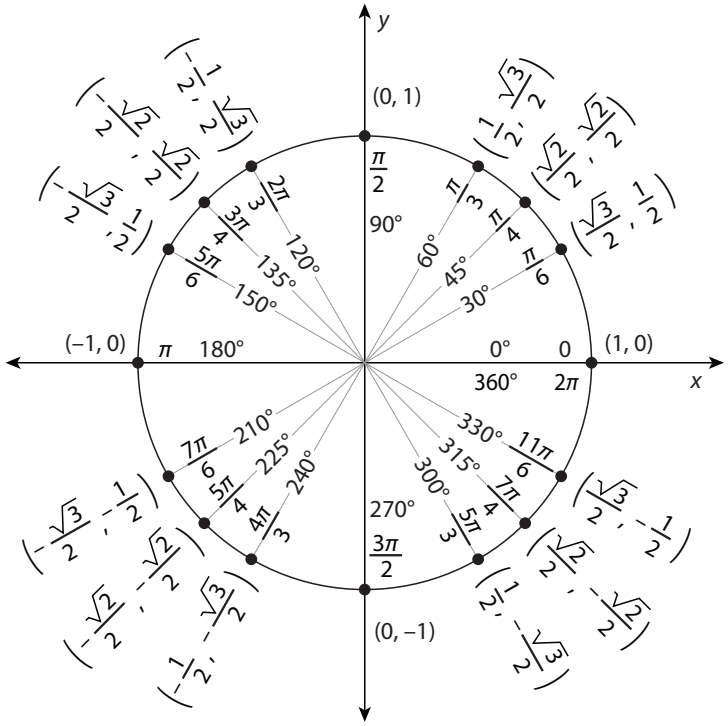
- The same constructions for angles of 30° and 60° (equivalent to $\frac{\pi}{6}$ and $\frac{\pi}{3}$ radians, respectively) can be made for points on the circumference of the circle in the first quadrant. For 30°, the coordinates of the point that lies on the unit circle are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, which is the same as (cos 30°, sin 30°). For 60°, the coordinates of the point that lies on the circle are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, which is the same as (cos 60°, sin 60°).

- The coordinates of the points formed by angles of 30°, 45°, and 60° can be applied to the other three quadrants, but the angle measures will change, and the signs of one or both of the coordinates will change.

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Instruction

- The following diagram shows the angle measures of θ and the related coordinates on the unit circle for the four quadrants:



- Angles measured counterclockwise around the unit circle are positive, whereas angles measured clockwise are negative.

- The coordinates of the points (0, 1), (–1, 0), (0, –1), and (1, 0) on the graph of the unit circle represent the cosine and sine ratios of angles that measure 90°, 180°, 270°, and 360° ($\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π radians, respectively) if the angle θ is measured counterclockwise from the positive *x*-axis. These points also represent the cosine and sine ratios of angles that measure –270°, –180°, –90°, and 0° ($-\frac{3\pi}{2}$, -2π, $-\frac{\pi}{2}$, and 0 radians) if the angle θ is measured clockwise from the positive *x*-axis.

- Adding any positive or negative integer multiple of 360° or 2π radians to an angle will produce equivalent coordinates.

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Instruction

Common Errors/Misconceptions

- confusing degrees and radians in the conversion formula 2π radians = 360°
- forgetting that the domain values of sine function graphs are expressed in radians unless otherwise indicated
- reversing the amplitude and period in the general sine function $f(x) = a \sin bx$
- plotting too few points when graphing the sine function
- using the incorrect sign for the sine or cosine function in any of the four quadrants on a coordinate plane

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Guided Practice 4A.3.1

Example 1

Sketch the graph of $f(x) = 2 \sin x$ over the restricted domain $[-2\pi, 2\pi]$.

1. Identify the amplitude and period of the function.

   This will determine the x- and y-axis scales.

   The coefficient of the sine term is 2, which is the amplitude. Therefore, the y-axis scale should run from at least $y = -2$ to $y = 2$. The coefficient of the sine argument is 1, so $x = 2\pi$, which means the period is 2π. If the graph shows the restricted domain of $[-2\pi, 2\pi]$, it will include two complete "cycles" of the function.

2. Identify any other coefficients or terms that would affect the shape of the graph.

   This is necessary to determine if the graph is translated horizontally or vertically, or if other mathematical terms affect the graph's shape.

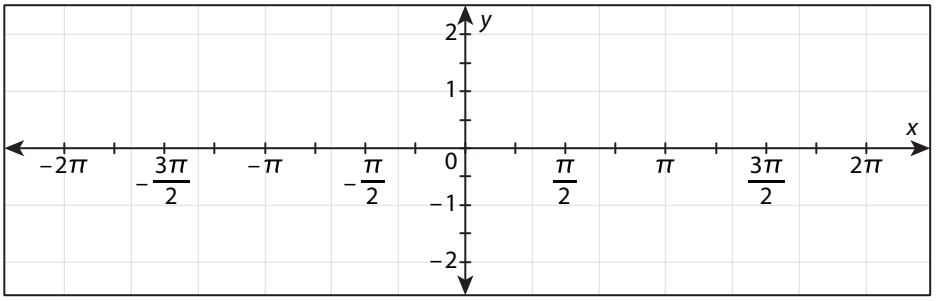
   The equation of the function, $f(x) = 2 \sin x$, has no other coefficients or terms that affect the shape or placement of the function's graph.

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Instruction

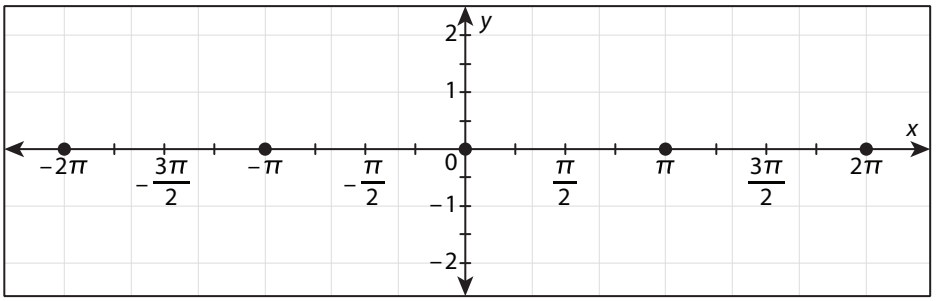
3. Draw and label the axes for your graph based on steps 1 and 2.

   The x-axis scale will range from -2π to 2π, and the y-axis scale will range from at least –2 to at least 2. Each axis should be divided into sufficient intervals to allow for enough points to be plotted to show the graph. Label the x-axis in increments of $\pm\frac{\pi}{2}$ and the y-axis in increments of $\frac{1}{2}$.



4. Determine the values of the restricted domain for which the function value(s) equal 0. List the points corresponding to these zeros, and plot them on the graph.

   The period of the function $f(x) = 2 \sin x$ is 2π. The function is equal to 0 at $x = \pm 2\pi, \pm\pi$, and 0. The corresponding points are $(-2\pi, 0)$, $(-\pi, 0)$, $(0, 0)$, $(\pi, 0)$, and $(2\pi, 0)$. Plot the five points as shown.



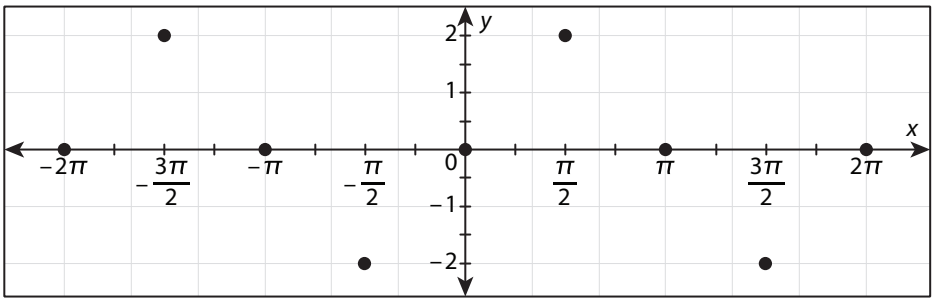
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5. Determine what values of the restricted domain are the maximum and minimum of the function value(s). List the points corresponding to these extremes, and plot them on the graph.

   The sine function values vary between the two extremes defined by the amplitude. In this case, the amplitude is 2, so the function values for the maximum and minimum values of $f(x) = 2 \sin x$ will vary between –2 and 2. These values occur at $x = \pm\frac{\pi}{2}$ and $\pm\frac{3\pi}{2}$.

   The points for the maximum and minimum values of $f(x) = 2 \sin x$ are $\left(-\frac{3\pi}{2}, 2\right)$, $\left(-\frac{\pi}{2}, -2\right)$, $\left(\frac{\pi}{2}, 2\right)$, and $\left(\frac{3\pi}{2}, -2\right)$. Add these points to the graph.



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6. Plot points for the sines of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ radians.

   Refer to the diagram of coordinates for given values of $m\angle\theta$ in the Key Concepts to find the sines, or use a graphing calculator.

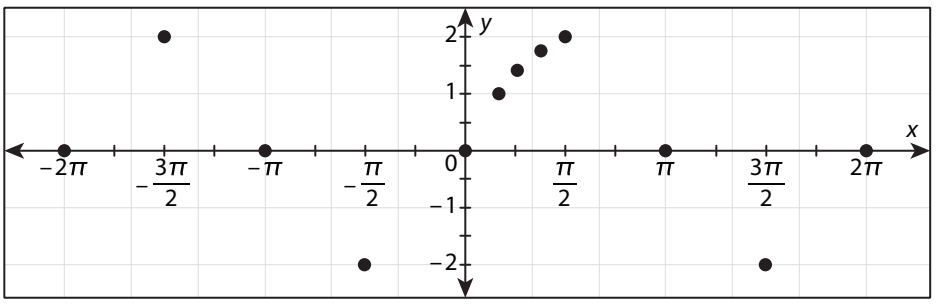
   These points will fall between the zero at (0, 0) and the function maximum at $\left(\frac{\pi}{2}, 2\right)$. Moreover, they will suggest the shape of a quarter-period "piece" of the sine curve, which can be reflected across the *y*-axis from the maximum point at $\left(\frac{\pi}{2}, 2\right)$ to the *x*-axis to produce the sine curve over the domain of $[0, \pi]$.

   From the diagram of coordinates for $m\angle\theta$, we see that $2\sin\left(\frac{\pi}{6}\right) = 1$ and $\frac{\pi}{6} \approx 0.5$, so its ordered pair is (0.5, 1). The coefficient 2 is specific to this function; multiply by the function values $\frac{\pi}{4}$ and $\frac{\pi}{3}$, respectively, to find the other two function values and points:

   $2\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \approx 1.4$ and $\frac{\pi}{4} \approx 0.8$; (0.8, 1.4)

   $2\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \approx 1.7$ and $\frac{\pi}{3} \approx 1.1$; (1.1, 1.7)

   Plot (0.5, 1), (0.8, 1.4), and (1.1, 1.7) on the graph between the points (0, 0) and $\left(\frac{\pi}{2}, 2\right)$.



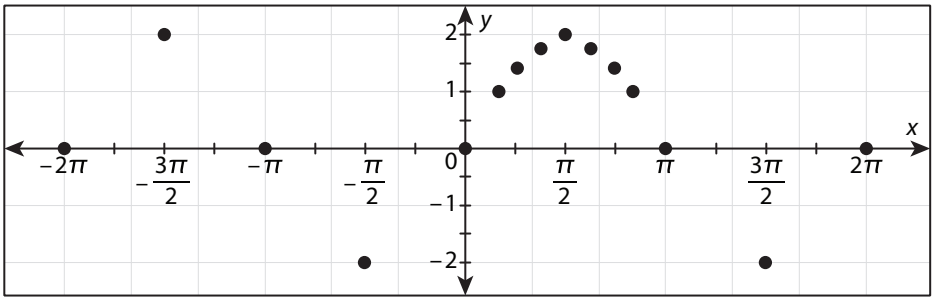
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7. Plot additional points.

Plot the points for the x-values $\frac{2\pi}{3}$, $\frac{3\pi}{4}$, and $\frac{5\pi}{6}$, which are between the points $\left(\frac{\pi}{2}, 2\right)$ and $(\pi, 0)$.

This will show how the function values are mirrored on either side of the altitude from the maximum point at $\left(\frac{\pi}{2}, 2\right)$ to the midline $y = 0$ (the x-axis) for domain values that differ by $\frac{\pi}{2}$.

For this function, $f\left(\frac{2\pi}{3}\right) = 2\sin\left(\frac{2\pi}{3}\right) = 2 \bullet \frac{\sqrt{3}}{2} \approx 1.7$, so the approximate point is given by $\left(\frac{2\pi}{3}, 1.7\right)$. The other function values can be determined by using a calculator or by finding the appropriate values in the diagram of coordinates for $m\angle\theta$ and multiplying by 2, giving the approximate points as $\left(\frac{3\pi}{4}, 1.4\right)$ and $\left(\frac{5\pi}{6}, 1\right)$. Plot these points.



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8. Compare the ordered pairs for the graphed points on either side of the x-axis.

   The points occur where $x = \frac{\pi}{6}$ and $\frac{5\pi}{6}$, $x = \frac{\pi}{4}$ and $\frac{3\pi}{4}$, and $x = \frac{\pi}{3}$ and $\frac{2\pi}{3}$.

   The function values for these point pairs are equal, which confirms that the points are mirrored on either side of the altitude from the maximum point at $\left(\frac{\pi}{2}, 2\right)$ to the midline at $y = 0$ (the x-axis) over the restricted interval $[0, \pi]$ for domain values that differ by $\frac{\pi}{2}$.

9. Determine the function values for each value of x over the restricted domain of $[\pi, 2\pi]$. Plot points on the graph for this part of the domain.

   This will show that the function values for values of x over the restricted domain $[\pi, 2\pi]$ are the opposite of those over the restricted domain $[0, \pi]$ for domain values that differ by $\frac{\pi}{2}$.

   Use a graphing calculator or refer to the diagram of coordinates for given values of $m\angle\theta$ in the Key Concepts to find the function values for the x-values over the domain $[\pi, 2\pi]$.

   The x-values are $\frac{7\pi}{6}$, $\frac{5\pi}{4}$, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$, $\frac{7\pi}{4}$, and $\frac{11\pi}{6}$.

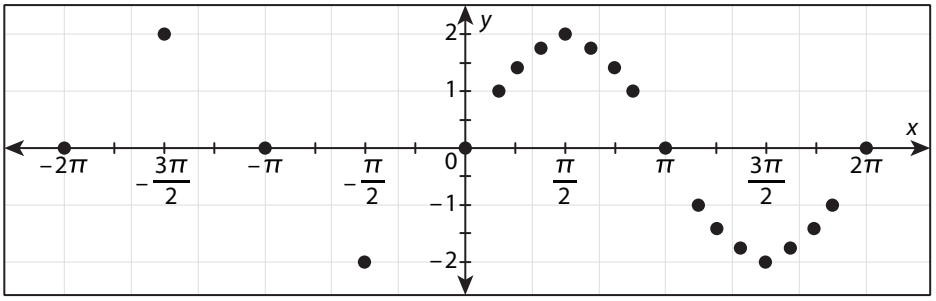
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The calculator will give function values that result in the approximate points $\left(\frac{7\pi}{6},-1\right)$, $\left(\frac{5\pi}{4},-1.4\right)$, $\left(\frac{4\pi}{3},-1.7\right)$, $\left(\frac{5\pi}{3},-1.7\right)$, $\left(\frac{7\pi}{4},-1.4\right)$, and $\left(\frac{11\pi}{6},-1\right)$.

Plot these points on the graph.



10. Compare the points over the two restricted domains $[0, \pi]$ and $[\pi, 2\pi]$.

    The points over the restricted domain $[\pi, 2\pi]$ graphed in step 9 are the points previously graphed for the restricted domain $[0, \pi]$, but flipped over the x-axis and reflected over the point $(\pi, 0)$. This reinforces the idea that the function values of the points over the restricted domain $[0, \pi]$ and points over the restricted domain $[\pi, 2\pi]$ are opposites for domain values that differ by $\frac{\pi}{2}$.

11. Predict what the shape of the function graph will be over the remainder of the domain, $(-2\pi, 0)$.

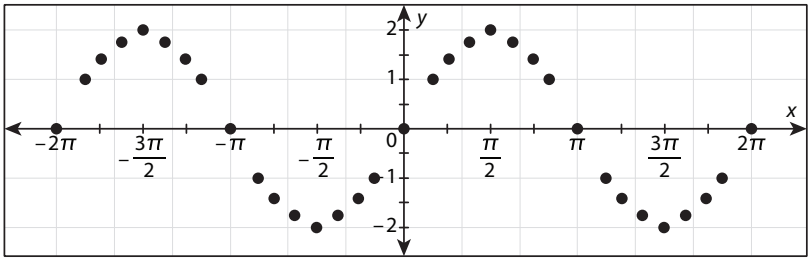
    Note that the function values change signs for every half period of domain values graphed. Therefore, the shape of the function graph over the domain $(-2\pi, 0)$ should be a reflection of the function graph over the domain $[0, 2\pi]$ across the midline $y = 0$ (the x-axis), followed by a reflection across the y-axis.

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12. Plot additional points on the graph to confirm your prediction. Then, draw a curve connecting the points across the domain $[-2\pi, 2\pi]$.

    At a minimum, points over the domain $(-\pi, 0)$ should be plotted for x-values that are the opposite of those used for the domain $(\pi, 0)$.



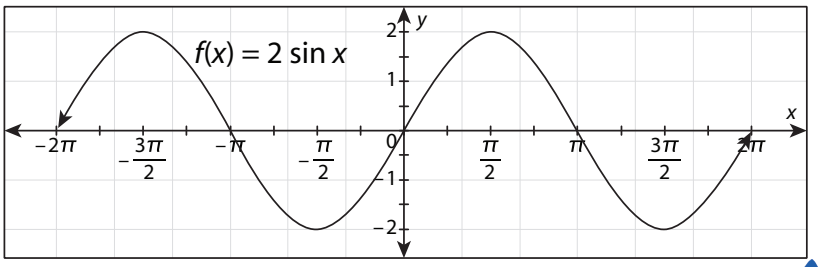
13. Verify the resulting graph using a graphing calculator.

    Enter the given function, $f(x) = 2 \sin x$, into your graphing calculator. Regardless of whether you are using the TI-83/84, the TI-Nspire, or a similar calculator, remember to adjust the viewing window values so that the x-axis endpoints include the function's domain and the y-axis endpoints include the amplitude. Use the known x- and y-values to determine a value for the x-axis scale that will provide a comprehensive view of the graphed function. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

    For this function:

    - Since the x-values are in increments of π, set the mode to radians.
    - The domain is $[-2\pi, 2\pi]$, so set the x-axis endpoints to -2π and 2π.
    - The amplitude is 2, so set the y-axis endpoints to at least -2 and 2.
    - For the x-axis scale, we know that the smallest interval between x-values is $\frac{\pi}{6}$; let this be the x-axis scale.

    The resulting graph on the calculator should confirm the accuracy of the sketched function graph.



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Example 2

How many complete cycles of the sine function $g(x) = \sin 3x$ are found in the restricted domain $[-180°, 180°]$?

1. Identify the period of $g(x)$.

   This will be needed to calculate the number of complete cycles of $g(x)$ that exist in the restricted domain.

   Set the argument of the function equal to the period of the parent sine function, $\sin x$, and solve for x: $3x = 2\pi$, so $x = \frac{2\pi}{3}$.

   The period of the function $g(x) = \sin 3x$ is $\frac{2\pi}{3}$.

2. Convert the period to degrees.

   Since the domain is given in degrees, the period must also be in degrees.

   1 radian $= \frac{180°}{\pi}$, so $\frac{2\pi}{3}$ radians is $\frac{2\pi}{3} \bullet \frac{180°}{\pi} = \frac{360\pi}{3\pi} = 120°$.

   The period of $g(x)$, converted to degrees, is 120°.

3. Determine the number of complete cycles $g(x)$ makes over the domain $[-180°, 180°]$.

   In order to determine the number of completed cycles, calculate the total number of degrees within the domain of the function, and then divide this number by the period.

   The domain covers $180° - (-180°)$ or 360°. Therefore, the number of complete cycles of $g(x)$ is 360° divided by the period, 120°: $360 \div 120 = 3$ complete cycles.

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4. Verify the result using a graphing calculator.

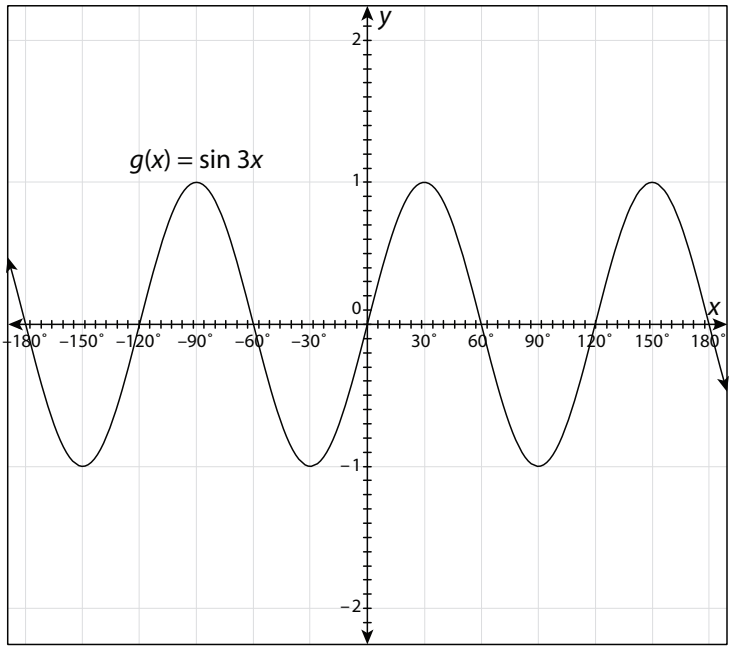
   Enter the given function, $g(x) = \sin 3x$, into your graphing calculator. Regardless of whether you are using the TI-83/84, the TI-Nspire, or a similar calculator, remember to adjust the viewing window values so that the x-axis endpoints include the function's domain and the y-axis endpoints include the amplitude. Use the known x- and y-values to determine a value for the x-axis scale that will provide a comprehensive view of the graphed function. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

   For this function:

   - Set the mode to degrees.
   - The domain is [–180°, 180°], so set the x-axis endpoints to –180 and 180.
   - The amplitude is 1, so set the y-axis endpoints to at least –1 and 1.
   - For the x-axis scale, choose a scale value that's a factor of the endpoints. Let the x-axis scale be 30 (a factor of –180 and 180).

   Graph the function and count the number of complete cycles.

   The resulting graph confirms our calculation that $g(x)$ has 3 complete cycles over the restricted domain [–180°, 180°].



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Instruction

Example 3

Determine the coordinates of the point(s) that represent a maximum positive function value of the function $h(x) = 3 \sin 2x$ over the restricted domain $\left[-\frac{3\pi}{4}, \frac{5\pi}{3}\right]$.

1. Identify the amplitude and period of the function.

   This information will be needed to determine the location and the magnitude of the maximum function value.

   The amplitude is 3 because 3 is the coefficient of the sine term. The period can be found by setting the argument of the function equal to 2π and solving for x: $2x = 2\pi$, so $x = \pi$.

2. Determine where the function will have maximum values over the domains of its period on either side of the origin.

   This will indicate the value(s) of x at which the function has a maximum.

   At the origin, $x = 0$, so the domains of the function's period on either side of the origin are $[-\pi, 0]$ and $[0, \pi]$.

   The parent sine function $f(x) = \sin x$ has a maximum at $x = \frac{\pi}{2}$ and at $x = -\frac{3\pi}{2}$, so $h(x) = 3 \sin 2x$ will have a maximum of 3 at $2x = \frac{\pi}{2}$ or at $x = \frac{\pi}{4}$. It will have another maximum of 3 at $2x = -\frac{3\pi}{2}$ or at $x = -\frac{3\pi}{4}$.

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Instruction

3. Determine how many maximums $h(x)$ has over the restricted domain $\left[-\frac{3\pi}{4}, \frac{5\pi}{3}\right]$.

The problem statement gives the restricted domain of $h(x) = 3 \sin 2x$ as $\left[-\frac{3\pi}{4}, \frac{5\pi}{3}\right]$.

One maximum value occurs at $x = -\frac{3\pi}{4}$, which is also the lower bound of the restricted domain. The other maximum occurs at $x = \frac{\pi}{4}$, which is less than the upper bound of the restricted domain at $x = \frac{5\pi}{3}$.

However, it might be useful to find the next maximum on the positive axis and compare it to the upper bound of the restricted domain. The next maximum for $h(x)$ occurs at $x = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$. This result is less than $x = \frac{5\pi}{3}$, so there are two maximums along the positive part of the x-axis that is in the restricted domain. The next maximum for $h(x)$ occurs at $\frac{\pi}{4} + 2\pi = \frac{9\pi}{4}$, which is greater than $x = \frac{5\pi}{3}$.

There are 3 maximum points over the restricted domain: $\left(-\frac{3\pi}{4}, 3\right)$, $\left(\frac{\pi}{4}, 3\right)$, and $\left(\frac{5\pi}{4}, 3\right)$.

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4. Check the result using a graphing calculator.

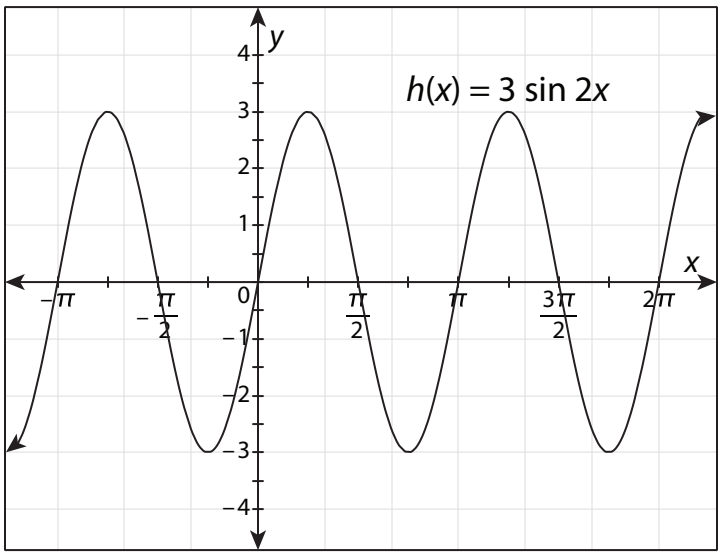
   Enter the given function, $h(x) = 3 \sin 2x$, into your graphing calculator. Adjust the viewing window values to include the function's domain and amplitude, with a suitable x-axis scale. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

   For this function:

   - Set the mode to radians.
   - The domain is $\left[-\frac{3\pi}{4}, \frac{5\pi}{3}\right]$, so set the x-axis endpoints to $-\frac{3\pi}{4}$ and $\frac{5\pi}{3}$.
   - The amplitude is 3, so set the y-axis endpoints to at least –3 and 3.
   - We know that the first function maximum occurs at $x = \frac{\pi}{4}$; let $\frac{\pi}{4}$ be the x-axis scale.

   Graph the function and count the number of maximums.

   The resulting graph confirms our calculation that $h(x) = 3 \sin 2x$ has three maximums over the restricted domain $\left[-\frac{3\pi}{4}, \frac{5\pi}{3}\right]$.



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Example 4

Sketch the graph of the function $a(x) = 1 + 2 \sin 3x$ over the restricted domain $[0, 2\pi]$.

1. Identify the amplitude of the function.

   The addition of the constant to the sine term will result in a vertical shift to the graph as well as to the relative locations of the maximum and minimum function values. The amplitude is 2, but the 1 added to the sine term will produce a maximum function value of 3 and a minimum function value of –1. The constant defines the midline, which is $y = 1$.

   Therefore, the maximum function value occurs at $y = 3$ and the minimum function value occurs at $y = -1$.

2. Determine the period of the function.

   Set the argument of the function equal to 2π, and then solve for x to find the period.

   $$3x = 2\pi$$

   $$x = \frac{2\pi}{3}$$

   The period is $\frac{2\pi}{3}$.

3. Determine how many cycles of the function can be shown over the domain $[0, 2\pi]$.

   Divide the domain by the period to find the number of cycles.

   $$2\pi \div \frac{2\pi}{3} = \cancel{2\pi} \cdot \frac{3}{\cancel{2\pi}} = 3$$

   There are 3 cycles over the domain $[0, 2\pi]$.

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Instruction

4. Determine the values of x at which the maximum and minimum occur over the domain of the cycle given by $\left[0, \frac{2\pi}{3}\right]$.

   The maximum function value of the parent sine function occurs at $x = \frac{\pi}{2}$, so set $3x = \frac{\pi}{2}$ and solve for x. The result is $x = \frac{\pi}{6}$.

   The minimum function value of the parent sine function occurs at $x = \frac{3\pi}{2}$, so set $3x = \frac{3\pi}{2}$ and solve for x. The result is $x = \frac{\pi}{2}$.

   For the domain of the cycle given by $\left[0, \frac{2\pi}{3}\right]$, the maximum occurs at $x = \frac{\pi}{6}$ and the minimum occurs at $x = \frac{\pi}{2}$.

5. Determine the coordinates of the points for the maximum and minimum values.

   The values found in the previous step represent the x-values of the coordinates.

   Maximum: $x = \frac{\pi}{6}$ Minimum: $x = \frac{\pi}{2}$

   The values found in step 1 represent the y-values of the coordinates.

   Maximum: $y = 3$ Minimum: $y = -1$

   Write these values as coordinates to determine the points of the maximum and minimum values.

   The maximum occurs at the point $\left(\frac{\pi}{6}, 3\right)$.

   The minimum occurs at the point $\left(\frac{\pi}{2}, -1\right)$.

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6. Use the period to determine the coordinates of the maximum and minimum points for the remaining two cycles in the restricted domain of the function.

Find the *x*-values of the coordinates by adding the period, $x=\frac{2\pi}{3}$, to the maximum and minimum function values found in step 4. The *y*-values will be the maximum and minimum function values found in step 1: 3 and –1, respectively.

The maximum function values occur at $x=\frac{\pi}{6}$ plus 1 period $\left(x=\frac{2\pi}{3}\right)$, and at $\frac{\pi}{6}$ plus 2 periods $\left(x=2\bullet\frac{2\pi}{3}=\frac{4\pi}{3}\right)$:

$\frac{\pi}{6}+\frac{2\pi}{3}$ or $x=\frac{5\pi}{6}$

$\frac{\pi}{6}+\frac{4\pi}{3}$ or $x=\frac{9\pi}{6}=\frac{3\pi}{2}$

The maximum *y*-value is 3, so the coordinates of these two maximum points are therefore $\left(\frac{5\pi}{6},3\right)$ and $\left(\frac{3\pi}{2},3\right)$.

Likewise, the minimum function values occur at $x=\frac{\pi}{2}$ plus 1 period, and at $\frac{\pi}{2}$ plus 2 periods:

$\frac{\pi}{2}+\frac{2\pi}{3}$ or $x=\frac{7\pi}{6}$

$\frac{\pi}{2}+\frac{4\pi}{3}$ or $x=\frac{11\pi}{6}$

The minimum *y*-value is –1; the coordinates of these two minimum points are therefore $\left(\frac{7\pi}{6},-1\right)$ and $\left(\frac{11\pi}{6},-1\right)$.

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7. Determine the y-intercept of the function.

   The y-intercept is also the equation of the midline of the function.

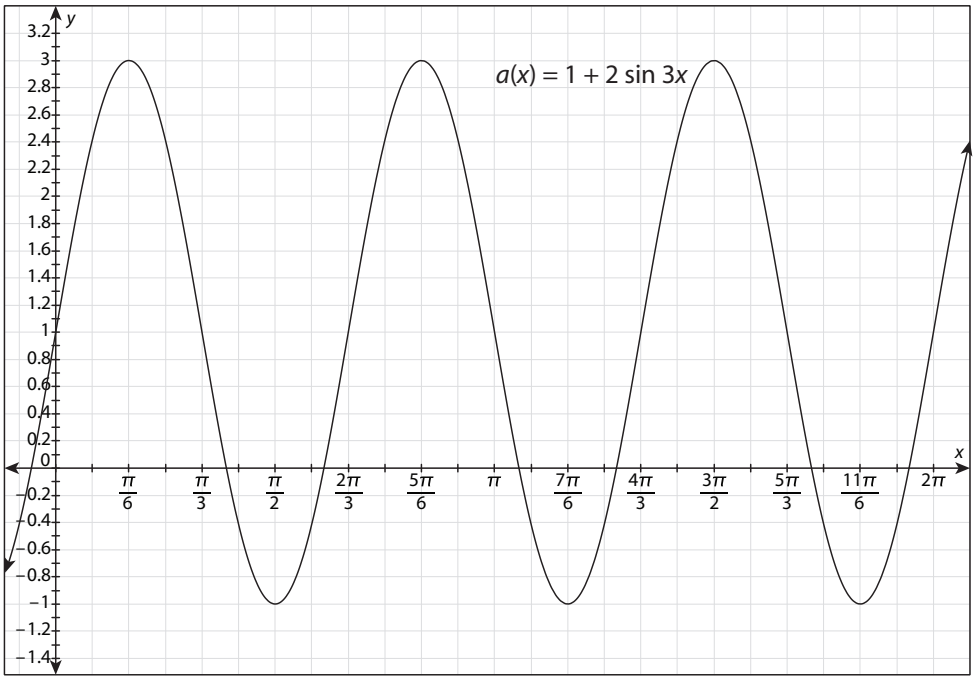
   Substitute $x = 0$ into the function, $a(x) = 1 + 2 \sin 3x$, and then solve.

   $a(0) = 1 + 2 \sin 3 \bullet (0) = 1 + 2 \bullet 0 = 1$

   The y-intercept of the function is at (0, 1).

8. Use the coordinates found in steps 5–7 to sketch the graph of the function.

   Sketch the graph of $a(x) = 1 + 2 \sin 3x$ by plotting the points on a coordinate plane and connecting them with a smooth curve, as shown.



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9. Confirm your sketch using a graphing calculator.

   Enter the given function, $a(x) = 1 + 2 \sin 3x$, into your graphing calculator. Adjust the viewing window values to include the function's domain and amplitude, with a suitable x-axis scale. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

   For this function:

   - Set the mode to radians.
   - The domain is $[0, 2\pi]$, so set the x-axis endpoints to 0 and 2π.
   - The amplitude of the function is 2 but the midline is $y = 1$, so add 1 to –2 and 2 to determine the y-axis endpoints: –1 and 3.
   - The first function maximum occurs at $x = \frac{\pi}{6}$; let $\frac{\pi}{6}$ be the x-axis scale.

   The resulting graph should confirm the accuracy of the sketch.

Example 5

Determine the coordinates of the points at which the first maximum and minimum function values occur for the function $c(x) = \sin\left(x + \frac{\pi}{3}\right)$ for values of $x > 0$.

1. Determine the period of the function.

   The variable part of the function's argument has no coefficients, so the function has a period of 2π. The presence of the constant $\frac{\pi}{3}$ shifts the graph horizontally, but does not affect the amplitude or period of the function.

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2. Determine the value of x at which the maximum function value occurs.

   The value of x at which the parent function reaches a maximum is $\frac{\pi}{2}$. However, the graph of this function is progressing rapidly toward reaching a maximum function value by the amount $\frac{\pi}{3}$, which is added to x. Therefore, this function will reach its maximum point at an x-value that is given by the difference $\frac{\pi}{2}-\frac{\pi}{3}$, or at $x=\frac{\pi}{6}$.

3. Determine the value of x at which the minimum function value occurs.

   The period of the function is the same as the parent function for sine, $f(x) = \sin x$, so the minimum occurs π units "past" the maximum at $x=\frac{\pi}{6}$. Add π to find the minimum:

   $$x=\frac{\pi}{6}+\pi=\frac{7\pi}{6}$$

4. Determine the maximum and minimum function values.

   The amplitude of the function is 1, so the maximum function value is 1 and the minimum function value is –1.

5. Write the coordinates of the points for the maximum and minimum function values.

   The point with the maximum function value for $x > 0$ is $\left(\frac{\pi}{6},1\right)$.

   The point with the minimum function value for $x > 0$ is $\left(\frac{7\pi}{6},-1\right)$.

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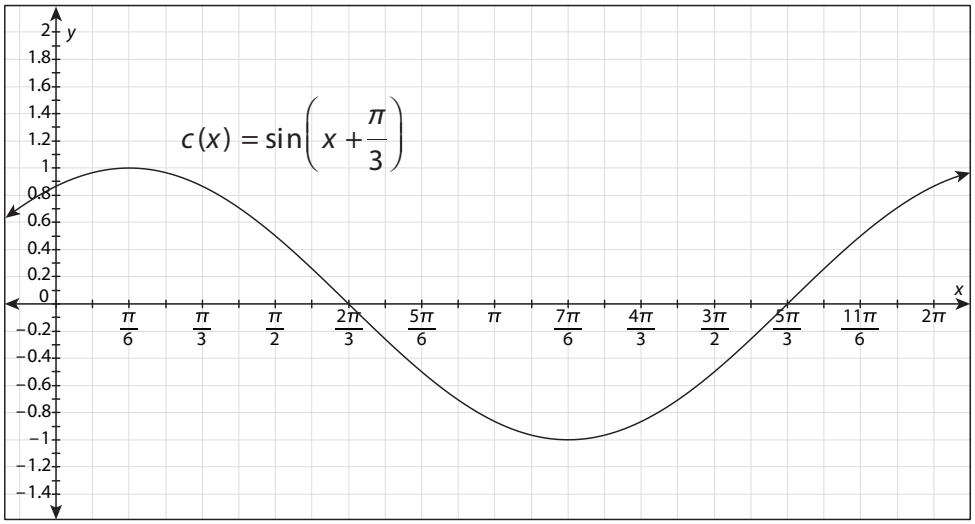
6. Verify the coordinates using a graphing calculator.

   Enter the given function, $c(x)=\sin\left(x+\frac{\pi}{3}\right)$, into your graphing calculator. Adjust the viewing window values to include the function's domain and amplitude, with a suitable *x*-axis scale. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

   For this function:

   - Set the mode to radians.
   - Set the *x*-axis endpoints to 0 and 2π.
   - The amplitude of the function is 1, so set the *y*-axis endpoints to at least –1 and 1.
   - The first function maximum occurs at $x=\frac{\pi}{6}$; let $\frac{\pi}{6}$ be the *x*-axis scale.

   Your result should resemble the following graph of $c(x)=\sin\left(x+\frac{\pi}{3}\right)$ for $x > 0$. Use your calculator's trace feature to confirm that the maximum is at $\left(\frac{\pi}{6},1\right)$ and the minimum is at $\left(\frac{7\pi}{6},-1\right)$.



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Problem-Based Task 4A.3.1: Going Through a Phase

SMP	
1	2 ✓
3	4
5	6
7 ✓	8 ✓

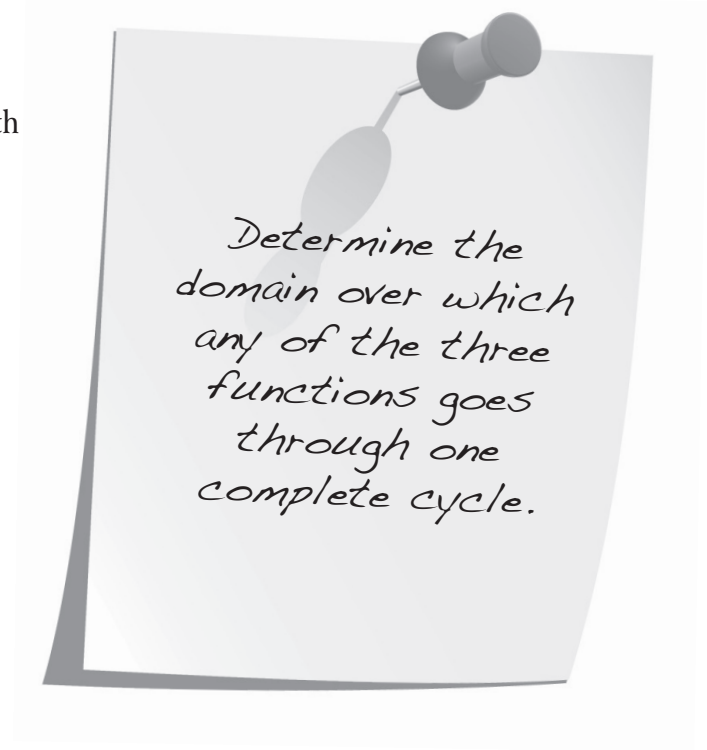
Electricity generated in the United States is often delivered by alternating current from generators. In most homes and small businesses, the voltage of this alternating current has a magnitude from 110 to 120 volts. However, such alternating current actually oscillates (or moves back and forth) rapidly each second between a maximum and minimum voltage value that is given by the function $V(t) = V_{\text{maximum}} \bullet \sin(2\pi ft)$, in which f is the frequency with which each cycle of the voltage V progresses from 0 to a maximum, down to a minimum, and back up to 0.

Since most household electrical current in the United States has a frequency, f, from 50 to 60 cycles per second, the function can be written as $V(t) = V_{\text{maximum}} \bullet \sin(100\pi t)$ or $V(t) = V_{\text{maximum}} \bullet \sin(120\pi t)$, depending on the frequency value.

The most common form of alternating current that comes from the generators is described as *three-phase* current, because the current produced by *each* of three coils in the generator is out of phase with the one that precedes it by $\frac{1}{3}$ of a complete rotation of the generator shaft.

Write functions for the three alternating-current voltages that make up the three-phase electrical energy that comes from such a generator with $f = 60$ cycles per second and $V_{\text{maximum}} = 170$ volts.

Then, sketch the graph of each function on a coordinate plane over the domain $\left[0, \frac{1}{10\pi}\right]$ with an x-axis interval of $\frac{1}{100\pi}$. Finally, determine the domain over which any of the three functions goes through one complete cycle.



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Problem-Based Task 4A.3.1: Going Through a Phase

Coaching

a. Rewrite the function $V(t) = V_{\text{maximum}} \bullet \sin(2\pi ft)$ to reflect the given maximum voltage and frequency of the generator.

b. Which part of this function will be affected by the fact that the other two generated voltages differ from the first—one by $\frac{1}{3}$ of a complete revolution of the generator shaft, the other by $\frac{2}{3}$ of a revolution?

c. What are the radian equivalents of $\frac{1}{3}$ and $\frac{2}{3}$ of a complete revolution of the generator shaft?

d. Will the results of part c be added to, subtracted from, or both added to *and* subtracted from the answer to part b to determine the simplest function? Explain your reasoning.

e. Write the functions for the other two alternating-current voltages created by the generator.

f. What characteristic of the sine function has to be calculated in order to find the domain over which one complete cycle—one maximum and one minimum—occurs for *any* of the three functions?

g. What is the formula for calculating the characteristic identified in part f? Calculate that value.

h. Why does the answer to part g apply to any of the three functions?

i. What are the values of the three functions at $t = 0$ and at the value of t at which one complete cycle of the first function ends?

j. What is the amplitude of any of the three functions?

k. Graph the three functions by hand or with a graphing calculator.

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Instruction

Problem-Based Task 4A.3.1: Going Through a Phase

Coaching Sample Responses

a. Rewrite the function $V(t) = V_{\text{maximum}} \bullet \sin(2\pi ft)$ to reflect the given maximum voltage and frequency of the generator.

Substitute 170 for V_{maximum} and 60 for f, and then simplify.

$V(t) = V_{\text{maximum}} \bullet \sin(2\pi ft)$

$V(t) = (170) \bullet \sin[2\pi(60)t]$

$V(t) = 170 \bullet \sin(120\pi t)$

b. Which part of this function will be affected by the fact that the other two generated voltages differ from the first—one by $\frac{1}{3}$ of a complete revolution of the generator shaft, the other by $\frac{2}{3}$ of a revolution?

Parts of a revolution imply parts of a complete revolution of a circle, which is measured in degrees as 360° or in radians as 2π radians. The argument of the sine function is affected by such phase shifts.

Therefore, the portion of the function $V(t) = 170 \bullet \sin(120\pi t)$ that will be affected by the differences in voltage is the argument, $(120\pi t)$.

c. What are the radian equivalents of $\frac{1}{3}$ and $\frac{2}{3}$ of a complete revolution of the generator shaft?

$\frac{1}{3}$ of a revolution is $\frac{1}{3}$ of 2π radians, or $\frac{2\pi}{3}$ radians; thus, $\frac{2}{3}$ of a revolution is $\frac{4\pi}{3}$ radians.

d. Will the results of part c be added to, subtracted from, or both added to *and* subtracted from the answer to part b to determine the simplest function? Explain your reasoning.

The function determined in part a, $V(t) = 170 \bullet \sin(120\pi t)$, could represent any one of the three voltages. In other words, any one of the three functions for the three voltages can be chosen as the simplest function, which means that the second and third functions can lag behind the first function by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, or they can be ahead of the first function by the same amounts.

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Or, the simplest function could fall between the other two—lagged by one function and ahead of the other function. Therefore, the arguments of the three sine functions could be written as follows:

- Case I (subtraction from the argument): $120\pi t - \frac{4\pi}{3}$, $120\pi t - \frac{2\pi}{3}$, and $120\pi t$
- Case II (addition): $120\pi t$, $120\pi t + \frac{2\pi}{3}$, and $120\pi t + \frac{4\pi}{3}$
- Case III (subtraction and addition): $120\pi t - \frac{2\pi}{3}$, $120\pi t$, and $120\pi t + \frac{2\pi}{3}$

e. Write the functions for the other two alternating-current voltages created by the generator.

Use the simplest function $V(t) = 170 \bullet \sin(120\pi t)$ and the results of part d to write the functions for the other two voltages created by the generator.

For Case I, subtraction from the argument, the three functions are as follows:

$$V_1(t) = 170 \bullet \sin(120\pi t)$$

$$V_2(t) = 170 \bullet \sin\left(120\pi t - \frac{2\pi}{3}\right)$$

$$V_3(t) = 170 \bullet \sin\left(120\pi t - \frac{4\pi}{3}\right)$$

Note: A case can be made that it is best to use sine function arguments that *add* the phase factors to the sine term. This will result in values for the function argument that are greater than 0, and thus more compatible with real-world phenomena. However, any of the three cases can be used as long as the real-world domain of $[0, +\infty)$ is kept in mind.

f. What characteristic of the sine function has to be calculated in order to find the domain over which one complete cycle—one maximum and one minimum—occurs for *any* of the three functions?

The period must be calculated, since it is the restricted domain over which one complete cycle of the sine curve repeats.

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Instruction

g. What is the formula for calculating the characteristic identified in part f? Calculate that value.

To calculate the period, set the part of the argument containing the variable t equal to 2π and solve for t:

$120\pi t = 2\pi$

$t = \frac{2\pi}{120\pi} = \frac{1}{60}$, or $\frac{1}{60}$ of a second

h. Why does the answer to part g apply to any of the three functions?

The part of the argument containing t is the same for each function found in part e. The added constants shift the functions along the t-axis, but do not change their periods, which are the same.

i. What are the values of the three functions at $t = 0$ and for the value of t at which one complete cycle of the first function ends?

Substitute 0 for t in the three functions identified in part e.

For Case I, at $t = 0$, the function values are as follows:

$V_1(0) = 170 \bullet \sin[120\pi(0)] = 0$

$V_2(0) = 170 \bullet \sin\left(120\pi(0) - \frac{2\pi}{3}\right) \approx -147$

$V_3(0) = 170 \bullet \sin\left(120\pi(0) - \frac{4\pi}{3}\right) \approx 147$

At the end of one cycle, when $t = \frac{1}{60}$, the function values are as follows:

$V_1\left(\frac{1}{60}\right) = 170 \bullet \sin\left[120\pi\left(\frac{1}{60}\right)\right] = 0$

$V_2\left(\frac{1}{60}\right) = 170 \bullet \sin\left[120\pi\left(\frac{1}{60}\right) - \frac{2\pi}{3}\right] \approx -147$

$V_3\left(\frac{1}{60}\right) = 170 \bullet \sin\left[120\pi\left(\frac{1}{60}\right) - \frac{4\pi}{3}\right] \approx 147$

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j. What is the amplitude of any of the three functions?

The amplitude can be read directly from the first function, $V_1(t) = 170 \bullet \sin(120\pi t)$. The amplitude is the coefficient of the sine term, 170.

k. Graph the three functions by hand or with a graphing calculator.

Use the results of part i to sketch the beginning and end of one period of each of the three functions by hand, along with the function value at half of the period. This value of t for $V_1(t)$ is $\frac{1}{2} \bullet \frac{1}{60} = \frac{1}{120}$, so the zero for $V_1(t)$ has the coordinates $\left(\frac{1}{120}, 0\right)$. For $V_2(t)$ and $V_3(t)$, the points would be offset due to the phase shift.

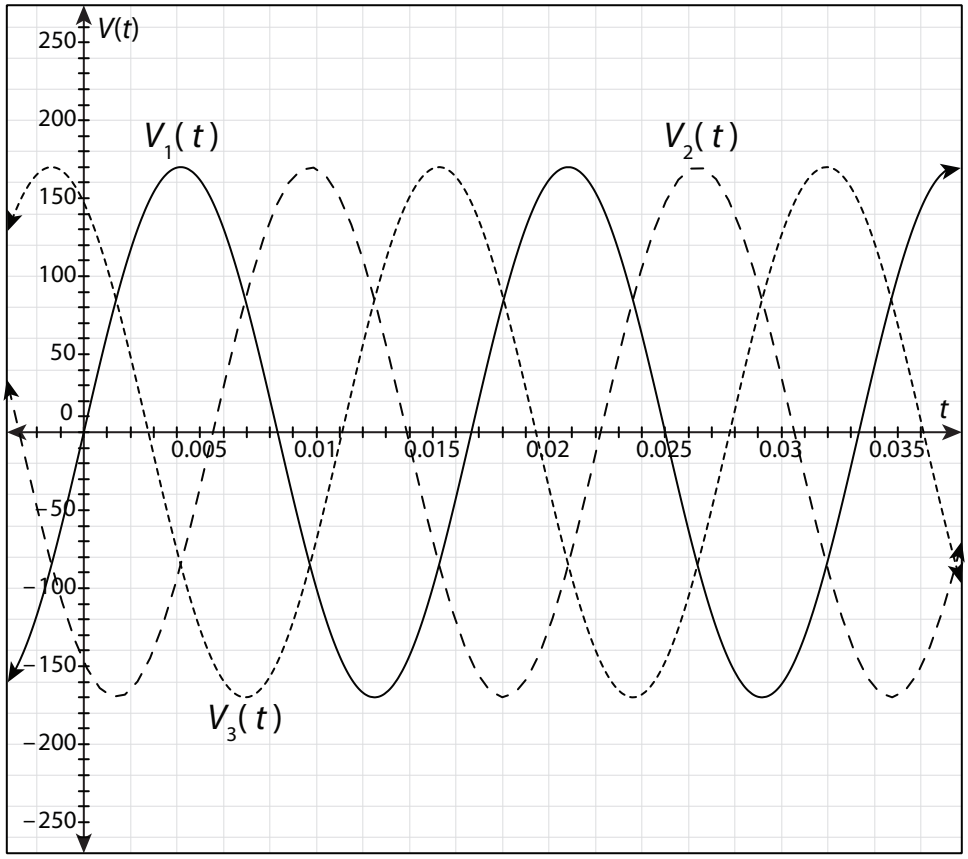
For $V_2(t)$, the zero occurs at $t = 120\pi\left(\frac{1}{120}\right) - \frac{2\pi}{3} = \frac{\pi}{3}$, so the zero has the coordinates $\left(\frac{\pi}{3}, 0\right)$.

For $V_3(t)$, the zero occurs at $t = 120\pi\left(\frac{1}{120}\right) - \frac{4\pi}{3} = -\frac{\pi}{3}$, which cannot be shown on this graph due to real-world constraints. (See the note at the end of part e about real-world compatible values of the sine function argument.)

The maximum and minimum function values of the three functions (170 and –170, respectively) occur at values of t that are $\frac{1}{2}$ and $\frac{3}{4}$ of the period. For $V_1(t)$, the first maximum function value occurs at a value of t of $\frac{1}{4} \bullet \frac{1}{60}$, which gives the coordinates $\left(\frac{1}{240}, 170\right)$. The first minimum value for $V_1(t)$ occurs at a value of t of $\frac{3}{4} \bullet \frac{1}{60}$, which gives the coordinates $\left(\frac{1}{80}, -170\right)$. The first maximum and minimum function values for $V_2(t)$ and $V_3(t)$ can be computed in the same way that the zeros were computed earlier.

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Graphing the functions using a graphing calculator provides the following result:



Recommended Closure Activity

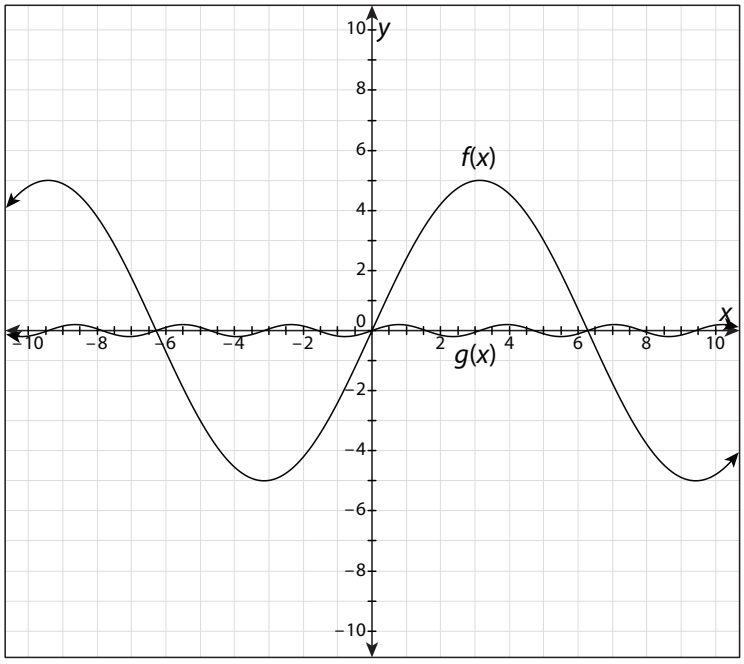
Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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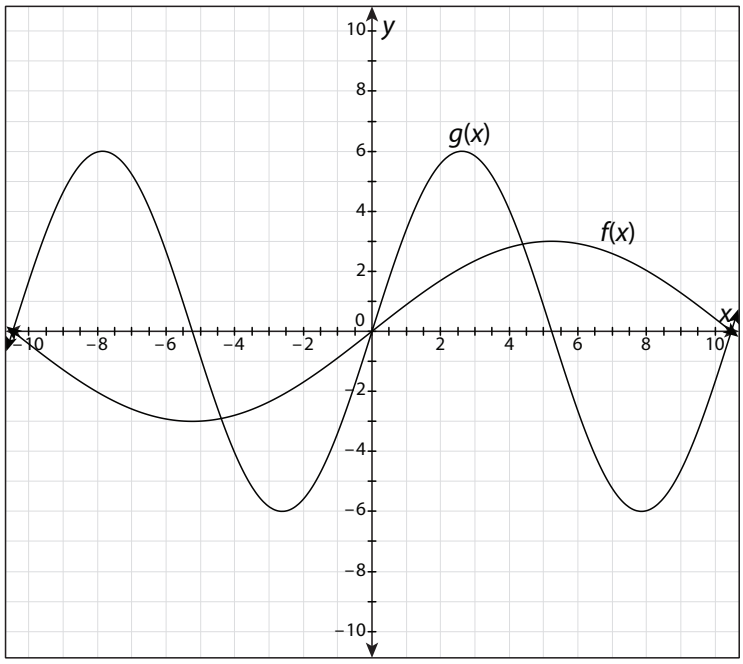
Practice 4A.3.1: Graphing the Sine Function

For problems 1–4, refer to the provided graph to complete each problem. (*Note*: Some graphs show only part of a complete cycle. The *x*-axis of each graph is expressed in radians.)

1. Which function has the greater amplitude, $f(x)$ or $g(x)$?



2. Which function has the greater period, $f(x)$ or $g(x)$?

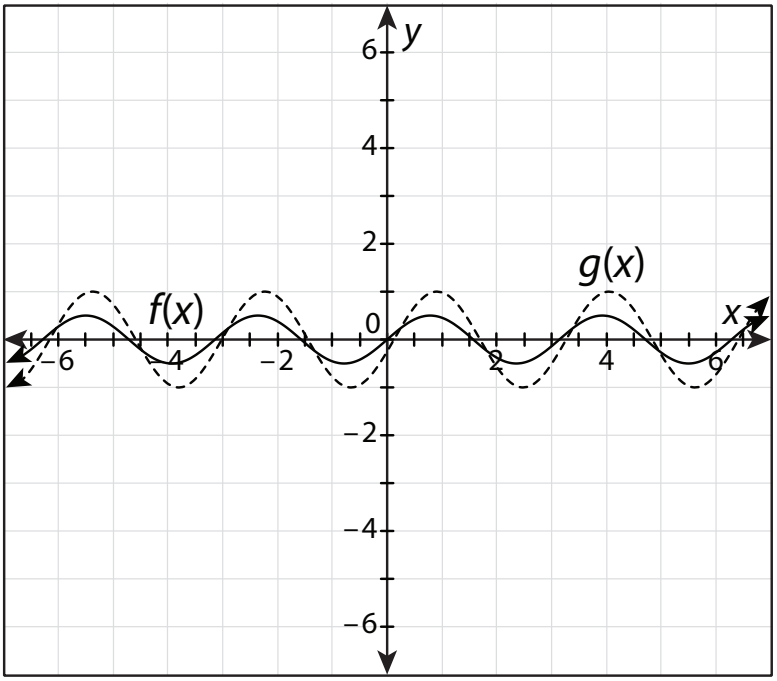


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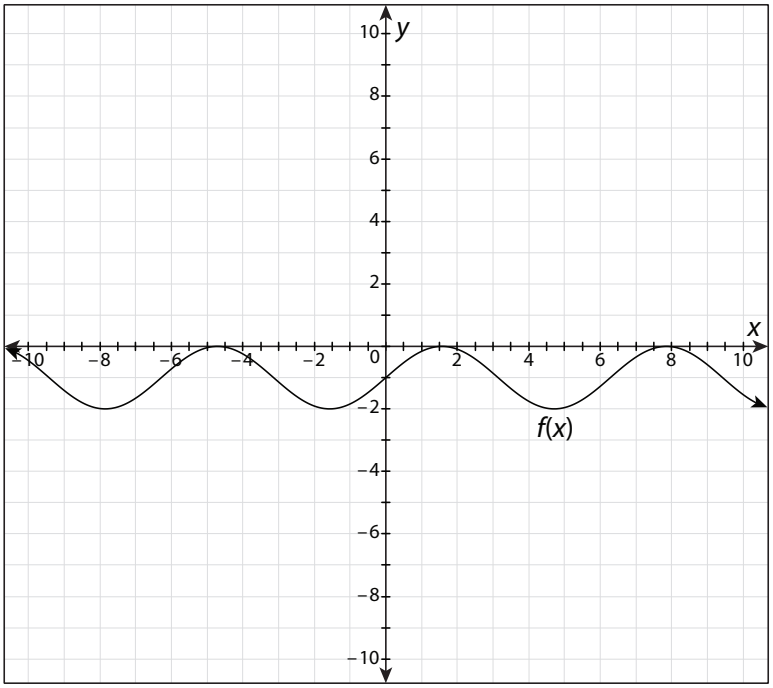
SWB p. 87; SRB Practice 4A.3.1 p. U4A-143

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3. Determine the amount by which the functions $f(x)$ and $g(x)$ are out of phase.



4. Write the simplest form of the sine function shown.



continued

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For problems 5–7, use the given information to find the requested values and coordinates.

5. At what value of $x > 0$ will the first maximum occur for the function $f(x) = 4 \sin 3x$? Determine the coordinates of the point for this value of x.

6. At what value of $x > 0$ will the first zero occur for the function $g(x) = 2 \sin 0.5x$? Determine the coordinates of the point for this value of x.

7. At what value of $x > 0$ will the first minimum occur for the function $h(x) = \sin (0.4x + 45°)$? Determine the coordinates of the point for this value of x.

Use your knowledge of sine functions to complete problems 8–10.

8. The frequency of a sound wave is 750 cycles per second. If the sound intensity can be modeled by the sine function $S(t) = 0.05 \sin 750t$, what is the period of the sound wave?

9. The voltage in an alternating current circuit can be modeled by the function $V(t) = 175 \sin (110\pi t)$. How often does the voltage reach a peak positive *or* negative value in 1 second?

10. The angle of a beam of light relative to a vertical line that passes through one material into a second material is related to the angle of the beam of light in the second material by the relationship $n_a \sin a = n_b \sin b$. The constants n_a and n_b differ for various materials. How are $\sin a$ and $\sin b$ related over the domain $\left(0, \frac{\pi}{2}\right)$ for the following situations?

    a. $n_a > n_b$

    b. $n_a = n_b$

    c. $n_a = 2n_b$, and angles a and b are complementary

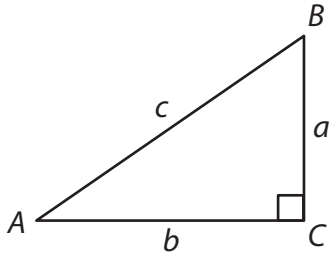
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Lesson 4A.3.2: Graphing the Cosine Function

Warm-Up 4A.3.2

Sofia is going to great lengths to convince the city council to build new ramps at the skate park. She knows her presentation is strong, but she wants to strengthen it with more mathematical justification for the new ramp lengths in order to prove her point to the council. To add to her presentation, Sofia wants to include a table and diagram for cosine values of the existing ramp angles to support her longer suggested ramp lengths. The diagram is finished, but the table is not.

In Sofia's diagram, the cosine ratio for angle A in right triangle ABC is defined as the ratio of the length of side b to the length of the hypotenuse, c.



Mathematically, the cosine ratio of angle A can be written as $\cos A = \frac{b}{c}$. Similarly, the cosine ratio of angle B can be written as $\cos B = \frac{a}{c}$. Use the diagram and the following questions to complete the table for Sofia's presentation.

$m\angle A$	$m\angle B$	$m\angle C$	a	b	c	$\cos A$	$\cos B$
30°			1				
45°			1				
60°			1				

1. What is the relationship between the sum of $m\angle A$ and $m\angle B$ in each row? Use this information to fill in the missing angle measures for angles B and C.

2. What is the relationship between the lengths of the three sides of right triangle ABC? Use this information to fill in the missing lengths for b and c. Simplify the resulting values for the rows for 30°, 45°, and 60° as much as possible without using a calculator.

3. Use the definition of the cosine ratio to compute the values of $\cos A$ and $\cos B$ in the table. What do you notice about the cosine ratios for complementary angles? (*Hint*: Complementary angles have measures that add up to the measure of a right angle.)

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Instruction

Lesson 4A.3.2: Graphing the Cosine Function

Common Core State Standard

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Warm-Up 4A.3.2 Debrief

1. What is the relationship between the sum of $m\angle A$ and $m\angle B$ in each row? Use this information to fill in the missing angle measures for angles *B* and *C*.

   In each row, $m\angle A + m\angle B = 90$; that is, the angles opposite the legs of the right triangle sum to 90°.

Let $m\angle A = 30$.	Let $m\angle A = 45$.	Let $m\angle A = 60$.
$m\angle A + m\angle B = 90$	$m\angle A + m\angle B = 90$	$m\angle A + m\angle B = 90$
$(30) + m\angle B = 90$	$(45) + m\angle B = 90$	$(60) + m\angle B = 90$
$m\angle B = 60$	$m\angle B = 45$	$m\angle B = 30$

   As shown in the diagram, $\angle C$ is a right angle and measures 90° in each case.

   Write these measures for angles *B* and *C* in the appropriate table cells. (Refer to the completed table in the response to problem 3.)

2. What is the relationship between the lengths of the three sides of right triangle *ABC*? Use this information to fill in the missing measures for *b* and *c*. Simplify the resulting values for the rows for 30°, 45°, and 60° as much as possible without using a calculator.

   Sides *a*, *b*, and *c* are related to one another by the Pythagorean Theorem, $a^2 + b^2 = c^2$.

   The table includes rows for three different right triangles: 30°–60°–90°, 45°–45°–90°, and 60°–30°–90°.

   In a 30°–60°–90° triangle, the side opposite the 30° angle is half the length of the hypotenuse. As stated in the table, $a = 1$, which gives the relationship $1^2 + b^2 = 2^2$. Solving for *b* results in $b = \sqrt{3}$.

   In a 45°–45°–90° triangle, sides *a* and *b* are equal in length, so the relationship among the sides is given by $1^2 + 1^2 = c^2$. Solving for *c* results in $c = \sqrt{2}$.

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Instruction

A 60°–30°–90° triangle is actually the same triangle as the 30°–60°–90° triangle. However, in this example, $a = 1$, and we do not know the values of b and c, but we do know that b is half the length of c. Therefore, we can substitute x for the unknown length of b and $2x$ for c to determine that the relationship among the three sides is $1^2 + x^2 = (2x)^2$. Solve this equation for x to determine the missing lengths of b and c.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$(1)^2 + (x)^2 = (2x)^2$	Substitute 1 for a, x for b, and $2x$ for c.
$1 + x^2 = 4x^2$	Apply the exponent.
$1 = 3x^2$	Subtract x^2 from both sides.
$\frac{1}{3} = x^2$	Divide by 3.
$\sqrt{\frac{1}{3}} = x$	Take the square root.
$\frac{1}{\sqrt{3}} = x$	Simplify the numerator.
$\frac{1}{\sqrt{3}} \bullet \frac{\sqrt{3}}{\sqrt{3}} = x$	Rationalize the denominator.
$\frac{1\sqrt{3}}{\sqrt{9}} = x$	Multiply.
$\frac{\sqrt{3}}{3} = x$	Simplify.

Since $x = b$, this means that the length of b is $\frac{\sqrt{3}}{3}$ and $c = 2 \bullet b = \frac{2\sqrt{3}}{3}$.

Write these measures for sides b and c in the appropriate table cells. (Refer to the completed table in the response to problem 3.)

3. Use the definition of the cosine ratio to compute the values of cos A and cos B in the table. What do you notice about the cosine ratios for complementary angles? (*Hint*: Complementary angles have measures that add up to the measure of a right angle.)

   As given in the problem, the cosine ratios are defined as $\cos A = \frac{b}{c}$ and $\cos B = \frac{a}{c}$.

   - For the 30°–60°–90° triangle:

     $\cos 30° = \frac{b}{c} = \frac{\sqrt{3}}{2}$ $\quad$ $\cos 60° = \frac{a}{c} = \frac{1}{2}$

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- For the 45°–45°–90° triangle:

$$\cos 45° = \frac{b}{c} = \frac{a}{c} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

This applies to both angles A and B since each has a measure of 45°.

- For the 60°–30°–90° triangle:

$$\cos 60° = \frac{b}{c} \qquad \cos 30° = \frac{a}{c}$$

$$= \frac{\frac{\sqrt{3}}{3}}{\frac{2\sqrt{3}}{3}} = \frac{\sqrt{3}}{3} \div \frac{2\sqrt{3}}{3} = \frac{\sqrt{3}}{3} \bullet \frac{3}{2\sqrt{3}} \qquad = \frac{1}{\frac{2\sqrt{3}}{3}} = 1 \div \frac{2\sqrt{3}}{3} = 1 \bullet \frac{3}{2\sqrt{3}}$$

$$= \left(\frac{3\sqrt{3}}{6\sqrt{3}}\right) = \frac{1}{2} \qquad = \frac{3}{2\sqrt{3}} \bullet \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$$

In the 30°–60°–90° and 60°–30°–90° right triangles, the cosine ratios for the 30° angles are the same as the cosine ratios for the 60° angles even though the lengths of the corresponding sides differ in the two triangles.

Write these measures for cos A and cos B in the appropriate table cells.

Completed table:

$m\angle A$	$m\angle B$	$m\angle C$	a	b	c	cos A	cos B
30°	60°	90°	1	$\sqrt{3}$	2	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45°	45°	90°	1	1	$\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	30°	90°	1	$\frac{\sqrt{3}}{3}$	$\frac{2\sqrt{3}}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

Connection to the Lesson

- Students will identify the relationships between the cosine values of complementary angles.
- Students will discover that the values of the cosine term (cos x) in the functions $f(x) = \cos x$ and $g(x) = a \cos x$ are equal, as shown by the different right-triangle calculations in the Warm-Up.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- converting from degrees to radians and vice versa (F–TF.1)
- finding the missing measurements of a right triangle using the Pythagorean Theorem (G–SRT.8)
- using radical operations to add and subtract radicals (N–RN.2)
- using radical operations to simplify radical expressions, including expressions that involve radicals raised to the second power (N–RN.2)
- identifying the cosine function values in an application involving triangles or from the ordered pairs of points on a unit circle graph on a coordinate plane (F–TF.2)

Introduction

Recall that **cosine** is a trigonometric function of an acute angle in a right triangle. It is the ratio of the length of the side adjacent to the length of the hypotenuse; the cosine of $\theta = \cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$. Graphing cosine functions is similar to graphing sine functions. In this lesson, you will explore the graphs of cosine functions.

Key Concepts

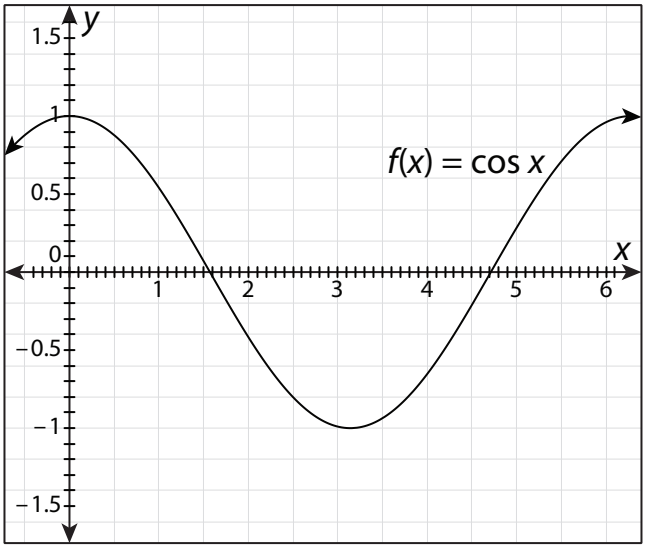
- A **cosine function** is a trigonometric function of the form $f(x) = a\cos bx$, in which a and b are constants and x is a variable defined in radians over the domain $(-\infty, \infty)$.

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Instruction

- The following graph shows the parent function $f(x) = \cos x$.

Cosine Parent Function, $f(x) = \cos x$



- The general form of a cosine function is $f(x) = a \cos bx$. The extended form is $g(x) = a \cos [b(x - c)] + d$.
- The parts of the equation of a cosine function and the relationships between those parts are nearly identical as those for a sine function:

Comparison of Sine and Cosine Functions

Form	Equation	Amplitude	Period	Argument
Parent	$f(x) = \sin x$	1	2π	x
	$g(x) = \cos x$	1	2π	x
General	$f(x) = a \sin bx$	$\lvert a \rvert$	$\frac{2\pi}{b}$	bx
	$g(x) = a \cos bx$	$\lvert a \rvert$	$\frac{2\pi}{b}$	bx
Extended	$f(x) = a \sin [b(x - c)] + d$	$\lvert a \rvert$	$\frac{2\pi}{b}$	$[b(x - c)]$
	$g(x) = a \cos [b(x - c)] + d$	$\lvert a \rvert$	$\frac{2\pi}{b}$	$[b(x - c)]$

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Instruction

- Just as with sines, the domain of a cosine function is generally expressed in radians, but can be converted to degrees to suit a given situation: 2π radians = 360°.
- Furthermore, the methods for determining the phase shift and period of a cosine function are also the same:
  - Phase shift: Set the argument equal to 0 and solve for x.
  - Period: Set the argument equal to 2π and solve for x.

Graphing Cosine Functions

- As with sine functions, in order to sketch the graph of a cosine function, it is necessary to know the amplitude and period of the parent function, $f(x) = \cos x$.
- Recall that the amplitude of a function is defined as the vertical distance from the midline to the highest point of the graph, and the period is the horizontal distance from the beginning of one complete cycle of the graph to the point at which its behavior or shape repeats.
- As with sine functions, any change in the amplitude in the equation of a cosine function has a *direct* effect on the amplitude of the graph, and any change in the period of the equation has an *inverse* effect on the period of the graph:
  - In the general form $g(x) = a \cos bx$, if the amplitude $|a|$ increases, the amplitude of the graph increases. If the value of b increases, the period of the graph decreases.
  - In the extended form $h(x) = a \cos [b(x - c)] + d$, if the amplitude $|a|$ increases, the value of the term $a \cos [b(x - c)]$ also increases if the other quantities stay the same. If the value of b increases, the period of the graph decreases.
  - Furthermore, in the expanded form, $h(x) = a \cos [b(x - c)] + d$, the constant d shifts the graph vertically, whereas the constant c shifts the graph horizontally.
- Recall that on a graph of a cosine or sine function of the form $f(x) = \sin x + d$ or $g(x) = \cos x + d$ the midline is a horizontal line of the form $y = d$ that bisects the vertical distance between the minimum and maximum function values.

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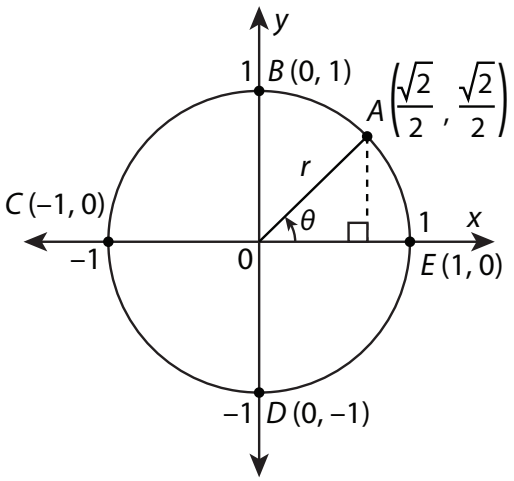
- The values of the cosine function at several critical values of x can be useful in sketching the graph of the cosine function. A few of these are listed here:

$$\cos 30° = \cos\left(\frac{\pi}{6}\text{ radian}\right) = \frac{\sqrt{3}}{2} \approx 0.87$$

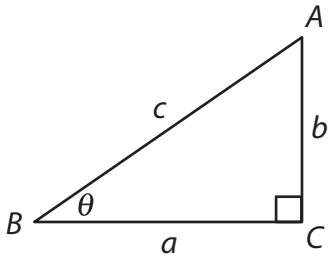
$$\cos 45° = \cos\left(\frac{\pi}{4}\text{ radian}\right) = \frac{\sqrt{2}}{2} \approx 0.71$$

$$\cos 60° = \cos\left(\frac{\pi}{3}\text{ radians}\right) = \frac{1}{2} = 0.5$$

- Recall that the unit circle can be used to visualize how cosine values are derived.



- Notice the right triangle formed by the radius:



- This triangle can be used to determine the formula for the cosine of angle B, which is defined as the ratio of the length of side a (the side that is adjacent to angle B) to the length of the hypotenuse, side c:

$$\cos B = \frac{a}{c} \text{ or } a = c \cos B$$

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- Recall that on the unit circle diagram, the horizontal leg of the right triangle is a and the vertical leg is b. Therefore, the coordinates of a point on the unit circle are (a, b) or $(c \cos \theta, c \sin \theta)$. For a unit circle, $c = 1$, so the coordinates of the point are $(\cos \theta, \sin \theta)$.
- For an angle measuring 30° $\left(\frac{\pi}{6} \text{ radian}\right)$ in the first quadrant of the unit circle, the coordinates of the point that lies on the unit circle are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, which is the same as $(\cos 30°, \sin 30°)$.
- Refer to the diagram of coordinates for values of $m\angle\theta$ in the first sub-lesson for the coordinates of the points formed by angles of 30°, 45°, and 60° in the four quadrants of the coordinate plane.

Common Errors/Misconceptions

- confusing degrees and radians in the conversion formula 2π radians = 360°
- forgetting that the domain values of cosine function graphs are expressed in radians unless otherwise indicated
- reversing the amplitude and period in the general cosine function $f(x) = a \cos bx$
- plotting too few points when graphing the cosine function
- using the incorrect sign for the cosine functions

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Guided Practice 4A.3.2

Example 1

Sketch the graph of $f(x) = 3 \cos x$ over the restricted domain $[-2\pi, 2\pi]$.

1. Identify the amplitude and period of the function.

   This will determine the x- and y-axis scales.

   The coefficient of the cosine term is 3, which is the amplitude. Therefore, the y-axis scale should run from at least $y = -3$ to $y = 3$. The coefficient of the cosine argument is 1, so $x = 2\pi$, which means the period is 2π. If the graph shows the restricted domain of $[-2\pi, 2\pi]$, it will include two complete cycles of the function.

2. Identify any other coefficients or terms that would affect the shape of the graph.

   This is necessary to determine if the graph is translated horizontally or vertically, or if other mathematical terms affect the graph's shape.

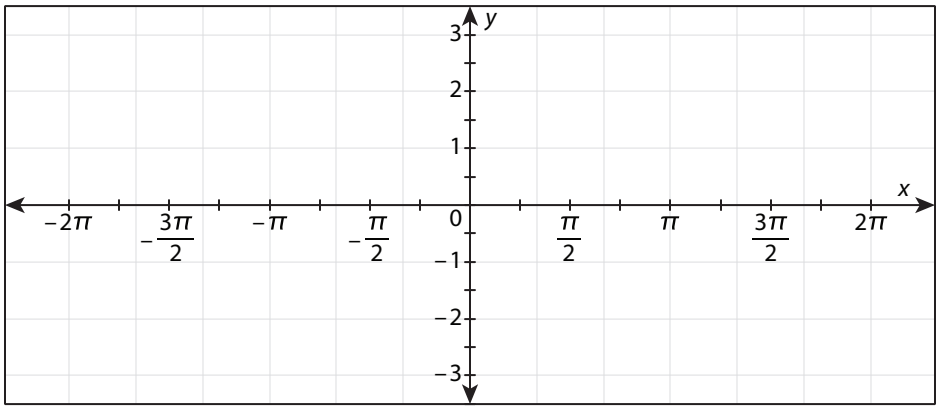
   The equation of the function, $f(x) = 3 \cos x$, has no other coefficients or terms that affect the shape or placement of the function's graph.

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3. Draw and label the axes for your graph based on steps 1 and 2.

   The *x*-axis scale will range from -2π to 2π, and the *y*-axis scale will range from at least –3 to at least 3. Each axis should be divided into sufficient intervals to allow for enough points to be plotted to show the graph. Label the *x*-axis in increments of $\pm\frac{\pi}{2}$ and the *y*-axis in increments of $\frac{1}{2}$.



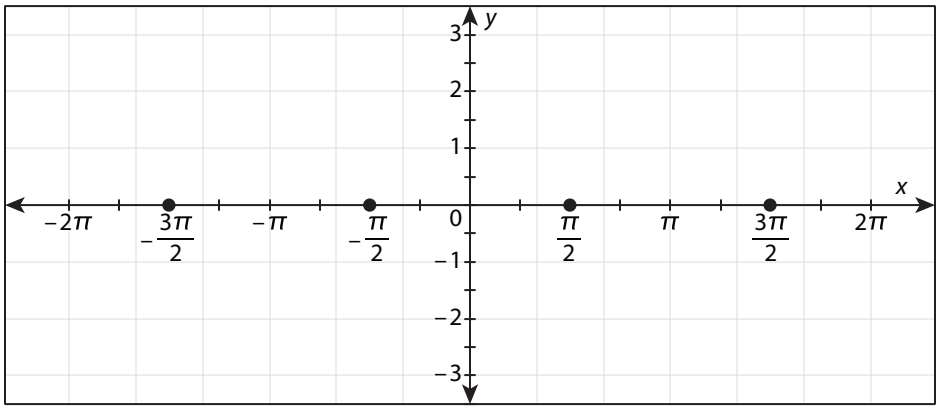
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4. Determine the values of the restricted domain for which the function value(s) equal 0. List the points corresponding to these zeros, and plot them on the graph.

   This will establish some points on the x-axis.

   Since the period of the function $f(x) = 3\cos x$ is 2π, the function is equal to 0 at $x = \pm\frac{\pi}{2}$ and $\pm\frac{3\pi}{2}$. The corresponding points are $\left(-\frac{3\pi}{2}, 0\right)$, $\left(-\frac{\pi}{2}, 0\right)$, $\left(\frac{\pi}{2}, 0\right)$, and $\left(\frac{3\pi}{2}, 0\right)$. Plot the four points as shown.



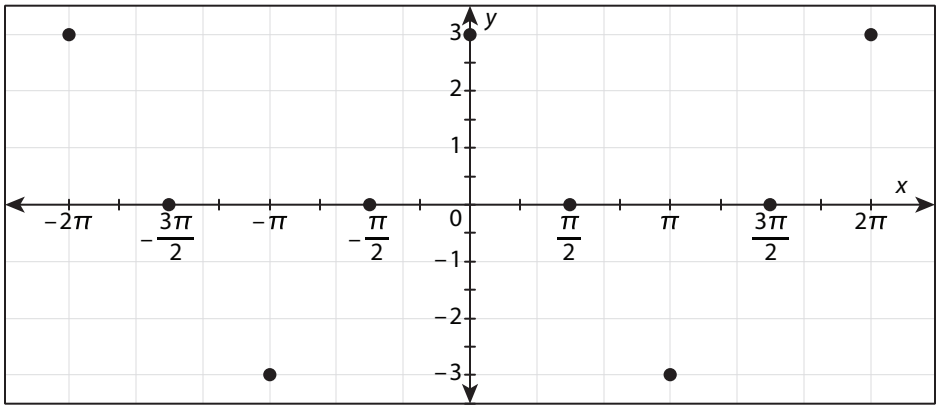
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5. Determine what values of the restricted domain are the maximum and minimum of the function value(s). List the points corresponding to these extremes, and plot them on the graph.

   The cosine function values vary between the two extremes defined by the amplitude. In this case, the amplitude is 3, so the function values for the maximum and minimum values of $f(x) = 3 \cos x$ will vary between –3 and 3. These values occur at $x = 0$, $x = \pm\pi$, and $x = \pm 2\pi$.

   The points for the maximum and minimum values of $f(x) = 3 \cos x$ are $(-2\pi, 3)$, $(-\pi, -3)$, $(0, 3)$, $(\pi, -3)$, and $(2\pi, 3)$. Add these points to the graph.



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6. Plot points for the cosines of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ radians.

Use a calculator to determine the cosines of the given values of $m\angle\theta$.

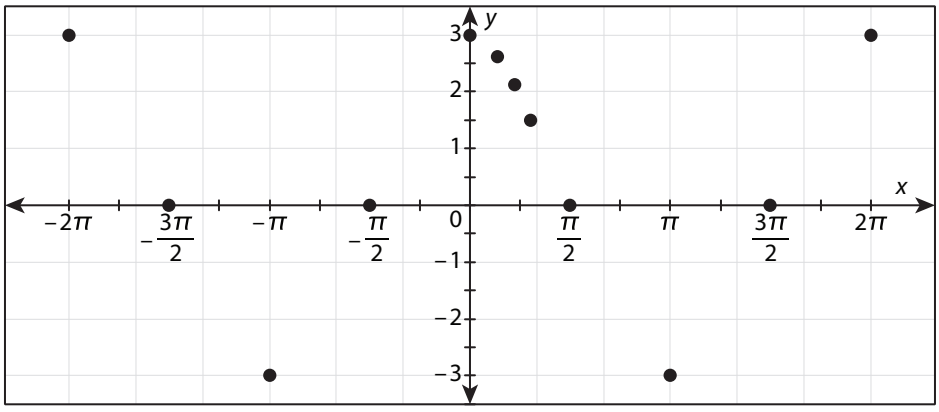
These points will fall between the function maximum at (0, 3) and the zero at $\left(\frac{\pi}{2}, 0\right)$. Moreover, they will suggest the shape of a quarter-period "piece" of the cosine curve, which can be reflected across the y-axis from the zero at $\left(-\frac{\pi}{2}, 0\right)$ to the x-axis to produce the cosine curve over the domain of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Using your calculator, we see that $3\cos\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2} \approx 2.6$ and $\frac{\pi}{6} \approx 0.5$, so its ordered pair is (0.5, 2.6). The coefficient 3 is specific to this function; multiply by the function values $\frac{\pi}{4}$ and $\frac{\pi}{3}$, respectively, to find the other two function values and points:

$3\cos\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} \approx 2.1$ and $\frac{\pi}{4} \approx 0.8$; (0.8, 2.1)

$3\cos\left(\frac{\pi}{3}\right) = 3 \bullet 0.5 = 1.5$ and $\frac{\pi}{3} \approx 1.1$; (1.1, 1.5)

Plot these three points on the graph between the points (0, 3) and $\left(\frac{\pi}{2}, 0\right)$.



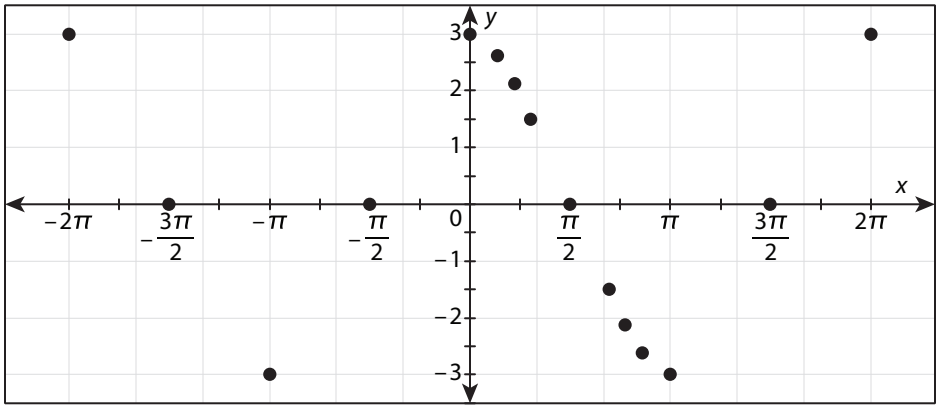
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7. Plot additional points.

Plot the points for the x-values $\frac{2\pi}{3}$, $\frac{3\pi}{4}$, and $\frac{5\pi}{6}$, which are between the points $\left(\frac{\pi}{2},0\right)$ and $\left(\frac{3\pi}{2},0\right)$.

This will show how the function values are reflected across the line $x=\frac{\pi}{2}$ and then reflected across the midline ($y = 0$) for domain values that differ by $\frac{\pi}{2}$.

For this function, $f\left(\frac{2\pi}{3}\right)=2\cos\left(\frac{2\pi}{3}\right)=3\bullet-0.5=-1.5$, so the point is given by $\left(\frac{2\pi}{3},-1.5\right)$. The other function values can be determined by using a calculator. They produce the following two points: $\left(\frac{3\pi}{4},-2.1\right)$ and $\left(\frac{5\pi}{6},-2.6\right)$. Plot these three points.



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8. Compare the ordered pairs for the graphed points on either side of the x-axis.

   The points occur where $x=\frac{\pi}{6}$ and $\frac{5\pi}{6}$, $x=\frac{\pi}{4}$ and $\frac{3\pi}{4}$, and $x=\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

   The function values for these point pairs are opposites, which confirms that the points are reflected twice across the line $x=\frac{\pi}{2}$ and across the function midline (at $y = 0$, the x-axis) over the restricted interval $[0, \pi]$.

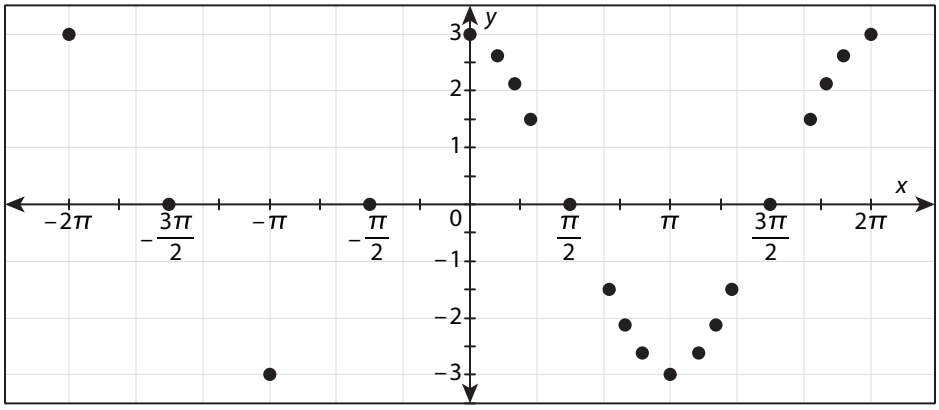
9. Determine the function values for each value of x over the restricted domain of $[\pi, 2\pi]$. Plot points on the graph for this part of the domain.

   This will show that the function values for values of x over the restricted domain $[\pi, 2\pi]$ are the opposite of those over the restricted domain $[0, \pi]$.

   Use a graphing calculator to find the function values for the x-values over the domain $[\pi, 2\pi]$.

   The x-values are $\frac{7\pi}{6}$, $\frac{5\pi}{4}$, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$, $\frac{7\pi}{4}$, and $\frac{11\pi}{6}$.

   The calculator will give function values that result in the approximate points $\left(\frac{7\pi}{6}, -2.6\right)$, $\left(\frac{5\pi}{4}, -2.1\right)$, $\left(\frac{4\pi}{3}, -1.5\right)$, $\left(\frac{5\pi}{3}, 1.5\right)$, $\left(\frac{7\pi}{4}, 2.1\right)$, and $\left(\frac{11\pi}{6}, 2.6\right)$. Plot these points on the graph.



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10. Compare the points over the two restricted domains $[0, \pi]$ and $[\pi, 2\pi]$.

    The points occur where $y \approx 1.5$ and -1.5, $y \approx 2.1$ and -2.1, and $y = 2.6$ and -2.6.

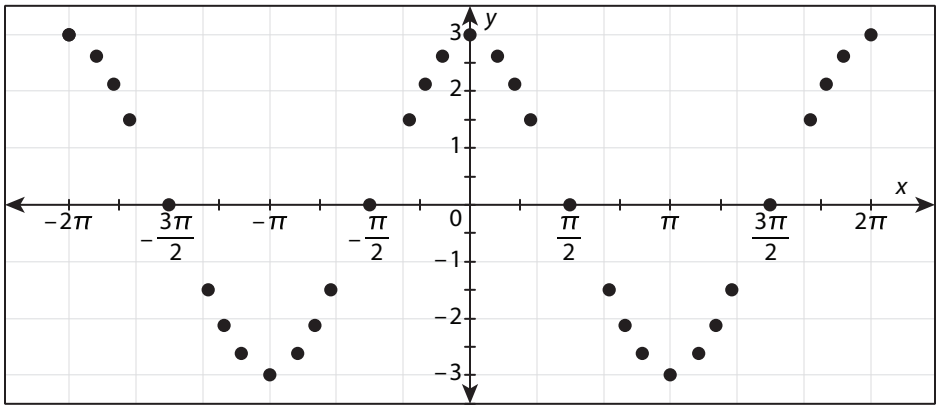
    This confirms that the function values of the points over the restricted domain $[0, \pi]$ and points over the restricted domain $[\pi, 2\pi]$ are opposites for specific domain values.

11. Predict what the shape of the function graph will be over the remainder of the domain, $(-2\pi, 0)$.

    Note that the function values change signs every half period of domain values graphed. Therefore, the shape of the function graph over the domain $(-2\pi, 0)$ should be two reflections of the function graph for the domain $[0, 2\pi]$ across the y-axis ($x = 0$) and across the midline $y = 0$ (the x-axis).

12. Plot additional points on the graph to confirm your prediction. Then, draw a curve connecting the points across the domain $[-2\pi, 2\pi]$.

    At a minimum, points over the domain $(-\pi, 0)$ should be plotted for x-values that are the opposite of those used for the domain $(\pi, 0)$.



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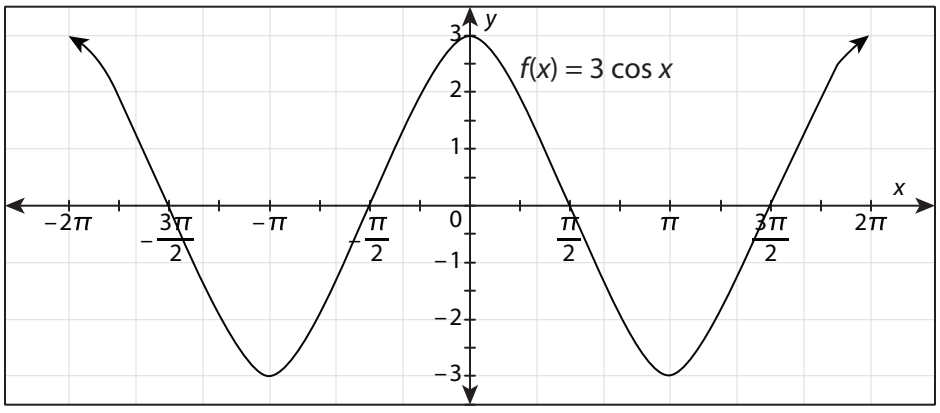
13. Verify the resulting graph using a graphing calculator.

Enter the given function, $f(x) = 3 \cos x$, into your graphing calculator. Regardless of whether you are using the TI-83/84, the TI-Nspire, or a similar calculator, remember to adjust the viewing window values so that the x-axis endpoints include the function's domain and the y-axis endpoints include the amplitude. Use the known x- and y-values to determine a value for the x-axis scale that will provide a comprehensive view of the graphed function. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

For this function:

- Since the x-values are in increments of π, set the mode to radians.
- The domain is $[-2\pi, 2\pi]$, so set the x-axis endpoints to -2π and 2π.
- The amplitude is 3, so set the y-axis endpoints to at least -3 and 3.
- For the x-axis scale, we know that the smallest interval between x-values is $\frac{\pi}{6}$; let this be the x-axis scale.

The resulting graph on the calculator should confirm the accuracy of the sketched function graph.



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Instruction

Example 2

How many complete cycles of the cosine function $g(x) = \cos 4x$ are found in the restricted domain $[-270°, 270°]$?

1. Identify the period of $g(x)$.

   This will be needed to calculate the number of complete cycles of $g(x)$ that exist in the restricted domain.

   Set the argument of the function equal to the period of the parent cosine function, $\cos x$, and solve for x: $4x = 2\pi$, so $x = \frac{\pi}{2}$.

   The period of the function $g(x) = \cos 4x$ is $\frac{\pi}{2}$.

2. Convert the period to degrees.

   Since the domain is given in degrees, the period must also be in degrees.

   1 radian $= \frac{180°}{\pi}$, so $\frac{\pi}{2}$ radians is $\frac{\pi}{2} \bullet \frac{180°}{\pi} = \frac{180\pi}{2\pi} = 90°$.

   The period of $g(x)$, converted to degrees, is 90°.

3. Determine the number of complete cycles $g(x)$ makes over the domain $[-270°, 270°]$.

   In order to determine the number of completed cycles, calculate the total number of degrees within the domain of the function, and then divide this number by the period.

   The domain covers $270° - (-270°)$ or 540°. Therefore, the number of complete cycles of $g(x)$ is 540° divided by the period, 90°:
   $540 \div 90 = 6$ complete cycles.

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4. Verify the result using a graphing calculator.

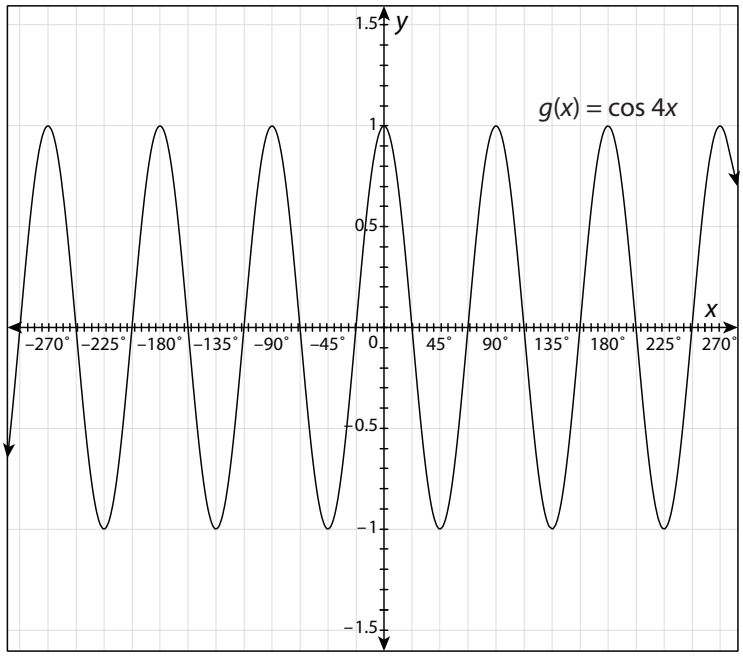
   Enter the given function, $g(x) = \cos 4x$, into your graphing calculator. Regardless of whether you are using the TI-83/84, the TI-Nspire, or a similar calculator, remember to adjust the viewing window values so that the x-axis endpoints include the function's domain and the y-axis endpoints include the amplitude. Use the known x- and y-values to determine a value for the x-axis scale that will provide a comprehensive view of the graphed function. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

   For this function:

   - Set the mode to degrees.
   - The domain is [–270°, 270°], so set the x-axis endpoints to –270 and 270.
   - The amplitude is 1, so set the y-axis endpoints to at least –1 and 1.
   - For the x-axis scale, choose a scale value that's a factor of the endpoints. Let the x-axis scale be 30 (a factor of –270 and 270).

   Graph the function and count the number of complete cycles.

   The resulting graph confirms our calculation that $g(x)$ has 6 complete cycles over the restricted domain [–270°, 270°].



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Instruction

Example 3

Determine the coordinates of the point(s) that represent a maximum positive function value of the function $h(x) = 2 \cos 3x$ over the restricted domain $\left[-\frac{3\pi}{2}, \frac{2\pi}{3}\right]$.

1. Identify the amplitude and period of the function.

   This information will be needed to determine the location and the magnitude of the maximum function value.

   The amplitude is 2 because 2 is the coefficient of the cosine term. The period can be found by setting the argument of the function equal to 2π and solving for x: $3x = 2\pi$, so $x = \frac{2\pi}{3}$.

2. Determine where the function will have maximum values over the domains of its period on either side of the origin.

   This will indicate the value(s) of x at which the function has a maximum.

   At the origin, $x = 0$, so the domains of the function's period on either side of the origin are $\left[-\frac{2\pi}{3}, 0\right]$ and $\left[0, \frac{2\pi}{3}\right]$.

   The function $h(x) = 2 \cos 3x$ has a maximum value of 2 at $x = \frac{2\pi}{3}$, $x = 0$, and $x = -\frac{2\pi}{3}$.

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Instruction

3. Compare the points at which $h(x)$ will have a maximum to the restricted domain of the problem.

The restricted domain of $h(x) = 2 \cos 3x$ is given as $\left[-\frac{3\pi}{2}, \frac{2\pi}{3}\right]$. One maximum value occurs at $x = \frac{2\pi}{3}$, which is also the upper bound of the restricted domain. However, the lower bound of the restricted domain is $x = -\frac{3\pi}{2}$, so the maximum at $x = -\frac{2\pi}{3}$ may not be the only maximum greater than the lower bound. Subtract the value of one cycle from $x = -\frac{2\pi}{3}$ to find another maximum:

$$x = -\frac{2\pi}{3} - \left(\frac{2\pi}{3}\right) = -\frac{4\pi}{3}$$

The maximum at $x = -\frac{4\pi}{3}$ is greater than the lower bound.

Determine whether there is another maximum by subtracting the value of one cycle from $x = -\frac{4\pi}{3}$:

$$x = -\frac{4\pi}{3} - \left(\frac{2\pi}{3}\right) = -2\pi$$

There is another maximum at $x = -2\pi$. This value is less than the lower bound, so there are no other maximum values over the restricted domain $\left[-\frac{3\pi}{2}, \frac{2\pi}{3}\right]$.

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Instruction

4. Determine how many maximum values $h(x)$ has over the restricted domain $\left[-\frac{3\pi}{2}, \frac{2\pi}{3}\right]$.

   Notice that the restricted domain includes the lower and upper bound.

   There are 4 maximum points over the restricted domain, at $x = \frac{2\pi}{3}$, $x = 0$, $x = -\frac{2\pi}{3}$, and $x = -\frac{4\pi}{3}$.

5. Check the result using a graphing calculator.

   Enter the given function, $h(x) = 2 \cos 3x$, into your graphing calculator. Adjust the viewing window values to include the function's domain and amplitude, with a suitable x-axis scale. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

   For this function:

   - Set the mode to radians.
   - The domain is $\left[-\frac{3\pi}{2}, \frac{2\pi}{3}\right]$, so set the x-axis endpoints to $-\frac{3\pi}{2}$ and $\frac{2\pi}{3}$.
   - The amplitude is 2, so set the y-axis endpoints to at least –2 and 2.
   - For the x-axis scale, choose a scale value that's a factor of the endpoints. Let the x-axis scale be $\frac{\pi}{6}$ (a factor of $-\frac{3\pi}{2}$ and $\frac{2\pi}{3}$).

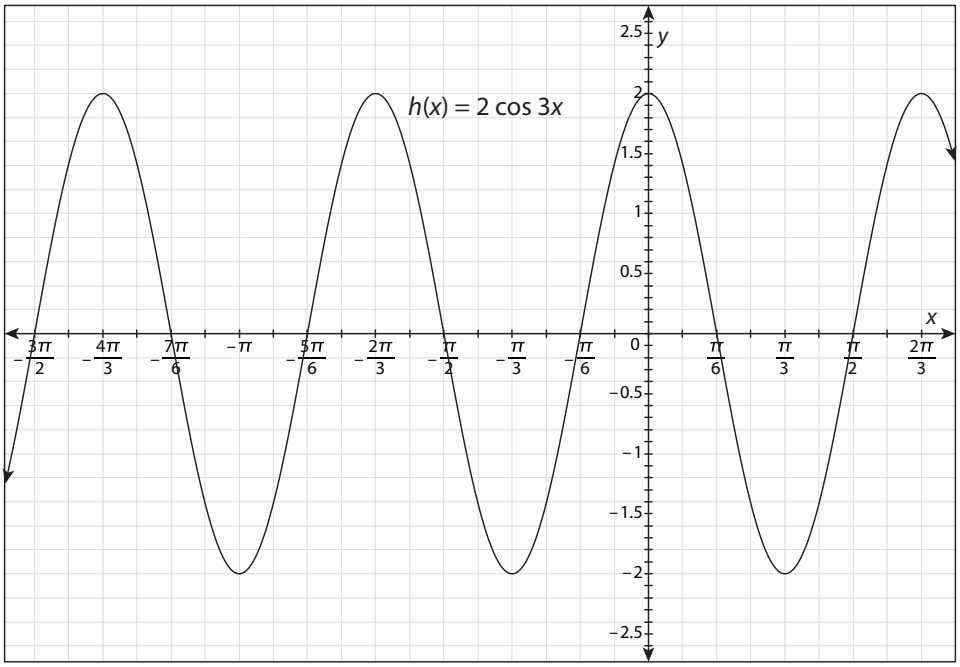
   Graph the function and count the number of maximums.

(*continued*)

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Instruction

The resulting graph confirms our calculation that $h(x) = 2 \cos 3x$ has four maximums over the restricted domain $\left[-\frac{3\pi}{2}, \frac{2\pi}{3}\right]$.



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Example 4

Sketch the graph of the function $a(x) = -2 + 4 \cos 2x$ over the restricted domain $[-\pi, \pi]$.

1. Identify the amplitude of the function.

   The addition of the constant to the cosine term will shift the graph and the relative locations of the maximum and minimum values of the function.

   The amplitude is 4, but the –2 added to the cosine term will produce a maximum function value of 2 and a minimum function value of –6. The constant defines the midline, which is $y = -2$.

   Therefore, the maximum function value occurs at $y = 2$ and the minimum function value occurs at $y = -6$.

2. Determine the period of the function.

   Set the argument of the function equal to 2π to find the period.

   $2x = 2\pi$

   $x = \pi$

   The period is equal to π.

3. Determine how many cycles of the function can be shown over the domain $[-\pi, \pi]$.

   First, determine the domain. Subtract the lower bound from the upper bound:

   $\pi - (-\pi) = 2\pi$

   Next, divide the domain by the period, π, to find the number of cycles.

   $$\frac{2\pi}{\pi} = 2$$

   There are 2 cycles over the domain $[-\pi, \pi]$.

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4. Determine the values of x at which the maximum and minimum occur over the domain of the cycle given by $[-\pi, \pi]$.

   The maximum function values occur at $x = 0$ and $x = \pi$ over the restricted domain $[0, \pi]$, so the minimum function value occurs at half of the distance from $x = 0$ to $x = \pi$, or at $x = \frac{\pi}{2}$.

   The maximum values over the restricted domain $[-\pi, 0]$ occur at $x = 0$ and $x = -\pi$, so the minimum function value occurs at half of the distance from $x = 0$ to $x = -\pi$, or at $x = -\frac{\pi}{2}$.

5. Determine the coordinates of the points for the maximum and minimum values determined in step 4.

   The values found in the previous step represent the x-values of the coordinates.

   Maximums: $x = -\pi$, 0, and π Minimums: $x = \frac{\pi}{2}$ and $-\frac{\pi}{2}$

   The values found in step 1 represent the y-values of the coordinates.

   Maximum: $y = 2$ Minimum: $y = -6$

   Write these values as coordinates to determine the points of the maximum and minimum values.

   The maximums occur at the points $(-\pi, 2)$, $(0, 2)$, and $(\pi, 2)$.

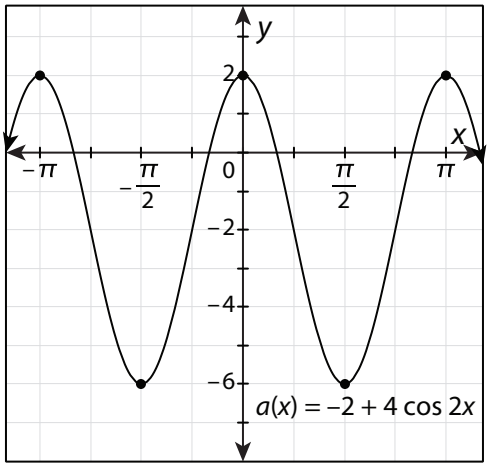
   The minimums occur at the points $\left(-\frac{\pi}{2}, -6\right)$ and $\left(\frac{\pi}{2}, -6\right)$.

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Instruction

6. Use the results from step 5 to sketch the graph of $a(x) = -2 + 4 \cos 2x$ over the restricted domain $[-\pi, \pi]$.

   Sketch the graph of $a(x) = -2 + 4 \cos 2x$ by plotting the points on a coordinate plane and connecting them with a smooth curve, as shown.



7. Confirm your sketch using a graphing calculator.

   Enter the given function, $a(x) = -2 + 4 \cos 2x$, into your graphing calculator. Adjust the viewing window values to include the function's domain and amplitude, with a suitable x-axis scale. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

   For this function:

   - Set the mode to radians.
   - The domain is $[-\pi, \pi]$, so set the x-axis endpoints to $-\pi$ and π.
   - The amplitude of the function is 4 but the midline is $y = -2$, so add –2 to –4 and 4 to determine the y-axis endpoints: –6 and 2.
   - We know that the smallest interval between the x-values of the graphed coordinates is $\frac{\pi}{2}$; let this be the x-axis scale.

   The resulting graph should confirm the accuracy of the sketch.

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Instruction

Example 5

Determine the coordinates of the points at which the first maximum and minimum function values occur for the function $c(x)=\cos\left(x-\frac{\pi}{4}\right)$ for values of $x > 0$.

1. Determine the period of the function.

   The variable part of the function's argument has no coefficients, so the function has a period of 2π. The presence of the constant $-\frac{\pi}{4}$ shifts the graph horizontally, but does not affect the amplitude or period of the function.

2. Determine the value of x at which the first maximum function value for $x > 0$ occurs.

   The first value of $x > 0$ at which the parent function reaches a maximum is 0. However, with this function, the graph *lags* reaching a maximum function value by the amount of $\frac{\pi}{4}$, which is subtracted from x. Therefore, this function will reach its first maximum point at an x-value that is given by the difference $\frac{\pi}{4}-\frac{\pi}{4}=0$, so that $\cos\left(\frac{\pi}{4}-\frac{\pi}{4}\right) = \cos 0 = 1$; therefore, the first maximum is at $x=\frac{\pi}{4}$. Because of the horizontal shift of $-\frac{\pi}{4}$, the function does not reach its first maximum point until $x=\frac{\pi}{4}$.

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Instruction

3. Determine the value of x at which the first minimum function value for $x > 0$ occurs.

   The period of the function is the same as the parent function for cosine, $f(x) = \cos x$, so the minimum occurs π units "past" the maximum at $x = \frac{\pi}{4}$. Add π to find the minimum:

   $$x = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

4. Determine the maximum and minimum function values.

   The amplitude of the function is 1, so the maximum function value is 1 and the minimum function value is –1.

5. Write the coordinates of the points for the maximum and minimum function values.

   The point with the first maximum function value for $x > 0$ is $\left(\frac{\pi}{4}, 1\right)$.

   The point with the first minimum function value for $x > 0$ is $\left(\frac{5\pi}{4}, -1\right)$.

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Instruction

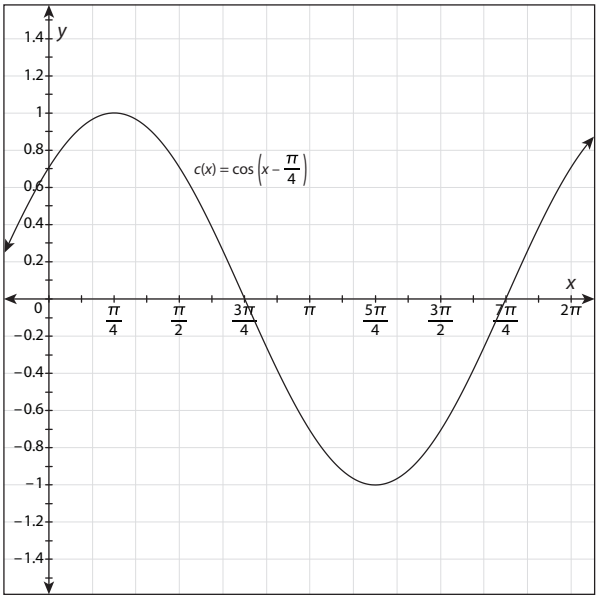
6. Verify the coordinates using a graphing calculator.

   Enter the given function, $c(x)=\cos\left(x-\frac{\pi}{4}\right)$, into your graphing calculator. Adjust the viewing window values to include the function's domain and amplitude, with a suitable x-axis scale. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

   For this function:

   - Set the mode to radians.
   - Set the x-axis endpoints to 0 and 2π.
   - The amplitude of the function is 1, so set the y-axis endpoints to at least –1 and 1.
   - The first function maximum occurs at $x=\frac{\pi}{4}$; let $\frac{\pi}{4}$ be the x-axis scale.

   Your result should resemble the following graph of $c(x)=\cos\left(x-\frac{\pi}{4}\right)$ for $x > 0$. Use your calculator's trace feature to confirm that the first maximum is at $\left(\frac{\pi}{4},1\right)$ and the first minimum is at $\left(\frac{5\pi}{4},-1\right)$.



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Problem-Based Task 4A.3.2: Picking Up the Beat(s)

SMP	
1 ✓	2 ✓
3	4 ✓
5	6 ✓
7 ✓	8

Cruz and Seth both play the six-string guitar. Seth's guitar is already in tune, so Cruz starts tuning his guitar to it. Seth plucks the low E string, the thickest string; Cruz plucks the low E string on his guitar and changes its tuning to match the frequency of Seth's E string. When the two strings sound like they are in tune, the two guitarists pluck the strings at the same time. Cruz listens for beats to determine the changing sound intensity of the tuned string. The number of beats per second indicates by what frequency the strings are out of tune. Therefore, though the strings may seem to be in tune, by counting beats, Cruz can confirm whether the two strings are exactly in tune.

The sound intensity of the first guitar's string (the reference string on Seth's guitar) varies with time like a cosine function, and can be given by $I_1(t) = A \cos 2\pi f_1 t$, in which A is the amplitude of the sound wave in millimeters at the instant the string is plucked, f_1 is the frequency of the wave, and t is the time in seconds. If the second guitar's string is plucked with exactly the same force, its sound intensity can be given by the function $I_2(t) = A \cos 2\pi f_2 t$.

When the two strings are plucked at the same time to tune one to the other, the two sound intensities cancel or reinforce each other according to the function $I_{\text{total}} = I_1 + I_2 = A \cos 2\pi f_1 t + A \cos 2\pi f_2 t$. According to the rules by which the sums of cosine functions can be written as products, the changing sound intensity $I_{\text{total}}(t)$ of the guitar string being tuned can also be given by the product-of-cosines function $I_{\text{total}}(t) = 2A\cos\left[2\pi t\left(\frac{f_1 + f_2}{2}\right)\right] \bullet \cos\left[2\pi t\left(\frac{f_1 - f_2}{2}\right)\right]$.

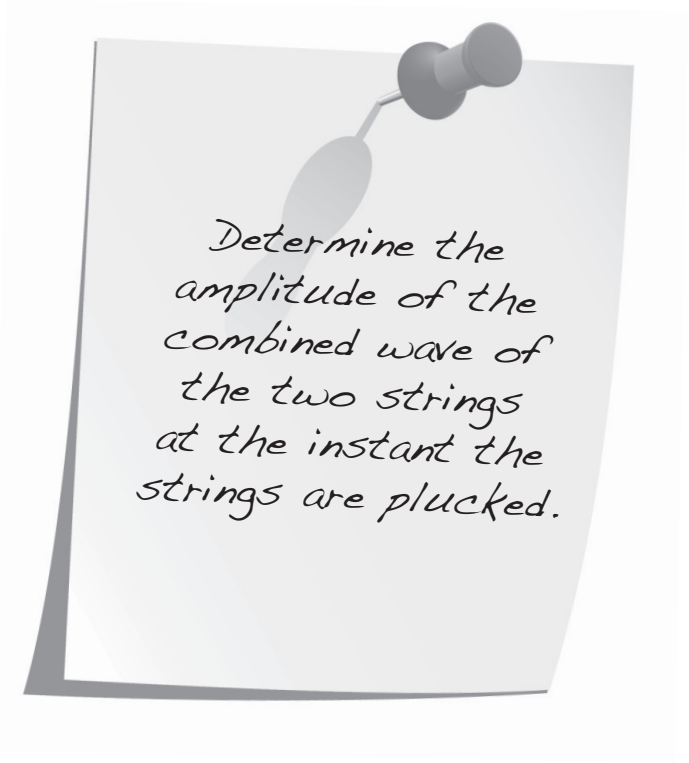
In this function, the cosine terms refer to two different components of the sound intensity:

- The term $2A\cos\left[2\pi t\left(\frac{f_1 + f_2}{2}\right)\right]$ determines how the sound intensity varies according to time. The amplitude of this sound wave varies between $2A$ and $-2A$ over each period of the cosine term.
- The term $\cos\left[2\pi t\left(\frac{f_1 - f_2}{2}\right)\right]$ contains the expression $f_1 - f_2$. The absolute value of this, $|f_1 - f_2|$, determines the number of beats per second due to the difference in frequency of the strings. $|f_1 - f_2|$ is called the *beat frequency*. The beat frequency causes the combined amplitude to vary in intensity by that number of times per second.

continued

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If the amplitude of each plucked string is 1 millimeter, determine the amplitude of the combined wave of the two strings at the instant the strings are plucked. Then, find the number of beats produced in 1 second if the reference string has a frequency of 438 cycles per second and the string being tuned has a frequency of 442 cycles per second.



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Problem-Based Task 4A.3.2: Picking Up the Beat(s)

Coaching

a. What is the amplitude of *each* guitar string if they are plucked at the same time with the same force?

b. What is the value of the sound intensity for *each* of the guitar strings at $t = 0$?

c. How does the answer to part b explain the amplitude of $2A$ for the product-of-cosines function?

d. What are the arguments of the cosine terms in the product-of-cosines function for $I_{total}(t)$, and what do they signify about the sound waves?

e. Does it matter which frequency is represented by f_1 and which is represented by f_2 in either of the cosine terms in the product-of-cosines function? Explain.

f. Find the number of beats produced in 1 second given the frequencies of the two strings.

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Problem-Based Task 4A.3.2: Picking Up the Beat(s)

Coaching Sample Responses

a. What is the amplitude of *each* guitar string if they are plucked at the same time with the same force?

In order to find the combined amplitude of the two sound waves, the individual amplitudes must be known. The cosine functions $I_1(t)$ and $I_2(t)$ have the same amplitude, A.

b. What is the value of the sound intensity for *each* of the guitar strings at $t = 0$?

The amplitude is given as 1 millimeter. Substitute $A = 1$ and $t = 0$ into the functions $I_1(t) = A \cos 2\pi f_1 t$ and $I_2(t) = A \cos 2\pi f_2 t$, and then solve each equation.

$I_1(t) = A \cos 2\pi f_1 t$ $\qquad$ $I_2(t) = A \cos 2\pi f_2 t$

$I_1(0) = (1) \cos 2\pi f_1(0)$ $\qquad$ $I_2(0) = (1) \cos 2\pi f_2(0)$

$I_1(0) = 1 \bullet 1 = 1$ $\qquad$ $I_2(0) = 1 \bullet 1 = 1$

c. How do the answers to part b explain the amplitude of $2A$ for the product-of-cosines function?

The first function for $I_{\text{total}}(t)$ equals the sum of $I_1(t)$ and $I_2(t)$. At $t = 0$, $I_1(0) + I_2(0) = A + A = 2A$. In the product-of-cosines function, $I_{\text{total}}(0) = 2A \bullet 1 \bullet 1 = 2A$, so the results are the same: $2A = 2(1 \text{ millimeter}) = 2 \text{ millimeters}$.

d. What are the arguments of the cosine terms in the product-of-cosines function for $I_{\text{total}}(t)$, and what do they signify about the sound waves?

The argument of the first term is $2\pi t\left(\frac{f_1 + f_2}{2}\right)$ and the argument of the second term is $2\pi t\left(\frac{f_1 - f_2}{2}\right)$. The factor $\frac{f_1 + f_2}{2}$ in the argument of the first term represents an average of the frequencies of the two waves, whereas the factor $\frac{f_1 - f_2}{2}$ in the argument of the second term represents half of the difference of the two wave frequencies.

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e. Does it matter which frequency is represented by f_1 and which is represented by f_2 in either of the cosine terms in the product-of-cosines function? Explain.

To determine this, first substitute $A = 1$, $f_1 = 438$ cycles per second, and $f_2 = 442$ cycles per second in the product-of-cosines function, and solve for $I_{\text{total}}(t)$.

$$I_{\text{total}}(t) = 2 \bullet 1 \cos\left[2\pi t\left(\frac{f_1 + f_2}{2}\right)\right] \bullet \cos\left[2\pi t\left(\frac{f_1 - f_2}{2}\right)\right]$$

$$I_{\text{total}}(t) = 2\cos\left\{2\pi t\left[\frac{(438)+(442)}{2}\right]\right\} \bullet \cos\left\{2\pi t\left[\frac{(438)-(442)}{2}\right]\right\}$$

$$I_{\text{total}}(t) = 2\cos(2\pi t \bullet 440) \bullet \cos(2\pi t \bullet -2)$$

$$I_{\text{total}}(t) = \left[2\cos(880\pi t)\right] \bullet \left[\cos(-4\pi t)\right]$$

Compare this result with that obtained by substituting $f_1 = 442$ cycles per second and $f_2 = 438$ cycles per second in the product-of-cosine functions, and solving for $I_{\text{total}}(t)$.

$$I_{\text{total}}(t) = \left[2\cos(880\pi t)\right] \bullet \left[\cos(-4\pi t)\right]$$

The only difference is the argument of the second cosine factor. How do cos $-4\pi t$ and cos $4\pi t$ compare? They are the same, because the period of both cosine factors is 2π in either direction on the x-axis. This means that the differences $f_1 - f_2$ and $f_2 - f_1$ can be represented as $|f_1 - f_2|$.

f. Find the number of beats produced in 1 second given the frequencies of the two strings.

The two frequencies are $f_1 = 438$ and $f_2 = 442$. Substitute these into the beat frequency formula, $|f_1 - f_2|$.

$$|f_1 - f_2| = |438 - 442| = 4$$

There are 4 beats per second.

Recommended Closure Activity

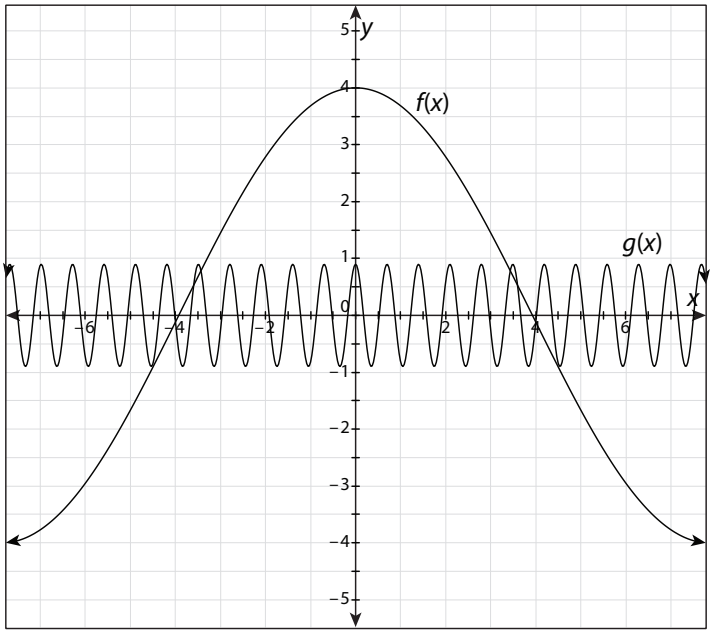
Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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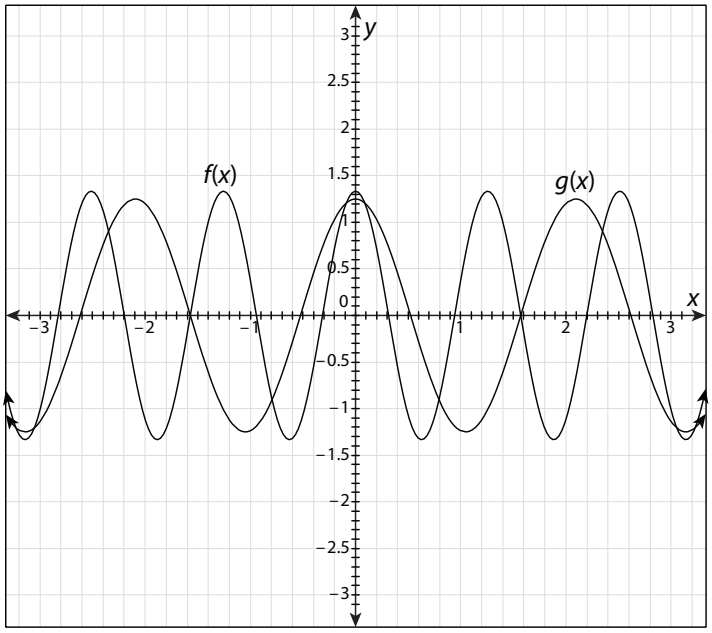
Practice 4A.3.2: Graphing the Cosine Function

For problems 1–4, refer to the provided graph to complete each problem. (*Note*: Some graphs show only part of a complete cycle. The *x*-axis of each graph is expressed in radians.)

1. Which function has the greater amplitude, $f(x)$ or $g(x)$?



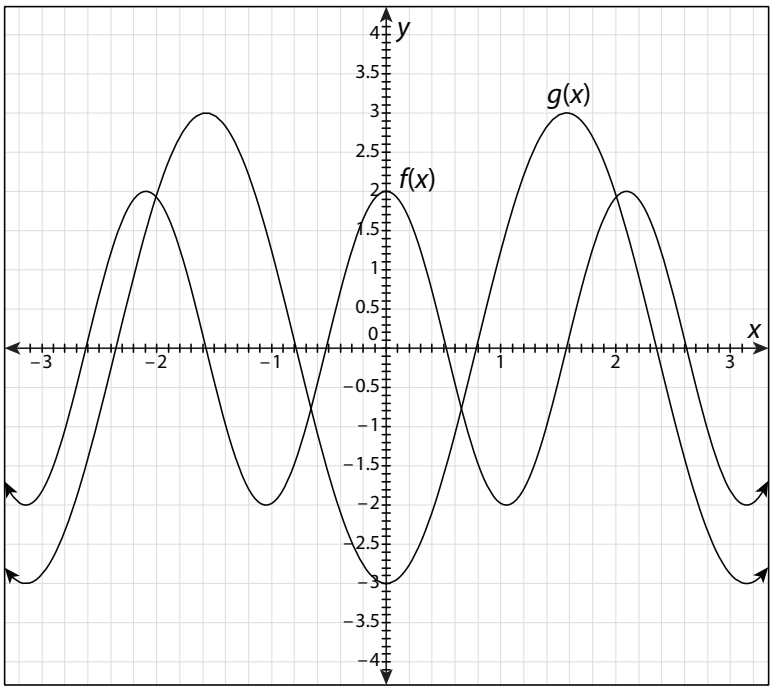
2. Which function has the greater period, $f(x)$ or $g(x)$?



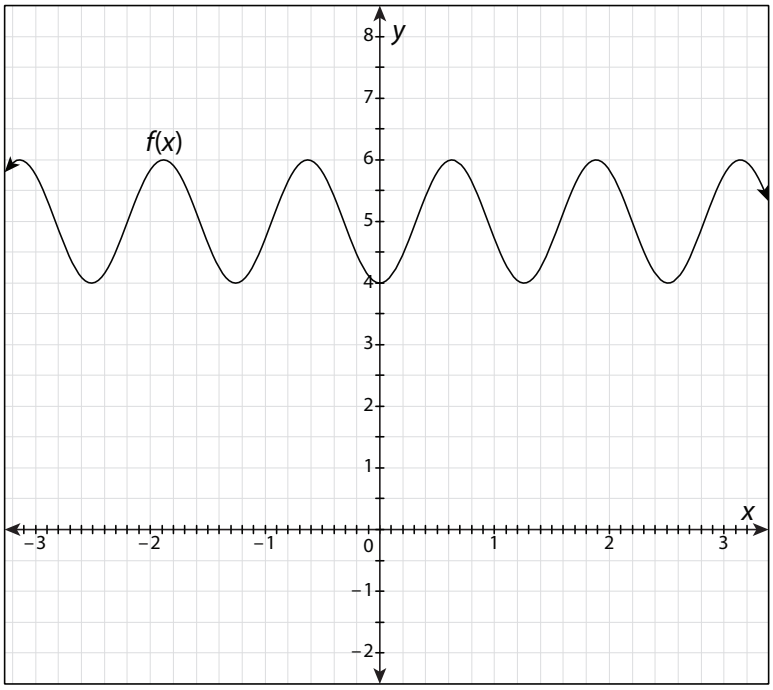
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3. Determine the amount by which the two functions $f(x)$ and $g(x)$ are out of phase.



4. Write the simplest form of the cosine function shown.



continued

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For problems 5–7, use the given information to find the requested values and coordinates.

5. At what value of $x > 0$ will the first maximum occur for the function $f(x) = 3 \cos 4x$? Determine the coordinates of the point for this value of x.

6. At what value of $x > 0$ will the first zero occur for the function $g(x) = 0.2 \cos 0.5x$? Determine the coordinates of the point for this value of x.

7. At what value of $x > 0$ will the first minimum occur for the function $h(x) = \cos (0.6x + 30°)$? Determine the coordinates of the point for this value of x.

Use your knowledge of cosine functions to complete problems 8–10.

8. The alarm in a smoke detector produces a high-pitched sound when smoke is detected. The intensity of the sound can be modeled by the function $I(t) = A \cos (3 \cdot 10^4 \cdot \pi \cdot t)$. What are the period and frequency of the sound intensity? The frequency is measured in cycles per second.

9. The horizontal distance a football travels, unaided by gravity, is given by $D(t) = v_0 \cdot t \cos A_0$, where v_0 is the horizontal velocity of the football as it leaves the kicker's foot, t is the football's "hang" time, and A_0 is the measure of the angle at which the football is kicked.

   a. If $D(t) = 150$ feet (50 yards on a football field) and the hang time is 5 seconds, what is the product $v_0 \cos A_0$ in feet per second?

   b. What is the range of the values of $\cos A_0$?

   c. What is v_0 if $A_0 = 60°$?

   d. Why can A_0 not equal 0° or 90°?

10. In an electric guitar, the pickup is the device that changes the vibrations of the strings into electrical current. The magnetic flux created by the permanent magnet in the pickup is given by the function $M(A, B, C) = AB \cos C$, in which A is the area enclosed by a loop of wire wrapped around the permanent magnet, B is the strength of the magnetic field, and C is the angle between the plane made by the loop of wire and the direction of the magnetic field relative to the loop of wire.

    a. At what value(s) of C is $M = 0$?

    b. At what value(s) of C is M a maximum?

    c. What is that maximum over the domain for C of [–180°, 180°]?

    d. State three possible combinations of values of A, B, and C that would make M what it is when it has a maximum value. (*Note*: $A > 0$, but B can be positive or negative.)

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Progress Assessment

Circle the letter of the correct answer.

1. What is the degree measure of an angle that is $\frac{2\pi}{3}$ radians?

   a. 240°
   c. 60°
   b. 120°
   d. 2°

2. What is the radian measure of an angle that is 135°?

   a. $\frac{3}{4}$ radians
   c. $\frac{4}{3}$ radians
   b. $\frac{3\pi}{4}$ radians
   d. $\frac{4\pi}{3}$ radians

3. What is the amplitude of the function $f(x) = 2 \sin 0.2x$?

   a. 10π
   c. 2
   b. 2π
   d. 0.2

4. What is the period of the function $g(x) = 5 \cos (10x - 15)$?

   a. 20π
   c. $\frac{\pi}{5} + \frac{3}{2}$
   b. $20\pi - 15$
   d. $\frac{\pi}{5}$

5. What is the period of the function $a(x) = 4\pi \sin (2\pi x)$?

   a. 1
   c. 2
   b. π
   d. 4π

6. What is the amplitude of the function $b(x) = -0.3 \cos 0.3x$?

   a. $-\frac{20\pi}{3}$
   c. 0.3
   b. –0.3
   d. $\frac{20\pi}{3}$

continued

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Assessment

7. What are the domain and range of $f(x) = 125 \sin 0.25x$?

   a. $(-\infty, \infty)$ and $[-0.25, 0.25]$

   b. $(-\infty, \infty)$ and $[-125, 125]$

   c. $[-0.25, 0.25]$ and $[-125, 125]$

   d. $[-125, 125]$ and $(-\infty, \infty)$

8. What are the domain and range of $g(x) = 13\pi \cos\left(\frac{2}{3}x\right)$?

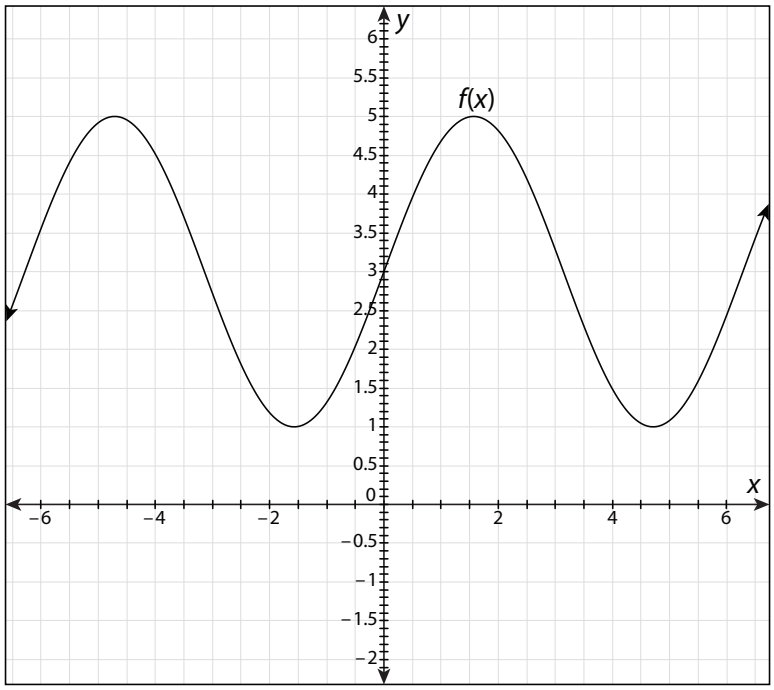
   a. $(-\infty, \infty)$ and $[-3\pi, 3\pi]$

   b. $(-\infty, \infty)$ and $[-3, 3]$

   c. $(-\infty, \infty)$ and $[-13, 13]$

   d. $(-\infty, \infty)$ and $[-13\pi, 13\pi]$

9. Which function does the graph represent?



   a. $f(x) = -\sin x + 3$

   b. $f(x) = -2 \sin x + 3$

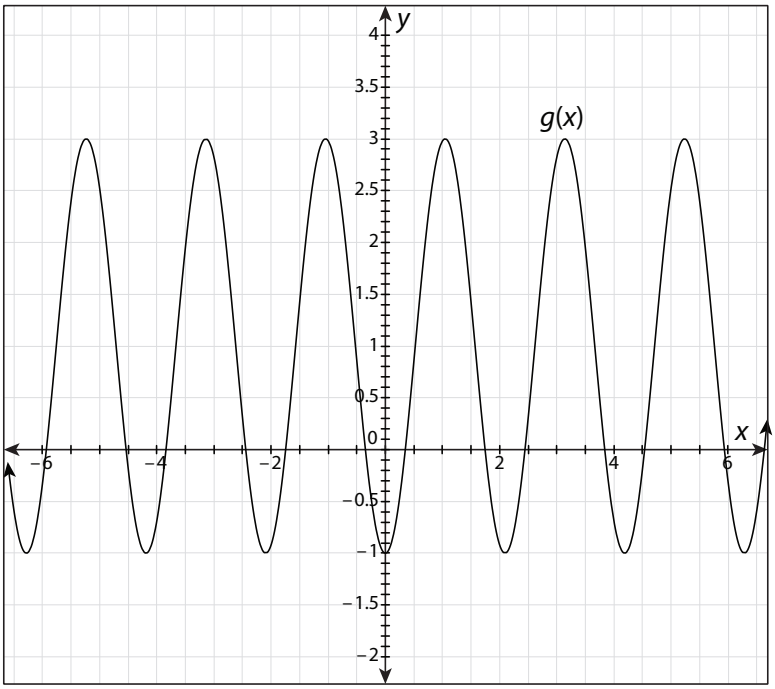
   c. $f(x) = \sin x + 3$

   d. $f(x) = 2 \sin x + 3$

continued

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10. Which function does the graph represent?



a. $g(x) = 1 - 2\cos 3x$

b. $g(x) = 1 - \cos 3x$

c. $g(x) = 1 - 2\cos x$

d. $g(x) = 1 - \cos x$

Use what you have learned about trigonometric functions to solve the following problem.

11. The amplitude of a sound wave y can be written as a function of x, the distance the sound wave travels, using the cosine function $y(x) = A\cos\left(\frac{2\pi}{B} \bullet x\right)$. The wavelength B represents one complete cycle of the function and x is the total distance the wave travels. Show that $x + B$ gives the same result as x in the argument of the function.

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Unit 4A Assessment

Circle the letter of the best answer.

1. What is the inverse of the quadratic function $f(x)=5x^2-1$?

   a. $f^{-1}(x)=\pm\sqrt{x+1}$

   b. $f^{-1}(x)=\pm\sqrt{5(x+1)}$

   c. $f^{-1}(x)=\pm\dfrac{\sqrt{5(x+1)}}{5}$

   d. $f^{-1}(x)=\pm\dfrac{1}{5}\sqrt{x+1}$

2. What is the inverse of the absolute-value function $g(x)=|3x+4|$?

   a. $g^{-1}(x)=\dfrac{\pm x-4}{3},\ x\geq-\dfrac{4}{3}$

   b. $g^{-1}(x)=\dfrac{1}{3}\bullet|x-4|,\ x\geq0$

   c. $g^{-1}(x)=3\bullet|x-4|,\ x\geq-\dfrac{4}{3}$

   d. $g^{-1}(x)=\pm3x-4,\ x\geq0$

3. What is the inverse of the exponential function $h(x)=2\bullet(3)^{4x}$?

   a. $h^{-1}(x)=4\log_3 x-4\log_3 2$

   b. $h^{-1}(x)=\dfrac{\log_3 x}{4\log_3 2}$

   c. $h^{-1}(x)=\dfrac{4\log_3 x}{\log_3 2}$

   d. $h^{-1}(x)=\dfrac{1}{4}\log_3 x-\dfrac{1}{4}\log_3 2$

4. What is the value of x in terms of common logarithms for the exponential function $c(x)=2\bullet6^{3x}$?

   a. $x=\dfrac{1}{3}\bullet\left(\dfrac{\log c-\log 2}{\log 6}\right)$

   b. $x=\dfrac{1}{3}\bullet\left(\dfrac{\log 2-\log c}{\log 6}\right)$

   c. $x=3\bullet\left(\dfrac{\log 2-\log c}{\log 6}\right)$

   d. $x = 3(\log 2 - \log c - \log 6)$

5. What is the value of x in the exponential equation $1-2e^{2x}=e^x$?

   a. $x = -\ln 2$

   b. $x = -1$ or $-\ln 2$

   c. $x=\dfrac{1}{2}$ or -1

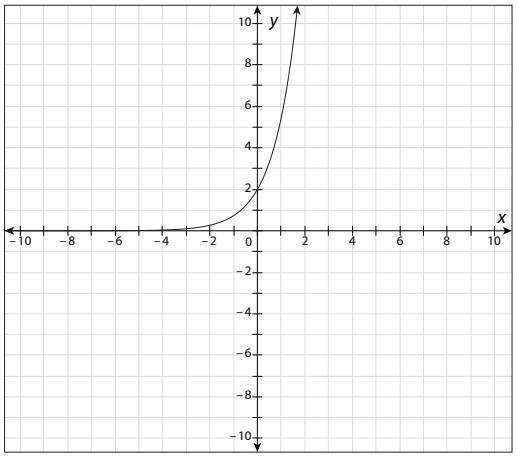
   d. $x=\dfrac{1}{2}$

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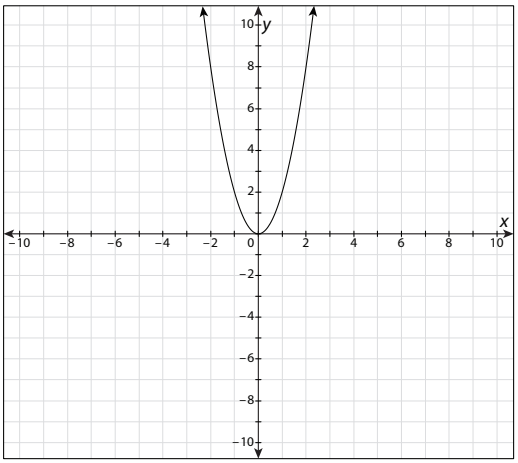
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6. Which graph represents a logarithmic function?

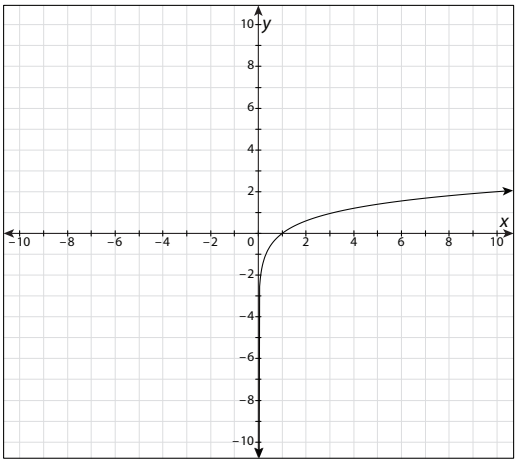
a.



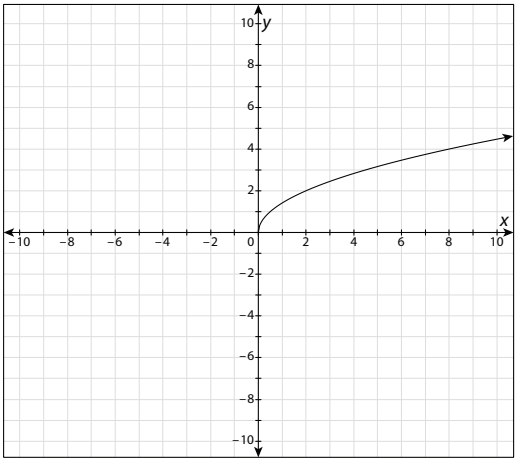
c.



b.



d.



7. A function for modeling earthquake intensity on the Richter scale is given by the equation $M_S = \log\left(\frac{A}{T}\right) + 1.\overline{6} \bullet \log d + 3.\overline{3}$, in which A is the amplitude of the ground-wave motion during the earthquake, T is the period of the ground wave, and d is the temperature in degrees. Which of the following shows another way that the logarithmic term in the function can be written?

a. $M_S = \log A + \log T + 1.\overline{6} \bullet \log d + 3.\overline{3}$

b. $M_S = \log\left(\frac{3.\overline{3} \bullet A \bullet d^{\frac{5}{3}}}{T}\right)$

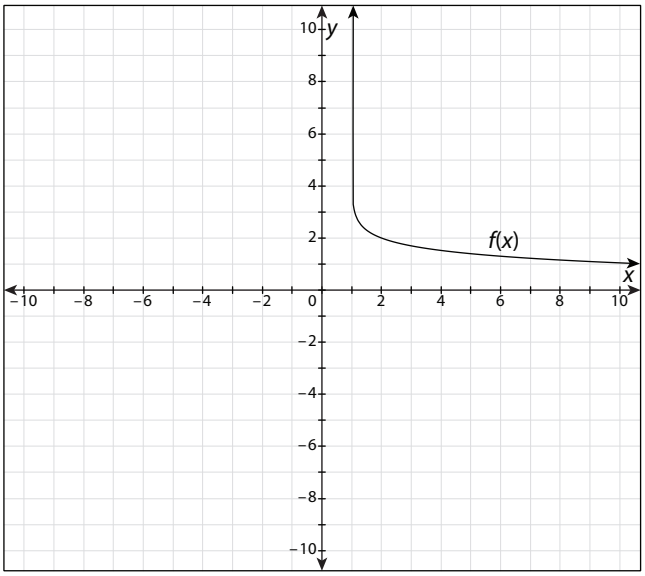
c. $M_S = \log\left(\frac{A \bullet d^{\frac{5}{3}}}{T}\right) + 3.\overline{3}$

d. $M_S = \log(3.\overline{3} \bullet A \bullet d^{\frac{5}{3}} - T)$

continued

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8. Which function does the graph represent?



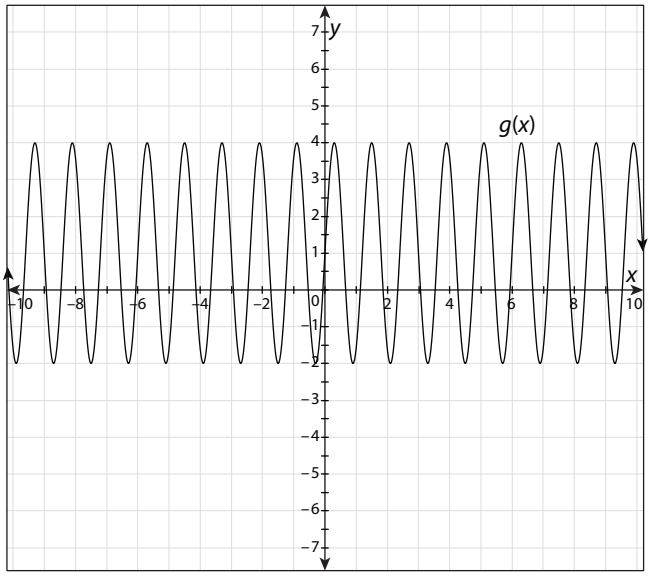
a. $f(x) = 1 + \log x$

b. $f(x) = 2 + \log (x - 1)$

c. $f(x) = 2 - \log (x - 1)$

d. $f(x) = 2 \bullet \log (x - 1)$

9. Which function does the graph represent?



a. $g(x) = 3 \bullet \sin 5x$

b. $g(x) = 1 + 3 \bullet \sin 5x$

c. $g(x) = -1 + 5 \bullet \sin 3x$

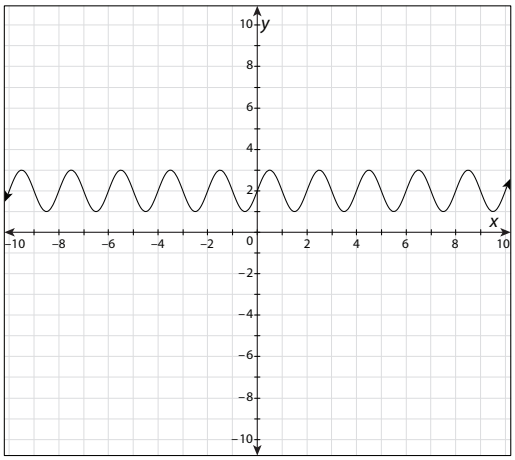
d. $g(x) = 1 + 5 \bullet \sin 3x$

continued

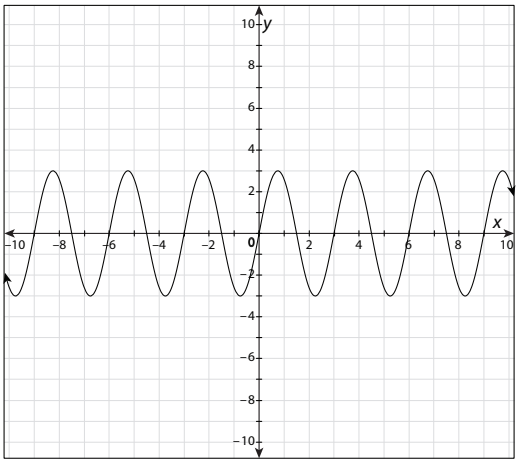
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10. Which graph represents $f(x) = 3 \cdot \sin 2x$?

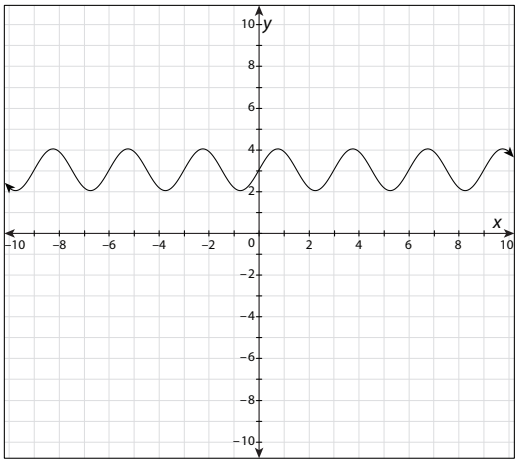
a.



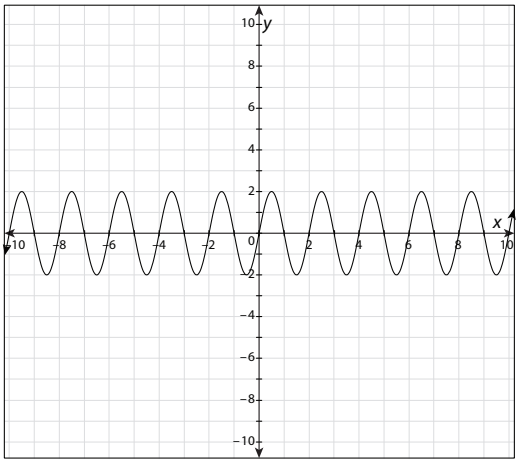
c.



b.



d.

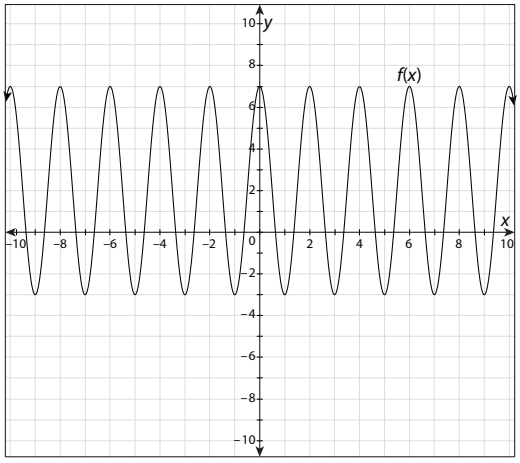


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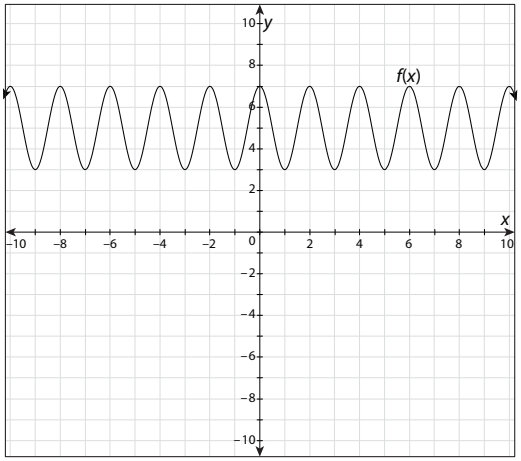
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11. Which graph represents $f(x) = 5 + 2 \bullet \cos 3x$?

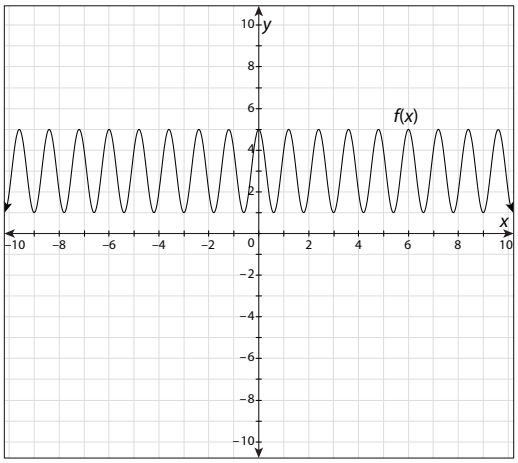
a.



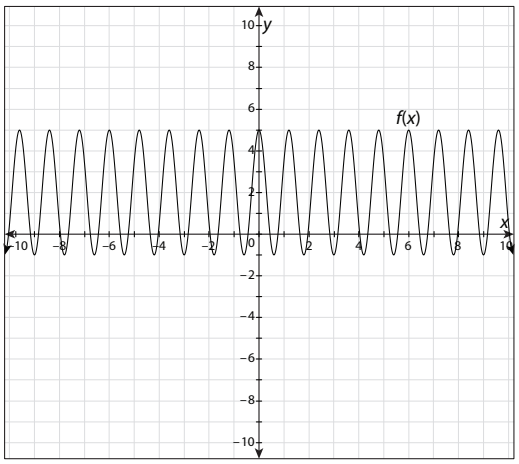
c.



b.



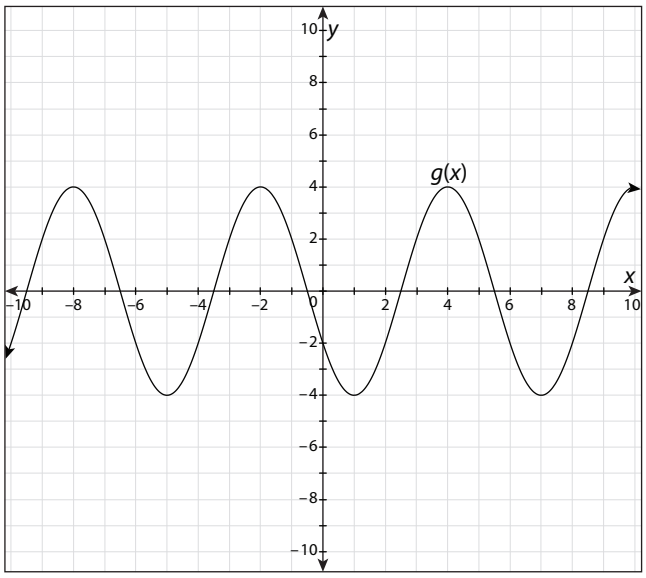
d.



continued

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12. Which function does the graph represent?



a. $g(x) = -4 \bullet \cos(x - 2)$

b. $g(x) = -4 \bullet \cos\left(x - \frac{\pi}{3}\right)$

c. $g(x) = -4 \bullet \cos\left(x + \frac{\pi}{3}\right)$

d. $g(x) = -4 \bullet \cos(x + 2)$

For problems 13 and 14, read the scenarios and use the information given in each to complete the problems.

13. The function that models how to find the focal length $f(x)$ of a contact lens is given by $\frac{1}{f(x)} = \frac{1}{x} + \frac{1}{0.1 - 10x}$, in which x is the distance from the surface of the contact lens to the object being viewed. How would you find the inverse function $f^{-1}(x)$, and what restrictions exist for x in this real-world situation?

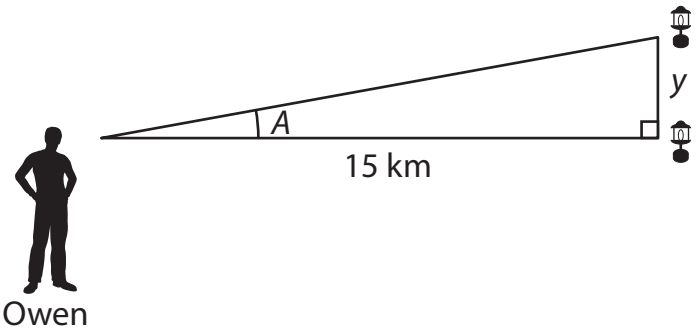
14. The population of native freshwater fish in a river has increased as the concentration of synthetic chemicals in the river has decreased over the past year. The concentration of these chemicals, $f(x)$, is given by the exponential function $f(x) = 10^{-x}$, where x is the number of years since the synthetic chemical began decreasing. Write the inverse function and describe what $f^{-1}(x)$ and x represent in this inverse function.

continued

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Read the scenario and use the diagram to complete problem 15.

Owen's house is on the shore of a lake. A few private islands are scattered throughout the lake. One night, Owen notices two green dock lights have been installed on one of the islands. The next night, there only appears to be one light. Owen takes note of this strange phenomenon each night, and realizes he can see the two green lights from his house when the air is very dry. However, when the air is humid, the two lights appear to be one light. The diagram that follows represents Owen's view of the two green lights on a dry night. In the diagram, y represents the distance between the two lights.



The function $A(n) = 1.22 \bullet \left[\frac{W}{n} \div D \right]$ gives the smallest angle $A(n)$ in radians between Owen's lines of sight and each of the two green lights when the two lights can be seen separately. The dimensionless quantity n is a measure of the humidity of the air; when it is very dry and both lights can be seen, $n \approx 1.35$. The wavelength of the green light, W, equals about $5 \bullet 10^{-7}$ meters. D is the diameter of the pupil of Owen's eye, which is 5 millimeters or $5 \bullet 10^{-3}$ meters.

15. Explain how Owen can use the sine and cosine of angle $A(n)$ to find the distance between the two lights, y. Then, explain how an increase in the humidity n in the argument of $\sin A(n)$ changes the magnitude of the angle $A(n)$. (*Note*: For values of $A(n) < 0.01$ radian, $\sin A(n) \approx A$ and $\cos A(n) \approx 1$.)

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Answer Key

Lesson 1: Inverses of Functions

Pre-Assessment, p. U4A-1

1. a
2. c
3. a
4. d
5. b

Warm-Up 4A.1.1, p. U4A-5

1. $x = \dfrac{8 \pm \sqrt{64 + 4A}}{2}$
2. This is a real-world problem, so the dimensions in feet have to be positive. The domain of x is given by $(8, +\infty)$.
3. This is a real-world problem, so the area has to be positive. The domain for A is given by $(0, +\infty)$.
4. Answers will vary, but all answers should have values of x greater than 8 feet. For example, if x is 40 feet, $x - 8$ would be 32 feet, and the area would be 1,280 square feet. A dance floor with a length of 40 feet and a width of 32 feet could reasonably be found in a large dance studio or hotel ballroom.
5. Answers will vary. Sample answer: For an arbitrarily chosen value of $A = 200$, x is approximately equal to 18.7 feet, and $x - 8$ is approximately equal to 10.7 feet. The area would be $A \approx 18.7(10.7)$, which would equal 200.09 square feet. This is very close to our choice of 200 square feet as a reasonable value for A.

Practice 4A.1.1: Determining Inverses of Quadratic Functions, p. U4A-35

1. $f^{-1}(x) = \pm\dfrac{\sqrt{3(x+5)}}{3}$; $(-\infty, 0]$ or $[0, +\infty)$
2. $g^{-1}(x) = \dfrac{7}{2} \pm \dfrac{\sqrt{49 - 4x}}{-2}$; $\left(-\infty, \dfrac{7}{2}\right]$ or $\left[\dfrac{7}{2}, +\infty\right)$
3. $h^{-1}(x) = \dfrac{1}{2} \pm \dfrac{\sqrt{9 + 4x}}{2}$; $\left(-\infty, \dfrac{1}{2}\right]$ or $\left[\dfrac{1}{2}, +\infty\right)$
4. $a^{-1}(x) = \pm\dfrac{\sqrt{2(x+3)}}{4}$; $\left(-\infty, 0\right]$ or $\left[0, +\infty\right)$
5. $b^{-1}(x) = \dfrac{5 \pm \sqrt{25 + 4x}}{2}$; $\left(-\infty, \dfrac{5}{2}\right]$ or $\left[\dfrac{5}{2}, +\infty\right)$
6. $c^{-1}(x) = \dfrac{1}{3} \pm \dfrac{\sqrt{4 + 3x}}{3}$; $\left(-\infty, \dfrac{1}{3}\right]$ or $\left[\dfrac{1}{3}, +\infty\right)$
7. $d^{-1}(x) = \pm\dfrac{1}{2}\sqrt{2(x-1)}$; $\left(-\infty, 0\right]$ or $\left[0, +\infty\right)$
8. The domain of $A^{-1}(x)$ is defined by $x \geq -225$. However, the realistic lower limit for this domain would be 0, so the realistic domain is $(0, +\infty)$. The real-world restricted domain of the function over which it is one-to-one with its inverse is also $(0, +\infty)$.
9. The domain of $K^{-1}(v)$ is defined by $2mv \geq 0 \rightarrow v \geq 0$, so the inverse of the function has the restricted domain $[0, +\infty)$.
10. a. $N^{-1}(y) = 1.7 \pm \dfrac{\sqrt{4225 + 100y}}{50}$
    b. $4225 + 100y \geq 0 \rightarrow y \geq -42.25$
    c. $[1.7, +\infty)$

Warm-Up 4A.1.2, p. U4A-36

1. Since the variable L is under a square root sign and since it is a real-world object, the lower bound of the domain of L is any value greater than 0. The upper bound of the domain at the instant the clock stopped can be found by evaluating the function at $T = 3$ seconds.
2. Yes, the function does have an inverse, because it shows one-to-one correspondence: each domain value of L is associated with a unique function value of T.
3. $T(L) = 2\pi\sqrt{\dfrac{L}{g}} \rightarrow L = 2\pi\sqrt{\dfrac{T^{-1}(L)}{g}} \rightarrow L^2 = \dfrac{4\pi^2 T^{-1}(L)}{g} \rightarrow$

   $gL^2 = 4\pi^2 T^{-1}(L) \rightarrow T^{-1}(L) = \dfrac{gL^2}{4\pi^2}$
4. The mathematical domain of L for the inverse is $(-\infty, +\infty)$, but the real-world restrictions on L range from greater than 0 to the length of a pendulum that has a period of 3 seconds. Therefore, the restricted domain is $(0, 3]$.
5. Both the function and its inverse share the domain that ranges from greater than 0 to the length of a pendulum that has a period of 3 seconds. The real-world domain of the pendulum at the instant it loses power is approximately $(0, 2.23]$.

Practice 4A.1.2: Determining Inverses of Other Functions, p. U4A-63

1. $f(x) = \dfrac{x-2}{x+3}; f^{-1}(x) = \dfrac{3x+2}{1-x}$
2. $f(x) = x(x-1) = x^2 - x;\ f^{-1}(x) = \dfrac{1 \pm \sqrt{1 + 4x}}{2}$
3. $f(x) = \sqrt{3x+1}; f^{-1}(x) = \dfrac{1}{3}(x^2 - 1)$

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4. $f(x)=\frac{1}{2}|5-2x|; f^{-1}(x)=\frac{1}{2}(5+|2x|)$
5. $(-\infty,0)\cup(0,1)\cup(1,+\infty)$
6. $(-\infty,0]$ and $[0,+\infty)$
7. $(-\infty,-\sqrt{2}]\cup[-\sqrt{2},0]\cup[0,\sqrt{2}]\cup[\sqrt{2},+\infty)$
8. $W^{-1}(t)=\frac{1}{2}+t\pm\frac{\sqrt{1+4t^2}}{2};\ t>\frac{1}{2}$
9. $v^{-1}(x)=\frac{400-x^2}{10}=4-\frac{x^2}{10};\ x\le 20$
10. $f^{-1}(x) = -x + 25$ and $f^{-1}(x) = x + 25,\ x \ge 0$

Progress Assessment, p. U4A-65

1. c
2. b
3. d
4. d
5. c
6. a
7. d
8. d
9. a
10. b
11. The concentration function is $f(60)=\frac{5(60)}{0.01(60)^2+5}\approx 7.3$ after an hour (60 minutes). The inverse of the concentration function is $f^{-1}(x)=\frac{250}{x}\pm\frac{\sqrt{25-0.2x^2}}{0.02x}$. The inverse is undefined at $x < 11.2$ because this value results in a negative radicand.

Lesson 2: Modeling Logarithmic Functions

Pre-Assessment, p. U4A-69

1. b
2. c
3. a
4. d
5. b

Warm-Up 4A.2.1, p. U4A-74

1. $0.1 = 10^{-1}$
2. The sound intensity of the rock concert is 10^{11} or 100 billion times greater than the threshold of human hearing.
3. $I(x) = 10^x$

Practice 4A.2.1: Logarithmic Functions as Inverses, p. U4A-94

1. $f(x)=10^{\frac{1}{2}(x+3)}-1$
2. $g(x)=3^{\left(\frac{5x+4}{x-2}\right)}$
3. $h(x) = 5^{-16x}$
4. domain: $(6,+\infty)$; range: $(-\infty,+\infty)$
5. domain: $(0,+\infty)$; range: $\left(\frac{1}{3},+\infty\right)$
6. domain: $(-\infty,2)$; range: $(6,+\infty)$
7. domain: $\left(\frac{1}{2},+\infty\right)$; range: $(-\infty,+\infty)$
8. The second sound is 4 times louder than the first. The second sound is 10,000 times more intense than the first.
9. The range of concentrations is from $10^{-3.45}$ to $10^{-3.55}$ for a concentration of about 0.000355 to 0.000282 units of mass per unit of volume of acid bath.
10. $\left(\frac{1}{6}\right)^{10}$; $C(n)=\left(\frac{1}{6}\right)^n = 6^{-n}$; $-n = \log_6 C(n)$ or $C^{-1}(n) = -\log_6 n$

Warm-Up 4A.2.2, p. U4A-96

1. $\log_{10} L(v)\approx\frac{1}{2}\log_{10}(9\bullet 10^{16}-v^2)-8.5$
2. The length of the object when $v = 10^3$ m/s is $L\approx 10^0\approx 1$.
3. In other words, at a speed of 1,000 meters per second, there is no change in length of this object for all practical purposes.

Practice 4A.2.2: Common Logarithms, p. U4A-112

1. $f^{-1}(x)=\frac{\log\left(\frac{x}{3}\right)}{\log 2}$
2. $g^{-1}(x)=\frac{\log\left(\frac{x}{4}\right)}{2\bullet\log 3}$
3. $h^{-1}(x)=-\frac{\log 2x}{\log 5}$
4. $a(x) = \log_4 x^2$
5. $b(x)=\frac{1}{\log_5 x^5}$
6. $c(x) = \log_3 x^2 - 4$
7. $d(x)=\frac{\log_4 x}{\log_3(x-1)}$
8. $R^{-1}(y)=\frac{\log\left(\frac{y}{3\bullet 10^9}\right)}{\log 2}$; domain: [0, 5]
9. $P^{-1}(n)=\frac{\log\left(\frac{n}{4}\right)}{\log 2.15}$
10. $C^{-1}(y) = 8 - \log y$

Warm-Up 4A.2.3, p. U4A-114

1. $a^1=\frac{4.437+1}{2}\approx 2.7185$
2. The value of a differs from e by about 0.00022.
3. The value for $r(1)$ might be an approximation, since e is an irrational number that is rounded by calculator algorithms. This would introduce error into the calculation, which uses a value of a that is given to four decimal places.

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Practice 4A.2.3: Natural Logarithms, p. U4A-134

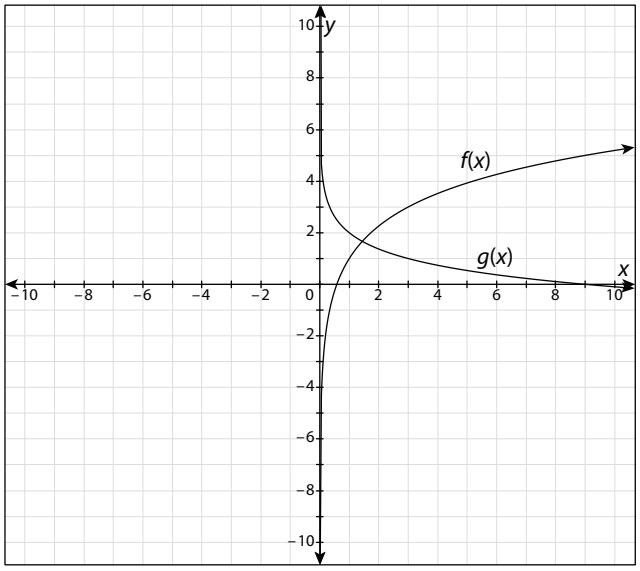
1. $f(x)=\dfrac{2 \bullet \ln 4x}{\ln 3}$
2. $g(x)=1-\dfrac{\ln(x-2)}{\ln 10}$
3. $f(x)=\dfrac{5 \bullet \ln(4-x)-3 \bullet \ln(x-2)}{\ln 6}$
4. $g(x)=\left[\dfrac{\ln(x+1)}{\ln 2}\right]^3$
5. $f(2) = 2 \bullet \log 2$; $g(2) = \log 2$; $f(2) > g(2)$
6. $f(2) = 4 \bullet \ln 1 = 0$; $g(2) = 2 \bullet \ln 2$; $g(2) > f(2)$
7. $f(2) = \log_3 (1) = 0$; $g(2) = \ln (1) = 0$; $f(2) = g(2)$
8. $N(3) = 3.75$, which rounds to 4 whole coins
9. $t = \ln 2 \approx 0.7$, or after about 0.7 seconds
10. $-\ln 1 + \ln 2 = 3.83 \bullet 10^{-12} \bullet t$; $t \approx 1.81 \bullet 10^{11}$ seconds, or about 5,740 years

Warm-Up 4A.2.4, p. U4A-136

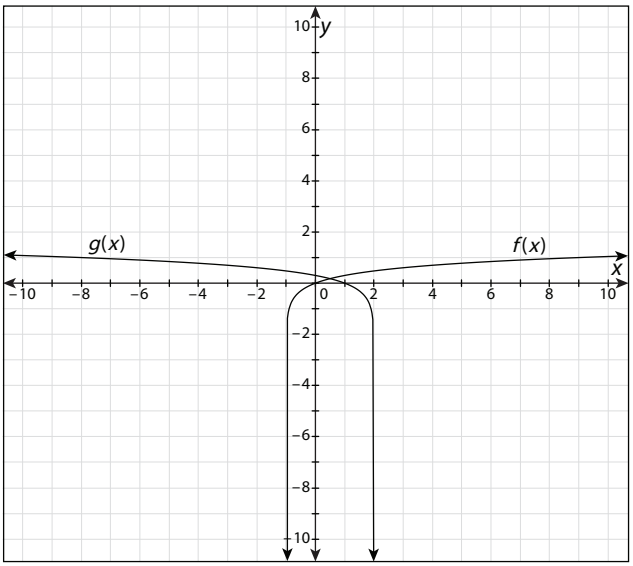
1. (0, 1)
2. $(1,+\infty)$
3. $f(x) = g(x)$ at $x = 1$
4. Both functions have a range of $(-\infty,+\infty)$.

Practice 4A.2.4: Graphing Logarithmic Functions, p. U4A-159

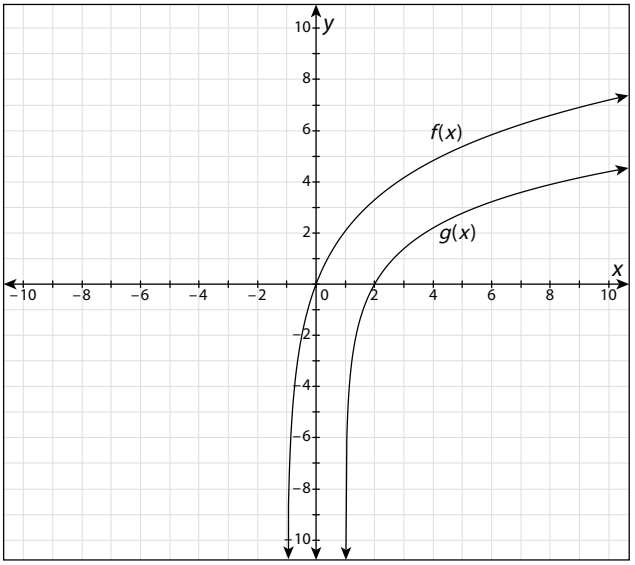
1. solution: $\left(\sqrt[3]{3},\dfrac{5}{3}\right)$



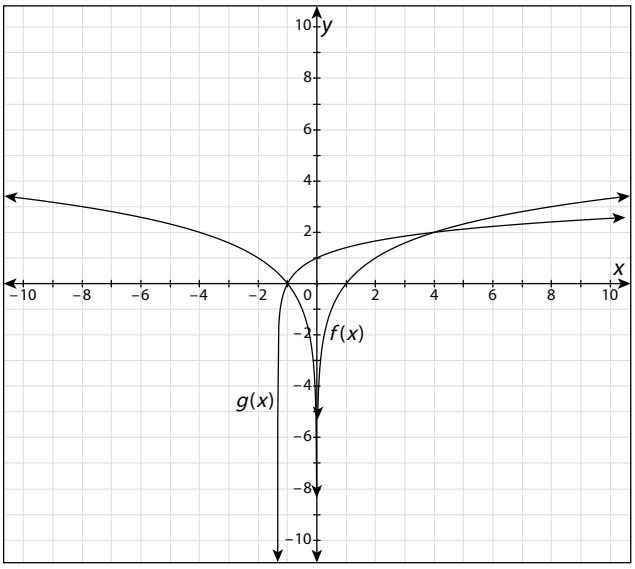
2. solution: (0.5, log 1.5)



3. no solution

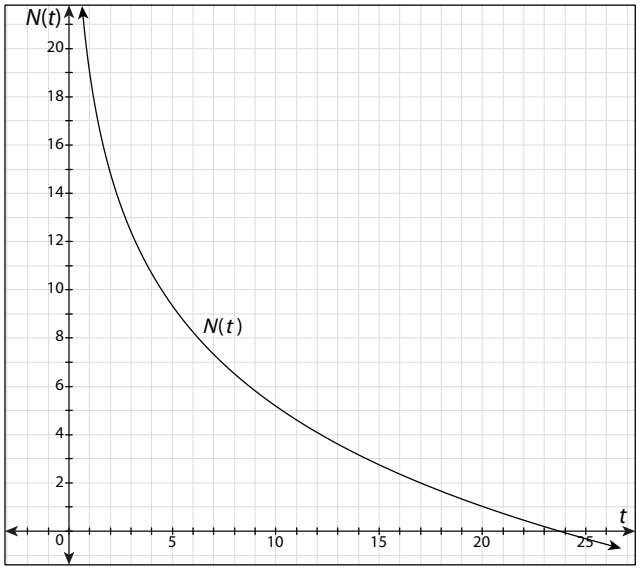


4. solutions: (4, 2) and (–1, 0)

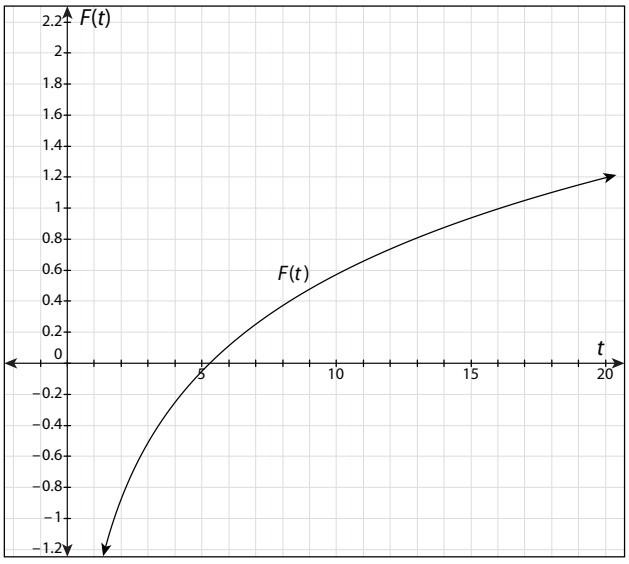


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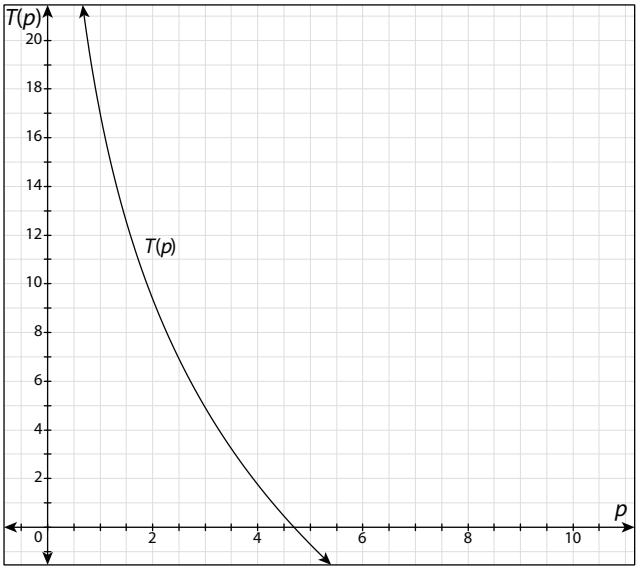
5. The domain of all three functions is $(0,+\infty)$; the range of all three functions is $(-\infty,+\infty)$.
6. The domain of $f(x)$ is $(1,+\infty)$; the domain of $g(x)$ is $(2,+\infty)$; the domain of $h(x)$ is $(-1,+\infty)$. The range of all three functions is $(-\infty,+\infty)$. The restricted domain over which all three functions are defined is $(2,+\infty)$.
7. The domain of $f(x)$ is $(-4,+\infty)$; the domain of $g(x)$ is $(-1,+\infty)$; the domain of $h(x)$ is $(-2,+\infty)$. The range of all three functions is $(-\infty,+\infty)$. The restricted domain over which all three functions are defined is $(-1,+\infty)$.
8. If $N(2) = 19 - 6 \cdot \ln 2 \approx 14.8$ and if $N(20) = 19 - 6 \cdot \ln 20 \approx 1.03$, then the range is [1.03, 14.8]. However, N is the number of cartridges, which is a nonzero positive whole number. Therefore, the range could be defined as (1, 15) to include the results at the bounds of the domain.



9. The minimum time elapsed after which a failure is predicted to occur is $t \approx 16.1$.



10. The domain is [0, 5) since $T(s)$ is negative for a value of s between 4 and 5 shots. The range is (0, 17) since the problem states the time as being *within* 17 minutes of the overtime period.

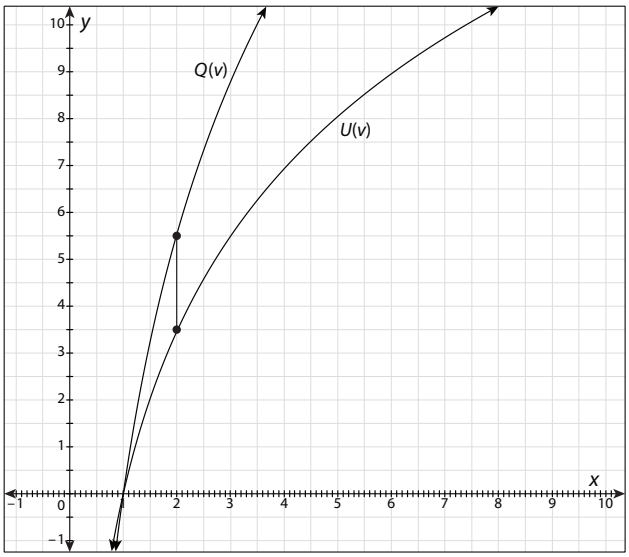


Warm-Up 4A.2.5, p. U4A-161

1. As the coefficient of the power of e decreases, the growth of the number of users decreases for any given year t. For example, the number of users of the social media website is greater than the number of users of the smartphone after 3 years, because $f(3) = e^3 - 1$ and $g(3) = e^{0.5 \cdot 3} - 1$. $f(3) > g(3)$ because $e^3 > e^{1.5}$.
2. For $f(t)$: $\ln (n + 1) = f^{-1}(n)$; for $g(t)$: $\ln (n + 1)^2 = g^{-1}(n)$; for $h(t)$: $\ln (n + 1)^{10} = h^{-1}(n)$.
3. The slower the growth of users of a technology, the larger the argument of the logarithmic function becomes. For example, for the social media website, the argument of the natural logarithm is $t + 1$, but for the tablet, the argument is $(t + 1)^{10}$.
4. Sample answer: The graphs depict the changing number of users per year for each technology. The inverse function is the number of years needed to see an increase in growth of a fixed number of users. Therefore, the logarithmic model implies that it will take more years as indicated by the inverse functions to see the same amount of user growth for a technology. For example, if the number of years elapsed is $n = 2$, then $\ln (3) = f^{-1}(2)$ for the social media website and $\ln (3)^2 = g^{-1}(2)$ for the smartphone. $g^{-1}(2) > f^{-1}(2)$ because $\ln 9 > \ln 3$, which implies that it will take the smartphone about twice as many years as the social-media website to realize the same increase in the number of users.

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Practice 4A.2.5: Interpreting Logarithmic Models, p. U4A-176

1. $A(1) = -18.5 + 30.5 \bullet \ln(1) = -18.5 + 0 = -18.5$, which does not have a physical meaning in the context of the problem because acreage is not a negative quantity.
2. domain: $(\approx 1.8, 10]$
3. range: $(0, \approx 52]$
4. The pH in a less-acidic juice has a concentration of acid-producing substances that is $0.5 \bullet c$, where c is the concentration of the original juice, which has a pH of 3.25. This gives a pH of the lower-acid juice of about 3.55, which is less acidic by about 0.3 on the pH scale.
5. 

6. $\Delta W = V(2) - U(2) = \ln 8$
7. Sample answers: $B = \frac{1}{10{,}000{,}000}$ and $H = \frac{1}{1000}$
8. $\log B = -7$; $\log H = -3$
9. Sample answers: $104 \bullet B = H$; $\log H - \log B = 4$; the beagle's sense of smell is 10,000 times more powerful than a human's, or 4 orders of magnitude.
10. For 6 trumpets, $90 = 10 \bullet \log\left(\frac{I}{I_0}\right)$; for 9 trumpets, replace I with $1.5 \bullet I$, which results in $D(9) = 10 \bullet \log\left(\frac{1.5I}{I_0}\right)$ ≈ 91.76 dB. 91.76 dB is not 50 percent greater than 90 dB, so the math club president is correct.

Progress Assessment, p. U4A-179

1. d
2. c
3. b
4. a
5. c
6. c
7. b
8. c
9. c
10. a
11. a. This step is correct. Correct operation or rule applied: the quotient-difference rule for logarithms.
    b. This step is incorrect. Incorrect operation or rule applied: $\log(a - b) = \log a - \log b$; this is incorrect because the rules of logarithms do not include a rule stating that $\log(a - b) = \log a - \log b$.
    c. This step is correct. Correct operation or rule applied: writing an inverse for a function involving exponential functions and logarithms.
    d. This step is correct. Correct operation or rule applied: division by 15.

Lesson 3: Modeling Trigonometric Functions

Pre-Assessment, p. U4A-183

1. c
2. a
3. d
4. a
5. c

Warm-Up 4A.3.1, p. U4A-187

1. The angles opposite the legs of the right triangle sum to 90°. Refer to the completed table in the response to problem 3.
2. Sides a, b, and c are related to one another by the Pythagorean Theorem, $a^2 + b^2 = c^2$. Refer to the completed table in the response to problem 3.
3. In the 30°–60°–90° and 60°–30°–90 right triangles, the sine ratios for the 30° angles are the same and the sine ratios for the 60° angles are the same, even though the lengths of the corresponding sides differ in the two triangles. Completed table:

$m\angle A$	$m\angle B$	$m\angle C$	a	b	c	$\sin A$	$\sin B$
30°	60°	90°	1	$\sqrt{3}$	2	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
45°	45°	90°	1	1	$\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	30°	90°	1	$\frac{\sqrt{3}}{3}$	$\frac{2\sqrt{3}}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

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Practice 4A.3.1: Graphing the Sine Function, p. U4A-227

1. $f(x)$
2. $g(x)$
3. The functions appear to be out of phase by $\frac{1}{8}$ or 0.125 radian.
4. $f(x) = \sin x - 1$
5. The first maximum for $x > 0$ occurs at $x = \frac{\pi}{6}$; the point is $\left(\frac{\pi}{6}, 4\right)$.
6. The first zero for $x > 0$ occurs at $x = 2\pi$; the point is $(2\pi, 0)$.
7. The first minimum for $x > 0$ occurs at about $x = 562°$; the point is $(562°, -1)$.
8. $\frac{\pi}{375}$ radians
9. The period of the voltage is $\frac{1}{55}$ second, so 55 cycles of the voltage occur per second, which is 55 maximum values and 55 minimum values, or 110 maximum *or* minimum values.
10. a. $\sin a < \sin b$
    b. $\sin a = \sin b$
    c. $\sin A = \frac{\sqrt{5}}{5}$ and $\sin B = \frac{2\sqrt{5}}{5}$, or $m\angle A \approx 27°$ and $m\angle B \approx 63°$

Warm-Up 4A.3.2, p. U4A-230

1. The angles opposite the legs of the right triangle sum to 90°. Refer to the completed table in the response to problem 3.
2. Sides a, b, and c are related to one another by the Pythagorean Theorem, $a^2 + b^2 = c^2$. Refer to the completed table in the response to problem 3.
3. In the 30°–60°–90° and 60°–30°–90° right triangles, the cosine ratios for the 30° angles are the same and the cosine ratios for the 60° angles are the same, even though the lengths of the corresponding sides differ in the two triangles. Completed table:

$m\angle A$	$m\angle B$	$m\angle C$	a	b	c	$\cos A$	$\cos B$
30°	60°	90°	1	$\sqrt{3}$	2	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45°	45°	90°	1	1	$\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	30°	90°	1	$\frac{\sqrt{3}}{3}$	$\frac{2\sqrt{3}}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

Practice 4A.3.2: Graphing the Cosine Function, p. U4A-265

1. $f(x)$
2. $g(x)$
3. The functions are not out of phase, because they have different periods with no constant term added to the variable term in the cosine argument.
4. $f(x) = 5 - \cos 5x$
5. The first maximum for $x > 0$ occurs at $x = \frac{\pi}{2}$; the point is $\left(\frac{\pi}{2}, 3\right)$.
6. The first zero for $x > 0$ occurs at $x = \pi$; the point is $(\pi, 0)$.
7. The first minimum for $x > 0$ occurs at $x = 270°$; the point is $(270°, -1)$.
8. period: $\frac{2}{3} \cdot 10^{-4}$ seconds; frequency: $\frac{3}{2} \cdot 10^{4}$ cycles per second
9. a. $v_0 \cdot \cos A_0 = 30$ feet per second
    b. $0 < \cos A < 1$
    c. $D(5) = 5 \cdot v_0 \cdot \cos 60° = 150$; $v_0 = \frac{30}{\cos 60°} = 60$ feet per second
    d. if $A_0 = 0$, the football moves horizontally during the 5 seconds, which is unrealistic; if $A_0 = 90°$, the football moves vertically during the 5 seconds, which is unrealistic because gravity would affect its motion.
10. a. The magnetic flux $M = 0$ when angle $C = 90°$.
    b. and c. The magnetic flux has a maximum value of $A \cdot B$ when $C = 0°$ and $B > 0$, and again when $C = 180°$ and $B < 0$.

    Sample answers: (a) if A is half as large, B is twice as large, and C is 0°, then $M = (0.5A)(2B)(1) = AB$; (b) if A is four times as large, B is the opposite of one fourth as large, and C is 180°, then $M = (4A)(-0.25B)(-1) = AB$; and (c) if A is six times as large, B is one sixth as large, and C is 360°, then $M = (6A)(0.1\overline{6}B)(1) = AB$.

Progress Assessment, p. U4A-268

1. b
2. b
3. c
4. d
5. a
6. c
7. b
8. d
9. d
10. a

11. Substitute $x + B$ for x into the given function to result in $y(x) = A\cos\left[\frac{2\pi}{B} \cdot (x + B)\right] = A\cos\left(\frac{2\pi x}{B} + 2\pi\right)$. This shows a phase shift of 2π, which would have the same value as y at $x = 0$.

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Unit Assessment

p. U4A-271

1. c
2. a
3. d
4. a
5. a
6. b
7. c
8. c
9. b
10. c
11. c
12. b

13. Find the inverse by solving the equation for $f(x)$, then switching the variables x and $f(x)$ and changing $f(x)$ to $f^{-1}(x)$. Restrictions: The distance x should be positive since this is a real-world situation.
14. The inverse is $f^{-1}(x) = -\log x$. In the inverse function, $f^{-1}(x)$ represents the number of years and x represents the concentration of the chemical.
15. The sine of angle $A(n)$ can be written as $\sin[A(n)] = \frac{y}{\sqrt{15^2 + y^2}}$ and $\cos[A(n)] = \frac{15}{\sqrt{15^2 + y^2}}$, so $\frac{\sin[A(n)]}{\cos[A(n)]} = \frac{y}{15}$. The angle $A(n)$ can also be calculated using the formula $A(n) = 1.22 \bullet \left(\frac{W}{n} \div D\right) = 1.22 \bullet \left(\frac{5 \bullet 10^{-7}}{1.35} \div 5 \bullet 10^{-3}\right) \approx 10^{-4}$. For this small angle, $\frac{\sin[A(n)]}{\cos[A(n)]} \approx \sin[A(n)] = \frac{y}{15}$ and $\sin A(n) \approx A(n)$. Therefore, $A(n) \approx \frac{y}{15}$ or $y \approx 10^{-4} \bullet 15 \approx 10^{-4} \bullet 15 \bullet 10^{3} \approx 1.5$, so the lights are about 1.5 meters apart. If n is increased from 1.35, it will make the sine of angle $A(n)$ larger, which means the minimum angle at which Owen can see two separate lights also increases. Therefore, Owen would have to be closer to the private island in order to see two separate lights.

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Student Book Answer Key

Lesson 1: Inverses of Functions

Practice 4A.1.1: Determining Inverses of Quadratic Functions, p. U4A-24

1. $f^{-1}(x)=\pm\frac{\sqrt{5(x+3)}}{5}$; $[-3,+\infty)$
2. $g^{-1}(x)=\frac{7}{2}\pm\frac{\sqrt{49+4x}}{2}$; $\left[-\frac{49}{4},+\infty\right)$
3. $h^{-1}(x)=-\frac{1}{2}\pm\frac{\sqrt{9+4x}}{2}$; $\left[-\frac{9}{4},+\infty\right)$
4. $a^{-1}(x)=\pm\frac{\sqrt{3(x+8)}}{3}$; $(-\infty,0]$ or $[0,+\infty)$
5. $b^{-1}(x)=\frac{-5\pm\sqrt{25-4x}}{-2}$; $(-\infty,2.5]$ or $[2.5,+\infty)$
6. $c^{-1}(x)=1\pm\sqrt{4+x}$; $(-\infty,1]$ or $[1,+\infty)$
7. $d^{-1}(x)=\pm\sqrt{x-2}$; $(-\infty,0]$ or $[0,+\infty)$
8. The domain of $A^{-1}(x)$ is defined by $x\geq-\frac{9}{4}$. However, the realistic lower limit for this domain would be 0, so the realistic domain is $(0,+\infty)$. The real-world restricted domain of the function over which it is one-to-one with its inverse is also $(0,+\infty)$.
9. The real-world domain of this function is $[0,+\infty)$ because time is positive. However, the function values have to be negative because the life preserver is falling toward the surface of the river away from the helicopter. This also preserves the positive value of time. The inverse function does not exist in this specific situation in which time is positive, gravity is negative, and the height is negative.
10. $P^{-1}(y)=\frac{3}{2}\pm\frac{\sqrt{2y+3}}{2}$; $2y+3\geq0\rightarrow y\geq-\frac{3}{2}$; $[1,+\infty)$

Practice 4A.1.2: Determining Inverses of Other Functions, p. U4A-43

1. $f(x)=\frac{2(x+1)}{2x-1}; f^{-1}(x)=\frac{x+2}{2(x-1)}$
2. $f(x)=(x-2)(x+2)=x^2-4;\ f^{-1}(x)=\pm\sqrt{x+4}$
3. $f(x)=\sqrt[3]{x^2-3}; f^{-1}(x)=\pm\sqrt{x^3+3}$
4. $f(x)=2|x+3|$; $f^{-1}(x)=\frac{1}{2}(|x|-6)$
5. $(-\infty,0)\cap\left(0,\frac{1}{2}\right)\cap\left(\frac{1}{2},+\infty\right)$
6. $(-\infty,+\infty)$
7. $\left(-\infty,-\sqrt{2}\right],\left[-\sqrt{2},0\right],\left[0,\sqrt{2}\right]$ and $\left[\sqrt{2},+\infty\right)$
8. $F^{-1}(t)=t-1\pm\sqrt{t^2+1}$
9. $v^{-1}(x)=\frac{x^2}{10}-10$; $x>10$
10. $h^{-1}(x)=5+|x|$; $[-3, 3]$

Lesson 2: Modeling Logarithmic Functions

Practice 4A.2.1: Logarithmic Functions as Inverses, p. U4A-60

1. $f(x)=\frac{1}{2}(1+10^{\frac{x-6}{4}})$
2. $g(x)=5^{\left(\frac{3-6x}{x+4}\right)}$
3. $h(x)=4^{9x}$
4. domain: $(-\infty,8)$; range: $(+\infty,-\infty)$
5. domain: $\left(0,\frac{1}{2}\right)\cup\left(\frac{1}{2},+\infty\right)$; range: $(-\infty,0)\cup(0,+\infty)$
6. domain: $(-1,+\infty)$; range: $(-\infty,+\infty)$
7. domain: $(-\infty,5)$; range: $(-\infty,+\infty)$
8. The intensity of the cafeteria sound is 10^{40} times greater than that of the whisper.
9. The concentration of stomach acid before taking the antacid is about 0.00178 units of mass per unit of volume. The concentration after taking the antacid is about 0.00007 units of mass per unit of volume. The stomach acid concentration was reduced by about 25 times.
10. $\left(\frac{1}{2}\right)^{12}$; $C(n)=\left(\frac{1}{2}\right)^{n}=2^{-n}$; $C^{-1}(n)=-\log_2 n$

Practice 4A.2.2: Common Logarithms,p. U4A-69

1. $f^{-1}(x)=\frac{\log\left(\frac{x}{4}\right)}{\log3}$
2. $g^{-1}(x)=\frac{\log\left(\frac{x}{2}\right)}{4\bullet\log3}$
3. $h^{-1}(x)=-\frac{\log5x}{\log4}$
4. $a(x)=8\bullet\log_2 x$
5. $b(x)=\frac{1}{7\bullet\log_7 x}$
6. $c(x)=4+\log_6 x$

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7. $d(x)=\dfrac{\log_2(x-2)}{\log_3 x}$

8. $N^{-1}(y)=\dfrac{\log\left(\dfrac{y}{92}\right)}{\log 1.5}$

9. $C^{-1}(y)=\dfrac{\log\left(\dfrac{y}{1{,}100}\right)}{\log 9-1}$

10. $G^{-1}(y)=\log y$

Practice 4A.2.3: Natural Logarithms, p. U4A-81

1. $f(x)=-\dfrac{3\ln 2x}{\ln 4}$

2. $g(x)=\dfrac{\ln(2x-1)+\ln 10}{\ln 10}$

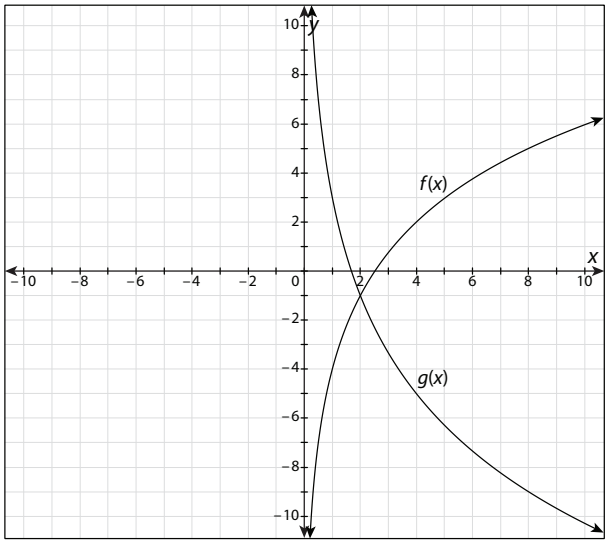
3. $f(x)=\dfrac{6\bullet\ln 3\bullet\ln(x+4)+\ln 5\bullet\ln(2-x)}{\ln 3\bullet\ln 5}$

4. $g(x)=\left[\dfrac{\ln(x-1)}{\ln 3}\right]^2$

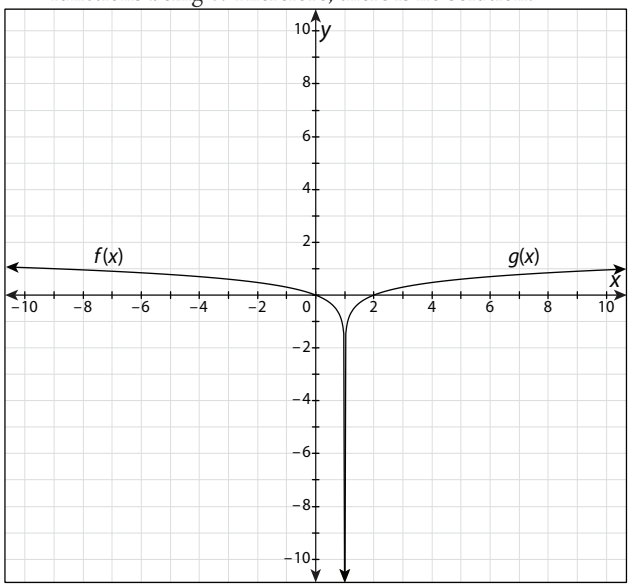
5. $f(5) = \log 5$; $g(5) = 0.1 \bullet \log 5$; $f(5) > g(5)$
6. $f(5) = 2 \bullet \ln 1 = 0$; $g(x) = 4 \bullet \ln 5$; $g(5) > f(5)$
7. $f(5) = \log_2(1) = 0$; $g(5) = \ln(1) = 0$; $f(5) = g(5)$
8. $N(2) = 2.2$, or about 2 cubes would be left (theoretically)
9. $t \approx 4$ days
10. $\ln 0.121 - \ln 0.231 = -1.21 \bullet 10^{-4} \bullet t$; $t \approx 5340$ years

Practice 4A.2.4: Graphing Logarithmic Functions, p. U4A-100

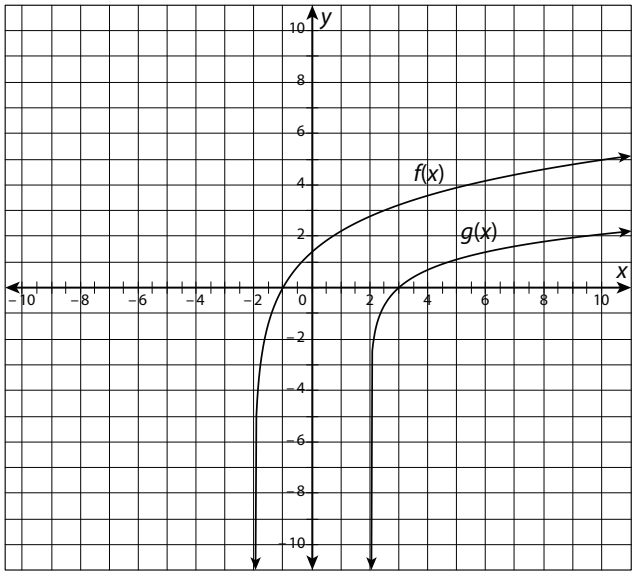
1. solution: (2, –1)



2. The solution $x = 1$ results in the argument of both functions being 0. Therefore, there is no solution.

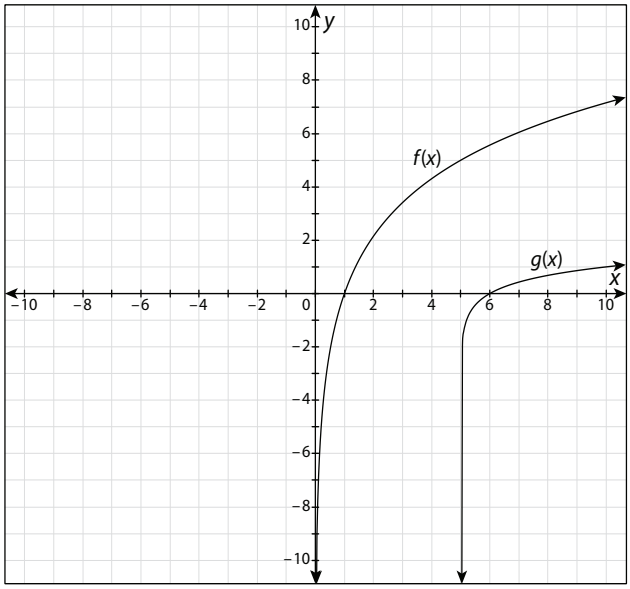


3. no solution

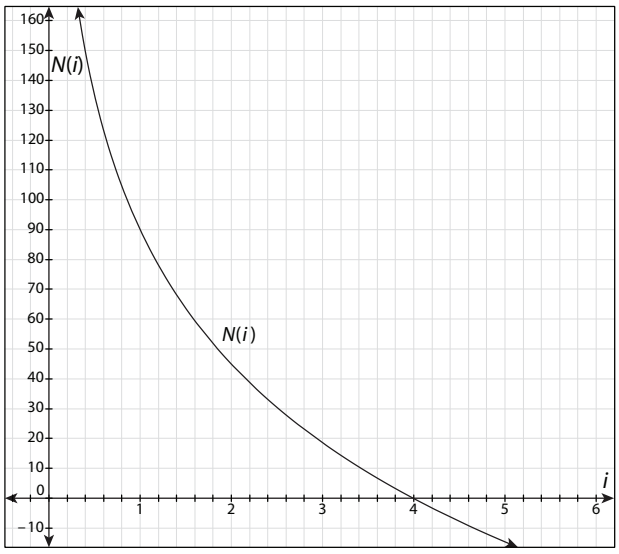


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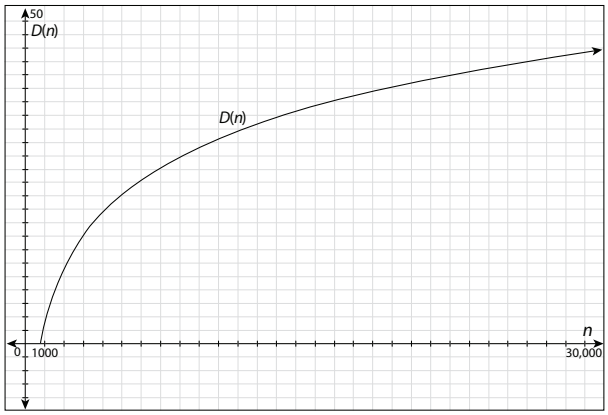
4. The only real solution occurs at about $x = -1.45$. However, this x-value produces a negative argument for both $f(x)$ and $g(x)$. Therefore, there is no solution, as the graph shows.



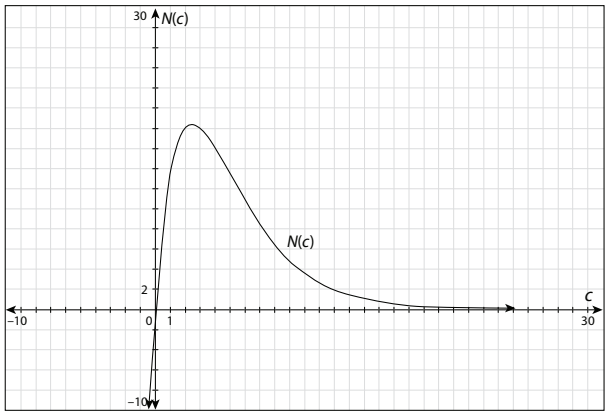
5. The domain of all three functions is $(0,+\infty)$; the range of all three functions is $(-\infty,+\infty)$.
6. The domain of $f(x)$ is $(-1,+\infty)$; the domain of $g(x)$ is $(-\infty,1)$; the domain of $h(x)$ is $(1,+\infty)$. The range of all three functions is $(-\infty,+\infty)$. There is no common domain for all three functions; however, $f(x)$ and $g(x)$ have a common restricted domain of $(-1, 1)$ and $f(x)$ and $g(x)$ have a common restricted domain of $(1, \infty)$.
7. The domain of $f(x)$ is $(3,+\infty)$; the domain of $g(x)$ is $(1,+\infty)$; the domain of $h(x)$ is $(-3,+\infty)$. The range of all three functions is $(-\infty,+\infty)$. The common restricted domain for all three functions is $(3,+\infty)$.
8. Both N and i are positive whole numbers so the domain and range must be positive whole numbers. The domain and range can be used to define the scale on the N and I axes for the graph of the function.



9. At $n = 1000$ cases produced, $D \approx 3$ defective cases; at $n = 2000$ cases produced, $D \approx 11$ defective cases. The increase in defects from $n = 1000$ to $n = 2000$ is about 367% or $3\frac{2}{3}$. At $n = 3000$ cases produced, the number of defective cases would about 16. The increase in defects from $n = 2000$ to $n = 3000$ is about 145% or $1\frac{5}{11}$. Other such calculations would indicate that the rate of defective cases decreases as the size of the production run increases.



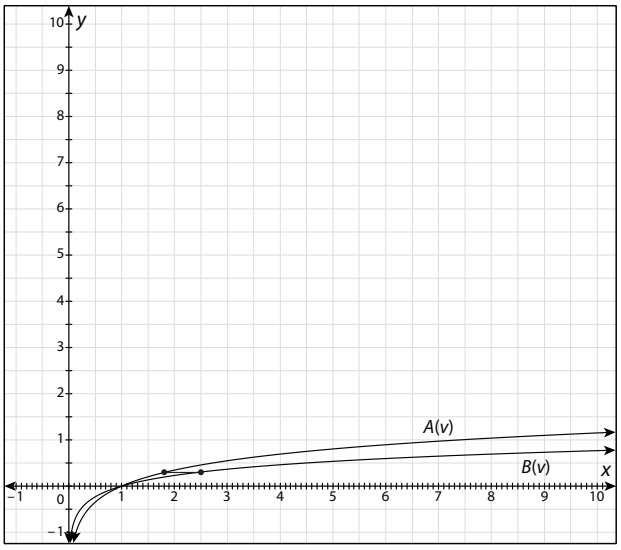
10. The function does not have an explicit inverse, so the best we can do is to rewrite the function in an equivalent form using the rules of logarithms. In this case, an equivalent expression is $\ln(N) = \ln(20c) - 0.4c$. If we could find the inverse, it would reverse the domain and range values in data points on the graph. Therefore, the domain of the inverse function would be $(1, N)$, where N is a real-world production rate of peppers per unit of carbon dioxide concentration. Notice that there is a value of the domain at which the value of N reaches a maximum on the inverse function graph, too. This is the concentration of carbon dioxide at which the most peppers will be produced per unit of carbon dioxide.



Practice 4A.2.5: Interpreting Logarithmic Models, p. U4A-110

1. $A(1) = 475 - 85 \cdot \ln 1 = 475$; this is the number of acres of salt marsh after one month of study.

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2. [1, 60], since t is measured in months and the span of time for the project is 5 years, and since any value of $t < 1$ will result in unrealistic values of $A(t)$.
3. Half of 475 is 237.5, so the function becomes $237.5 = 475 - 85 \bullet \ln t$, which simplifies to $\ln t \approx 2.8$ or $t \approx 16.3$ months.
4. The reduction of acreage by $\frac{1}{2}$ in less than a year and a half seems somewhat excessive. The model's accuracy is questionable.
5. $pH_{before} = 6.15 = -\log c_{before}$, which gives $c_{before} = 10^{-6.15}$; $pH_{after} = 6.9 = -\log c_{after}$, which gives $c_{after} = 10^{-6.9}$; the percentage change in concentration can be defined as $100\left(\frac{c_{after} - c_{before}}{c_{before}}\right)$, or about –82.2, which means that the acidic substances have been reduced by 82.2%.
6. 

7. The horizontal line represents the two volumes at which the same amount of work is done at the two different temperatures.
8. $A(v) = \frac{1}{2} \bullet \ln v_A$, so $0.3 = \frac{1}{2} \bullet \ln v_A$ or $0.6 = \ln v_A$, which gives $v_A = e^{0.6}$; $B(v) = \frac{1}{3} \bullet \ln v_B$, so $0.3 = \frac{1}{3} \bullet \ln v_B$ or $0.9 = \ln v_B$, which gives $v_B = e^{0.9}$.
9. $v_B = e^{0.9} = e^{0.6} \bullet e^{0.3} = v_A \bullet e^{0.3}$
10. The power of e is the work performed at this volume value.

Lesson 3: Modeling Trigonometric Functions

Practice 4A.3.1: Graphing the Sine Function, p. U4A-143

1. $g(x)$
2. $f(x)$
3. The functions appear to be out of phase by 2 radians.
4. $f(x) = 3 + \sin 2x$
5. The first minimum for $x > 0$ occurs at $x = \frac{3\pi}{8}$; the point is $\left(\frac{3\pi}{8}, -3\right)$.
6. The first zero for $x > 0$ occurs at $x = 5\pi$; the point is $(5\pi, 0)$.
7. The first maximum for $x > 0$ occurs at $x = 120°$; the point is $(120°, 1)$.
8. $\frac{\pi}{125}$ radian
9. The period of the current is $\frac{1}{60}$ second, so 60 cycles of the current occur per second, which is 60 maximum values and 60 minimum values, or 120 maximum *or* minimum values.
10. $P(4) = A_P \sin 4c_P$, so $4 \bullet c_P = 2\pi$ or $c_P = \frac{\pi}{2}$, which means that $P(t)$ can be written as $A_P \sin\left(\frac{\pi}{2} \bullet t\right)$.

    $R(20) = A_R \sin 20c_P$, so $20 \bullet c_R = 2\pi$ or $c_R = \frac{\pi}{10}$, which means that $R(t)$ can be written as $A_R \sin\left(\frac{\pi}{10} \bullet t\right)$.

Practice 4A.3.2: Graphing the Cosine Function, p. U4A-171

1. $g(x)$
2. $g(x)$
3. The functions appear to be out of phase by 0.1.
4. $f(x) = \cos(3x) + 3$
5. The first minimum for $x > 0$ occurs at $x = 2\pi$; the point is $(2\pi, -2)$.
6. The first zero for $x > 0$ occurs at $x = \frac{\pi}{12}$; the point is $\left(\frac{\pi}{12}, 0\right)$.
7. The first maximum for $x > 0$ occurs at $x = 60°$; the point is $(60°, 1)$.
8. period: $\frac{1}{3} \bullet 10^{-4}$ seconds; frequency: $3 \bullet 10^4$ cycles per second
9. a. $v_0 \bullet \cos A_0 = 120$ feet per second
   b. $0 < \cos A < 1$
   c. $D(5) = 5 \bullet v_0 \bullet \cos 60° = 600$; $v_0 = \frac{120}{\cos 45°} = 120\sqrt{2}$ feet per second
10. a. If the average power is half of the maximum average power, then $\cos A = 0.5$ which implies that A is 60°.
    b. This also implies that the phase factor between the average current and voltage is 60° *if* the frequencies of the average current and voltage are the same, which they usually are for "household" current delivered by a public utility in the United States at 50 – 60 cycles per second and at an average of 110 – 120 volts.

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Instruction

Goal: To guide students to an understanding of inverse functions, including finding and graphing a function's inverse

Common Core State Standard

F–BF.4 Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example, $f(x) = 2x^3$ or $f(x) = (x + 1)/(x – 1)$ for $x \neq 1$.*

Student Activities Overview and Answer Key

Station 1

Working with groups, students find the inverses of various functions.

Answers

1. c
2. c
3. c
4. b
5. $f(f^{-1}(x)) = 3 - \left(\frac{-2x+6}{2}\right)$
$= 3 + x - 3$
$= x$
6. $f(f^{-1}(x)) = \left((x+2)^{\frac{1}{3}}\right)^3 - 2$
$= x + 2 - 2$
$= x$
7. $f(f^{-1}(x)) = \left(\frac{\frac{1}{2}}{(2x-6)}\right) + 3$
$= \frac{2x-6}{2} + 3$
$= x - 3 + 3$
$= x$

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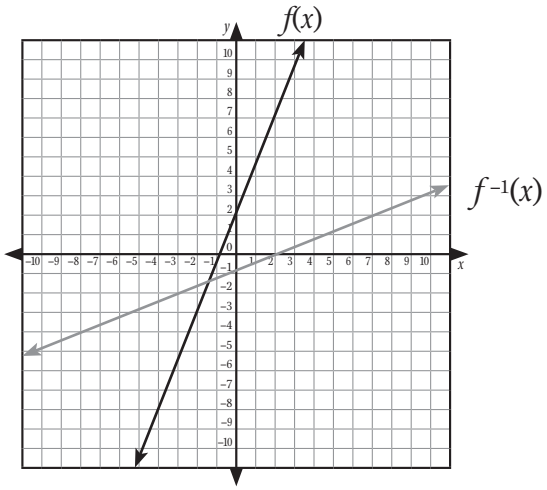
Instruction

Station 2

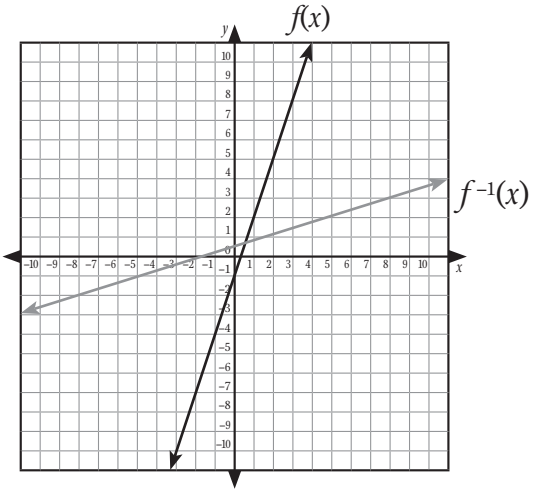
Students find the inverses of linear functions and graph each function with its inverse.

Answers

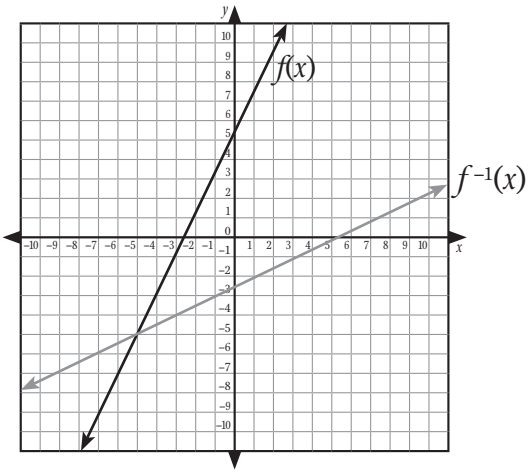
1. $f^{-1}(x)=\frac{2}{5}(x-2)$



2. $f^{-1}(x)=\frac{1}{3}(x+1)$

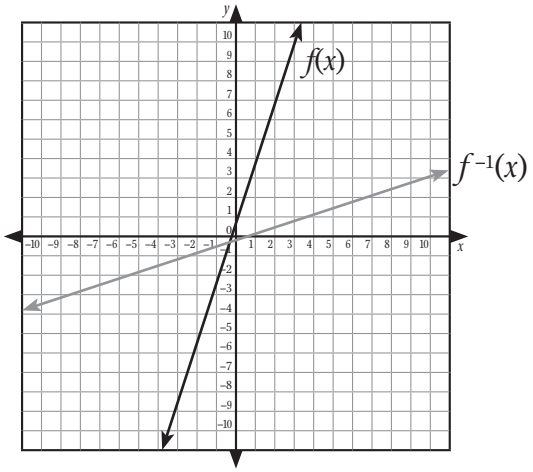


3. $f^{-1}(x)=\frac{1}{2}(x-5)$

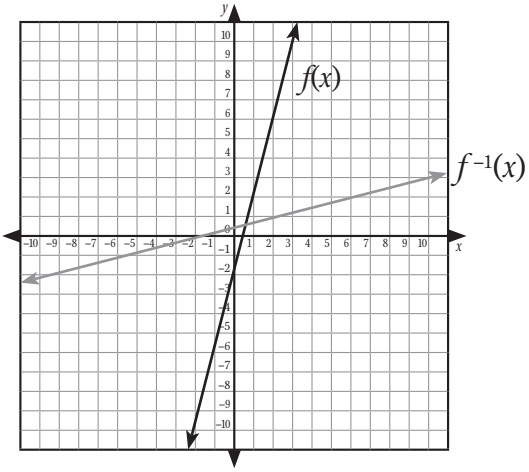


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4. $f^{-1}(x)=\frac{1}{3}x-\frac{1}{6}$



5. $f^{-1}(x)=\frac{1}{4}x+\frac{1}{2}$



6. They are symmetrical over the line $y = x$.

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Instruction

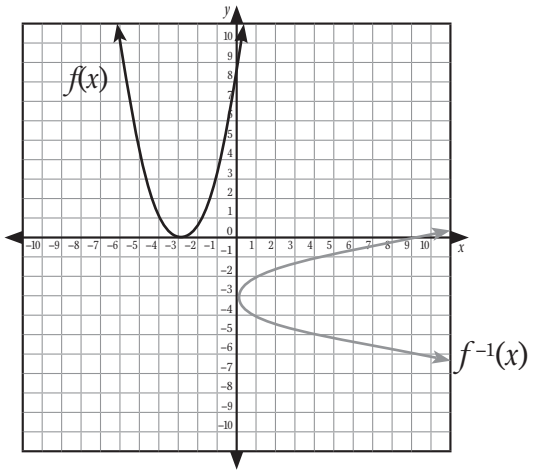
Station 3

Working with groups, students find the inverses of quadratic and power functions and graph each function with its inverse. They also assess whether each original function is one-to-one.

Answers

1. a. $f^{-1}(x)=\pm\sqrt{x}-3$

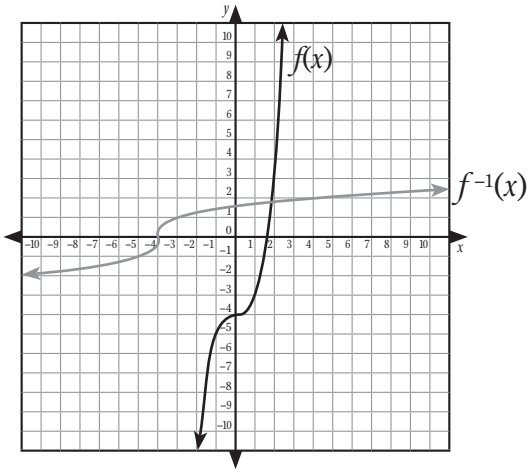
   b.



   c. no

2. a. $f^{-1}(x)=(x+4)^{\frac{1}{3}}$

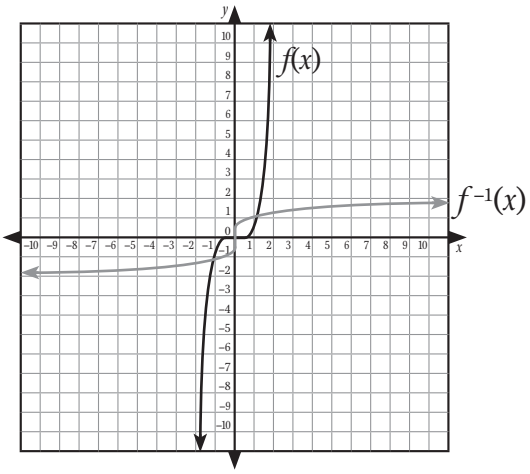
   b.



   c. yes

3. a. $f^{-1}(x)=(2x)^{\frac{1}{5}}$

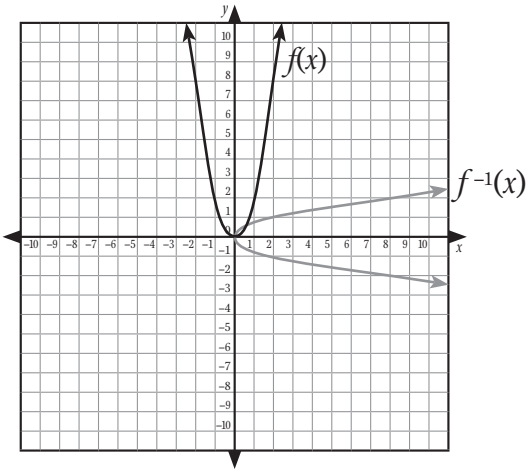
   b.



   c. yes

4. a. $f^{-1}(x)=\pm\dfrac{\sqrt{2x}}{2}$

   b.



   c. no

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Instruction

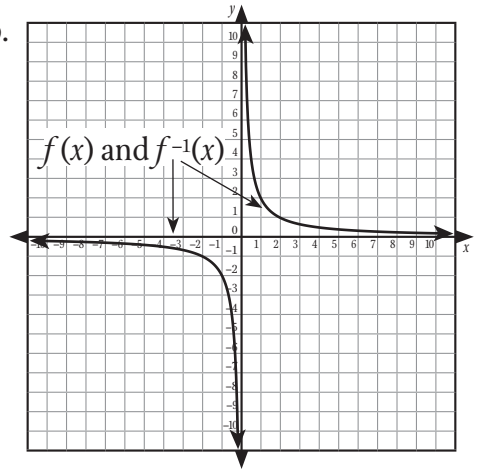
Station 4

Working with groups, students find the inverses of functions in the form $f(x)=\frac{a}{x}$ and graph each function with its inverse. They also assess whether each original function is one-to-one.

Answers

1. a. $f^{-1}(x)=\frac{2}{x}$

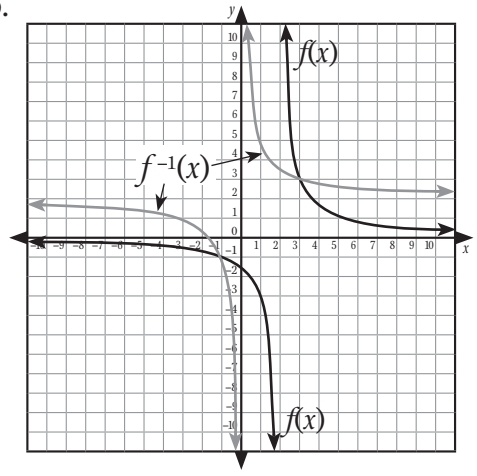
b.



c. yes

2. a. $f^{-1}(x)=\frac{3}{x}+2$

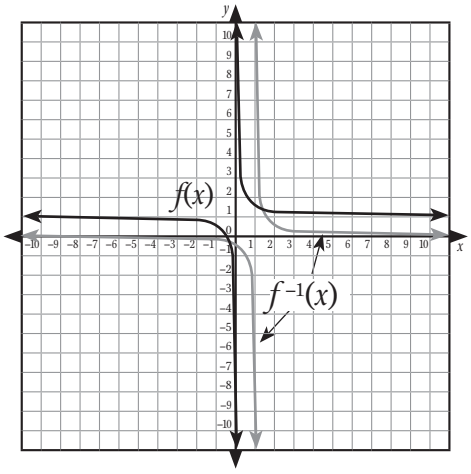
b.



c. yes

3. a. $f^{-1}(x)=\frac{1}{4(x-1)}$

b.



c. yes

Materials List/Setup

No materials needed

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. What is a function?
2. What is a reflection?
3. Why might it be important to know the inverse of a function?
4. What is a one-to-one function?

Think, Pair, Share

Have students jot down their own responses to questions. Pair students from different station groups and have them discuss their answers. Finally, discuss answers as a whole class.

Suggested Appropriate Responses

1. A function is a description of a mathematical operation in which each value in the domain is mapped to exactly one value in the range. The vertical line test can be used to determine if a graph represents a function.
2. A reflection is the graph of the function across an axis of symmetry. Each point on the graph is the same distance from the axis of symmetry as the corresponding point on the reflection.
3. You can use an inverse function to "undo" what a function has done.
4. It is a function in which each output value is mapped to one and only one input value.

Possible Misunderstandings/Mistakes

- Not understanding how a function's inverse works
- Not understanding the steps of finding a function's inverse
- Incorrectly manipulating numbers, variables, or exponents
- Not understanding how to use composition to verify the inverse of a function
- Confusing the inverse with the conjugate or with the reciprocal
- Misunderstanding how whole-number exponents relate to their fraction inverses
- Not understanding the vertical-line test
- Misunderstanding inverses and their relationship to one-to-one functions

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Station 1

Work with your group to complete each problem.

For problems 1–4, find the inverse of each given function. Circle the letter of the correct answer.

1. $f(x)=3x+2$

   a. $f^{-1}(x)=\dfrac{1}{3x-2}$

   b. $f^{-1}(x)=3x+2$

   c. $f^{-1}(x)=\dfrac{x-2}{3}$

2. $f(x)=x^2+3$

   a. $f^{-1}(x)=\dfrac{1}{x^2+3}$

   b. $f^{-1}(x)=3x+2$

   c. $f^{-1}(x)=\pm\sqrt{x-3}$

3. $f(x)=\dfrac{x^4}{2}$

   a. $f^{-1}(x)=2x^{\frac{1}{4}}$

   b. $f^{-1}(x)=\dfrac{x^2}{4}$

   c. $f^{-1}(x)=\pm(2x)^{\frac{1}{4}}$

4. $f(x)=\dfrac{3}{x}$

   a. $f^{-1}(x)=\dfrac{x}{3}$

   b. $f^{-1}(x)=\dfrac{3}{x}$

   c. $f^{-1}(x)=x^3$

continued

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Use the given information to complete problems 5–7.

5. If $f(x)=3-\frac{x}{2}$, prove that $f^{-1}(x)=-2x+6$.

6. If $f(x)=x^3-2$, prove that $f^{-1}(x)=\left(x+2\right)^{\frac{1}{3}}$.

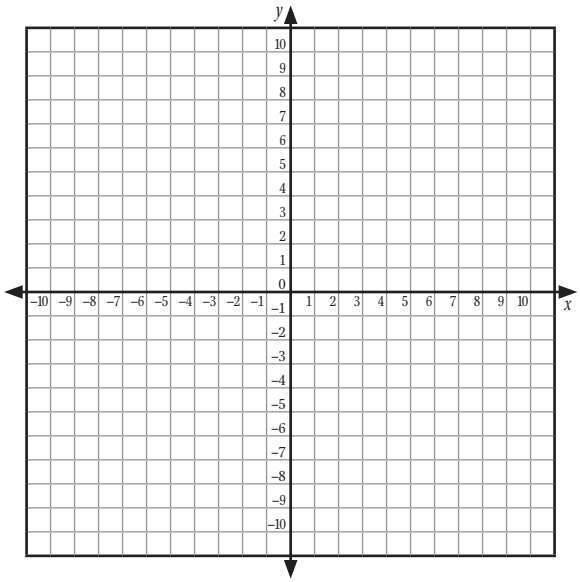
7. If $f(x)=\frac{1}{2x}+3$, prove that $f^{-1}(x)=\frac{1}{2x-6}$.

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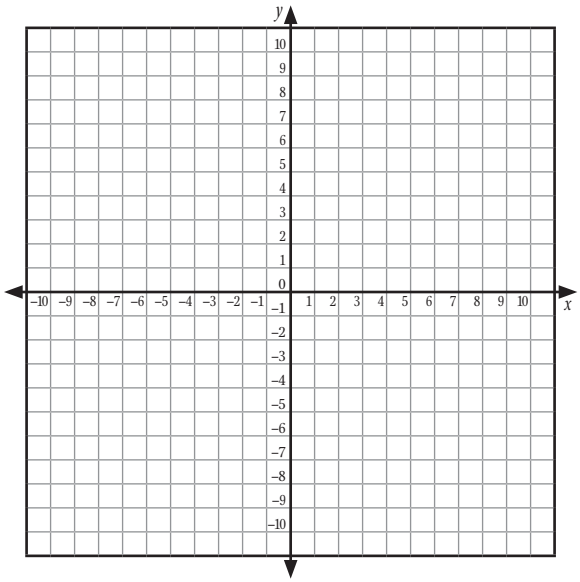
Station 2

Work with your group to find the inverse of the given function. Graph both $f(x)$ and $f^{-1}(x)$.

1. $f(x)=\frac{5}{2}x+2$



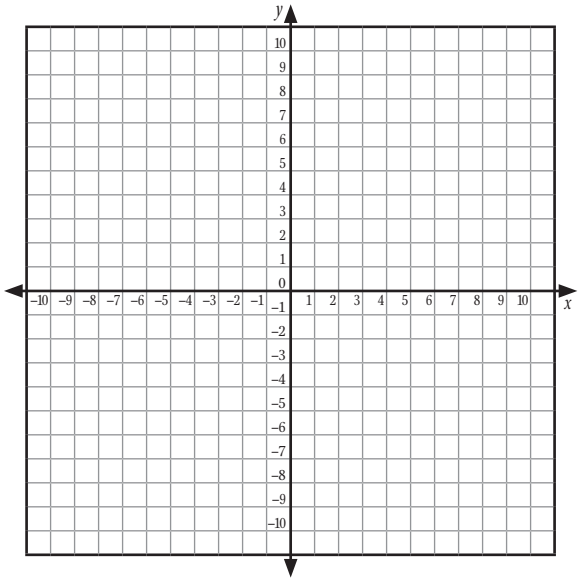
2. $f(x)=3x-1$



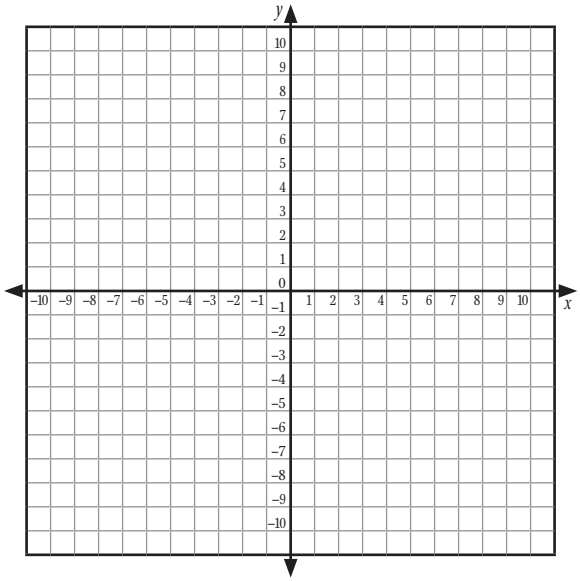
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3. $f(x)=2x+5$



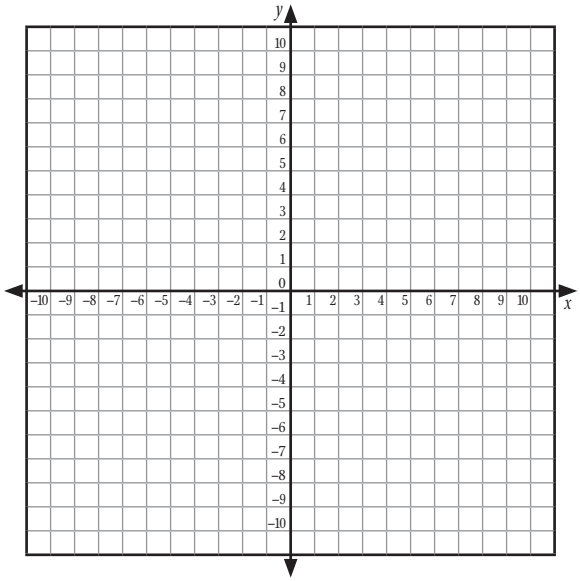
4. $f(x)=3x+\frac{1}{2}$



continued

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5. $f(x)=4x-2$



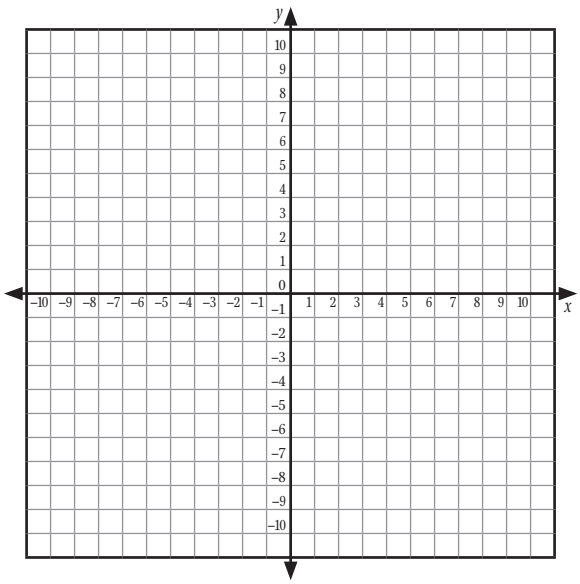
6. What is the relationship between the graph of $f(x)$ and the graph of $f^{-1}(x)$?

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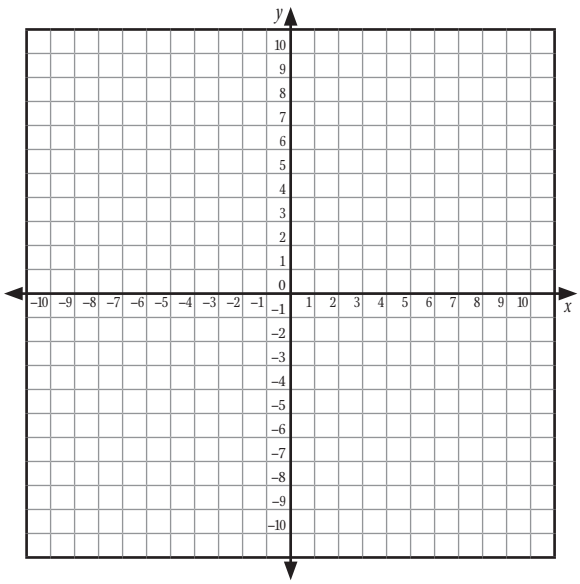
Station 3

Work with your group. For each function, a) provide the inverse; b) graph the function and its inverse; and c) state whether the original function is one-to-one.

1. $f(x)=(x+3)^2$



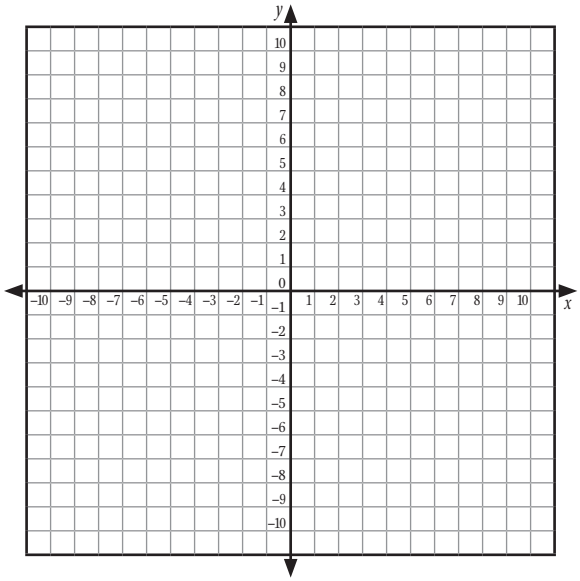
2. $f(x)=x^3-4$



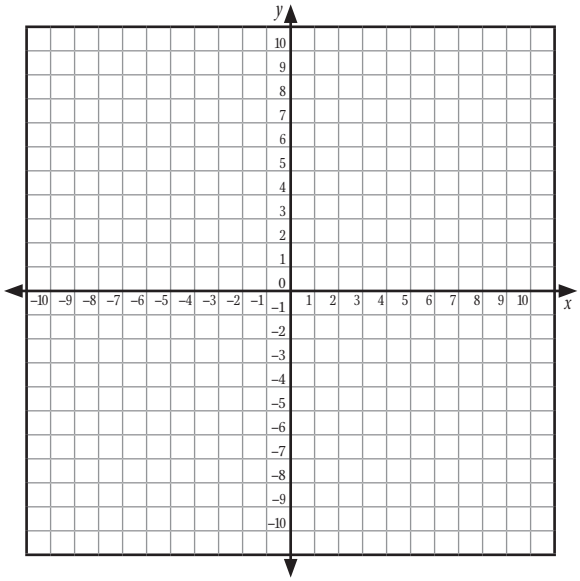
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3. $f(x)=\frac{x^5}{2}$



4. $f(x)=2x^2$

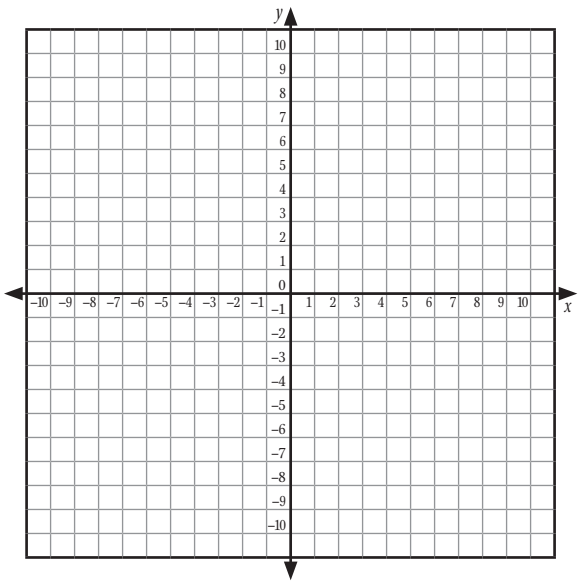


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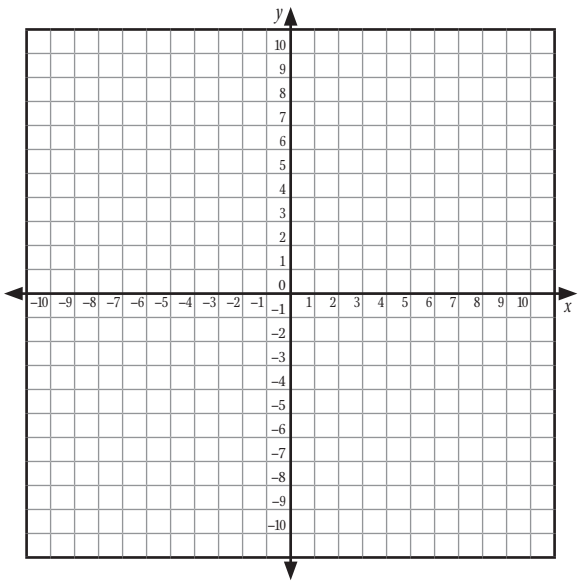
Station 4

Work with your group. For each function, a) provide the inverse; b) graph the function and its inverse; and c) state whether the original function is one-to-one.

1. $f(x)=\frac{2}{x}$



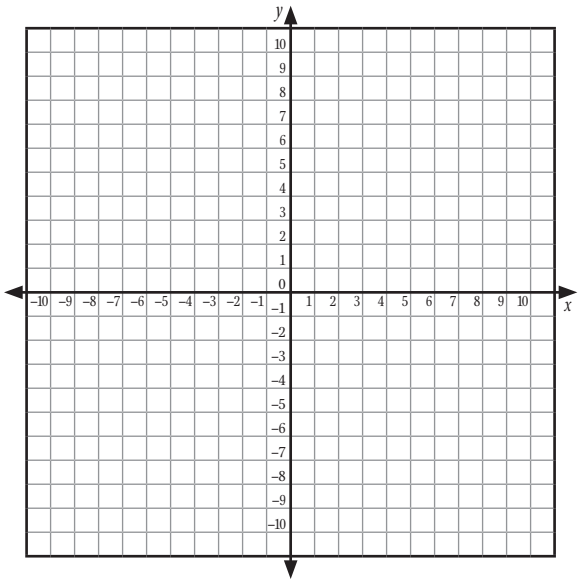
2. $f(x)=\frac{3}{x-2}$



continued

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3. $f(x)=\frac{1}{4x}+1$



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APPENDIX

Problem-Based Task Implementation Guides

Introduction

This addendum contains a collection of Implementation Guides (IGs) designed to support the use of the Problem-Based Tasks found throughout this unit.

Each Implementation Guide includes an overview and suggestions for introducing and facilitating the task. These contain specific recommendations for selected Standards for Mathematical Practice (SMPs) to highlight, guidance on addressing common errors and misconceptions associated with the task, and ideas for monitoring and coaching students as they work through the task. Finally, there is guidance for debriefing the task, with special attention to the SMPs and relevant Key Concepts from the instruction, as well as information on suitable extensions and alternate strategies.

Throughout the Implementation Guides, suggestions are provided in a question/prompt format to support your efforts to elicit student thinking and facilitate classroom discourse.

We developed these Implementation Guides for our Problem-Based Tasks at the request of math educators, and with advice and feedback from mathematics supervisors and teachers. Please let us know if they are helpful. Visit us at www.walch.com, follow us on Twitter (@WalchEd), or e-mail suggestions to customerservice@walch.com.

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Implementation Guide

Problem-Based Task 4A.1.1: Sliding Rocks

Common Core State Standard

F–BF.4 Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *or* $f(x) = (x + 1)/(x - 1)$ *for* $x \neq 1$.

Task Overview

Focus

What are the relationships between the domains and ranges of a function and its inverse? How can functions and their inverses be applied in real-world scenarios? In this task, students will analyze the domain and range of a quadratic function representing the motion of an overhang formed by rocks in a rock formation. They will also analyze the function's inverse and analyze its domain and range in the context of the real-world scenario.

This activity will provide practice with:

- identifying the domain and range of a quadratic function and its inverse
- writing the inverse of a quadratic function
- comparing domains and ranges of a function and its inverse
- applying the quadratic formula to solve an equation representing an inverse function
- interpreting solutions in the context of the scenario
- converting units of measurement
- substituting a value into a function and getting a resulting output value

Introduction

This task should be used to explore or to apply the skill of solving an equation of the form $f(x) = c$ for a simple function f that has an inverse and writing an expression for the inverse. Students should already be familiar with identifying the domain of a quadratic function and applying the quadratic formula.

Begin by reading the problem and clarifying the meaning of *domain, inverse function, quadratic function,* and *range.*

domain the set of all input values (x-values) that satisfy the given function without restriction

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inverse function	the function that may result from switching the x- and y-variables in a given function; the inverse of $f(x)$ is written as $f^{-1}(x)$
quadratic function	a function defined by a second-degree expression of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants. The graph of any quadratic function is a parabola.
range	the set of all outputs of a function; the set of y-values that are valid for the function

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they recognize that they will need to understand and apply properties of functions and inverse functions to accomplish the goal of the task. They will reason that the domain and range of the function representing the motion of the overhang and its inverse are affected by the constraints of the context of the given scenario. They will identify the quadratic formula as a method of finding the inverse function, and they will interpret the results in terms of what is applicable to the scenario. They will reason quantitatively as they substitute the proper terms into the Quadratic formula and simplify and solve the formula, as well as when they substitute the value for the given distance in order to determine the number of decades it will take for the overhang of rocks to move a distance of 1 meter.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure of process for finding the inverse of a function and determine the inverse of the function representing the motion of the overhang of rocks. They will also recognize that the structure of the quadratic formula can be used in order to find the equation representing the inverse function. Once they have determined the applicable components of the inverse function, they will use this in order to find the number of decades it will take for the rock formation to move a distance of 1 meter.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  Students will express regularity in repeated reasoning as they explain and justify their steps involved in determining the inverse of the function representing the motion of the overhang of rocks. They will follow the steps necessary to find the inverse, which include the steps of applying the quadratic formula. They will also use repeated reasoning as they determine the domain and range of the original function as well as its inverse, and explain how the domains and ranges are affected by the context of the scenario.

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Implementation Guide

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- leaving the variables unchanged when rewriting a function as its inverse

  Before beginning the task, review with students the process for constructing a possible inverse function for a quadratic function. Have them practice with a few sample functions in order to ensure they understand and can apply the process.

- identifying the domain in a real-world problem involving a quadratic function $f(x)$ and its inverse $f^{-1}(x)$ without taking into account the real-world restrictions on domain values

  Discuss with students the scenario presented in the task. Remind them that the domain of the function, represented by the variable x, represents the time in decades, and time must be positive in the context of this scenario.

- mistakenly identifying the symbol $f^{-1}(x)$ as the reciprocal of a function instead of as the function's inverse

  Remind students that the symbol $f^{-1}(x)$ does *not* mean to raise the function $f(x)$ to the negative one power, but instead it is the notation for the inverse function of $f(x)$.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, review the concepts of domain and range with them. Ask them, "What are other names and/or terms for the domain and range of a function? Describe what the domain and range mean in words." (**Answer:** Other names for the domain of a function are the input variable, or the independent variable. Other names for the range of a function are the output variable and the dependent variable. The domain of a function is all possible input values for a function that satisfy it without restriction, and the range is all possible output values that result from the input values.)

- Before students begin the task, ask them to discuss the quadratic function given to represent the motion of the overhang of rocks. Ask them, "The function given to represent the motion of the overhang of rocks is $f(x) = 10x + x^2$. What is the shape of the graph of this function and which way does it open? Explain." (**Answer:** The shape of the graph of a quadratic function is a parabola. The graph of this function will open upward, because the coefficient for the variable a on the squared term of the function is positive.)

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Implementation Guide

- As students begin working on the task, ensure their understanding of the domain and range of the original function representing the motion of the overhang. Ask them, "What do the domain and range represent in the context of the scenario, and what does this tell you about the restrictions on the domain and range?" (**Answer:** The domain is the values for the variable x, which is the time in decades it takes the overhang of rocks to move a certain distance. The range is the values representing the distance that the overhang moves. Since time is a positive quantity, the domain must contain positive values, i.e. any values greater than or equal to 0. The distance is also a positive quantity in this scenario, so the range must also contain positive values.)
- As students begin working on the task, ask them, "What is another form of notation that can be used to represent the inverse, other than the notation of $f^{-1}(x)$, given in the problem, and how could the inverse be found using this alternate notation?" (**Answer:** Instead of using the notation $f^{-1}(x)$ to represent the inverse, the $f(x)$ in the original function could have been replaced by y, and then for inverse notation, the x and y in the function can be switched. To find the inverse, solve for y in the new function, y: $f(x) = 10x + x^2$; $y = 10x + x^2$; $x = 10y + y^2$.)
- As students are working through the task and applying the quadratic formula to find the inverse of the original function, ask them, "When the values of 1, 10 and $–x$ are used to represent the values of a, b, and c, in the quadratic formula, what does the $–x$ represent? Why is the value for c not just a –1?" (**Answer:** The term "$–x$" represents the constant term in the quadratic function. Since this is the entire term representing the constant term, the variable x is considered part of the constant term.)
- As students are working to interpret the two solutions for the inverse function given when the quadratic formula is applied, ask them, "What are some examples of real-life scenarios where distance is considered 'negative' or 'decreasing'?" (**Answer:** Answers will vary. In football, if a running back or a quarterback, for example, is pushed backwards instead of making forward progress, the amount of yards gained is stated as negative yards. Another example of "negative" distance is the amount of meters a scuba diver dives below sea level.)
- As students are analyzing the two solutions, $-5 \pm \sqrt{25 + x}$, ask them to explain why the domain of the inverse is $25 + x \geq 0$. Ask them, "Once the quadratic formula has been applied to find two solutions, $-5 \pm \sqrt{25 + x}$, how are these solutions interpreted such that the domain of the inverse is given as $25 + x \geq 0$, or $x \geq –25$? Also explain why $x \geq –25$ is not actually the domain." (**Answer:** When the solutions are found as $-5 \pm \sqrt{25 + x}$, the expression under the radical must be positive, as imaginary numbers and complex solutions do not apply in this real-world scenario. Therefore, the values of x that will ensure that the expression under the radical is positive are the values greater than or equal to –25. However, since x must be positive in the context of this problem, the values of x are greater than or equal to 0.)

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- As students are determining how many decades it will take the overhang of rocks to move 1 meter, ask them, "Why is the input value of 100 used in the function instead of the value of 1?" (**Answer:** In the original function, $f(x) = 10x + x^2$, the problem states that $f(x)$ represents the distance that the rock overhang moves in units of centimeters. Since the problem asks for the time in decades that it will take the rock overhang to move 1 meter, this value must be converted to centimeters in order to keep the units consistent with the units given for the function. Since there are 100 centimeters in 1 meter, the value of 100 is substituted in for x in the inverse function in order to determine the number of decades it will take the rock overhang to move 1 meter.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss what the domain and range of the original function represent, and ask students to explain why these components must represent positive values. Also ask students to explain each step of the process of finding the inverse function, as well as why the quadratic formula could be applied. Specifically discuss why the coefficients of 1, 10 and $-x$ were used in the formula. Discuss the results of using the quadratic formula in terms of what the solutions meant about the direction of motion of the overhang. Also discuss how the number of decades it will take for the overhang to move a distance of 1 meter was found. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to explain how the variables in the original function were substituted and replaced by the notation representing the inverse function. Discuss how the context of the given scenario caused the restrictions on the domains and ranges of both the original function and its inverse. Also discuss the use of the terms "disappearing" and "decreasing" in terms of the motion of the overhang. Focus on the use of precise mathematical language and clarity, specifically when discussing function notation, as well as inverse notation. Also remind students to use the correct terminology when discussing the domains and ranges of the functions in the context of the scenario, specifically what the input and output values represent.

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Implementation Guide

Connecting to Key Concepts

Make explicit connections to key concepts:

- Determining the inverse of a function often requires examining the original function's domain and range.

  In this task, students will interpret and analyze the context of the given scenario, and they will examine the domain and range of the original function and its inverse accordingly.

- Note that the $f^{-1}(x)$ notation does *not* mean to take the –1 power of $f(x)$. Both notations, $f^{-1}(x)$ and $g(x)$, are used to represent an inverse.

  Students will use the notation $f^{-1}(x)$ to represent the inverse of the function representing the motion of the overhang of rocks.

- Recall that a quadratic function is a second-degree function of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants.

  Students will recognize that the function representing the motion of the overhang of rocks is a quadratic function.

- It is possible for a quadratic function to have an inverse, given a restricted domain.

  Students will identify the restrictions on the domain and range of the original function, and they will conclude that the inverse of the function does exist and is limited to the values allowed in the context of the scenario.

Extending the Task

- To extend the task, ask students to graph the original function in the task for the part of the function that has a domain of $[0, +\infty)$. Then ask them to graph the part of the inverse function that also has the same domain. Ask students to compare and contrast the two graphs and discuss the similarities and differences between the two. Ask for volunteers to discuss the graphs.

- Another option for extending the task is to ask students to work with a partner and have each pair of students create a new quadratic function to represent a new real-world scenario. Ask each pair to find the inverse of their new function, and ask them to identify the domains and ranges of the original function and the function's inverse. Ask for volunteers to explain the steps and procedures involved in finding the inverse of their functions, as well as how the context of their real-life scenario put restrictions on the domains and ranges of the original and inverse functions.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first recognized that I needed to apply properties of functions and inverse functions to accomplish the goal of the task. I reasoned that the domain and range of the function representing the motion of the overhang and its inverse were affected by the constraints of the context of the given scenario. I identified the quadratic formula as a method of finding the inverse function, and I interpreted the results in terms of what was applicable to the scenario.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I reasoned quantitatively as I substituted the proper terms into the Quadratic formula and simplified and solved the formula, as well as when I substituted the value for the given distance in order to determine the number of decades it will take for the overhang of rocks to move a distance of 1 meter.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I looked for and made use of structure when solving this problem by applying the structure of process for finding the inverse of a function and determining the inverse of the function representing the motion of the overhang of rocks. I also recognized that the structure of the quadratic formula could be used in order to find the equation representing the inverse function. Once I determined the applicable components of the inverse function, I used this in order to find the number of decades it will take for the rock formation to move a distance of 1 meter.)
- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** I looked for and expressed regularity in repeated reasoning as I explained and justified my steps involved in determining the inverse of the function representing the motion of the overhang of rocks. I followed the steps necessary to find the inverse, which included the steps of applying the quadratic formula. I also used repeated reasoning as I determined the domain and range of the original function as well as its inverse, and I explained how the domains and ranges were affected by the context of the scenario.)

Alternate Strategies or Solutions

Instead of using the $f^{-1}(x)$ function notation, some students may have chosen to replace $f(x)$ with y and use this variable when determining the inverse: $f(x) = 10x + x^2$; $y = 10x + x^2$; $x = 10y + y^2$. Remind students that the goal is still to "solve for y", so they will need to move the "x" term to the other side and use the quadratic formula in order to solve for y.

Technology

Students can use scientific calculators for this task.

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Implementation Guide

Problem-Based Task 4A.2.1: Healing the Waters

Common Core State Standards

F–BF.4 Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *or* $f(x) = (x + 1)/(x - 1)$ *for* $x \neq 1$.

F–LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$, where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.★

Task Overview

Focus

How can logarithmic functions be used to represent real-world scenarios? How can the pH level of acidity of a river be represented by a logarithmic function? In this task, students will analyze the changes in the acidity levels of a river and determine if cleanup efforts are causing the river to become less acidic over the time frame of a year.

This activity will provide practice with:

- rewriting a formula using properties of logarithms and exponents
- evaluating a power of ten with a calculator
- substituting values into a logarithmic function
- interpreting mathematical results
- analyzing data in order to draw conclusions
- calculating the projected pH level of a river

Introduction

This task should be used to explore or to apply the skill of using exponential models of logarithmic functions and expressing and evaluating a logarithm in the form of $ab^{ct} = d$, where a, c, and d are numbers and the base b is 2, 10, or e. Students should already be familiar with recognizing that a logarithmic function is the inverse of the related exponential function and evaluating powers of 10.

Begin by reading the problem and clarifying the meaning of *logarithmic function, pH scale,* and *power*.

logarithmic function	the inverse of an exponential function; for the exponential function $f(x) = 5^x$, the inverse logarithmic function is $g(x) = \log_a x$
pH scale	a scale ranging from 0 to 14 that measures the acidity or alkalinity of water soluble substances, with 7 as the neutral (or middle) point
power	the result of raising a base to an exponent; 32 is a power of 2 since $2^5 = 32$

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine what the pH of a river will be after a year of education and clean-up activities in an effort to make the river less acidic. They will find the concentration of the acid-producing chemical by rewriting the given pH formula, which is given as a logarithmic equation, and they will evaluate the power of ten using a calculator in order to find the amount of the chemical in milligrams per liter. Then they will determine the new concentration of the chemical after one month of progress on the river clean-up. Once they have interpreted these results and determined if the river is becoming more or less acidic, they will calculate the pH of the river after one year, based on the rate of change of the pH after one month. They will analyze these results and draw conclusions about the condition of the river in terms of its acidity.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled with logarithmic functions. They will recognize that the river's acidity level can be measured by a pH scale, which measures the amount of acidity of a body of water in terms of a logarithmic scale. They will rewrite the given pH formula in order to determine the amount of the chemical that is causing the river to be acidic. They will use properties of exponents and logarithms in order to rewrite this formula with a power of 10 so that the actual number of milligrams per liter of the chemical can be calculated. They will also make predictions about the river's acidity after one year by applying an expression that represents the change in the pH of the river after one year.

- **SMP 6:** Attend to precision.

  Students will use precision as they rewrite the given pH formula that represents the concentration of the acid-producing chemical that is found in the river. They will rewrite this formula as an exponential function with a base of 10, and then they will evaluate the power of 10 with a calculator. They will also use precision as they compare the concentration of the acid-producing chemical in the river after one month to the initial measurement of the chemical. Then, they will create an expression that represents the amount of concentration that will be present in the river after one year, based on the rate of change for one month. Encourage them to use the correct terminology throughout the task, specifically when referring to the components of the pH formula, as well as the units of measurement for the acid-producing chemical found in the river.

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Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- confusing the base with the power in expressing an exponential function as a logarithmic function, or vice versa

  Have students write the original equation for the pH formula which represents the concentration of the chemical responsible for the acidity level in the river. Ask them to label the parts of the formula in terms of the base and the exponent.

- misinterpreting the coefficients of a base and of a variable in a power in an exponential function when writing the inverse logarithmic function

  Review the parts of an exponential expression with students. Remind them which parts are the base and the exponent, and then review how these parts are changed accordingly when the inverse logarithmic function is written.

- misapplying the rules of exponents and logarithms in rewriting exponential and logarithmic functions

  Review the process of writing an inverse logarithmic function from an exponential function using simple samples. Remind students of the basic rules of exponents and logarithms and have them list these rules at the top of their papers.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, discuss the effects of acid rain on aquatic environments such as rivers, streams, and lakes. Ask them, "Why do you think it is important to monitor the levels of acidity in rivers? What are some of the effects of high levels of acidity?" (**Answer:** Rivers that contain high levels of acidity can cause plants and animals to be unhealthy and possibly die. It is important to know what the acidity level is in a river in order to know which types of plants and animals can survive and thrive.)

- Before students begin the task, review the concept of the pH scale and its meaning. Ask students, "What is the pH scale and what does it measure? What are some common everyday items that can be measured on the pH scale?" (**Answer:** The pH scales ranges from 0 to 14 and measures the amount of acidity of a body of water in terms of a logarithmic scale. The pH scale is important to understand how acidic or alkaline (basic) a substance is. Some examples include measuring the pH of a body of water, such as a river, lake or stream, measuring the pH of flavors of food, measuring the pH of soil in a garden, measuring the pH of pool water, and measuring the pH of the battery in your car.)

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- As students are beginning the task, ensure their understanding of the components of a logarithmic function. Ask them, "In the original pH formula, $pH = \log_{10}\left(\frac{1}{C}\right)$, which parts represent the base, the argument, and the logarithm of the argument to the base?" (**Answer:** In the formula $pH = \log_{10}\left(\frac{1}{C}\right)$, the base is 10, the argument is $\left(\frac{1}{C}\right)$, and the logarithm of the argument to the base is the value of the pH.)
- Ask students, "When evaluating the concentration of the acid-producing chemical in units of milligrams per liter for both the original pH and the pH of the river after one month, what is important about the number of decimal places used to represent each amount?" (**Answer:** It is important that when the pH is evaluated with a calculator, the results for each concentration amount are rounded to the same place value so that they can be easily compared. Both amounts are given in terms of millionths of a milligram per liter.)
- As students are describing the change over the first month in terms of the pH and in terms of the acid-producing chemical concentration, ask them to describe the change in an alternate way than what was given in the task. Ask them, "What is the percent of change of the amount of concentration of the chemical from the beginning to after one month? Explain how you found this value and how it relates to a reduction factor of one-third." (**Answer:** The percent of change can be found by using the formula: $\frac{\text{new amount} - \text{original amount}}{\text{original amount}} \times 100$; $\frac{0.000006 - 0.000004}{0.000006} \times 100 \approx 33\%$. The percent of change is 33%, which relates to the reduction factor of one-third, since 33% written as a fraction is one-third.)
- After students have calculated the change in the river's acidity after one month, ask them, "In terms of the pH scale, how can you explain the river's change in acidity after one month?" (**Answer:** On the pH scale, a lower value indicates greater acidity. Since the amount of the concentration of the acid-producing chemical decreased after one month, then the value on the pH scale was higher after one month, which means the river was less acidic after one month.)
- Ask students to explain their conclusions about the river's projected pH level after one year of the clean-up and education project. Ask them, "According to the projections for the pH of the river after one year, do you think the clean-up project will be considered a success? Explain." (**Answer:** If the projections are correct, then the pH of the river will be approximately 7.6, which is very close to the neutral value of 7 on the pH scale. Based on this value, the clean-up project will appear to be a success.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

Ask for volunteers to discuss their thoughts on how to accomplish the goal of the task. Discuss the concept of a pH scale and how it relates to the use of logarithms. Also discuss factors that could affect the acidity of a river. Ask students to explain how they rewrote the pH formula and then evaluated the rewritten expression using the amount of pH the river had at the beginning of the project to determine the amount of concentration of the acid-producing chemical. Also ask them how they determined the amount of concentration of the chemical after one month. Discuss how an expression was created to determine the amount of pH in the river after a year, based on the rate of change of the amount of concentration of the chemical over one month. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.

Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and their steps for rewriting the pH formula, and how they used this rewritten formula to determine the amount of concentration of the acid-producing chemical in terms of milligrams per liter. Ensure students' understanding of the pH scale in terms of how the values represent the varying degrees of acidity. Discuss how logarithms and exponents were applied in this task, in the context of the type of functions, as well as how they related to the context of the scenario. Focus on the use of precise mathematical language and clarity, specifically when discussing the components of the pH scale, as well as the units used to represent the concentration of the acid-producing chemical.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Like other functions, exponential functions have inverses, which are called logarithmic functions. A **logarithmic function** is the inverse of an exponential function.

  In this task, students will apply a logarithmic equation which represents the concentration of the chemical responsible for the acidity level in a river, in order to accomplish the goal of the task. They will recognize that this logarithmic equation is the inverse of an exponential function.

- In a logarithmic equation, $\log_a b = c$, a is the base, b is the argument, and c is the logarithm of b to the base a.

  In this task, students will identify the components of the logarithmic equation given to represent the concentration of the chemical responsible for the level of acidity in the river.

- The **base** is the quantity that is being raised to an exponent in an exponential expression, such as a in the expression a^x, or the quantity that is raised to an exponent which is the value of the logarithm, such as 2 in the function $\log_2 g(x) = 3 - x$.

  In the equation given to represent the concentration of the chemical responsible for the level of acidity in the solution, $\text{pH} = \log_{10}\left(\frac{1}{C}\right)$, students will identify the base as 10, as this quantity will be raised to the exponent when the equation is rewritten.

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Extending the Task

To extend the task provide a different value for the pH of the river at the beginning of the project. Also give them a lower value for the pH of the river after one month of the project. Ask students to work through the task again with these different values. As a further extension, given the new value for the pH of the river at the beginning of the project, ask students to determine the rate per month that the pH levels of the river need to change in order to get it as close as possible to a pH value of 7 at the end of one year of the project. Ask for volunteers to discuss their calculations and explain their results. Discuss any difficulty that students may have encountered while working through this extended task, and allow students to help each other correct any errors.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of this problem by first recognizing that the goal of the task is to determine what the pH of a river will be after a year of education and clean-up activities, in an effort to make the river less acidic. First I found the concentration of the acid-producing chemical by rewriting the given pH formula, which was given as a logarithmic equation, and then I evaluated the power of ten using a calculator in order to find the amount of the chemical in milligrams per liter. Then I determined the new concentration of the chemical after one month of progress on the river clean-up. Next, I calculated the pH of the river after one year, based on the rate of change of the pH after one month. Then I analyzed these results and drew conclusions about the condition of the river in terms of its acidity.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by recognizing that the scenario could be modeled by the use of logarithmic functions. I recognized that the river's acidity level could be measured by a pH scale, which measures the amount of acidity of a body of water in terms of a logarithmic scale. I rewrote the given pH formula in order to determine the amount of the chemical that was causing the river to be acidic. I used properties of exponents and logarithms in order to rewrite this formula with a power of 10 so that the actual number of milligrams per liter of the chemical could be calculated. I also made predictions about the river's acidity after one year by applying an expression that represented the change in the pH of the river after one year.)

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- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision as I first rewrote the given pH formula that represented the concentration of the acid-producing chemical that is found in the river. I rewrote this formula as an exponential function with a base of 10, and then I evaluated the power of 10 with a calculator. I also used precision as I compared the concentration of the acid-producing chemical in the river after one month to the measurement of the chemical taken at the beginning of the project. Then, I created an expression that represented the amount of concentration that will be present in the river after one year, based on the rate of change for one month. I was careful to use the correct terminology throughout the task, specifically when referring to the components of the pH formula, as well as the units of measurement for the acid-producing chemical found in the river.)

Alternate Strategies or Solutions

Some students may choose to rewrite the original pH formula in a different manner than presented in the task. They may choose to rewrite it as follows:

$$\text{pH} = \log_{10}\left(\frac{1}{C}\right);\ 10^{\text{pH}} = \frac{1}{C};\ 10^{5.2} = \frac{1}{C};\ C = \frac{1}{10^{5.2}};\ C = 10^{-5.2}$$

Tell students that although this way is also correct, it is more lengthy than rewriting it using properties of logarithms and exponents.

Technology

Students can use scientific calculators for this task.

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Implementation Guide

Problem-Based Task 4A.2.3: Watch IT Grow!

Common Core State Standards

F–IF.8	Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
F–LE.4	For exponential models, express as a logarithm the solution to $ab^{ct} = d$, where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.★

Task Overview

Focus

How can natural logarithmic functions be applied to real-world scenarios? How are exponential and logarithmic functions related to natural logarithmic functions? In this task, students will use a graphing calculator to generate a natural logarithmic function in order to describe the exponential growth of global consumer Internet traffic in a specific time frame.

This activity will provide practice with:

- identifying the domain and range of a natural logarithmic function in the context of a real-world scenario
- using a graphing calculator to generate a natural logarithmic function
- substituting domain and range values into a function
- creating an natural logarithmic function as the inverse of an exponential function
- evaluating an exponential function
- analyzing results in the context of a real-world scenario

Introduction

This task should be used to explore or to apply the skill of writing a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. Students should already be familiar with applying the definitions and rules of logarithms and using a graphing or scientific calculator to solve problems and graph functions expressed as logarithms.

Begin by reading the problem and clarifying the meaning of the following terms:

e	an irrational number with an approximate value of 2.71828; e is the base of the natural logarithm ($\ln x$ or $\log_e x$)
exponential function	a function that has a variable in the exponent, such as $f(x) = 5^x$

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natural logarithm	a logarithm whose base is the irrational number e; usually written in the form "ln," which means "log$e$"
petabyte	a unit of measurement referring to memory or data storage capacity, which is the equivalent of one million gigabytes; one petabyte is one quadrillion

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they recognize that they will need to create a natural logarithmic function to describe the exponential growth of global consumer Internet traffic from the year 2006 to 2014. They will reason that they will need to first determine the domain and range of the function, and then they will input these values into a graphing calculator. They will apply the values given from the graphing calculator and substitute them into the natural logarithmic equation. Finally, they will interpret and explain these results in the context of the scenario. They will reason quantitatively as they assign numerical values to represent the domain of the function, which is each year in the time frame of the study. They will also reason quantitatively as they substitute the given values from the calculator into the natural logarithmic equation and use the equation to evaluate the number of petabytes per month in a given year.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled with a natural logarithmic function representing the exponential growth of global consumer Internet traffic from the years 2006 to 2104. They will recognize that the natural logarithmic function is the inverse of the exponential function, and the exponential function with a base of the irrational number e is commonly used when describing types of population growth. They will use the model of the natural logarithmic function generated by their graphing calculator in order to describe the exponential growth of the global consumer Internet traffic from the years 2006 to 2014.

- **SMP 5:** Use appropriate tools strategically.

  Students will use a graphing calculator, or other graphing technology, in order to create a natural logarithmic function that can be used to describe the exponential growth of global consumer Internet traffic. They will follow the steps and procedures outlined in the task, and they will enter domain and range values into the calculator, in order to receive values from the calculator that represent the variables a and b in the equation form of the natural logarithmic function. They will then use this equation in order to determine specific output values for the growth of the Internet traffic for a specific year.

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Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly calculating common and natural logarithms by confusing the bases of each type when writing their inverse exponential functions

  Have students write down the general formats for both the natural logarithmic function and the exponential function. Also review the format of a common logarithm with them.

- applying the rules of logarithms differently for natural logarithms than for logarithms with other bases

  Remind students that the rules of logarithms apply for any type of logarithmic function. Review the rules with them and ask them to write them down at the top of their papers.

- forgetting to switch the domain and range of the logarithmic function from the exponential function

  Review inverse functions with students, specifically what an inverse function means in regards to the domain and range values. Remind them that the inverse of an exponential function is the natural logarithmic function, so the values for the domain and range must be switched from one function to the other.

- incorrectly entering values into the calculator

  Before students begin the task, use a sample problem and review the calculator steps for inputting data, specifically the domain and range values, into the calculator. Ask students to check with other students to ensure the correct steps are followed. Encourage students to ask questions if they are uncertain about the use of their calculators.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, discuss the unit of measurement of "petabytes" that is used in the problem. Ask them, "What other units of measurement are you familiar with in terms of computer memory or data storage? What does the term 'byte' mean? What is a 'petabyte'?" (**Answer:** Answers may vary. Some students may be familiar with gigabytes or megabytes to describe the size of computer memory. A byte is a unit of measurement used to describe the amount of digital information. A petabyte is one million gigabytes or 10^{15} bytes.)
- Before students begin the task, ask them to recall their prior knowledge about linear versus exponential functions. Ask them, "If a population growth was said to be linear over a course of ten years, what does this mean in comparison to a population growth that is exponential over a course of ten years?" (**Answer:** If a population grows linearly over a course of ten years, then it increases at the same rate each year. If it grows exponentially, then the growth can be represented by an exponential function, and the population is increasing at a growing rate each year over the course of ten years.)

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- As students begin working on the task, ask them, "What is the original domain and range of the exponential function?" (**Answer:** The domain of the exponential function is the year of growth, which is designated by a numerical value starting at 1 to represent the first year, 2006. The range is the number of petabytes in the given year.)
- As students are working through the task, review the term "line of best fit" with them. Ask them, "Recall the term 'line of best fit'. What type of graph uses a line of best fit? How does the term apply to the scenario in this task?"(**Answer:** If data points are given and graphed as a scatter plot, a line of best fit can be drawn so that an equation can be written to represent the trend of the data points. By inputting data values for the domain and range into the calculator, a natural logarithmic equation can be generated to represent the trend of the exponential growth of global consumer Internet traffic. This equation is similar to a "line of best fit".)
- As students substitute in the values for x and for y in the equation representing the natural logarithmic function, ask them, "Why do you think the value when $x = 2280$ is not exactly 1 when substituted into the equation?" (**Answer:** It is possible that the 2280 petabytes was a rounded number for the year 2006, and according to the results of evaluating this in the natural logarithmic equation, the value of 2280 was reached before 2006, so possibly in late 2005.)
- Discuss the effects of rounding in a problem such as this one. Ask students, "What is one way that the results could have been given more precisely and accurately when working through the calculations in the problem?" (**Answer:** When a calculator is used to evaluate an expression, if all of the digits of the resulting values are listed, as opposed to rounding the results to a specific place value, then the results will be more precise and accurate. In other words, the more that the values are rounded, the less precise the actual values will be.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss how exponential functions and their inverse natural logarithmic functions can be applied to real-world scenarios such as the one presented in the task. Ask them to explain how the domain and range of the function were determined, specifically focusing on why the values for the domain and range were switched, as well as why values were designated by numbers representing the specific years. Carefully review the calculator steps necessary for generating the natural logarithmic function, and also review which data values are entered. Discuss how the values generated by the calculator were applied in the equation form of the function, and also discuss how the output value was calculated when the input value of 2280 was substituted into the equation. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.

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- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and their steps determining the domain and range values to input into the calculator. Also ask them to explain how they applied the values for a and b that were generated by the calculator. Discuss how the format of the equation for the natural logarithmic function was applied, as well as how the results were interpreted in the context of the scenario. Focus on the use of precise mathematical language and clarity, specifically when discussing the types of functions and the components of each function, as well as when discussing the domain and range values that are input into the calculator.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Exponential functions with base e and natural logarithms have been found useful in describing a number of important real-world phenomena, including population growth and extinction, as well as radioactive decay of the isotopes of elements.

  In this task, students will recognize that an exponential function with base e and its inverse natural logarithmic function, are applied to a real-world scenario of describing population growth in terms of global consumer Internet traffic.

- A natural logarithm is a logarithm with a base of e. The irrational number e is approximately equal to 2.71828. A natural logarithm can be written as $\log_e x$ or, more commonly, $\ln x$.

  Students will recognize that the format of the natural logarithmic function generated by the calculator is in the form using $\ln x$. They will also recognize the base of e when applying the exponential function.

- A graphing calculator can be used to perform calculations and display graphs of natural logarithm functions and the exponential functions to which they are related.

  Students will use a graphing calculator, or other graphing technology, to create a natural logarithmic function which describes exponential growth of global consumer Internet traffic. They will also apply the inverse exponential function in order to calculate the growth, given a specific year.

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Extending the Task

To extend the task, ask students to determine the values of the exponential function for the remaining years, i.e. 2007 through 2014. Ask students to create a table with their results and include a column indicating the difference between the values given in the task for the years 2010 and 2014 and the calculated values from the exponential function. Also ask them to find the predicted number of petabytes for the year 2020. Ask for volunteers to discuss their calculations and their results, and discuss whether or not students feel that this exponential model is representative of the growth of global consumer Internet traffic.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first recognized that I needed to create a natural logarithmic function to describe the exponential growth of global consumer Internet traffic from the year 2006 to 2014. I reasoned that first I needed to determine the domain and range of the function, and then I inputted these values into a graphing calculator. Then I applied the values given from the graphing calculator and substituted them into the natural logarithmic equation. Finally, I interpreted and explained these results in the context of the scenario.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I reasoned quantitatively as I assigned numerical values to represent the domain of the function, which was each year in the time frame of the study. I also reasoned quantitatively as I substituted the given values from the calculator into the natural logarithmic equation and used the equation to evaluate the number of petabytes per month in a given year.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by first recognizing that this scenario could be modeled with a natural logarithmic function representing the exponential growth of global consumer Internet traffic from the years 2006 to 2104. I recognized that the natural logarithmic function is the inverse of the exponential function, and the exponential function with a base of the irrational number e is commonly used when describing types of population growth. I used the model of the natural logarithmic function generated by my graphing calculator in order to describe the exponential growth of the global consumer Internet traffic from the years 2006 to 2014.)

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- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** I used appropriate tools strategically as I used a graphing calculator to create a natural logarithmic function that could be used to describe the exponential growth of global consumer Internet traffic. I followed the steps and procedures outlined in the task, and I entered domain and range values into the calculator, in order to receive values from the calculator that represented the variables *a* and *b* in the equation form of the natural logarithmic function. Then I used this equation in order to determine specific output values for the growth of the Internet traffic for a specific year.)

Alternate Strategies or Solutions

Some students may choose not to round the results of their calculations as presented in the task. They may choose to use as many digits as possible when substituting in values, or in evaluating the exponential equation for specific values. Remind them that their results will differ slightly from the results given in the task. Discuss the advantages and disadvantages of not having a specific place value for rounding for each step of the process outlined in the task. Encourage students to draw conclusions about the importance of consistency in rounding.

Technology

Students can use a graphing calculator, or other graphing technology, for this task.

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Problem-Based Task 4A.2.5: When Will It Beep?

Common Core State Standards

F–IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F–IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F–IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

Task Overview

Focus

How can logarithmic models be applied to represent real-world scenarios? How can an exponential function and its inverse natural logarithmic function be used to model radioactive decay? In this task, students will determine the amount of time, in years, that a smoke detector is expected to function properly based on the decay of the radioactive element present in the particular smoke detector model.

This activity will provide practice with:

- interpreting written information in the context of the scenario
- calculating the value of a constant in an exponential function
- determining the inverse of an exponential function
- converting a percent to a decimal
- substituting values into a function and simplifying
- writing a natural logarithmic function
- analyzing results in the context of a real-world scenario

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Implementation Guide

Introduction

This task should be used to explore or to apply the skill of relating the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. Students should be familiar with writing a logarithmic function from an exponential function and vice versa, translating what a function represents in a real-world problem, using the rules of exponentials and logarithms to simplify functions, and relating different types of functions to one another.

Begin by reading the problem and clarifying the meaning of the following terms:

exponential function	function that has a variable in the exponent, such as $f(x) = 5^x$
half-life	time taken for the radioactivity of a specified isotope to fall to half its original value
natural logarithm	logarithm whose base is the irrational number e; usually written in the form "ln," which means "$\log_e$"
radioactive decay	spontaneous transformation of an unstable atomic nucleus into a lighter one, in which radiation is released in the form of alpha particles, beta particles, gamma rays, and other particles

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine how many years it will take for a smoke detector to stop working properly, based on how long it takes for the radioactive element inside of the smoke detector to drop to 0.2% of its original radioactivity level. To accomplish this goal, first they will recognize that the exponential function given represents the amount of radioactivity. Then, they will use the half-life data to find the value of the constant c in the exponent of the exponential term of the function. Next, they will write the inverse of the exponential function, which is the natural logarithmic function, to solve for the constant c, which represents the radioactive element in the smoke detector. Finally, they will write a natural logarithmic function and solve for the number of years it will take for the radioactive element to decay to the point that the smoke detector stops working. They will analyze these results and discuss other factors related to changing a smoke detector battery on an annual basis.

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- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled with an exponential function and its inverse natural logarithmic function. They will recognize that the scenario represents a real-world application of the half-life and radioactive decay of an element, specifically one found in a smoke detector. They will recognize that the original exponential function represents the original amount of radioactivity of the radioactive element that is present in the smoke detector. They will first use the inverse of this function to solve for the constant c in the exponential function, and then they will use this value to solve for the number of years that it will take the radioactive element in the smoke detector to decay to the point that the smoke detector stops working. Once students have calculated the values of the unknown quantities, they will analyze and interpret the results in the context of the scenario.

- **SMP 6:** Attend to precision.

  Students will use precision as they rewrite, simplify, and substitute values into the equations representing the exponential and inverse natural logarithmic functions. For example, they will use substitution to eliminate one of the variables in the original exponential function. Then, they will rewrite the exponential function by using natural logarithms, and they will use this form to solve for the constant c. They will also use precision as they substitute known values into the natural logarithmic function, simplify the function, and then solve for the number of years it will take the radioactive element of the smoke detector to decay to the point that the smoke detector stops working. Encourage them to use the correct terminology throughout the task, specifically when referring to the components of both the exponential function and the natural logarithmic function, as well as the terminology of radioactivity, half-life, and radioactive decay.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly interpreting the representation of a logarithmic function in a problem

  Before students begin the task, thoroughly discuss the real-world scenario presented in the problem, and help students connect the purpose of using a logarithmic function to represent the scenario. Remind students that the half-life of a radioactive element can be represented by an exponential function. Then, remind them that the inverse of an exponential function is a natural logarithmic function.

- when working with two or more functions, choosing the less-appropriate function to model a real-world problem and its solution

  Remind students that exponential and natural logarithmic functions are inverses of each other. Also remind them that the function representing the amount of radioactive decay in the element in the smoke detector is exponential, so the inverse natural logarithmic function can be used to solve for the constant in the function.

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- incorrectly writing the inverse natural logarithmic function and/or incorrectly simplifying it

  Review with students the procedure for writing a natural logarithmic function from the format of an exponential function using a different, simpler example than what is given in the task. Encourage students to practice with additional examples if they are not confident in the procedure.

- incorrectly interpreting the results of the amount of time it will take for the radioactive element in the smoke detector to decay to the point that the smoke detector stops working, such that the conclusion is drawn that smoke detectors never need to be replaced

  Remind students that although the amount of time that it will take the radioactive element in the smoke detector to decay is approximately 4,000 years, the battery in the smoke detector could be viewed as an independent factor in the overall operation of the smoke detector. Discuss how other components of the smoke detector depend on the battery.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, discuss the terms "radioactive decay" and half-life with them. Ask them, "What does it mean if a radioactive element is decaying? What does its half-life mean?" (**Answer:** A decaying radioactive element is an element whose atomic nucleus is unstable, and therefore the element is losing energy. The half-life of an element is the time it takes for the radioactivity of a specified element to fall to half its original value of radioactivity.)
- As an extension to the previous question, ask students, "Besides radioactive elements found in some smoke detectors, what are some other real-life examples of where radioactive elements are found?" (**Answer:** Answers will vary. Some examples include in watches and clocks, in fertilizer, in medical imaging systems, in certain types of food, and in ceramic materials.)
- As students begin working on the task, ask them, "How can the half-life data be used to isolate the variable constant c in the original exponential function?" (**Answer:** Since the half-life represents half of the amount of the original radioactivity, the value of 0.5 multiplied by the original amount of radioactivity, R_0, can be substituted into the value for R_t in the original exponential function. Then, since the value of $t = 450$ years was given for the element, when this value is substituted into the original function, the only remaining variable left in the function is the constant c.)
- As an extension to the previous question, ensure students' understanding of how the inverse of the exponential function was used to find the value of the constant c. Ask them, "How was the natural logarithmic function used to find the value of the constant c in the exponential equation?" (**Answer:** Since the natural logarithmic function is the inverse of the exponential function, the exponential function can be rewritten using natural logarithm notation. Then, the variable c was isolated and its value was determined by dividing the natural log of 2 by 450.)

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- As students continue working through the task, ask them to explain what 0.2% represents in the context of the problem. Ask them, "What does 0.2% represent in the context of the problem, and how is this value used to find the number of years it will take for the radioactive element inside the smoke detector to decay to the point that the smoke detector stops working?" (**Answer:** The 0.2% represents the level of radioactivity that the radioactive element will drop to that will make the smoke detector stop working. This value, represented as a decimal, is substituted into the exponential function to represent the level of radioactivity, and is one of the two known values in the exponential function. Once this value and the value for the constant *c* are substituted into the function, the only unknown variable remaining is *t,* which represents the number of years it will take for the radioactive element inside the smoke detector to decay to the point that the smoke detector stops working.)
- Ask students, "Why is it necessary to convert 0.2% to a decimal, and what is the process of converting a percent to a decimal?" (**Answer:** Since percents are based on the number 100, they are used in decimal format when solving problems containing equations. To convert 0.2% to a decimal, the percent sign is removed, and the decimal is moved two places to the left for a result of 0.002.)
- Once students have completed solving for the value of *t*, ask them, "Do you think the radioactive element in the smoke detector has any effect on how often the battery needs to be changed? Why or why not?" (**Answer:** Since the number of years it will take for the radioactive element inside the smoke detector to decay to the point that the smoke detector stops working is approximately 4,000 years, it is highly unlikely that the radioactive element and the battery have any relation to each other. If the number of years was small, even less than one year, then the battery and the radioactive element might have some connection to each other.)
- Ask students, "Why do you think it is necessary to change the battery in a smoke detector once a year? Explain in terms of the components of a smoke detector." (**Answer:** The battery in a smoke detector keeps other components working, such as the sound mechanism that alerts a person if there is a fire, and the light that blinks intermittently. The lifespan of the battery is probably only good for a year, so it is important to change the batteries in smoke detectors annually.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss how exponential functions and their inverse natural logarithmic functions can be applied to real-world scenarios such as the one presented in the task. Discuss the concept of radioactive decay and half-life with them. Ask them to explain the components of the original exponential function

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representing the original amount of radioactivity of the radioactive element in the smoke detector. Discuss how values were substituted into parts of the functions and how these values allowed for the solutions of the unknown variables to be found. Carefully review how students manipulated the formats of the functions in order to achieve the desired outcomes. Discuss the specific calculations used to find the number of years it will take the radioactive element of the smoke detector to decay to the point that the smoke detector stops working. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.

Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and their steps for rewriting the original exponential function so that one of the variables could be eliminated. Also ask them to explain how the inverse of the exponential function was written and simplified in order to find how long it will take for the radioactive decay to cause the smoke detector to stop working. Discuss the realistic implications of this result in terms of how the use of a smoke detector is affected, and in terms of what factors might contribute to the necessity of changing the smoke detector's battery on an annual basis.

Focus on the use of precise mathematical language and clarity, specifically when discussing the types of functions and the components of each function, as well as when discussing the terminology associated with the real-world smoke detector scenario.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Expressing or solving logarithmic functions in terms of exponential function models is one technique for solving real-world problems.

  In this task, students will perform operations and calculations with both exponential functions and natural logarithmic functions in order to model the half-life of a radioactive element in a smoke detector. They will write new functions based on the inverse functions, and they will use the forms of these functions to solve for missing components.

- The logarithmic function is the inverse of an exponential function and vice versa.

  Students will apply the fact that the inverse of an exponential function is a natural logarithmic function as they rewrite an exponential function as a natural logarithmic function and use it to solve for unknown values.

- Apply the basic rules of exponentials and logarithms when rewriting equations representing exponential and logarithmic functions.

  Students will rewrite the given exponential function using rules of exponents and logarithms, as the inverse function can also be written using the rules for rewriting an exponential function using natural logarithmic notation.

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Extending the Task

To extend the task, ask students to work through the task again, but this time have them use the value of 0.5% for the level of radioactivity that the radioactive element drops to in order to make the smoke detector not work properly. Once they have worked through the calculations with this new value, ask them to compare this new value of t to the value of t that was found in the original task. Ask for volunteers to share their results and discuss the conclusions that they made based on this higher value for the level of radioactivity. Discuss any difficulty that students may have encountered while working through this extended task, and allow students to help each other correct any errors.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of this problem by first recognizing that the goal of the task is to determine how many years it will take for a smoke detector to stop working properly, based on how long it takes for the radioactive element inside of the smoke detector to drop to 0.2% of its original radioactivity level. To accomplish this goal, first I recognized that the exponential function given represented the amount of radioactivity. Then, I used the half-life data to find the value of the constant c in the exponent of the exponential term of the function. Next, I wrote the inverse of the exponential function, which is the natural logarithmic function, in order to solve for the constant c, which represented the radioactive element in the smoke detector. Finally, I wrote an equation using the natural logarithmic function and solved for the number of years it will take for the radioactive element in the smoke detector to decay to the point that the smoke detector stops working. I analyzed these results and discussed other factors related to changing a smoke detector battery on an annual basis.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by first recognizing that this scenario could be modeled with an exponential function and its inverse natural logarithmic function. I recognized that the scenario represented a real-world application of the half-life and radioactive decay of an element, specifically one found in a smoke detector. I also recognized that the original exponential function represented the original amount of radioactivity of the radioactive element that was present in the smoke detector. First I used the inverse of this function to solve for the constant c in the exponential function, and then I used this value to solve for the number of years that it will take the radioactive element in the smoke detector to decay to the point that the smoke detector stops working. Once I calculated the values of the unknown quantities, I analyzed and interpreted the results in the context of the scenario.)

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- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision as I rewrote, simplified, and substituted values into the equations representing the exponential and inverse natural logarithmic functions. For example, I used substitution to eliminate one of the variables in the original exponential function. Then, I rewrote the exponential function by using natural logarithms, and I used this form to solve for the constant *c*. I also used precision as I substituted known values into the natural logarithmic function, simplified the function, and then solved for the number of years it will take the radioactive element of the smoke detector to decay to the point that the smoke detector stops working. I was careful to use correct terminology throughout the task, specifically when referring to the components of both the exponential function and the natural logarithmic function, as well as the terminology of radioactivity, half-life, and radioactive decay.)

Alternate Strategies or Solutions

Some students may have chosen to rewrite and manipulate the original exponential function in a different order than what was presented in the task. In other words, they may have chosen to isolate the constant *c* before they substituted any known values or rewritten expressions. Encourage them to double-check their work with the work presented in the task to ensure they achieve the same results. Some students may also choose to round to different place values than the values rounded to in the task. Remind them that their results will differ slightly from the results given in the task.

Technology

Students can use scientific calculators which contain logarithmic functions for this task.

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Problem-Based Task 4A.3.1: Going Through a Phase

Common Core State Standard

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Task Overview

Focus

How can trigonometric functions be applied to represent real-world scenarios? How can the graphs of trigonometric functions display key features of the functions? In this task, students will write functions representing three different phases of household electrical current that contain the sine trigonometric function, and they will also graph these functions given a specific amplitude, period, frequency, and phase shift.

This activity will provide practice with:

- substituting values into a sine function and simplifying
- identifying radian equivalents of fractions of a revolution
- writing sine functions to represent a real-world scenario
- calculating the period of a sine function
- analyzing the domain of a sine function over a specific period
- graphing sine functions by hand or with a graphing calculator

Introduction

This task should be used to explore or to apply the skill of graphing trigonometric functions, showing period, midline, and amplitude. Students should already be familiar with the trigonometric function sine, converting degrees to radians and vice versa, and identifying the sine function values in an application involving triangles or from the ordered pairs of points on a unit circle graph on a coordinate plane.

Begin by reading the problem and clarifying the meaning of *amplitude, argument, cycle, period, phase shift,* and *sine function.*

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amplitude	the coefficient a of the sine or cosine term in a function of the form $f(x) = a \sin bx$ or $g(x) = a \cos bx$; on a graph of the cosine or sine function, the vertical distance from the y-coordinate of the maximum point on the graph to the midline of the cosine or sine curve
argument	the term $[b(x - c)]$ in a cosine or sine function of the form $f(x) = a \sin [b(x - c)] + d$ or $g(x) = a \cos [b(x - c)] + d$
cycle	the smallest representation of a cosine or sine function graph as defined over a restricted domain; equal to one repetition of the period of a function
period	in a cosine or sine function graph, the horizontal distance from a maximum to a maximum or from a minimum to a minimum; one repetition of the period of a function is called a *cycle*
phase shift	on a cosine or sine function graph, the horizontal distance by which the curve of a parent function is shifted by the addition of a constant or other expression in the argument of the function
sine function	a trigonometric function of the form $f(x) = a \sin bx$, in which a and b are constants and x is a variable defined in radians over the domain $(-\infty,\infty)$

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they recognize that they will need to write functions for the three alternating-current voltages that make up the three-phase electrical energy that comes from a household generator. They will recognize that they will need to identify and apply the components of a sine function, specifically the amplitude, period, and phase shift. They will reason that three different sine functions can be written to represent the three alternating-current voltages, and that the amplitudes and periods for all three functions are the same. They will reason that the phase shift is the difference in the three functions. They will apply these features when graphing the three functions. They will reason quantitatively as they substitute the given voltage and frequency into the original voltage function in order to rewrite it, as well as when they calculate the period of the three functions. They will also reason quantitatively as they substitute values for t into the function in order to determine coordinates of the points necessary for creating the graphs of the three sine functions.

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- **SMP 7:** Look for and make use of structure.

  Students will apply the structure of the sine function in order to identify the key features of the functions. They will recognize that the structure of the original function representing voltage can be used to find the amplitude, period, and phase shifts of the three functions that they will write. They will also recognize and apply the structure for graphing the three functions, as they will graph each component of the function according to its value given in the format and structure of the function. They will also observe a pattern in the sine wave for each function as it is graphed, recognizing that the difference between the three graphs is the phase shift.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  Students will express regularity in repeated reasoning as they explain and justify their steps involved in calculating the period and concluding that it is the same value for all three sine functions. They will also use repeated reasoning as they conclude that the amplitude will be the same value for all three functions. They will express regularity as they calculate the first minimum and maximum values for $V_2(t)$ and $V_3(t)$, since these values can be calculated in the same way that the zeros for the functions are computed. Overall, they will determine that the same reasoning and regularity can be applied from one function to the other two functions, as they are working with the three alternating-current voltage functions.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- forgetting that the domain values of sine function graphs are expressed in radians unless otherwise indicated

  Remind students that the domain of the function in the task is expressed in radians, and they will need to apply units of radians in the appropriate calculations.

- reversing the amplitude and period in the general sine function $f(x) = a \sin bx$

  Have students write the definitions of amplitude and period at the top of their papers. Also have them write the general form of the sine function and make note of which variable in the general form represents the amplitude and which represents the period.

- plotting too few points when graphing the sine function

  Remind students of the necessity of graphing enough points in order to get a full, representative picture of a sine function on the graph. Also remind them to determine the values of the function at several critical values of x.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, review the general extended format of the sine function with them and ensure their understanding of its components. Ask them, "The general extended format of the sine function is $f(x) = a \sin [b(x - c)] + d$. What do the variables a, b, c, d, and x represent?" (**Answer:** The variable a represents the amplitude, b represents the period, which is found by the formula $\frac{2\pi}{b}$, c represents the horizontal phase shift of the function, d represents the vertical shift of the function, and x is a variable defined in radians.)
- As an extension to the previous question, ask students, "What do the amplitude, period, frequency, and phase shift mean in your own words?" (**Answer:** The amplitude of a function is its height, or the vertical distance from the center of the graph to its minimum or maximum point. The period is the distance it takes for the function to complete one full cycle. The frequency is the number of complete waves of the function in a given period. The phase shift is the horizontal distance that a graph has moved from its parent function, or in other words, the number of units it has moved left or right from the original starting point of a cycle.)
- As students are working through the task, ensure their understanding of the domain. Ask them, "In regards to the domain, why do you think the most practical operation for the phase factors is to add them to the sine function arguments, as opposed to subtracting them from the sine function arguments?" (**Answer:** When the phase factors are added to the sine function arguments, the resulting values will be greater than 0. Since the scenario has a real-world application, having positive output values is desired and is makes the data more compatible with the actual scenario.)
- As students are calculating the period of the function, ask them, "Why is it unnecessary to calculate the period for the other two functions, after the period for the first function is found?" (**Answer:** Although the three sine functions differ in the phase shift, they all have the same argument part of the function. To calculate the period, the argument part of the function is set equal to 2π and solved for the variable t. This value of t will be the same for all three functions.)
- Ask students, "What does the amplitude of the functions represent in the context of the scenario? How is this amplitude represented on the graph?" (**Answer:** The amplitude represents the maximum voltage from the generator. This is displayed on the graph as the maximum and minimum heights of the three sine functions.)

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- As students are working on sketching the functions and determining the values for the zeros, ask them, "For $V_3(t)$, the point representing the zero is $\left(-\frac{\pi}{3}, 0\right)$. Why will this value not be shown on the graph? (**Answer:** Due to real-world constraints and based on the given scenario, the portions of the graph that are applicable to the problem are the portions displayed in the first and fourth quadrants of the graph. The point $\left(-\frac{\pi}{3}, 0\right)$ is located in the second quadrant of the graph.)
- Ask students, "If these graphs are sketched by hand instead of using a graphing calculator, what are some of the challenges that you might face?" (**Answer:** It may be difficult to find the precise locations of the zeros on the *x*-axis because the values are decimal values in the thousandths and hundredths place, and the requested intervals on the *x*-axis are small. Also, it may be time-consuming to calculate enough values so that there are an adequate number of points to plot so that the shape of the graphs is representative enough of the function.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss the general format of a sine function and what each variable represents. Also sketch the graph of the parent sine function and have students explain how the amplitude, period, and phase shifts are represented in graph of the parent function. Discuss what is meant by a three-phase electrical current and how that can be represented by three different functions. Ask them to explain how the known values were substituted into the original voltage function to reflect the given maximum voltage and frequency of the generator. Discuss how the three functions were created in terms of how the differing parts of a revolution affected each function. Ask them to explain how the period was calculated, as well as how the amplitude and period were applied, along with the phase shift, in order to graph the three functions. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.

Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process as they first wrote and then graphed the three sine functions representing the alternating-current voltages. Ask them to explain how and why the radian equivalents of a complete revolution of the generator shaft were either added or subtracted to the function representing the maximum voltage and frequency of the generator. Discuss how the real-world domain affected the choices for adding or subtracting the phase factors. Ask them to explain their steps and procedures

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for graphing the three functions, specifically how they denoted the values on the *x*- and *y*-axes, as well as how they represented the key features of each sine function on the graph. Focus on the use of precise mathematical language and clarity, specifically when discussing the features of the sine functions, such as the amplitude, period, phase shift, and frequency. Also focus on the correct terminology when discussing the terms associated with the real-world household electrical current scenario.

Connecting to Key Concepts

Make explicit connections to key concepts:

- A **sine function** is a trigonometric function of the form $f(x) = a \sin bx$, in which a and b are constants and x is a variable defined in radians over the domain $(-\infty,\infty)$.

  In this task, students will apply the sine function, given in the form $f(x) = a \sin bx$, in a real-world scenario involving house electrical current. They will identify and apply the features of the function, represented by the variables a, b, and x, and they will analyze the given domain and apply it when graphing the function.

- In order to sketch the graph of a sine function, it is necessary to know the amplitude and period of the parent function.

  Students will identify the amplitude and calculate the period of the sine functions representing the voltage of the three-phase currents, and they will apply these values when creating the graphs.

- The amplitude of the general sine function $f(x) = a \sin [b(x - c)] + d$ is $|a|$.

  Students will identify the amplitude of the sine functions, based on the value of the coefficient in front of the "sin". They will identify this value as the maximum and minimum height of the graph from the midline.

- A useful formula for finding the period of any sine function of the form $f(x) = a \sin [b(x - c)] + d$ is to set the argument of the sine function, $[b(x - c)]$, equal to 2π and solve for x. The result is the period of the function.

  Students will apply the formula for finding the period of a sine function, and they will set the argument of the function equal to 2π and solve for the unknown variable t, in this scenario.

- The domain of a sine function is often expressed in radians.

  Students will recognize that the given domain for the sine function is stated in radians, and they will apply this fact as they are creating their graphs and establishing the values on the *x*- and *y*-axes.

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Extending the Task

To extend the task, ask students to work through the task again, but this time have them use the values of 200 volts for the maximum voltage of the generator and 70 cycles per second for the frequency of the generator. Once they have worked through the task again, with the new values for the maximum voltage and frequency, ask them to graph the three functions. Ask them to compare the period and amplitude of their functions to that of the period and amplitude of the functions from the original task. Also ask them to compare the shape of the graphs to the graphs from the original task. Ask for volunteers to share their results and discuss the conclusions that they made based on this revised task. Discuss any difficulty that students may have encountered while working through this extended task, and allow students to help each other correct any errors.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first recognized that I needed to write functions for the three alternating-current voltages that make up the three-phase electrical energy that comes from a household generator. I recognized that I needed to identify and apply the components of a sine function, specifically the amplitude, period, and phase shift. I reasoned that three different sine functions could be written to represent the three alternating-current voltages, and that the amplitudes and periods for all three functions were the same. I also reasoned that the phase shift was the difference in the three functions. I applied these features when graphing the three functions.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I reasoned quantitatively as I substituted the given voltage and frequency into the original voltage function in order to rewrite it, as well as when I calculated the period of the three functions. I also reasoned quantitatively as I substituted values for t into the function in order to determine coordinates of the points necessary for creating the graphs of the three sine functions.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I looked for and made use of structure when solving this problem as I applied the structure of the sine function in order to identify the key features of the functions. I recognized that the structure of the original function representing voltage could be used to find the amplitude, period, and phase shifts of the three functions that I needed to write. I also recognized and applied the structure for graphing the three functions, as I graphed each component of the function according to its value given in the format and structure of the function. I also observed a pattern in the sine wave for each function as it was graphed.)

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- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** I looked for and expressed regularity in repeated reasoning as I explained and justified my steps involved in calculating the period and concluding that it was the same value for all three sine functions. I also used repeated reasoning as I concluded that the amplitude was the same value for all three functions. I expressed regularity as I calculated the first minimum and maximum values for $V_2(t)$ and $V_3(t)$, since these values could be calculated in the same way that the zeros for the functions were computed. Overall, I determined that the same reasoning and regularity could be applied from one function to the other two functions as I worked with the three alternating-current voltage functions.)

Alternate Strategies or Solutions

Some students may have chosen different points to plot on their graphs for the three functions. Also, some students may choose to sketch the graphs of the functions using radian values on the *x*-axis. Encourage students to carefully consider their calculations when determining the zeros of each function in radians. Also encourage them to follow the steps outlined in the task for graphing and compare their graphing results to those presented in the task.

Technology

Students can use graphing calculators or other graphing technology for this task.

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Common Core State Standards

Mathematics III

Integrated Pathway

Teacher Resource

Unit 4B

WALCH

INTEGRATED MATH

This program was developed and reviewed by experienced math educators who have both academic and professional backgrounds in mathematics. This ensures: freedom from mathematical errors, grade level appropriateness, freedom from bias, and freedom from unnecessary language complexity.

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Standards Correlations

Each lesson in this Integrated Pathway: Mathematics III program was written specifically to address the Common Core State Standards. Each lesson lists the standards covered in all the sub-lessons, and each sub-lesson lists the standards addressed in that particular section. In this section, you'll find a comprehensive list mapping the sub-lessons to the CCSS.

Guide to Common Core State Standards Annotation

As you use this program, you will come across a symbol included with the Common Core standards for some of the lessons and activities. These symbols are explained below.

Symbol: ⋆

Denotes: Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (⋆).

From http://www.walch.com/CCSS/00003

Symbol: (+)

Denotes: College and Career Readiness Standards

Advanced mathematics standards that are required in higher-level courses such as advanced statistics may also be included in lower-level courses. These additional standards are denoted by (+). According to the Common Core State Standards Initiative, "the evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6–8."

From http://www.walch.com/CCSS/00004

Connections to Future Courses

This section provides a map between topics introduced in each unit of this course and subsequent courses where each topic is revisited and built upon.

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CCSS INTEGRATED PATHWAY: MATHEMATICS III STANDARDS CORRELATIONS

Unit 4B: Mathematical Modeling and Choosing a Model				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Creating Equations**			
	4B.1.1	Creating Equations in One Variable	A–CED.1★	U4B-7
	4B.1.2	Representing and Interpreting Constraints	A–CED.3★	U4B-36
	4B.1.3	Rearranging Formulas	A–CED.4★	U4B-56
Lesson 2	**Transforming a Model and Combining Functions**			
	4B.2.1	Transformations of Parent Graphs	F–BF.3	U4B-90
	4B.2.2	Recognizing Odd and Even Functions	F–BF.3	U4B-116
	4B.2.3	Combining Functions	F–BF.1b★	U4B-131
Lesson 3	**Comparing Properties Within and Between Functions**			
	4B.3.1	Reading and Identifying Key Features of Real-World SituationGraphs	F–IF.4★ F–IF.5★ F–IF.6★	U4B-155
	4B.3.2	Calculating Average Rates of Change	F–IF.6★	U4B-188
	4B.3.3	Comparing Functions	F–IF.6★ F–IF.9	U4B-210

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Lesson	Sub-lesson number	Title	Standard(s)	Pages
Lesson 4	**Choosing a Model**			
	4B.4.1	Linear, Exponential, and Quadratic Functions	A–CED.2★ F–IF.4★ F–IF.5★ F–BF.3	U4B-246
	4B.4.2	Piecewise, Step, and Absolute Value Functions	F–IF.4★ F–IF.5★ F–IF.7b★ F–BF.3	U4B-270
	4B.4.3	Square Root and Cube Root Functions	F–IF.4★ F–IF.5★ F–IF.7b★ F–BF.3	U4B-293
Lesson 5	**Geometric Modeling**			
	4B.5.1	Two-Dimensional Cross Sections of Three-Dimensional Objects	G–GMD.4 G–MG.1★	U4B-326
	4B.5.2	Density	G–MG.2★	U4B-344
	4B.5.3	Design	G–MG.3★	U4B-364

TOPICS FROM FUTURE COURSES: CCSS INTEGRATED PATHWAY MATHEMATICS III

Math III	Topic introduced	Course/unit where addressed	How addressed
Unit 4B	Using knowledge of function families to generate models that fit real-world phenomena	Precalculus, Calculus, or Math IV	exploring odd and even functions with respect to trigonometric functions; studying and graphing more complex logarithmic and trigonometric functions

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Conceptual Activities

Use these interactive open education and/or Desmos resources to build conceptual understanding of mathematical ideas. (*Note*: Activity links will be monitored and repaired or replaced as necessary.)

- Desmos. "Card Sort: Exponentials."

  http://www.walch.com/ca/01044

  In this activity, students practice what they've learned about exponential functions by matching equations to properties of the graphs the functions will produce. Students will then use their knowledge of transforming exponential functions to pair equations with graphs.

- Desmos. "Polygraph: Exponential Functions."

  http://www.walch.com/ca/01046

  This activity is designed to spark vocabulary-rich conversations about exponential functions. Key vocabulary terms that may appear in student questions include *increasing*, *decreasing*, *asymptote*, *quadrant*, and *axis*.

- Desmos. "Polygraph: Parent Functions."

  http://www.walch.com/ca/01051

  This activity is designed to spark vocabulary-rich conversations about graphs of parent functions. Key vocabulary terms that may appear in student questions include *increasing*, *decreasing*, *linear*, *quadratic*, *cubic*, *absolute value*, *exponential*, *logarithmic*, *rational*, *radical*, *axis*, *intercept*, and *coordinate*.

- Desmos. "Polygraph: Twelve Functions."

  http://www.walch.com/ca/01053

  This activity is designed to spark vocabulary-rich conversations about various functions. Key vocabulary terms that may appear in student questions include *linear*, *quadratic*, *exponential*, *cubic*, *absolute value*, *rational*, *radical*, *sinusoid*, and *step*.

- Desmos. "Writing Rules: Linear, Quadratic, and Exponential."

  http://www.walch.com/ca/01047

  In this activity, students have an opportunity to deepen their understanding of linear, quadratic, and exponential functions by making connections between their tables, graphs, and equations.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. A farmer would like to give 4 bushels of apples from his orchard to each family in his town. Which equation describes the maximum number of different families (n) to which the farmer can distribute 1,215 bushels of apples?

   a. $1215 - 4n > 0$
   b. $4n - 1215 > 0$
   c. $1215 + 4n > 0$
   d. $4n - 1215 = 0$

2. The function that describes how far an object falls under the influence of gravity is given by $h(t) = h_0 + v_0 \bullet t + \frac{1}{2} g \bullet t^2$, in which h_0 and v_0 are the object's initial height and velocity, respectively, and g is the object's acceleration due to gravity. Which equation correctly represents the quantities in the function *and* gives the time it takes for an object to fall from a height of 64 feet to the ground if the acceleration due to gravity is –32 feet per second squared?

   a. $0 = -64 + 0 + \frac{1}{2}(-32) \bullet t^2$
   b. $0 = 64 + 0 + \frac{1}{2}(-32) \bullet t^2$
   c. $64 = 0 + 0 + \frac{1}{2}(-32) \bullet t^2$
   d. $64 = 0 + 0 + \frac{1}{2}(32) \bullet t^2$

3. A 40-watt energy-saving light bulb will operate on an average alternating current voltage between 110 and 120 volts. If the wattage rating is equal to the product of the average voltage and the average current in amps, what is the range of current values at which the light bulb will operate?

   a. $\frac{1}{3}$ to $\frac{4}{11}$ amp
   b. $\frac{4}{11}$ to 3 amps
   c. 150 to 160 amps
   d. 4,400 to 4,800 amps

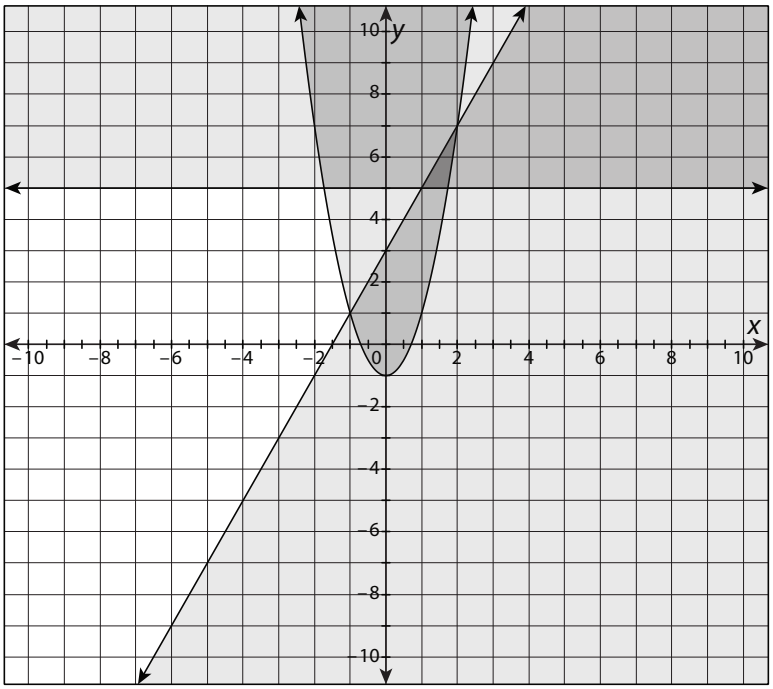
4. The surface area of a cylinder is given by the function $S(r, h) = 2\pi r^2 + 2\pi rh$, in which r is the radius of the cylinder and h is the cylinder's height. Which formula gives the cylinder's height if the surface area and the radius are known?

   a. $h = S(r, h) - 2\pi r$
   b. $h = \frac{S(r, h)}{2\pi r} - r$
   c. $h = \frac{S(r, h) - r}{2\pi r}$
   d. $h = S(r, h) + 2\pi r^2$

continued

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5. Which restricted domain of x contains common solutions for all three inequalities shown on the graph?



a. (0, 1]

b. $\left(0,\sqrt{3}\right]$

c. $\left[1,\sqrt{3}\right]$

d. [1, 2]

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Lesson 1: Creating Equations

Instruction

Common Core State Standards

SMP	
1 ✓	2 ✓
3 ✓	4
5	6 ✓
7 ✓	8 ✓

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

A–CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*★

A–CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R.*★

Essential Questions

1. How can you tell from looking at a graph or table that an equation in one variable is represented?
2. How can you tell from looking at an equation or equations in one variable how the dependent and independent variables are related?
3. What are the conditions in which an inequality statement in one variable should be written instead of an equality statement?
4. How do you identify the constraints on a function and its variable in a real-world problem?
5. What steps are necessary to isolate a variable in a multi-variable formula?

WORDS TO KNOW

constraint	a limit or restriction on the domain, range, and/or solutions of a mathematical or real-world problem
data fitting	the process of assigning a rule, usually an equation or formula, to a collection of data points as a method of predicting the values of new dependent variables that result from new independent-variable values

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Instruction

data point	a point (x, y) on the coordinate plane that represents the value of an independent variable (x) that results in a specific dependent variable value (y). The term also refers to solutions for an equation or inequality in one variable that originate from a real-world situation. A data point is also called an *ordered pair*.
dependent variable	labeled on the y-axis; the quantity that is based on the input values of the independent variable; the output variable of a function
domain	the set of all input values (x-values) that satisfy the given function
exponential equation	an equation that has a variable in the exponent
formula	a mathematical statement of the relationship among two or more variables
half plane	a region containing all points on one side of a boundary, which is a line or curve that continues in both directions infinitely. The line or curve may or may not be included in the region. A half plane can be used to represent a solution to an inequality statement.
independent variable	labeled on the x-axis; the quantity for which values are chosen; the input variable of a function
inequality	a mathematical statement that compares the value of an expression in one independent variable to the value of a dependent variable using the comparison symbols $>$, $<$, $\geq$, and $\leq$
linear equation	an equation that can be written in the form $ax + by = c$, where a, b, and c are constants; can also be written as $y = mx + b$, in which m is the slope and b is the y-intercept. The graph of a linear equation is a straight line; its solutions are the infinite set of points on the line.
logarithmic equation	an equation that includes a logarithmic expression

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Instruction

ordered pair	a point (x, y) on a two-dimensional coordinate plane that represents the value of an independent variable (x) that results in a specific dependent variable value (y). The term also refers to solutions for an equation or inequality in one variable. An ordered pair is also called a *data point*.
parent function	a function that represents a family of functions. The graphs of the functions in the family have the same general shape as the parent function.
quadratic equation	an equation that can be written in the form $y = ax^2 + bx + c$, where x is the independent variable, y is the dependent variable, a, b, and c are constants, and $a \neq 0$
range	the set of all outputs of a function; the set of y-values that are valid for the function
restricted domain	a subset of a function's defined domain
solution set	the set of ordered pairs that represent all of the solutions to an equation or a system of equations

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Instruction

Recommended Resources

- IXL Learning. "Graph a Linear Inequality in One Variable."

  http://www.walch.com/rr/00236

  This site allows users to practice graphing linear inequalities in one variable.

- Khan Academy. "The Quadratic Formula (Quadratic Equation)."

  http://www.walch.com/rr/00237

  This series of videos presents the quadratic equation as well as the quadratic formula. The videos feature various examples and applications.

- Khan Academy. "Solving for a Variable."

  http://www.walch.com/rr/00238

  These videos include topics such as rearranging formulas and solving for an isolated variable, with accompanying examples.

- Monterey Institute. "Writing, Solving, and Graphing Inequalities in One Variable."

  http://www.walch.com/rr/00239

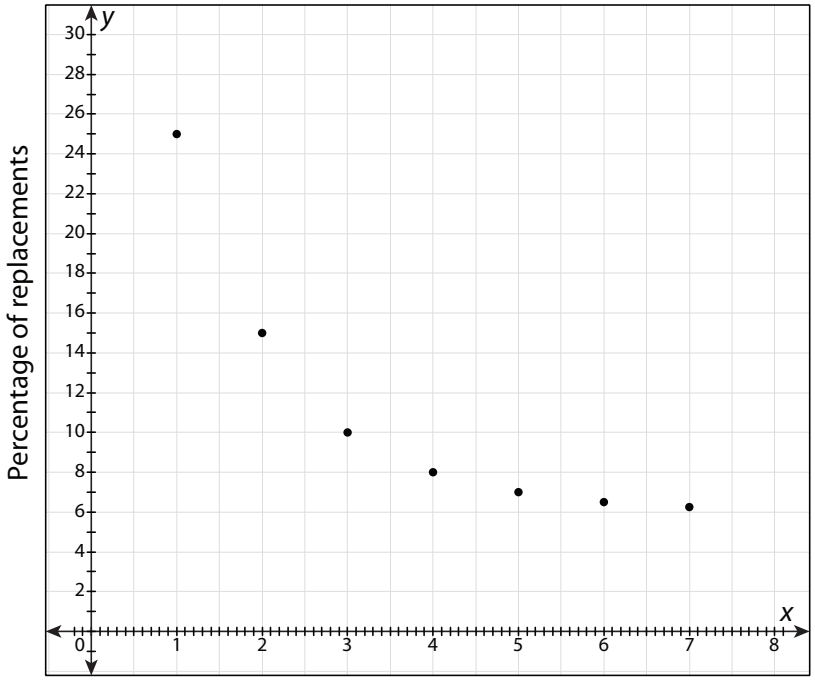
  This website features a comprehensive tutorial on solving inequalities in one variable by applying properties of inequalities. It includes solving one-step and two-step inequalities with examples.

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Lesson 4B.1.1: Creating Equations in One Variable

Warm-Up 4B.1.1

An aquarium-supply manufacturer conducts yearly follow-up customer surveys in order to track the performance of its products. The graph shows the percentage of aquarium heating elements that customers reported had to be replaced over the course of 7 years.



1. How does the independent variable (time) relate to the dependent variable (percentage of replacements)?

2. What type of equation(s)—linear, exponential, quadratic, and/or logarithmic—could this graph represent? Explain each in terms of the graphed relationship between the time and the percentage of replacements.

3. What are some of the limitations on the values of the percentage and time variables in this example?

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Instruction

Lesson 4B.1.1: Creating Equations in One Variable

Common Core State Standard

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

Warm-Up 4B.1.1 Debrief

1. How does the independent variable (time) relate to the dependent variable (percentage of replacements)?

   The dependent variable decreases as the independent variable increases. The independent variable increases by 1 each time, but the dependent variable does not decrease by the same amount each time.

2. What type of equation(s)—linear, exponential, quadratic, and/or logarithmic—could this graph represent? Explain each in terms of the graphed relationship between the time and the percentage of replacements.

   The graph does not appear to represent a linear equation. The slope between pairs of consecutive points on the graph changes, so the points do not form a straight line.

   The graph could be exponential or quadratic if the domain has a lower bound of 0, which should be the case if the domain variable represents time (since time cannot be negative).

   The graph approaches but does not intersect the y-axis, so the relationship of the variables could be logarithmic since the argument of a logarithmic function is undefined if it is less than 0.

   Therefore, the graph could represent an exponential, quadratic, or logarithmic equation.

3. What are some of the limitations on the values of the percentage and time variables in this example?

   The domain of the time variable is positive because it represents a real-world quantity and, as noted, time cannot be negative. The data also limit the maximum amount of time to 7 years. The range of the percentage values is theoretically [0, 100], but the data limit the range to the maximum and minimum values shown on the graph for the points at which $t = 1$ year and $t = 7$ years, respectively.

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Instruction

Connection to the Lesson

- Students will identify relationships that *appear* to be nonlinear based on a graph of independent and dependent variables.
- Students will recognize that nonlinear relationships can be exponential, logarithmic, or quadratic, and that additional information is sometimes needed to understand the relationships with more certainty.
- Students will identify dependent and independent variables, and work with cases where the variables have real-world contexts that have general (e.g., time as a positive quantity) and specific (e.g., data-defined) constraints.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- understanding the basic conditions under which a nonlinear relationship can be described as exponential, logarithmic, or quadratic, based on knowledge of the parent functions of these function types and their standard forms (F-LE.1)
- identifying dependent and independent variables and recognizing that real-world limitations, or constraints, exist on the domain and range of those variables, such as continuous-versus-discrete quantities, data limitations, percentages, probabilities, and time (F-IF.2, F-LE.5)

Introduction

Often, the solutions to equations in real-world situations depend on a single variable. For example, if you are selling lollipops for $1 each for a prom fund-raiser, the amount of money you raise depends only on how many lollipops you sell. Creating equations in one variable to describe a mathematical or real-world situation typically requires analyzing data or viewing a visual display of the data, such as a graph on a coordinate plane. Sometimes you may need to analyze both the data and a visual display of the information.

Key Concepts

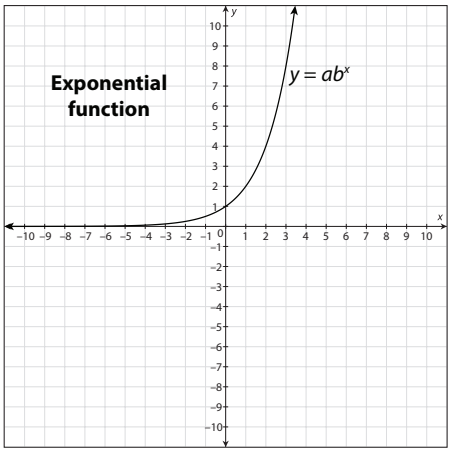
- An **independent variable**, labeled on the x-axis, is the quantity that changes based on values chosen. It is also referred to as the input variable of an equation or function.
- A **dependent variable**, labeled on the y-axis, is the quantity that changes based on the input values of the independent variable. The dependent variable is often referred to as the output variable of an equation or function.
- Mathematically, the number of data points needed to create an equation in one variable depends on the type of function that is being created.
- A **data point** (or **ordered pair**) is a point (x, y) on a two-dimensional coordinate plane that represents the value of an independent variable (x) that results in a specific dependent variable value (y).
- Ordered pairs also result from solutions for an equation or inequality in one variable that originate from a real-world situation.
- Recall that an **inequality** is a mathematical statement that compares the value of an expression in one independent variable to the value of a dependent variable using the comparison symbols $>$, $<$, $\geq$, and $\leq$.
- The data points or ordered pairs that make the inequality a true statement are the solutions of the inequality statement.

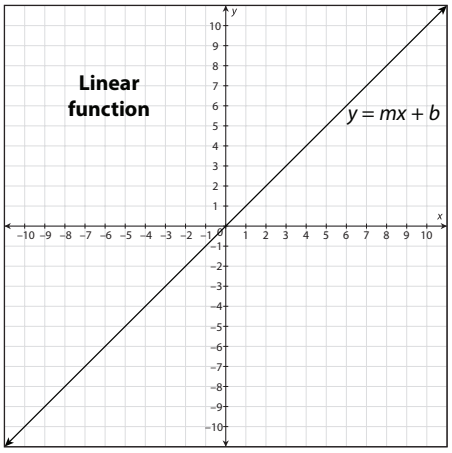
SRB p. U4B-5

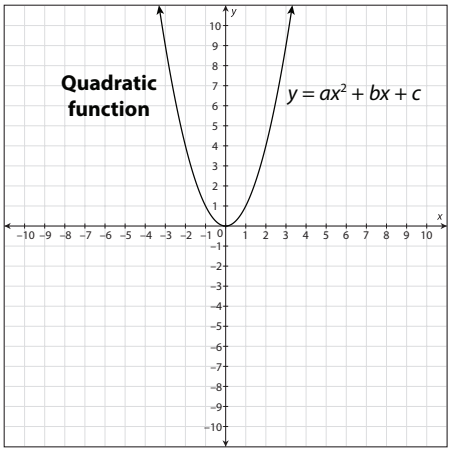
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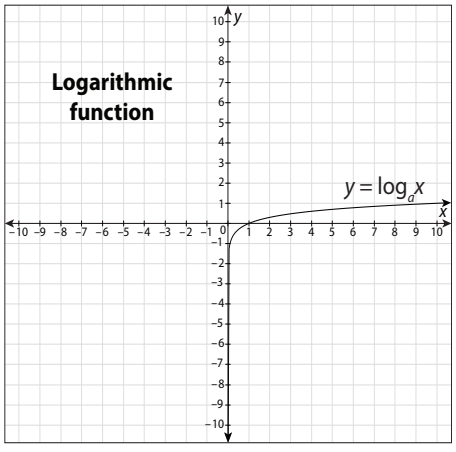
Instruction

- Recall that a **linear equation** is an equation that can be written in the form $ax + by = c$, where a, b, and c are constants, or $y = mx + b$, in which m is the slope and b is the y-intercept. The graph of a linear equation is a straight line, and its solutions are the infinite set of points on the line.
- Only two data points are needed to write a linear equation in one variable because the graph of a linear equation is a straight line that is determined by two points. Therefore, the x- and y-intercepts and the slope of a straight line can be found using only two data points.
- Recollect that a **quadratic equation** is an equation that can be written in the form $y = ax^2 + bx + c$, where x is the independent variable, y is the dependent variable, a, b, and c are constants, and $a \neq 0$. The graph of a quadratic equation is a parabola on a coordinate plane. The values of a, b, and c can be determined if three data points are known. These data points will create three linear equations in three unknowns that can be solved.
- Also recall that an **exponential equation** is an equation that has a variable in the exponent.
- A **logarithmic equation** is an equation that includes a logarithmic expression.
- The following graphs depict several types of one-variable equations.









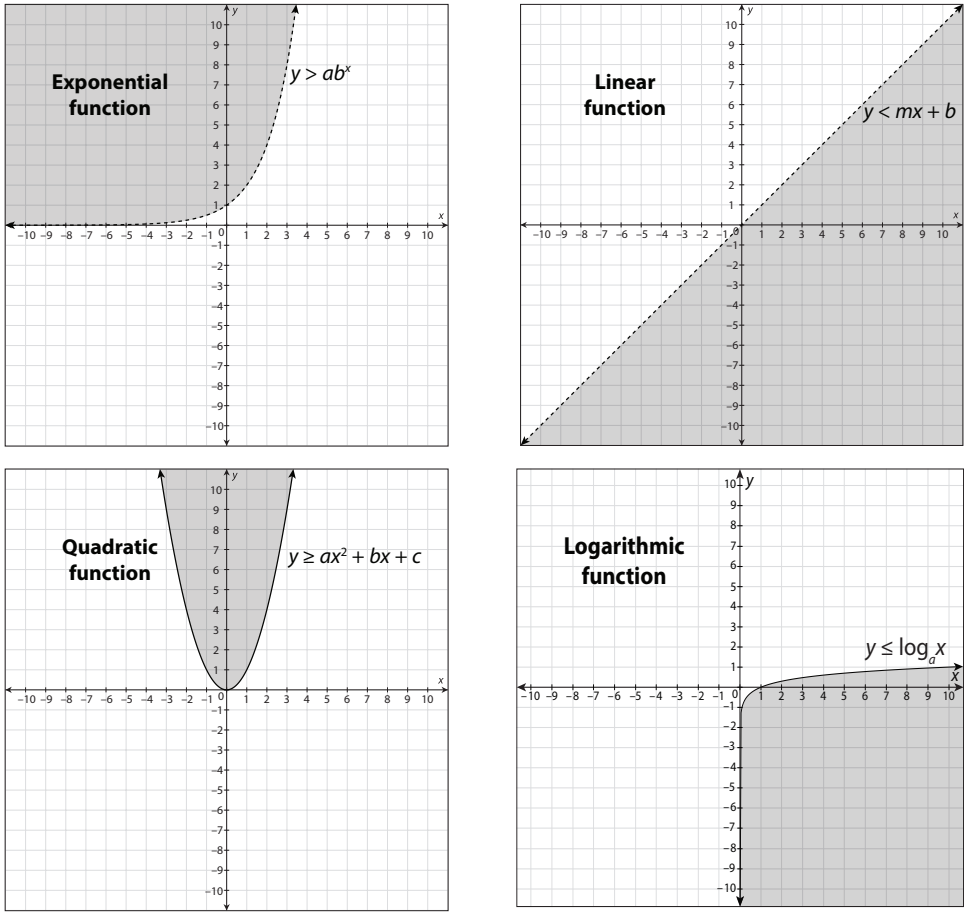
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Instruction

- A **parent function** is a function with a simple algebraic rule that represents a family of functions. The graphs of functions in the family have the same general shape as the parent function.
- When analyzing the graph of an equation or function, it is important to consider the mathematical constraints on the graph. **Constraints** are limits or restrictions on the domain, range, and/or solutions of a mathematical or real-world problem. Constraints can result from mathematical and real-world considerations. Recall that the **domain** is the set of all input values (x-values) that satisfy the given equation or function without restriction, and the **range** is the set of all outputs (y-values) that are valid for the equation or function.
- Later in the lesson, real-world constraints will be applied to one-variable equations and their parent functions.
- Formulas can be useful in determining how variables relate to one another. Formally, a **formula** is a mathematical statement of the relationship between two or more variables. The variables' values are sometimes constrained by mathematical or real-world conditions.
- An inequality in one variable can be written for exponential, linear, logarithmic, and quadratic equations in one variable. The solutions to the inequality are defined as the set of data points or ordered pairs that make the inequality true in the coordinate plane.
- A **half plane** is a region containing all points on one side of a boundary, which is a line or curve that continues in both directions infinitely. The line or curve may or may not be included in the region. A half plane can be used to represent a solution to an inequality statement.
- A half plane is used to represent a linear inequality of the form $ax + by > c$ instead of the straight line that represents a linear equation of the form $ax + by = c$. (The linear inequality would be $y > mx + b$ for a linear equation in slope-intercept form, $y = mx + b$.) In this case, the straight line is a boundary for the half plane, but the solution for the inequality does not include the points on the line. The inequality is greater than (>), not greater than or equal to (≥). The graphs that follow show the one-variable equations from the previous graphs, now changed to inequalities that use the four different types of inequality conditions (>, <, ≥, and ≤).

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Instruction



- A graphing calculator can be used to compare the different kinds of equations in one variable that can be written on the basis of a finite number of data points or ordered pairs. This process is sometimes called **data fitting** because an equation is "fitted" to a collection of data points as a way to produce a rule (the equation) that predicts the values of new dependent variables that result from new independent-variable values in the same situation.

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Instruction

- To fit an equation to the graph of a set of data points, follow the directions appropriate to your calculator model. Either calculator will return values for the constants that can be substituted into the equation $y = ax + b$ or $y = ax^2 + bx + c$, depending on the type of equation chosen. For linear equations, these values can be verified by calculating the slope and y-intercept of a line passing through the points.

On a TI-83/84:

Step 1: Press [STAT] to bring up the statistics menu. The first option, 1: Edit, will already be highlighted. Press [ENTER].

Step 2: Arrow up to L1 and press [CLEAR], then [ENTER], to clear the list. Repeat this process to clear L2 as needed.

Step 3: From L1, press the down arrow to move your cursor into the list. Enter the x-value of the first ordered pair. Press [ENTER]. Repeat until all x-values have been entered.

Step 4: Press the right arrow key and enter the y-value of the first ordered pair next to its x-value. Press [ENTER]. Repeat until all y-values have been entered.

Step 5: Press [2ND][Y=] to bring up the STAT PLOTS menu.

Step 6: The first option, Plot 1, will already be highlighted. Press [ENTER].

Step 7: Under Plot 1, select ON if it isn't selected already.

Step 8: Arrow over to Plot 2 and repeat. Check that "Xlist:" is set to "L1" and "Ylist:" is set to "L2." Press [ENTER] to save any changes.

Step 9: Press [ZOOM] and select 6: ZStandard to produce a four-quadrant grid. Notice the graphed data points.

Step 10: To fit an equation to the data points, press [STAT] and arrow over to the CALC menu. Then, select 4: LinReg(ax+b) or 5: QuadReg, depending on the type of graph desired.

Step 11: Press [2ND][1] to type "L1" for Xlist. Arrow down to Ylist and press [2ND][2] to type "L2" for Ylist, if not already shown.

Step 12: Arrow down to "Calculate" and press [ENTER].

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On a TI-Nspire:

Step 1: Press the [home] key. Arrow over to the spreadsheet icon, the fourth icon from the left, and press [enter].

Step 2: To clear the lists in your calculator, arrow up to the topmost cell of the table to highlight the entire column, then press [menu]. Choose 3: Data, then 4: Clear Data. Repeat for each column as necessary.

Step 3: Arrow up to the topmost cell of the first column, labeled "A." Press [X][enter] to type *x*. Then, arrow over to the second column, labeled "B." Press [Y][enter] to type *y*.

Step 4: Arrow down to cell A1 and enter the first *x*-value. Press [enter]. Enter the second *x*-value in cell A2 and so on.

Step 5: Move over to cell B1 and enter the first *y*-value. Press [enter]. Enter the second *y*-value in cell B2 and so on.

Step 6: To see a graph of the points, press the [home] key. Arrow over to the graphing icon, the second icon from the left, and press [enter]. Press [menu], then select 3: Graph Type, and 4: Scatter Plot.

Step 7: At the bottom of the screen, use the pop-up menus to enter "*x*" for the *x*-variable and "*y*" for the *y*-variable. Press [enter]. The data points are displayed.

Step 8: If needed, adjust the viewing window. Press [menu], then select 4: Window/Zoom, and then 1: Window Settings. Change the settings as appropriate.

Step 9: To fit an equation to the data points, first press [ctrl] and the up arrow key to display the open windows. Highlight the table of *x*- and *y*-values and press [enter]. Press [menu] and select 4: Statistics, and then 1: Stat Calculations. Select the desired type of equation from the list. The coefficients of the desired equation will be listed in a table.

- *Note*: A word of caution is needed in using a graphing calculator to fit a quadratic equation based on given data. The generated equation is of the "best fit," which means that it may not exactly fit the data points. Real-world measurements always have some uncertainty in them, because no measuring tool is perfect. However, if the fitted function repeatedly diverges from the data values by large distances, or if there appears to be a pattern among the values that don't fit, a different type of function may fit the data better.

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Instruction

Common Errors/Misconceptions

- misidentifying the dependent and independent variables
- switching the dependent and independent variables in data points or ordered pairs
- using too few data points to determine a line of best fit in the cases of linear or quadratic equations
- forgetting the domain restrictions on parent functions, such as when working with logarithms
- forgetting to include or incorrectly including boundary lines for the solutions of inequalities

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Instruction

Guided Practice 4B.1.1

Example 1

Aaron wants to have his company's logo printed on tablet computer cases, so that he can give away the cases as part of a marketing campaign. A specialty printing company will charge Aaron a $750 fee to design and print the personalized cases, plus the cost of the actual cases. The price of each case is $3. How many personalized cases can Aaron purchase if his budget is $1,200?

1. Write an equation in words for determining the total cost to produce the personalized cases.

   Review the problem statement to determine the given information and the information that is needed to solve the problem.

   The *total cost* of buying the cases includes a *fee* plus the *cost of the actual cases*.

   The cost of the cases is determined by the *price of each case* multiplied by the *number of cases*.

   Summarize this information as an equation in words:

   The total cost is the fee plus the price of each case multiplied by the number of cases.

2. Write an equation for the cost for n cases.

   Let C represent the cost of the cases.

   The cost, C, of n cases is equal to $C(n)$.

   From the problem statement, we know that n cases will cost $3 each.

   We also know that the fee is $750.

   In the word equation written for step 1, substitute $C(n)$ for the total cost, $750 for the fee, n for the number of cases, and $3 for the price of each case:

total cost	is	fee	plus	price of each case	multiplied by	number of cases
↓	↓	↓	↓	↓	↓	↓
$C(n)$	=	$750	+	$3	•	n

   The mathematical equation for the cost, C, of n cases is $C(n) = 750 + 3n$.

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Instruction

3. Determine how much money Aaron will have left to spend on cases after paying the fee.

   Aaron's budget is $1,200. Subtract the fee of $750 from the amount Aaron has to spend.

   $1200 - 750 = 450$

   After the fee, Aaron has $450 to spend on cases.

4. Determine the number of cases Aaron can buy.

   Aaron has $450 to spend on cases and the price of each case is $3. Therefore, divide $450 by $3 to determine the number of cases he can afford.

   $$\frac{450}{3} = 150$$

   Aaron can buy 150 cases.

5. Use the equation to check the result.

   Check the result by substituting 1,200 for $C(n)$ and 150 for n in the equation determined in step 2.

$C(n) = 750 + 3n$	Equation determined in step 2
$(1200) = 750 + 3(150)$	Substitute 1,200 for $C(n)$ and 150 for n.
$1200 = 750 + 450$	Multiply.
$1200 = 1200$	Add.

   The answer results in a true statement. Aaron can buy 150 cases with his budget of $1,200.

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Instruction

Example 2

Chloe is driving back home from college for summer vacation. She fuels up her gas tank and then drives for a certain amount of time before passing a roadside attraction. She drives on without stopping, and 3 hours after leaving her college, she has driven 120 miles past the attraction. Seven hours after leaving her college, she has driven 400 miles past the attraction. Write a linear equation in one variable for the distance Chloe covers in t hours, and describe the domain of the linear equation. Assuming that Chloe travels at a constant speed without stopping, use the equation to determine her speed. Then, determine how far she had traveled before she passed the roadside attraction.

1. Name a dependent variable and an independent variable based on the given data.

   Let the independent variable be time in hours, t.

   Let the dependent variable be the distance in miles, d.

2. Write ordered pairs for the data in the problem based on the identified variables.

   The ordered pairs will be of the form (t, d), since the independent variable is on the x-axis and the dependent variable is on the y-axis. Therefore, the ordered pairs for the given data are (3, 120) and (7, 400).

3. Write an equation for the speed in terms of the distance and time.

   This is a distance-rate-time problem in which the rate or speed, r, is the distance divided by the time.

   $$\text{rate} = \frac{\text{distance}}{\text{time}} \text{ or } r = \frac{d}{t}$$

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4. Use the two ordered pairs from step 2 and the formula for the slope of a linear equation, $m=\frac{y_2-y_1}{x_2-x_1}$, to find the rate.

   The slope formula can be used to find the rate because it is of the same form as the formula $r=\frac{d}{t}$.

   Recall that the two-point slope formula is $m=\frac{y_2-y_1}{x_2-x_1}$ for points (x_1, y_1) and (x_2, y_2). We can think of this as $r=\frac{d_2-d_1}{t_2-t_1}$ for points (t_1, d_1) and (t_2, d_2). Thus, the slope m would also be the rate r at which Chloe drives.

   Let (3, 120) represent (t_1, d_1) and (7, 400) represent (t_2, d_2).

   Substitute these values into the formula for slope.

$r=\frac{d_2-d_1}{t_2-t_1}$	Slope formula written in terms of the rate, r
$r=\frac{(400)-(120)}{(7)-(3)}$	Substitute (3, 120) for (t_1, d_1) and (7, 400) for (t_2, d_2).
$r=\frac{280}{4}$	Simplify.
$r=70$	

   Chloe's speed during her 7-hour drive was 70 miles per hour.

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5. Use the point-slope formula, $y - y_1 = m(x - x_1)$, to write the linear equation in one variable.

   In the point-slope formula, $y - y_1 = m(x - x_1)$, m is the slope and (x_1, y_1) is a point on the line. Use the value $r = 70$ from the previous step for the slope and either of the given ordered pairs for (x_1, y_1) in this formula.

   Let's use (3, 120). Simplify and solve for d.

$y - y_1 = m(x - x_1)$	Point-slope formula
$(d) - (120) = (70)[(t) - (3)]$	Substitute (3, 120) for (x_1, y_1), d for y, r for m, and t for x.
$d - 120 = 70t - 210$	Distribute.
$d = 70t - 90$	Simplify.

6. Find the value of d when $t = 0$.

   $t = 0$ represents the time at which Chloe left college for the drive home.

$d = 70t - 90$	Equation from the previous step
$d = 70(0) - 90$	Substitute 0 for t.
$d = -90$	Simplify.

   When $t = 0$, $d = -90$.

7. Interpret the results based on the information given in the problem to determine how far Chloe drove before passing the roadside attraction.

   If Chloe drove at a constant speed of 70 miles per hour, the ordered pair (3, 120) implies that she had traveled 3 • 70 or 210 miles from her starting point. The distance 120 in the ordered pair implies that Chloe was 210 – 120 or 90 miles past that starting point when she passed the roadside attraction. Therefore, Chloe drove 90 miles before passing the roadside attraction.

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Example 3

Write a quadratic equation in one variable that is true for the three data points (0, 0), (1, 2), and (2, 8) by solving a system of three equations based on the standard form of a quadratic equation, $y = ax^2 + bx + c$. Then, use your equation to find the y-value for a fourth point on the same graph that has an x-value of 3.

1. Use the given data points to write three equations using the standard form of a quadratic equation.

   Substitute the x- and y-values from each of the three data points into $y = ax^2 + bx + c$ and simplify.

For (0, 0):	**For (1, 2):**	**For (2, 8):**
$y = ax^2 + bx + c$	$y = ax^2 + bx + c$	$y = ax^2 + bx + c$
$(0) = a(0)^2 + b(0) + c$	$(2) = a(1)^2 + b(1) + c$	$(8) = a(2)^2 + b(2) + c$
$c = 0$	$2 = a + b + c$	$8 = 4a + 2b + c$

   Substitute $c = 0$ into the other two equations to produce two equations in two unknowns.

For (1, 2):	**For (2, 8):**
$2 = a + b + c$	$8 = 4a + 2b + c$
$2 = a + b + (0)$	$8 = 4a + 2b + (0)$
$2 = a + b$	$8 = 4a + 2b$
	$4 = 2a + b$

   We now have two separate equations. We must continue to simplify until only one equation that is true for all three given points remains.

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2. Examine the resulting equations for any common traits that can be used to simplify them.

   Compare $2 = a + b$ to $4 = 2a + b$.

   Notice that the constants in both equations are multiples of 2.

   Make the left sides of the two equations equal by multiplying the terms in the first equation by 2.

   $2 = a + b$

   $2 \bullet 2 = 2 \bullet (a + b)$

   $4 = 2a + 2b$

3. Use the results to solve for another variable.

   Set the right sides of the two equations equal to each other and solve for any variable possible in order to eliminate another variable from the system of equations.

   $4 = 2a + 2b$ Revised first equation

   $4 = 2a + b$ Second equation

   Set the right sides equal to each other and solve.

   $2a + 2b = 2a + b$

   $2b - b = 2a - 2a$

   $b = 0$

   Substitute the value found for b into either of the equations $4 = 2a + 2b$ or $4 = 2a + b$. This will allow the value of the third variable to be found.

   Solve for a in $4 = 2a + b$ by substituting 0 for b.

   $4 = 2a + b$

   $4 = 2a + (0)$

   $4 = 2a$

   $a = 2$

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4. Use the values determined for a, b, and c to write the quadratic equation that is true for the three data points.

   Substitute $a = 2$, $b = 0$, and $c = 0$ into the standard form of a quadratic equation and simplify.

$y = ax^2 + bx + c$	Standard form of a quadratic equation
$y = (2)x^2 + (0)x + (0)$	Substitute 2 for a, 0 for b, and 0 for c.
$y = 2x^2$	Simplify.

   Check the resulting equation by substituting the x- and y-values from each of the three data points to see if a true statement results. All three data points should produce true statements.

For (0, 0):	**For (1, 2):**	**For (2, 8):**
$y = 2x^2$	$y = 2x^2$	$y = 2x^2$
$(0) = 2(0)^2$	$(2) = 2(1)^2$	$(8) = 2(2)^2 = 2(4)$
$0 = 0$	$2 = 2$	$8 = 8$

   All three data points produce true statements for $y = 2x^2$.

5. Use the resulting quadratic equation model for the three given points to predict the location of a fourth point for which $x = 3$.

   Substitute 3 for x in the equation and solve for y.

   $y = 2x^2$

   $y = 2(3)^2 = 2(9) = 18$

   The fourth point is (3, 18).

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Example 4

The data shows the current i in milliamps (mA) in a cell phone circuit in fractions of a second after a cell-tower signal is received.

Time, t (s)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Current, i (mA)	2	3	5	8	13	21	34	55	89

Graph these points on a graphing calculator. Use the graph to write an exponential equation of the general form $y = ab^x$ that approximately fits the data. Then, rewrite the resulting equation so that it includes a power of 10.

1. Name an independent variable and a dependent variable.

   In this case, the current increases as a function of the elapsed time.

   Dependent variable: i (current in milliamps)

   Independent variable: t (time in seconds)

2. Use the table of values to determine the data points.

   Follow the format (t, i), using the convention that the independent variable is listed first.

   The data points are (0.1, 2), (0.2, 3), (0.3, 5), (0.4, 8), (0.5, 13), (0.6, 21), (0.7, 34), (0.8, 55), and (0.9, 89).

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3. Plot the points on a graphing calculator and use the graph to find the exponential function for the data.

   The graph of the points will allow you to estimate values for the constants a and b in the general form $y = ab^x$. With these values, you can write an equation that approximately fits the given data.

   On a TI-83/84:

   Step 1: Press [STAT] to bring up the statistics menu. The first option, 1: Edit, will already be highlighted. Press [ENTER].

   Step 2: Arrow up to L1 and press [CLEAR], then [ENTER], to clear the list. Repeat this process to clear L2 as needed.

   Step 3: From L1, press the down arrow to move your cursor into the list. Enter the t-value of the first ordered pair. Press [ENTER]. Repeat until all t-values have been entered.

   Step 4: Press the right arrow key and enter the i-value of the first ordered pair next to its t-value. Press [ENTER]. Repeat until all i-values have been entered.

   Step 5: Press [2ND][Y=] to bring up the STAT PLOTS menu.

   Step 6: The first option, Plot 1, will already be highlighted. Press [ENTER].

   Step 7: Under Plot 1, select ON if it isn't selected already.

   Step 8: Arrow over to Plot 2 and repeat. Check that "Xlist:" is set to "L1" and "Ylist:" is set to "L2." Press [ENTER] to save any changes.

   Step 9: Press [WINDOW] to set the viewing window to display the data: Xmin = 0, Xmax = 1, Xscl = 0.1, Ymin = 0, Ymax = 90, Yscl = 10.

   Step 10: Press [GRAPH] to see the data graph.

   Step 11: To fit an equation to the data points, press [STAT] and arrow over to the CALC menu. Then, select 0: ExpReg.

   Step 12: Press [2ND][1] to type "L1" for Xlist. Arrow down to Ylist and press [2ND][2] to type "L2" for Ylist, if not already shown.

   Step 13: Arrow down to "Calculate" and press [ENTER]. The resulting equation is of the general form $y = ab^x$, in which a and b are constants.

(*continued*)

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On a TI-Nspire:

Step 1: Press the [home] key. Arrow over to the spreadsheet icon, the fourth icon from the left, and press [enter].

Step 2: To clear the lists in your calculator, arrow up to the topmost cell of the table to highlight the entire column, then press [menu]. Choose 3: Data, then 4: Clear Data. Repeat for each column as necessary.

Step 3: Arrow up to the topmost cell of the first column, labeled "A." Press [X][enter] to type x. Then, arrow over to the second column, labeled "B." Press [Y][enter] to type y.

Step 4: Arrow down to cell A1 and enter the first t-value. Press [enter]. Enter the second t-value in cell A2 and so on.

Step 5: Move over to cell B1 and enter the first i-value. Press [enter]. Enter the second i-value in cell B2 and so on.

Step 6: To see a graph of the points, press the [home] key. Arrow over to the graphing icon, the second icon from the left, and press [enter]. Press [menu], then select 3: Graph Type, and 4: Scatter Plot.

Step 7: At the bottom of the screen, use the pop-up menus to enter "x" for the t-variable and "y" for the i-variable. Press [enter]. The data points are displayed.

Step 8: If needed, adjust the viewing window. Press [menu], then select 4: Window/Zoom, and then 1: Window Settings. Change the settings as appropriate: XMin = 0, XMax = 1, XScale = 0.1, YMin = 0, YMax = 90, YScale = 10. Select "OK" or press [enter] to view the adjusted graph.

Step 9: To fit an equation to the data points, first press [ctrl] and the up arrow key to display the open windows. Highlight the table of t- and i-values and press [enter]. Press [menu] and select 4: Statistics, and then 1: Stat Calculations. Select A: Exponential Regression.

(*continued*)

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Instruction

Step 10: At the "Exponential Regression" settings screen, use the pop-up menus to select "x" for X List and "y" for Y List. Press [enter]. The coefficients in the quadratic equation $y = ab^x$ should be listed in the table with the data points.

Either calculator will return approximate results for a and b of $a \approx 1.2$ and $b \approx 120$. Substituting these values into the general form $y = ab^x$ results in the equation $y = 1.2 \bullet 120^x$. Note that the constants are approximate.

4. Rewrite the base to include a power of 10.

The base of the exponent variable can be rewritten as a power of 10, and then simplified using the properties of exponents.

$y = 1.2 \bullet 120^x$	Equation from the previous step
$y = 1.2 \bullet (1.2 \bullet 10^2)^x$	Rewrite 120 as a power of 10.
$y = 1.2 \bullet (1.2)^x \bullet (10)^{2x}$	Apply the Power of Powers Property.
$y = 1.2^{x+1} \bullet 10^{2x}$	Rewrite using the Product of Powers Property.

The exponential equation written in base 10 that fits the data in the table is $y = 1.2^{x+1} \bullet 10^{2x}$.

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Problem-Based Task 4B.1.1: How Long Will the Fertilizer Last?

SMP
1 ✓ 2
3 ✓ 4
5 6
7 ✓ 8

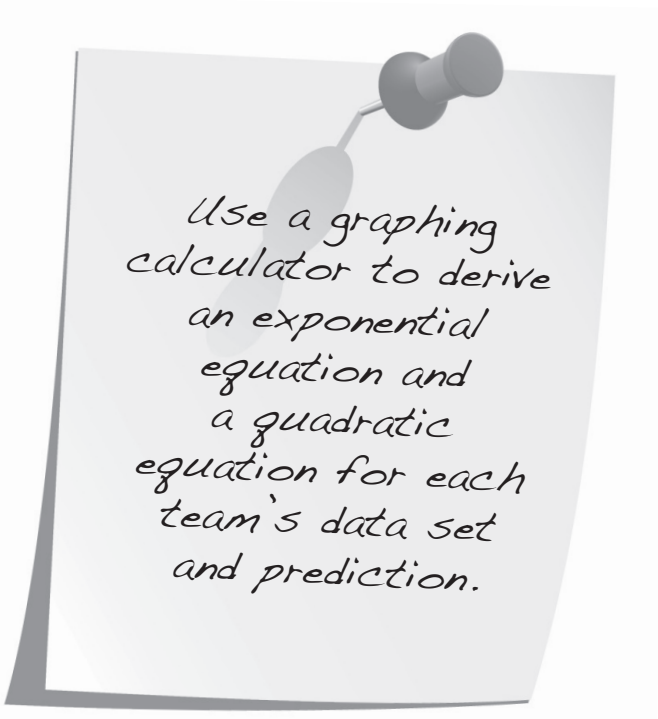
Two teams from the state agriculture department are conducting a long-term study of how long nitrogen-based fertilizer stays in the soil. Over a number of years beginning in 1982 (Year 1), members of each team took soil samples from a field to which the fertilizer was applied, and measured the amount of remaining nitrogen-15 isotope (N-15) from the fertilizer. Both teams' samples contained identical amounts of N-15 each year. However, when the teams used their existing data to predict the amount of N-15 that will be present in the soil in the year 2056, their predictions were different: Team A predicted that 10^{-5} kg of N-15 will remain, but Team B predicted that all the fertilizer will be gone by 2056.

The data in the table shows the amount in kilograms of N-15 remaining in a soil sample for given years. Each amount is represented as a decimal fraction of the original amount applied to the field in 1982. The original amount of the sample is the same for each team, so it is assigned a value of 1; the amounts in subsequent years are fractions of that amount.

Remaining Nitrogen-15 (kg)

	Year 1 (1982)	Year 2 (1983)	Year 25 (2006)	Year 75 prediction (2056)
Team A	1	$\frac{1}{2}$	$\frac{1}{12}$	10^{-5}
Team B	1	$\frac{1}{2}$	$\frac{1}{12}$	0

The two teams would like to come up with an equation in one variable to describe how the amount of nitrogen-15 decreases as a function of the number of years since the application of the fertilizer. Team A thinks an exponential equation would be the most accurate model, but Team B thinks a quadratic equation would be better. The teams agree to derive both forms of equations and decide which one is a better predictor of the amount of N-15 that will be present in 2056. Each team plans to use its own prediction for 2056 to write its equations.



Use a graphing calculator to derive an exponential equation and a quadratic equation for each team's data set and prediction. Discuss the advantages and the limitations of the equations for the data sets, including comments about the domain (years) and the range (amount of N-15) of the equations for the years after 2056.

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Problem-Based Task 4B.1.1: How Long Will the Fertilizer Last?

Coaching

a. What should be done prior to using a graphing calculator to derive the equations for each team?

b. Use a graphing calculator to derive the exponential equations for both Team A and Team B.

c. Use a graphing calculator to derive the quadratic equations for both Team A and Team B.

d. How could you transform the equations generated by the graphing calculator to reflect the variables in the problem?

e. How can any differences between the two teams' data and equations be explained?

f. How would you check to see how accurate the equations are in giving the correct amount of N-15 for a specific year?

g. Use the method you described for part f to check the equations for each team. Compare your results with the values in the table.

h. How can each team refine their equations to more accurately predict the amount of N-15 left after t years?

i. Which team's preferred equation format is best for predicting the amount of fertilizer that will be present in 2056? Why?

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Instruction

Problem-Based Task 4B.1.1: How Long Will the Fertilizer Last?

Coaching Sample Responses

a. What should be done prior to using a graphing calculator to derive the equations for each team?

The independent and dependent variables need to be identified first, followed by a list of the ordered pairs for each data set. The amount of nitrogen-15 in the soil sample depends on how much time has elapsed since the fertilizer was applied, so the amount of N-15 is the dependent variable. Time is the independent variable, with the year of first fertilizer application (Year 1) represented as $x = 1$.

- Team A: (1, 1), (2, 0.5), $\left(25, \frac{1}{12}\right)$, and $(75, 10^{-5})$
- Team B: (1, 1), (2, 0.5), $\left(25, \frac{1}{12}\right)$, and (75, 0)

b. Use a graphing calculator to derive the exponential equations for both Team A and Team B.

Enter the data for Team A's ordered pairs into the calculator to derive the exponential equation (of the general form $y = ab^x$) for Team A from the resulting graph.

The values of a and b for Team A are $a \approx 1.3$ and $b \approx 0.863$, so the approximately fitted exponential equation is $y = 1.3 \bullet 0.863^x$.

When the process is repeated with Team B's ordered pairs, the graphing calculator produces an error message: "ERR:DOMAIN." This indicates that there is no exponential equation that fits Team B's data.

c. Use a graphing calculator to derive the quadratic equations for both Team A and Team B.

Derive the quadratic equation of the general form $y = ax^2 + bx + c$ for Team A.

The values of a, b, and c for Team A are $a \approx 0.0004$, $b \approx -0.04$, and $c \approx 0.8$, so the approximately fitted quadratic equation is $y = 0.0004x^2 - 0.04x + 0.8$.

Repeating the process with Team B's data yields the values $a \approx 0.0004$, $b \approx -0.04$, and $c \approx 0.8$; therefore, Team B's quadratic equation is $y = 0.0004x^2 - 0.04x + 0.8$, which is the same as the quadratic equation for Team A.

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d. How could you transform the equations generated by the graphing calculator to reflect the variables in the problem?

Use $A(t)$ for y and t for x and write Team A's exponential and quadratic equations.

For Team A's data, the exponential equation $y = 1.3 \cdot 0.86^x$ becomes $A(t) = 1.3 \cdot 0.86^t$, and the quadratic equation $y = 0.0004x^2 - 0.04x + 0.8$ becomes $A(t) = 0.0004t^2 - 0.04t + 0.8$.

Use $B(t)$ for y and t for x to write Team B's exponential and quadratic equations.

For Team B's data, an exponential equation could not be produced. The quadratic equation is the same as for Team A: $B(t) = 0.0004t^2 - 0.04t + 0.8$.

e. How can any differences between the two teams' data and equations be explained?

The most obvious difference between the two teams' data is the prediction for the amount of nitrogen-15 remaining in the soil 50 years from 2006. Team A estimates the amount to be nonzero (10^{-5} kg, which simplifies to 0.00001 kg), whereas Team B estimates that 0 kg will remain in 2056.

The other question concerns the graphing calculator error message that appeared for the Team B data points when attempting to derive an exponential equation. The message suggests that there is some sort of problem with the "domain" of the data for Team B's exponential equation. In fact, the error has to do with the point (75, 0) lying on the asymptote of the exponential equation. By definition, no values of the equation can lie on the asymptote, so attempting to create an equation with a point that would lie on the asymptote will cause problems.

Both teams produced the same quadratic equation.

f. How would you check to see how accurate the equations are in giving the correct amount of N-15 for a specific year?

Substitute one of the values of the domain into the exponential and quadratic functions and compare the resulting calculation to the values of $A(t)$ given in the table.

g. Use the method you described for part f to check the equations for each team. Compare your results with the values in the table.

For $t = 25$, the exponential equation $A(t) = 1.3 \cdot 0.86^t$ gives $A(25) = 1.3 \cdot 0.86^{(25)} \approx 0.03$, which is off by a factor of between 2 and 3 from the amount in the table, $\frac{1}{12}$ or $0.0833\overline{3}$.

For $t = 25$, the quadratic equation $A(t) = 0.0004t^2 - 0.04t + 0.8$ gives $A(25) = 0.0004(25)^2 - 0.04(25) + 0.8 \approx 0.05$, which is about $\frac{5}{8}$ of the amount in the table, $\frac{1}{12}$ or $0.0833\overline{3}$.

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h. How can each team refine their equations to more accurately predict the amount of N-15 left after t years?

The biggest sources of error for the equations used are the values for the constants, which are rounded. The values for the constants could be rounded to more places. Team B might want to re-examine its predictions about the amount of nitrogen-15 that remains after 75 years. They might also want to review key features of exponential functions since the error message produced by the graphing calculator was related to the location of the asymptote.

i. Which team's preferred equation format is best for predicting the amount of fertilizer that will be present in 2056? Why?

Team A's choice of the exponential equation could be defended as "best," but only if the data does not include any built-in contradictions of the exponential model, as Team B's data does with $A(75) = 0$. Team B preferred the quadratic equation, which works with both sets of data and more closely approximates the values of $A(t)$ found in the table. Since both teams arrived at the same quadratic equation, and it fits both teams' data sets without resulting in exponential contradictions, Team B's preference is the best choice.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4B.1.1: Creating Equations in One Variable

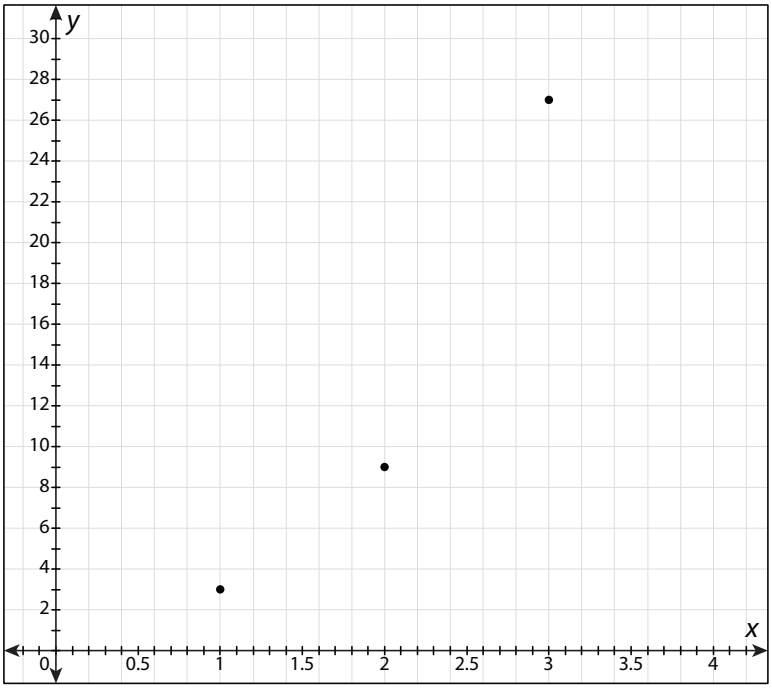
For problems 1–3, write an equation in one variable that fits the data points *exactly* without using a calculator.

1. (–2, –3) and (3, 2)

2. (2, 12), (3, 24), and (0, 0)

3. (–6, 2), (2, 2), and (0, –4)

For problems 4–7, write an equation in one variable in the simplest form of the equation type listed, using all three of the graphed data points.



4. an exponential equation

5. a linear equation

6. a logarithmic equation

7. a quadratic equation

continued

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Read the scenario that follows, and use the information in it to complete problems 8–10.

> Dane and his friend Gale rode their in-line skates to a friend's barbecue. They traveled the same route. Dane took off from a starting point (0, 0) and skated at a constant speed of 20 kilometers per hour for half an hour. Then, he skated at a constant speed of 10 kilometers per hour for another quarter of an hour. Gale skated from the same starting point (0, 0), was at the same point as Dane after the first half hour, and arrived at the same destination at the same time as Dane. The distance Gale skated is modeled by a quadratic equation for the route's entire duration.

8. Write a linear equation for the first segment of Dane's trip.

9. Write a linear equation for the second segment of Dane's trip.

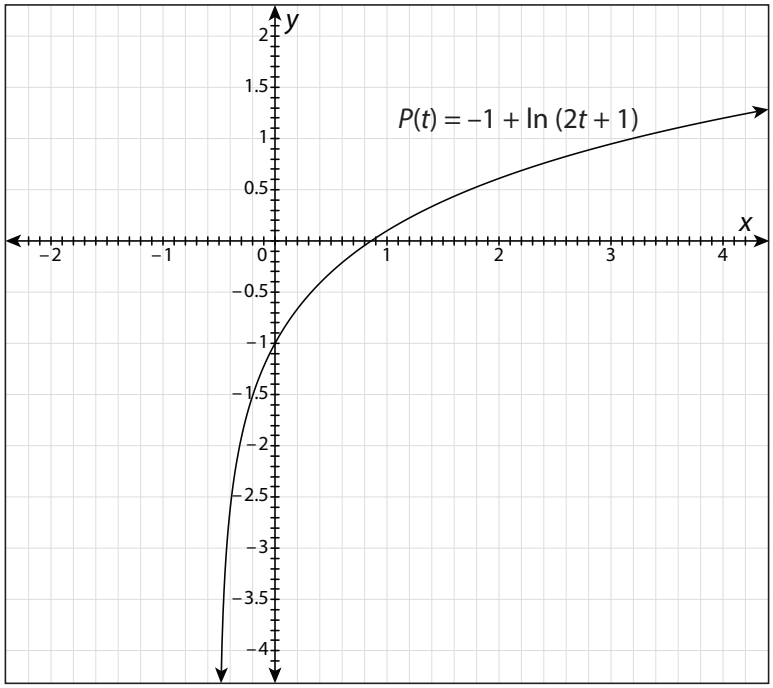
10. Write a quadratic equation for Gale's distance traveled as a function of time.

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Lesson 4B.1.2: Representing and Interpreting Constraints

Warm-Up 4B.1.2

A chemist finds that the logarithmic function $P(t) = -1 + \ln(2t + 1)$ models the probability that a certain compound will be produced in the presence of a catalyst. In the equation of the function, $P(t)$ represents the probability that the compound will be produced, and t represents the elapsed time in seconds after the catalyst is introduced to the reactor. The graph represents the equation. Use the equation and its graph to solve the problems that follow.



1. What constraints exist for the argument of the natural logarithm?

2. What constraints exist for the dependent variable, $P(t)$?

3. What real-world constraints exist for the independent variable, t?

4. What is the minimum value of t for which $P(t) \geq 0$?

5. How does this value of t compare to the answer to problem 1, and what does this value mean in the context of the problem?

6. What are the equations of the lines that represent the constraints found in problems 1–5?

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Lesson 4B.1.2: Representing and Interpreting Constraints

Common Core State Standard

A–CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*★

Warm-Up 4B.1.2 Debrief

1. What constraints exist for the argument of the natural logarithm?

   The natural logarithm is only defined if $2t + 1 > 0$ or $t > -\frac{1}{2}$.

2. What constraints exist for the dependent variable, $P(t)$?

   The value of a probability is defined by the interval [0, 1], so the constraints for $P(t)$ are defined by the interval [0, 1].

3. What real-world constraints exist for the independent variable, t?

   The variable t represents time, so $t > 0$.

4. What is the minimum value of t for which $P(t) \geq 0$?

   Solve the equation $-1 + \ln(2t + 1) \geq 0$ to find the minimum value of t for which $P(t) \geq 0$:

$-1 + \ln(2t + 1) \geq 0$	Original inequality
$\ln(2t + 1) \geq 1$	Add 1 to both sides.
$2t + 1 \geq e^1$	Convert to exponential form.
$2t \geq e^1 - 1$	Subtract 1 from both sides.
$t \geq \frac{1}{2}(e - 1)$	Divide.

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5. How does this value of t compare to the answer to problem 1, and what does this value mean in the context of the problem?

   The two quantities are related by the inequality $\frac{1}{2}(e-1) > -\frac{1}{2}$. This means that $t \geq \frac{1}{2}(e-1)$ satisfies the mathematical and real-world constraints on the independent variable in the problem.

6. What are the equations of the lines that represent the constraints found in problems 1–5?

   The constraint lines are the horizontal and vertical lines that contain the values of the dependent and independent variables for which the equation is defined.

   The horizontal lines are $P(t) \geq 0$ and $P(t) \leq 1$. The vertical line is $t \geq \frac{1}{2}(e-1)$. These three lines define the region on the coordinate plane for which the equation is defined.

Connection to the Lesson

- Students will identify, compare, and contrast mathematical constraints for given problems, such as domain and range.
- Students will identify real-world constraints in problems, such as probability and time.
- Students will combine the mathematical and real-world constraints to define the overall constraints on the dependent and independent variable in a given problem.

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Prerequisite Skills

This lesson requires the use of the following skills:

- identifying the dependent and independent variables in a mathematical or real-world problem (F-IF.2)
- calculating and/or naming the domain and range of the variables in an equation or inequality model of a real-world problem (F-IF.5)
- identifying the values of a dependent and/or independent variable for which an equation or inequality is undefined (F-IF.5)
- solving equations and inequalities in one variable or in systems of variables involving exponential, linear, logarithmic, and quadratic equations (A-REI.1, A-REI.11, A-REI.12)

Introduction

Constraints on mathematical problems can be a function of the mathematical expressions and relationships used. Limitations on the domain of the independent variable in the argument of a logarithm are examples of this. Constraints can also be related to the features of a real-world problem. For example, if time is being measured, it is usually a positive quantity. A problem with a real-world context can combine both mathematical and real-world constraints.

Key Concepts

- Recall that a constraint is a limit or restriction on the domain, range, and/or solutions of a mathematical or real-world problem.
- A problem can have constraints that result in a restricted domain for the independent variable in the model or a real-world limitation on the mathematical domain of the model variable(s). A **restricted domain** is a subset of a function's defined domain. The subset is limited by mathematical conditions and/or real-world constraints.
- Constraints can also affect the dependent variable by limiting its range. The range itself might also have constraints regardless of the constraints on the domain of the independent variable used in the model.
- When a constraint is placed on an inequality statement, a subset of the inequality's solution set can result. A **solution set** is the set of ordered pairs that represent all of the solutions to an equation or system of equations. For example, the solution set of the inequality statement $y > 4x - 1$ is all values
  (x, y) that make the statement true. If x is restricted to values of x that are integers between $113 < x < 137$, a different set of values of y make the inequality true than if x is unrestricted.

SRB p. U4B-25

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- Real-world problems are sometimes modeled with systems of equations or inequalities when there are several different variables, or when it is easier to use different variables than to keep track of a variety of constraints placed on one variable in a model.
- Systems of inequalities have solutions that can be visualized as partial planes that overlap with many common solutions, or that share a point in common.
- Sometimes, a system of inequalities that has no solution is nevertheless the model of a real-world problem.
- Graphing calculators can help you visualize solutions to systems of equations or inequalities. They can also provide the solutions for real-world problems which are sometimes less accessible by manual techniques.

Common Errors/Misconceptions

- forgetting to include mathematical and/or real-world constraints in giving the domain, range, and/or solution(s) of a problem
- confusing the dependent and independent variable constraints
- stating the solution for a system of inequalities as the coordinates of the point(s) of intersection without also including the regions of the coordinate plane within which the solution is defined for the whole system

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Guided Practice 4B.1.2

Example 1

Meredith runs the souvenir store for a minor-league baseball team. She checked into how much it would cost to have smartphone cases made that feature an image of the team's mascot. One manufacturer charges a $250 fee to design and print the personalized cases, plus the cost of the actual cases. The first 50 cases cost $5 each, and the next 100 cases cost $3 each. Write an equation to determine how many cases Meredith can purchase with a budget of $750. Determine the constraints on the terms of the equation based on the situation, then apply the constraints to solve for the number of cases that can be purchased.

1. Write an equation in words for determining the total cost to produce the personalized cases.

   Review the problem statement to determine the given information and the information that is needed to solve the problem.

   The *total cost* of buying the cases includes a *fee* plus the *cost of the actual cases*.

   The cost of the cases is determined by the *price of each case* multiplied by the *number of cases*.

   Summarize this information as an equation in words:

   The total cost is the fee plus the price of each case multiplied by the number of cases.

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2. Write an equation for the cost of n cases if n is 50 or less.

   Let C represent the cost of the cases.

   The cost, C, of n cases is equal to $C(n)$.

   From the problem statement, we know that n cases will cost \$5 each if 50 or fewer are purchased.

   We also know that the fee is \$250.

   In the word equation written for step 1, substitute $C(n)$ for the total cost, \$250 for the fee, n for the number of cases, and \$5 for the price of each case when $n \leq 50$:

total cost	is	fee	plus	price of each case	multiplied by	number of cases
↓	↓	↓	↓	↓	↓	↓
$C(n)$	=	\$250	+	\$5	•	n

   The mathematical equation for the cost, C, of n cases when n is 50 or less is $C(n) = 250 + 5n$.

3. Write an equation for the cost $C(n, m)$ if n is the number of cases that are \$5 each and m is the number of cases that are \$3 each.

   Modify the equation for $C(n)$. Let m represent the number of cases that are \$3 each. Since this quantity represents the cost of additional cases, it is being added to the equation for $C(n)$ to become the equation for $C(n, m)$.

   $C(n, m) = 250 + 5n + 3m$

4. List any constraints on the value of the expression m.

   Within the context of the problem statement, m and n are both positive whole numbers because they represent whole smartphone cases being purchased. The expression m only exists if $n > 50$. Also, since the problem statement specified that the \$3 price applies only to the next 100 cases, $m \leq 100$.

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5. Write an equation for the amount of money Meredith has available to spend as a function of the cost equation identified in step 3.

   Meredith's budget is \$750. Substitute this amount for $C(n, m)$ and simplify.

$C(n, m) = 250 + 5n + 3m$	Equation from step 3
$(750) = 250 + 5n + 3m$	Substitute 750 for $C(n, m)$.
$500 = 5n + 3m$	Subtract 250 from both sides.

   Based on Meredith's budget of \$750, the cost equation is $500 = 5n + 3m$.

6. List any constraints on the terms on the right side of the resulting equation.

   For the term $3m$ to exist, $5n$ has to exist and have a maximum value, namely \$5 per case • 50 cases, which equals \$250.

7. Substitute this constraint into the equation from step 5 and simplify.

   Since the maximum value of $5n$ is \$250, which is within Meredith's budget, we know that she will have enough money for more than 50 cases. Now we must determine how many more cases she can purchase. Substitute 50 for n to find m, the number of additional cases Meredith can buy for \$3.

$500 = 5n + 3m$	Equation from step 5
$500 = 5(50) + 3m$	Substitute 50 for n.
$500 = 250 + 3m$	Simplify.
$250 = 3m$	Subtract 250 from both sides.
$m = 83.\overline{3}$	Divide both sides by 3.

   Since Meredith cannot buy a fraction of a case, round down to the nearest whole number.

   Meredith can buy up to 83 additional cases at \$3 each.

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8. Apply any remaining constraints to the value of m and determine the total number of cases Meredith can purchase for \$750.

   Since this scenario has a real-world context, an unstated assumption about m and n is that both are positive whole numbers. Also, m exists when $m \leq 100$ and $n = 50$.

   Therefore, Meredith can afford 50 cases for \$5 each and 83 additional cases for \$3 each.

   Meredith can buy a total of 133 personalized cases with her \$750 budget.



Example 2

Certain medical tests require that patients be injected with liquids containing trace amounts of radioactive elements in order to track the movement of blood in the circulatory system. The concentration of the radioactive tracer substance diminishes in the human body over time according to the function $C(t)=a\left(\frac{2-t}{1+4t-t^2}\right)$, in which a is a constant unique to the tracer ($a > 0$), t is time in hours, and $C(t)$ is the concentration of the tracer in milligrams per liter. Identify real-world and mathematical constraints on t, the time that the tracer is in the body, which allow $C(t)$ to be defined in the context of the situation.

1. Identify a real-world condition that might be placed on the time variable, t.

   Time cannot be negative, so the values of t must be non-negative.

2. Determine any specific mathematical constraints on t.

   Since the term with t is a rational number, $\left(\frac{2-t}{1+4t-t^2}\right)$, the denominator will have to be nonzero in order for this rational term of the function to exist.

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3. Identify any real-world constraints on the function values.

   Notice that concentration cannot be negative; this means that $C(t)=a\left(\frac{2-t}{1+4t-t^2}\right)>0$, which places an additional constraint on t.

   There are two cases in which $C(t) > 0$: one in which the numerator and denominator are both positive, and one in which both are negative. It is immediately obvious that the numerator will be positive whenever $t < 2$, and negative whenever $t > 2$. To find where the denominator will be positive or negative, use the quadratic formula to find the zeros of the expression and analyze the results.

4. Use the quadratic formula to find value(s) of t for which the denominator is positive and negative.

   The zeros of the denominator determine both where the function is undefined and where the sign of the denominator switches from positive to negative. Find the zeros, then analyze the expression to find what t-values yield a positive value in the denominator. Use the quadratic formula to find the zeros of $C(t)$.

   In the quadratic formula for equations of the form $y = ax^2 + bx + c$, the value of x is $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

   Rearrange the expression $1 + 4t – t^2$ to more readily identify values for a, b, and c: $–t^2 + 4t + 1$.

   Thus, for the denominator of the function $C(t)=a\left(\frac{2-t}{1+4t-t^2}\right)$, $a = –1$, $b = 4$, and $c = 1$. Substitute these values for a, b, and c into the quadratic formula, with t in place of x.

(*continued*)

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Instruction

$t=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	Quadratic formula with t in place of x
$t=\frac{-(4)\pm\sqrt{(4)^2-4(-1)(1)}}{2(-1)}$	Substitute –1 for a, 4 for b, and 1 for c.
$t=\frac{-4\pm\sqrt{16+4}}{-2}$	Simplify.
$t=\frac{4\pm\sqrt{20}}{2}$	Simplify.
$t=\frac{4}{2}\pm\frac{\sqrt{20}}{2}$	Rewrite the equation as the sum of two fractions.
$t=2\pm\frac{2\sqrt{5}}{2}$	Simplify.
$t=2\pm\sqrt{5}$	Simplify.
$t\approx 2\pm 2.24$	Simplify using a calculator.

Negative values of t can be ignored, because t cannot be negative in the context of this problem. Therefore, the only t-value of concern is $t \approx 4.24$.

Now check to see where the denominator will be positive and negative. The value of the denominator must be positive for t-values on one side of $t \approx 4.24$, and negative for t-values on the other side. Check which side is which by choosing values to the left and right of $t \approx 4.24$ on a number line, and then substituting each value of t into the denominator of the function to see if the result is positive or negative. Choose $t = 0$ and $t = 5$ to make calculation easy.

(*continued*)

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For $t = 0$:

$1 + 4t - t^2$	Denominator of the function $C(t)$
$1 + 4(0) - (0)^2$	Substitute 0 for t.
1	Simplify.

The denominator is positive for $t = 0$, which means that t-values to the left of $t \approx 4.24$ make the denominator positive.

For $t = 5$:

$1 + 4t - t^2$	Denominator of the function $C(t)$
$1 + 4(5) - (5)^2$	Substitute 5 for t.
$1 + 20 - 25$	Simplify.
-4	Simplify.

The denominator is negative for $t = 5$, which means that t-values to the right of $t \approx 4.24$ make the denominator negative.

5. Determine what values of t make the function positive.

   In previous steps, we found that the function will be positive if the numerator and denominator are both either positive or negative. The numerator is positive when $t < 2$, and the denominator is positive when $t < 4.24$. The numerator is negative when $t > 2$, and the denominator is negative when $t > 4.24$. Therefore, the function will be positive when $t < 2$ or $t > 4.24$.

6. Combine the results of the previous steps to determine a realistic domain for the time in this problem.

   Recall that $t \geq 0$ and $C(t) > 0$. The function must start at $t = 0$ and continue uninterrupted until it reaches 0 to accurately measure concentration. Previously, we found that the function is positive when $t < 2$ or $t > 4.24$. Since the domain must be uninterrupted, the domain for t in the tracer-concentration model must be $[0, 2]$.

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Example 3

A sporting goods store is running a sale to eliminate inventory of a discontinued tennis racket. The price of the racket can be marked down as much as necessary to make sure all of the rackets are sold. However, the total revenue from selling all of the rackets cannot be less than $7,500. The shop has already sold 50 of the 62 rackets at the full price of $130 per racket. Write an equation to find the minimum price at which the remaining rackets can be sold while meeting the sales quota. Then, find the maximum percentage reduction by which the remaining rackets can be marked down, assuming the store can only mark down items in units of 5%. Use this number to determine the actual sale price of the remaining rackets and the total revenue generated once all the rackets have sold.

1. Use the given information to write a function for the total revenue generated by racket sales.

   Set up an equation that relates the total revenue generated by racket sales to the number of rackets sold at different prices. This will be the sum of the different prices multiplied by the number of rackets sold at each price:

   total revenue = full price • number of rackets sold at full price + discount price • number of rackets remaining

   Let $R(d)$ represent the total revenue generated by sales of the rackets, let n_1 represent the number of rackets sold at full price, let n_2 represent the number of rackets sold at a discount, and let d represent the discounted price.

   We know that the full price is $130, and that the number of rackets sold at this price is 50. We also know there were 62 rackets to start, so there must be 12 remaining.

   Rewrite the word problem using the assigned values.

$R(d) = 130n_1 + dn_2$	Revenue function
$R(d) = 130(50) + d(12)$	Substitute 50 for n_1 and 12 for n_2.
$R(d) = 6500 + 12d$	Simplify.

   The function for the total revenue generated by racket sales is $R(d) = 6500 + 12d$.

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2. Determine the value of d when the total revenue generated by racket sales is \$7,500.

   Recall that the total revenue for selling all of the rackets cannot be less than \$7,500.

   When the total revenue generated by racket sales is \$7,500, then $R(d) = 7500$. Therefore, substitute 7500 for $R(d)$ and solve for d to find the lowest price for which the rackets can be sold.

$R(d) = 6500 + 12d$	Equation from the previous step
$7500 = 6500 + 12d$	Substitute 7500 for $R(d)$.
$1000 = 12d$	Subtract 6500 from both sides.
$d = 83.33$	Divide both sides by 12.

   The minimum selling price for rackets that will meet the revenue quota is \$83.33.

3. Calculate the percentage decrease from the full price that this minimum selling price represents.

   To calculate the percentage decrease, take the difference of the full price and the reduced price, then divide by the full price. Next, multiply by 100.

$\text{percentage decrease} = 100 \cdot \frac{n_1 - n_2}{n_1}$	Formula for percentage decrease from n_1 to n_2
$\text{percentage decrease} = 100 \cdot \frac{(130) - (83.33)}{(130)}$	Substitute 130 for n_1 and 83.33 for n_2.
$\text{percentage decrease} = 100 \cdot \frac{46.67}{130}$	Simplify.
$\text{percentage decrease} = 100 \cdot 0.359$	Simplify the fraction.
$\text{percentage decrease} = 35.9\%$	Multiply.

   The minimum selling price of \$83.33 represents a 35.9% discount.

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Instruction

4. Given that discounts must be in increments of 5%, determine the maximum discount that can be applied to the rackets while still meeting the minimum revenue requirement.

   Because discounts can only be applied in units of 5%, the maximum discount will need to be the largest discount that can be applied that is less than 35.9%. Therefore, the maximum discount that can be applied is 35% percent.

5. Use the maximum discount percentage to calculate the minimum sale price for the rackets. Then, confirm that the resulting revenue will be no less than $7,500.

   To calculate the sale price, find 35% of the original sale price and subtract it from the original sale price.

   $\$130 - 0.35 \bullet \$130 = \$84.50$

   To find the total revenue generated by racket sales with this discount applied, substitute the actual sale price into the total revenue function.

$R(d) = 6500 + 12d$	Revenue function
$R(84.5) = 6500 + 12(84.5)$	Substitute 84.5 for d.
$R(84.5) = 6500 + 1014$	Simplify.
$R(84.5) = 7514$	

   The total revenue generated by racket sales at the 35% discounted price of $84.50 is $7,514. This is greater than the minimum revenue of $7,500 called for in the problem statement.

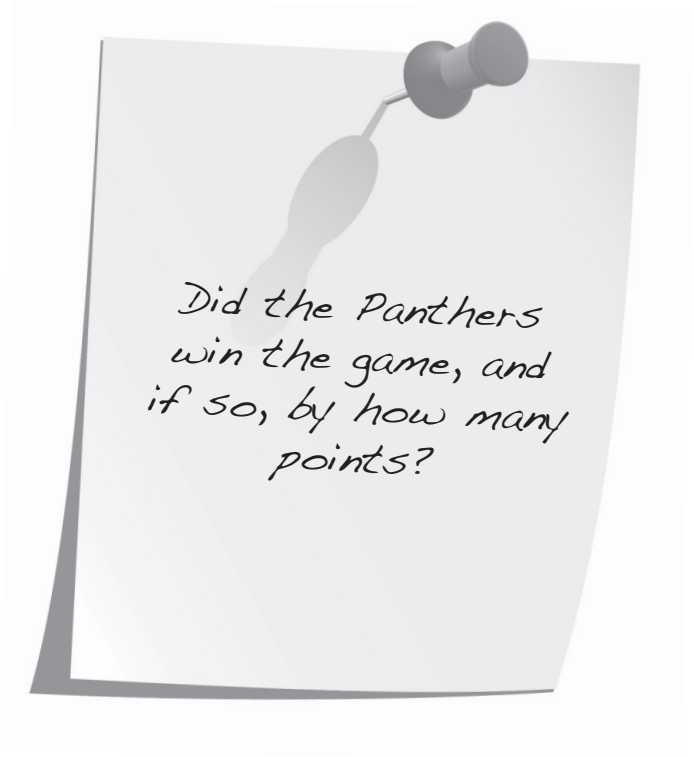
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Problem-Based Task 4B.1.2: Home-Team Hoops

SMP	
1 ✓	2 ✓
3	4
5	6 ✓
7 ✓	8

The home basketball team, the Panthers, scored 85 points, which consisted of x two-point goals, y three-point goals, and z one-point free throws. The data point describing the two- and three-point goals and the one-point free throws is (x, y, z). Describe any restrictions on the type of numbers represented by x, y, and z. Write a linear equation for the total score in terms of x, y, and z. Determine three possible data points that would result in the Panthers' score of 85 points. Then, determine possible values of x, y, and z for three teams that lost to the Panthers, assuming that the Panthers scored 85 in each of the three games.

Suppose that the Panthers scored 3 fewer two-point goals than a fourth team, the Vikings, but then the Panthers also scored 2 more three-point goals and twice as many one-point free throws. Did the Panthers win the game, and if so, by how many points? Assume that no game ended in a tied score.



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Problem-Based Task 4B.1.2: Home-Team Hoops

Coaching

a. What kind of number is represented by x, y, and z?

b. Write expressions for the total number of points from two-point goals, three-point goals, and one-point free throws.

c. Use the results of part b to write a word equation for the Panthers' score in terms of the three types of point-scoring throws.

d. Write the word equation of the Panthers' score as an algebraic equation.

e. What kind of an inequality has to be written for the three losing teams? Explain your answer in terms of tied scores. Then, write the inequality.

f. Determine three possible data points that would result in a score of 85 for the Panthers, and three possible data points and resulting scores for each of the losing teams.

g. Write an equation for the Panthers' score for their game against the Vikings in terms of x, y, and z. Then, write an equation for the Vikings' score.

h. Use the equations from part g to determine by how many points the Panthers won or lost.

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Problem-Based Task 4B.1.2: Home-Team Hoops

Coaching Sample Responses

a. What kind of number is represented by x, y, and z?

The variables x, y, and z represent the number of each kind of basketball score; therefore, they have to be whole numbers.

b. Write expressions for the total number of points from two-point goals, three-point goals, and one-point free throws.

If x is the number of two-point shots, then the total points from two-point shots will be the product of 2 and x or $2x$. If y is the number of three-point shots, then the total points from three-point shots will be the product of 3 and y or $3y$. If z is the number of one-point free throws, then the total points from free throws will be 1 and z or $1z$, which simplifies to z.

c. Use the results of part b to write a word equation for the Panthers' score in terms of the three types of point-scoring throws.

The word equation will represent the number of each of the three types of goals that will yield a total sum of 85 points.

The total score is the sum of two-point goals plus three-point goals plus one-point free throws.

d. Write the word equation of the Panthers' score as an algebraic equation.

Replace the key terms of the word problem with the known information for the total score, the algebraic expressions from the response to part b, and operator symbols.

total score	is	two-point goals	plus	three-point goals	plus	one-point free throws
↓	↓	↓	↓	↓	↓	↓
85	=	$2x$	+	$3y$	+	z

The algebraic equation is $85 = 2x + 3y + z$.

e. What kind of an inequality has to be written for the three losing teams? Explain your answer in terms of tied scores. Then, write the inequality.

The inequality comparing the Panthers' score of 85 to a losing team's score has to be "greater than," not "greater than or equal to" because games do not end in a tied score. Therefore, rewrite $85 = 2x + 3y + z$ into an inequality using the $>$ symbol: $85 > 2x + 3y + z$.

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f. Determine three possible data points that would result in a score of 85 for the Panthers, and three possible data points and resulting scores for each of the losing teams.

Answers will vary. Sample answers:

- Panthers: (30, 8, 1), (32, 5, 6), (40, 0, 5)
- Losing teams: (10, 10, 10), 60 points; (30, 5, 5), 80 points; (35, 4, 2), 84 points

g. Write an equation for the Panthers' score for their game against the Vikings in terms of x, y, and z. Then, write an equation for the Vikings' score.

Let x, y, and z represent the number of two-point, three-point, and one-point shots scored by the Vikings. The Panthers scored 3 fewer two-point goals, 2 more three-point goals, and twice as many one-point goals as the Vikings. Write the Panther's score in terms of x, y, and z.

The score for the Panthers is given by the following equation:

Panthers' Score = $2(x - 3) + 3(y + 2) + 2z$

The score for the Vikings is given by the following equation:

Vikings' Score = $2x + 3y + z$

h. Use the equations from part g to determine by how many points the Panthers won or lost.

Compare the scores by simplifying the expressions for the number of points the Panthers scored.

- Panthers: $2(x - 3) + 3(y + 2) + 2z = 2x - 6 + 3y + 6 + 2z = 2x + 3y + 2z$
- Vikings: $2x + 3y + z$

The expressions are the same except that the Panthers scored twice as many one-point free throws as the Vikings. Therefore, the Panthers won by two times the number of free-throw points made by the Vikings because z is a positive whole number.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4B.1.2: Representing and Interpreting Constraints

For problems 1–3, determine the restricted domain that corresponds to the range constraint on the dependent variable in the equation.

1. $y = -3x + 5$ if $y > -1$

2. $y = 2x^2 - 3$ if $y \leq 11$

3. $y = 2 + 3^x$ if $y < 4$

For problems 4–7, use the given constraints on the variable(s) to determine the restricted domain and range of the system.

4. $x + y > 3$, $x - y < 2$ if $x > 1$

5. $2x - y \leq 3$, $x - 2y \geq 3$ if $x \leq -2$

6. $x + 2y < 3$, $3x - 2y > 1$ if $x < 0$ and $y > 1$

7. $x \leq 2y$, $y \geq 2x$ if $x < 2$ and $y > 2$

Use the given information to solve problems 8–10.

8. The distance an object falls under the influence of gravity is given by $h(t) = \frac{1}{2}gt^2$, where $h(t)$ is distance in meters and t is time in seconds. If $g = -9.8$ meters per second squared, what are the constraints on the values of the variables $h(t)$ and t?

9. A school club is selling canvas tote bags to raise money for a trip to an amusement park. The total cost of the bags is $750. How many bags do the club members have to sell at $12 per bag before they start making money for their trip?

10. A state environmental protection agency considers the presence of a toxic substance in soil "negligible" when the amount of the substance reaches 0.1% of the amount originally measured. The equation that measures the reduction in the toxic substance is given by $A(t) = A_0 \bullet e^{-t}$, in which A_0 is the original amount of the toxic substance and t is the time in years. What is the domain of the time variable t for the amount of the toxic substance to be declared "negligible"?

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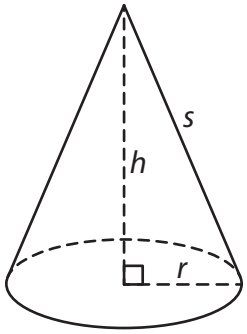
Lesson 4B.1.3: Rearranging Formulas

Warm-Up 4B.1.3

Mahomet is an artist who creates cast-iron geometric sculptures. The developers of an upscale office building want him to make a sculpture for the building's front lobby. Mahomet has determined the sculpture will be a right circular cone with a volume of 12π cubic meters, a slant height s of 5 meters, and a radius r of 3 meters. He needs to determine the cone's actual height to make sure the sculpture will fit in the lobby, which has a height of 5 meters.

The volume of a right circular cone is given by the formula $V = \frac{1}{3}\pi r^2 h$, in which h is the height of the cone and r is the radius of the cone's base. The height h is perpendicular to the base, and the slant height s is the distance from the vertex of the cone to a point on the circumference of its circular base.

Use the formula and the given information to complete the problems that follow.



1. Rewrite the given formula in terms of the height, h.

2. Use the known values and the rewritten formula to solve for the cone's height, h.

3. Using the variables s, h, and r, how could the Pythagorean Theorem be used to determine the height, h? Write this formula in terms of h.

4. Using this alternate formula, what is the cone's height, h?

5. Compare your results from problems 2 and 4. Are they the same? Explain.

6. Based on your results, will Mahomet's cone fit in the lobby?

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Lesson 4B.1.3: Rearranging Formulas

Common Core State Standard

A–CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.*★

Warm-Up 4B.1.3 Debrief

1. Rewrite the given formula in terms of the height, *h*.

   Rearrange the volume formula to solve for the height, *h*.

   $V = \frac{1}{3}\pi r^2 h$ Formula for the volume of a cone

   $3V = \pi r^2 h$ Multiply both sides by 3.

   $\frac{3V}{\pi} = r^2 h$ Divide both sides by π.

   $\frac{3V}{\pi r^2} = h$ Divide both sides by r^2.

   Solved for *h*, the rewritten volume formula is $h = \frac{3V}{\pi r^2}$.

2. Use the known values and the rewritten formula to solve for the cone's height, *h*.

   We are given that the sculpture's volume, *V*, will be 12π cubic meters, and its radius, *r*, will be 3 meters.

   Substitute these values into the rearranged volume formula, $h = \frac{3V}{\pi r^2}$.

   $h = \frac{3V}{\pi r^2}$ Formula for volume in terms of *h*

   $h = \frac{3(12\pi)}{\pi(3)^2}$ Substitute 12π for *V* and 3 for *r*.

   $h = \frac{36\pi}{9\pi}$ Multiply.

   $h = 4$ Divide.

   The height of the cone is 4 meters.

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3. Using the variables *s*, *h*, and *r*, how could the Pythagorean Theorem be used to determine the height, *h*? Write this formula in terms of *h*.

   Since *s*, *h*, and *r* form a right triangle, the Pythagorean Theorem can be used to solve for the height.

   Substitute the variables *s*, *h*, and *r* into the formula for the Pythagorean Theorem, and then solve for *h*.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$(r)^2 + (h)^2 = (s)^2$	Substitute *r* for *a*, *h* for *b*, and *s* for the hypotenuse, *c*.
$h^2 = s^2 - r^2$	Subtract r^2 from both sides.

   The formula $h^2 = s^2 - r^2$ can also be used to solve for the cone's height, *h*.

4. Using this alternate formula, what is the cone's height, *h*?

   We are given that the slant height, *s*, is 5 and the radius, *r*, is 3. Substitute these values into the rewritten Pythagorean Theorem formula and solve for the height, *h*.

$h^2 = s^2 - r^2$	Rewritten formula
$h^2 = (5)^2 - (3)^2$	Substitute 5 for *s* and 3 for *r*.
$h^2 = 25 - 9$	Apply the exponents.
$h = \sqrt{25-9}$	Take the square root of both sides.
$h = \sqrt{16}$	Subtract.
$h = 4$	Simplify.

   Using the Pythagorean Theorem, the height of the cone is 4 meters.

5. Compare your results from problems 2 and 4. Are they the same? Explain.

   Using the volume formula, $h = 4$. Using the Pythagorean Theorem, $h = 4$. Yes, the results are the same. Either method can be used to determine that the cone's height is 4 meters.

6. Based on your results, will Mahomet's cone fit in the lobby?

   The height of the cone is 4 meters and the height of the lobby is 5 meters. The lobby is 1 meter higher than the cone, so Mahomet's cone will fit in the lobby.

Connection to the Lesson

- Students will substitute values for variables in a formula.
- Students will express variables in terms of other variables in a formula.
- Students will isolate a variable in a formula by performing mathematical operations.

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Prerequisite Skills

This lesson requires the use of the following skills:

- using the mathematical operation(s) of addition, exponentiation, finding roots, multiplying, and/or using inverses to solve equations (N.RN.2)
- naming constraints or restrictions on variables in a formula based on mathematical and/or real-world conditions in a problem (F-LE.5)

Introduction

Mathematical formulas that involve exponents, logarithms, powers, and roots have constraints on the values that can be assigned or associated with two or more variables. Real-world formulas involving measurements like area, length, time, and volume have additional restraints, whereas others (such as current, direction, and voltage) can have positive and negative values in addition to the limits imposed by real-world relationships. Rearranging variables in a formula can result in changes to these constraints, so the result of rearranging variables has to be checked and redefined in those cases. Formulas can often be analyzed and visualized in the same way that equations and functions can, and the data resulting from formulas can be presented as data in tables or in graphs.

Key Concepts

- A formula relates two or more variables in a mathematical or a real-world problem context.
- The values of the variables in a formula can be constrained or limited by mathematical or real-world conditions. For example, the formula $A = \frac{1}{2}bh$ for the area of a triangle contains three quantities that are positive. If a triangle is graphed on a coordinate plane and its orientation on the plane results in either b or h being assigned a negative value, then the other variable must also be assigned a negative value so that the area of the triangle is positive.

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- Rearranging a formula can leave the constraints on variable values the same or it can change them. For example, in the distance-rate-time formula $d = rt$, time is constrained by the condition $t \geq 0$, with $t = 0$ signifying the time at which motion begins. The formula can be rewritten as $r = \frac{d}{t}$; however, $r = \frac{d}{t}$ is undefined at $t = 0$, so in this case $t > 0$. This corresponds to the real-world condition that $r \neq 0$ only when $t \neq 0$, since at $t = 0$, there is no distance covered and no motion.
- Formulas can be rearranged to express one or more variables in terms of another variable. A rearranged formula can reduce the number of variables to be calculated.

Common Errors/Misconceptions

- using the wrong operation (e.g., dividing instead of multiplying) in the process of isolating a variable in a formula
- forgetting to check that the constraints on a variable are the same after the variable is isolated or relocated in a formula
- confusing a mathematical constraint with a real-world constraint in a formula
- forgetting to check both the mathematical *and* real-world constraints on the variable(s) in a rearranged formula

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Guided Practice 4B.1.3

Example 1

The pressure and temperature inside an insulated hot-beverage bottle is related to the volume of the bottle and the amount of beverage in it by a real-world form of the ideal gas law. The ideal gas law is given by the formula $PV = nRT$, in which n is the number of moles (a unit of counting) of the gas in a container, P is the pressure the gas exerts on the container, V is the volume of the container, and T is the temperature in degrees Kelvin. The only constant in the formula is R, which is the ideal, or universal, gas constant. Rearrange the formula to show how the temperature T is affected by doubling each variable n, P, and V. (*Note*: All of the quantities in the formula are nonzero.)

1. Isolate temperature, T, in the given formula, $PV = nRT$.

   Use division to isolate T on one side of the equation.

$PV = nRT$	Given formula
$\frac{PV}{nR} = \frac{\cancel{n}\cancel{R}T}{\cancel{n}\cancel{R}}$	Divide both sides of the formula by nR.
$\frac{PV}{nR} = T$	Simplify.
$T = \frac{PV}{nR}$	Apply the Symmetric Property of Equality.

The formula $PV = nRT$, isolated for T, is $T = \frac{PV}{nR}$.

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2. Determine how T is affected in the rearranged formula if n is doubled and P and V stay the same.

   We are given that R is a constant, so it will not be affected by changes to the other variables.

   Let the original value of n be n_1. If n_1 is doubled, the resulting value of n is $2n_1$.

   Write the revised ideal gas law from step 1 for both conditions.

   Formula rewritten for n_1 (the original value of n):

   $$T_1 = \frac{PV}{n_1 R}$$

   Formula rewritten for $2n_1$ (twice the original value of n):

   $$T_2 = \frac{PV}{2n_1 R}$$

   Factor out $\frac{1}{2}$ from $T_2 = \frac{PV}{2n_1 R}$:

   $$T_2 = \frac{1}{2} \bullet \frac{PV}{n_1 R}$$

   Notice that T_2 is half of T_1. In other words, *doubling* the value of n reduces the temperature by *half*. In general, increasing n will decrease T if P and V stay the same.

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3. Determine how T is affected in the rearranged formula if P is doubled and n and V stay the same.

   Let the original value of P be P_1. If P_1 is doubled, the resulting value of P is $2P_1$.

   Write the revised ideal gas law from step 1 for both conditions.

   Formula rewritten for P_1 (the original value of P):

   $$T_1 = \frac{P_1 V}{nR}$$

   Formula rewritten for $2P_1$ (twice the original value of P):

   $$T_2 = \frac{2P_1 V}{nR}$$

   Factor out 2 from $T_2 = \frac{2P_1 V}{nR}$:

   $$T_2 = 2 \bullet \frac{P_1 V}{nR}$$

   Notice that T_2 is twice T_1. In other words, *doubling* the value of P *doubles* the temperature. In general, increasing P increases T if n and V stay the same.

4. Determine how T is affected in the rearranged formula if V is doubled and n and P stay the same.

   Note that V, like P, is a factor of the numerator of the formula $T = \frac{PV}{nR}$.

   Since both V and P are in the numerator, changes in the volume, V, would have the same effect on temperature, T, as changes in pressure, P. In other words, if we were to double the volume instead of the pressure, the temperature would still be doubled.

   Therefore, increasing V increases T if n and P stay the same.

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Example 2

The distance, d, from an object's starting location relative to its location at time t can be described by the formula $d = vt + \frac{1}{2}at^2$, in which v is the velocity of the object when timing starts, a is the acceleration of the object when timing starts, and t is the duration of the motion. Rearrange the formula for d so that it can be solved for t using the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. In this problem, the quantities a, d, and v are constants that can be positive, negative, or 0. Apply the condition $t > 0$ to the resulting values of t and interpret the result(s) for real-world motion if the following values are given: $a = -2$ meters per second squared, $d = 10$ meters from a reference point, and $v = 7$ meters per second. Describe the motion of the object using these values.

1. Rewrite the given formula for d so that all of the terms are one side of the equation.

   Rewriting the formula so all terms are on one side of the equation corresponds to having 0 on one side of the equation. Results that are in the form $ax^2 + bx + c = 0$ reveal the values to be substituted into the quadratic formula.

$d = vt + \frac{1}{2}at^2$	Given equation
$0 = \frac{1}{2}at^2 + vt - d$	Subtract d from both sides of the equation.
$0 = at^2 + 2vt - 2d$	Multiply both sides of the equation by 2.
$at^2 + 2vt - 2d = 0$	Apply the Symmetric Property of Equality.

$at^2 + 2vt - 2d = 0$ is a quadratic in the form $ax^2 + bx + c = 0$.

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2. Identify the values of the resulting equation that correspond to a, b, c, and x in the quadratic formula.

The quadratic formula is $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. For the equation $at^2+2vt-2d=0$, $a=a$, $b=2v$, $c=-2d$, and $x=t$.

3. Use the quadratic formula and the values determined in the previous step to write an equation that can be solved for t.

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	Quadratic formula
$(t)=\frac{-(2v)\pm\sqrt{(2v)^2-4(a)(-2d)}}{2(a)}$	Substitute a for a, $2v$ for b, $-2d$ for c, and t for x.
$t=-\frac{v}{a}\pm\frac{\sqrt{4v^2+8ad}}{2a}$	Simplify.
$t=-\frac{v}{a}\pm\frac{2\sqrt{v^2+2ad}}{2a}$	Factor out 2 from under the radical sign.
$t=-\frac{v}{a}\pm\frac{\sqrt{v^2+2ad}}{a}$	Divide.

The resulting equation is $t=-\frac{v}{a}\pm\frac{\sqrt{v^2+2ad}}{a}$.

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4. Find t for the given values of $a = -2$ meters per second squared, $d = 10$ meters from a reference point, and $v = 7$ meters per second.

   Substitute these values into the simplified quadratic equation for t.

   $t = -\frac{v}{a} \pm \frac{\sqrt{v^2 + 2ad}}{a}$ Simplified quadratic equation

   $t = -\frac{(7)}{(-2)} \pm \frac{\sqrt{(7)^2 + 2(-2)(10)}}{(-2)}$ Substitute –2 for a, 10 for d, and 5 for v.

   $t = 3.5 \pm \frac{\sqrt{9}}{2}$ Simplify.

   $t = 3.5 \pm 1.5$

   $t = 2$ or $t = 5$

   These results give $t = 2$ or $t = 5$. Since $t > 0$, both values of t are relevant.

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5. Use the values of t found in the previous step to calculate d. Describe the motion of the object in terms of the real-world constraints.

For $t = 2$:

$d = vt + \frac{1}{2}at^2$	Original formula
$d = (7)(2) + \frac{1}{2}(-2)(2)^2$	Substitute –2 for a, 2 for t, and 7 for v.
$d = 10$	Simplify.

For $t = 5$:

$d = vt + \frac{1}{2}at^2$	Original formula
$d = (7)(5) + \frac{1}{2}(-2)(5)^2$	Substitute –2 for a, 5 for t, and 7 for v.
$d = 10$	Simplify.

Interpreting these values in the context of the problem, the object began its journey in a positive direction from its starting point, passed a point 10 meters from the starting point, then reversed direction and passed the 10-meter point again. The acceleration and velocity result in motion that eventually takes the object back (a negative distance) toward the reference point.



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Example 3

The formula for a standard earthquake-body wave scale, m_b, is given by $m_b = \log\left(\frac{A}{T}\right) + Q$, in which A is the amplitude of the ground motion in microns (10^{-6} meter), T is the period of the wave, and Q is a correction constant. Determine a formula for the frequency of the earthquake wave if the frequency F is defined as the reciprocal of the wave period. Rearrange the formula to find the range of F values for when the range of T values is [4, 5] seconds. Then, rearrange the formula to find the range of values of A for when the range of values of m_b is [6, 9], the range of T values is [4, 5] seconds, and $Q = 2$.

1. Substitute an expression for F in place of T in the formula.

   Frequency F can be written as $F = \frac{1}{T}$. The argument of the logarithm $\frac{A}{T}$ can be written as $A\left(\frac{1}{T}\right)$. Therefore, the argument of the logarithm can be written as AF.

   $m_b = \log\left(\frac{A}{T}\right) + Q$ Original formula

   $m_b = \log(AF) + Q$ Substitute AF for $\frac{A}{T}$.

   The formula, using F in place of T, is $m_b = \log(AF) + Q$.

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2. Use the definition of the logarithm to rewrite the formula with an exponential term. Then, solve it for F.

   Isolate the logarithm from the rest of the formula, then simplify the result and solve it for F.

$m_b = \log(AF) + Q$	Modified formula
$m_b - Q = \log(AF)$	Subtract Q from both sides.
$10^{m_b - Q} = AF$	Definition of the logarithm
$\frac{10^{m_b}}{10^Q} = AF$	Simplify the exponential term.
$F = \frac{10^{m_b}}{A \bullet 10^Q}$	Divide both sides by A.

   The rewritten formula is $F = \frac{10^{m_b}}{A \bullet 10^Q}$.

3. Use the relationship between the period and the frequency to calculate the range of frequency values.

   The period T is related to the frequency F by the relationship $F = \frac{1}{T}$.

   If the range of T is [4, 5] seconds, then the range of F values is the reciprocal of the period interval values, or $\left[\frac{1}{5}, \frac{1}{4}\right]$.

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4. Write the formula for the earthquake-magnitude scale so that the factors with known values are isolated from A.

   To find the range of values of A for when the range of values of m_b is [6, 9] and $Q = 2$, first substitute 2 for Q in the rewritten formula, $F = \frac{10^{m_b}}{A \bullet 10^{Q}}$, and then rearrange the formula so that it's in terms of A.

$F = \frac{10^{m_b}}{A \bullet 10^{Q}}$	Rewritten formula
$F = \frac{10^{m_b}}{A \bullet 10^{(2)}}$	Substitute 2 for Q.
$F = \frac{10^{m_b - 2}}{A}$	Simplify using the rules of exponents.
$AF = 10^{m_b - 2}$	Multiply both sides by A.
$A = \frac{10^{m_b - 2}}{F}$	Divide both sides by F.
$A = \frac{10^{m_b - 2}}{\left(\frac{1}{T}\right)}$	Substitute $\frac{1}{T}$ for F.
$A = T \bullet 10^{m_b - 2}$	Simplify.

   The formula, written in terms of A when $Q = 2$, is $A = T \bullet 10^{m_b - 2}$.

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5. Use the result of the previous step to write the maximum and minimum values of the range of values for A.

   Use the formula, $A = T \bullet 10^{m_b - 2}$, to determine the minimum and maximum values using the given ranges of T and m_b.

   The range values for T are given as [4, 5], so the minimum value for T is 4 and the maximum value is 5.

   The range values for m_b are given as [6, 9]. Therefore, the minimum value for m_b is 6 and the maximum value is 9.

   Substitute the minimum values, $T = 4$ and $m_b = 6$:

   $A_{\text{minimum}} = T \bullet 10^{m_b - 2}$ Formula for the minimum value of A

   $A_{\text{minimum}} = (4) \bullet 10^{(6)-2}$ Substitute 4 for T and 6 for m_b.

   $A_{\text{minimum}} = 4 \bullet 10^{4}$ Simplify.

   The minimum value of A is $4 \bullet 10^4$, or 40,000 microns, which is 4 centimeters.

   Substitute the maximum values, $T = 5$ and $m_b = 9$:

   $A_{\text{maximum}} = T \bullet 10^{m_b - 2}$ Formula for the maximum value of A

   $A_{\text{maximum}} = (5) \bullet 10^{(9)-2}$ Substitute 5 for T and 9 for m_b.

   $A_{\text{maximum}} = 5 \bullet 10^{7}$ Simplify.

   The maximum value of A is $5 \bullet 10^7$ microns, or 50 meters.



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Example 4

The diffusion rate of a gas in the combustion chamber of a diesel engine is directly proportional to the square root of the molecular mass of the gas. This relationship is given by the formula $r = k \bullet \sqrt{m}$, in which k is a constant. Find the diffusion rate for two gases A and B if the molecular mass of gas A is three more than two times the molecular mass of gas B. Describe how the resulting two rates are related.

1. Use the variables m_A, m_B, r_A, and r_B to represent the variables and write the formulas for gas A and gas B.

   Substitute the variables and write the formulas.

   For gas A:

   $r = k \bullet \sqrt{m}$ Original formula

   $r_A = k \bullet \sqrt{m_A}$ Represent r as r_A and m as m_A.

   For gas B:

   $r = k \bullet \sqrt{m}$ Original formula

   $r_B = k \bullet \sqrt{m_B}$ Represent r as r_B and m as m_B.

2. Write an equation for the molecular mass of gas A in terms of the molecular mass of gas B.

   We are given that the mass of gas A is three more than two times the mass of gas B.

   This can be written as $m_A = 2 \bullet m_B + 3$.

3. Substitute the result of step 2 into the formula for gas A.

   $r_A = k \bullet \sqrt{m_A}$ becomes $r_A = k \bullet \sqrt{2 \bullet m_B + 3}$.

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4. Rearrange the formulas for r_A and r_B to isolate k.

For gas A:

$r_A = k \bullet \sqrt{2 \bullet m_B + 3}$ — Rewritten formula for r_A

$\frac{r_A}{\sqrt{2 \bullet m_B + 3}} = k$ — Divide both sides of the formula by the radical expression.

For gas B:

$r_B = k \bullet \sqrt{m_B}$ — Formula for r_B

$\frac{r_B}{\sqrt{m_B}} = k$ — Divide both sides of the formula by the radical expression.

5. Set the resulting formulas equal to each other.

Since both formulas are equivalent to k, they are equivalent to each other.

$$\frac{r_A}{\sqrt{2 \bullet m_B + 3}} = \frac{r_B}{\sqrt{m_B}}$$

6. Rearrange the resulting equation to solve for r_A.

$\frac{r_A}{\sqrt{2 \bullet m_B + 3}} = \frac{r_B}{\sqrt{m_B}}$ — Equation from the previous step

$r_A = \frac{r_B \bullet \sqrt{2 \bullet m_B + 3}}{\sqrt{m_B}}$ — Multiply both sides by the denominator on the left side of the equation to isolate r_A.

$r_A = r_B \bullet \sqrt{\frac{2m_B + 3}{m_B}}$ — Simplify.

$r_A = r_B \bullet \sqrt{2 + \frac{3}{m_B}}$ — Continue to simplify.

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Instruction

7. Describe how the two rates are related.

The rate of diffusion of gas A is greater than the rate of gas B by a factor of $\sqrt{2+\frac{3}{m_B}}$. This is another way of writing the ratio of the square root of the molecular mass of gas A to the square root of the molecular mass of gas B.



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Problem-Based Task 4B.1.3: A Math Melody Mash-Up

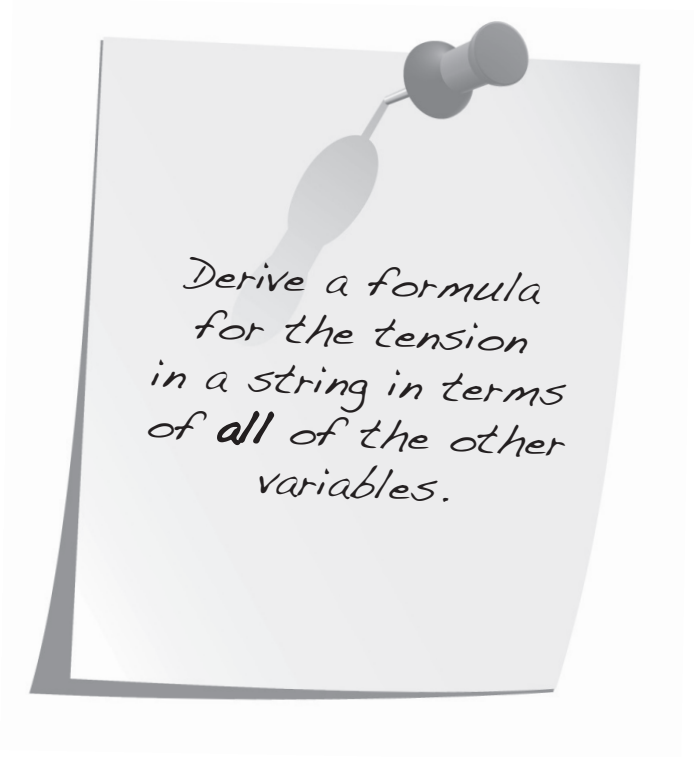
SMP	
1 ✓	2 ✓
3	4
5	6
7	8 ✓

The school band director also teaches physics. He has asked some of the school orchestra to learn more about the science behind their instruments. In one class, he explained how these three formulas apply to stringed musical instruments such as guitars, violins, and pianos:

$$\mu = \frac{m}{L} \qquad v = \sqrt{\frac{F}{\mu}} \qquad v = f \bullet \lambda$$

In the formulas, the quantity μ (the lowercase Greek letter *mu*) is a constant. It is the mass m per unit length of a string of length L. The quantity v is the velocity of a sound wave produced by the string, and is equal to the product of the frequency f of the sound and its wavelength λ (the lowercase Greek letter *lambda*). The quantity F is the tension in the string.

Derive a formula for the tension in a string in terms of *all* of the other variables. Then, explain how the tension can stay the same for a guitar string if the frequency is changed from 275 cycles per second to 400 cycles per second. What is the wavelength of the waves corresponding to these frequencies if the length of the string is 1 meter and the string has a mass of 100 grams?



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Problem-Based Task 4B.1.3: A Math Melody Mash-Up

Coaching

a. Which single quantity should be by itself on one side of the formula?

b. Which formula contains this variable?

c. What are the necessary steps for isolating the variable in this formula? Perform the operation(s) to isolate this variable.

d. Does the resulting formula contain *all* of the variables in the given formulas? If not, rewrite the formula so that all variables are represented.

e. If F stays the same when the frequency changes, what variables change?

f. How are F and λ related if $f = 275$ cycles per second?

g. How are F and λ related if $f = 400$ cycles per second?

h. What information is needed to eliminate the unknowns from the results of parts f and g so that the wavelength λ can be calculated?

i. Substitute the known information from the problem statement into the equations written for parts f and g. Does this provide enough information to find the wavelength λ? If not, what other information is needed?

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Instruction

Problem-Based Task 4B.1.3: A Math Melody Mash-Up

Coaching Sample Responses

a. Which single quantity should be by itself on one side of the formula?

The problem asks for the tension, which is represented by *F*, so *F* should be by itself on one side of the formula.

b. Which formula contains this variable?

The formula $v = \sqrt{\frac{F}{\mu}}$ contains the variable *F*.

c. What are the necessary steps for isolating the variable in this formula? Perform the operation(s) to isolate this variable.

First, both sides of the formula $v = \sqrt{\frac{F}{\mu}}$ have to be squared.

Squaring the formula $v = \sqrt{\frac{F}{\mu}}$ gives $v^2 = \frac{F}{\mu}$.

Then, cross-multiplying gives $F = \mu \bullet v^2$.

d. Does the formula contain *all* of the variables in the formulas? If not, rewrite the formula so that all variables are represented.

No, the formula $F = \mu \bullet v^2$ does not contain all of the variables; λ, *L*, and *m* are missing.

Substitute $\mu = \frac{m}{L}$ for μ and $v = f \bullet \lambda$ for *v* into $F = \mu \bullet v^2$:

$$F = \mu \bullet v^2$$

$$F = \left(\frac{m}{L}\right) \bullet (f \bullet \lambda)^2$$

$$F = \frac{m}{L} \bullet f^2 \bullet \lambda^2$$

The rewritten formula $F = \frac{m}{L} \bullet f^2 \bullet \lambda^2$ contains all the variables from the three given formulas.

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Instruction

e. If F stays the same when the frequency changes, what variables change?

On a single instrument, the mass and length of the string are the same when the frequency changes. If the tension also stays the same, the only quantities that can change are f and λ. Those two variables are factors of the velocity given by $v = f \bullet \lambda$.

f. How are F and λ related if $f = 275$ cycles per second?

Use the formula written for part d: $F = \frac{m}{L} \bullet f^2 \bullet \lambda^2$. Remember that $\frac{m}{L}$ is constant for a single instrument. Substitute 275 for f in the formula, then rearrange the formula so that the numerical term is placed before the other variables.

$$F = (275)^2 \bullet \frac{m}{L} \bullet \lambda^2$$

g. How are F and λ related if $f = 400$ cycles per second?

Again, use the formula from part d: $F = \frac{m}{L} \bullet f^2 \bullet \lambda^2$. Remember that $\frac{m}{L}$ is constant. Substitute 400 for f in the formula, then rearrange the formula so that the numerical term is placed before the other variables.

$$F = (400)^2 \bullet \frac{m}{L} \bullet \lambda^2$$

h. What information is needed to eliminate the unknowns from the results of parts f and g so that the wavelength λ can be calculated?

The force (F), the mass (m), and the length (L), are needed to find λ.

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Instruction

i. Substitute the known information from the problem statement into the equations written for parts f and g. Does this provide enough information to find the wavelength λ? If not, what other information is needed?

The mass, *m*, is 100 grams or 0.1 kilogram and the length, *L*, is 1 meter. Substituting this information gives the formulas $F = (275)^2 \bullet \frac{0.1}{1} \bullet \lambda^2 = 0.1 \bullet (275)^2 \bullet \lambda^2$ and $F = (400)^2 \bullet \frac{0.1}{1} \bullet \lambda^2 = 0.1 \bullet (400)^2 \bullet \lambda^2$. If *F* is the same for both frequencies, the two formulas can be set equal to each other: $0.1 \bullet (275)^2 \bullet \lambda_{275}^2 = 0.1 \bullet (400)^2 \bullet \lambda_{400}^2$, in which λ_{275} is the wavelength corresponding to a frequency of 275 cycles per second and λ_{400} is the wavelength corresponding to a frequency of 400 cycles per second. This last equation can be rewritten as $\frac{275}{400} = \frac{\lambda_{400}}{\lambda_{275}}$, which means that the *ratio* of the wavelengths of the two sounds is the reciprocal of the ratio of the two frequencies. No other specific calculations can be made unless information is provided about the magnitude of the tension, *F*, or the velocity of the wave, *v*.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4B.1.3: Rearranging Formulas

For problems 1–4, rearrange the given formulas to complete the problems.

1. If $V = I \bullet R$ and $P = V \bullet I$, write P in terms of R.

2. If $k = Ae^{-\frac{E_a}{RT}}$, write a formula for E_a.

3. Given that $[H^+]$ and $[OH^-]$ are ions, find $[H^+]$ and $[OH^-]$ if $pH^+ + pOH^- = 14$, $pH^+ = 2 \bullet pOH^-$, $pH^+ = -\log [H^+]$, and $pOH^- = -\log [OH^-]$.

4. Which is greater, the surface tension factor for a ring of diameter l and a circumference C, or the surface tension factor for a wire of length l? Use the formulas $s_{\text{ring}} = \frac{F}{2 \bullet C_{\text{ring}}}$ and $s_{\text{wire}} = \frac{F}{2 \bullet l_{\text{wire}}}$ to justify your answer.

continued

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Use the given information to complete problems 5–10. Rearrange the formulas if necessary.

5. How is L_i related to L_v if $c > v$ and $L_i > 0$ in the formula $L_v = L_i \bullet \sqrt{1 - \frac{v^2}{c^2}}$?

6. The magnetic field, B, near a wire carrying a current, I, is given by the formula $B = \frac{\mu_0 \bullet I}{4\pi r}$, in which r is the distance from the wire to a point in space at which the field is measured. What happens to the values of B if I varies between –1 and 1 ampere and r decreases by 5 meters?

7. What is the range of values for f_c if $D = \frac{RT}{N_A \bullet f_c}$ and $10^3 < D < 10^5$?

8. The formula for the ideal gas law, $PV = nRT$, relates the number of moles of a gas, n, to its pressure P, temperature T, and the volume V of the container in which it is held. The quantity R is a constant. The quantity V_m is the molar volume and is defined by the formula $V_m = \frac{V}{n}$. All of the quantities are positive. Use the formula for the ideal gas law to write the molar volume in terms of P, R, and T.

9. Find a if $K_{eq} = \frac{C^c \bullet D^d}{A^a \bullet B^b}$ and $K_{eq} > 1$.

10. The surface area of a cylinder is given by the formula $A(r, h) = 2\pi r^2 + 2\pi rh$. Find h if $A(r, h) = 9\pi$.

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Progress Assessment

Circle the letter of the best answer.

1. Two medical technologists are running a panel of blood tests. One technologist takes 2 minutes less than the other. Which equation would give the amount of time, t, it would take to run the tests if the technologists work together to split the work on the different test types for each panel?

   a. $t = \frac{1}{x} + \frac{1}{x-2}$

   b. $\frac{1}{t} = \frac{1}{x} + \frac{1}{x-2}$

   c. $t = x + x - 2$

   d. $\frac{1}{t} = \frac{1}{x} - \frac{1}{x-2}$

2. Danica is on the high school welcome committee, and is mixing fruit punch for a reception for the incoming middle school students. She would like to mix a total of 5 gallons of punch using two types of fruit juice: one that's 50% pineapple juice, and another that's 25% pineapple juice. Danica would like the resulting mixture to be 40% pineapple juice. Which equation describes how to find the amount of each of the two types of fruit juice that Danica will need?

   a. $0.5x + 0.25(5 - x) = 2$

   b. $0.5x + 0.25(5 - x) = 0.4$

   c. $0.5x - 0.25(5 - x) = 2$

   d. $5x - 25(5 - x) = 40$

3. The equation $v^2 = v_0^2 + 2ad$ describes how fast a vehicle is moving after accelerating at a constant rate, a, from a starting speed of v_0 over a specific distance, d. Which equation describes the motion of a vehicle accelerating at a rate of 33.75 miles per hour squared and that reaches a speed of 75 miles per hour?

   a. $v^2 = v_0^2 + (33.75)^2 d$

   b. $75 = v_0 + 33.75d$

   c. $v^2 = 75^2 + 67.5d$

   d. $75^2 = v_0^2 + 67.5d$

4. A forester is studying a section of the woods that measures 1 square mile. The section suffered a fire 100 years before the study. The forester found that the average diameter of fir trees in the section exponentially approaches a limit of 50 centimeters. Which equation could describe this growth?

   a. $d = 50(1 - e^{-0.25t})$

   b. $d = 50(1 - e^{0.25t})$

   c. $d = 50\left(1 + e^{-0.25t}\right)$

   d. $d = 50e^{-0.25t}$

continued

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Assessment

5. Power, *P*, is dissipated as heat from a load, *R*, in an air-conditioning circuit. If the circuit voltage, *V*, times the circuit current, *I*, gives the power, and the voltage is the product of the current and the load, *R*, which formula gives the load, *R*?

   a. $R = \frac{I}{P^2}$

   b. $R = \frac{V}{I^2}$

   c. $R = \frac{P}{I^2}$

   d. $R = \frac{V^2}{I}$

6. A neuroscientist conducting a sleep study administered memory tests to subjects who had slept a given number of hours the night before. The lowest score was associated with 4 hours of sleep per night and the maximum score was associated with 7 hours of sleep. Scores decreased again for more than 7 hours of sleep per night. The amount of change in the scores varied between consecutive hours. Which type of equation model best describes this data?

   a. exponential; increasing from 4 to 7 hours or more

   b. linear; decreasing from 4 to 7 hours or more

   c. logarithmic; decreasing from 4 to 7 hours or more

   d. quadratic; increasing and then decreasing from 4 to 7 hours or more

7. The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$, in which *r* is the radius of the sphere. The surface area of a sphere is given by $A = 4\pi r^2$. Which equation describes *A* in terms of *V*?

   a. $A = 4\pi V^2$

   b. $A = \frac{3V}{r}$

   c. $A = 3rV$

   d. $A = \sqrt{\frac{3V}{r}}$

8. What restrictions exist for the variable *c* in the formula $a = \sqrt{b^2 + (c-d)^3}$?

   a. $c < -d - \sqrt[3]{-b^2}$

   b. $c < -d - \sqrt[3]{b^2}$

   c. $c > d + \sqrt[3]{-b^2}$

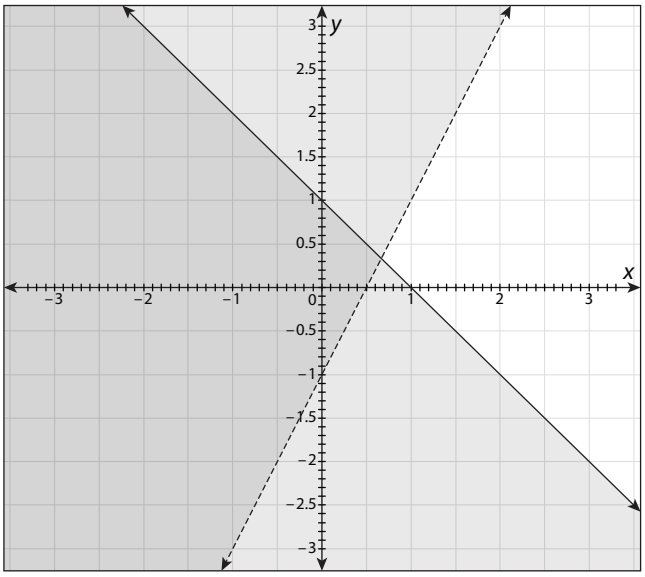
   d. $c < d - \sqrt[3]{b^2}$

continued

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Assessment

9. Which restricted domain of x contains common solutions for the equations shown on the graph?



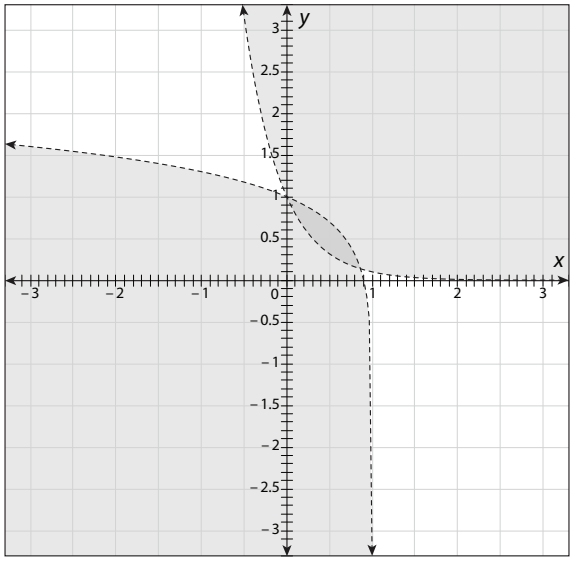
a. $x < \frac{2}{3}$

b. $x > \frac{2}{3}$

c. $x \leq \frac{1}{3}$

d. $x \geq \frac{1}{3}$

10. Which restricted domain of x contains common solutions for the inequalities shown on the graph?



a. $\{0 < x < 1\}$

b. $\{0 < x < 0.86\}$

c. $\{0.1 < x < 1\}$

d. $\{0.86 < x < 1\}$

continued

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Assessment

Use what you have learned about creating equations to answer the following question.

11. The speed of calculations for supercomputers is measured in petaflops (quadrillions of calculations per second). In 2013, the Chinese Tianhe-2 supercomputer reached a record speed of approximately 33.9 petaflops. The increase in the speed of supercomputers is given by the following data covering eight months prior to Tianhe-2 reaching this record.

Month	1	2	3	4	5	6	7	8
Speed (in petaflops)	1	2	2.5	8	11	17	18	34

What type of equation can model this data and why?

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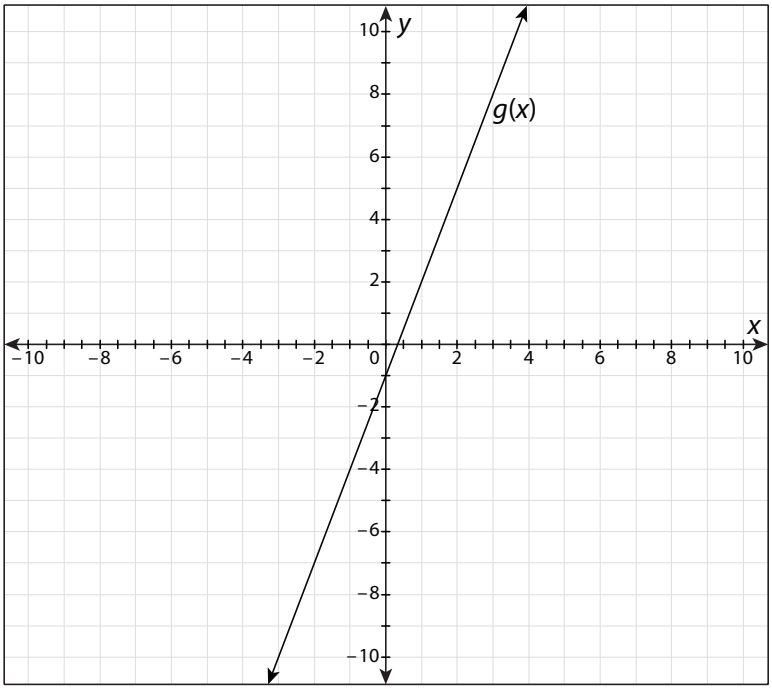
Pre-Assessment

Circle the letter of the best answer.

1. Which function is a transformation of the parent function $f(x) = x^2$ of the form $g(x) = k \bullet f(x)$?

   a. $g(x) = -4 + x^2$
   b. $g(x) = -4x^2$
   c. $g(x) = (x - 4)^2$
   d. $g(x) = 4(x - 4)^2$

2. The graphed function shown results from the transformation(s) of the parent function $f(x) = x$. What is the equation of the graphed function?



   a. $g(x) = \frac{1}{3}x$
   b. $g(x) = \frac{x}{3} - 1$
   c. $g(x) = 3x$
   d. $g(x) = -1 + 3x$

3. Which function is even?

   a. $f(x) = 3x^3 - x + 1$
   b. $f(x) = 3x^3 - x$
   c. $f(x) = 3x^4 - x^2 + 1$
   d. $f(x) = 3x^4 - x^3 + 1$

continued

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Assessment

4. If $f(x) = \log(x + 2)$ and $g(x) = \log(x + 1)$, which function represents $f(x) + g(x)$?

   a. $h(x) = \log(2x + 3)$

   b. $h(x) = \log 2x + \log 3$

   c. $h(x) = \log 6x$

   d. $h(x) = \log(x + 2)(x + 1)$

5. The volume of a cylindrical inflatable tunnel used by soccer players to exit the field is given by $V(r, h) = 2\pi rh$. The tunnel is inflated just before the end of each half of the match. The rates at which the height and radius increase when the tunnel is inflated are given by the functions $h = 4t$ and $r = 6t$. Which equation represents the volume of the tunnel as a function of the inflating time, t?

   a. $V(t) = 48\pi t^2$

   b. $V(t) = 48\pi t$

   c. $V(t) = 48t^2$

   d. $V(t) = 48rht^2$

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Lesson 2: Transforming a Model and Combining Functions

Instruction

Common Core State Standards

SMP	
1 ✓	2
3 ✓	4 ✓
5	6 ✓
7 ✓	8 ✓

F–BF.1 Write a function that describes a relationship between two quantities.★

b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*

F–BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Essential Questions

1. How can you tell if a parent function, $f(x)$, has been changed by the transformations $g(x) = k \bullet f(x)$ or $g(x) = f(x) + k$?
2. How do the transformations $g(x) = f(kx)$ and $g(x) = f(x + k)$ differ in their effect on a parent function, $f(x)$?
3. How do you distinguish between odd and even functions based on their algebraic expressions and their graphs?
4. What function results when you add, subtract, multiply, or divide two or more functions?
5. What function results when you use a function, $f(x)$, as the independent variable in another function, $g(x)$?

WORDS TO KNOW

combination of functions the process of combining two or more functions using the operations of addition, subtraction, multiplication, or division to create a new function

composition of functions the process of substituting one function for the independent variable of another function to create a new function

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Instruction

even function	a function that, when evaluated for $-x$, results in a function that is the same as the original function; $f(-x) = f(x)$
family of functions	a set of functions whose graphs have the same general shape as their parent function. The parent function is the function with a simple algebraic rule that represents the family of functions.
odd function	a function that, when evaluated for $-x$, results in a function that is the opposite of the original function; $f(-x) = -f(x)$
parent function	a function with a simple algebraic rule that represents a family of functions. The graphs of the functions in the family have the same general shape as the parent function.

Recommended Resources

- MathIsFun.com. "Operations with Functions."

  http://www.walch.com/rr/00240

  This site uses a scaffolded, step-by-step approach to review basic operations with functions.

- Purplemath.com. "Function Notation: Even and Odd."

  http://www.walch.com/rr/00241

  This site provides a review of how to determine whether a function is even, odd, or neither using algebraic methods and by evaluating the function's graph. The sample functions are accompanied by illustrations that would be helpful to visual learners.

- SheLovesMath.com. "Parent Functions and Transformations."

  http://www.walch.com/rr/00242

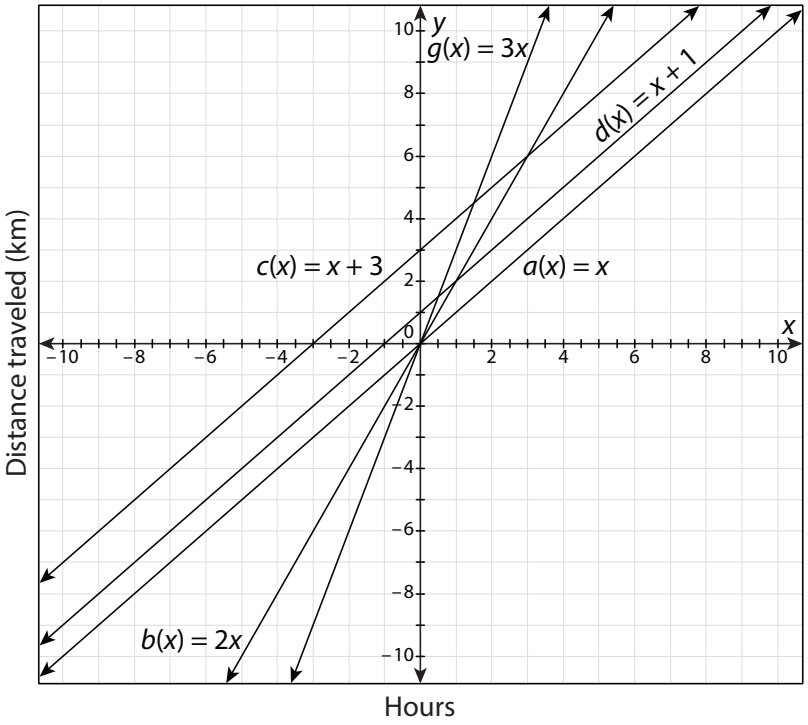
  This comprehensive site reviews basic parent functions and transformations of those functions. The site provides a balanced mix of scaffolded examples and visuals that will help users with a variety of learning-style preferences.

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Lesson 4B.2.1: Transformations of Parent Graphs

Warm-Up 4B.2.1

Five bicyclists—Anh, Brayden, Caroline, Damian, and Ganesh—are training for a race. The following graph shows the distance traveled in kilometers by each bicyclist relative to a landmark (0, 0) over the course of x hours. The graphed functions for the five bicyclists are $a(x)$, $b(x)$, $c(x)$, $d(x)$, and $g(x)$, respectively. Anh, Brayden, and Ganesh took off from the landmark at (0, 0). Caroline and Damian started their rides from their respective houses, represented by the y-intercept for their functions, $c(x)$ and $d(x)$. Let the function $a(x)$ of the first bicyclist, Anh, be the parent function $a(x) = x$, and the four other functions, $b(x)$, $c(x)$, $d(x)$, and $g(x)$, be transformations of the parent function. Use this information and the graph to answer the questions that follow.



1. Which graph or graphs represent the transformation $k \cdot a(x)$?

2. Which graph or graphs represent the transformation $a(kx)$?

3. Which graph or graphs represent the transformation $a(x + k)$?

4. Which graph or graphs represent the transformation $a(x) + k$?

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Instruction

Lesson 4B.2.1: Transformations of Parent Graphs

Common Core State Standard

F–BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Warm-Up 4B.2.1 Debrief

1. Which graph or graphs represent the transformation $k \bullet a(x)$?

   The function $b(x) = 2x$ can be written as $b(x) = 2 \bullet a(x)$.

   The function $g(x) = 3x$ can be written as $g(x) = 3 \bullet a(x)$.

   Therefore, the graphs of $b(x)$ and $g(x)$ represent the transformation $k \bullet a(x)$.

2. Which graph or graphs represent the transformation $a(kx)$?

   The function $b(x) = 2x$ can be written as $b(x) = a(2x)$.

   The function $g(x) = 3x$ can be written as $g(x) = a(3x)$.

   Thus, the graphs of $b(x)$ and $g(x)$ represent the transformation $a(kx)$.

3. Which graph or graphs represent the transformation $a(x + k)$?

   The function $c(x) = x + 3$ can be written as $c(x) = a(x + 3)$.

   The function $d(x) = x + 1$ can be written as $d(x) = a(x + 1)$.

   Hence, the graphs of $c(x)$ and $d(x)$ represent the transformation $a(x + k)$.

4. Which graph or graphs represent the transformation $a(x) + k$?

   The function $c(x) = x + 3$ can be written as $c(x) = a(x) + 3$.

   The function $d(x) = x + 1$ can be written as $d(x) = a(x) + 1$.

   Therefore, the graphs of $c(x)$ and $d(x)$ represent the transformation $a(x) + k$.

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Instruction

Connection to the Lesson

- Students will identify the transformations $g(x) = f(kx)$, $g(x) = k \bullet f(x)$, $g(x) = f(x + k)$, and $g(x) = f(x) + k$.
- Students will learn how to determine which transformations result in changes to dependent and independent variables in a function $f(x)$ that result in the transformed function $g(x)$.
- Students will recognize that different transformations of a function $f(x)$ can possibly result in the same transformed function $g(x)$.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- identifying the parent functions or graphs of exponential, linear, logarithmic, and quadratic functions (F-BF.3)
- identifying the dependent and independent variables in a function (F-IF.2)
- identifying the domain and range of a function and how they might be changed by a transformation (F-IF.5, F-BF.3)
- calculating the change in a function value when a constant is added to or multiplied by the independent variable in the function equation (F-BF.3)

Introduction

Functions can be transformed, or changed, by performing operations on the dependent and independent variable. For example, adding a constant k to the *independent* variable of a function results in a new function of the form $g(x) = f(x + k)$. On the other hand, adding a constant to the *dependent* variable of a function results in a new function of the form $g(x) = f(x) + k$. Similarly, multiplying the independent variable by a constant yields a new function of the form $g(x) = f(kx)$, while multiplying the dependent variable by the constant yields a new function of the form $g(x) = k \bullet f(x)$. The operation performed, and the variable upon which it is performed, can result in various changes to the graph of the function.

Key Concepts

- Recall that a parent function is a function with a simple algebraic rule that represents a family of functions. The parent function is the simplest form of a function that is changed by the addition or multiplication of constants.
- For linear and quadratic functions, the parent functions can be defined as $f(x) = x$ and $f(x) = x^2$, respectively.
- A **family of functions** is a set of functions whose graphs have the same general shape as their parent function.
- For example, the functions $f(x) = 25x$, $g(x) = 75x$, and $h(x) = 110x$ constitute a family of functions, that are of the general forms $a(x) = k \bullet b(x)$ or $c(x) = b(kx)$.
- Transformations of the dependent or independent variables of a function can shift the location of the function on the coordinate plane. Or, they can make the end behavior of the function more pronounced or less pronounced.

SRB p. U4B-57

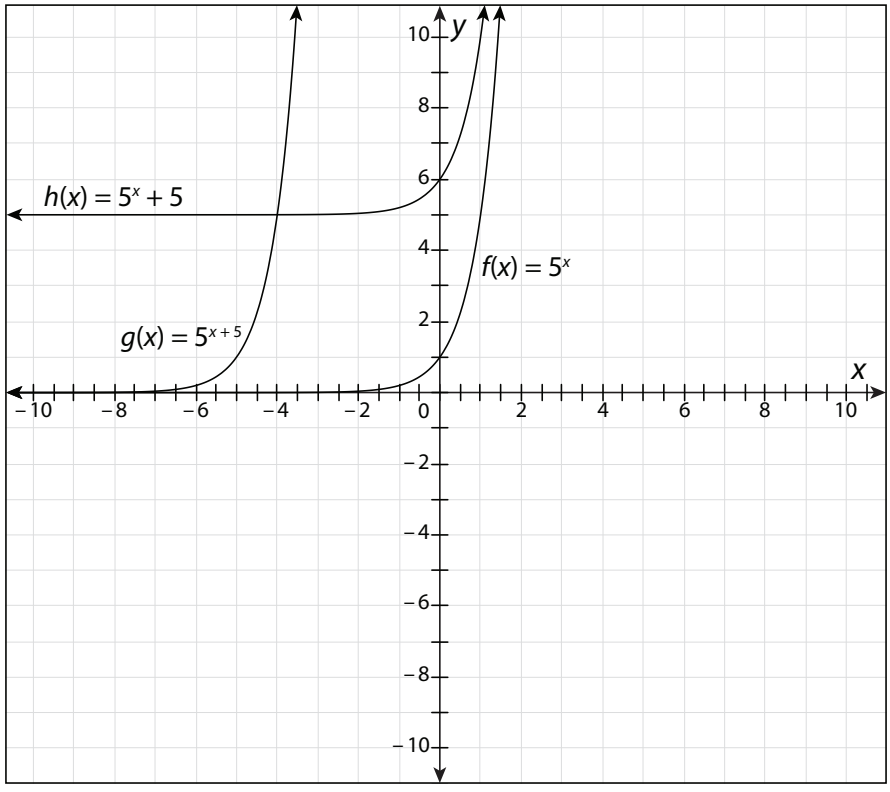
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Instruction

- Transformations can also change the domain and range of the parent function. Solutions to real-world problems involving transformations should include a check of the domain and range values to ensure that the solutions continue to reflect the realities of the problem conditions.
- Graphing a transformed function and its corresponding parent function on a coordinate plane can help show the similarities and differences of the characteristics of each function.
- Three types of transformations can be demonstrated algebraically or graphically.

Case 1: Addition of a Constant

- A constant can be added to the dependent or independent variable.
- The transformation $g(x) = f(x + k)$ adds a constant, k, to the value of the independent variable, x.
- The transformation $g(x) = f(x) + k$ adds a constant, k, to the value of the dependent variable, $f(x)$.
- The functions $g(x) = 5^{x+5}$ and $h(x) = 5^x + 5$ are such transformations of the function $f(x) = 5^x$. All three functions are shown on the following graph. Notice how the transformed functions $g(x)$ and $h(x)$ have been shifted along the axes.

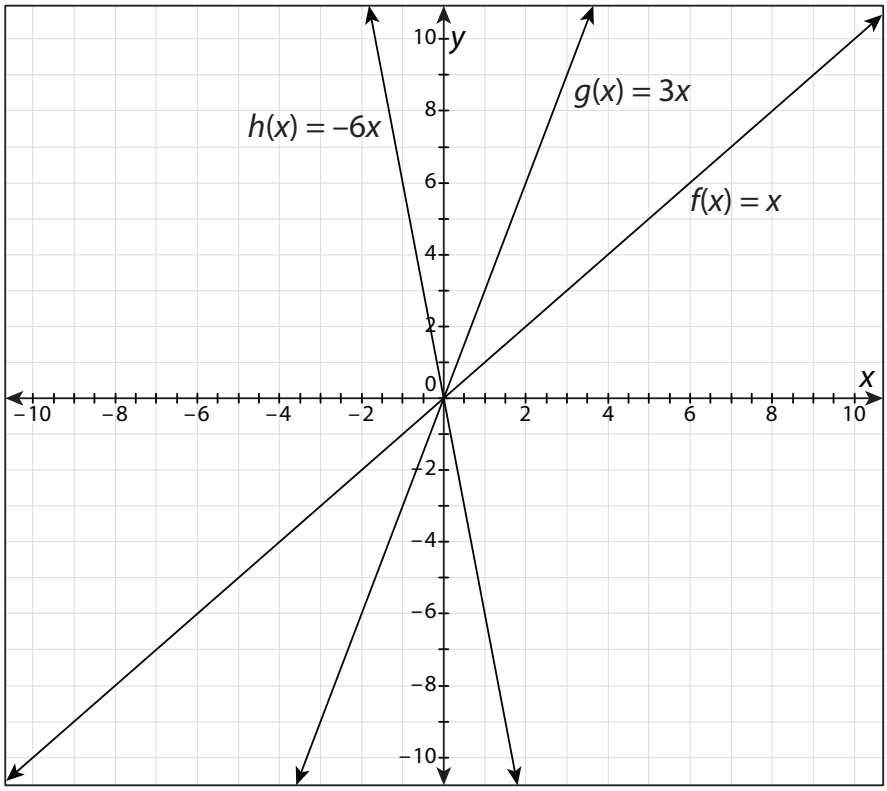


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Instruction

Case 2: Multiplication by a Constant

- The dependent or independent variable can be multiplied by a constant.
- The transformation $g(x) = f(kx)$ multiplies the independent variable, x, by a constant, k.
- The transformation $g(x) = k \bullet f(x)$ multiplies the value of the dependent variable, $f(x)$, by a constant, k.
- The functions $g(x) = 3x$ and $h(x) = -6x$ are such transformations of the function $f(x) = x$. All three functions are shown on the following graph.

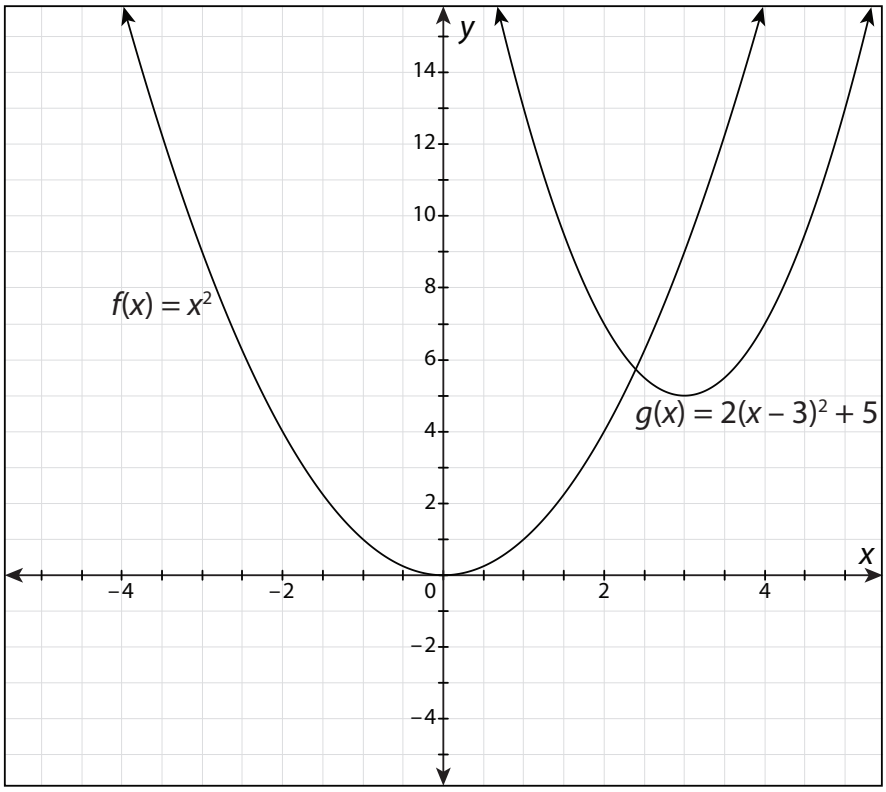


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Instruction

Case 3: Addition of a Constant and Multiplication by a Constant

- The first two cases can be combined to show both addition and multiplication.
- For example, the quadratic function $g(x) = 2(x - 3)^2 + 5$ represents two addition transformations and a multiplication transformation of the parent function $f(x) = x^2$.
  - First, 5 is added to the function:
    $f(x) + 5 = x^2 + 5$
  - Then, 3 is subtracted from x:
    $f(x - 3) + 5 = (x - 3)^2 + 5$
  - Finally, the x-squared term is multiplied by 2:
    $2[f(x - 3)] + 5 = 2[(x - 3)^2] + 5$
  - The resulting function is renamed $g(x)$:
    $g(x) = 2[f(x - 3)] + 5 = 2(x - 3)^2 + 5$
- The two functions $f(x)$ and $g(x)$ are shown on the following graph.



- A graphing calculator also provides an efficient way to change coefficients as well as add terms to functions. The following directions demonstrate how to do this on the TI-83/84 and TI-Nspire calculators.

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Instruction

On a TI-83/84:

Step 1: Graph the family of functions of the form $f(x) = ax + b$ for three values of a (a = 1, 2, and 3) and one value of b (b = 1). Then, graph the family of functions for three values of b (b = 2, 4, and 6) and one value of a (a = 1).

Step 2: Press [Y=]. Then, enter the *form* of the function using the "list" tool for the coefficient a at the cursor next to "\Y1=" and enter the three values of a in the "list" space in the equation: [2ND][(][1][,][2][,][3][2ND][)][X, T, θ, n][+][1].

Step 3: Press [ENTER] to save the equations. Then, press [GRAPH].

Step 4: Next, press [Y=] and move the cursor to the "\Y2=" line. Enter the *form* of the second family of functions using the "list" tool for the constant that is added to the x term: [X, T, θ, n][+][2ND][(][2][,][4][,][6][2ND][)].

Step 5: Press [ENTER] to save the equations. Then, press [GRAPH].

On a TI-Nspire:

Step 1: Graph the family of functions of the form $f(x) = ax + b$ for three values of a (a = 1, 2, and 3) and one value of b (b = 1). Then, graph the family of functions for three values of b (b = 2, 4, and 6) and one value of a (a = 1).

Step 2: Press [home]. Arrow down to the graphing icon and press [enter].

Step 3: Enter the *form* of the function using the "list" tool for the coefficient a at the cursor next to "$f1(x)$ =" and enter the three values of a in the "list" space in the equation: [ctrl][)][1][,][2][,][3]. Then, press the right arrow key to move your cursor outside of the "list" field and continue entering the equation.

Step 4: Press [enter] to graph the equations.

Step 5: To graph the second family of functions, press [menu] and arrow down to 3: Graph Type, then arrow right to 1: Function. Press [enter].

Step 6: Enter the *form* of the second family of functions at the cursor next to "$f2(x)$ =" using the "list" tool for the constant that is added to the x term: [X][+][ctrl][)][2][,][4][,][6]. Then, press the right arrow key to move your cursor outside of the "list" field and continue entering the equation.

Step 7: Press [enter] to graph the equations.

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Instruction

Common Errors/Misconceptions

- mistaking the effect(s) that a constant k has on a dependent and an independent variable; e.g., mixing up addition and multiplication
- misidentifying the direction in which a transformation of the form $x + k$ or $f(x) + k$ shifts the graph of a function
- incorrectly determining the effect (e.g., dilation or shrinkage) that a transformation of the form kx or $k \cdot f(x)$ has on the graph of a function
- forgetting to compare the domain and range of a transformed function to those of the parent function

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Instruction

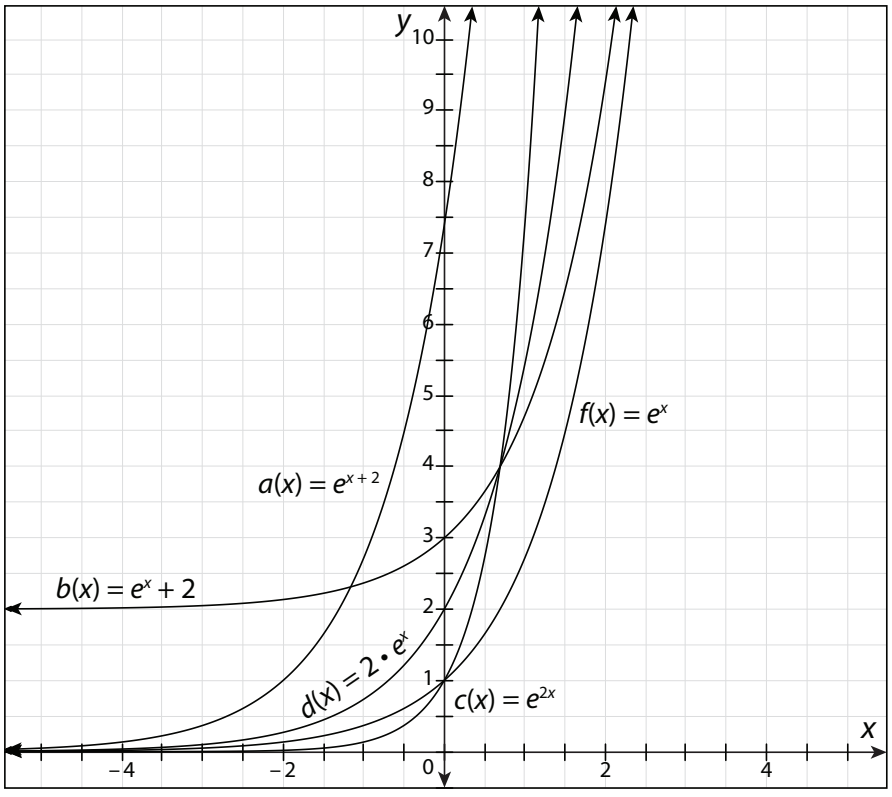
Guided Practice 4B.2.1

Example 1

Compare the graphs of the exponential functions $a(x) = e^{x+2}$, $b(x) = e^x + 2$, $c(x) = e^{2x}$, and $d(x) = 2 \cdot e^x$. Compare each graph's x- and y-intercepts, domain, and range to those of the parent function, $f(x) = e^x$. Finally, determine the value of x at the intersection point of the functions $a(x)$ and $b(x)$.

1. Graph each of the given functions on a graphing calculator.

   The resulting graph should appear as follows.



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2. Determine and compare the domains and ranges of the functions.

The domain and range values can be inferred from the table of x- and y-values of the graphing calculator or by looking at the graph of each function.

Function	Domain	Range
$f(x) = e^x$	$(-\infty, \infty)$	$(0, \infty)$
$a(x) = e^{x+2}$	$(-\infty, \infty)$	$(0, \infty)$
$b(x) = e^x + 2$	$(-\infty, \infty)$	$(2, \infty)$
$c(x) = e^{2x}$	$(-\infty, \infty)$	$(0, \infty)$
$d(x) = 2 \bullet e^x$	$(-\infty, \infty)$	$(0, \infty)$

Each transformed function has the same domain as the parent function, $f(x)$. All but $b(x)$ have the same range as the parent function.

3. Determine and compare the y-intercepts of the functions.

To determine the y-intercept of each function, find the value of each function at $x = 0$.

It can be seen from the graph that there are four different y-intercepts, but the values can also be shown algebraically by substituting 0 for x in each function.

$f(x) = e^x$	Given function
$f(0) = e^{(0)}$	Substitute 0 for x.
$f(0) = 1$	Simplify.

The y-intercept of $f(x) = e^x$ is (0, 1).

Follow the same process for the transformed functions.

$a(x) = e^{x+2}$	$b(x) = e^x + 2$	$c(x) = e^{2x}$	$d(x) = 2 \bullet e^x$
$a(0) = e^{[(0)+2]}$	$b(0) = e^{(0)} + 2$	$c(0) = e^{2(0)}$	$d(0) = 2 \bullet e^{(0)}$
$a(0) = e^2$	$b(0) = 1 + 2$	$c(0) = e^0$	$d(0) = 2 \bullet 1$
$a(0) \approx 7.4$	$b(0) = 3$	$c(0) = 1$	$d(0) = 2$

(*continued*)

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Instruction

For each transformed function, write the y-intercept.

- $a(x) = e^{x+2}$: approximately (0, 7.4)
- $b(x) = e^x + 2$: (0, 3)
- $c(x) = e^{2x}$: (0, 1)
- $d(x) = 2 \cdot e^x$: (0, 2)

The y-intercept of $c(x)$ is the same as that of the parent function, $f(x)$; all the other transformations have been shifted up along the y-axis.

4. Determine and compare the x-intercepts of the functions.

   To find the x-intercepts, solve each function for $y = 0$.

   None of the functions exist at $y = 0$. The function $b(x)$ does not exist for values of y that are less than or equal to 2. It can be seen from the graph that the other functions approach but do not reach $y = 0$.

5. Determine the value of x at the intersection point of the functions $a(x)$ and $b(x)$.

   At their intersection point, $a(x) = b(x)$. Therefore, set the right sides of the equations for $a(x)$ and $b(x)$ equal and solve for x.

$e^{x+2} = e^x + 2$	Set $a(x)$ equal to $b(x)$.
$e^{x+2} - e^x = 2$	Subtract e^x from both sides.
$e^x(e^2 - 1) = 2$	Rewrite using the Distributive Property.
$e^x = \frac{2}{e^2 - 1}$	Divide both sides by $e^2 - 1$.
$x = \ln\left(\frac{2}{e^2 - 1}\right)$	Solve for x using natural logarithms.
$x = \ln 2 - \ln(e^2 - 1)$	Simplify.
$x \approx -1.16$	

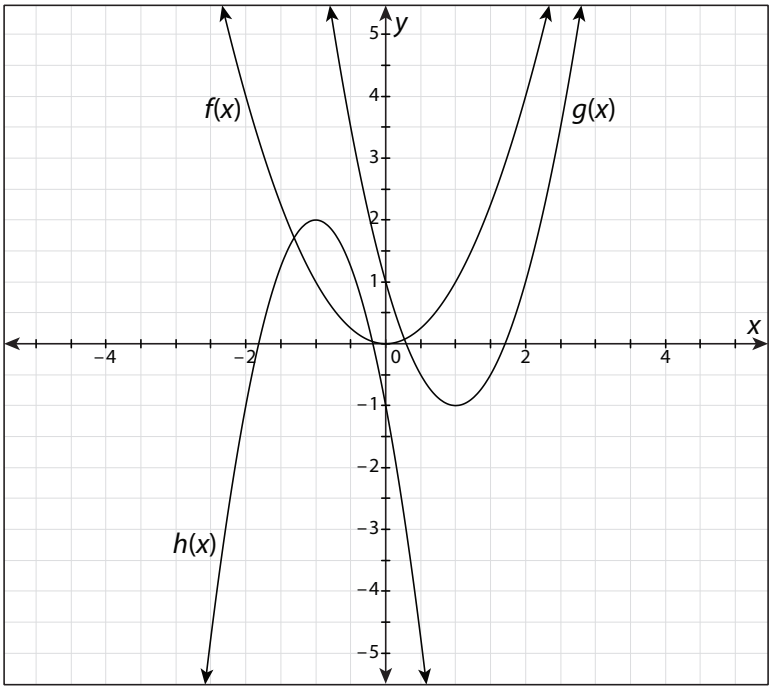
The value of x at the intersection point of the functions $a(x)$ and $b(x)$ is approximately –1.16.

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Instruction

Example 2

The graph shows a parent quadratic function $f(x) = x^2$ and two quadratic functions, $g(x)$ and $h(x)$, derived from it. Use the maximum or minimum point of the quadratic functions to derive their equations, and write the functions in a form that indicates the transformation(s) of the parent function. Then, describe what transformation(s) the parent function underwent to result in each transformed function.



1. Use the vertex form of a quadratic function, $f(x) = a(x - h)^2 + k$, to determine the equation of the transformed function $g(x)$.

   Replace $f(x)$ in the equation of the vertex form with y:

   $y = a(x - h)^2 + k$

   Recall that the vertex of the graph of the quadratic function is also known as the maximum or minimum point of the function and is represented by the coordinate (h, k). (x, y) represents some point on the graph of the function.

   To determine the equation of $g(x)$, first identify the vertex of the graph of $g(x)$.

   It can be seen from the graph that the vertex of $g(x)$ is $(1, -1)$.

(*continued*)

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Instruction

Next, identify an additional point on the graph of $g(x)$.

It can be seen from the graph that an additional point on the graph of $g(x)$ is (0, 1).

Use the vertex (1, –1) and the additional point (0, 1) to find the value of a in the standard form of the equation.

$y = a(x - h)^2 + k$	Standard form of a quadratic equation
$(1) = a[(0) - (1)]^2 + (-1)$	Substitute (1, –1) for (h, k) and (0, 1) for (x, y).
$1 = a(-1)^2 - 1$	Simplify.
$1 = a - 1$	Apply the exponent.
$2 = a$	Add 1 to both sides.

The value of a is 2.

Use the value of a and the coordinate of the vertex to write the equation of the transformed function $g(x)$.

$y = a(x - h)^2 + k$	Standard form of a quadratic equation
$[g(x)] = (2)[x - (1)]^2 + (-1)$	Let $a = 2$, $y = g(x)$, and (h, k) be (1, –1).
$g(x) = 2(x - 1)^2 - 1$	Simplify.

The equation of the transformed function $g(x)$ in vertex form is $g(x) = 2(x - 1)^2 - 1$.

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Instruction

2. Use the standard form $y = a(x - h)^2 + k$ to determine the equation of the transformed function $h(x)$.

   It can be seen from the graph of $h(x)$ that the vertex is (–1, 2) and an additional point is (0, –1).

   Use the vertex (–1, 2) and the additional point (0, –1) to find the value of a in the standard form equation.

$y = a(x - h)^2 + k$	Standard form of a quadratic equation
$(-1) = a[(0) - (-1)]^2 + (2)$	Substitute (–1, 2) for (h, k) and (0, –1) for (x, y).
$-1 = a(1)^2 + 2$	Simplify.
$-1 = a + 2$	Apply the exponent.
$-3 = a$	Subtract 2 from both sides.

   The value of a is –3.

   Use the value of a and the coordinate of the vertex to write the equation of the transformed function $h(x)$.

$y = a(x - h)^2 + k$	Standard form of a quadratic equation
$[h(x)] = (-3)[x + (1)]^2 + (2)$	Let $a = -3$, $y = h(x)$, and (h, k) be (–1, 2).
$h(x) = -3(x + 1)^2 + 2$	Simplify.

   The equation of the transformed function $h(x)$ is $h(x) = -3(x + 1)^2 + 2$.

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Instruction

3. Describe the transformation(s) of the parent function that resulted in the functions $g(x)$ and $h(x)$.

   The transformed functions are $g(x) = 2(x - 1)^2 - 1$ and $h(x) = -3(x + 1)^2 + 2$.

   To describe the transformation(s) of the parent function $f(x) = x^2$, start inside the parentheses and work outward.

   For $g(x)$:

   - 1 was subtracted from x to give $x - 1$ in place of x.
   - Then, 2 was multiplied by the "new" squared term, $(x - 1)^2$.
   - Lastly, 1 was subtracted from the new squared term and its new coefficient to give the final result, $g(x) = 2(x - 1)^2 - 1$.

   For $h(x)$:

   - 1 was added to x to give $x + 1$ in place of x.
   - Then, -3 was multiplied by the "new" squared term, $(x + 1)^2$.
   - Finally, 2 was added to the new squared term and its new coefficient to give the final result, $h(x) = -3(x + 1)^2 + 2$.



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Instruction

Example 3

Identify the transformation(s) of the parent function $f(x) = \log x$ that result in the function $g(x) = 1 + 2 \cdot \log (3x + 4)$. Describe the effect of the transformation(s) on the domain and range of the function $g(x)$. Finally, determine the y-intercept(s) of $g(x)$.

1. Identify the transformation(s) of the parent function.

   To determine how the parent function $f(x) = \log x$ was transformed, compare it to $g(x) = 1 + 2 \cdot \log (3x + 4)$.

   - First, note that in $g(x)$, the constant 1 is added to $2 \cdot \log (3x + 4)$. Adding a constant to the function value is the transformation $g(x) = f(x) + k$.
   - Also notice that 2 is multiplied by $\log (3x + 4)$. Multiplying the function value by a constant is the transformation $g(x) = k \cdot f(x)$.
   - The coefficient 3 is in the argument of the logarithm, $\log (3x + 4)$. Multiplying the independent variable by a constant is the transformation $g(x) = f(kx)$.
   - Furthermore, note the addition of 4 in the argument of the logarithm, $\log (3x + 4)$. Adding a constant to the independent variable in the argument of the logarithm is the transformation $g(x) = f(x + k)$.

   Therefore, four transformations—$g(x) = f(x) + k$, $g(x) = k \cdot f(x)$, $g(x) = f(kx)$, and $g(x) = f(x + k)$—have been performed on the parent function $f(x) = \log x$.

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Instruction

2. Compare the domain and range of the parent function to those of the transformed function $g(x)$.

   The domain of the parent function is $(0, \infty)$.

   Since $g(x)$ is a logarithmic function, the domain of $g(x)$ is defined by the argument of the logarithmic term, $3x + 4$. The condition $3x + 4 > 0$ or $x > -\frac{4}{3}$ exists on the argument. Therefore, the domain of $g(x)$ is $\left(-\frac{4}{3}, \infty\right)$.

   The range of both functions is $(-\infty, \infty)$.

   The transformations of $f(x)$ that result in $g(x)$ alter the rate at which the function values change, but not their upper and lower bounds.

3. Calculate the y-intercept of $g(x)$.

   The y-intercept of $g(x)$ is defined by the function value $g(0)$. Substitute 0 for x and solve for $g(0)$:

   $g(x) = 1 + 2 \bullet \log(3x + 4)$

   $g(0) = 1 + 2 \bullet \log[3(0) + 4]$

   $g(0) = 1 + 2 \bullet \log 4$

   $g(0) \approx 2.2$

   The y-intercept of $g(x) = 1 + 2 \bullet \log(3x + 4)$ is approximately 2.2, or (0, 2.2).

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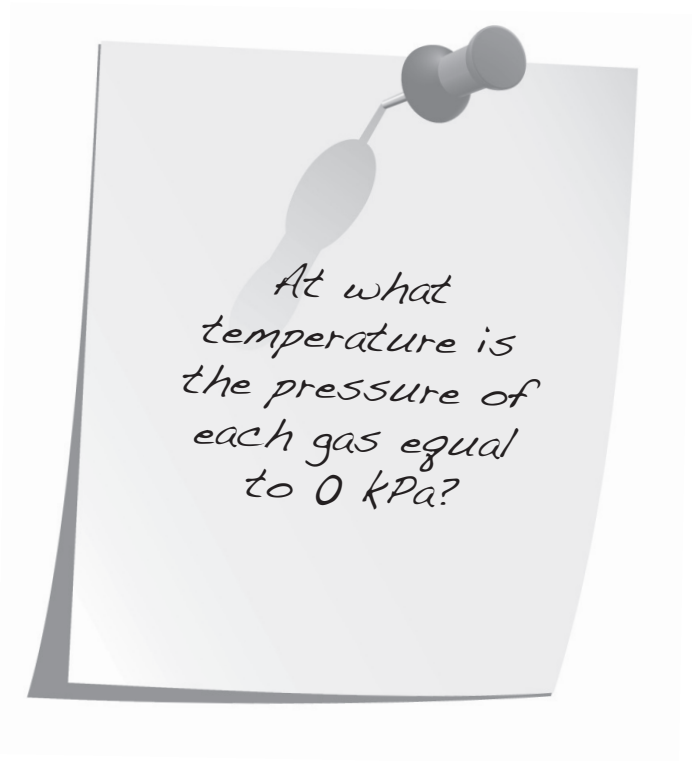
Problem-Based Task 4B.2.1: An Absolute Gas?

SMP	
1	2
3	4 ✓
5	6
7 ✓	8 ✓

Hydraulic fracturing, or "fracking," is the process of injecting pressurized fluid into shale rocks in order to fracture them, which releases natural gas trapped by the rocks. A particular fracking well has three 400-liter gas tanks, each containing a different formulation of natural gas extracted from the ground. A technician checked the pressure and temperature gauges on the three tanks twice during his shift, and recorded the results in the table shown. The temperature is measured in degrees Celsius (°C) and the pressure is measured in kilopascals (kPa).

	Pressure 1 (kPa)	Temperature 1 (°C)	Pressure 2 (kPa)	Temperature 2 (°C)
Gas A	80	0	109	100
Gas B	60	0	82	100
Gas C	100	0	137	100

Assume that the relationship between the pressure P (the dependent variable) and the temperature t (the independent variable) can be represented by a linear function. Derive the family of linear functions for this situation, and state the transformation(s) of the parent function $f(x) = x$ that result in the form of the three linear functions in the family. At what temperature is the pressure of each gas equal to 0 kPa?



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Problem-Based Task 4B.2.1: An Absolute Gas?

Coaching

a. Use the data points in the table to graph each linear function.

b. How are the graphs of each type of gas alike and different?

c. Determine the slope of each graphed line.

d. What does the slope mean for each of the gases?

e. Write a linear function relating pressure P and temperature t for each gas.

f. Describe the transformation(s) of the parent linear function represented by this family of pressure-temperature functions.

g. At what temperature is the pressure of each gas equal to 0 kPa?

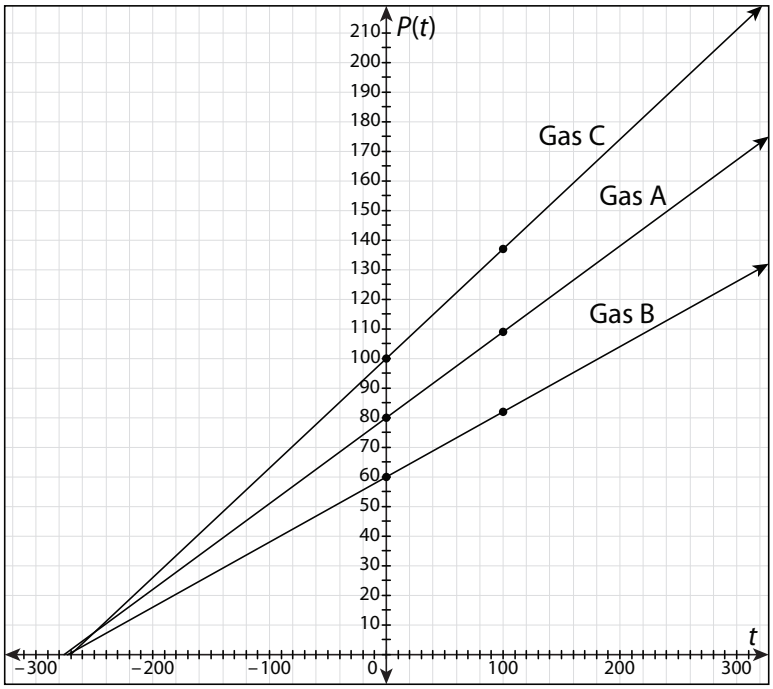
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Problem-Based Task 4B.2.1: An Absolute Gas?

Coaching Sample Responses

a. Use the data points in the table to graph each linear function.

The linear functions are shown in the following graph.



b. How are the graphs of each type of gas alike and different?

The graphs of each type of gas appear to intersect at the same point along the negative x-axis. Each graph also has a positive slope; however, the slopes are unequal.

c. Determine the slope of each graphed line.

The slope, m, of a straight line can be found by substituting any two points (x_1, y_1) and (x_2, y_2) on the line into the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$. Use the values in the data table to calculate each slope.

- For Gas A: $m_A = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(109) - (80)}{(100) - (0)} = 0.29$
- For Gas B: $m_B = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(82) - (60)}{(100) - (0)} = 0.22$
- For Gas C: $m_C = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(137) - (100)}{(100) - (0)} = 0.37$

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d. What does the slope mean for each of the gases?

The differing slopes mean that the pressure of each gas is affected differently by the same temperature change, which is a function of the type of gas.

e. Write a linear function relating pressure P and temperature t for each gas.

Use the point-slope form of a linear function, $y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) is a point on the line, to write a linear function for each gas.

For Gas A, let $m = 0.29$ and $(x_1, y_1) = (0, 80)$.

$y - y_1 = m(x - x_1)$

$y - (80) = (0.29)[x - (0)]$

$y - 80 = 0.29x$

$y = 0.29x + 80$

The function relating the pressure and temperature of Gas A is $P_A(t) = 0.29t + 80$.

For Gas B, let $m = 0.22$ and $(x_1, y_1) = (0, 60)$.

$y - y_1 = m(x - x_1)$

$y - (60) = (0.22)[x - (0)]$

$y - 60 = 0.22x$

$y = 0.22x + 60$

The function relating the pressure and temperature of Gas B is $P_B(t) = 0.22t + 60$.

For Gas C, let $m = 0.37$ and $(x_1, y_1) = (0, 100)$.

$y - y_1 = m(x - x_1)$

$y - (100) = (0.37)[x - (0)]$

$y - 100 = 0.37x$

$y = 0.37x + 100$

The function relating the pressure and temperature of Gas C is $P_C(t) = 0.37t + 100$.

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f. Describe the transformation(s) of the parent linear function represented by this family of pressure-temperature functions.

The functions $P_A(t)$, $P_B(t)$, and $P_C(t)$ are all transformations where the variable t was multiplied by a constant (0.22, 0.29, and 0.37, respectively); also, a constant was added to each function (80, 60, and 100).

g. At what temperature is the pressure of each gas equal to 0 kPa?

Use the equations for $P_A(t)$, $P_B(t)$, and $P_C(t)$ to determine the temperature at which the pressure of each gas will reach 0 kPa.

Let $P_A(t) = 0$.

$P_A(t) = 0.29t + 80$

$(0) = 0.29t + 80$

$-80 = 0.29t$

$t \approx -275.86$

The temperature at which the pressure of Gas A reaches 0 kPa is approximately –276°C.

Let $P_B(t) = 0$.

$P_B(t) = 0.22t + 60$

$(0) = 0.22t + 60$

$-60 = 0.22t$

$t \approx -272.72$

The temperature at which the pressure of Gas B reaches 0 kPa is approximately –273°C.

Let $P_C(t) = 0$.

$P_C(t) = 0.37t + 100$

$(0) = 0.37t + 100$

$-100 = 0.37t$

$t \approx -270.27$

The temperature at which the pressure of Gas C reaches 0 kPa is approximately –270°C.

Recommended Closure Activity

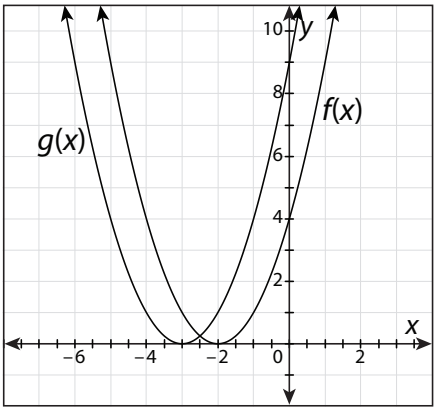
Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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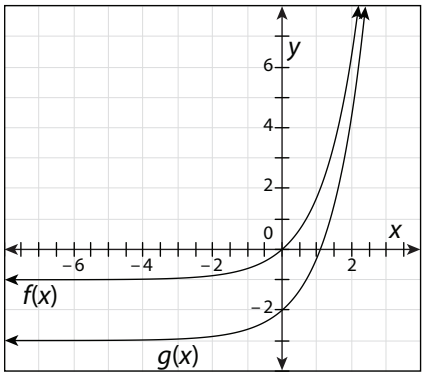
Practice 4B.2.1: Transformations of Parent Graphs

For problems 1–3, use the graph of functions $f(x)$ and $g(x)$ to determine the value of the constant for the given general form of both functions. Then, write the equations of $f(x)$ and $g(x)$ in the given form.

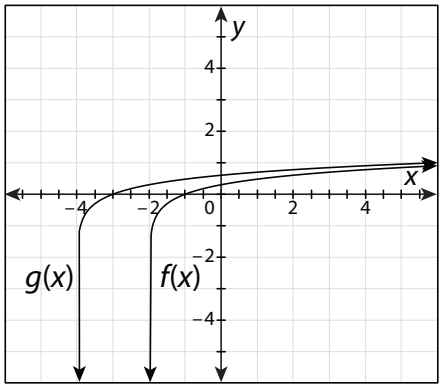
1. general form: $(x + a)^2$



2. general form: $e^x + a$



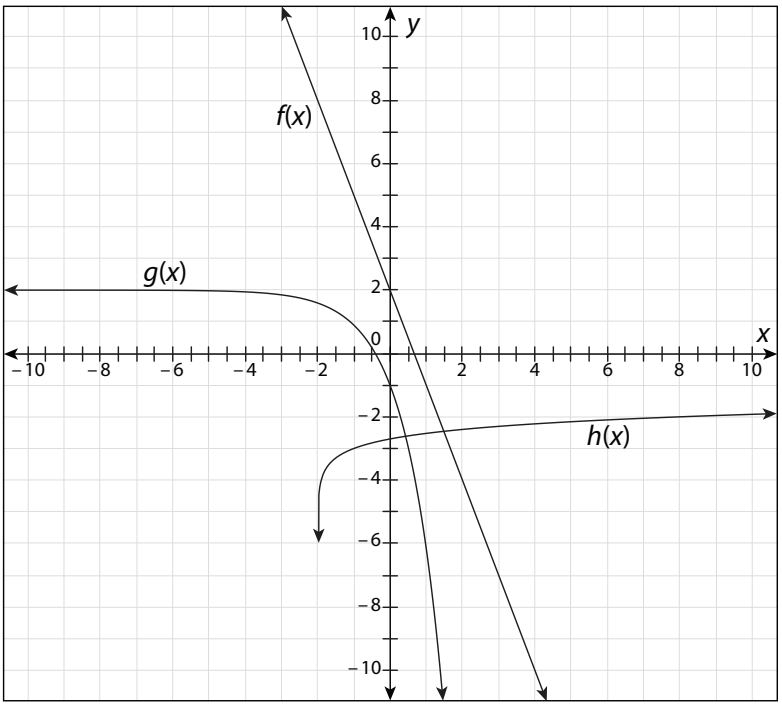
3. general form: $\log(x + a)$



continued

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For problems 4–6, use the graph to determine the domain and range for each given function. Write your answers in interval notation.



4. $f(x)$

5. $g(x)$

6. $h(x)$

continued

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Use the information given in each problem to complete problems 7–10.

7. The area of a rectangle is given by the function $A(x) = x(x - 2)$. What is the value of x if the area is doubled?

8. A vehicle's distance d in miles is given by the function $d(r, t) = r \bullet t$, in which r is the rate in miles per hour and t is the time in hours. What happens to the value of d if the rate is tripled and the time increases by 3 hours?

9. The growth factor for one generation of a bacteria culture is given by the function $g(t) = \frac{\ln 2}{t}$, in which t is the time it takes for the generation to mature to its full size based on the energy resources and environmental conditions available in the culture. What happens to t if $g(t)$ is cut in half?

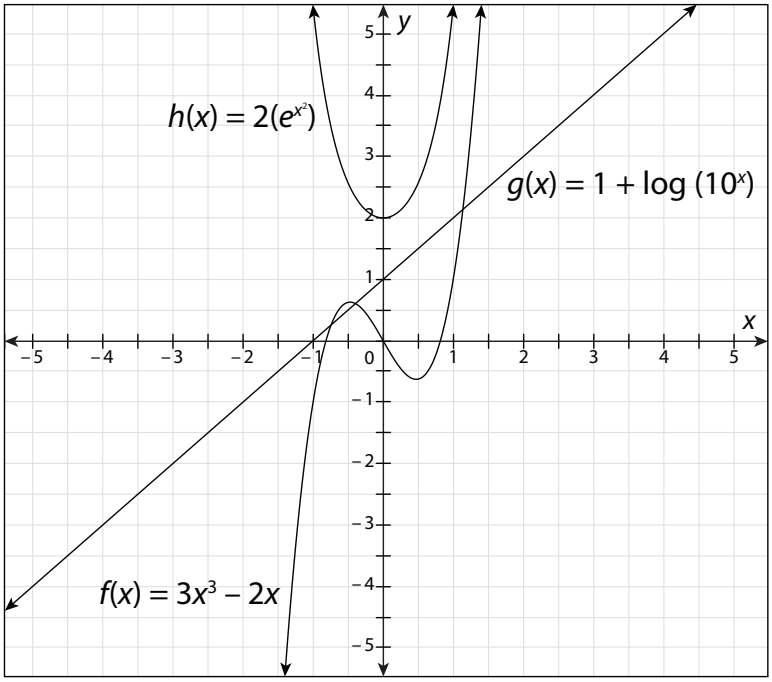
10. The profit p from selling n units of a product is given by the function $p(n) = 250n - 75{,}000$. What is the domain of n if the range of $p(n)$ is (0, 5000)? How does the domain of n change if the upper bound of the range of $p(n)$ is decreased by 2,500?

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Lesson 4B.2.2: Recognizing Odd and Even Functions

Warm-Up 4B.2.2

The functions $f(x) = 3x^3 - 2x$, $g(x) = 1 + \log(10^x)$, and $h(x) = 2\left(e^{x^2}\right)$ are models for changes in the population of three groups of kangaroo rats at a time x in months in the presence of predators ($x < 0$) and in the absence of predators ($x > 0$). The population is the dependent variable. Use the graph to answer the questions that follow.



1. How do the function values of $f(x)$ and $f(-x)$ compare?

2. How do the function values of $g(x)$ and $g(-x)$ compare?

3. How do the function values of $h(x)$ and $h(-x)$ compare?

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Instruction

Lesson 4B.2.2: Recognizing Odd and Even Functions

Common Core State Standard

F–BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Warm-Up 4B.2.2 Debrief

1. How do the function values of $f(x)$ and $f(-x)$ compare?

   First, substitute specific opposite values of x into the parent function, $f(x) = 3x^3 - 2x$. Then, solve each equation. Convenient values to use are 1 and –1.

Let $x = 1$.	Let $x = -1$.
$f(x) = 3x^3 - 2x$	$f(x) = 3x^3 - 2x$
$f(1) = 3(1)^3 - 2(1)$	$f(-1) = 3(-1)^3 - 2(-1)$
$f(1) = 3 - 2$	$f(-1) = -3 + 2$
$f(1) = 1$	$f(-1) = -1$

   For $x = 1$, $f(1) = 1$. For the opposite value of x, where $x = -1$, $f(-1) = -1$. Therefore, $f(x) = -f(-x)$; substituting other opposite values of x into the function will confirm this result.

2. How do the function values of $g(x)$ and $g(-x)$ compare?

   Substitute specific opposite values of x into the parent function, $g(x) = 1 + \log(10^x)$, and solve each equation. Let's use 1 and –1.

Let $x = 1$.	Let $x = -1$.
$g(x) = 1 + \log(10^x)$	$g(x) = 1 + \log(10^x)$
$g(1) = 1 + \log[10^{(1)}]$	$g(-1) = 1 + \log[10^{(-1)}]$
$g(1) = 1 + 1$	$g(-1) = 1 - 1$
$g(1) = 2$	$g(-1) = 0$

   For $x = 1$, $g(1) = 2$. For the opposite value of x, where $x = -1$, $g(-1) = 0$. Therefore, $g(x) \neq -g(-x)$; substituting other opposite values of x into the function will confirm this result.

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Instruction

3. How do the function values of $h(x)$ and $h(-x)$ compare?

   Once more, substitute 1 and –1 for x into the parent function, $h(x)=2\left(e^{x^2}\right)$, and solve.

Let $x = 1$.	Let $x = -1$.
$h(x)=2\left(e^{x^2}\right)$	$h(x)=2\left(e^{x^2}\right)$
$h(1)=2\left[e^{(1)^2}\right]$	$h(-1)=2\left[e^{(-1)^2}\right]$
$h(1) = 2e$	$h(-1) = 2e$

   For $x = 1$, $h(1) = 2e$. For the opposite value of x, where $x = -1$, $h(-1) = 2e$. Therefore, $h(x) = h(-x)$; substituting other opposite values of x into the function will confirm this result.

Connection to the Lesson

- Students will use the ideas in the warm-up to recognize even and odd functions.
- Students will expand upon these ideas to identify the type of function (even, odd, or neither) resulting from a function transformation.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- evaluating a function value by substituting a number or its opposite into the function and solving (F-IF.2)
- identifying the domain conditions under which a function value is equal to or the opposite of a given value (F-IF.5)
- using a function transformation to create a new function (F-BF.3)

Introduction

When working with functions, it can be useful to recognize when a specific domain value or its opposite changes a function value in a particular way. For some functions, $f(x) = f(-x)$; that is, evaluating a function for $-x$ results in a function equation that is the same as the original function. Such a function is defined as an **even function**. An **odd function** is a function that, when evaluated for $-x$, results in a function equation that is the opposite of the original function; $f(-x) = -f(x)$. Even and odd functions can be changed into other functions using transformations. In some cases, these changed functions are still even or odd functions, but in other cases they no longer meet the criteria for being even or odd.

Key Concepts

- A function $f(x)$ for which $f(x) = f(-x)$ is defined as an even function.
- A function $f(x)$ for which $f(x) = -f(-x)$ is defined as an odd function.
- Whether a function is even, odd, or neither can be determined algebraically by substituting $-x$ for x and comparing the result to the original function.
- The transformations $g(x) = f(kx)$ and $g(x) = k \bullet f(x)$ of an even function result in an even function.
- The transformation $g(x) = f(x) + k$ of an even function results in an even function.
- The transformation $g(x) = f(x) + k$ of an odd function results in a new function that is neither even nor odd.
- A function that contains terms with both odd and even exponents is neither an even nor an odd function.

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Instruction

- To quickly determine if a function is even, odd, or neither, look at its graph:
  - The graph of an even function is symmetric about the *y*-axis.
  - The graph of an odd function is symmetric about the origin.
  - The graph of a function that is neither odd nor even is not symmetric about either the *y*-axis or the origin.
- The graphs of trigonometric functions can also be described as even or odd. For example, sine functions are odd, whereas cosine functions are even.
- A graphing calculator can be especially useful for evaluating evenness or oddness for higher-degree polynomials or other non-polynomial functions.

Common Errors/Misconceptions

- identifying an even function as odd or vice versa
- identifying an even or odd function as neither type
- identifying a function that is neither even nor odd as one of the two types
- identifying an even or odd function as even or odd after a transformation that makes the function neither even nor odd

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Guided Practice 4B.2.2

Example 1

Show that the function $f(x) = 2x^2 - x$ is neither even nor odd by using two opposite values of x.

1. Evaluate $f(x)$ at a value of x greater than 0.

   For convenience, use a positive single-digit integer, such as $x = 3$.

$f(x) = 2x^2 - x$	Original function
$f(3) = 2(3)^2 - (3)$	Substitute 3 for x.
$f(3) = 18 - 3$	Simplify.
$f(3) = 15$	

   For $x = 3$, the function $f(x) = 2x^2 - x$ is equal to 15.

2. Evaluate $f(x)$ at a value of x that is the opposite of the value of x used in the previous step.

   The value of x used in step 1 is 3; therefore, the opposite value is –3.

$f(x) = 2x^2 - x$	Original function
$f(-3) = 2(-3)^2 - (-3)$	Substitute –3 for x.
$f(-3) = 18 + 3$	Simplify.
$f(-3) = 21$	

   For $x = -3$, the function $f(x) = 2x^2 - x$ is equal to 21.

3. Summarize your findings.

   Use the results from steps 1 and 2 to determine what type of function $f(x)$ is.

   Recall that in an even function, $f(x) = f(-x)$.

   For $f(x) = 2x^2 - x$ to be even, then the result of $f(3)$ must equal the result of $f(-3)$. Since $15 \neq 21$, $f(x) = 2x^2 - x$ is not an even function.

   In an odd function, $f(x) = -f(-x)$.

   For $f(x) = 2x^2 - x$ to be odd, then the result of $f(3)$ must equal the result of $-f(-3)$.

   Since $15 \neq -21$, $f(x) = 2x^2 - x$ is not an odd function.

   Notice that for $f(x) = 2x^2 - x$, $f(x) \neq f(-x)$ and $f(x) \neq -f(-x)$; therefore, the function $f(x)$ is neither even nor odd.

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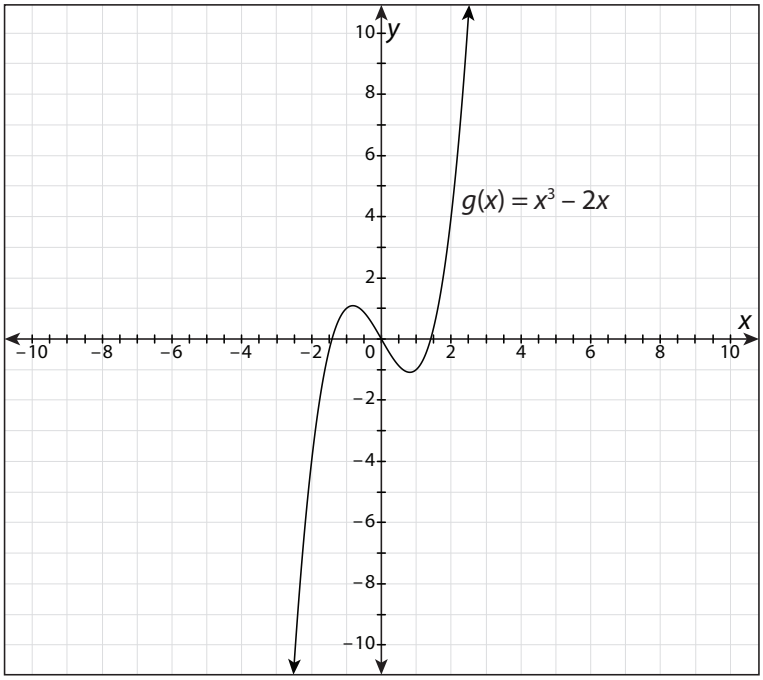
Instruction

Example 2

Describe how the graph of the function $g(x) = x^3 - 2x$ can be used to determine if the function is even or odd.

1. Graph the function by hand or using a graphing calculator.

   The graph of $g(x) = x^3 - 2x$ is as follows.



2. Visually compare the parts of the graph to the right and left of the y-axis.

   From the graph, it can be seen that graph is symmetric about the origin.

   Notice that when $x = 1$, $y = -1$, or $(1, -1)$.

   Also notice that when $x = -1$, $y = 1$, or $(-1, 1)$.

   Therefore, it can be hypothesized that opposite values of x result in opposite values of y.

3. Summarize your findings.

   The function $g(x)$ is an odd function since it can be seen from the graph that $g(x) = -g(-x)$.

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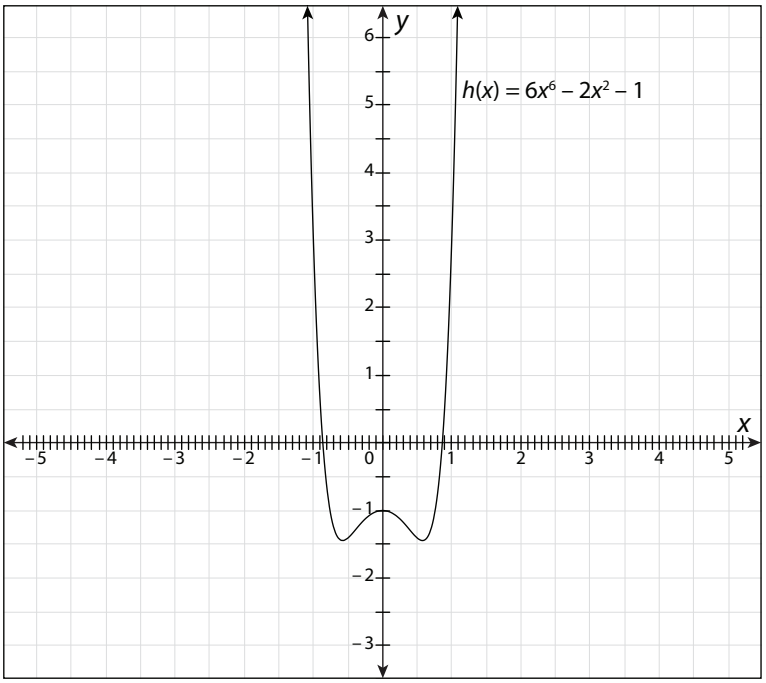
Example 3

Describe how the graph of the function $h(x) = 6x^6 - 2x^2 - 1$ can be used to determine if the function is even or odd.

1. Graph the function by hand or using a graphing calculator.

   With higher-degree polynomials such as this one, it is more efficient to use a graphing calculator.

   The graph of $h(x) = 6x^6 - 2x^2 - 1$ is as follows.



2. Visually compare the parts of the graph to the right and left of the y-axis.

   The graph is symmetric about the y-axis. The values of x on either side of the y-axis are opposites. For example, when $y = 0$, $x = 1$ and $x = -1$.

   It can be hypothesized that opposite values of x result in the same value of y.

3. Describe the function $h(x)$ as an even or an odd function.

   The function $h(x)$ is an even function since it can be seen from the graph that $h(x) = h(-x)$.

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Instruction

Example 4

Compare the functions $f(x) = \log x^2$ and $g(x) = 2 \bullet \log x$ using the definitions of even and odd functions.

1. Determine the domain values for which $f(x)$ exists.

   The domain of $f(x)$ is $(-\infty, 0) \cap (0, \infty)$ since the argument of the logarithm squares the value of x.

2. Describe $f(x)$ using the definitions of even and odd functions.

   The function $f(x) = \log x^2$ can be described as an even function since it can be written as $f(x) = f(-x) = \log x^2$. (Recall that the square of a negative term is positive.)

3. Determine the domain values for which $g(x)$ exists.

   $g(x)$ is a logarithmic function; therefore, the domain of $g(x)$ is $(0, \infty)$.

4. Describe $g(x)$ using the definitions of even and odd functions.

   Notice that the function $f(x) = \log x^2$ can be written as $f(x) = 2 \bullet \log x$, which is the same as $g(x) = 2 \bullet \log x$.

   The function $f(x) = 2 \bullet \log x$ is neither even nor odd since $f(x) \neq f(-x) \neq -f(-x)$; therefore, $g(x)$ is also neither even nor odd.

5. Summarize your findings.

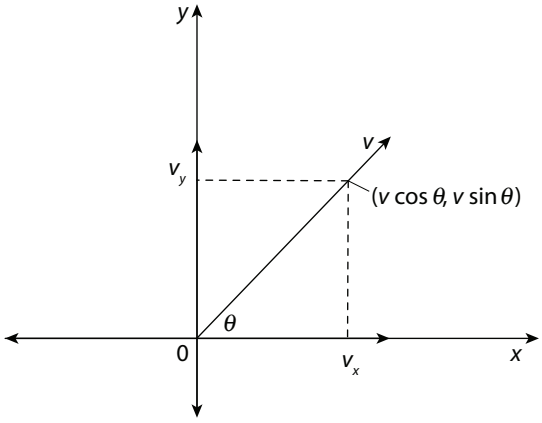
   The function $f(x) = \log x^2$ is an even function, but its rewritten form, $g(x)$, is not.

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Problem-Based Task 4B.2.2: Odd and Even Velocity Functions

SMP	
1 ✓	2
3 ✓	4
5	6
7 ✓	8 ✓

The velocity of a plane or ship has a direction and a magnitude. For example, a plane can be described as traveling in a northeast direction from a starting point at 350 miles per hour. On a coordinate plane, this velocity can be described as traveling at a specific velocity to the east and at a specific velocity to the north that result in the "total" velocity to the northeast, as shown in the following graph.



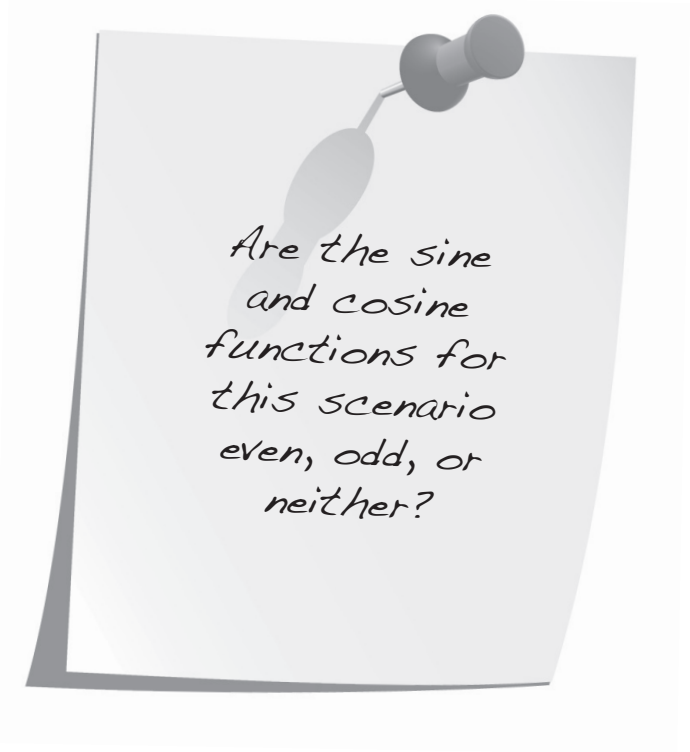
Notice that the velocity in the northeast direction is in the first quadrant and is labeled v. Its eastern part lies along the positive x-axis and is labeled v_x. Its northern part lies along the positive y-axis and is labeled v_y. The angle θ is measured counterclockwise from the positive x-axis. If the plane or ship were to travel in a different direction at the same speed from the starting point, its velocity would stay the same, but its direction would change and so would the values of v_x and v_y.

Notice that v_x and v_y form the sides of a right triangle with v as the hypotenuse. Recall the cosine and sine functions of an angle θ, where θ is a positive angle that measures less than 90°:

$$\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$$

$$\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$$

Thus, for this situation, the ratios can be written as $\cos\theta = \frac{v_x}{v}$ and $\sin\theta = \frac{v_y}{v}$. Are the sine and cosine functions for this scenario even, odd, or neither?



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Problem-Based Task 4B.2.2: Odd and Even Velocity Functions

Coaching

a. Solve $\cos\theta = \frac{v_x}{v}$ for v_x.

b. Solve $\sin\theta = \frac{v_y}{v}$ for v_y.

c. What are the coordinates of the endpoint of v?

d. If θ is a positive angle that measures less than 90° and is measured counterclockwise from the positive x-axis, how is $-\theta$ defined?

e. In what quadrants of the coordinate plane are θ and $-\theta$ located?

f. What are the signs of the values of v_x and v_y for the quadrant in which $-\theta$ is located?

g. What is the sign of the value of v in *any* of the quadrants?

h. How is the cosine of $-\theta$ defined?

i. How is the sine of $-\theta$ defined?

j. What are the signs of the values of the cosine and sine of $-\theta$?

k. Is $v_x(\theta)$ even or odd? Explain.

l. Is $v_y(\theta)$ even or odd? Explain.

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Problem-Based Task 4B.2.2: Odd and Even Velocity Functions

Coaching Sample Responses

a. Solve $\cos\theta = \frac{v_x}{v}$ for v_x.

Cross multiply to solve for v_x.

$$\cos\theta = \frac{v_x}{v}$$
$$v \bullet \cos\theta = v_x$$

b. Solve $\sin\theta = \frac{v_y}{v}$ for v_y.

Cross multiply to solve for v_y.

$$\sin\theta = \frac{v_y}{v}$$
$$v \bullet \sin\theta = v_y$$

c. What are the coordinates of the endpoint of v?

The coordinates of the endpoint of v are v_x for the x-coordinate and v_y for the y-coordinate: (v_x, v_y). Since expressions that are equivalent to v_x and v_y are known as well, this can also be written as $(v \bullet \cos\theta, v \bullet \sin\theta)$.

d. If θ is a positive angle that measures less than 90° and is measured counterclockwise from the positive x-axis, how is $-\theta$ defined?

The angle $-\theta$ should be measured clockwise from the positive x-axis and thus into Quadrant IV.

e. In what quadrants of the coordinate plane are θ and $-\theta$ located?

Since each angle is less than 90°, angle θ is located in Quadrant I and angle $-\theta$ is located in Quadrant IV.

f. What are the signs of the values of v_x and v_y for the quadrant in which $-\theta$ is located?

In Quadrant IV, v_x is positive and v_y is negative.

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Instruction

g. What is the sign of the value of v in *any* of the quadrants?

The velocity v is the hypotenuse of the triangle, so it is defined as being positive. Therefore, $v > 0$ in any of the quadrants.

h. How is the cosine of $-\theta$ defined?

The cosine of $-\theta$ is defined as $\cos(-\theta) = \frac{v_x}{v}$.

i. How is the sine of $-\theta$ defined?

The sine of $-\theta$ is defined as $\sin(-\theta) = \frac{v_y}{v}$.

j. What are the signs of the values of the cosine and sine of $-\theta$?

$\cos(-\theta) > 0$ because $v_x > 0$ and $v > 0$; therefore, the sign of these values is positive.

$\sin(-\theta) < 0$ because $v_y < 0$ and $v > 0$; therefore, the sign of these values is negative.

k. Is $v_x(\theta)$ even or odd? Explain.

The function $v_x(\theta)$ is an even function because $v_x(\theta) = v_x(-\theta)$. It meets the criteria that $f(x) = f(-x)$.

l. Is $v_y(\theta)$ even or odd? Explain.

The function $v_y(\theta)$ is an odd function because $v_y(\theta) = -v_y(-\theta)$. It meets the criteria that $f(x) = -f(-x)$.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4B.2.2: Recognizing Odd and Even Functions

For problems 1–4, change or remove terms to first rewrite each original function as an even function, then as an odd function.

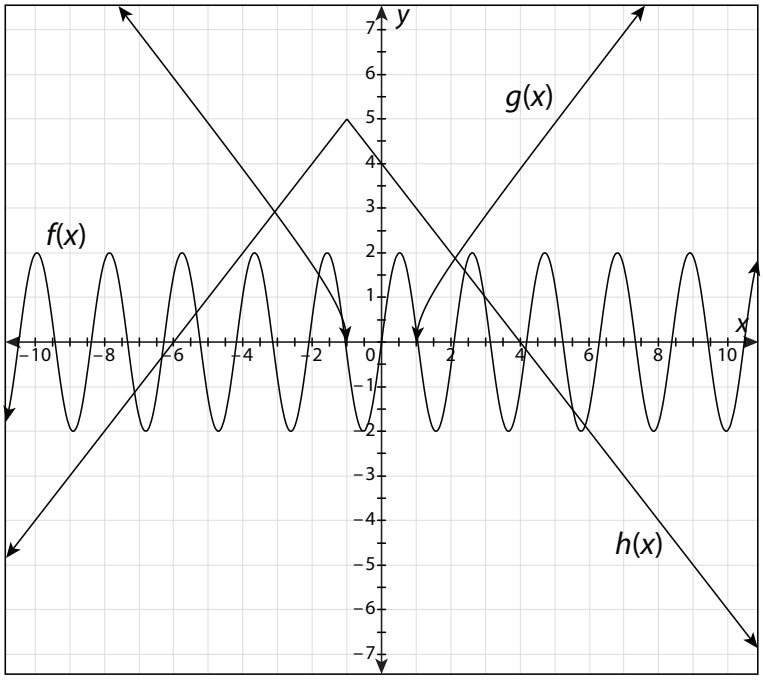
1. $a(x) = 5 - 2x + x^2$

2. $b(x) = -2 \bullet |3 - 4x|$

3. $c(x) = x^2 \bullet \log(x^2 + x)$

4. $d(x) = \cos x + \sin x$

For problems 5–7, use the following graph to identify the function as even, odd, or neither.



5. $f(x)$

6. $g(x)$

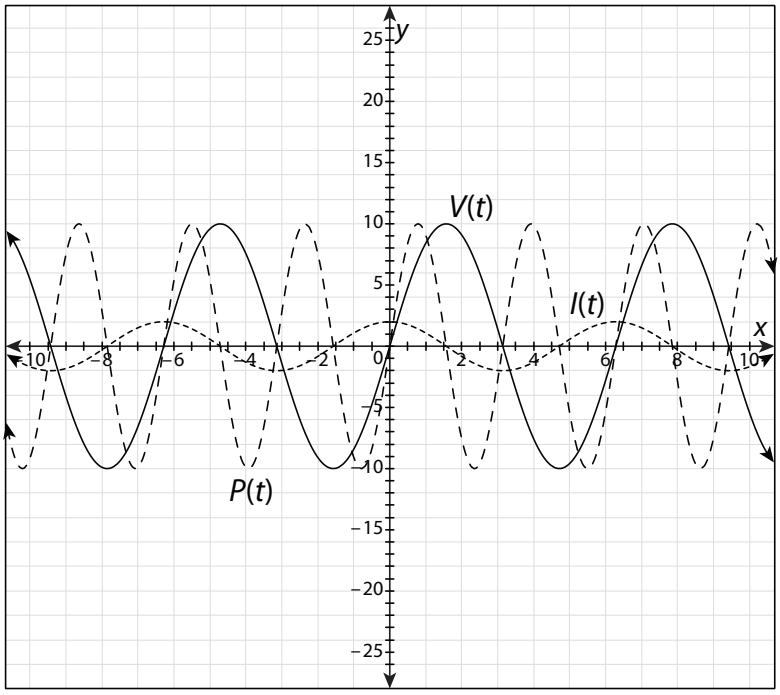
7. $h(x)$

continued

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Use the information and the graph that follows to complete problems 8–10.

In a microelectronics circuit, the current I and voltage V fluctuate with time t according to the functions $I(t) = 2 \cdot \cos at$ and $V(t) = 10 \cdot \sin at$, as shown on the graph. The third curve on the graph, $P(t)$, is the power produced by the circuit. The power function $P(t)$ is the product of the current and voltage functions.



8. Are the functions $I(t)$ and $V(t)$ even, odd, or neither?

9. What does the graph of $P(t)$ indicate about its classification as even, odd, or neither? Is the graph of $P(t)$ the graph of a cosine or a sine function? Explain your answers.

10. Recall that the period of a cosine or a sine function is the distance (which is time t in this problem) between successive maximum or minimum points on a graph. How does the period of $P(t)$ compare to the period of the current and voltage functions?

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Lesson 4B.2.3: Combining Functions

Warm-Up 4B.2.3

Ella is an artist who uses a three-dimensional printer to produce custom-designed plastic toys for specialty retailers. Ella charges a $1,500 set-up fee to program and use the printer for a specific design, and $25 for the first toy produced. The cost for each additional toy is discounted by 0.5 percent of the original cost of the first toy.

1. Without including the set-up charge, what would be the cost of producing 50 toys?

2. Write a general function for the cost $c(n)$ of producing n toys without the set-up charge.

3. Write a function for the total cost $T(c)$ of producing n toys, including the set-up charge and the cost $c(n)$ of n toys.

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Instruction

Lesson 4B.2.3: Combining Functions

Common Core State Standard

F–BF.1 Write a function that describes a relationship between two quantities.★

b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*

Warm-Up 4B.2.3 Debrief

1. Without including the set-up charge, what would be the cost of producing 50 toys?

   The charge for each toy produced after the first one is discounted by 0.5 percent of the original cost of the first toy; 0.5 percent = 0.005.

   The cost of the first toy is $25.

   After the first toy, we have 50 – 1 or 49 toys.

   The cost of each of these 49 toys is 25 – 0.005(25), or $24.875 for each toy. Therefore, the total cost of 50 toys is 25 + 49(24.875) = 1,243.875, or approximately $1,243.88.

2. Write a general function for the cost $c(n)$ of producing n toys without the set-up charge.

   The cost of producing n toys without a set-up charge can be generalized from the expression used in problem 1 to find the cost for 50 toys, or $c(50)$: 25 + 49(24.875).

   In that expression, the cost of the first toy was $25. Let this toy be represented by $c(1)$.

   For $c(50)$, we found the number of toys remaining by subtracting 1 from 50: 50 – 1 = 49 toys. Therefore, for n toys, we can write the number of toys after the first as $n – 1$. We also found that the cost of each toy after the first was $24.875; use this figure in the general function $c(n)$.

   Therefore, $c(50) = 25 + 49(24.875)$ can be rewritten as $c(n) = 25 + (n – 1) \bullet (24.875)$. Simplify this equation to find the general cost function.

   $c(n) = 25 + (n – 1) \bullet (24.875)$

   $c(n) = 25 + 24.875n – 24.875$

   $c(n) = 24.875n + 0.125$

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Instruction

3. Write a function for the total cost $T(c)$ of producing n toys, including the set-up charge and the cost $c(n)$ of n toys.

   The set-up charge is \$1,500. The total cost function, $T(c)$, can be found by adding the set-up charge to the function for $c(n)$:

$T(c) = 1500 + c(n)$	Set-up charge + $c(n)$
$T(c) = 1500 + (24.875n + 0.125)$	Substitute $24.875n + 0.125$ for $c(n)$.
$T(c) = 1500.125 + 24.875n$	Simplify.

   The function for the total cost $T(c)$ of producing n toys is $T(c) = 1500.125 + 24.875n$.

Connection to the Lesson

- Students apply the conditions of a real-world problem to the parts of an equation or function that models the problem.
- Students see one equation or function substituted into another equation or function and the effect that this substitution has on the variables in the model for the problem.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- translating real-world descriptions in a problem to mathematical terms (F-BF.1a)
- combining the mathematical terms in a problem into an equation or function that models the problem (F-BF.1a)
- substituting one expression for another in a mathematical statement (F-IF.2)
- checking a mathematical statement for accuracy and revising the statement when errors are found (7.EE.3)

Introduction

Functions can be combined in many ways, such as by being added, subtracted, multiplied, divided, and substituted into each other. Some real-world problems do not lend themselves to modeling by simple functions of one type and/or in one variable with predictable domains and ranges, except as approximations or estimates of solutions. Combinations of functions are useful in modeling such problems and their solutions.

Key Concepts

- The **combination of functions** is the process of adding, subtracting, multiplying, or dividing two or more functions to produce a new function. This table shows how two functions $f(x)$ and $g(x)$ can be combined through addition, subtraction, multiplication, or division to produce a new function $h(x)$.

Combination	Description
$h(x) = (f + g)(x)$	$(f + g)(x) = f(x) + g(x)$; also, $(f + g)(x) = (g + f)(x) = g(x) + f(x)$
$h(x) = (f - g)(x)$	$(f - g)(x) = f(x) - g(x)$; $(f - g)(x) \neq (g - f)(x)$
$h(x) = (g - f)(x)$	$(g - f)(x) = g(x) - f(x)$; $(g - f)(x) \neq (f - g)(x)$
$h(x) = (f \bullet g)(x)$	$(f \bullet g)(x) = f(x) \bullet g(x)$; also, $(f \bullet g)(x) = (g \bullet f)(x) = g(x) \bullet f(x)$
$h(x) = \left(\frac{f}{g}\right)(x)$	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$; $\left(\frac{f}{g}\right)(x) \neq \left(\frac{g}{f}\right)(x)$
$h(x) = \left(\frac{g}{f}\right)(x)$	$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$; $\left(\frac{g}{f}\right)(x) \neq \left(\frac{f}{g}\right)(x)$

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- The **composition of functions** is the process of substituting one function for the independent variable of another function to create a new function.
- The notation $g(f(x))$ or $(g \circ f)(x)$ means that the function $f(x)$ is substituted for the independent variable x in the function $g(x)$. Similarly, the notation $f(g(x))$ or $(f \circ g)(x)$ means that the function $g(x)$ is substituted for the independent variable x in the function $f(x)$.
- A special case of function composition is the inverse function. A function $y = g(x)$ is an inverse of $y = f(x)$ if $g(f(x)) = x$ and $f(g(y)) = y$ and the function and its inverse are defined over a common domain.
- A domain and range of a function resulting from the combination or composition of two or more functions should be compared to the domain and range of each of the functions used in the combination or composition; the resulting domain and range of each function may not be the same as they were.

Common Errors/Misconceptions

- substituting the wrong function into a second function during a composition of functions
- assuming that the domain and range of a function resulting from a combination or a composition will be the same as the domain and range of the functions that are combined
- assuming that switching the order of functions being divided or subtracted will result in the same function

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Guided Practice 4B.2.3

Example 1

Find $(f+g)(x)$ if $f(x)=\frac{x}{x-1}$ and $g(x)=\frac{x-1}{x+2}$. Determine the domains over which $f(x)$, $g(x)$, and $(f+g)(x)$ are defined.

1. To find $(f+g)(x)$, find $f(x)+g(x)$.

$(f+g)(x)=f(x)+g(x)$ Rule for finding $(f+g)(x)$

$(f+g)(x)=\left(\frac{x}{x-1}\right)+\left(\frac{x-1}{x+2}\right)$ Substitute $\frac{x}{x-1}$ for $f(x)$ and $\frac{x-1}{x+2}$ for $g(x)$.

$(f+g)(x)=\frac{x(x+2)+(x-1)^2}{(x-1)(x+2)}$ Simplify.

$(f+g)(x)=\frac{2x^2+1}{(x-1)(x+2)}$

2. Determine the domain of $f(x)$.

The domain of $f(x)$ is $(-\infty, 1)$ and $(1, \infty)$, or $(-\infty, 1)\cup(1, \infty)$, because $f(x)$ is undefined at $x = 1$.

Note: Parentheses around the value mean that domain values go up to but do not include the value. Brackets mean that the domain values include the value. For example, the domain [1, 6) means all the values between 1 and 6, including 1, but not including 6.

3. Determine the domain of $g(x)$.

The domain of $g(x)$ is $(-\infty, -2)$ and $(-2, \infty)$, or $(-\infty, -2)\cup(-2, \infty)$, because $g(x)$ is undefined at $x = -2$.

4. Determine the domain of the combined function $(f+g)(x)$.

The domain of $(f+g)(x)$ is $(-\infty, -2)\cup(-2, 1)\cup(1, \infty)$ because $(f+g)(x)$ is undefined at $x = -2$ *and* at $x = 1$.



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Example 2

Find $(f \bullet g)(x)$ if $f(x)=\sqrt{x-3}$ and $g(x)=\sqrt{4-x}$. Determine the domains over which $f(x)$, $g(x)$, and $(f \bullet g)(x)$ are defined.

1. Find $(f \bullet g)(x)$.

   To find $(f \bullet g)(x)$, find $f(x) \bullet g(x)$.

   $(f \bullet g)(x) = f(x) \bullet g(x)$ — Rule for multiplying functions

   $(f \bullet g)(x)=\left(\sqrt{x-3}\right)\bullet\left(\sqrt{4-x}\right)$ — Substitute $\sqrt{x-3}$ for $f(x)$ and $\sqrt{4-x}$ for $g(x)$.

   $(f \bullet g)(x)=\sqrt{-x^2+7x-12}$ — Simplify.

2. Determine the domain of $f(x)$.

   The domain of $f(x)$ has to meet the condition $x-3\geq 0$ or $x\geq 3$, so the domain is $[3,\infty)$.

3. Determine the domain of $g(x)$.

   The domain of $g(x)$ has to meet the condition $4-x\geq 0$ or $x\leq 4$, so the domain is $(-\infty,4]$.

4. Determine the domain of $(f \bullet g)(x)$.

   The domain of $(f \bullet g)(x)$ has to meet the condition $-x^2 + 7x - 12 \geq 0$, so the domain is $(-\infty,4]\cup[3,\infty)$, which reduces to [3, 4] since any value of x less than 3 or greater than 4 results in $(f \bullet g)(x)$ being undefined.

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Example 3

Find the function that results from the composition $(f \circ g)(x)$ if $f(x) = x^2$ and $g(x) = \sqrt{2-x}$. Determine the domains of $f(x)$, $g(x)$, and $(f \circ g)(x)$.

1. To find $(f \circ g)(x)$, find $f(g(x))$.

   Begin by setting $(f \circ g)(x)$ equal to $f(g(x))$, then substitute the value of $g(x)$ into $f(g(x))$ and solve.

   $(f \circ g)(x) = f(g(x))$ Composition Rule

   $(f \circ g)(x) = f(\sqrt{2-x})$ Substitute $\sqrt{2-x}$ for $g(x)$.

   $(f \circ g)(x) = (\sqrt{2-x})^2$ Substitute $\sqrt{2-x}$ for x in $f(x)$.

   $(f \circ g)(x) = 2 - x$ Simplify.

2. Determine the domain of $f(x)$.

   The domain of $f(x)$ is all real numbers. Therefore, the domain of $f(x)$ is $(-\infty, \infty)$.

3. Determine the domain of $g(x)$.

   The domain of $g(x)$ is restricted by the condition $x \leq 2$. Therefore, the domain of $g(x)$ is $(-\infty, 2]$.

4. Determine the domain of the composition $(f \circ g)(x)$.

   The domain of $(f \circ g)(x)$ is restricted by the condition $x \leq 2$ because x has to satisfy $g(x)$ first, and $f(x)$ can take any number that comes out of $g(x)$. Therefore, the domain of $(f \circ g)(x)$ is $(-\infty, 2]$.

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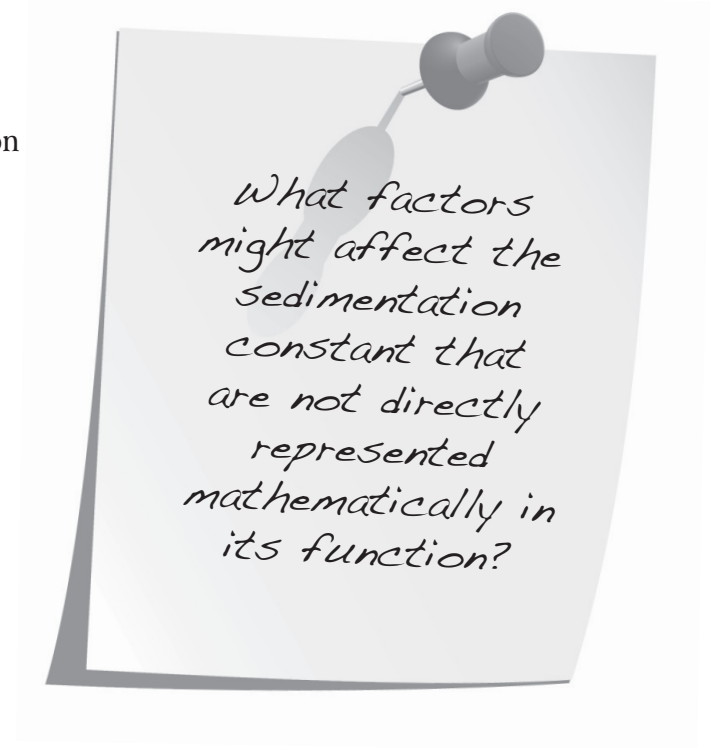
Problem-Based Task 4B.2.3: Blender Compositions

SMP	
1 ✓	2
3	4 ✓
5	6 ✓
7	8

Ice cream shops often use an industrial-strength high-speed blender to mix various flavorings or toppings with milk and ice cream to create milkshakes. Unlike traditional household blenders, these powerful machines feature a long blending apparatus that mixes the ingredients. Blender manufacturers use a function called the *sedimentation constant* to quantify how fast a blender can mix a customer's request in order to market their blenders to restaurant owners. The sedimentation constant also gives a qualitative measure of how creamy or thick the blend is.

The sedimentation constant c is given by the function $c(\omega)=\frac{v}{\omega^2 s}$, in which v is the speed (in cm per second) at which a solid topping particle travels from the blending apparatus to the edge of the mixing container, s is the distance in centimeters the solid topping particle travels from the blending apparatus to the edge of the mixing container, and the Greek lowercase letter ω (*omega*) is the angular or rotational speed of the mixture. The rotational speed ω is measured in radians per second and is given by the function $\omega(t)=\frac{2\pi}{t}$, in which t is the rotational time in seconds.

Write a composition function for the sedimentation constant c that is a function of the rotational time t. Then, use the relationship $s = v \bullet m$ to write the sedimentation constant c in terms of the rotational time t and the "topping migration time" m (in seconds) from the blending apparatus to the edge of the mixing container. Finally, determine the units of the sedimentation constant. What factors might affect the sedimentation constant that are not directly represented mathematically in its function?



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Problem-Based Task 4B.2.3: Blender Compositions

Coaching

a. Which of the following compositions will lead to an expression for c as a function of t: $(c \circ \omega)(t)$ or $(\omega \circ c)(t)$? Explain your reasoning.

b. Which function should be substituted into the other according to the answer to part a? Simplify this function.

c. Use the relationship $s = v \bullet m$ to simplify the composition function from part b.

d. What constants and variables are left in the result for part c?

e. What are the units of part c?

f. What factors might affect the sedimentation constant that are not directly represented mathematically in its function?

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Instruction

Problem-Based Task 4B.2.3: Blender Compositions

Coaching Sample Responses

a. Which of the following compositions will lead to an expression for c as a function of t: $(c \circ \omega)(t)$ or $(\omega \circ c)(t)$? Explain your reasoning.

The composition $(c \circ \omega)(t)$ leads to the composition function for the sedimentation constant c that is a function of the rotational time t because $\omega(t)$ is found in the function $c(\omega)$ and not vice versa.

b. Which function should be substituted into the other according to the answer to part a? Simplify this function.

The composition $(c \circ \omega)(t)$ will substitute the function $\omega(t)$ for the independent variable ω in the function $c(\omega) = \frac{v}{\omega^2 s}$. Recall that $c[\omega(t)] = (c \circ \omega)(t)$.

Thus, the substitution is as follows:

$$c(\omega) = \frac{v}{\omega^2 s}$$

$$(c \circ \omega)(t) = \frac{v}{\left(\frac{2\pi}{t}\right)^2 s}$$

$$(c \circ \omega)(t) = \frac{t^2 \bullet v}{4\pi^2 \bullet s}$$

c. Use the relationship $s = v \bullet m$ to simplify the composition function from part b.

The relationship $s = v \bullet m$ can be used to simplify the composition function. Substitute $v \bullet m$ for s and then simplify as shown:

$$(c \circ \omega)(t) = \frac{t^2 \bullet v}{4\pi^2 \bullet s}$$

$$(c \circ \omega)(t) = \frac{t^2}{4\pi^2} \bullet \frac{v}{v \bullet m}$$

$$(c \circ \omega)(t) = \frac{t^2}{4\pi^2 \bullet m}$$

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d. What constants and variables are left in the result for part c?

The term $4\pi^2$ is a constant. The variables are t and m.

e. What are the units of part c?

The result of part d shows that time squared is divided by time, so the unit is time.

f. What factors might affect the sedimentation constant that are not directly represented mathematically in its function?

Factors that are not directly represented in the function could include the density of the ice cream and milk mixture, the mass and shape of the toppings used and their effect on rotational speed, and the dimensions of the mixing container and the design of the blender blade.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4B.2.3: Combining Functions

For problems 1–3, identify $h(x)$ as a combination, a composition, or both.

1. $h(x)=\left[f(x)\right]^2$

2. $h(x)=\left(f\circ f\right)(x)$

3. $h(x)=\left(f^2\circ f\right)(x)$

For problems 4–7, $f(x)=\dfrac{3}{x+1}$ and $g(x)=\dfrac{1}{x}$. Write the equation of the combination or composition given and determine its domain.

4. $\left(\dfrac{f}{g}\right)(x)$ and $\left(\dfrac{g}{f}\right)(x)$

5. $\left(f-g\right)(x)$ and $\left(g-f\right)(x)$

6. $\left(f\circ g\right)(x)$ and $\left(g\circ f\right)(x)$

7. $\left(f\circ f\right)(x)$ and $\left(g\circ g\right)(x)$

continued

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Use the given information to complete problems 8–10.

An animator using CGI software to create a 3-D animation starts out by drawing a wire frame, or grid, to represent the object being animated. The wire frame outlines the basic geometric shape that surrounds all the parts of the object. The animator's latest project is to create an animation of a hobby rocket that is launched and follows an erratic path above the ground until it burns out after 10 seconds. The part of the wire frame that represents the space formed by the rocket's path from launch to landing can be modeled by a rectangular solid. The solid's base has a length of x and a width of $x + 20$; the solid's height is given by $3x - 1$. The rectangular solid's wire frame is no larger than the flight path—that is, there is no extra margin of space around the path. After t seconds, the rocket is at the corner of the rectangular space that is diagonally across from the corner of the rectangular space from which it was launched. The variable x is related to the time of flight by the function $x(t) = 50 \cdot t$. (*Note*: The time for the rocket to fall back to Earth is not a factor in this problem.)

8. Write a function $V(x)$ for the volume of the wire space formed by the rocket.

9. Write a function $V(t)$ for the volume of the wire space as a function of the flight time of the rocket.

10. What is the volume of the wire space in which the rocket could have flown at $t = 5$ seconds? What is the maximum volume of the space during the rocket's flight?

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Progress Assessment

Circle the letter of the best answer.

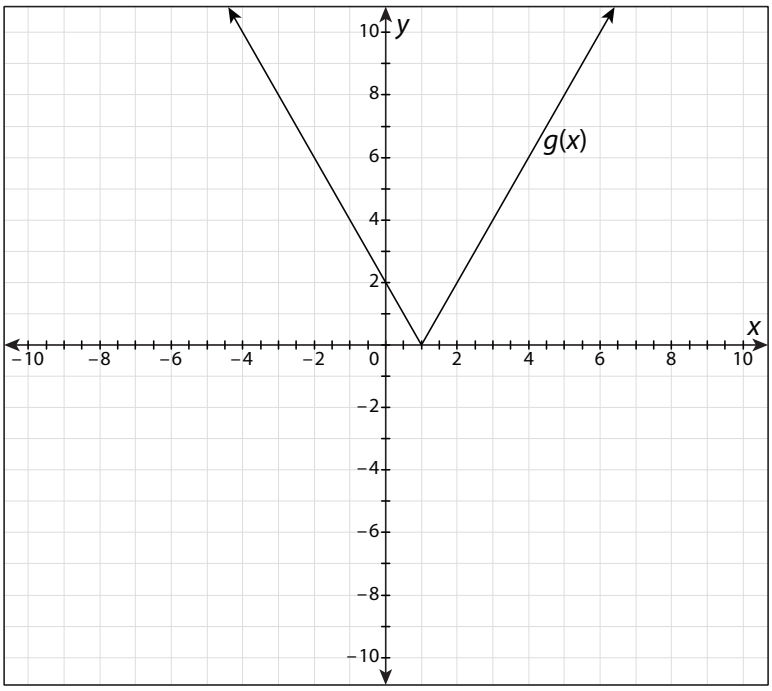
1. Which function is a transformation of the parent function $f(x) = \cos x$ of the form $g(x) = f(x + k)$?

   a. $g(x) = \cos x - 0.5$
   b. $g(x) = \cos (x - 0.5)$
   c. $g(x) = -0.5 \cdot \cos x$
   d. $g(x) = \cos (-0.5x)$

2. Which function is a transformation of the parent function $g(x) = \log x$ of the form $h(x) = g(kx)$?

   a. $h(x) = \log 4 \cdot \log x$
   b. $h(x) = 4 \cdot \log x$
   c. $h(x) = \log 4 + \log x$
   d. $h(x) = \log (x + 4)$

3. The graphed function shown results from the transformation(s) of the parent function $f(x) = |x|$. What is the equation of the graphed function?

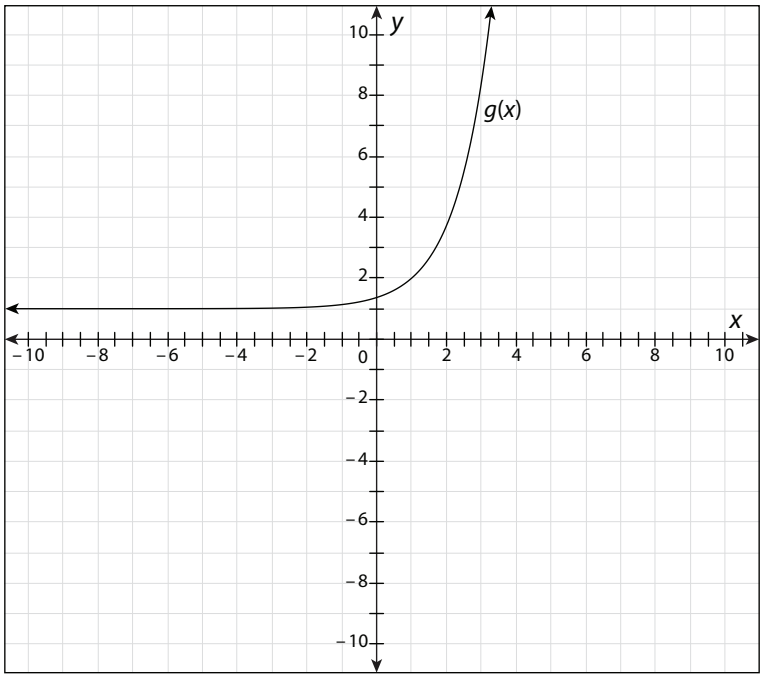


   a. $g(x) = 2 \cdot |x - 1|$
   b. $g(x) = -2 \cdot |x - 1|$
   c. $g(x) = \frac{1}{2} \cdot |x - 1|$
   d. $g(x) = \frac{1}{2} \cdot |1 - x|$

continued

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4. The graphed function shown results from the transformation(s) of the parent function $f(x) = e^x$. What is the equation of the graphed function?



a. $g(x) = -e^{x-1}$

b. $g(x) = e^{x-1}$

c. $g(x) = -1 + e^{x-1}$

d. $g(x) = 1 + e^{x-1}$

5. Which function is neither even nor odd?

a. $f(x) = x^4 - x^2 - 1$

b. $f(x) = x^3 - x$

c. $f(x) = x^3 - x^2 - 1$

d. $f(x) = 1 - x^2$

6. Which function is odd?

a. $g(x) = \cos x$

b. $g(x) = \cos(90° - x)$

c. $g(x) = \cos x + \sin x$

d. $g(x) = \sin x \bullet \sin x$

7. Which choice represents $f(x) \bullet g(x)$ if $f(x) = 2 \bullet e^x$ and $g(x) = 3 \bullet e^{3x}$?

a. $6 \bullet e^{3x^2}$

b. $6 \bullet e^{-2x}$

c. $6 \bullet e^{4x^2}$

d. $6 \bullet e^{4x}$

continued

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Assessment

8. Which choice represents $\frac{g(x)}{f(x)}$ if $f(x) = \log(x-1)$ and $g(x) = \log(x+1)$?

   a. $\log_{x-1}(x+1)$
   b. $\log_{x+1}(x-1)$
   c. $\log(x+1) - \log(x-1)$
   d. $\log 2$

9. Mason built his young cousin a playhouse in the shape of a regular pyramid with a square base. The edge of the base b is equal to the slant height s of any one of the four faces of the pyramid. Which choice gives the height h of the playhouse in terms of b? (*Hints*: What is the length of a diagonal of the base? What are the lengths of the segments formed by the intersection of the slant height with the base of the pyramid?)

   a. $h = \frac{1}{4}\sqrt{3b}$
   b. $h = \frac{b}{2}\sqrt{3}$
   c. $h = \frac{3}{4}\sqrt{b}$
   d. $h = \frac{3b}{4}$

10. The quantities pH and pOH are measures of how acidic or alkaline (basic) a solution is. They are related by the function $\text{pH}(c) + \text{pOH}(d) = 14$, in which c and d are the concentrations of each type of acid and alkali ions, respectively. The functions for $\text{pH}(c)$ and $\text{pOH}(d)$ are given by $\text{pH}(c) = -\log c$ and $\text{pOH}(d) = -\log d$. What is the domain of c and d?

    a. (1, 14)
    b. (0, 14)
    c. $\left(10^{-14}, 1\right)$
    d. $\left(10^{-14}, 10^{1}\right)$

Use what you have learned about transforming models and combining functions to complete all parts of the following problem.

11. Visible-light curing allows a dentist to harden restorative substances on teeth in a matter of seconds. The intensity of LED light used for curing is about 1 watt per centimeter squared (10^{-4} watt per square meter). The energy E in the light is related to the wattage W by the function $E(t) = W \bullet t$, in which time t is in seconds. The energy E is related to the temperature increase ΔT in the tooth in degrees Kelvin by the function $E(t) = kA \bullet \left(\frac{\Delta T}{d}\right)t$. The constant k is the thermal conductivity of the tooth, A is the area exposed to the light in square meters, d is the thickness of the tooth expressed in meters, and t is the time in seconds.

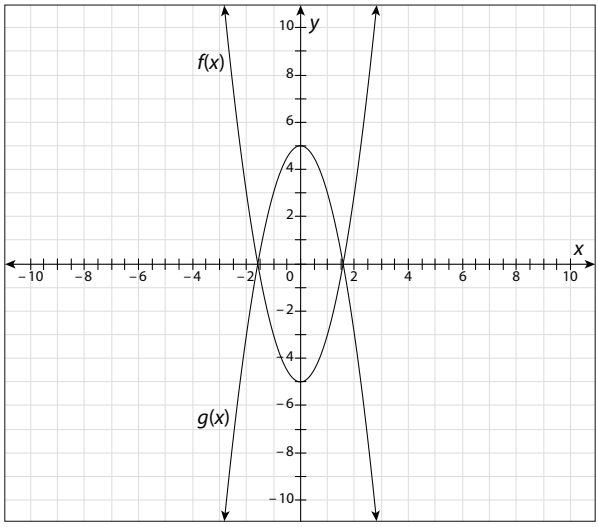
    a. Rewrite the function so that ΔT is expressed in terms of d, W, k, and A.
    b. Use the function from part a to find the temperature increase for a 3-millimeter-thick tooth if $k = 0.000047$, the area exposed to the light is 25 mm^2, and $W = 10^{-4}$.

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Pre-Assessment

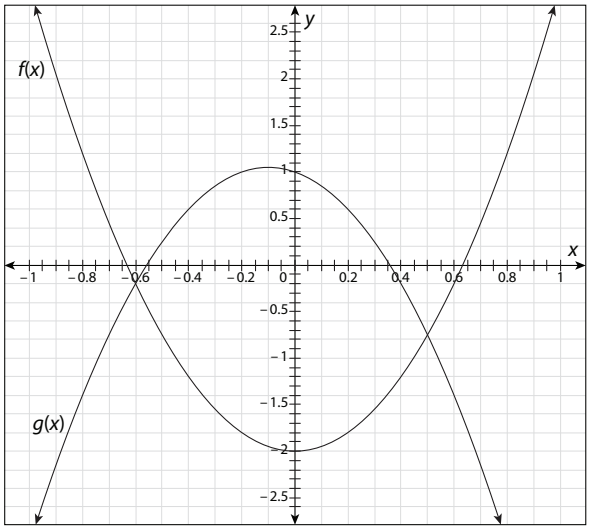
Circle the letter of the best answer.

1. Which key feature is the same for the following graphed functions?



   a. the maximum point
   b. the minimum point
   c. the *x*-intercepts
   d. the *y*-intercepts

2. Over what interval do both of the following functions decrease?

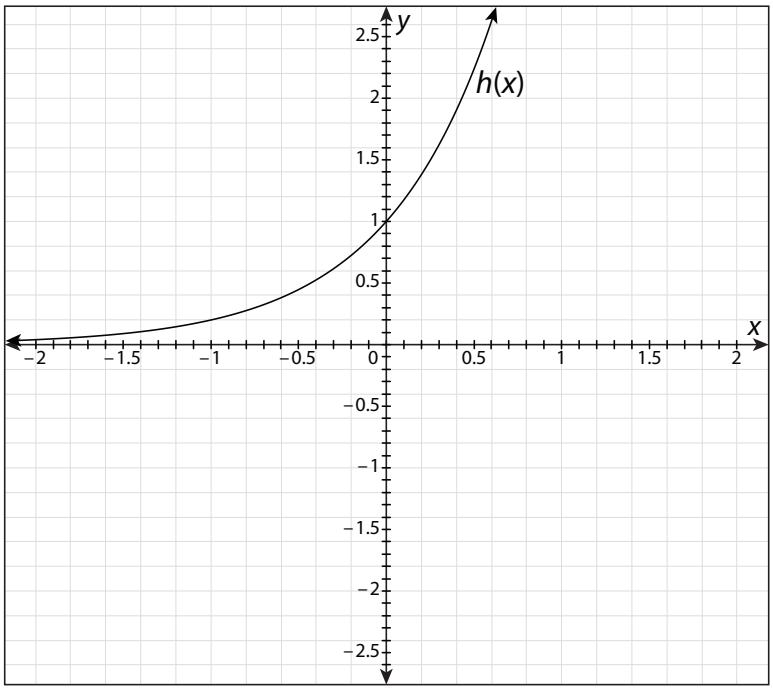


   a. $[0, \infty)$
   b. $[0.1, \infty)$
   c. $(-0.1, 0)$
   d. $(-\infty, -2] \cap [1, \infty)$

continued

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3. Which statement describes the graph of the function $h(x)=5^x$ for the function values between $x = -1$ and $x = 0$?



a. The function values are increasing at the same rate as the function values between $x = 0$ and $x = 1$.

b. The function values are increasing at the same rate as the function values between $x = -2$ and $x = -1$.

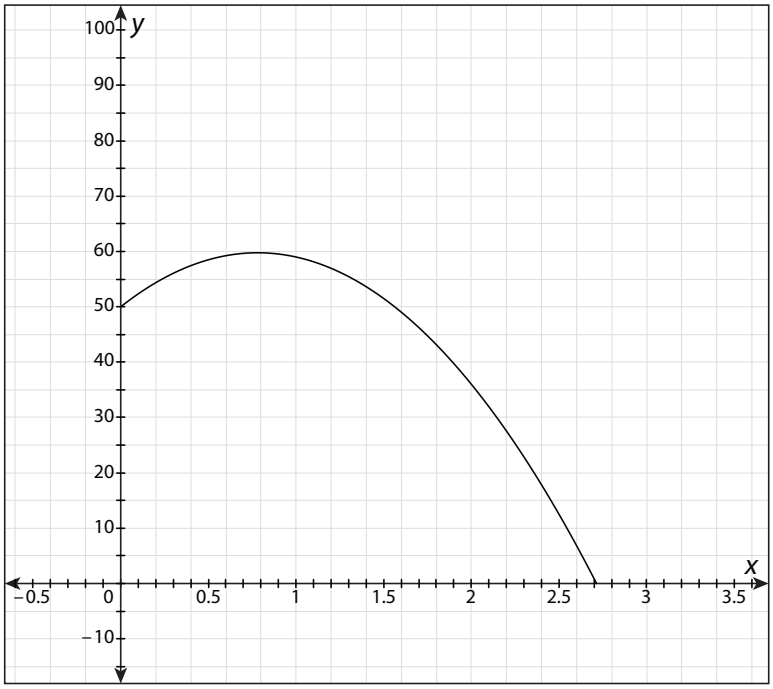
c. The function values are increasing at a faster rate than the function values between $x = 0$ and $x = 1$.

d. The function values are increasing at a slower rate than the function values between $x = 0$ and $x = 1$.

continued

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4. The following graph shows the path that an arrow travels after being shot into the air from a point on the *y*-axis, reaching a maximum height, and then falling to the ground. Assume that there is no wind resistance. What is the *x*-interval for which the speed is decreasing?



a. approximately [0, 2.7]

b. approximately [0, 0.8]

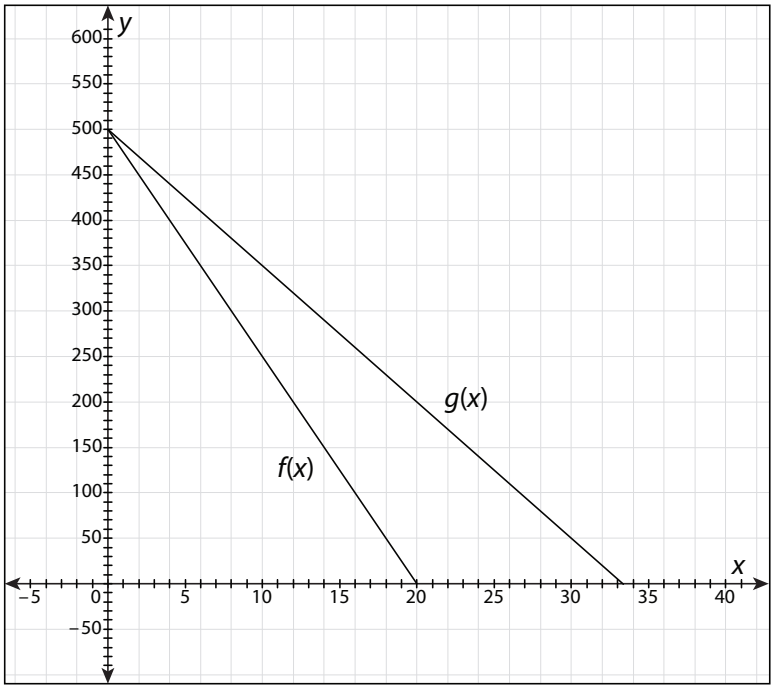
c. approximately [0.8, 2.7]

d. [0, 100]

continued

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5. The following graph shows two functions that describe the average rate at which two wholesalers sell $500,000 worth of inventory. What does the difference between the *x*-intercepts of the two graphs represent?



a. the difference between the rates at which each wholesaler is reducing inventory

b. the difference between the dollar amount of inventory the two wholesalers have left after a specific amount of time

c. the difference between the time it takes one wholesaler to sell out of the inventory in comparison to the time it takes for the second wholesaler to do so

d. the difference between the time it takes for each wholesaler to sell out of $250,000 of inventory

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Lesson 3: Comparing Properties Within and Between Functions

Instruction

Common Core State Standards

SMP	
1 ✓	2 ✓
3	4 ✓
5 ✓	6
7 ✓	8

F–IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F–IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F–IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

F–IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Essential Questions

1. What is the significance of a difference in *x*- or *y*-intercepts of different forms of the same function or in comparing two or more functions in real-world situations?
2. In a real-world problem, how does a function model change when different domains result in positive and negative values?
3. When are maximum and minimum function values relevant to solving or visualizing real-world problems?
4. What are the limitations on the domain of a real-world function and how do these limitations affect function values and possible solutions to real-world problems?
5. How do variations in the rate of change of a function value affect the interpretation of real-world problems in which rate of change is a factor?

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Instruction

WORDS TO KNOW

asymptote	an equation that represents sets of points that are not allowed by the conditions in a parent function or model; a line that a function gets closer and closer to as one of the variables increases or decreases without bound
average rate of change	the ratio of the difference of output values to the difference of the corresponding input values: $\frac{f(b)-f(a)}{b-a}$; a measure of how a quantity changes over some interval
boundary condition	a constraint or limit on a function or domain value based on real-world conditions or restraints in the problem or its solution
***delta* (Δ)**	a Greek letter commonly used to represent the change in a value
extrema	the minima or maxima of a function
initial condition	a constraint or limit on a function or domain value that exists in the form of a *y*-intercept or other starting-point mathematical restraint in a real-world problem or solution
local maximum	the greatest value of a function for a particular interval of the function; also known as a *relative maximum*
local minimum	the smallest value of a function for a particular interval of the function; also known as a *relative minimum*
periodic function	a function whose values repeat at regular intervals
rate of change	a ratio that describes how much one quantity changes with respect to the change in another quantity; also known as the slope of a line
relative maximum	the greatest value of a function for a particular interval of the function; also known as a *local maximum*
relative minimum	the smallest value of a function for a particular interval of the function; also known as a *local minimum*
symmetry of a function	the property whereby a function exhibits the same behavior (e.g., graph shape, function values, etc.) for specific domain values and their opposites

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Recommended Resources

- Interactivate. "Data Flyer."

  http://www.walch.com/rr/00243

  This applet allows users to create graphs by entering data points, then derive the function equation that fits the graph. Graphs can be manipulated to see how changing the constant and/or coefficient affects the shape of the graph.

- OnlineMathLearning.com. "Compare Two Functions."

  http://www.walch.com/rr/00244

  This site offers users access to several videos comparing the properties and features of different types of functions.

- Virtual Nerd. "How Do You Make an Approximate Graph from a Word Problem?"

  http://www.walch.com/rr/00245

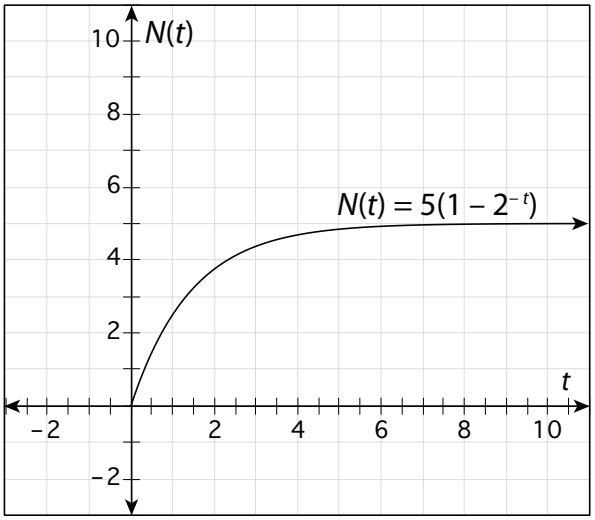
  This video tutorial covers how to sketch the graph of a problem using information derived from a word problem.

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Lesson 4B.3.1: Reading and Identifying Key Features of Real-World Situation Graphs

Warm-Up 4B.3.1

The exponential function $N(t) = 5(1 - 2^{-t})$, graphed as follows, describes the number of breeding pairs of cougars that can be supported by a nature preserve. The independent variable t represents the time in years. Use the function equation and the graph to complete the problems.



1. What is the domain of t?

2. Based on the graph and the domain, describe what happens to the function values as time increases.

3. Rewrite the function with a positive exponent.

4. Use the result of problem 3 to describe what happens to the function values as t increases.

5. What are the maximum and minimum function values? Explain what these values mean within the context of the problem.

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Lesson 4B.3.1: Reading and Identifying Key Features of Real-World Situation Graphs

Common Core State Standards

F–IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F–IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F–IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

Warm-Up 4B.3.1 Debrief

1. What is the domain of t?

   The mathematical domain of t is $(-\infty,\infty)$. However, t represents time, which has values greater than or equal to 0, so the modified mathematical domain is $[0, \infty)$. The problem also assumes that no breeding pairs can be in the preserve until the time is greater than 0; at $t = 0$, $N(t) = 0$. This value of $N(0)$ conforms to the requirement that $N(t)$ should yield a whole number. The real world domain is therefore $[0, \infty)$.

2. Based on the graph and the domain, describe what happens to the function values as time increases.

   The function values increase continuously as time increases. However, the rate at which the function values increase is decreasing as time goes on. The function also appears to approach a maximum value as time increases.

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3. Rewrite the function with a positive exponent.

   In order to change a negative exponent to a positive exponent, the base and the exponent must be moved to the denominator (that is, rewrite the exponential term as its reciprocal).

   $N(t) = 5(1 - 2^{-t})$ Given function

   $N(t) = 5\left(1 - \frac{1}{2^t}\right)$ Rewrite the exponential term as its reciprocal.

   The original function can be rewritten with a positive exponent as $N(t) = 5\left(1 - \frac{1}{2^t}\right)$.

4. Use the result of problem 3 to describe what happens to the function values as t increases.

   As t increases, the value of the fraction $\frac{1}{2^t}$ becomes smaller and approaches 0. This causes the function value to approach 5 • 1 – 0, or a value of 5.

5. What are the maximum and minimum function values? Explain what these values mean within the context of the problem.

   For the real-world domain found in problem 1, [0, ∞), the minimum function value would be 0, which corresponds to no breeding pairs of cougars at a time of 0 years. The maximum function value is close to 5, but since the function is limited to whole numbers, it will have to be rounded down to 4.

Connection to the Lesson

- Students will identify the differences in domains of mathematical problems and real-world problems.
- Students will recognize that mathematically unbounded (infinite or zero) domains or function values are sometimes bounded (finite) in real-world problems.
- Students will compare and contrast increasing or decreasing function values with the rate of change of function values.

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Prerequisite Skills

This lesson requires the use of the following skills:

- identifying the mathematical domain and range of a function (F-IF.5)
- calculating and/or describing how a function increases or decreases in value for different domain values (F-IF.6)
- comparing and contrasting mathematical and real-world function and domain values (F-LE.5)
- comparing how function values change for two or more functions with the same domain values (F-IF.9)

Introduction

Mathematical models can be useful in solving real-world problems. However, it is sometimes the case that the features of a real-world mathematical model are limited or specialized forms of the "pure" mathematics counterpart. Visual representations or graphs of the two versions can highlight the differences and refine the model so that it accurately portrays the problem and/or its solution. The graphs and the accompanying data tables produced by graphing calculators can be invaluable in interpreting function features in real-world applications.

Key Concepts

- The domain and range of a mathematical function represented by the intervals $(-\infty,\infty)$ sometimes include values of the domain or the range that do not have meaning in the context of a given problem.
- For example, time and other properties of matter (including area, number of units, and volume) usually have values that are at least 0. Other physical quantities, such as electrical properties, profit, speed, and temperature, can have both positive and negative values, but they sometimes have interval values that have finite upper and lower values.
- Function values in mathematical and real-world models of problems can be described as increasing, decreasing, or constant. Some real-world situations, such as those involving biological growth or radioactive decay, can be described with only one of these terms in the absence of other factors. For example, natural radioactive decay of a specific isotope of an element is a one-way process (i.e., decreasing) because the reverse process would represent a violation of the accepted physical laws describing the behavior of radioactive elements.
- Mathematical and real-world function models are sometimes restricted to certain unchanging constants. Examples include the values of e and π, the speed of light in a vacuum (often denoted c), and the temperature value of absolute zero.

SRB p. U4B-91

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- Other constants limit function values in a specific problem context. For example, empirical evidence might suggest that only a specific number of polar bears can be supported in an area that has a finite supply of resources the polar bears need to live and multiply.
- The intercepts of a mathematical model have special meanings in real-world problems and/or solutions. At times, the *y*-intercept represents an **initial condition**—that is, a constraint or limit on the domain or function values. An initial condition is a "starting-point" mathematical constraint that dictates function values over other parts of the model of the domain of an independent variable. Typically, the initial condition occurs at a value of 0 for time measurements; however, it might also occur at a value greater than 0. For example, a store may have a certain number of tote bags that need to be sold in a 24-hour period.
- A **boundary condition** is a constraint or limit on a function or domain value based on real-world conditions or restraints in the problem or its solution. Sometimes it represents a physical or quantitative limit. A boundary condition can be represented numerically or with another function.
- Boundary conditions often occur at the asymptotes of exponential, logarithmic, and rational functions. An **asymptote** is an equation that represents sets of points that are not allowed by the conditions in a parent function or model. On a graph, an asymptote is a line that a function gets closer and closer to as one of the variables increases or decreases without bound. Asymptotes represent specific domain values at which mathematical or real-world models do not exist and/or are undefined.
- The **extrema** of a function are the minima or maxima of the function.
- The domain value(s) at which a real-world model has a maximum or a minimum function value are of importance in a wide variety of physical applications.
- With some complicated functions, maximum and minimum function values occur locally; that is, over a restricted domain interval of the function, which may or may not have physical meaning in a real-world model. For example, a function of time given by $f(t)=\frac{t}{t+1}$ only exists for values of $t > 0$.
- Such localized values are called the **local maximum** or **relative maximum** (the greatest value of a function for a particular interval of the function) and the **local minimum** or **relative minimum** (the smallest value of a function for a particular interval of the function).
- The **symmetry of a function** is the property whereby a function exhibits the same behavior (such as graph shape and function values) for specific domain values and their opposites.

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- Symmetry can be useful in determining if all of the solutions of a model have been identified and if they have relevance for the solution of the real-world context being modeled.
- Symmetries can exist across axes or across lines representing linear equations such as $f(x) = x$, as is the case with functions and their inverses. A special type of symmetry exists for periodic functions. A **periodic function** is a function whose values repeat at regular intervals—that is, it exhibits the same function values or properties over repeating restricted domains.
- The behavior of functions for very large and very small domain values can be important in solving real-world problems and describing restrictions on the domain and range values over which a mathematical model can describe a real-world situation.
- For example, compare the functions $f(t)=\frac{t}{t+1}$ and $g(t)=\frac{t}{t+2}$. For small values of t such as $t = 1$, the function values can vary by significant amounts: $f(1) = 0.5$ and $g(1)=0.\overline{3}$. For very large values of t, both functions approach a value of 1; for example, for $t = 100$, $f(100)$ is about 0.9901, whereas $g(100)$ is about 0.9804 The difference between these two models for "large" values of t is about 1 percent. On the other hand, for very small values of t, such as $t = 0.01$, $f(0.01)$ is about 0.01 whereas $g(0.01)$ is about 0.005; the difference between the function values for this "small" value of t is about 100 percent.

Common Errors/Misconceptions

- applying function characteristics in mathematical models to real-world problems and solutions that cannot be supported by the real-world conditions found in the problems
- not identifying constraints and limits on function characteristics for both mathematical models and real-world models
- using mathematical models that inaccurately reflect the real-world conditions of the problems; e.g., using rational functions when exponential or logarithmic functions would be more appropriate

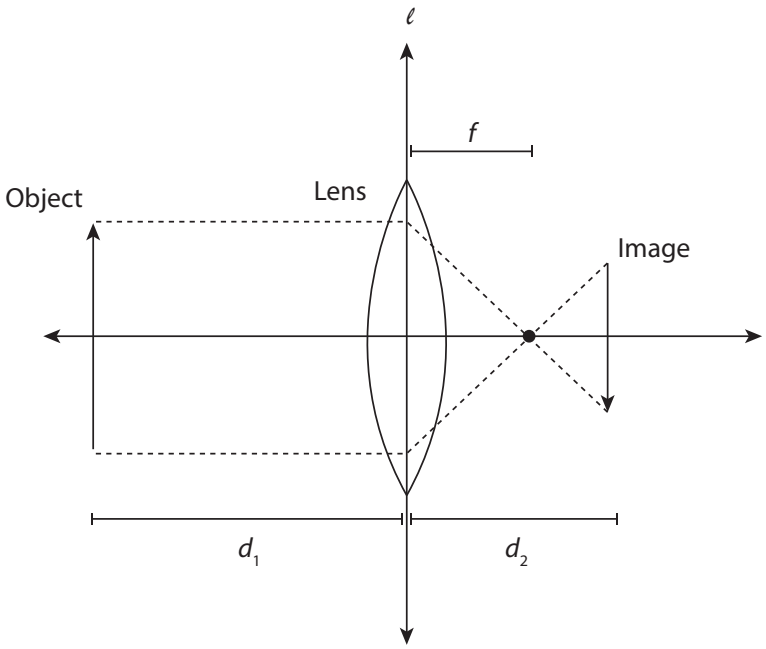
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Guided Practice 4B.3.1

Example 1

The function for finding the amount of time it takes two workers to complete a job is given by $\frac{1}{t_{\text{both}}}=\frac{1}{t_1}+\frac{1}{t_2}$. Compare this to the function $\frac{1}{f}=\frac{1}{d_1}+\frac{1}{d_2}$, which relates the focal length f of a lens to the distances between the lens and an object (d_1) and between the lens and the *image* of the object created by the lens (d_2). The lens is two-sided, so if the lens is centered at the origin, the values of f, d_1, and d_2 can be positive or negative, depending on the position of the object. How are the domains and ranges of these two functions alike and different? Solve each function for the function variable on the left side of each equation. How do the variable relationships differ?



1. State the restrictions, if any, on the domain and range of the first function, t_{both}.

   None of the variables can equal 0 since they are in the denominators of fractions. The variables measure time, so the values of all three variables are assumed to be greater than 0. Therefore, for t_{both}, the restricted domain is $t_1 > 0$ and $t_2 > 0$, and the range is $t_{\text{both}} > 0$.

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2. State the restrictions, if any, on the domain and range of the second function, f.

   None of the variables can equal 0 since they are in the denominators of fractions. According to the diagram, the variables can be positive or negative. The location of the image and/or the object being magnified could be "inside" the focal length f; i.e., between the focal point F and the origin. Therefore, for f, the restricted domain is $d_1 \neq 0$ and $d_2 \neq 0$, and the range is $f \neq 0$.

3. Solve the function $\frac{1}{t_{\text{both}}} = \frac{1}{t_1} + \frac{1}{t_2}$ for t_{both}.

$\frac{1}{t_{\text{both}}} = \frac{1}{t_1} + \frac{1}{t_2}$	Given function
$1 = \frac{1}{t_1} \bullet t_{\text{both}} + \frac{1}{t_2} \bullet t_{\text{both}}$	Multiply both sides by t_{both}.
$1 = \frac{t_{\text{both}}}{t_1} + \frac{t_{\text{both}}}{t_2}$	Simplify.
$1 = \frac{t_{\text{both}} \bullet t_2}{t_1 \bullet t_2} + \frac{t_{\text{both}} \bullet t_1}{t_2 \bullet t_1}$	Create fractions with common denominators.
$1 = \frac{t_{\text{both}} \bullet t_2 + t_{\text{both}} \bullet t_1}{t_1 \bullet t_2}$	Add.
$1 = \frac{t_{\text{both}}(t_2 + t_1)}{t_1 \bullet t_2}$	Apply the Distributive Property.
$t_1 \bullet t_2 = t_{\text{both}}(t_2 + t_1)$	Multiply both sides by $t_1 \bullet t_2$.
$\frac{t_1 \bullet t_2}{t_2 + t_1} = t_{\text{both}}$	Divide both sides by $t_2 + t_1$.
$t_{\text{both}} = \frac{t_1 \bullet t_2}{t_2 + t_1}$	Apply the Symmetric Property of Equality.

The function $\frac{1}{t_{\text{both}}} = \frac{1}{t_1} + \frac{1}{t_2}$ solved for t_{both} is $t_{\text{both}} = \frac{t_1 \bullet t_2}{t_2 + t_1}$.

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4. Solve the function $\frac{1}{f}=\frac{1}{d_1}+\frac{1}{d_2}$ for f.

$\frac{1}{f}=\frac{1}{d_1}+\frac{1}{d_2}$	Given function
$1=\frac{1}{d_1}\bullet f+\frac{1}{d_2}\bullet f$	Multiply both sides by f.
$1=\frac{f}{d_1}+\frac{f}{d_2}$	Simplify.
$1=\frac{f\bullet d_2}{d_1\bullet d_2}+\frac{f\bullet d_1}{d_2\bullet d_1}$	Create fractions with common denominators.
$1=\frac{f\bullet d_2+f\bullet d_1}{d_1\bullet d_2}$	Add.
$1=\frac{f(d_2+d_1)}{d_1\bullet d_2}$	Apply the Distributive Property.
$d_1\bullet d_2=f\left(d_2+d_1\right)$	Multiply both sides by $d_1 \bullet d_2$.
$\frac{d_1\bullet d_2}{d_2+d_1}=f$	Divide both sides by d_2+d_1.
$f=\frac{d_1\bullet d_2}{d_2+d_1}$	Apply the Symmetric Property of Equality.

The function $\frac{1}{f}=\frac{1}{d_1}+\frac{1}{d_2}$ solved for f is $f=\frac{d_1\bullet d_2}{d_2+d_1}$.

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5. Use the restrictions on the domain and range of each function to compare the relationships between the variables in the functions.

   For t_{both}, the restricted domain is $t_1 > 0$ and $t_2 > 0$, and the range is $t_{both} > 0$. As a result of these restrictions, the variables in $t_{both} = \frac{t_1 \bullet t_2}{t_2 + t_1}$ are all positive, so the function value will always be positive.

   For f, the restricted domain is $d_1 \neq 0$ and $d_2 \neq 0$, and the range is $f \neq 0$. As a result of these restrictions, the variables in $f = \frac{d_1 \bullet d_2}{d_2 + d_1}$ could be positive or negative, so f could be positive or negative.

   Also, the condition $d_2 + d_1 \neq 0$ must be observed in order for the function f to exist.

   In summary, the domain and range values for t_{both} will always be positive. Values of the range of the second function, f, can be positive or negative, but the function is only valid for domain values that do not sum to 0 ($d_2 + d_1 \neq 0$).



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Example 2

Working together, carpenters Benjamin and Ava can build a table in 4 hours. Use the function $t_{\text{both}} = \frac{t_1 \bullet t_2}{t_1 + t_2}$ to create a graph and to find the time required for each carpenter to build a table separately if Benjamin takes t hours and Ava takes $t + 2$ hours.

1. Define the variables in the function.

   Using the given information, let $t_1 = t$ hours, $t_2 = t + 2$ hours, and $t_{\text{both}} = 4$ hours.

2. Substitute the defined variables into the given function and simplify.

$t_{\text{both}} = \frac{t_1 \bullet t_2}{t_1 + t_2}$	Given function
$(4) = \frac{(t) \bullet (t+2)}{(t) + (t+2)}$	Substitute t for t_1, $t + 2$ for t_2, and 4 for t_{both}.
$4 = \frac{t^2 + 2t}{t + (t+2)}$	Simplify the numerator.
$4 = \frac{t^2 + 2t}{2t + 2}$	Simplify the denominator.
$4(2t + 2) = t^2 + 2t$	Multiply both sides by $2t + 2$.
$8t + 8 = t^2 + 2t$	Distribute.
$0 = t^2 - 6t - 8$	Rewrite the equation so that all terms are on one side.

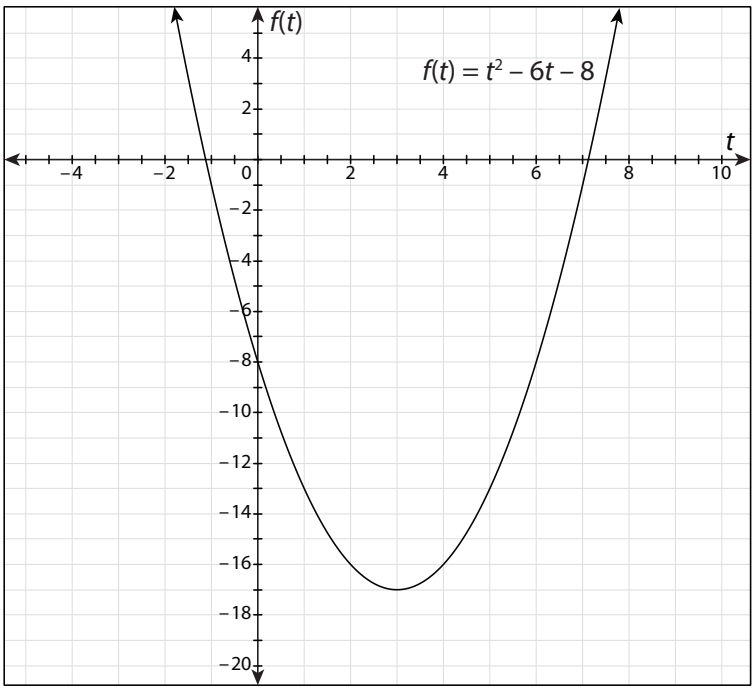
The simplified function can be written as $0 = t^2 - 6t - 8$.

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3. Graph the simplified equation.

   Graph the equation $0 = t^2 - 6t - 8$ either by hand or using a graphing calculator. The equation can be written as $f(t) = t^2 - 6t - 8$.



4. Describe the domain of the function and what that implies for the number of possible solutions for t.

   Since t represents time, the domain of t is values of t that are greater than 0. Therefore, only the t-intercept on the positive t-axis (the horizontal axis) gives a real-world solution for t.

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5. Solve $0 = t^2 - 6t - 8$ to find the indicated domain value(s) of t and determine the amount of time Benjamin and Ava each take to build a table on their own.

   To determine the time required for each carpenter to build the table when working separately, use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to solve $0 = t^2 - 6t - 8$ for t.

   Let $a = 1$, $b = -6$, $c = -8$, and $x = t$.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Quadratic formula
$(t) = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-8)}}{2(1)}$	Substitute 1 for a, –6 for b, –8 for c, and t for x.
$t = \frac{6 \pm 2\sqrt{17}}{2}$	Simplify.
$t = 3 \pm \sqrt{17}$	Divide.

   Only the positive value of the radical, $t = 3 + \sqrt{17}$, will result in a positive value of t; therefore, $t = 3 + \sqrt{17} \approx 7.1$ hours.

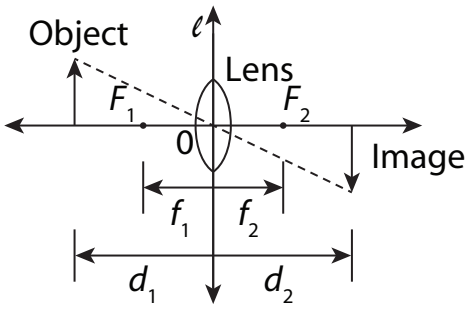
   The variable t represents Benjamin's time, so he can build the table by himself in approximately 7.1 hours. Since $t + 2$ represents Ava's time, substitute 7.1 for t and solve to determine that Ava can build the table by herself in approximately 9.1 hours.

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Example 3

The diagram shows the distance (d_1) between an object and a lens and the distance (d_2) between the lens and the image it creates of the object. F_1 and F_2 represent the focal points of the two-sided lens. (Note that these points are symmetric about the lens at line ℓ.) Use the diagram and the function $f_1 = \frac{d_1 \cdot d_2}{d_1 + d_2}$ to find d_1 and d_2 if the focal length f_1 of the lens is –1 and $d_2 = -d_1 + 2$.



1. Define the variables in the function.

   Using the given information, let $f_1 = -1$ and $d_2 = -d_1 + 2$.

2. Substitute the defined variables into the given function and simplify.

   Substitute each of the defined variables into the given function.

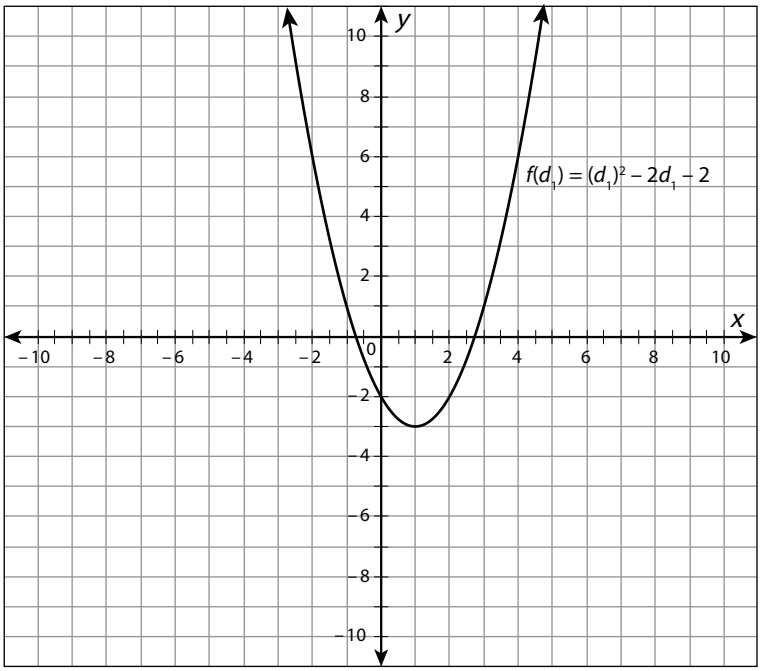
$f_1 = \frac{d_1 \cdot d_2}{d_1 + d_2}$	Given function
$(-1) = \frac{d_1 \cdot (-d_1 + 2)}{d_1 + (-d_1 + 2)}$	Substitute –1 for f_1 and $-d_1 + 2$ for d_2.
$-1 = \frac{-(d_1)^2 + 2 \cdot d_1}{d_1 + (-d_1 + 2)}$	Distribute the numerator.
$-1 = \frac{-(d_1)^2 + 2 \cdot d_1}{2}$	Simplify the denominator.
$-2 = -(d_1)^2 + 2 \cdot d_1$	Multiply both sides by 2.
$0 = (d_1)^2 - 2 \cdot d_1 - 2$	Set all terms on one side with a positive leading coefficient.

The simplified function can be written as $0 = (d_1)^2 - 2d_1 - 2$.

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3. Graph the simplified equation.

   Graph the equation $0 = (d_1)^2 - 2d_1 - 2$ by hand or using a graphing calculator. This equation can be written as $f(d_1) = (d_1)^2 - 2d_1 - 2$.



4. Describe the domain of the function and what that implies for the number of possible solutions for d_1 and d_2.

   The variables cannot equal 0, but they can be positive or negative. The lens diagram shows that d_1 and d_2 can be positive or negative, depending on where the object is located relative to the lens.

   The object could be located between the focal point F_2 and the lens or anywhere to the left of the focal point F_2, as shown in the diagram. This creates the possibility of an infinite number of possible solutions for d_1 and d_2.

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5. Solve $0 = (d_1)^2 - 2d_1 - 2$ to find the value(s) of d_1 as described in step 4.

To solve $0 = (d_1)^2 - 2d_1 - 2$, use the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Let $a = 1$, $b = -2$, $c = -2$, and $x = d_1$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic formula

$$(d_1) = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)}$$ Substitute 1 for a, –2 for b, –2 for c, and d_1 for x.

$$d_1 = 1 \pm \frac{\sqrt{12}}{2}$$ Simplify.

$$d_1 = 1 \pm \sqrt{3}$$

The solutions are $d_1 = 1 + \sqrt{3} \approx 2.73$ or $d_1 = 1 - \sqrt{3} \approx -0.73$.

6. Find d_2 and interpret the solutions in terms of the locations of the object and image in relation to the lens.

The two distances are related by the formula $d_2 = -d_1 + 2$. Since $d_1 \approx 2.73$ or –0.73, substitute each of these values into the formula and solve for d_2.

$d_2 = -d_1 + 2$	$d_2 = -d_1 + 2$
$d_2 \approx -(2.73) + 2$	$d_2 \approx -(-0.73) + 2$
$d_2 \approx -0.73$	$d_2 \approx 2.73$

The focal length, f_1, was given as –1, which means it is measured to the left of the lens.

A negative value for d_2 means the image is on the left side of the lens. A positive value for d_2 means the image is on the right side of the lens.

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Example 4

The table shows the number of earthquakes of six different average magnitudes detected over the course of a year by the Earthquake Hazards Program of the U.S. Geological Survey.

Magnitude, *m*	6.0	6.5	7.0	7.5	8.0	8.5
Number, *N*	210	56	15	3	1	0

Use a graphing calculator to find a natural logarithm function that fits this data. Then choose the scales for a graph of the data based on real-world factors. (*Note*: The earthquake magnitude is the power of an exponent, which can be positive, negative, or 0.) Describe the shape of the resulting graph.

1. Identify the dependent and independent variables.

   The number of earthquakes N depends on the magnitude m, so N is the dependent variable and m is the independent variable.

2. Use these variables to write the general form of the natural logarithm function.

   The parent function of the natural logarithm function is $f(x) = \ln x$, so this function will be of the form $N(m) = \ln m$.

3. Use a graphing calculator to generate a natural logarithm function from the table data.

   You will need to refer to the resulting table of values later, so keep your work on the calculator screen.

   On a TI-83/84:

   Step 1: Press [2ND][Y=] to bring up the STAT PLOTS menu.

   Step 2: The first option, Plot 1, will already be highlighted. Press [ENTER].

   Step 3: Under Plot 1, select ON if it isn't selected already.

(*continued*)

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Step 4: Press [STAT]. Select 1: Edit... to open a table. Arrow up to L1, then press [CLEAR][ENTER] to clear the list. Repeat as needed. Press [ENTER].

Step 5: From L1, press the down arrow to move your cursor into the list. Enter the values from the table for each magnitude, pressing [ENTER] after each number to navigate to the next blank spot in the list.

Step 6: Arrow over to the L2 column. Enter the corresponding values representing the number of earthquakes for each magnitude.

Step 7: To fit an equation to the data points, press [STAT] and arrow over to the CALC menu. Then, select 9: LnReg, "natural logarithm regression."

Step 8: Press [2ND][1] to type "L1" for Xlist. Press [,], then arrow down to Ylist and press [2ND][2] to type "L2" for Ylist, if not already shown.

Step 9: Arrow down to "Calculate" and press [ENTER] to generate a natural logarithm function of the form $y = a + b \cdot \ln x$.

On a TI-Nspire:

Step 1: Press [home]. Arrow over to the spreadsheet icon, the fourth icon from the left, and press [enter].

Step 2: To clear the lists in your calculator, arrow up to the topmost cell of the table to highlight the entire column, then press [menu]. Choose 3: Data, then 4: Clear Data. Repeat for each column as necessary.

Step 3: Arrow over to column A. Name the column "m" and enter the values from the table for each magnitude, pressing [enter] to navigate to the next blank cell.

Step 4: Arrow over to column B. Name the column "n" and enter the corresponding values representing the number of earthquakes for each magnitude.

(*continued*)

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Step 5: To generate a natural logarithm function with the data, press [menu]. Select 4: Statistics, then 1: Stat Calculations, and then B: Logarithmic Regression.... Press [enter].

Step 6: Use the pop-up menus to enter "m" in the X List field and "n" in the Y List field. Tab to OK and press [enter]. This will generate a natural logarithm function of the form $y = a + b \cdot \ln x$. The coefficients appear on the right side of the screen in column D.

Either calculator will return approximate results of $a \approx 1075.78$ and $b \approx -520.92$.

4. Write the function and enter it on the graphing calculator.

   To write a natural logarithm function of the form $y = a + b \cdot \ln x$, let $y = N(m)$, $a = 1075.78$, $b = -520.92$, and $x = m$.

   The function that approximately fits the table data is $N(m) = 1075.78 - 520.92 \cdot \ln m$.

   Enter the function on the graphing calculator.

5. Select a scale for the dependent and independent variable axes on a graph of the data.

   The m values range from 6.0 to 8.5, so an independent variable scale could be from $m = 0$ to $m = 10$. The N values range from 0 to 210, so a dependent variable scale could be from $N = 0$ to $N = 250$ with a scale of 25 for each interval on the axis. Other scales are acceptable as long as they show all of the data points in the table.

6. Use the results to adjust the scales of the graphing calculator's viewing window, then graph the natural logarithm function and table data.

   On a TI-83/84:

   Step 1: Press [WINDOW] and change the x and y scales to the values identified earlier: Xmin = 0, Xmax = 10, Xscl = 1, Ymin = 0, Ymax = 250, and Yscl = 25.

   Step 2: Press [GRAPH] to view the data plot and function.

(*continued*)

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On a TI-Nspire:

Step 1: Press [home]. Arrow over to the graphing icon, the second icon from the left, and press [enter].

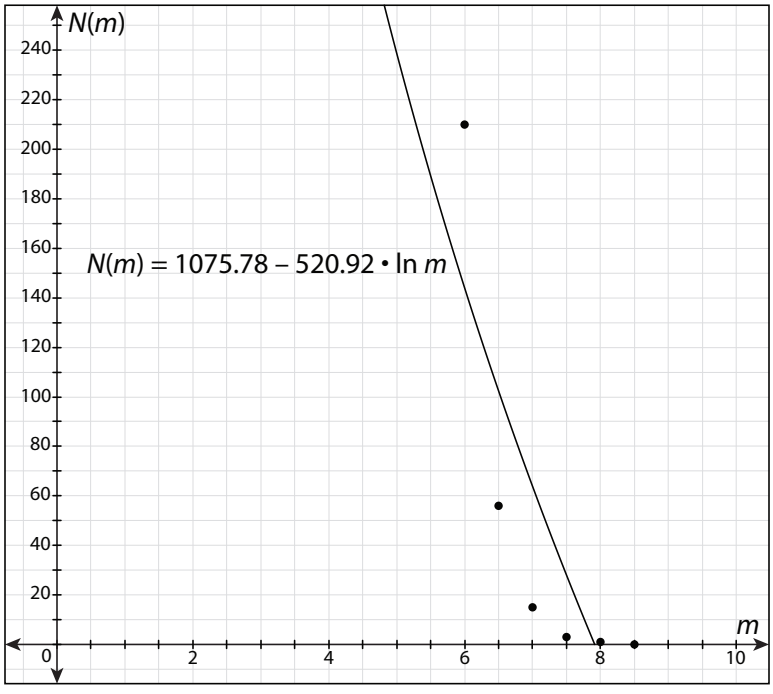
Step 2: Press [menu], then select 4: Window/Zoom, and then 1: Window Settings. Change the settings as identified earlier: XMin = 5, XMax = 10, XScale = Auto, YMin = 0, YMax = 250, and YScale = 25. Tab to "OK" and press [enter] to save the settings.

Step 3: Press [menu] when the new window appears and select 3: Graph Type and 4: Scatter Plot.

Step 4: At the bottom of the graph window, beside *x* type *m*. Press [tab].

Step 5: Beside *y*, type *n*. Press [enter]. The points will appear on the graph.

The result on either calculator includes the data points and the natural logarithm function. (*Note*: Save this data for use with Example 5.)



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7. Describe the shape of the graph in terms of how the function values change as the earthquake magnitude increases from 6.0 to 8.5 and in terms of changes in N as m changes.

   The function value decreases from $m = 6.0$ to $m = 8.5$, and the *rate* of increase also decreases from $m = 6.0$ to $m = 8.5$.

   The value of N decreases with each step in the value of $m(0.5)$, but the amount of decrease is constant over each interval. For example, N decreases from 210 to 56 for m from 6.0 to 6.5, and N decreases from 56 to 15 for m from 6.5 to 7.



Example 5

Modify the earthquake data from Example 4 (repeated below) by adding a second dependent variable in the graphing calculator table for the natural logarithm of the number of earthquakes—ln N—of each value of m. Graph this new variable and the earthquake magnitude m and describe the resulting graph. Suggest an explanation for the shape and domain of the new graph given that the earthquake magnitude m is defined as the power of an exponent. What restrictions exist on m?

Magnitude, *m*	6.0	6.5	7.0	7.5	8.0	8.5
Number, *N*	210	56	15	3	1	0

1. Modify the graphing calculator data from Example 4 to add values for ln N of each value of m.

   On your graphing calculator, return to the table of values for m and N that you created in Example 4. (If you need to enter the values again, refer to the calculator directions in that example.)

   On a TI-83/84:

   Note: Recall that L1 represents the independent variable m and L2 represents the dependent variable N; therefore, L3 will represent the third variable, ln N.

   Step 1: Press [2ND][Y=] to bring up the STAT PLOTS menu.

   Step 2: Select 1: Plot 1..., then arrow down to Data List and press [2ND][3] to change the variable to L3. Press [ENTER] to save the change.

(*continued*)

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Step 3: Press [STAT]. Select 1: Edit... and press [ENTER].

Step 4: The first *N* value from column L2 is cell L2(1). To enter the natural logarithm of this value, arrow to the first blank cell in L3. Press [LN][2ND][2][(][1][)]. Press [ENTER].

Step 5: Repeat Step 4 for the other nonzero numbers in the L2 column, following the pattern [LN][2ND][2] and changing the number in parentheses depending on the row.

Step 6: Delete the data from cells L1(6) and L2(6) by pressing [DEL] in each of those cells so the calculator can produce the graph for the variables L1 and L3.

On a TI-Nspire:

Step 1: Press [ctrl] and then press up on the NavPad to view all open documents. Select the table document from Example 4.

Step 2: Navigate to the first cell in column C and press up on the NavPad to highlight the column. Press [clear] to delete the data. Repeat for column D.

Step 3: Highlight the top cell in column C. Use your keypad to enter a name for the natural logarithm of *N*, such as "lnn." (*Note*: The "ln" operation key cannot be used for this since it is a variable name.)

Step 4: The first *N* value from column B is cell B1. To enter the natural logarithm of this value, arrow to the first blank cell in column C. Press [=][ctrl][e^x][B][1]. Press [enter].

Step 5: Repeat Step 4 for the other nonzero numbers in the B column, following the pattern [=][ctrl][e^x][B] and changing the last number depending on the row. Do not repeat for the last cell, B6, because the logarithm of 0 is undefined.

Step 6: Clear the data from cells A6 and B6 so the calculator can calculate the graph for the *m* and "lnn" variables.

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2. Select a scale for the dependent and independent variable axes on a graph of the revised data.

   The settings for the independent variable *m* can be the same as those used in Example 4, since those did not change. However, the values of the dependent variable "lnn" vary between 0 and 5.3471, so this scale should be reset to values from –1 to 6. Other scales are acceptable as long as they show all of the data points.

   On a TI-83/84:

   Step 1: Press [WINDOW] and change the *x* and *y* scales as needed to reflect the modified values: Xmin = –1, Xmax = 10, Xscl = 1, Ymin = –1, Ymax = 6, and Yscl = 1.

   Step 2: Press [GRAPH] to view the data plot.

   On a TI-Nspire:

   Step 1: Press [home]. Arrow over to the graphing icon, the second icon from the left, and press [enter].

   Step 2: Press [menu], then select 4: Window/Zoom, and then 1: Window Settings. Change the settings as needed to reflect the modified values: XMin = –1, XMax = 10, XScale = Auto, YMin = –1, YMax = 6, and YScale = Auto. Tab to "OK" and press [enter] to save the settings.

   Step 3: Press [menu] when the new window appears and select 3: Graph Type and 4: Scatter Plot.

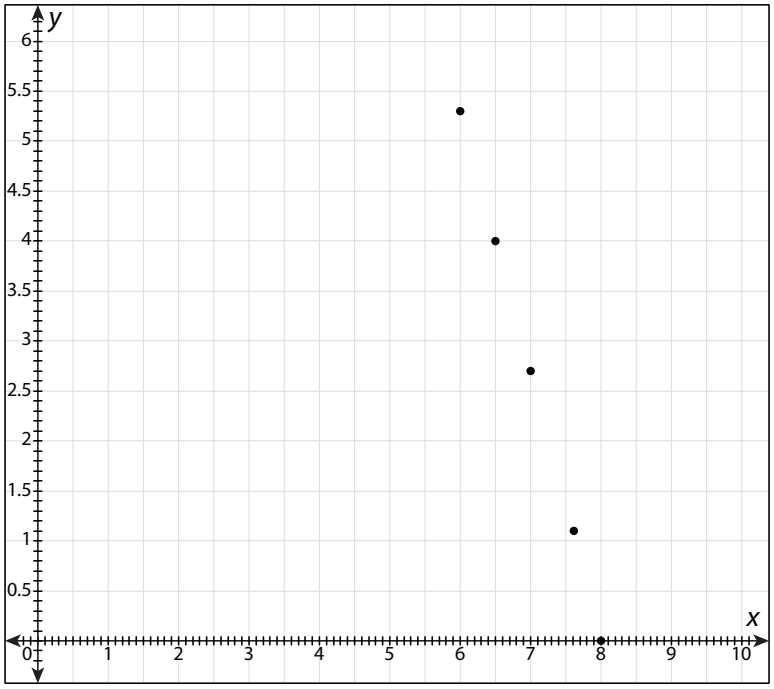
   Step 4: At the bottom of the graph window, beside *x* type *m*. Press [tab].

   Step 5: Beside *y*, type *lnn*. Press [enter]. The points will appear on the graph.

(*continued*)

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The resulting graph is as follows.



3. Describe the shape of the graph.

   The graph is a straight line with a negative slope.

4. Suggest an explanation for the shape and domain of the new graph based on the definition of m from the problem statement.

   The variable m is described as "the power of an exponent," which is also a definition of a logarithm. This means that in this graph, one logarithm is plotted against another logarithm plus a constant. The average rate of change of one logarithm relative to the other logarithm is constant in this case, which results in a straight line.

   Mathematically, since ln N is undefined for $m = 8.5$, the domain of the quake magnitudes is restricted to values of m that are less than 8.5. However, since this is a logarithm, recall that the independent variable m will not be less than or equal to 0, as evidenced by the point (8.5, 0). So, since for this real-world situation ln N cannot be less than 0, the domain of the quake magnitudes is restricted to values of m that are less than or equal to 8.5.

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Problem-Based Task 4B.3.1: Tracking Carbon Dioxide

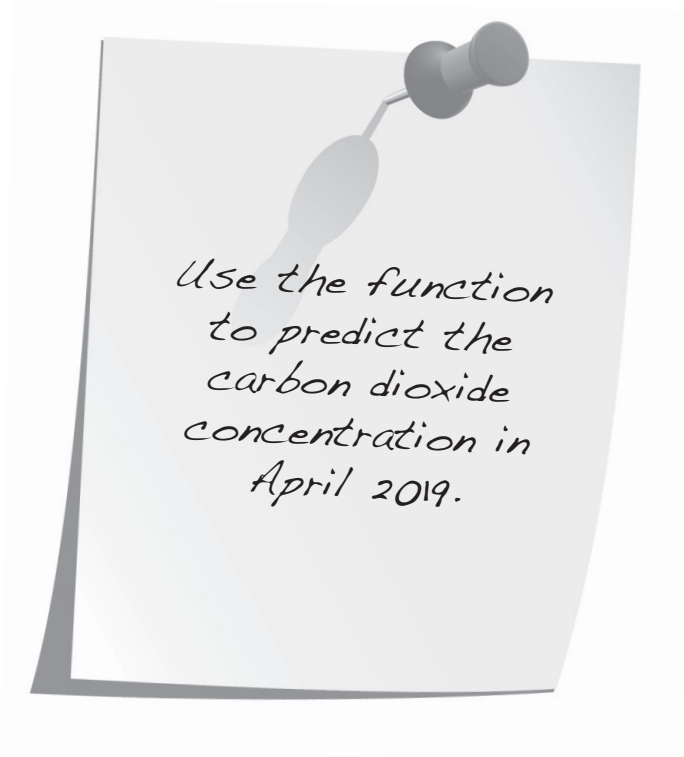
SMP	
1	2 ✓
3	4 ✓
5 ✓	6
7 ✓	8

Scientists at the Mauna Loa Observatory in Hawaii have been tracking the levels of carbon dioxide in the atmosphere since the 1950s. The table shows the scientists' measurements of the amount of carbon dioxide in the atmosphere in parts per million over a six-decade period.

Year, *t*	1960	1970	1980	1990	2000	2010
Carbon dioxide, *C*	315	324	337	352	368	388

The concentration of carbon dioxide in the atmosphere can be modeled by the function $C(t) = at + b + c \cdot \sin 20\pi t$, in which a is the concentration in 1960 ($t = 1$), b is the concentration when measurements started, and c is a constant that indicates by how much the carbon dioxide concentration varies during each year of the decade. The argument of the sine function shows that the carbon dioxide fluctuates from a maximum to minimum value annually.

Use a graphing calculator to analyze the data in the table and fit an equation to it. Graph the table data and compare it to the given function, $C(t) = at + b + c \cdot \sin 20\pi t$. How are they alike and how are they different? Modify the function for $C(t)$ by adding a sine term in which the annual fluctuation of the carbon dioxide concentration is 3 parts per million. Finally, use the function to predict the carbon dioxide concentration in April 2019.



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Problem-Based Task 4B.3.1: Tracking Carbon Dioxide

Coaching

a. Plot the points from the table on a graphing calculator.

b. What kind of function(s) will fit the data points on the graph?

c. Use a graphing calculator to find the function named in part b that fits the data in the table.

d. Compare the function from part c with the given modeling function, $C(t) = at + b + c \bullet \sin 20\pi t$. How are they alike and how are they different?

e. How can you combine the calculator-generated function with a sine function to produce the modeling function given in the problem?

f. What are the amplitude and period of the sine term in the function, $C(t) = at + b + c \bullet \sin 20\pi t$?

g. Use the results of parts e and f to write the combined function for the carbon dioxide concentration trend.

h. Check your work by graphing the function written in part g and checking the concentration values for the values of t given in the table.

i. Predict what the carbon dioxide concentration will be in April 2019.

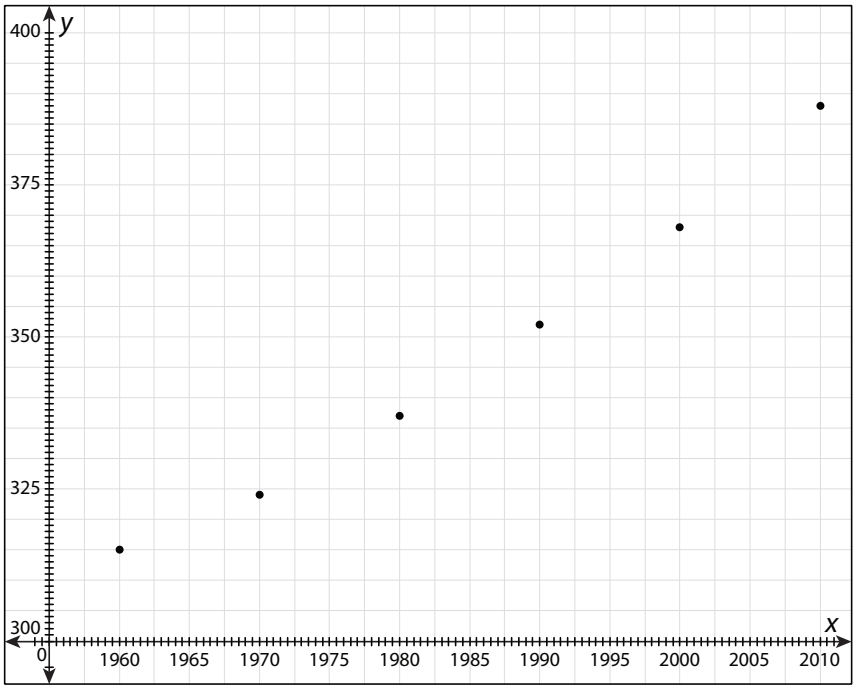
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Problem-Based Task 4B.3.1: Tracking Carbon Dioxide

Coaching Sample Responses

a. Plot the points from the table on a graphing calculator.

The points should appear on a graph as follows.



b. What kind of function(s) will fit the data points on the graph?

The data points appear to form a fairly straight line, indicating a linear function.

c. Use a graphing calculator to find the function named in part b that fits the data in the table.

Use the graph created for part a to create the function on a graphing calculator. Round the coefficients to three significant figures, since that is the number of significant places in the data. The result should be in the form $y = ax + b$.

The calculator shows values for a and b of $a = 14.6286$ and $b = 296.133$.

The linear function that approximately fits the table data is $C(t) = 14.6 \bullet t + 296$.

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d. Compare the function from part c with the given modeling function, $C(t) = at + b + c \cdot \sin 20\pi t$. How are they alike and how are they different?

The graphing calculator function is approximately $C(t) = 14.6 \cdot t + 296$. The modeling function from the problem statement has the form $C(t) = at + b + c \cdot \sin 20\pi t$. The graphing calculator function is missing the sine term that shows the annual variation in carbon dioxide concentration. In the graphing calculator function, the value 14.6 corresponds to the coefficient a in the model, and the value 296 corresponds to the coefficient b in the model.

e. How can you combine the calculator-generated function with a sine function to produce the modeling function given in the problem?

Substitute the coefficients generated by the calculator in the combined function model: $C(t) = at + b + c \cdot \sin 20\pi t$ becomes $C(t) = 14.6t + 296 + c \cdot \sin 20\pi t$.

f. What are the amplitude and period of the sine term in the function, $C(t) = at + b + c \cdot \sin 20\pi t$?

The problem states that the carbon dioxide concentration fluctuates by 3 parts per million annually. If this is taken as the difference between the maximum and the minimum value of the sine terms, then the amplitude is 1.5. The period is found by solving the equation $20\pi t = 2\pi$, which results in 0.1. This period corresponds to a frequency of 10 times in a decade, or 1 time each year.

g. Use the results of parts e and f to write the combined function for the carbon dioxide concentration trend.

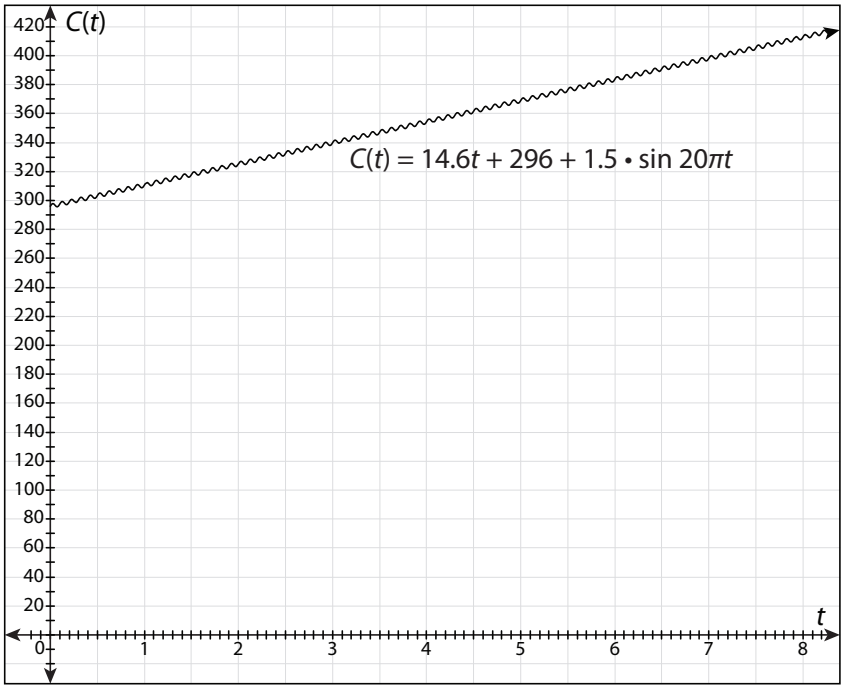
Combining the results of parts e and f gives a combined function for the increasing carbon dioxide concentration over the six-decade period:

$C(t) = 14.6t + 296 + 1.5 \cdot \sin 20\pi t$

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h. Check your work by graphing the function written in part g and checking the concentration values for the values of t given in the table.

Use a graphing calculator to graph $C(t) = 14.6t + 296 + 1.5 \cdot \sin 20\pi t$.



The $C(t)$ values for carbon dioxide concentration in parts per million (ppm) in the decades from $t = 1$ (1960) to $t = 6$ (2010) are as follows:

- $C(1) \approx 311$ ppm
- $C(2) \approx 325$ ppm
- $C(3) \approx 340$ ppm
- $C(4) \approx 354$ ppm
- $C(5) \approx 369$ ppm
- $C(6) \approx 384$ ppm

These values are very close to those given in the table. Therefore, the model predicts the values of $C(t)$ accurately.

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i. Predict what the carbon dioxide concentration will be in April 2019.

April 2019 is nine years and four months into the decade of 2010 ($t = 6$), so t will be the sum of six decades (6), nine-tenths of a decade (0.9), and $\frac{1}{3}$ of a year (which is 0.1 of a decade, so $\frac{1}{3} \bullet 0.1 = 0.0\overline{3}$). Therefore, for April 2019, $t = 6.9\overline{3}$.

Substitute $t = 6.9\overline{3}$ into the function from part g, $C(t) = 14.6t + 296 + 1.5 \bullet \sin 20\pi t$. The sine term becomes $1.5 \bullet \sin\left(20\pi \bullet 6.9\overline{3}\right)$ or approximately 1.5 • 0.969.

The predicted approximate carbon dioxide concentration for April 2019 is approximately 399 parts per million.

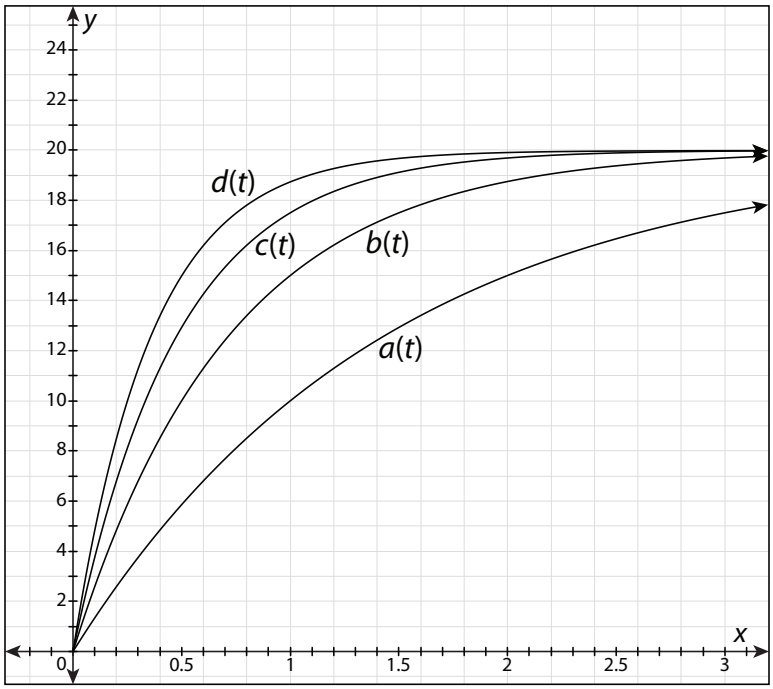
Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4B.3.1: Reading and Identifying Key Features of Real-World Situation Graphs

The following graph shows the rate at which different-sized groups of 2, 4, 8, and 16 fashion designers—$a(t)$, $b(t)$, $c(t)$, and $d(t)$, respectively—created 20 different but specific designs for a new line of clothing made from biodegradable natural fabrics and recyclable synthetic materials. The vertical axis represents the number of designs completed, and the horizontal axis represents the time in days. Use the graph to complete problems 1–3.



1. Describe the change in the number of designs completed for all four groups from time $t = 0$ to $t = 1$.

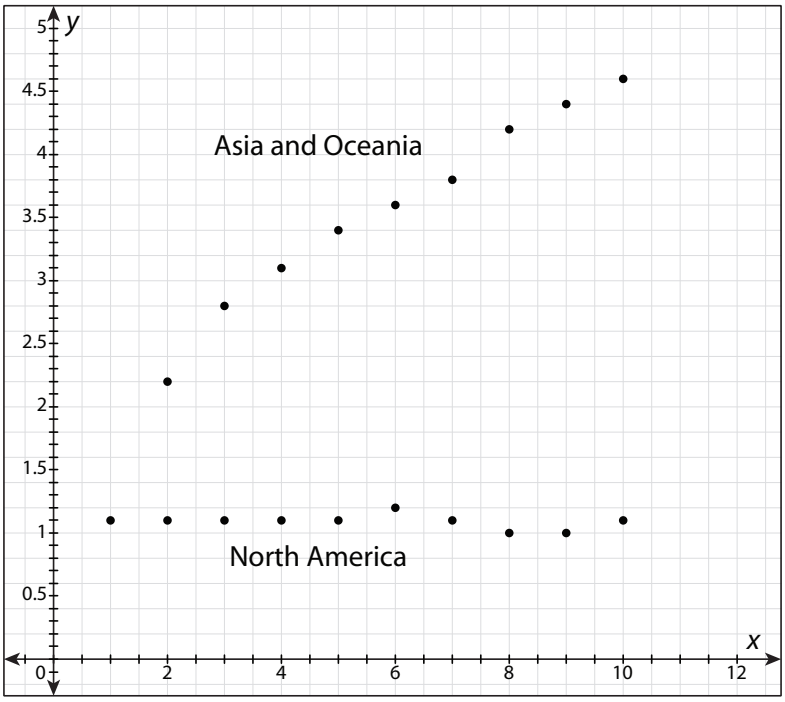
2. Describe the change in the number of designs completed for all four groups from time $t = 2$ to $t = 3$.

3. Describe how the number of completed designs changes as the group size increases.

continued

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The following graph shows the annual consumption of coal by Asian nations and Oceania from 2001 through 2010 in comparison to coal consumption in North America over the same time span. Use the graph to complete problems 4–6.



4. What kind of function would best represent both sets of data?

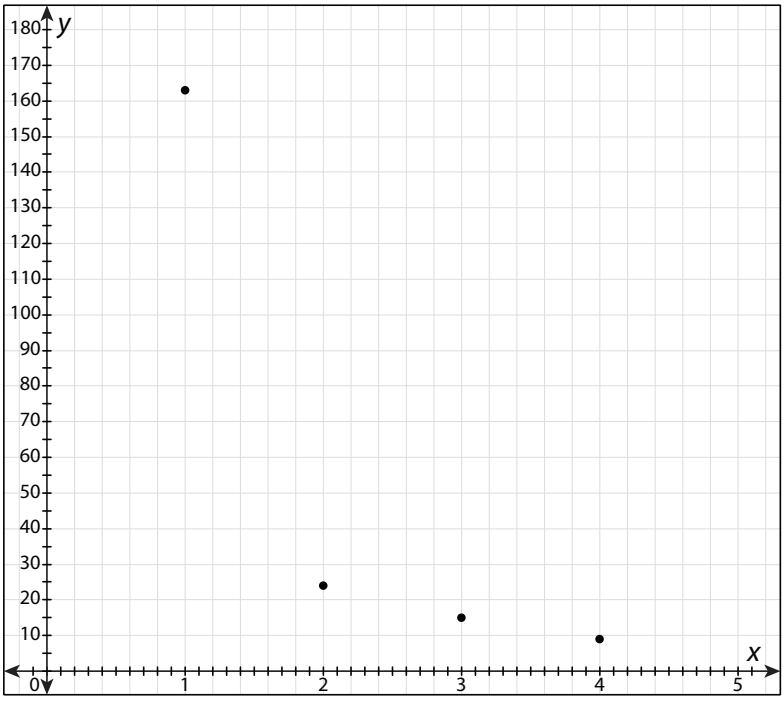
5. Describe the change in function values across the time interval.

6. At what time was the coal consumption of Asian nations and Oceania 3 times that of North America?

continued

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Scientists at the Jet Propulsion Laboratory at the California Institute of Technology track the number of asteroids passing close to Earth as detected by an unmanned satellite called the Wide-Field Infrared Survey Explorer (WISE). Such near-Earth objects (NEOs) are grouped into categories according to their diameters. The following graph shows a plot of four categories of NEOs. The table shows the NEO categories c, the diameter ranges in meters, and the number n of NEOs in hundreds detected over a given time period. Use the graph and table to complete problems 7–10.



Category, *c*	1	2	3	4
NEO diameter range (meters)	100–300	300–500	500–1,000	> 1,000
Number, *n* (hundreds)	163	24	15	9

7. What does the graph imply about the number of detected NEOs that are less than 100 meters in diameter?

8. What does the graph imply about the number of detected NEOs that are greater than 1,000 meters in diameter?

9. Explain how and why both an exponential and a linear function can be used to describe this data.

10. Compare the exponential and the linear model for this data and draw a conclusion about which function would "best" describe the data.

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Lesson 4B.3.2: Calculating Average Rates of Change

Warm-Up 4B.3.2

For a given week at a local gas station, the average price of gasoline was $3.50 per gallon, and the average price of diesel was $3.30 per gallon. The following table shows the price in dollars for various purchases of equal amounts of gasoline and diesel, where x represents the number of gallons purchased, $g(x)$ is the function for gasoline purchases, and $d(x)$ is the function for diesel purchases.

x	5	10	15	20
$g(x)$	17.50	35	52.50	70
$d(x)$	16.50	33	49.50	66

1. How does the value of $g(5) - g(15)$ compare to the value of $g(15) - g(5)$?

2. How does the value of $d(5) - d(15)$ compare to the value of $d(15) - d(5)$?

3. Compare $g(x) - d(x)$ to the value of $d(x) - g(x)$ for two of the given values of x.

4. How do the values of $g(x)$ and $d(x)$ change when x increases by 1?

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Lesson 4B.3.2: Calculating Average Rates of Change

Common Core State Standard

F–IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

Warm-Up 4B.3.2 Debrief

1. How does the value of $g(5) - g(15)$ compare to the value of $g(15) - g(5)$?

   Substitute the values from the table and subtract.

   $g(5) - g(15) = 17.50 - 52.50 = -35$

   $g(15) - g(5) = 52.50 - 17.50 = 35$

   $g(5) - g(15)$ is equal to $-[g(15) - g(5)]$.

2. How does the value of $d(5) - d(15)$ compare to the value of $d(15) - d(5)$?

   Again, substitute the values from the table and subtract.

   $d(5) - d(15) = 16.50 - 49.50 = -33$

   $d(15) - d(5) = 49.50 - 16.50 = 33$

   $d(5) - d(15)$ is equal to $-[d(15) - d(5)]$.

3. Compare $g(x) - d(x)$ to the value of $d(x) - g(x)$ for two of the given values of x.

   Choose any two values of x for both functions. Let's use $x = 10$ and $x = 20$.

For $x = 10$:	**For $x = 20$:**
$g(10) - d(10) = 35 - 33 = 2$	$g(20) - d(20) = 70 - 66 = 4$
$d(10) - g(10) = 33 - 35 = -2$	$d(20) - g(20) = 66 - 70 = -4$

   The results show that $g(x) - d(x)$ is the opposite of $d(x) - g(x)$; symbolically, $g(x) - d(x) = -[d(x) - g(x)]$.

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4. How do the values of $g(x)$ and $d(x)$ change when x increases by 1?

   The given x-values are in increments of 5; to determine the change in values for when x increases by 1, calculate the difference in function values between two given values of x for each of the functions, then divide that difference by 5.

   - $g(5) = 17.50$ and $g(10) = 35$, which is a difference of 17.50 when x increases by 5. $17.5 \div 5 = 3.5$; therefore, values of $g(x)$ increase by 3.5 when x increases by 1.
   - $d(5) = 16.50$ and $d(10) = 33$, which is a difference of 16.50 when x increases by 5. $16.5 \div 5 = 3.3$; therefore, values of $d(x)$ increase by 3.3 when x increases by 1.

Connection to the Lesson

- Students will compare and contrast function values of the same function at different domain values in order to find an average rate of change for the function values.
- Students will expand upon their understanding of this type of comparison in order to compare values of two or more functions that are the result of combining those functions.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- finding the value of a function at a specific domain value (F-IF.2)
- calculating the difference of two function values at different domain values (F-IF.2)
- comparing the values of two different functions at the same domain value (F-IF.9)
- identifying an increasing or decreasing trend in function values as domain values increase or decrease (F-IF.4)
- applying the rules for combining functions using the addition and subtraction operations to produce a new function (F-BF.1b)

Introduction

The **rate of change** is a ratio that describes how much one quantity changes with respect to the change in another quantity. For a function, the rate of change is the amount by which function values change for each change in domain values. When the rate of change of a function increases by the same amount for every consecutive domain value, the rate of change is said to be constant. Linear functions exhibit this trait; however, not all functions have a constant rate of change.

Key Concepts

- The rate of change of a single function is dependent on the change that occurs between two points (ordered pairs) on the graph of that function. Recall that this is the same as the slope when the function is a line.

- The rate of change of a function is the difference between the values of the function at two different domain values; mathematically, this can be represented by the formula $r = \frac{f_1(x) - f_2(x)}{x_1 - x_2} = \frac{f_2(x) - f_1(x)}{x_2 - x_1} = \frac{\Delta f(x)}{\Delta x}$, in which $f_1(x)$ is the function value at the domain value x_1 and $f_2(x)$ is the function value at the domain value x_2.

- The uppercase Greek letter ***delta*** **(Δ)** is commonly used to represent the "change" in a value; for example, $\Delta f(x)$ can be read as "change in f of x." Therefore, $\Delta f(x)$ and Δx are more concise ways of representing the numerator and denominator, respectively, in the rate of change formula.

- Note that the order in which the function values are compared must be the same as the order in which the domain values are compared. Mixing up the order of the values in the numerator with the order of the values in the denominator will result in an incorrect rate calculation.

SRB p. U4B-115

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Instruction

- The **average rate of change** is the average of all of the different rates of change of a function value over a restricted domain. Formally, it is the ratio of the difference of output values to the difference of the corresponding input values: $\frac{f(b)-f(a)}{b-a}$.
- The average rate of change of a function or a specific variable is useful in those mathematical or real-world problems in which the average rate of change of different function values varies over equal restricted domain intervals.
- The average rate of change is often calculated by substituting the function values that correspond to the endpoints or bounds of a restricted domain interval into the formula for calculating the rate of change.
- The rate of change of a function or variable in a real-world situation can be positive, negative, 0, undefined, or a constant, depending on the situation. In particular, the rate of change of a variable can be negative or 0 even if the variable values are always positive, such as in problems involving area, mass, time, volume, and any number of similar quantities.
- A special case of the constant rate of change can be found with linear functions. For a linear function $ax + by = c$, the rate of change of the function values y is a constant given by the quantity $-\frac{a}{b}$. This quantity is the slope of the straight-line graph that represents the linear function. In the slope-intercept form of the linear function $f(x) = mx + b$, the constant m is the rate of change of the function values.
- The average rate of change of a quantity can have special meaning in situations in which the restricted domain over which it is computed is very small or approaches 0. A graphing calculator can be helpful in approximating or visualizing the rate of change in such applications.

Common Errors/Misconceptions

- reversing the order of the function values and domain values in a rate of change calculation
- switching the order of two function values in the numerator of a rate of change calculation from that of the two domain values in the denominator
- neglecting to check a rate of change calculation, especially in real-world problems, for accuracy or reasonableness in terms of the problem assumptions, conditions, and restrictions

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Instruction

Guided Practice 4B.3.2

Example 1

Compare the average rate of change of the function $f(x) = \sin x$ over the restricted domain [0°, 45°] with the average rate of change of the same function over the restricted domain [45°, 90°].

1. Calculate the values of $f(0°)$ and $f(45°)$.

   Substitute 0° and 45° for x.

   $f(x) = \sin x$ $\qquad$ $f(x) = \sin x$

   $f(0°) = \sin(0°)$ $\qquad$ $f(45°) = \sin(45°)$

   $f(0°) = 0$ $\qquad$ $f(45°) = \dfrac{\sqrt{2}}{2}$

2. Convert degree measures to radian measures.

   In order for the rate of change formula to compare like units, all the measures must be expressed in the same units. 0° is the same as 0 radians, and $\dfrac{\sqrt{2}}{2}$ is already expressed in radians. Therefore, the remaining degree measure, 45°, must be converted to radians.

   Recall the conversion formula, $1 \text{ degree} = \dfrac{\pi}{180} \text{ radian}$.

   Therefore, 45° is $45° \cdot \dfrac{\pi}{180}$ radian or $\dfrac{\pi}{4}$ radian.

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3. Calculate the rate of change over the restricted domain [0°, 45°].

Substitute the results from the previous steps into the formula for the rate of change.

$r = \dfrac{f_1(x) - f_2(x)}{x_1 - x_2}$ Rate of change formula

$r = \dfrac{\left(\dfrac{\sqrt{2}}{2}\right) - (0)}{\left(\dfrac{\pi}{4}\right) - (0)}$ Substitute $\dfrac{\sqrt{2}}{2}$ for $f_1(x)$, 0 for $f_2(x)$, $\dfrac{\pi}{4}$ for x_1, and 0 for x_2.

$r = \dfrac{4\sqrt{2}}{2\pi}$ Simplify.

$r = \dfrac{2\sqrt{2}}{\pi}$

$r \approx 0.9$

The rate of change over the restricted domain [0°, 45°] is approximately 0.9 radian.

4. Calculate the rate of change over the restricted domain [45°, 90°].

As determined in step 1, $f(45°) = \dfrac{\sqrt{2}}{2}$.

Calculate the value of $f(90°)$.

$f(x) = \sin x$

$f(90°) = \sin(90°)$

$f(90°) = 1$

Convert degree measures to radian measures.

As found in step 2, 45° is equal to $\dfrac{\pi}{4}$ radian.

(*continued*)

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Using the conversion formula, 90° is $90° \cdot \frac{\pi}{180}$ radian or $\frac{\pi}{2}$ radians.

Substitute the results of the previous steps into the formula for the rate of change.

$r = \frac{f_1(x) - f_2(x)}{x_1 - x_2}$ Rate of change formula

$r = \frac{\left(\frac{\sqrt{2}}{2}\right) - (1)}{\left(\frac{\pi}{4}\right) - \left(\frac{\pi}{2}\right)}$ Substitute $\frac{\sqrt{2}}{2}$ for $f_1(x)$, 1 for $f_2(x)$, $\frac{\pi}{4}$ for x_1, and $\frac{\pi}{2}$ for x_2.

$r = \frac{\frac{\sqrt{2}-2}{2}}{\frac{\pi - 2\pi}{4}}$ Simplify.

$r = \frac{4\left(\sqrt{2}-2\right)}{-2\pi}$ Continue to simplify.

$r = \frac{2\left(2-\sqrt{2}\right)}{\pi}$

$r \approx 0.37$

The rate of change over the restricted domain [45°, 90°] is approximately 0.37 radian.

5. Summarize the results of steps 1–4 for the two average rates of change for $f(x) = \sin x$.

   The function is increasing over both restricted domains. However, the rate of change of the increase is greater over the interval [0°, 45°] than over the interval [45°, 90°].

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Instruction

Example 2

Two school clubs are selling plants to raise money for a trip. The juggling club started out with 250 plants and had 30 left after selling them for 7 days. The mock trial club started out with 450 plants and had 50 left after selling them for 8 days. Compare the rates at which the two clubs are selling their plants. If the two clubs started selling plants at the same time, which club will sell out of plants first? Explain.

1. Determine how many plants the juggling club has sold.

   The juggling club has sold 250 – 30 or 220 plants.

2. Calculate the average number of plants the juggling club sold per day.

   The juggling club sold 220 plants over 7 days. Divide to determine the average.

   $$\frac{220}{7} \approx 31$$

   The juggling club sold an average of approximately 31 plants each day.

3. Determine how many plants the mock trial club has sold.

   The mock trial club has sold 450 – 50 or 400 plants.

4. Calculate the average number of plants the mock trial club sold per day.

   The mock trial club sold 400 plants over 8 days. Divide to determine the average.

   $$\frac{400}{8} = 50$$

   The mock trial club sold an average of 50 plants each day.

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5. Describe the rates at which the two clubs sold their plants. Compare the kinds of numbers that represent days, plants, and rates.

   The variables "days" and "plants" are positive whole numbers, whereas "rate" can be a negative number. In this situation, the rates for the two clubs are negative, because the rates reflect a reduction in the number of plants each club has in stock. The rate for the juggling club is approximately –31 plants per day because the quantity of plants is being reduced by an average of about 31 per day; similarly, the rate for the mock trial club is –50 plants per day.

6. Calculate how long it will take the juggling club to sell its remaining plants.

   The juggling club has 30 plants left and is selling them at a rate of about –31 plants per day, so the juggling club's plants will sell out in 1 day.

7. Calculate how long it will take the mock trial club to sell its remaining plants.

   The mock trial club has 50 plants left and is selling them at a rate of –50 plants per day, so the mock trial club's plants will sell out in 1 day.

8. Explain which club will sell out of plants first.

   Both clubs will sell off their remaining stock in 1 day. However, assuming each club started selling the plants on the same day, the juggling club will sell out first because it took 1 day less to sell off its initial stock of plants. (Recall that the juggling club sold plants for 7 days and the mock trial club sold plants for 8 days.)

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Instruction

Example 3

Find the average rate of change of the function $f(x) = x^3 + 2x^2 - x - 2$ over the restricted intervals $[-2, -1]$ and $[-1, 1]$. Determine if the rate of change over both intervals is positive, negative, 0, undefined, and/or constant. Use the graph of $f(x)$ to explain how the average rate of change relates to the behavior of the graph across its whole domain.

1. Calculate the average rate of change of $f(x)$ over the interval $[-2, -1]$.

   First find the values of $f(-2)$ and $f(-1)$.

   Calculate $f(-2)$.

$f(x) = x^3 + 2x^2 - x - 2$	Original function
$f(-2) = (-2)^3 + 2(-2)^2 - (-2) - 2$	Substitute -2 for x.
$f(-2) = -8 + 8 + 2 - 2$	Simplify.
$f(-2) = 0$	

   Calculate $f(-1)$.

$f(x) = x^3 + 2x^2 - x - 2$	Original function
$f(-1) = (-1)^3 + 2(-1)^2 - (-1) - 2$	Substitute -1 for x.
$f(-1) = -1 + 2 + 1 - 2$	Simplify.
$f(-1) = 0$	

   Substitute the values for $f(-2)$ and $f(-1)$ into the rate of change formula and solve.

$r = \frac{f_1(x) - f_2(x)}{x_1 - x_2}$	Rate of change formula
$r = \frac{(0) - (0)}{(-2) - (-1)}$	Substitute 0 for $f_1(x)$, 0 for $f_2(x)$, -2 for x_1, and -1 for x_2.
$r = 0$	Simplify.

   The rate of change of $f(x)$ over the interval $[-2, -1]$ is 0.

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Instruction

2. Calculate the average rate of change of $f(x)$ over the interval $[-1, 1]$.

The value of $f(-1)$ is 0, as found in the previous step.

Find the value of $f(1)$.

Calculate $f(1)$.

$f(x) = x^3 + 2x^2 - x - 2$	Original function
$f(1) = (1)^3 + 2(1)^2 - (1) - 2$	Substitute 1 for x.
$f(1) = 1 + 2 - 1 - 2$	Simplify.
$f(1) = 0$	

Substitute the values for $f(-1)$ and $f(1)$ in the rate of change formula.

$r = \frac{f_1(x) - f_2(x)}{x_1 - x_2}$	Rate of change formula
$r = \frac{(0) - (0)}{(-1) - (1)}$	Substitute 0 for $f_1(x)$, 0 for $f_2(x)$, -1 for x_1, and 1 for x_2.
$r = 0$	Simplify.

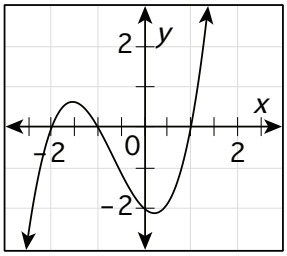
The rate of change of $f(x)$ over the interval $[-1, 1]$ is 0.

3. Determine if the rate of change over both intervals is positive, negative, 0, undefined, and/or constant.

The rate of change that occurred over each of the two intervals is 0 and constant.

4. Graph $f(x)$.

Use a graphing calculator to view the graph of $f(x)$.



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5. Describe what happens to the function values over the restricted domains [–2, –1] and [–1, 1].

   The function value goes from 0 to a maximum value and back to 0 over the restricted domain [–2, –1].

   The function value goes from 0 to a minimum value and back to 0 over the restricted domain [–1, 1]. For the domain interval $(-\infty, -2)$, the function value range is given by the interval $(-\infty, 0)$; for the domain interval $(1, +\infty)$, the function value range is given by the interval $(0, +\infty)$.

Example 4

Three linear functions—$f(x)$, $g(x)$, and $h(x)$—share a common intersection point of (1, 1) on a coordinate plane. The function $f(x)$ also passes through the point (–2, –3), while the function $g(x)$ passes through the point (–2, 3), and the function $h(x)$ passes through the point (2, 3). Which function has the greatest rate of change? What are the equations of the functions?

1. Calculate the rate of change for $f(x)$.

   The rate of change formula can be applied to the two points that satisfy the function $f(x)$: (1, 1) and (–2, –3).

   $r = \dfrac{f_1(x) - f_2(x)}{x_1 - x_2}$ Rate of change formula

   $r = \dfrac{(1) - (-3)}{(1) - (-2)}$ Substitute 1 for $f_1(x)$, –3 for $f_2(x)$, 1 for x_1, and –2 for x_2.

   $r = \dfrac{4}{3}$ Simplify.

   The rate of change for $f(x)$ is $\dfrac{4}{3}$.

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2. Calculate the rate of change for $g(x)$.

Substitute the two points that satisfy $g(x)$, (1, 1) and (–2, 3), into the rate of change formula.

$r = \frac{f_1(x) - f_2(x)}{x_1 - x_2}$ Rate of change formula

$r = \frac{(1) - (3)}{(1) - (-2)}$ Substitute 1 for $f_1(x)$, 3 for $f_2(x)$, 1 for x_1, and –2 for x_2.

$r = -\frac{2}{3}$ Simplify.

The rate of change for $g(x)$ is $-\frac{2}{3}$.

3. Calculate the rate of change for $h(x)$.

Substitute the two points that satisfy $h(x)$, (1, 1) and (2, 3), into the rate of change formula.

$r = \frac{f_1(x) - f_2(x)}{x_1 - x_2}$ Rate of change formula

$r = \frac{(1) - (3)}{(1) - (2)}$ Substitute 1 for $f_1(x)$, 3 for $f_2(x)$, 1 for x_1, and 2 for x_2.

$r = 2$ Simplify.

The rate of change for $h(x)$ is 2.

4. Compare the rates of change for the three functions.

The rate of change for $f(x)$ is $\frac{4}{3}$, the rate of change for $g(x)$ is $-\frac{2}{3}$, and the rate of change for $h(x)$ is 2. This can be written symbolically using the inequality $r_{h(x)} > r_{f(x)} > r_{g(x)}$. Therefore, $h(x)$ has the greatest rate of change.

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5. Use the point-slope form of a linear equation to write the equation of each function.

   The point-slope form of a linear equation is $y - y_1 = m(x - x_1)$, in which m is the rate of change and x_1 and y_1 are coordinates of a point that makes the function a true statement. Use the shared point (1, 1) for the point for each function.

   For *f*(*x*):

$y - y_1 = m(x - x_1)$	Point-slope formula
$f(x) - (1) = \left(\frac{4}{3}\right)[x - (1)]$	Substitute 1 for x_1, 1 for y_1, and $\frac{4}{3}$ for m.
$f(x) = \frac{4}{3}x - \frac{1}{3}$	Simplify.

   For *g*(*x*):

$y - y_1 = m(x - x_1)$	Point-slope formula
$g(x) - (1) = \left(-\frac{2}{3}\right)[x - (1)]$	Substitute 1 for x_1, 1 for y_1, and $-\frac{2}{3}$ for m.
$g(x) = -\frac{2}{3}x + \frac{5}{3}$	Simplify.

   For *h*(*x*):

$y - y_1 = m(x - x_1)$	Point-slope formula
$h(x) - (1) = (2)[x - (1)]$	Substitute 1 for x_1, 1 for y_1, and 2 for m.
$h(x) = 2x - 1$	Simplify.

   The equations for the three functions are $f(x) = \frac{4}{3}x - \frac{1}{3}$, $g(x) = -\frac{2}{3}x + \frac{5}{3}$, and $h(x) = 2x - 1$.



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Example 5

The table gives the function values for $f(x) = \log x$ over the restricted domain [0.001, 1,000]. Use the data to calculate and compare the rates of change of $f(x)$ for the intervals [0.001, 0.01], [0.01, 0.1], [0.1, 1], and [1, 10]. Describe how the rates change as the values of x become larger and smaller.

x	0.001	0.01	0.1	1	10	100	1,000
$f(x)$	–3	–2	–1	0	1	2	3

1. Calculate the rate of change of $f(x)$ over the interval [0.001, 0.01].

   Substitute the table values for x and $f(x)$ into the rate of change formula, $r = \dfrac{\Delta f(x)}{\Delta x}$.

   $r = \dfrac{\Delta f(x)}{\Delta x}$ Rate of change formula

   $r = \dfrac{(-3)-(-2)}{(0.001)-(0.01)}$ Substitute (–3) – (–2) for $\Delta f(x)$ and (0.001) – (0.01) for Δx.

   $r = \dfrac{-1}{-0.009}$ Simplify.

   $r = 111.\overline{1}$

   The rate of change of $f(x)$ over the interval [0.001, 0.01] is $111.\overline{1}$.

2. Calculate the rate of change of $f(x)$ over the interval [0.01, 0.1].

   Substitute the table values for x and $f(x)$ into $r = \dfrac{\Delta f(x)}{\Delta x}$.

   $r = \dfrac{\Delta f(x)}{\Delta x}$ Formula

   $r = \dfrac{(-2)-(-1)}{(0.01)-(0.1)}$ Substitute (–2) – (–1) for $\Delta f(x)$ and (0.01) – (0.1) for Δx.

   $r = \dfrac{-1}{-0.09}$ Simplify.

   $r = 11.\overline{1}$

   The rate of change of $f(x)$ over the interval [0.01, 0.1] is $11.\overline{1}$.

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3. Calculate the rate of change of $f(x)$ over the interval [0.1, 1].

Substitute the table values for x and $f(x)$ into $r = \frac{\Delta f(x)}{\Delta x}$.

$r = \frac{\Delta f(x)}{\Delta x}$	Formula
$r = \frac{(-1)-(0)}{(0.1)-(1)}$	Substitute (–1) – (0) for $\Delta f(x)$ and (0.1) – (1) for Δx.
$r = \frac{-1}{-0.9}$	Simplify.
$r = 1.\overline{1}$	

The rate of change of $f(x)$ over the interval [0.1, 1] is $1.\overline{1}$.

4. Calculate the rate of change of $f(x)$ over the interval [1, 10].

Substitute the table values for x and $f(x)$ into $r = \frac{\Delta f(x)}{\Delta x}$.

$r = \frac{\Delta f(x)}{\Delta x}$	Formula
$r = \frac{(0)-(1)}{(1)-(10)}$	Substitute (0) – (1) for $\Delta f(x)$ and (1) – (10) for Δx.
$r = \frac{-1}{-9}$	Simplify.
$r = 0.\overline{1}$	

The rate of change of $f(x)$ over the interval [1, 10] is $0.\overline{1}$.

5. Describe how the rate of change will change as the values of the domain interval increase.

Compare the rate of change r for each interval of interest for $f(x)$:

- For [0.001, 0.01], $r = 111.\overline{1}$.
- For [0.01, 0.1], $r = 11.\overline{1}$.
- For [0.1, 1], $r = 1.\overline{1}$.
- For [1, 10], $r = 0.\overline{1}$.

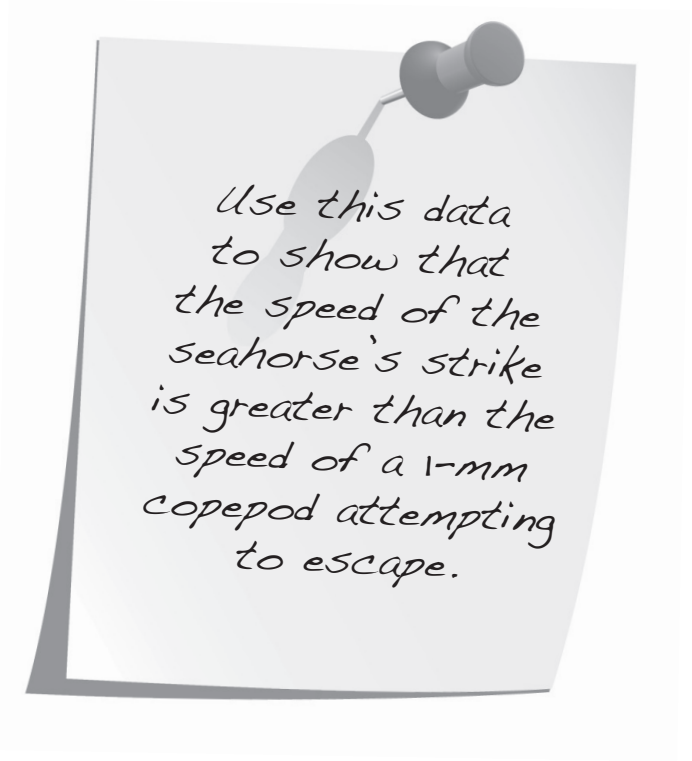
As the values of x increase, the rate of change decreases.

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Problem-Based Task 4B.3.2: When Seahorses Attack

SMP	
1 ✓	2 ✓
3	4 ✓
5	6
7 ✓	8

The dwarf seahorse is an efficient predator of one of its favorite foods—copepods, which are tiny relatives of crayfish, lobsters, and shrimp. The seahorse's success rate when hunting copepods is about 94 percent. Copepods move with an average speed of 500 body lengths per second; they vary from 1 to 2 millimeters in length. The ergonomic shape of the dwarf seahorse's head limits turbulence as the seahorse moves toward its prey. When the dwarf seahorse gets to within 1 millimeter of a stationary copepod, it strikes, with a strike time of 1 millisecond. Use this data to show that the speed of the seahorse's strike is greater than the speed of a 1-mm copepod attempting to escape.



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Problem-Based Task 4B.3.2: When Seahorses Attack

Coaching

a. What quantities in the problem represent the time it would take a copepod to dart away from a striking seahorse?

b. What is the rate of change formula for calculating speed?

c. What quantities in the problem represent the speed at which a striking seahorse can snatch a copepod? Calculate this speed.

d. How can you compare the speed of the copepod to that of the seahorse?

e. Use the results of parts c and d to show that the speed of a striking seahorse is greater than the speed of a 1-mm copepod trying to escape.

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Problem-Based Task 4B.3.2: When Seahorses Attack

Coaching Sample Responses

a. What quantities in the problem represent the time it would take a copepod to dart away from a striking seahorse?

The copepod's speed (500 body lengths per second) and its length (1 millimeter) are given. The quantity "500 body lengths" can be multiplied by the copepod size to give a distance traveled of 500 millimeters per second.

b. What is the rate of change formula for calculating speed?

The average speed is the distance traveled divided by the time it takes to travel the distance, or $r = \frac{d}{t}$.

c. What quantities in the problem represent the speed at which a striking seahorse can snatch a copepod? Calculate this speed.

The problem states that a seahorse strikes when it gets within 1 millimeter of a copepod, and its strike time is 1 millisecond. The formula $r = \frac{d}{t}$ gives the average speed, which is distance divided by time. Therefore, the seahorse's speed is 1 millimeter divided by 1 millisecond, which is the same as 1 meter per second.

d. How can you compare the speed of the copepod to that of the seahorse?

The speeds have to be in the same units. The seahorse's speed is in meters per second, but the copepod's speed is in millimeters per second. Convert the copepod's speed to meters.

500 millimeters is equal to 0.5 meter, so the copepod's speed in meters is 0.5 meter per second.

e. Use the results of parts c and d to show that the speed of a striking seahorse is greater than the speed of a 1-mm copepod trying to escape.

In part c, the seahorse's speed was calculated to be 1 meter per second. In part d, the copepod speed was calculated to be 0.5 meter per second. The seahorse is twice as fast as a 1-mm copepod.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4B.3.2: Calculating Average Rates of Change

For problems 1–4, find the average rate of change between $x = 1$ and $x = 2$ for the function given.

1. $a(x) = 0.1x - 0.2$

2. $b(x) = 0.2x - 0.1$

3. $c(x) = 10^{-x}$

4. $d(x) = 1 - 10^{-x}$

For problems 5–7, determine the average rate of change over the domain intervals [0.01, 0.1] and [0.1, 1] for the function given.

5. $f(x) = 1 - 0.01x$

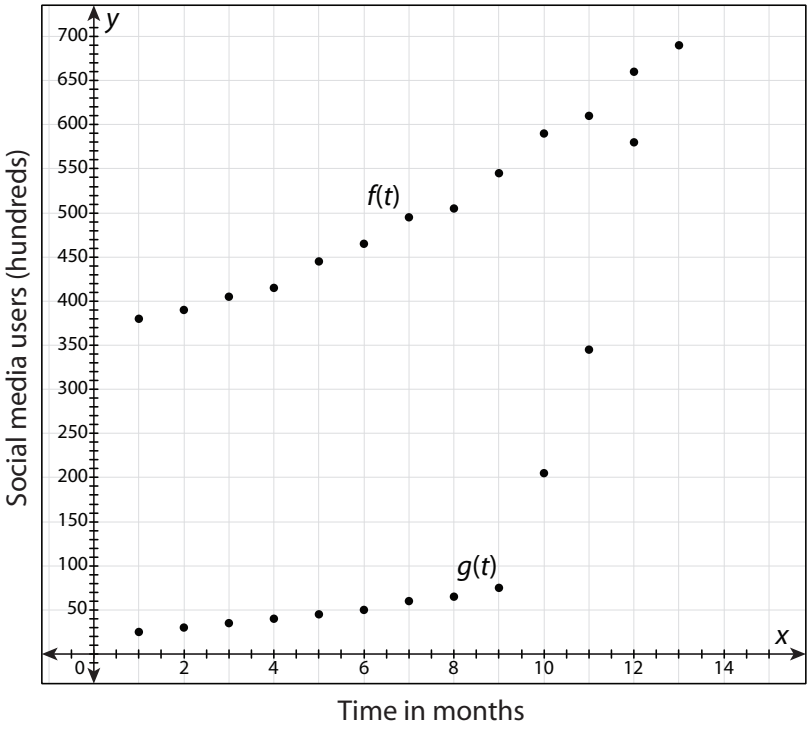
6. $g(x) = 1 - 0.01x^3$

7. $h(x) = \sqrt{x^2 + 0.04}$

continued

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A nonprofit organization commissioned a study of the number of its members who are also members of two different social network websites over a 13-month period. The following graph shows the number of organization members in hundreds who used the two social networks each month, with the function $f(t)$ representing users of one social network and $g(t)$ representing users of the second social network. Use the graph to complete problems 8–10.



8. Describe the average rate(s) of change of the function $f(t)$ over a 10-month period of the study and the overall average rate of change in the function value from $t = 1$ to $t = 10$.

9. Describe the average rate(s) of change of the function $g(t)$ over the 10-month period of the study and the overall average rate of change in the function value from $t = 1$ to $t = 10$.

10. Compare the average rates of change for both functions over the period of the study, and compare the number of members who use each social network.

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Lesson 4B.3.3: Comparing Functions

Warm-Up 4B.3.3

Oceanographers use software to forecast ocean wave height. Such wave forecasts can be used to warn vessels to stay in port, or let surfers know when to grab their boards. Ocean forecasting software relies on mathematical models.

Nyala is a graduate student studying oceanography. One of her assignments for her Introduction to Ocean Modeling class is to determine and compare the average rates of change in the heights of two particular ocean waves, represented by the functions $f(x) = 3 \sin 2x$ and $g(x) = 2 \sin 3x$. Each wave's height is measured in meters. Use this information to solve the problems that follow.

1. Calculate the rate of change of the function $f(x)$ over the interval $\left[0, \frac{\pi}{12}\right]$.

2. Calculate the rate of change of the function $g(x)$ over the interval $\left[0, \frac{\pi}{12}\right]$.

3. Compare the results for the two functions.

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Lesson 4B.3.3: Comparing Functions

Common Core State Standards

F–IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

F–IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Warm-Up 4B.3.3 Debrief

1. Calculate the rate of change of the function $f(x)$ over the interval $\left[0, \frac{\pi}{12}\right]$.

   Use the rate of change formula, $r = \frac{\Delta f(x)}{\Delta x}$, and the values of the sine function to calculate the rate of change of the function $f(x)$.

   $r = \frac{\Delta f(x)}{\Delta x}$ Rate of change formula

   $r = \frac{\left[3\sin(2 \bullet 0)\right] - \left[3\sin\left(2 \bullet \frac{\pi}{12}\right)\right]}{(0) - \left(\frac{\pi}{12}\right)}$ Substitute $3\sin(2 \bullet 0) - 3\sin\left(2 \bullet \frac{\pi}{12}\right)$ for $\Delta f(x)$ and $(0) - \left(\frac{\pi}{12}\right)$ for Δx.

   $r = \frac{0 - 3\sin\left(\frac{\pi}{6}\right)}{-\frac{\pi}{12}}$ Simplify.

   $r = \frac{-\frac{3}{2}}{-\frac{\pi}{12}}$ Simplify the numerator.

   $r = \frac{18}{\pi}$ Simplify.

   The rate of change for the function $f(x)$ over the interval $\left[0, \frac{\pi}{12}\right]$ is $\frac{18}{\pi}$.

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2. Calculate the rate of change of the function $g(x)$ over the interval $\left[0, \frac{\pi}{12}\right]$.

   Again, use the rate of change formula, $r = \frac{\Delta f(x)}{\Delta x}$, and the values of the sine function to calculate the rate of change of the function $g(x)$.

$r = \frac{\Delta f(x)}{\Delta x}$	Formula
$r = \frac{\left[2\sin(3 \bullet 0)\right] - \left[2\sin\left(3 \bullet \frac{\pi}{12}\right)\right]}{(0) - \left(\frac{\pi}{12}\right)}$	Substitute $2\sin(3 \bullet 0) - 2\sin\left(3 \bullet \frac{\pi}{12}\right)$ for $\Delta f(x)$ and $(0) - \left(\frac{\pi}{12}\right)$ for Δx.
$r = \frac{0 - 2\sin\left(\frac{\pi}{4}\right)}{-\frac{\pi}{12}}$	Simplify.
$r = \frac{-\frac{2\sqrt{2}}{2}}{-\frac{\pi}{12}}$	Simplify the numerator.
$r = \frac{12\sqrt{2}}{\pi}$	Simplify.

The rate of change for the function $g(x)$ over the interval $\left[0, \frac{\pi}{12}\right]$ is $\frac{12\sqrt{2}}{\pi}$.

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3. Compare the results for the two functions.

   Use a graphing or scientific calculator to calculate an approximation for each rate of change.

Function	Exact rate	Approximate rate
$f(x)$	$\frac{18}{\pi}$	≈ 5.7
$g(x)$	$\frac{12\sqrt{2}}{\pi}$	≈ 5.4

   Notice that the average rates of change for the two functions over the interval $\left[0, \frac{\pi}{12}\right]$ are very close in value, but not equal.

Connection to the Lesson

- Students will understand the importance of making accurate calculations of similar quantities of two or more functions.
- Students will recognize differences and similarities of function characteristics.
- Students will make comparisons of "like" functions that have different values of important function features.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- evaluating a function at a specific domain value (F-IF.2)
- identifying and comparing a specific function characteristic for different values of the function domain and range (F-IF.4)
- comparing and contrasting the appropriateness of two or more "unlike" functions that might be used as models for real-world problems, and choosing the one that best models the problem and/or its solution (F-IF.9)

Introduction

Functions can be compared for a variety of purposes. Functions of the same type—for example, quadratic functions—can be compared on the basis of maximum or minimum function values, symmetry about an axis or other vertical line on the coordinate plane, or on the basis of how well the function fits real-world data.

Functions of different types—such as polynomial functions and rational functions—can likewise be compared on the basis of one or more characteristics, such as the number of real solutions or the rate of change over a restricted domain interval.

Key Concepts

- The function characteristics that are used to compare and contrast function types can depend on the purpose of the comparison.
- Recall that the extrema of a function, or the points on a graph of a function at which a maximum or minimum function value occurs, can apply to the function across its entire domain or they can be localized, as in the case of polynomial functions of degree three and higher.
- For example, a cubic equation might be used to model a real-world problem that has a single solution. Graphing a proposed model for the problem can quickly show whether the model meets the criteria of having only one real solution.
- The rate of change of a function value can be calculated to determine which of several different functions (or functions of the same form) best represent a condition of decreasing or increasing function values.
- The symmetry exhibited by a function can be about a vertical line on a coordinate plane or it can be exhibited across a line of symmetry that takes the form of a linear function.

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Instruction

- Symmetry of domain and function values can also be observed in the algebraic form of a function, such as in even or odd functions.
- Domains and ranges can also be used to compare the validity of a function serving as a model for real-world phenomena. In some cases, only a restricted domain or range of a mathematical model is relevant to a specific problem.
- A function's behavior at domain values that are very large or small can be relevant to some real-world problems. Functions might serve as reliable models over a narrowly restricted mathematical domain, but might fail to describe real-world phenomena when domain values increase or decrease by large values.

Common Errors/Misconceptions

- misunderstanding the variable or variable value(s) that are relevant to a comparison of like or unlike functions
- confusing mathematical features with real-world features and values of functions in comparing functions
- not using (or not choosing) the most appropriate function model to describe a data set

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Guided Practice 4B.3.3

Example 1

Two functions, $f(x) = -2(x - 5)^2 + 3$ and $g(x) = -4x^2 + 8x + 3$, each represent the volume of a rectangular solid. Which solid's function, $f(x)$ or $g(x)$, has the greater maximum volume?

1. Determine the coordinates of the maximum point of $f(x)$.

   To determine the coordinates of the maximum point of a function, first determine the type of function and compare it to the standard form for that function.

   The function $f(x) = -2(x - 5)^2 + 3$ is a quadratic function.

   The vertex form for a quadratic function is $f(x) = a(x - h)^2 + k$, where the coordinates of the maximum ($a < 0$) or minimum ($a > 0$) function value are (h, k).

   Since $f(x) = -2(x - 5)^2 + 3$ is already written in vertex form, it can be seen that for $f(x)$, $h = 5$, $k = 3$ and $a = -2$, which is less than 0; therefore, the maximum point of $f(x)$ is located at (5, 3).

2. Determine the coordinates of the maximum point of $g(x)$.

   $g(x) = -4x^2 + 8x + 3$ is also a quadratic function, but it is written in standard form, $f(x) = ax^2 + bx + c$, where $a = -4$, $b = 8$, and $c = 3$.

   Recall that the x-coordinate of the maximum or minimum of a quadratic in this form is $x = -\frac{b}{2a}$.

   $x = -\frac{b}{2a}$ Formula for the x-coordinate

   $x = -\frac{(8)}{2(-4)}$ Substitute -4 for a and 8 for b.

   $x = -\frac{8}{-8}$ Simplify.

   $x = 1$

(*continued*)

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The x-coordinate of the maximum point of $g(x)$ is 1. To determine the y-coordinate of the maximum, substitute the value of x into the function $g(x)$.

$g(x) = -4x^2 + 8x + 3$	Given function
$g(1) = -4(1)^2 + 8(1) + 3$	Substitute 1 for x.
$g(1) = -4 + 8 + 3$	Simplify.
$g(1) = 7$	

The y-coordinate of the maximum point of $g(x)$ is 7.

The maximum point of $g(x)$ is (1, 7).

3. Determine which function models the greatest maximum volume of its associated rectangular solid.

   The function value of the maximum point of $f(x)$ is 3, and the function value of the maximum point of $g(x)$ is 7. Therefore, $g(x)$ represents the function that models the greatest maximum volume for the associated rectangular solid.

Example 2

Sand dune "booming" or singing refers to the audible, tone-specific sounds made by shifting sand dunes under the influence of wind. A team of biophysicists compared the loudness of several sand dunes' booms with the decibel ratings (or sound intensity) of the booms. The table shows some of the team's data, with x representing the loudness x in joules (energy) for each sound and $f(x)$ representing the decibel rating.

x	–0.999	–0.99	–0.9	0	9	99
$f(x)$	–1	0	1	2	3	4

Describe how the domain and function values change over the interval [–0.999, 99]. Then compare the rate of change of the function values over the restricted intervals of [–0.999, –0.99] and [9, 99].

1. Describe how the domain values change over the interval.

   Inspect the table to determine how the domain values change. Reading the table values from left to right, the domain values are increasing because each domain value is greater than the one preceding it.

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2. Describe how the function values (or range) change over the domain interval.

   The function or range values increase at a constant rate of 1 unit for each of the domain values.

3. Calculate the rate of change for the intervals [–0.999, –0.99] and [9, 99].

   Substitute the table values for x and $f(x)$ into the rate of change formula, $r = \frac{\Delta f(x)}{\Delta x}$.

   For the interval [–0.999, –0.99]:

   $r = \frac{\Delta f(x)}{\Delta x}$ Rate of change formula

   $r = \frac{(-1)-(0)}{(-0.999)-(-0.99)}$ Substitute (–1) – (0) for $\Delta f(x)$ and (–0.999) – (–0.99) for Δx.

   $r = \frac{-1}{-0.009}$ Simplify.

   $r = 111.\overline{1}$

   The rate of the change for the interval [–0.999, –0.99] is $111.\overline{1}$.

   For the interval [9, 99]:

   $r = \frac{\Delta f(x)}{\Delta x}$ Rate of change formula

   $r = \frac{(3)-(4)}{(9)-(99)}$ Substitute (3) – (4) for $\Delta f(x)$ and (9) – (99) for Δx.

   $r = \frac{-1}{-90}$ Simplify.

   $r = 0.0\overline{1}$

   The rate of the change for the interval [9, 99] is $0.0\overline{1}$.

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4. Compare the rates of change of the function values for the two intervals.

   Both rates of change are positive, but the rate of change over the interval [–0.999, –0.99] is 10,000 times greater than the rate of change over the interval [9, 99].



Example 3

Use the data from the table in Example 2 (shown in the table that follows) to find the function that fits the data if the general form of the function is $f(x) = a + \log(x + b)$.

x	–0.999	–0.99	–0.9	0	9	99
$f(x)$	–1	0	1	2	3	4

1. Determine the number of data points that will be needed to find the function.

   The general form has four variables—a, b, $f(x)$, and x. However, each data point has both an x and an $f(x)$ value, which leaves two unknowns, a and b. Therefore, two equations in a and b will be needed to find the function.



2. Pick two points from the table and substitute them into the general form of the function to write equations for the points.

   Any two points can be chosen. Let the two points be (9, 3) and (99, 4).

   For (9, 3):

$f(x) = a + \log(x + b)$	General form
$(3) = a + \log[(9) + b]$	Substitute 9 for x and 3 for $f(x)$.
$3 = a + \log(9 + b)$	Simplify.

(*continued*)

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For (99, 4):

$f(x) = a + \log(x + b)$	General form
$(4) = a + \log[(99) + b]$	Substitute 99 for x and 4 for $f(x)$.
$4 = a + \log(99 + b)$	Simplify.

The equations for the data points (9, 3) and (99, 4) are $3 = a + \log(9 + b)$ and $4 = a + \log(99 + b)$.

3. Solve the system of resulting equations. Begin by eliminating one of the unknown variables (a or b).

   Either variable can be eliminated; let's eliminate a.

   To eliminate a, subtract one equation from the other.

$[4 = a + \log(99 + b)] - [3 = a + \log(9 + b)]$	Set up the subtraction equation.
$1 = \log(99 + b) - \log(9 + b)$	Subtract like terms.
$1 = \log\left(\frac{99 + b}{9 + b}\right)$	Simplify using the quotient rule of logarithms.
$10^1 = \frac{99 + b}{9 + b}$	Simplify using the definition of a common logarithm.
$10 \bullet (9 + b) = 99 + b$	Multiply both sides by $(9 + b)$.
$90 + 10b = 99 + b$	Distribute.
$10b = 9 + b$	Subtract 90 from both sides.
$9b = 9$	Subtract b from both sides.
$b = 1$	Divide by 9.

The value of b is 1.

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4. Determine the value of the second unknown variable.

   Substitute the value of b into either equation $4 = a + \log(99 + b)$ or $3 = a + \log(9 + b)$; let's use the latter equation.

$3 = a + \log(9 + b)$	Equation
$3 = a + \log[9 + (1)]$	Substitute 1 for b.
$3 - a = \log 10$	Simplify.
$3 - a = 1$	Simplify the logarithmic term.
$-a = -2$	Subtract 3 from both sides.
$a = 2$	Divide by –1.

   The value of a is 2.

5. Write the function that represents the data in Example 2.

   The general form of the equation is $f(x) = a + \log(x + b)$. Let $a = 2$ and $b = 1$. The resulting function is $f(x) = 2 + \log(x + 1)$.

Example 4

Members of a Florida high school's nature club tracked the estimated number of tree frogs living in a nearby cypress tree for 9 weeks. The estimates are based on the recorded volume of the frogs' singing over a 24-hour period on the same day of each week. The table shows the club's estimates for the number of frogs n per week, with the independent variable t representing time in weeks.

Estimated number of frogs, *n*	10	14	25	27	29	32	35	36	38
Weeks, *t*	1	2	3	4	5	6	7	8	9

Use a graphing calculator to graph the data. Then, use the graph to find both an exponential function of the form $f(x) = a \cdot b^x$ and a power function of the form $g(x) = a \cdot x^b$ to fit the data. Compare how closely each function equation models the data on the graph.

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Instruction

1. Create a scatter plot of the table values using a graphing calculator.

   Follow the instructions appropriate to your calculator model.

   On a TI-83/84:

   Step 1: Press [STAT] to bring up the statistics menu. The first option, 1: Edit, will already be highlighted. Press [ENTER].

   Step 2: Arrow up to L1 and press [CLEAR], then [ENTER], to clear the list (or repeatedly hit the DEL key until all old data is deleted). Repeat this process to clear L2 and L3 if needed.

   Step 3: From L1, press the down arrow to move your cursor into the list. Enter each t value from the table into L1, pressing [ENTER] after each number to navigate down to the next blank spot in the list.

   Step 4: Arrow over to L2 and enter the n values as listed in the table.

   Step 5: Press [GRAPH] to see how n varies with t.

   Step 6: To adjust the viewing window, press [WINDOW]. Choose settings as appropriate; for example, the x-axis could run from 1 to 10 and the y-axis could run from 0 to 40.

   On a TI-Nspire:

   Step 1: Press [home]. Arrow over to the spreadsheet icon, the fourth icon from the left, and press [enter].

   Step 2: To clear the lists in your calculator, arrow up to the topmost cell of the table to highlight the entire column, then press [menu]. Use the arrow key to choose 3: Data, then 4: Clear Data, then press [enter]. Repeat for each column as necessary.

   Step 3: Select the top cell in column A and enter "t" since it is the independent variable. Then select the top cell in column B and enter "n" for the dependent variable.

   Step 4: Enter the ordered pairs in the cells A1 through A9 and B1 through B9.

   Step 5: To see the graph, press [menu], and then select 3: Graph Type and 4: Scatter Plot.

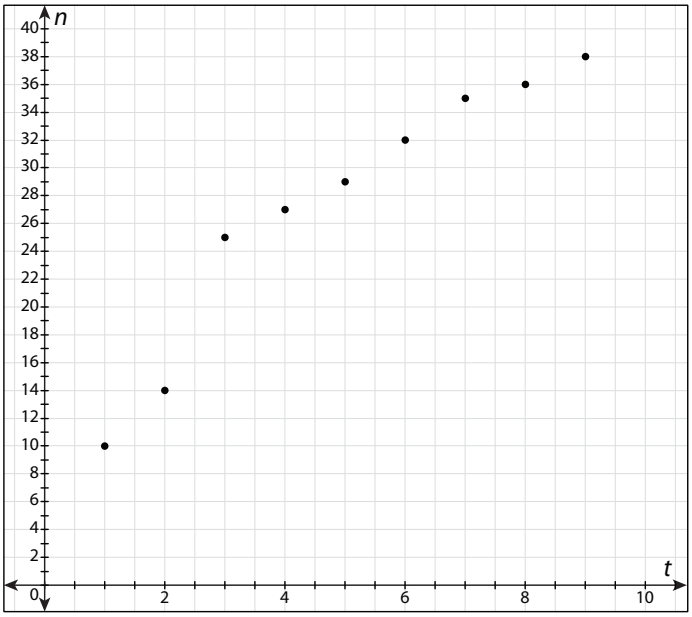
   Step 6: Select "t" from the x pop-up menu and "n" from the y pop-up menu to see the points.

(*continued*)

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Step 7: To adjust the viewing window, press [menu] and then select 4: Window/Zoom and 1: Window Settings. Choose settings as appropriate; for example, the x-axis could run from 1 to 10 and the y-axis could run from 0 to 40.

The resulting scatter plot should resemble the following.



2. Next, use the graphing calculator to fit the data to an exponential function and a power function.

   Follow the instructions appropriate to your calculator model. Round the values for a and b to two significant figures.

   On a TI-83/84:

   Step 1: Press [STAT] and arrow over to the CALC menu.

   Step 2: For an exponential function, arrow down to 0: ExpReg. Press [ENTER]. Enter the L1 variable ([2ND][1]) followed by a comma [,] and the L2 variable ([2ND][2]). Press [ENTER]. The results will be in the form $f(x) = a \cdot b^x$, with $a \approx 12$ and $b \approx 1.2$.

   Step 3: For a power function, select A: PwrReg from the CALC menu as described in Step 1. Repeat Step 2. The results will be in the form $g(x) = a \cdot x^b$, with $a \approx 10$ and $b \approx 0.6$.

(*continued*)

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On a TI-Nspire:

Step 1: Go back to the data table for n and t. Press [menu] and select 4: Statistics and then 1: Stat Calculations.

Step 2: For an exponential function, select A: Exponential Regression.

Step 3: For X List, select the t variable from the pop-up menu. For Y List, select the n variable from the pop-up menu. Tab to "OK" to see the equation of the function.

Step 4: The results will be in the form $f(x) = a \cdot b^x$, with $a \approx 12$ and $b \approx 1.2$.

Step 5: For a power function, repeat Step 1, then select 9: Power Regression from the Stat Calculations menu.

Step 6: Repeat Step 3.

Step 7: The results will be in the form $g(x) = a \cdot x^b$, with $a \approx 10$ and $b \approx 0.6$.

3. Write the equations of the calculated exponential and power functions from using the variables n and t.

   Substitute n, t, and the rounded values for a and b into the general form of each type of function.

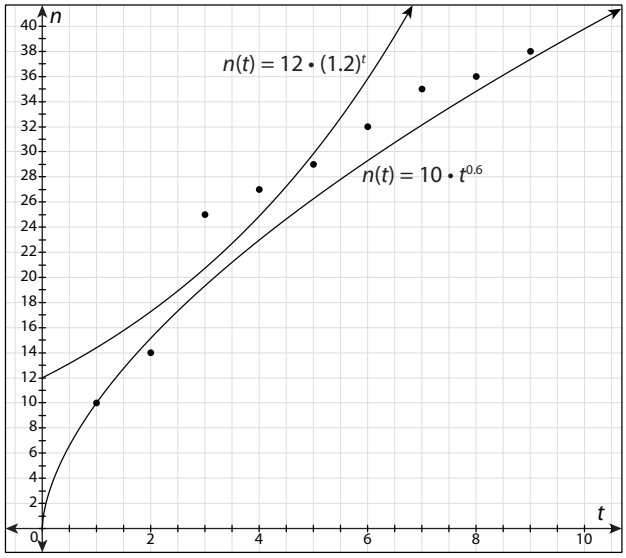
	Exponential function $f(x) = a \cdot b^x$	**Power function** $g(x) = a \cdot x^b$
TI-83/84	$n(t) = 12 \cdot (1.2)^t$	$n(t) = 10 \cdot t^{0.6}$
TI-Nspire	$n(t) = 12 \cdot (1.2)^t$	$n(t) = 10 \cdot t^{0.6}$

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Instruction

4. Plot the exponential and power functions from the previous step on the scatter plots produced by the tree-frog data.

   Return to your graph of the scatter plot and enter the equations for the functions $f(x)$ and $g(x)$ from step 3 to add them to the graph. The resulting combined graph should resemble the following.



5. Describe how well the exponential function fits the data points.

   The exponential function $n(t) = 12 \cdot (1.2)^t$ best fits the data from about $t = 3$ to $t = 6$. Before $t = 3$, the exponential function value is greater than the scatter plot function values. For $t > 6$, the exponential function values diverge from and are greater than the scatter plot function values.

6. Describe how well the power function fits the data points.

   The power function $n(t) = 10 \cdot t^{0.6}$ approximately fits the data from $t = 1$ to $t = 9$. The power function values are consistently less than the scatter plot function values from $t = 3$ to $t = 9$, but they follow the trend of the increasing scatter plot function values.

7. Determine which function type, exponential or power, best fits the data.

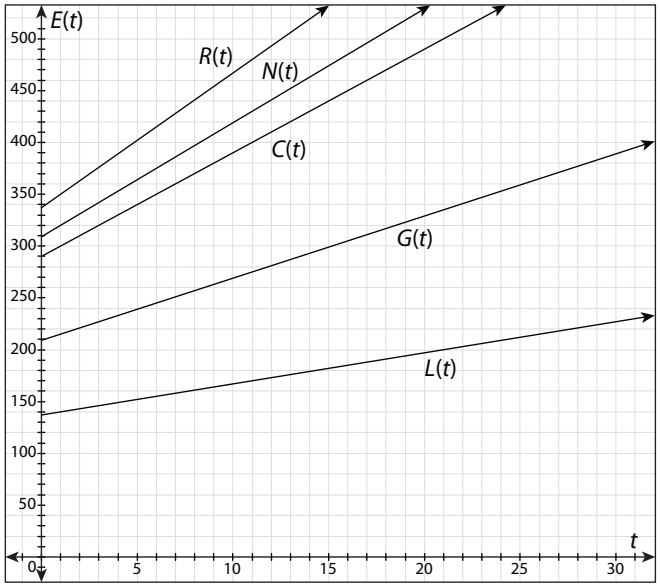
   The function values of the power function $n(t) = 10 \cdot t^{0.6}$ fit the scatter plot data better over this domain interval [1, 9].

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Problem-Based Task 4B.3.3: Comparing Energy Sources

SMP	
1 ✓	2
3	4 ✓
5	6
7 ✓	8

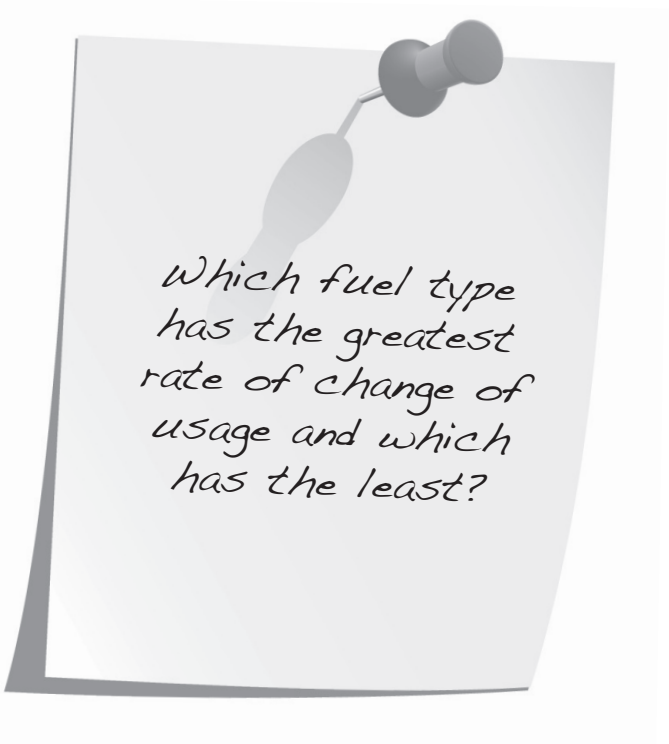
The U.S. Energy Information Administration has collected data on global energy use of five types of energy sources since 1990: renewables (such as biomass, hydroelectric, solar, tidal, and wind energy), nuclear energy, coal, natural gas, and liquid hydrocarbons (i.e., liquid natural gas and petroleum). It has also projected the use of those types of energy to the year 2020 as an aid to those developing federal energy law and policy. The graph shows the actual and projected usage amounts for each type of energy source.



Renewables	$R(t) = 13t + 337$
Nuclear energy	$N(t) = 11t + 309$
Coal	$C(t) = 10t + 290$
Natural gas	$G(t) = 6t + 209$
Liquid hydrocarbons	$L(t) = 3t + 137$

The horizontal axis runs from 1990 ($t = 0$) to 2020 ($t = 30$). The vertical axis is the amount of energy $E(t)$ used or projected in quadrillions of BTUs, a unit of energy measurement. The specific equation for each graphed function is listed in the table.

Which fuel type has the greatest rate of change of usage and which has the least? At what time will any one of the fuel types overtake the usage of any one of the other fuel types? How does the combined use of coal and natural gas compare with the use of renewables? At what time is the use of renewables greater than the combined use of coal and natural gas, if ever?



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Problem-Based Task 4B.3.3: Comparing Energy Sources

Coaching

a. How do you calculate the rate of change of each function?

b. What are the rates of change of each function?

c. Under what conditions are the function values of any pair of functions equal?

d. Check pairs of functions to find a domain value that makes the two functions equal.

e. What can be concluded from the result(s) of part d?

f. How can the usage of two or more functions be combined to give a meaningful result for their representation of energy usage?

g. Combine the coal and natural gas functions.

h. Describe how the result of part g compares to the renewables function.

i. How would you find the domain value at which the value of the combination of the coal and natural gas functions is greater than the value of the renewables function?

j. Write the relationship suggested by your answer to part i, solve it, and explain what it means.

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Instruction

Problem-Based Task 4B.3.3: Comparing Energy Sources

Coaching Sample Responses

a. How do you calculate the rate of change of each function?

One method of finding the rate of change is to use the rate of change formula, $r = \frac{\Delta f(x)}{\Delta x}$.

Substitute the values of t and $E(t)$ that are represented by two points of the linear functions on the graph.

Another method of finding the rate of change is to use the coefficient of the t term in each function, which is the rate of change of $E(t)$.

The two rates may differ since the linear functions provide the exact rate of change, and the result from using the coordinates of the two points is approximate.

b. What are the rates of change of each function?

From the provided graph, estimate the coordinates of points on each function.

For $R(t)$, two points on the line appear to be (0, 337) and (11, 480). Use these values to calculate the rate of change.

$$r = \frac{\Delta f(x)}{\Delta x} = \frac{480 - 337}{11 - 0} = 13$$

For $N(t)$, two points on the line appear to be (0, 309) and (20, 529).

$$r = \frac{\Delta f(x)}{\Delta x} = \frac{529 - 309}{20 - 0} = 11$$

For $C(t)$, two points on the line appear to be (0, 290) and (23, 520).

$$r = \frac{\Delta f(x)}{\Delta x} = \frac{520 - 290}{23 - 0} = 10$$

For $G(t)$, two points on the line appear to be (0, 209) and (30, 389).

$$r = \frac{\Delta f(x)}{\Delta x} = \frac{389 - 209}{30 - 0} = 6$$

For $L(t)$, two points on the line appear to be (0, 137) and (21, 200).

$$r = \frac{\Delta f(x)}{\Delta x} = \frac{200 - 137}{21 - 0} = 3$$

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The rates of change of the linear functions are:

- $R(t)$ = 13 quadrillion BTUs per year
- $N(t)$ = 11 quadrillion BTUs per year
- $C(t)$ = 10 quadrillion BTUs per year
- $G(t)$ = 6 quadrillion BTUs per year
- $L(t)$ = 3 quadrillion BTUs per year

c. Under what conditions are the function values of any pair of functions equal?

Any two functions $f(x)$ and $g(x)$ are equal if a value of x can be found for which $f(x) = g(x)$.

d. Check pairs of functions to find a domain value that makes the two functions equal.

For $R(t) = 13t + 337$ and $N(t) = 11t + 309$, the equality condition is $13t + 337 = 11t + 309$. The solution is $t = -14$, which is not within the domain of the functions. Therefore, there is no value in the domain of the function for which these functions will be equal.

Calculations with other pairs of functions will give the same result.

e. What can be concluded from the result(s) of part d?

The five functions' values are diverging across the domain for which the functions are defined, so there will be no value of t in the domain at which they will be equal. Based on this, at no time will the usage for one fuel type overtake the usage of any one of the other fuel types.

f. How can the usage of two or more functions be combined to give a meaningful result for their representation of energy usage?

They can be added, since adding will preserve the units of the function and still give a meaningful usage value at any time in the domain.

g. Combine the coal and natural gas functions.

$C(t) + G(t) = (10t + 290) + (6t + 209) = 16t + 499$

The combined coal-and-gas function is $(C + G)(t) = 16t + 499$.

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h. Describe how the result of part g compares to the renewables function.

The renewables function is $R(t) = 13t + 337$. The combined coal-and-gas function is $(C + G)(t) = 16t + 499$. At $t = 0$, $(C + G)(t) > R(t)$. The rate of change of $(C + G)(t)$ is also greater than the rate of change of $R(t)$. The rate of change of the combined use of coal and natural gas is greater than the rate of change of renewable sources of energy.

i. How would you find the domain value at which the value of the combination of the coal and natural gas functions is greater than the value of the renewables function?

Use the condition $(C + G)(t) > R(t)$ and find what values of t make it true.

j. Write the relationship suggested by your answer to part i, solve it, and explain what it means.

Substitute the expressions for $(C + G)(t)$ and $R(t)$ into the condition $(C + G)(t) > R(t)$. Simplify the resulting inequality:

$16t + 499 > 13t + 337$

$3t > -162$

$t > -54$

The functions are undefined for the interval $[-54, 0)$, but they exist for the domain $[0, 30]$, which is a subset of the solution to the inequality. This means that for any time in the domain, the combined coal-and-natural gas function will have a greater value than the renewables function. Similar calculations and comparisons can be made for other function combinations. The rate of change for the combined energy sources of coal and natural gas will always be greater than that for renewable sources of energy. Based on current data, the use of renewable sources will never be greater than the use of coal and natural gas combined.

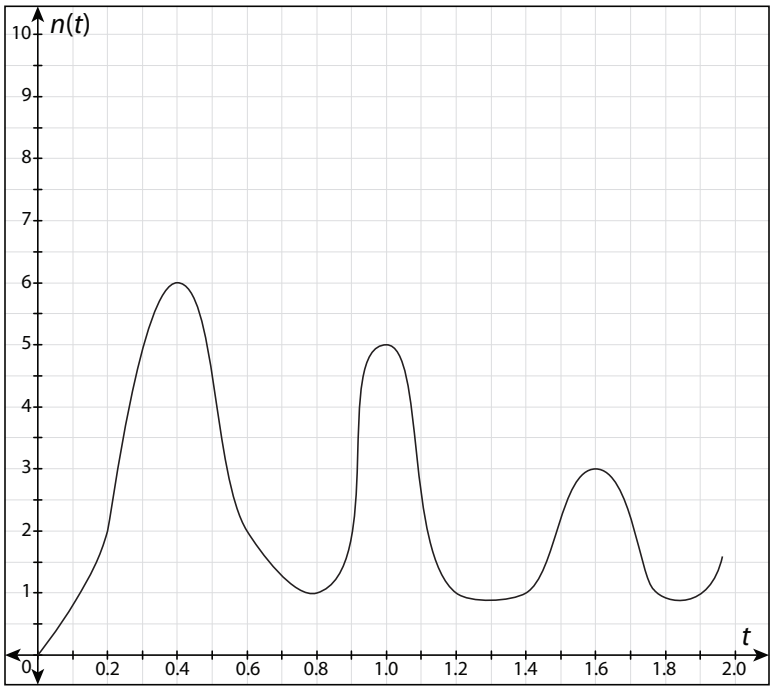
Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4B.3.3: Comparing Functions

The following graph shows the number of gallons of popcorn $n(t)$ sold during a basketball game. The game began at $t = 0.5$ and ended at $t = 1.5$. The time t covers from 0 to 2 hours in 0.1-hour intervals. Use the graph to complete problems 1–3.



1. Not counting the endpoints, how many different maximum function values occur during the game?

2. Not counting the endpoints, how many different minimum function values occur during the game?

3. At what time(s) during the game do the maximum values occur?

continued

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Use the given information to complete problems 4–7.

> For a class project, a group of physics students dropped a bag of sand from the top of a parking garage while another member of the group shot a video of the falling bag from the ground. Afterward, the students reviewed the video to determine how fast the bag was moving 1 second after it was dropped. It took the bag 3 seconds to hit the ground. The function $h(t) = -16t^2$ describes the height from which the bag fell in feet, with t representing time in seconds. Find the average rate of speed of the bag in feet per second for the intervals in problems 4–6, then use these rates to answer problem 7.

4. [1, 1.1]

5. [1, 1.01]

6. [1, 1.001]

7. What is the speed of the bag at a time of $t = 1$ second? What does the speed function at any time t appear to be, and how are its features related to those of the distance function for any time t?

For problems 8–10, compare the average rates of change over the interval [1, 1.1] for each function pair. Use a calculator to estimate the rates.

8. $a(x) = -1 + e^x$; $b(x) = \sin x$

9. $c(x) = x^2$; $d(x) = x$

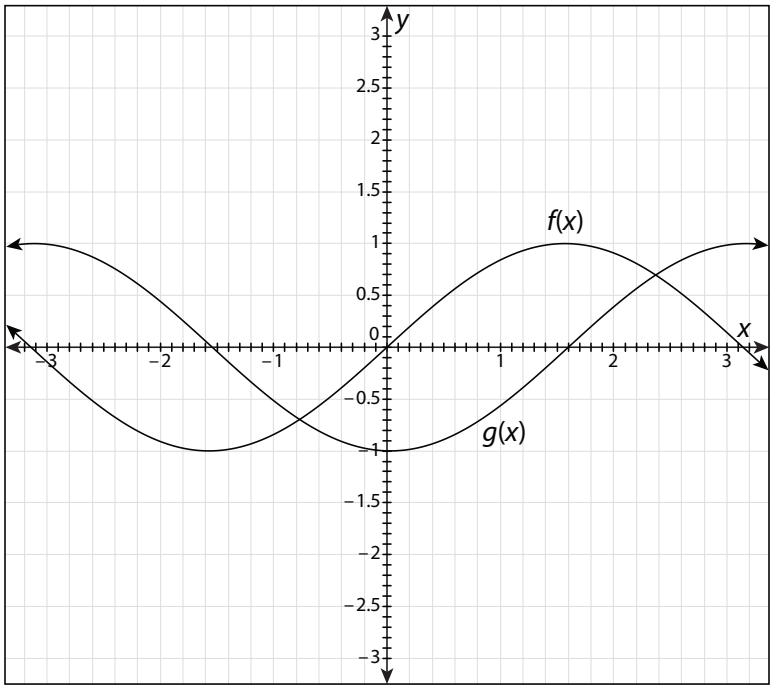
10. $e(x) = 1 + \log x$; $f(x) = \sqrt{x}$

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Progress Assessment

Circle the letter of the best answer.

1. Which statement is true about the functions $f(x)$ and $g(x)$?



   a. Over the domain [–3, 3], the maximum function values are different.

   b. Over the domain [–3, 3], the minimum function values are different.

   c. Over the domain [–3, 3], the *x*-intercepts are the same but the *y*-intercepts are different.

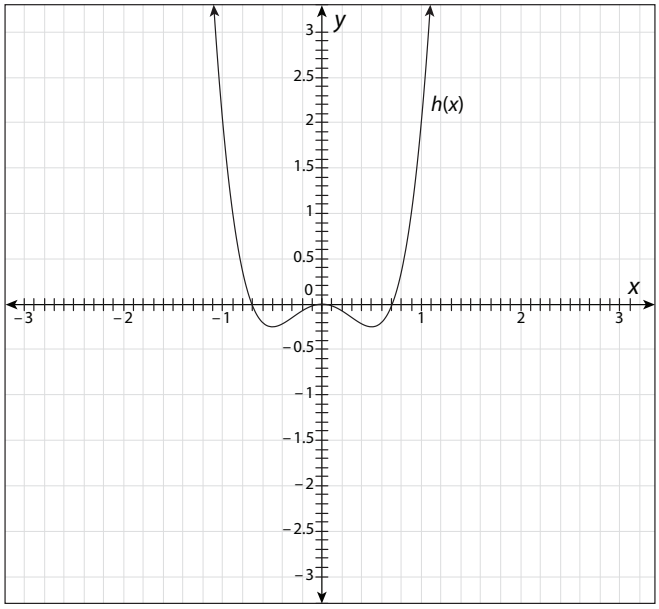
   d. Over the domain [–3, 3], the *x*- and *y*-intercepts are different.

continued

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Assessment

2. How many local maxima and minima points does the following graph of the function $h(x)$ show?



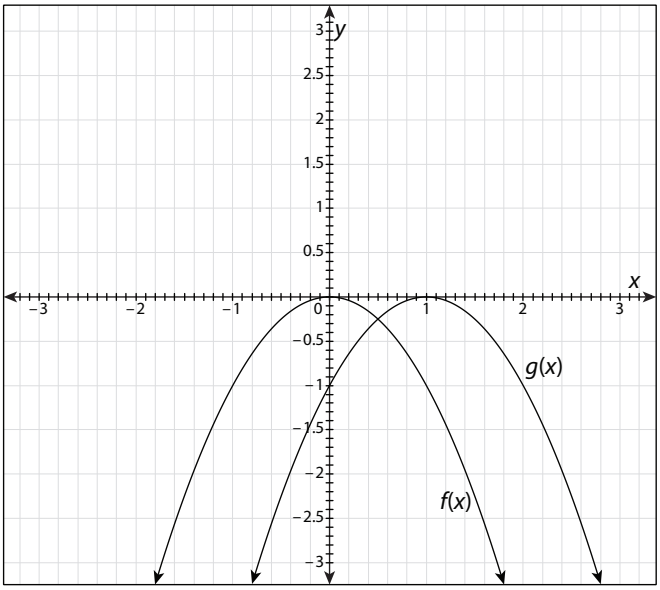
a. no maximum and no minimum

b. 1 maximum and 1 minimum

c. 1 maximum and 2 minima

d. 2 maxima and 2 minima

3. Over which of the following intervals are $f(x)$ and $g(x)$ both decreasing?



a. (0, 1)

b. (1, ∞)

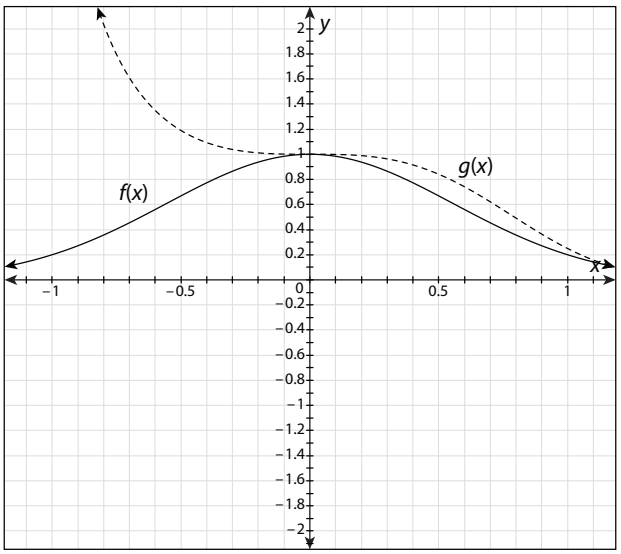
c. (–∞, 1)

d. (–∞, ∞)

continued

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4. At what domain value(s) are neither $f(x)$ nor $g(x)$ increasing?



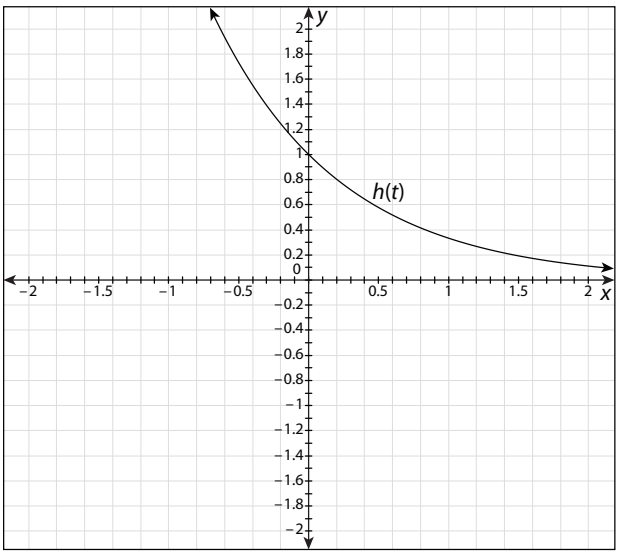
a. $x = 0$

b. $x \geq 0$

c. $x \geq 0$

d. $-1 < x < 1$

5. The following is the graph of the function $h(x) = 3^{-x}$. Which statement is true for the function values between $x = -1$ and $x = 0$?



a. The function values are decreasing at a faster rate than between $x = 0$ and $x = 1$.

b. The function values are decreasing at a slower rate than between $x = 0$ and $x = 1$.

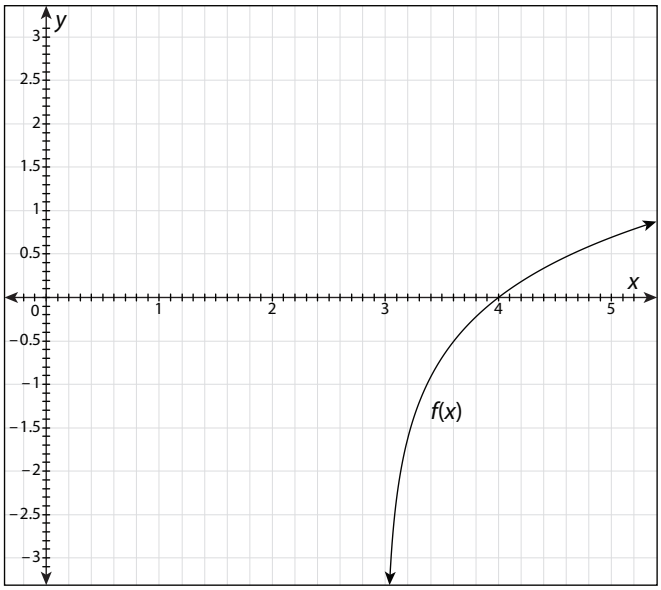
c. The function values are increasing at a faster rate than between $x = 0$ and $x = 1$.

d. The function values are increasing at a slower rate than between $x = 0$ and $x = 1$.

continued

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6. The following is a graph of the function $f(x) = \ln(x - 3)$. Which statement is true about the rate of change of the function values of consecutive decimal domain values (3.1 and 3.2, 3.2 and 3.3, etc.) between $x = 3$ and $x = 4$?



a. The rate of change is negative and decreasing.

b. The rate of change is negative and increasing.

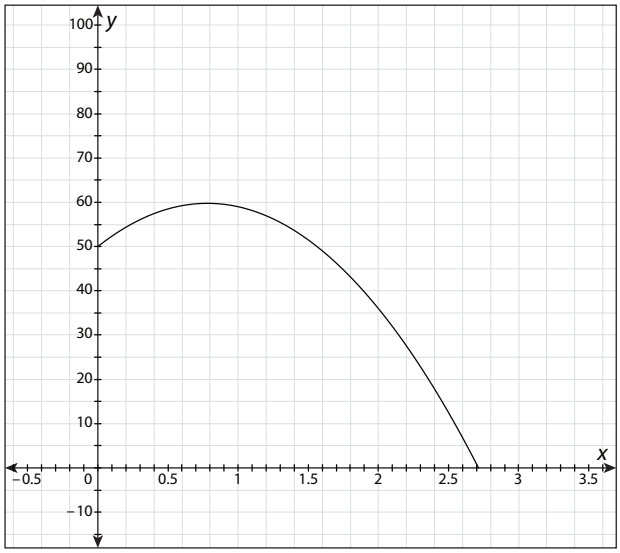
c. The rate of change is positive and increasing.

d. The rate of change is positive and decreasing.

continued

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7. The following graph shows the path an arrow follows after being shot into the air from a tree stand (the *y*-axis), reaching a maximum height, and falling back to the ground (the *x*-axis). Assume that there is no wind resistance. Over which interval is the rate of change of the speed increasing?



a. approximately [0, 0.8]

b. approximately [0.8, 2.7]

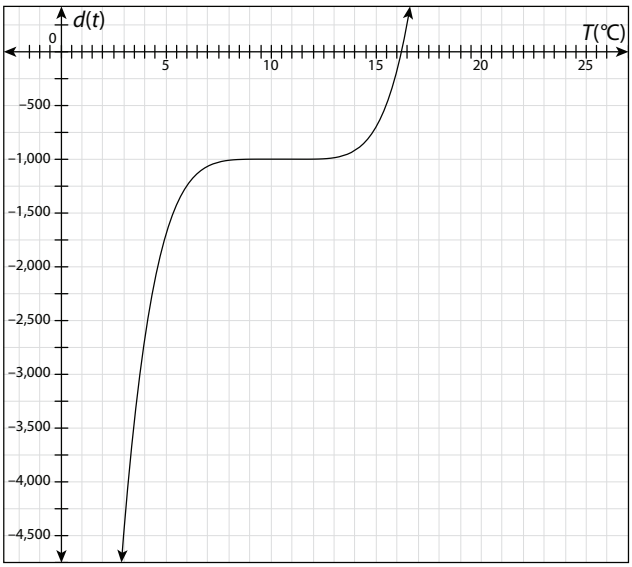
c. approximately [0, 2.7]

d. [0, 3]

continued

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8. The following graph shows how the water temperature of different layers of ocean water varies with ocean depth. What can be determined about the rate of change of the temperature shown by the "flat" part of the graph?



   a. The rate of change decreases as the temperature rises.

   b. The rate of change neither decreases nor increases as the temperature rises.

   c. The rate of change increases as the temperature rises.

   d. The rate of change increases but then decreases as the temperature rises.

9. The rates at which ice forms on a pond surface when the temperature drops by two different amounts are given by the exponential functions $f(t) = 1 - 2^{-t}$ and $g(t) = 1 - 2^{-3t}$. For very large values of t in hours, the function values approach 1, which means that the pond is completely covered in ice. If the surface of the pond represented by $f(t)$ is 75% covered with ice, what percent of the pond's surface represented by $g(t)$ is covered with ice?

   a. about 98%

   b. about 2%

   c. about 25%

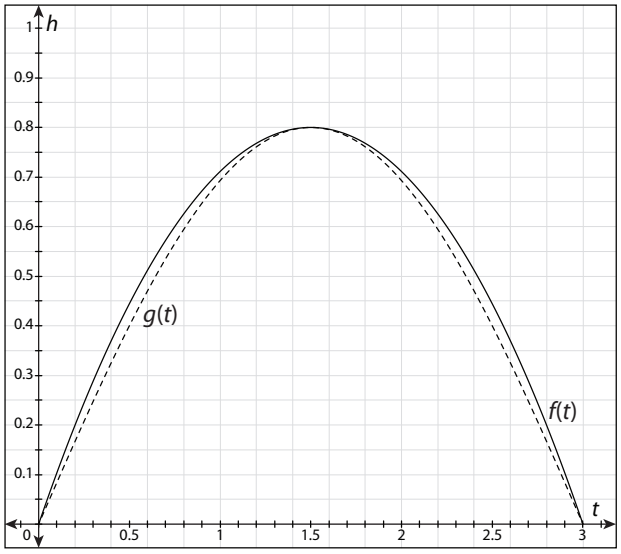
   d. 12.5%

continued

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Assessment

10. The two functions $f(t)=-\frac{32}{90}t^2+\frac{32}{30}t$ and $g(t)=0.8\bullet\sin\left(\frac{\pi}{3}t\right)$, shown in the following graph, model the height of a football as it travels through the air a distance, *d*, of 30 meters and reaches a height, *h*, of 8 meters. The time, *t*, the ball is in the air is 3 seconds, and the acceleration of gravity, *g*, is –32 feet per second squared. Which of the following is correct?



a. $f(t)=3dgt^2+dgt$ and $g(t)=10h\bullet\sin\left(\frac{\pi}{3}t\right)$

b. $f(t)=\frac{g}{3d}t^2+\frac{32}{d}t$ and $g(t)=\frac{h}{10}\bullet\sin\left(\frac{\pi}{3}t\right)$

c. $f(t)=gt^2+dt$ and $g(t)=0.8\bullet\sin(3\pi t)$

d. $f(t)=dgt^2+3dgt$ and $g(t)=\frac{h}{10}\bullet\sin(3t)$

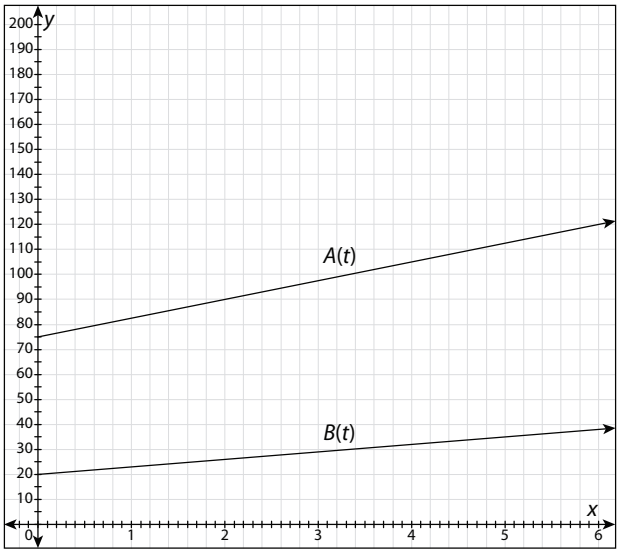
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Assessment

Use what you have learned about comparing properties within and between functions to complete all parts of the following problem.

11. The graph shows two functions that describe the average rate at which two construction companies A and B are awarded incentives to complete separate projects on time. The bid amount for company A's project is \$750,000, and the incentive pay is \$7,500 per month. The bid amount for company B's project is \$200,000, and the incentive pay is \$3,000 per month.



a. Compare the incentive rates for the projects.

b. Describe how to find the time at which the dollar amounts the two companies earn will be the same.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. The table shows the data points from a study of the contract price for the fuel used by a public transportation system. The variable $L(m)$ is the dependent variable and m is the independent variable. Which general form of a quadratic function models the data from the table?

m	0	2	4	6
$L(m)$	0	6	24	54

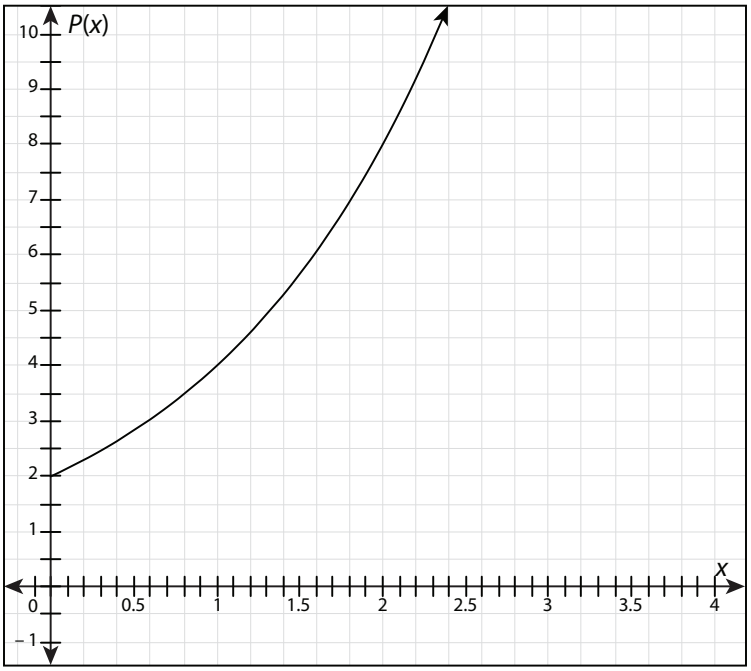
a. $f(x)=a-bx^2; a>0, b\neq 0$

b. $f(x)=a-bx-cx^2; a>0, b\neq 0, c>0$

c. $f(x)=-ax^2; a>0$

d. $f(x)=ax^2; a>0$

2. The graph shows the growth of a ground squirrel population in an ecosystem in which there are no predators. Which function models this data?



a. $P(x)=2^{x+1}$

b. $P(x)=2^x$

c. $P(x)=2+2^x$

d. $P(x)=2^{x-1}$

continued

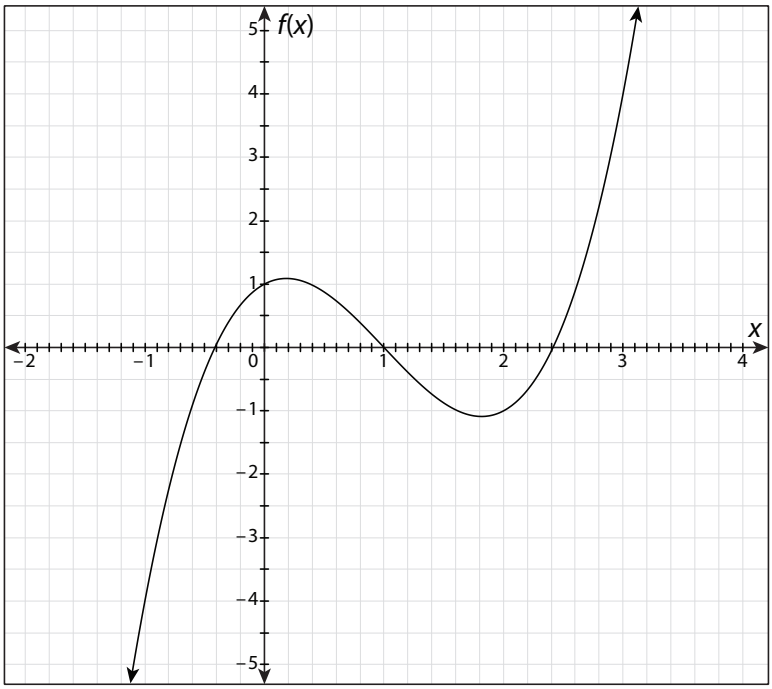
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Assessment

3. An object without a parachute is dropped from a plane. For a few seconds, the object falls with a speed that is modeled by the function $v(t)=\sqrt{16t}$, in which v is the velocity and t is the time before the object reaches its terminal speed. The terminal speed is the constant speed reached by the object due to air resistance. It is described by the function $v_f(t)=\frac{h}{t}$, in which h is the height at which the terminal speed is reached, and t is the time it takes for the object to fall to the ground. Which combined function represents this situation?
   a. $(v + v_f)(t)$ over the domain of t
   b. $v(t)$ for one restricted domain of t and $v_f(t)$ for another restricted domain of t
   c. $(v - v_f)(t)$ over the domain of t
   d. $v(t)$ for the domain of t, less $v_f(t)$ for a restricted domain of t

4. An absolute value function has the general form $f(x)=|x+a|+b$. What is the specific function if $f(1) = 2$?
   a. $f(x)=|x-1|+2$
   b. $f(x)=|x+1|+2$
   c. $f(x)=|x+2|+1$
   d. $f(x)=|x-2|-1$

5. What is the degree of the function shown? Assume that the function has no other solutions.



   a. 4
   b. 3
   c. 2
   d. 1

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Lesson 4: Choosing a Model

Instruction

Common Core State Standards

SMP
1 ✓ 2 ✓
3 ✓ 4 ✓
5 ✓ 6 ✓
7 ✓ 8

A–CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

F–IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F–IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

F–BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

SRB p. U4B-143

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Instruction

Essential Questions

1. How do you determine the scale to use on the axes of a graph that shows the relationship between two or more variables?
2. How do you interpret the features of a data plot that will determine what kind of function should be used to model the data?
3. How do you decide what restrictions exist on the domain and range variables in a model that represents a real-world problem and/or its solution?
4. How do you sketch a function model so that it represents the relationships in a real-world problem?
5. How do you use graphing calculator technology to explore the effect of constants added to or multiplied by a domain variable in a function that represents a real-world problem?

WORDS TO KNOW

absolute value function	a function of the form $f(x)=\lvert ax+b\rvert+c$, where x is the independent variable and a, b, and c are real numbers
continuous function	a function that does not have a break in its graph across a specified domain
cube root function	a function that contains the cube root of a variable. The general form is $f(x)=a\sqrt[3]{(x-h)-k}$, where a, h, and k are real numbers.
discontinuous function	a function that has a break, hole, or jump in the graph
discrete function	a function in which every element of the domain is individually separate and distinct
exponential function	a function of the form $f(x) = ab^{cx}$, in which a, b, and c are constants; a function that has a variable in the exponent, such as $f(x) = 5^x$
linear function	a function that can be written in the form $ax + by = c$, where a, b, and c are constants; can also be written as $f(x) = mx + b$, in which m is the slope and b is the y-intercept. The graph of a linear function is a straight line; its solutions are the infinite set of points on the line.

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Instruction

piecewise function	a function that is defined by two or more expressions on separate portions of the domain
quadratic function	a function defined by a second-degree expression of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants. The graph of any quadratic function is a parabola.
scale	the numbers representing the interval of a variable and the increments into which it is subdivided; usually includes the interval endpoints and the increments of the basic unit of the variable
square root function	a function that contains a square root of a variable. The general form is $f(x)=\sqrt{ax^2+bx+c}$, where a, b, and c are real numbers.
step function	a function that is a series of disconnected constant functions

Recommended Resources

- Algebra-Class.com. "Step Functions."

  http://www.walch.com/rr/00246

  This site gives a thorough review of the parts of step functions and how to read them.

- MathIsFun.com. "Piecewise Functions."

  http://www.walch.com/rr/00247

  This site offers easy-to-follow models for building an understanding of absolute value functions, floor (step) functions, and general piecewise functions.

- Twining Mathematics, LLC. "Graphs of Cube Roots."

  http://www.walch.com/rr/00248

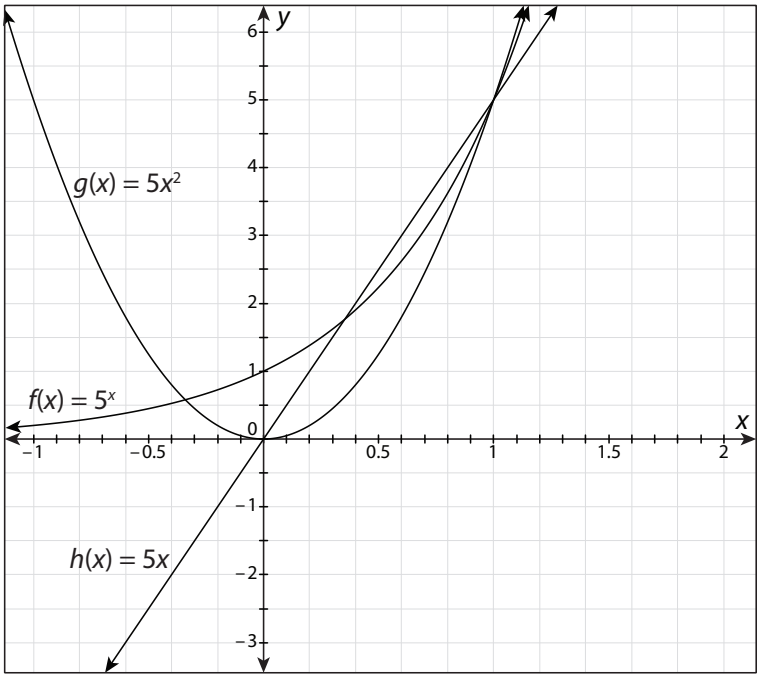
  This tutorial provides audio-assisted slides and animations that cover how to perform translations of cube root functions.

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Lesson 4B.4.1: Linear, Exponential, and Quadratic Functions

Warm-Up 4B.4.1

The graph shows three functions, $f(x) = 5^x$, $g(x) = 5x^2$, and $h(x) = 5x$, which are competing models used by a field zoologist in the Great Plains to describe the increase in a prairie dog population. In each function, x represents the number of years and the function values represent the prairie dog population in hundreds. The zoologist would like to know which model best describes the increasing population. Use the graph to complete the problems that follow.



1. Determine the average rate of change for each of the three functions over the domain interval [0, 1], which corresponds to 1 year.

2. Compare the y-intercepts of the functions. What do they mean in the context of the problem?

3. What is the solution for the system of all three functions and what is its meaning in the context of the problem?

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Instruction

Lesson 4B.4.1: Linear, Exponential, and Quadratic Functions

Common Core State Standards

A–CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

F–IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F–IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F–BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Warm-Up 4B.4.1 Debrief

1. Determine the average rate of change for each of the three functions over the domain interval [0, 1], which corresponds to 1 year.

   Recall the rate of change formula, $r = \frac{\Delta f(x)}{\Delta x}$, where $\Delta f(x)$ is the change in $f(x)$ and Δx is the change in x.

   For $f(x) = 5^x$:

   $r = \frac{\Delta f(x)}{\Delta x}$ Rate of change formula

   $r = \frac{f(0) - f(1)}{(0) - (1)}$ Substitute $f(0) - f(1)$ for $\Delta f(x)$ and $(0) - (1)$ for Δx.

   $r = \frac{(5^0) - (5^1)}{0 - 1}$ Substitute 5^0 for $f(0)$ and 5^1 for $f(1)$.

   $r = \frac{1 - 5}{-1}$ Simplify.

   $r = 4$

   The rate of change for $f(x)$ over the interval [0, 1] is 4.

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Instruction

For $g(x) = 5x^2$:

$r = \frac{\Delta g(x)}{\Delta x}$	Rate of change formula rewritten with $g(x)$ in place of $f(x)$
$r = \frac{g(0) - g(1)}{(0) - (1)}$	Substitute $g(0) - g(1)$ for $\Delta g(x)$ and $(0) - (1)$ for Δx.
$r = \frac{\left[5(0)^2\right] - \left[5(1)^2\right]}{0 - 1}$	Substitute $5(0)^2$ for $g(0)$ and $5(1)^2$ for $g(1)$.
$r = \frac{0 - 5}{-1}$	Simplify.
$r = 5$	

The rate of change for $g(x)$ over the interval [0, 1] is 5.

For $h(x) = 5x$:

$r = \frac{\Delta h(x)}{\Delta x}$	Rate of change formula rewritten with $h(x)$ in place of $f(x)$
$r = \frac{h(0) - h(1)}{(0) - (1)}$	Substitute $h(0) - h(1)$ for $\Delta h(x)$ and $(0) - (1)$ for Δx.
$r = \frac{\left[5(0)\right] - \left[5(1)\right]}{0 - 1}$	Substitute $5(0)$ for $h(0)$ and $5(1)$ for $h(1)$.
$r = \frac{0 - 5}{-1}$	Simplify.
$r = 5$	

The rate of change for $h(x)$ over the interval [0, 1] is 5.

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Instruction

2. Compare the *y*-intercepts of the functions. What do they mean in the context of the problem?

   The *y*-intercept is the function value at $x = 0$. In the context of this problem, it means that at the beginning of the study ($x = 0$), two of the models show a population of 0 prairie dogs and one model shows a population of 100 prairie dogs. Therefore, the *y*-intercept of $f(x)$ is at $y = 1$, and the *y*-intercepts of $g(x)$ and $h(x)$ are at $y = 0$.

3. What is the solution for the system of all three functions and what is its meaning in the context of the problem?

   The solution for the system of all three functions is the point at which their graphs intersect. As seen on the graph, the point of intersection is (1, 5). In the context of the problem, the solution shows the time at which the prairie dog population is equal for all three models. After 1 year, the prairie dog population will be 500 for each of the three models.

Connection to the Lesson

- Students will calculate rates of change and other quantities associated with exponential, linear, or quadratic functions, or systems of combinations of those functions, to solve problems.
- Students will identify characteristics of exponential, linear, or quadratic functions (e.g., asymptotes and intercepts).
- Students will find solutions of exponential, linear, or quadratic functions and check whether the solutions are reasonable in the context of the problem.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- identifying exponential, linear, and quadratic function models that can be used to represent tables of data points, data graphs, or symbolic expressions of data (F-LE.2, F-BF.1a)
- finding the solution(s) to a function used to model a problem and checking the solution(s) for accuracy and reasonableness (7.EE.3)
- identifying important characteristics of exponential, linear, and quadratic functions (e.g., domain, range, slope, intercepts, asymptotes, and zeros) (F-IF.4)

Introduction

Exponential, linear, and quadratic functions are used to represent a variety of real-world problems. In some cases, the accuracy and validity of using these function models for a real-world problem is limited to a restricted domain and/or range because of the finite nature of the problem situation; the model and its results apply to a specific set of data points. The key to selecting a function model is an accurate analysis of the data in a problem or its visual representation. Variables such as the domain of the data set and the rate at which the range or function value of a model is changing are critical to selecting a model that best fits a data set. Graphing calculators and other computer-assisted tools can be useful in experimenting with data sets to find the best model to represent that data. Asking "what if" questions about models and their features can help refine those models for best results in depicting a real-world problem and its solution.

Key Concepts

Exponential Functions

- Recall that **exponential functions** are functions that have a variable in the exponent. Exponential functions have the form $f(x) = ab^{cx}$, in which a, b, and c are constants.
- Two special categories of exponential functions are those in which $b = 10$, which is the base of common logarithms, and $b = e$, which is the base of natural logarithms.
- Real-world applications in which $c > 0$ include such subjects as tracking population growth. If $c < 0$, exponential functions can be used to model biological or radioactive decay.
- Combinations of exponential and linear functions can be used to describe the behavior of currents and voltages in electronic circuits, such as those found in cell phones and tablet computers.

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Instruction

Linear Functions

- A **linear function** is a function that can be written in the form $ax + by = c$, where a, b, and c are constants. A linear function can also be written as $f(x) = mx + b$, in which m is the slope and b is the y-intercept. The graph of a linear function is a straight line where the function's solutions are the infinite set of points on the line.
- In large multiple-variable systems of linear equations, the general form of a linear equation might be represented as $ax + by + cz + \ldots = k$, in which the constants are represented by a, b, c, etc., and k; the variables are represented by x, y, z, and so on. Such systems of linear equations are frequently found in large-scale industrial design processes. For example, designers of a new airplane might rely on large systems of linear equations to help account for the complex behavior of air moving over the plane's wing surface.

Quadratic Functions

- Recall that the general form of a **quadratic function** is $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants. The graph of any quadratic function is a parabola.
- The vertex form of the quadratic function, $g(x) = a(x - h)^2 + k$, can be used to express the function in terms of the coordinates of its maximum or minimum point, or vertex, (h, k).
- Other types of second-degree equations and relations can be used to represent real-world situations. For example, the motion of planets about a star or the motion of a moon about a planet can be described by a simple second-degree equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. However, the models used in this lesson will all be quadratic functions of the first two types.

Choosing Appropriate Models

- Visual representation of domain and range variables for real-world problems may require calculating conversions between units of distance, mass, time, etc. This will require that an appropriate scale be used for graphs and other visual aids.
- On a graph, **scale** refers to the numbers representing the interval of a variable and the increments into which it is subdivided.
- The scale consists of two components—the interval end points and the increments of the basic unit of the variable. For example, if a time variable is to be shown on the x-axis covering the interval of 0 to 15 seconds in increments of 0.1 second, the scale should be drawn with 0 and 15 as endpoints and 10 markings between each consecutive second (e.g., 1.0, 1.1, 1.2, and so on).

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- Sometimes, the available space for such a scale might not be possible, so the interval between the whole units has to be shown as close as possible to variable increments. Using the previously described example, it might only be possible to show half seconds between consecutive whole seconds, in which case the location of a time like 3.7 seconds would have to be approximated.
- Sometimes it is necessary to combine common function types in order to best model the situation. One such combination is the constant-exponential function model, which has the general form $f(x) = a + e^{b-x}$. In the general form of the combination constant-exponential function model, $f(x) = a + e^{b-x}$, a represents the horizontal asymptote and b is a constant.
- The choice of these functions as models for real-world problems can sometimes be justified using graphing calculator techniques. The data-fitting tools of graphing calculators offer a variety of options for comparing function models.
- In other real-world problem contexts, the nature of the variables and the behavior of the data provide the best justification for choosing a model.
- Complicated real-world problems are often difficult to model using purely mathematical functions with well-defined domains and behavior.
- Having the ability to think critically and compare competing function models to describe a real-world problem is often more important than having a thorough knowledge of the functions' characteristics and how they apply to a problem.

Common Errors/Misconceptions

- applying a function model to a real-world problem without checking to see if the model's features and solutions actually explain the problem's data
- confusing exponential functions with various "power" functions (e.g., quadratics, cube root functions, square root functions, and other functions of the form $f(x) = ax^b$)

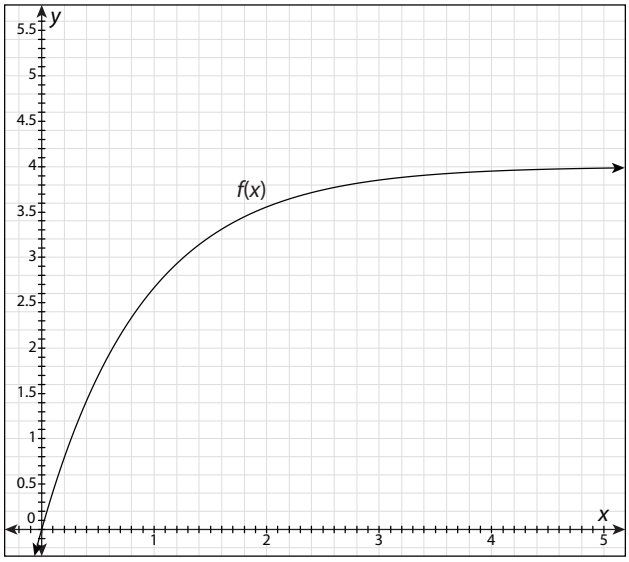
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Guided Practice 4B.4.1

Example 1

A function commonly used to represent bounded growth of a real-world function value is the exponential-linear combination function of the general form $f(x) = a(1 - b^{-x})$, in which a represents the horizontal asymptote $y = a$ toward which the values of $f(x)$ converge. Use the following graph to determine a and b using the point $\left(1, \frac{8}{3}\right)$, then write an exponential-linear combination function that models the graph.



1. Use the graph to identify the asymptote toward which the function values converge as the values of x increase.

   The asymptote shown is a horizontal asymptote that appears to be $y = 4$.

2. Determine the value of a in the general form of the combination function.

   The variable a represents the horizontal asymptote. Therefore, $a = 4$.

3. Update the general form of the combination function with the value of a.

   The general function is $f(x) = a(1 - b^{-x})$. The substitution of 4 for a results in the updated function, $f(x) = 4(1 - b^{-x})$.

SWB p. 105; SRB p. U4B-151

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4. Use the point $\left(1, \frac{8}{3}\right)$ to find the value of b.

Substitute the x- and y-coordinates of the point $\left(1, \frac{8}{3}\right)$ into the equation $f(x) = 4(1 - b^{-x})$.

$f(x) = 4(1 - b^{-x})$	Updated function
$y = 4(1 - b^{-x})$	Let $y = f(x)$.
$\left(\frac{8}{3}\right) = 4\left[1 - b^{-(1)}\right]$	Substitute 1 for x and $\frac{8}{3}$ for y.
$\frac{8}{3} = 4\left(1 - \frac{1}{b}\right)$	Simplify using the Negative Exponent Property.
$\frac{2}{3} = 1 - \frac{1}{b}$	Divide both sides by 4.
$-\frac{1}{3} = -\frac{1}{b}$	Subtract 1 from both sides.
$b = 3$	Solve for b.

The value of b is 3.

5. Write the exponential-linear combination function.

Using the original function, $f(x) = a(1 - b^{-x})$, let $a = 4$ and $b = 3$.

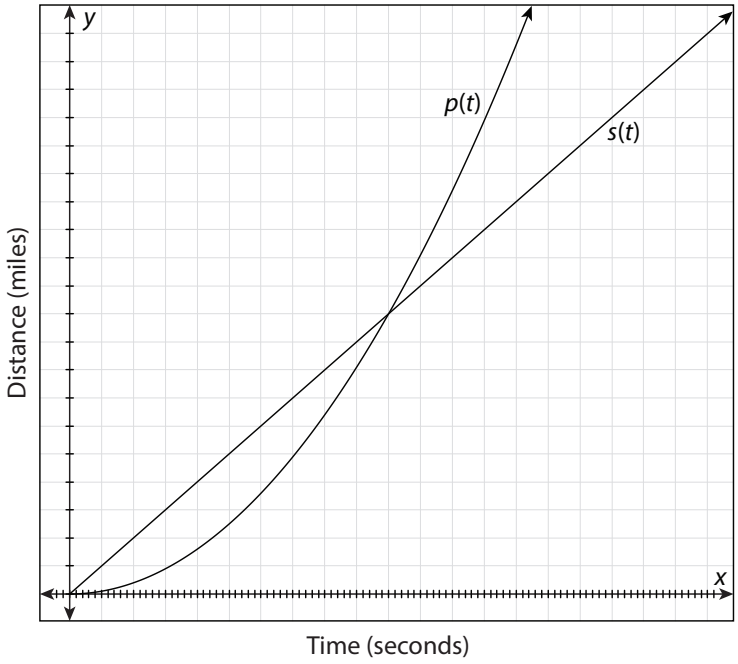
The exponential-linear combination function that models the given graph is $f(x) = 4(1 - 3^{-x})$.

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Example 2

The graph models the distance, $s(t)$, traveled by a speeding car moving at a constant rate r of 80 miles per hour, and the distance covered by a state patrol car, $p(t)$, until the patrol car intercepts the speeding car after 5 seconds (t). The general equations for the cars are $s(t) = rt$ for the speeding car and $p(t) = \frac{1}{2}at^2$ for the state patrol car, in which a is the acceleration in miles per second squared. Write the specific equations for the two distances, and determine appropriate horizontal and vertical axis scales on the graph to reflect the given problem conditions.



1. Determine how to calculate the distance traveled by the speeding car in 5 seconds.

   The function $s(t) = rt$ is used to calculate the speed, with time t in seconds. However, the speeding car's rate is given as 80 miles per hour. Therefore, this rate has to be converted to seconds.

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2. Calculate the speed of the speeding car in miles per second.

One hour is equal to 60 minutes, which is 3,600 seconds. This means that the conversion between hours and seconds can be represented by the ratios $\frac{1 \text{ hour}}{3600 \text{ seconds}}$ and $\frac{3600 \text{ seconds}}{1 \text{ hour}}$.

Use the conversion ratios to determine the speed of the speeding car in miles per second.

$\text{speed} = \frac{80 \text{ miles}}{\text{hour}} \cdot \frac{1 \text{ hour}}{3600 \text{ seconds}}$ Multiply the speed of the car by the conversion factor.

$\text{speed} = \frac{80 \text{ miles}}{\cancel{\text{hour}}} \cdot \frac{1 \ \cancel{\text{hour}}}{3600 \text{ seconds}}$ Cancel like units.

$\text{speed} = \frac{1 \text{ mile}}{45 \text{ seconds}}$ Simplify.

The speeding car is traveling at 1 mile every 45 seconds, or $\frac{1}{45}$ of a mile per second.

3. Use the result of step 2 to calculate the distance covered by the speeding car in 5 seconds.

Substitute $\frac{1}{45}$ for s and 5 for t in the function $s(t) = rt$.

$s(t) = rt$ Original function

$s(5) = \left(\frac{1}{45}\right)(5)$ Substitute $\frac{1}{45}$ for s and 5 for t.

$s(5) = \frac{1}{9}$ Simplify.

The speeding car travels $\frac{1}{9}$ of a mile in 5 seconds.

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4. Use the distance covered by the speeding car in 5 seconds to find the acceleration of the state patrol car.

   The distance that the patrol car travels to intercept the speeder and the time of 5 seconds are the same as in step 3. Substitute these values into the state patrol car's function, and then solve for the acceleration, a.

   $p(t)=\frac{1}{2}at^2$ State patrol car's function

   $\left(\frac{1}{9}\right)=\frac{1}{2}a(5)^2$ Substitute $\frac{1}{9}$ for $p(t)$ and 5 for t.

   $\frac{1}{9}=\frac{25a}{2}$ Simplify.

   $a=\frac{2}{225}$ Solve for a.

   The state patrol car's acceleration is $\frac{2}{225}$ of a mile per second squared.

5. Write the specific equation for the speeding car's function in terms of the value for r found in step 2.

   The rate r found for the speeding car is $\frac{1}{45}$ of a mile per second.

   Substitute this value into the given function for the speeding car,

   $s(t) = rt$.

   The equation for the speeding car's function, at any time t, is $s(t)=\frac{1}{45}t$; the distance covered is still in miles, but the speed is in miles per second, which is compatible with the time units.

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6. Write the specific equation for the state patrol car's function in terms of the result of step 4.

   The acceleration a found for the state patrol car is $\frac{2}{225}$ of a mile per second squared. Substitute this value into the given function for the patrol car, $p(t)=\frac{1}{2}at^2$.

   The equation for the patrol car is $p(t)=\frac{1}{2}\left(\frac{2}{225}\right)t^2=\frac{1}{225}t^2$. The distance is in miles and the acceleration is in miles per second squared.

7. Determine an appropriate scale for the graph's horizontal axis in order to fit the conditions of the problem.

   The horizontal axis t measures time in seconds; the scale is from 0 seconds to 10 seconds in increments of 1 second.

   The maximum value of t in the problem is $t = 5$, so the scale does not have to be greater than that value. Therefore, the axis could be limited from 0 to 5. Only whole seconds are used, so the one-second intervals do not need to be subdivided into smaller intervals.

8. Determine an appropriate scale for the vertical axis in order to fit the conditions of the problem.

   The vertical axis measures distance in miles; the scale is from 0 miles to 10 miles in increments of 1 mile.

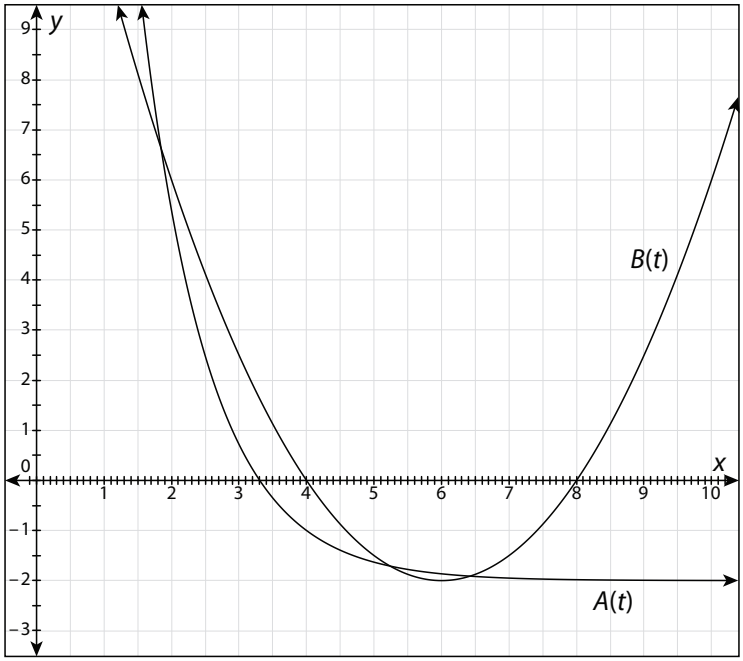
   From step 3, the maximum distance represented should be $\frac{1}{9}$ of a mile. One possible scale for the vertical axis would be from 0 to $\frac{1}{9}$, divided into 10 intervals of $\frac{1}{90}$ each. The vertical scale could retain its present scale (from 0 to 10) if the axis title were changed to "Distance $\left(\frac{1}{90}\text{ths of a mile}\right)$."

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Example 3

Cloud storage refers to keeping data on a remote database instead of your own computer or mobile device in order to save hard drive space. Cloud storage users pay for this service either as part of a data plan subscription or separately. Two market research teams used function models to describe the declining cost of cloud-based computer-storage services. Team A used an exponential function, represented in the graph by $A(t)$; Team B used a quadratic function, represented by $B(t)$. The graphs of the functions are shown over a 10-month period with "the present" represented by the region at which the quadratic function reaches a minimum function value. The y-axis represents the price in dollars per gigabyte of storage. Domain values that are greater than those represented by the minimum point are projections based on the models. The portion of range that is less than or equal to 0 represents the increase in free cloud storage, which is paid for by the data storage companies. Describe how well the two models fit the data of the changing cost of cloud storage. Use function characteristics to compare the models. Describe the limitations of these models for predicting future changes in the cost of cloud services.



1. State the coordinates of the minimum point of Team B's quadratic function model.

   According to the graph, the coordinates of the minimum appear to be (6, –2).

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2. Use the vertex form of a quadratic function, $f(x) = a(x - h)^2 + k$, to write a specific quadratic function $B(t)$ with the identified minimum point for the model used by Team B.

   In the form $f(x) = a(x - h)^2 + k$, (h, k) represents the minimum point of the parabola. Therefore, substituting the coordinates of the minimum point at $(6, -2)$, the function can be written as $B(t) = a(t - 6)^2 - 2$.

3. Determine another point on the graph of Team B's function that can be used to find the value of the constant, a, in the function for $B(t)$.

   Choose a point from the graph. Let $(t, B(t))$ be $(4, 0)$. Substitute these coordinates into the function for Team B and solve for a.

$B(t) = a(t - 6)^2 - 2$	Function for Team B
$(0) = a[(4) - 6]^2 - 2$	Substitute 4 for t and 0 for $B(t)$.
$0 = a(-2)^2 - 2$	Simplify the expression in the brackets.
$0 = 4a - 2$	Apply the exponent.
$2 = 4a$	Add 2 to both sides.
$a = 0.5$	Divide both sides by 4.

   The value of a is 0.5.

4. Write the quadratic function that represents Team B's model.

   Using the vertex form of a quadratic function, $f(x) = a(x - h)^2 + k$, the minimum point $(6, -2)$, and the value 0.5 for a, the quadratic function that represents Team B's model is $B(t) = 0.5(t - 6)^2 - 2$.

5. Determine the horizontal asymptote for Team A's exponential function model from the graph.

   The horizontal asymptote for Team A's function model appears to be $y = -2$.

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6. Use the general form of the combination constant-exponential function model, $f(x) = a + e^{b-x}$, to write a specific function model for $A(t)$.

   In the general form of the combination constant-exponential function model, $f(x) = a + e^{b-x}$, a represents the horizontal asymptote. Therefore, the function can be written as $A(t) = -2 + e^{b-t}$.

7. Find the value of the constant in the resulting function using a point on the graph of Team A's model.

   The constant in the function $A(t) = -2 + e^{b-t}$ is b.

   Choose a point from the graph. Let $(t, A(t))$ be $(4, -1)$.

$A(t) = -2 + e^{b-t}$	Function for Team A
$(-1) = = -2 + e^{b-(4)}$	Substitute 4 for t and –1 for $A(t)$.
$1 = e^{b-4}$	Add 2 to both sides.

   Recall that a number raised to an exponent of 0 will result in 1. Therefore, determine the value for which the exponent of b is 0. When b is 4, the exponent will be 4 – 4, or 0.

8. Write the combination function that represents Team A's model.

   Using the general function, $f(x) = a + e^{b-x}$, the value –2 for a, and the value 4 for b, the function that represents Team A's model is $A(t) = -2 + e^{4-t}$.

9. Describe what Team A's model shows for the first six months displayed on the graph and what it projects for the remaining four months of the domain interval [0, 10].

   Team A's model describes a declining cost of cloud services for the first six months until "the present," after which the cost is projected to be constant for the remaining four months.

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10. Describe what Team B's model shows for the first six months displayed on the graph and what it projects for the remaining four months of the domain interval [0, 10].

    Team B's model describes a declining cost of cloud services for the first six months until "the present," after which the cost is projected to increase for the remaining four months.

11. Describe the limitations of these models for predicting future changes in the cost of cloud services.

    It is difficult to predict what will happen to the cost of cloud services in the future without additional information. However, both models suggest a declining rate of cost changes over the interval [0, 6]. This favors the exponential model used by Team A, but it is not known if the asymptote $y = -2$ represents a "floor" for the costs or if the costs will decline to a lower-value asymptote with a value of $y < -2$. The future projection of increasing cloud-services cost estimates suggested by Team B's quadratic model does not seem realistic in light of other cost trends in the computer and information sciences field; typically, these costs go down as newer technologies become available and older ones are phased out, or as the technology becomes more widely available and, therefore, more competitively priced.

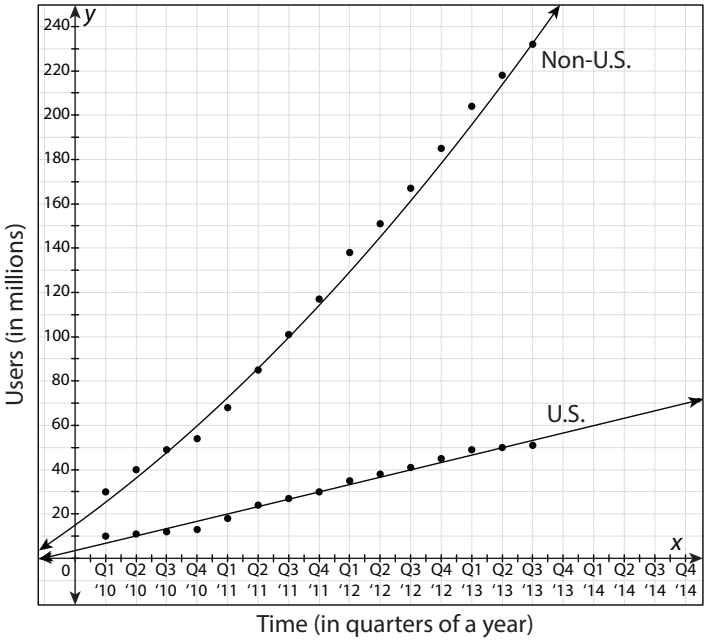
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Problem-Based Task 4B.4.1: Comparing Social Media Growth

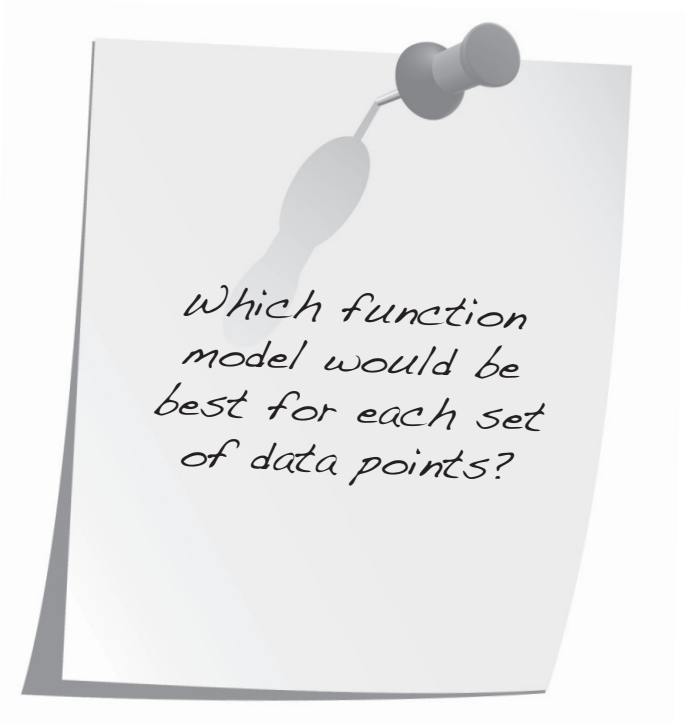
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An Internet market research company compared the growth in users of a popular social media website in the United States with the site's user-base growth in the rest of the world. The graph and table show the number of active site users (in millions) for each quarter of a year from 2010 through the third quarter of 2013.

The market researchers want to compare and project the growth in the two markets using exponential, linear, and quadratic function models. Use the table data and a graphing calculator to generate exponential, linear, and quadratic function models for the United States users and the non-United States users. Then, graph the models and see how they compare to the data points shown on the graph. Which function model would be best for each set of data points?



Quarter and year	Users in millions	
	U.S.	Non-U.S.
Q1 2010	10	30
Q2 2010	11	40
Q3 2010	12	49
Q4 2010	13	54
Q1 2011	18	68
Q2 2011	24	85
Q3 2011	27	101
Q4 2011	30	117
Q1 2012	35	138
Q2 2012	38	151
Q3 2012	41	167
Q4 2012	45	185
Q1 2013	49	204
Q2 2013	50	218
Q3 2013	51	232



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Problem-Based Task 4B.4.1: Comparing Social Media Growth

Coaching

a. What is the exponential function for the United States data?

b. What is the linear function for the U.S. data?

c. What is the quadratic function for the U.S. data?

d. Graph all three U.S. function models and the U.S. data points from the table on the same graph.

e. Which function or functions fit the U.S. data points best and why?

f. What is the exponential function for the non-United States data?

g. What is the linear function for the non-U.S. data?

h. What is the quadratic function for the non-U.S. data?

i. Graph all three non-U.S. function models and the non-U.S. data points from the table on the same graph.

j. Which function or functions fit the non-U.S. data points best and why?

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Problem-Based Task 4B.4.1: Comparing Social Media Growth

Coaching Sample Responses

a. What is the exponential function for the United States data?

The exponential function will be of the form $y = ab^x$. Enter the data points for United States users into a graphing calculator and find values for a and b to determine the exponential function that fits this data.

For the United States data, $a \approx 9.3$ and $b \approx 1.1$, which results in the exponential function $y = 9.3 \cdot 1.1^x$.

b. What is the linear function for the U.S. data?

Use the table values already entered into the graphing calculator to determine the equation of the linear function that fits this data.

The linear function will be of the form $y = ax + b$. Find values for a and b.

For the U.S. data, $a \approx 3.354$ and $b \approx 3.4$, which results in the linear equation $y = 3.354x + 3.4$.

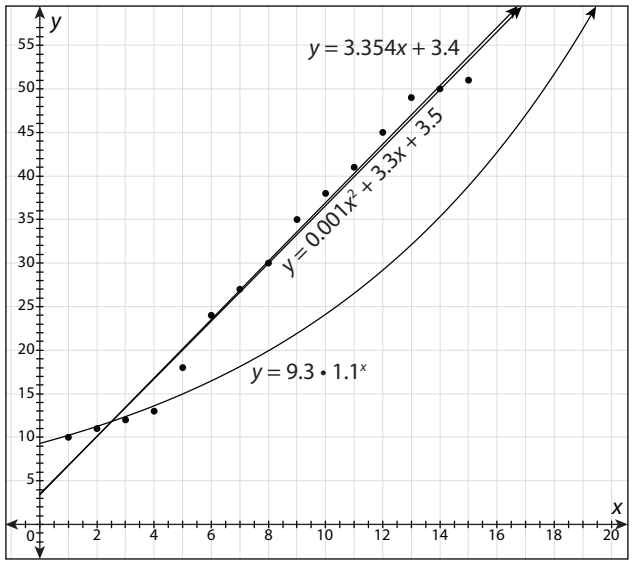
c. What is the quadratic function for the U.S. data?

Use the entered data to find values for a, b, and c for a quadratic function of the form $y = ax^2 + bx + c$.

For the U.S. data, $a \approx 0.001$, $b \approx 3.3$, and $c \approx 3.5$, which results in the quadratic function $y = 0.001x^2 + 3.3x + 3.5$.

d. Graph all three U.S. function models and the U.S. data points from the table on the same graph.

Again, use the graphing calculator to graph the data points and the generated exponential, linear, and quadratic functions.



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e. Which function or functions fit the U.S. data points best and why?

The linear and quadratic function models provide the best fit, because their graphs pass close to most of the data points; of the two, the quadratic function provides a slightly better fit, since the linear function passes close to but higher than some of the data points.

The exponential function is not a good fit for the data points because most of the graph diverges from the data points after passing close to the first four.

f. What is the exponential function for the non-United States data?

The exponential function will be of the form $y = ab^x$. Enter the data points for non-United States users into the graphing calculator and find values for a and b.

For the non-United States data, $a \approx 32$ and $b \approx 1.2$, which results in the exponential function $y = 32 \cdot 1.2^x$.

g. What is the linear function for the non-U.S. data?

Use the table values already entered into the graphing calculator to determine the equation of the linear function.

The linear function will be of the form $y = ax + b$. Find values for a and b.

For the non-U.S. data, $a \approx 15$ and $b \approx 1.3$, which results in the linear equation $y = 15x + 1.3$.

h. What is the quadratic function for the non-U.S. data?

Use the entered data to find values for a, b, and c for a quadratic function of the form $y = ax^2 + bx + c$.

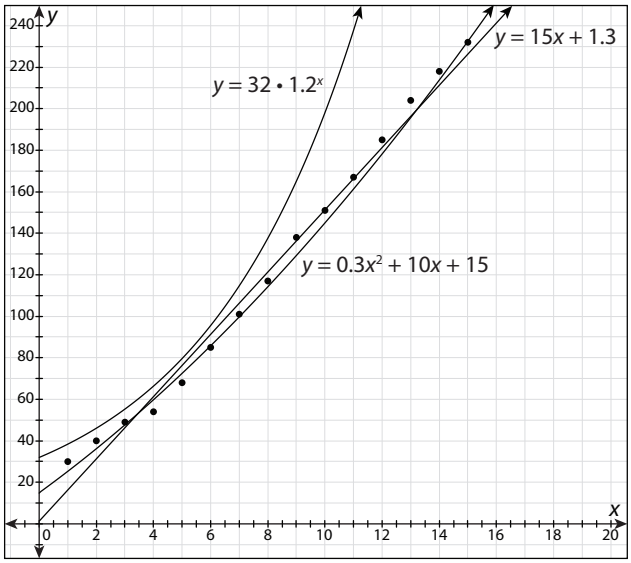
For the non-U.S. data, $a \approx 0.3$, $b \approx 10$, and $c \approx 15$, which results in the quadratic function $y = 0.3x^2 + 10x + 15$.

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i. Graph all three non-U.S. function models and the non-U.S. data points from the table on the same graph.

Again, use the graphing calculator to graph the data points and the generated exponential, linear, and quadratic functions.



j. Which function or functions fit the non-U.S. data points best and why?

The linear and quadratic function models provide the best fit, because their graphs pass close to most of the data points. The exponential function does not provide as good a fit with the data points since it passes close to only a few of the data points before diverging away from the remainder of the points. The linear function provides the best fit for both data sets. The quadratic function provides as good a fit as the linear function does for the non-U.S. user data.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4B.4.1: Linear, Exponential, and Quadratic Functions

A study of Wi-Fi data speeds and the distance at which the signals can be detected produced the average results shown in the table. ("Distance" refers to how far the Wi-Fi signal traveled from the device transmitting it to the device receiving the signal.) Each speed value s can be written as an ordered pair $(0, s)$ and each distance value d can be written as an ordered pair $(d, 0)$. Use the table to complete problems 1–3.

Wi-Fi speed (megabits per second)	Distance (meters)
450	18
110	55
50	91

1. If a line is drawn connecting the pairs of distances and speeds on a graph, what would the slope of that line represent? (*Hint*: It is an average rate of change.)

2. Calculate the slopes of the lines connecting each distance-speed pair of values, and explain what the different values mean in the context of the problem.

3. Which function model—exponential, linear, or quadratic—would best describe the relationship between distance and speed? Explain your answer.

For problems 4–10, determine the type of function—exponential, linear, or quadratic—that is most appropriate to model the situation described. Then, use the chosen function model to complete the problem.

4. A credit card company compounds annual interest daily. What is the domain for a calculation of the interest rate for 1 year that is not a leap year?

continued

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5. The school physical education program requires students to be timed for a 50-yard run. What is the range of the average student speeds?

6. A bowler starts out with a league average of 160 points per game and then has an average of 164 and 166 points per game after the next two weeks, respectively. What is the asymptote of the bowler's average if this trend continues and only whole numbers are used to express the average?

7. A magma bomb is a solid chunk of hot semi-solid rock ejected by an active volcano. What factors affect the height reached by the magma bomb? Assume that only the acceleration of gravity acts on the bomb once it is ejected.

8. In baseball, batting averages are expressed as decimal parts of a whole; for example, a batting average of 0.300 is said to be a batting average of 300. A baseball player's original batting average of 250 decreased by 5 points per game during a 10-game slump. Describe the domain and range of a function that relates the batting average to the number of games.

9. Meteorologists use the term "Saskatchewan screamer" to refer to a fast-moving arctic cold front originating near the province of Saskatchewan, Canada. One day, such a cold front caused the temperature in Chicago, Illinois, to drop from 30°F to –5°F in 3 hours. What are the domain of the temperature change and the average temperature change per quarter hour?

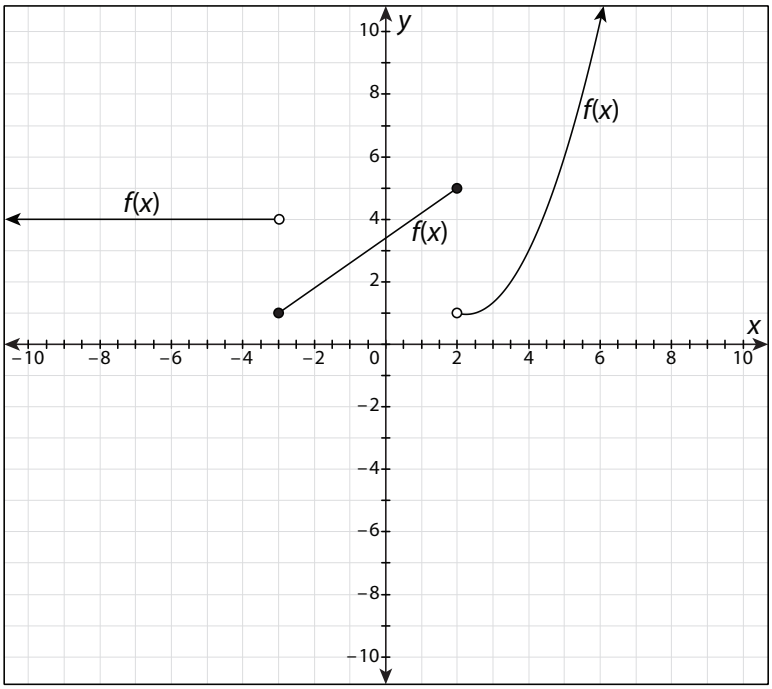
10. The dimensions of a piece of paper used to wrap a rectangular package are x and $2x - 1$ inches. What is the function with the dimensions that result in the maximum area for the wrapping paper?

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Lesson 4B.4.2: Piecewise, Step, and Absolute Value Functions

Warm-Up 4B.4.2

A piece of electronic test equipment delivers a different output voltage depending on the magnitude and polarity (positive or negative) of the input voltage. The output voltage is the dependent variable $f(x)$ and the input voltage is the independent variable. The function $f(x)$ that models this behavior is shown on the graph. The curved part of $f(x)$ is defined by the quadratic function model $f(x)=\frac{2}{3}x^2-3x+\frac{13}{3}$.



1. The function representing the curve in the first quadrant is given: $f(x)=\frac{2}{3}x^2-3x+\frac{13}{3}$. What is the restricted domain of this function?

2. What function represents the ray in the second quadrant?

3. What is the restricted domain of this function?

4. What function represents the line segment that intersects the y-axis?

5. What is the restricted domain of this function?

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Instruction

Lesson 4B.4.2: Piecewise, Step, and Absolute Value Functions

Common Core State Standards

F–IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F–IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

F–BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Warm-Up 4B.4.2 Debrief

1. The function representing the curve in the first quadrant is given: $f(x)=\frac{2}{3}x^2-3x+\frac{13}{3}$. What is the restricted domain of this function?

   Notice that the endpoint of the graph at $x = 2$ is an open circle, which means that the restricted domain for his curve begins at but does not include $x = 2$. Therefore, the restricted domain of $f(x)=\frac{2}{3}x^2-3x+\frac{13}{3}$ is $(2,\infty)$.

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2. What function represents the ray in the second quadrant?

The ray is part of a horizontal line. The general equation of a linear function in slope-intercept form is $f(x) = mx + b$, where m is the slope and b is the y-intercept. Since there is no change in the slope, the slope of the horizontal ray is 0. The y-intercept would be 4 if the line was extended to the y-axis. Therefore, substitute 0 for m and 4 for b into the general form, and simplify to determine the equation of the function.

$f(x) = mx + b$	General form of a linear function in slope-intercept form
$f(x) = (0)x + (4)$	Substitute 0 for m and 4 for b.
$f(x) = 4$	Simplify.

The function that represents the ray in the second quadrant is $f(x) = 4$.

3. What is the restricted domain of this function?

The open circle at $x = -3$ means "up to but not including" $x = -3$. Therefore, the restricted domain for $f(x) = 4$ is $(-\infty, -3)$.

4. What function represents the line segment that intersects the y-axis?

To determine the function for the line segment, first calculate the slope.

The slope of a line is given by the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$. Substitute the two endpoints of the line segment, (–3, 1) and (2, 5), into this formula and solve for m.

$m = \frac{y_2 - y_1}{x_2 - x_1}$	Slope formula
$m = \frac{(1) - (5)}{(-3) - (2)}$	Substitute 2 for x_1, –3 for x_2, 5 for y_1, and 1 for y_2.
$m = \frac{4}{5}$	Simplify.

The slope of the line is $m = \frac{4}{5}$.

Though we could use the slope-intercept form to determine the function as in the answer to problem 1, it is difficult to precisely approximate the y-intercept from the graph. Therefore, to more accurately find the equation of the line, substitute either of the previously determined endpoints along with the slope into the point-slope formula, $y - y_1 = m(x - x_1)$. Then, simplify the equation.

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Let's use the endpoint (2, 5).

$y - y_1 = m(x - x_1)$	Point-slope formula
$y-(5)=\left(\frac{4}{5}\right)[x-(2)]$	Substitute $\frac{4}{5}$ for m, 2 for x_1, and 5 for y_1.
$5(y - 5) = 4(x - 2)$	Multiply both sides by 5.
$5y - 25 = 4x - 8$	Distribute.
$5y = 4x + 17$	Add 25 to both sides.
$y=\frac{4}{5}x+\frac{17}{5}$	Divide both sides by 5.

The function that represents the line segment that intersects the y-axis is $f(x)=\frac{4}{5}x+\frac{17}{5}$.

5. What is the restricted domain of this function?

   Notice that the endpoints are solid circles; this means that the x-values of –3 and 2 are included in the restricted domain. Therefore, the restricted domain is given by [–3, 2].

Connection to the Lesson

- Students will identify the restricted domains of a function from a graph of the function.
- Students will use formulas and rules to determine the algebraic form of a function using coordinates or data points of the function.

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Prerequisite Skills

This lesson requires the use of the following skills:

- identifying restricted domains of parts of functions from observing a graph that has curves, line, and segments with open or solid endpoints of the graphs of those parts of functions (F-IF.5)
- using rules for calculating maximum or minimum points, slopes, *x*-intercepts, and *y*-intercepts to write equations for functions from the graphs of those functions (F-IF.4, F-LE.2)

Introduction

Many real-world applications of mathematics involve the use of two or more functions to describe relationships among several variables. These functions sometimes apply to the problem over different domains that correspond to different real-world conditions. For example, a quadratic function model might describe a real-world relationship between two variables over the interval $(a, b]$, but an exponential function model might better describe the relationship over the interval (b, c). In other words, portions of the same overall function can be defined by different equations over different domain intervals. The ability to apply a function model to a collection of data points depends, in part, on the ability to identify trends among data values and to associate those with an appropriate function type.

Key Concepts

- A **piecewise function** is a function that is defined by two or more expressions on separate portions of the domain.
- Such functions can be **continuous**, with no break in the graph of the function across a specified domain, or **discontinuous**, in which the graph of the function has a break, hole, or jump.
- A **discrete function** is a function in which every element of the domain is individually separate and distinct.
- Use a brace to show two (or more) pieces of the same function over different restricted domains. When stating the domain of the entire piecewise function, write the domain for each individual piece next to the appropriate expression. For example, for the piecewise function $a(x)$:

$$a(x)=\begin{cases}4.57 \bullet x; x \le 10\\ 3.43 \bullet x; x > 10\end{cases}$$

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- One piece of the function, $a(x) = 4.57 \bullet x$, represents the portion of the function over the restricted domain $x \leq 10$. The other piece, $a(x) = 3.43 \bullet x$, represents the portion of the function for which $x > 10$.
- An **absolute value function** is a function of the form $f(x) = |ax + b| + c$, where x is the independent variable and a, b, and c are real numbers. An absolute value function is a special type of piecewise function because the function values on either side of its maximum or minimum point are equal; for example, if $f(x) = |x - 6|$, then $f(3) = f(9)$. In general terms, an absolute value function can be represented by $f(x) = |x + a| + b$, where the maximum or minimum function value occurs at the point $(-a, b)$.
- For a value of $x = m$ in which $m < a$, or a value of $x = m$ in which $m > a$, the function value $f(m)$ is the same because the absolute value of the quantity $a + m$ is the same under both conditions, namely $|a + m| + b$.
- A **step function** (sometimes called a floor function or a postage function) is another special type of piecewise function that is discontinuous. A step function is a combination of one or more functions that are defined over restricted intervals and which may be undefined at other domain points or over other restricted domain intervals. In other words, a step function is a series of disconnected constant functions.
- A common type of step function is one in which the function values are restricted to integer domain values, such as the number of students who join the computer-gaming club—each student must be represented by a whole number (integer) value.
- By contrast, a function that models temperature changes throughout the day would not be restricted to integer domain values, since the temperature fluctuates by fractions of degrees.
- Just as with exponential, linear, and quadratic functions, domain and range variables for real-world problems involving piecewise functions may require adjustments to the axis scales when creating graphs and other visual aids.

Common Errors/Misconceptions

- applying a function model to a real-world problem without checking to see if the model's features and solutions actually explain the problem's data
- thinking of a piecewise function as a special type of step function rather than the other way around
- forgetting to check to see if the restricted domains of several functions making up a piecewise function together cover the complete domain of the piecewise function and its component parts

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Instruction

Guided Practice 4B.4.2

Example 1

A wholesale store sells paper plates for $3.27 per package, up to and including 50 packages purchased. If a customer buys more than 50 packages, the price of the entire purchase drops to $2.73 per package. Write a piecewise function for this pricing structure, and determine the price a deli owner would pay for 75 packages of plates.

1. Write a function for the price paid for 50 or fewer packages of plates.

   Let the total price be t, the price per package be p, and the number of packages be n. Thus, the total price, t, of n packages is $t(n)$.

   The general function $t(n) = p \bullet n$ represents the price paid for 50 or fewer packages of plates.

   We are given that the price per package is $3.27, up to and including 50 packages purchased; therefore, let $p = 3.27$.

   This function can be written as $t(n) = 3.27 \bullet n$, if $n \leq 50$.

2. Write a function for the price paid for more than 50 packages of plates.

   Again, let the total price be t, the price per package be p, the number of packages be n, and the total price, t, of n packages be $t(n)$.

   Thus, the general function from step 1, $t(n) = p \bullet n$, also represents the price paid for more than 50 packages of plates.

   However, the given price per package for more than 50 packages is $2.73; therefore, let $p = 2.73$.

   Substituting the new price of $2.73 for p, this function can be written as $t(n) = 2.73 \bullet n$, if $n > 50$.

3. Combine the two functions to show the "pieces" of the piecewise function and the restricted domain for each.

   Write the functions and restrictions for the two different pricing structures using appropriate notation.

$$t(n) = \begin{cases} 3.27 \bullet n; n \leq 50 \\ 2.73 \bullet n; n > 50 \end{cases}$$

SWB p. 117; SRB p. U4B-165

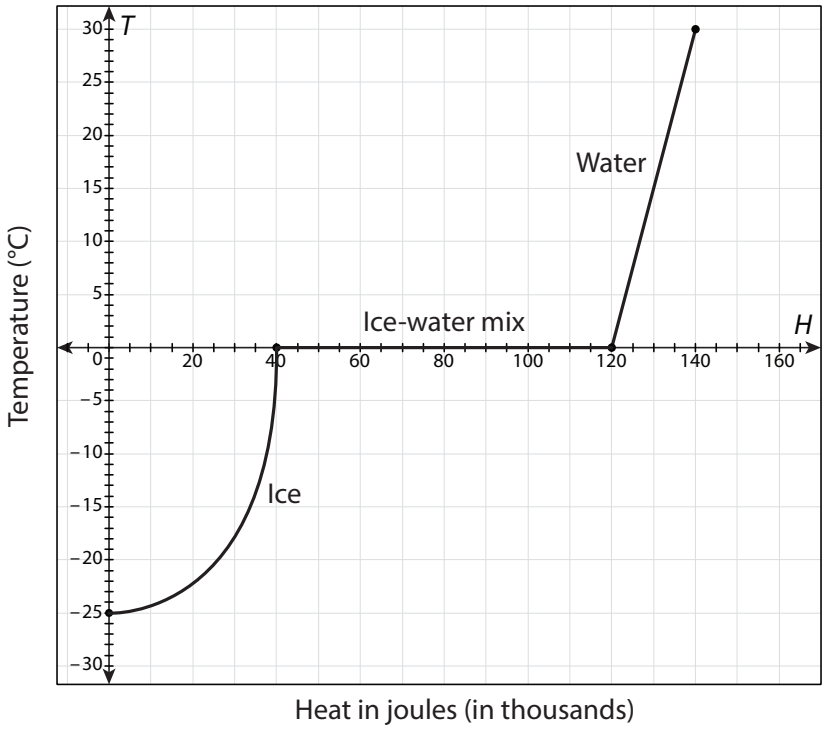
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4. Calculate the price paid for 75 packages of plates.

   The part of the piecewise function, $t(n)$, that applies to $n = 75$ is $t(n) = 2.73 \bullet n$. Therefore, the price for 75 packages of plates is $t(75) = 2.73 \bullet 75$ or \$204.75.

Example 2

When heat energy is added to ice that has a temperature less than the freezing point of water, the heat goes into raising the temperature of the ice until it starts to melt. When the ice starts to melt, all of the heat goes into melting the ice, not raising the temperature of the ice-water mixture. After the ice is melted, the heat goes back into raising the temperature of the water that results from the ice melting. The following graph shows these three distinct phenomena.



In this situation, the heat energy is added to the ice in such a way that it can be described by a quadratic function model. The "ice-water mix" and "water" parts of the heat-temperature function and graph are linear functions. Use a graphing calculator to derive a function for each of the three parts of this piecewise function. Then, write the complete function and its domain.

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Instruction

1. Determine the values of the three data points for the "ice" part of the function.

   Notice that the *H* or horizontal axis numbers have to be multiplied by 1,000 to yield the "true" values of *x* needed for finding the specific function equation. From the graph, it can be seen that the first point is at (0, –25). The second point appears to have an *H*-coordinate that is located at two thirds of the interval from 0 to 40 multiplied by 1,000. Two thirds of 40,000 is approximately 26,667. The *T*-coordinate of this point is –20; therefore, the second point is at approximately (26,667, –20). The third point has an *H*-coordinate of 40 times 1,000, so the third point's coordinates are (40,000, 0).

   The locations of the three data points can be approximated as (0, –25), (26,667, –20), and (40,000, 0).

2. Use the three data points for the "ice" part of the function to find the equation of the quadratic function.

   Use a graphing calculator to determine the equation of this part of the function. Follow the directions appropriate to your calculator model.

   On a TI-83/84:

   Step 1: Press [STAT] to bring up the statistics menu. The first option, 1: Edit, will already be highlighted. Press [ENTER].

   Step 2: Arrow up to L1 and press [CLEAR], then [ENTER], to clear the list. Repeat this process to clear L2 and L3 if needed.

   Step 3: Enter the three ordered pairs from example step 1 in the L1 and L2 lists. Make sure to enter the *H*-coordinates in L1 and the *T*-coordinates in L2.

   Step 4: Press [STAT]. Arrow to the CALC menu. Select 5: QuadReg.

   Step 5: Press [2ND][1] to type "L1" for Xlist. Arrow down to Ylist and press [2ND][2] to type "L2" for Ylist, if not already shown.

   Step 6: Arrow down to "Calculate" and press [ENTER].

(*continued*)

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On a TI-Nspire:

Step 1: Press [home]. Arrow over to the spreadsheet icon, the fourth icon from the left, and press [enter].

Step 2: To clear the lists in your calculator, arrow up to the topmost cell of the table to highlight the entire column, then press [menu]. Choose 3: Data, then 4: Clear Data. Repeat for each column as necessary.

Step 3: Arrow up to the topmost cell of the first column, labeled "A." Press [X][enter] to type x. Then, arrow over to the second column, labeled "B." Press [Y][enter] to type y.

Step 4: Arrow down to cell A1 and enter the first H-value from the ordered pairs in example step 1. Press [enter]. Enter the second H-value in cell A2 and so on.

Step 5: Move over to cell B1 and enter the first T-value. Press [enter]. Enter the second T-value in cell B2 and so on.

Step 6: To fit an equation to the data points, press [menu] and select 4: Statistics, then 1: Stat Calculations, then 6: Quadratic Regression. Select "x" from the X List pop-up menu. Press [tab] to move to the Y List pop-up menu, then select "y" from the options. Tab to "OK" and press [enter].

Either calculator will return values for the constants that can be substituted into the equation $y = ax^2 + bx + c$.

The result is the quadratic equation $y = (3.28135 \cdot 10^{-8})x^2 + (-6.875 \cdot 10^{-4})x - 25$.

3. Rewrite the resulting equation as a function using the variables H and T, and list the restricted domain over which the function is defined.

   The equation generated by the calculator for the "ice" part of the function can be written as shown:

   $T(H) = (3.28135 \cdot 10^{-8})H^2 + (-6.875 \cdot 10^{-4})H - 25$

   From the graph, it can be seen that the domain for this function is [0, 40,000].

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4. Write the equation for the "ice-water mix" part of the piecewise function, and list the restricted domain over which the function is defined.

   The ice-water mix is a horizontal line with a slope of 0 that would intersect the y-axis at 0. Therefore, the equation for this piece of the function can be written as $T(H) = 0$.

   An additional point appears to have an H-coordinate that is located at one-third of the interval from 120 to 160 multiplied by 1,000. One-third of 40,000 is approximately 13,333, so 120,000 + 13,333 is 133,333. The T-coordinate of this point is 0; therefore, the additional data point is at approximately (133,333, 0).

   From the graph, it can be seen that the domain for this function is approximately [40,000, 133,333].

5. Write the two points defining the "water" part of the function graph.

   From the graph, it can be seen that one point is (120,000, 0) and the other is (140,000, 30).

6. Write the equation for the "water" part of the function.

   First, calculate the slope of the line using the formula for slope, $m = \frac{y_2 - y_1}{x_2 - x_1}$. Substitute the points determined from the previous step, (140,000, 30) and (120,000, 0), and then solve for the slope, m.

   $$m = \frac{y_2 - y_1}{x_2 - x_1}$$ Slope formula

   $$m = \frac{(0) - (30)}{(120{,}000) - (140{,}000)}$$ Substitute 140,000 for x_1, 120,000 for x_2, 30 for y_1, and 0 for y_2.

   $$m = \frac{-30}{-20{,}000}$$ Simplify.

   $$m = 1.5 \bullet 10^{-3}$$

   The slope of the line is $m = 0.0015 \bullet 0.001$.

(*continued*)

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Use the point-slope formula, $y - y_1 = m(x - x_1)$, along with the slope and either of the two points from the previous step to write the equation for the "water" part of the function.

Let's use (120,000, 0).

$y - y_1 = m(x - x_1)$	Point-slope formula
$y - (0) = (0.0015 \bullet 0.001)[x - (120{,}000)]$	Substitute 120,000 for x_1, 0 for y_1, and $0.0015 \bullet 0.001$ for m.
$y = (0.0015 \bullet 0.001)x - 180$	Simplify.

The equation for the "water" part of the function is $y = (0.0015 \bullet 0.001)x - 180$.

7. Rewrite the resulting equation as a function using the variables H and T, and list the restricted domain over which the function is defined.

   The equation can be written as $T(H) = 0.0015H - 180$.

   From the graph, it can be seen that the domain for this function is [120,000, 140,000].

8. Write the complete piecewise function for $T(H)$, and determine its domain.

$$T(H) = \begin{cases} (3.28135 \bullet 10^{-8})H^2 + (-6.875 \bullet 10^{-4})H - 25; 0 \le H \le 40{,}000 \\ 0; 40{,}000 < H < 120{,}000 \\ (0.0015)H - 180; 120{,}000 \le H \le 140{,}000 \end{cases}$$

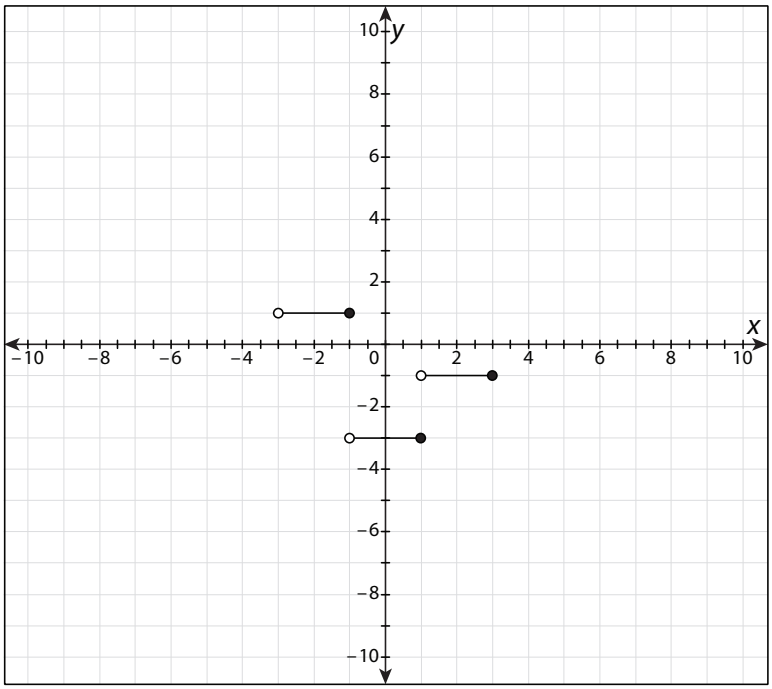
   Comparing the three pieces of the function and their restricted domains, the lowest domain value is 0, and the highest is 140,000. Both are included, as shown by the ≤ symbols. Therefore, the domain for the piecewise function $T(H)$ is [0, 140,000].

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Example 3

The graph shows a quadratic step function, $f(x)=\frac{3}{4}x^2-2x-\frac{7}{4}$. What domain restriction defines each step? Use the graph to write the equations of the steps of the quadratic function, and determine each step's restricted domain. Then, use this information to determine the domain of the quadratic function.



1. Write an equation of the form $y = a$ for each of the three function steps shown.

   The step equations are $y = 1$ (Quadrant II), $y = –3$ (Quadrants III and IV), and $y = –1$ (Quadrant IV).

2. Write the restricted domain for each equation.

   For $y = 1$, there is a closed circle at $x = –1$ and an open circle at $x = –3$. Therefore, the domain of $y = 1$ is $(–3, –1]$.

   For $y = –1$, there is an open circle at $x = 1$ and a closed circle at $x = 3$. Therefore, the domain of $y = –1$ is $(1, 3]$.

   For $y = –3$, there is an open circle at $x = –1$ and a closed circle at $x = 1$. Therefore, the domain of $y = –3$ is $(–1, 1]$.

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3. Use the restricted domains of the step functions to determine the rule that defines the domain of the quadratic function $f(x)$.

   The value of the quadratic function is given by the upper bound or limit of the interval over which each step is defined. Therefore, the domain of $f(x)$ is defined by the rule $\{x = -1, 1, \text{and } 3\}$.

4. Write the full function $f(x)$ along with the domain of the quadratic function.

   Use the pieces of the function, the given quadratic function, and each of the domains to write the step function $f(x)$.

$$f(x)=\begin{cases}\frac{3}{4}x^2-2x-\frac{7}{4};x=-1,1,3\\1;-3<x\le -1\\-3;-1<x\le 1\\-1;1<x\le 3\end{cases}$$

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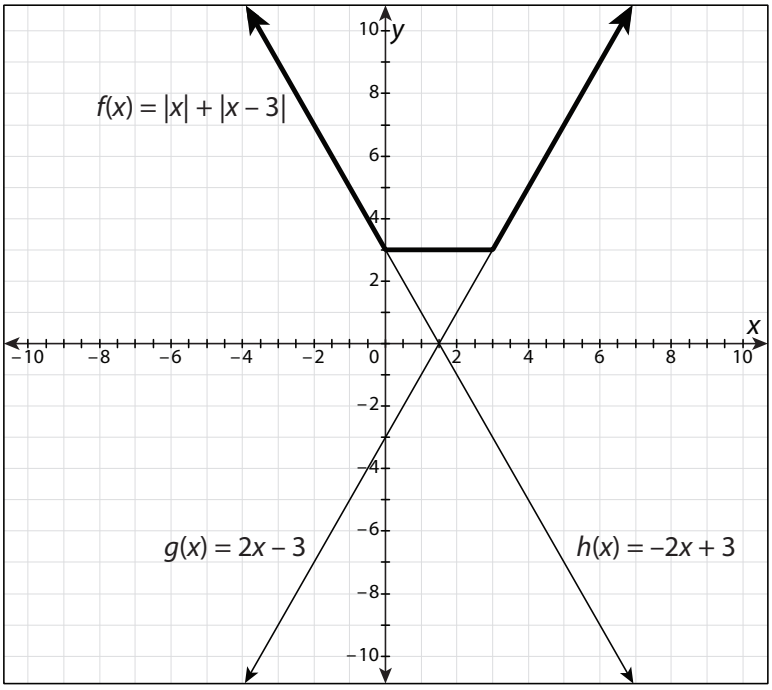
Instruction

Example 4

Use a graphing calculator to compare the graphs of the three functions $f(x)=|x|+|x-3|$, $g(x) = 2x - 3$, and $h(x) = -2x + 3$. How are the functions related? How are they alike and different?

1. Graph the functions using a graphing calculator.

   Enter each function into your graphing calculator. Graph all three functions on the same coordinate plane.



2. Describe the equations and domains of the three pieces of the piecewise function $f(x)$.

   Refer to the graph and the given equations.

   The piecewise function $f(x)$ has one piece that is defined by the rule $f(x) = -2x + 3$ for the restricted domain $(-\infty, 0)$, another that is defined by the rule $f(x) = 3$ for the restricted domain $[0, 3]$, and a third that is defined by the rule $f(x) = 2x - 3$ for the restricted domain $(3, \infty)$. All three rules comprise $f(x)$.

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3. Describe what the function $y = 3$ represents for $f(x)$.

   The minimum function value is 3, which occurs over the restricted domain [0, 3]. Therefore, $y = 3$ represents the minimum, or lowest value, of the function $f(x)$.

4. Determine the intersection point of the graphs of $g(x)$ and $h(x)$, and describe what it represents.

   Based on the graph of the functions, the intersection point is (1.5, 0). This point represents the domain value at which the function values are equal.

5. Describe the meaning of the overlapping portions of the graphs of the function pair $f(x)$ and $g(x)$ as well as the function pair $f(x)$ and $h(x)$.

   The overlap means that the function pairs share the same points for those domain values. It also means that the overlapping domains are solutions for the pairs of functions.

6. Determine the solutions for the function pair $f(x)$ and $g(x)$ and the function pair $f(x)$ and $h(x)$.

   Based on the graphs and the overlapping domains, the solution set for $f(x)$ and $g(x)$ is given by the domain interval $[3, \infty)$. The solution set for $f(x)$ and $h(x)$ is given by the domain interval $(-\infty, 0]$.

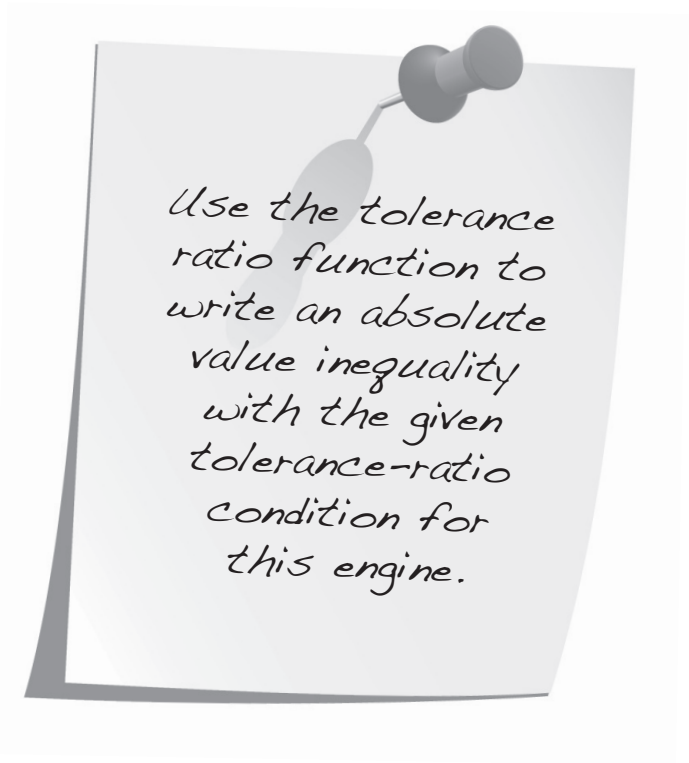
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Problem-Based Task 4B.4.2: Motorcycle Engine Tolerances

SMP	
1 ✓	2 ✓
3	4 ✓
5	6 ✓
7 ✓	8

Aliza is a mechanical engineer who designs motorcycle engines. She uses a mathematical model to maximize the combustion and compression efficiency of each engine she designs. Since the piston of the latest engine Aliza designed will be manufactured at a different factory from the combustion chamber into which it fits, there will be small but significant variations in the radii of the pistons and the combustion chambers. An absolute-value function, called the tolerance ratio, compares the amount of allowable variation, r, in the radius of the piston to the amount of allowable variation in the radius of the combustion chamber. These allowable variations are equal for all practical purposes, but are accounted for in Aliza's function model. (*Note*: r is in millimeters.)

Aliza has decided that the tolerance ratio for the new engine has to be less than 0.9 in order for the engine to meet quality expectations. Use the tolerance ratio function, $T(r)=\frac{|r-0.001|}{|r+0.002|}$, to write an absolute value inequality with the given tolerance-ratio condition for this engine. Then, calculate the range of values of r that meet the criteria.



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Problem-Based Task 4B.4.2: Motorcycle Engine Tolerances

Coaching

a. How can you rewrite the simple tolerance-ratio function $T(r)$ in order to incorporate the quality expectations without using the given equation of the function?

b. Combine the given equation for $T(r)$ with the result of part a to write a mathematical statement for the tolerance-ratio condition of this engine.

c. How many different values are possible for each absolute-value term?

d. How many combinations of values result from part c?

e. Use the results of parts b and d to write the mathematical statements needed to determine the possible values for r.

f. Solve each mathematical statement from part e for r.

g. Which outcome or outcomes make sense in terms of meeting the quality expectations for the engine? Explain.

h. How can you combine the results of part g to determine a range of values of r that will meet the quality expectations for the engine?

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Problem-Based Task 4B.4.2: Motorcycle Engine Tolerances

Coaching Sample Responses

a. How can you rewrite the simple tolerance-ratio function $T(r)$ in order to incorporate the quality expectations without using the given equation of the function?

The tolerance ratio has to be less than 0.9 in order for the engine to meet its quality expectations. Therefore, the simple tolerance ratio function can be written as $T(r) < 0.9$.

b. Combine the given equation for $T(r)$ with the result of part a to write a mathematical statement for the tolerance-ratio condition of this engine.

Since $T(r)$ in the tolerance-ratio equation $T(r)=\frac{|r-0.001|}{|r+0.002|}$ must be less than 0.9 for this engine, the mathematical statement for this condition is $\frac{|r-0.001|}{|r+0.002|}<0.9$.

c. How many different values are possible for each absolute-value term?

Each absolute-value term can be positive or negative; also, the absolute-value quantity in the numerator could be 0, but the quantity in the denominator cannot if the function is to exist.

d. How many combinations of values result from part c?

There are four possibilities:

For the numerator: $r - 0.001 > 0$ or $r - 0.001 < 0$

For the denominator: $r + 0.002 > 0$ or $r + 0.002 < 0$

The numerator can also be written as $|r-0.001|=r-0.001$ or $0.001 - r$, and the denominator can be written as $|r+0.002|=r+0.002$ or $-r - 0.002$.

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e. Use the results of parts b and d to write the mathematical statements needed to determine the possible values for r.

Create a table using the four cases from part d to efficiently organize the results.

$\lvert r-0.001\rvert$	$\lvert r-0.002\rvert$	Function ratio	Simplified statement
>0	>0	$\frac{r-0.001}{r+0.002}<0.9$	$r-0.001<0.9(r+0.002)$
>0	<0	$\frac{r-0.001}{-r-0.002}<0.9$	$r-0.001<0.9(-r-0.002)$
<0	<0	$\frac{-r+0.001}{-r-0.002}<0.9$	$0.001-r<0.9(-r-0.002)$
<0	>0	$\frac{-r+0.001}{r+0.002}<0.9$	$0.001-r<0.9(r+0.002)$

f. Solve each mathematical statement from part e for r.

For $r-0.001<0.9(r+0.002)$, the value is $r<0.028$ mm.

For $r-0.001<0.9(-r-0.002)$, the value is $r<-0.000421$ mm.

For $0.001-r<0.9(-r-0.002)$, the value is $r>0.028$ mm.

For $0.001-r<0.9(r+0.002)$, the value is $r>-0.000421$ mm.

g. Which outcome or outcomes make sense in terms of meeting the quality expectations for the engine? Explain.

Alone, none of the answers makes physical sense, because each implies an open-ended domain of r-values: $(-\infty,0.028)$, $(-\infty,-0.000421)$, $(0.028,\infty)$, and $(-0.000421,\infty)$. However, by combining the results, domains can be produced that do make physical sense.

h. How can you combine the results of part g to determine a range of values of r that will meet the quality expectations for the engine?

The restricted domains can be combined to determine a range of r-values expressed by the compound inequality -0.000421 mm $< r < 0.028$ mm. Remember that r is the range of variation in the radius of the piston or combustion chamber that will meet the engine's tolerance-ratio criteria, expressed by $T(r) < 0.9$. It is not the actual radius of the piston or combustion chamber.

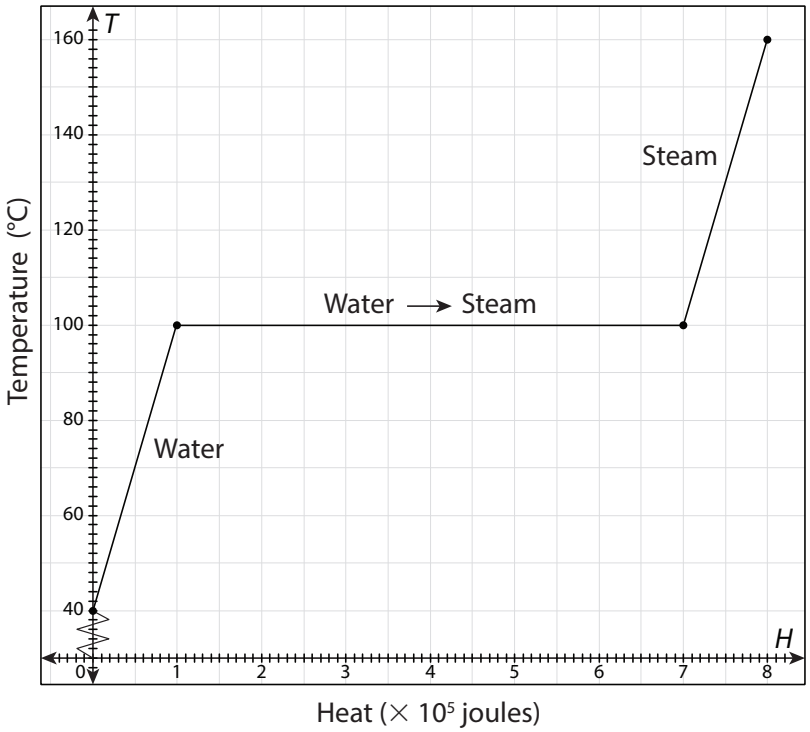
Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4B.4.2: Piecewise, Step, and Absolute Value Functions

The graph shows the temperature change that occurs when water is heated and changes from a liquid to a gas (steam), where temperature is T and heat is H. Notice that the vertical T-axis begins at 40°C, which is the initial temperature of the water. Use the graph to complete problems 1–4.



1. Describe the pieces of this piecewise function.

2. Derive the temperature function $T(H)$ for the heat domain interval when the water temperature rises from 40°C to 100°C.

3. Determine the equation for the heat domain values for $T = 100$°C.

4. Derive the temperature function $T(H)$ for the heat domain interval when the water temperature rises above 100°C.

continued

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A square wave is a type of periodic wave function whose amplitude changes suddenly instead of gradually (like a sine wave does), resulting in a "square" graph instead of smooth curves. The data in the table represent the output voltage of a square wave used for switching an electric circuit off and on automatically. The function model $V(t)$ represents the voltage, and time t is the independent variable. Dashes represent values of t for which the function is undefined. Use this information and the table to complete problems 5–7.

t	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	π
$V(t)$	—	4	4	4	—	–4	–4	–4

5. Explain how the results in the table imply that the function model is a step function.

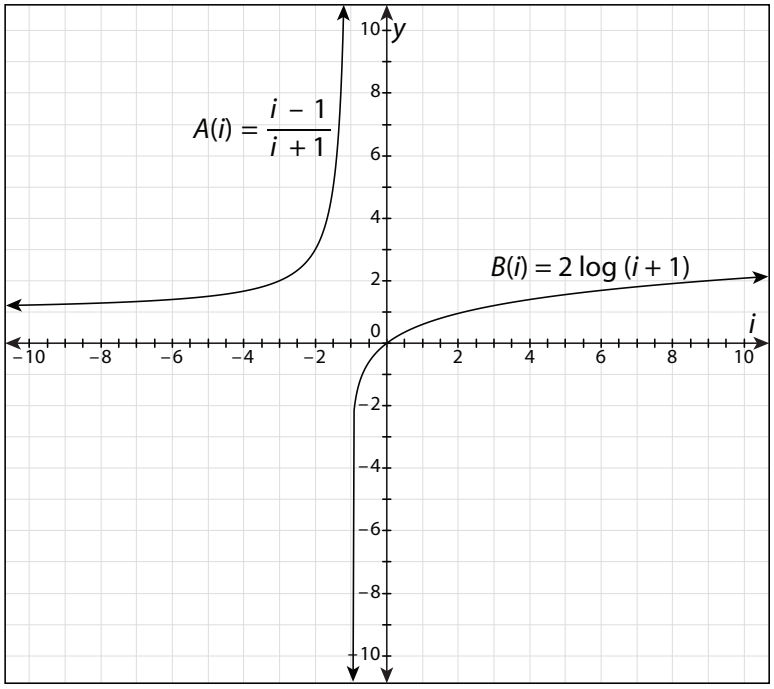
6. The output voltage is periodic. What is its period?

7. The function that generated the values in the table is $V(t)=\frac{4\bullet|\sin 2t|}{\sin 2t}$. Show how this function could generate the table function values.

continued

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The graph shows how the efficiency of an audio speaker varies with sound intensity. Efficiency is the speaker's ability to transform power into sound. Use the graph and the information that follows to complete problems 8–10.



The vertical asymptote $i = -1$ describes the intensity at which the ideal efficiency of the speaker for its designed frequency range is achieved. The functions shown represent models for how the efficiencies $A(i)$ and $B(i)$ vary at intensities above and below the efficiency ideal at $i = -1$. The function $A(i)$ applies to the intensity interval $(-\infty,-1)$ and models the efficiency of the speaker for very loud sounds. The function $B(i)$ applies to the intensity interval $(-1,\infty)$ and models the efficiency of the speaker for very faint sounds. (*Note*: The "efficiency" is a dimensionless number that is related to a variety of design and engineering factors that can have large positive and negative values for highly efficient speakers over a variety of frequency and power input-output ranges.)

8. How does $A(i)$ change as the intensity approaches –1?

9. How does $B(i)$ change as the intensity approaches –1?

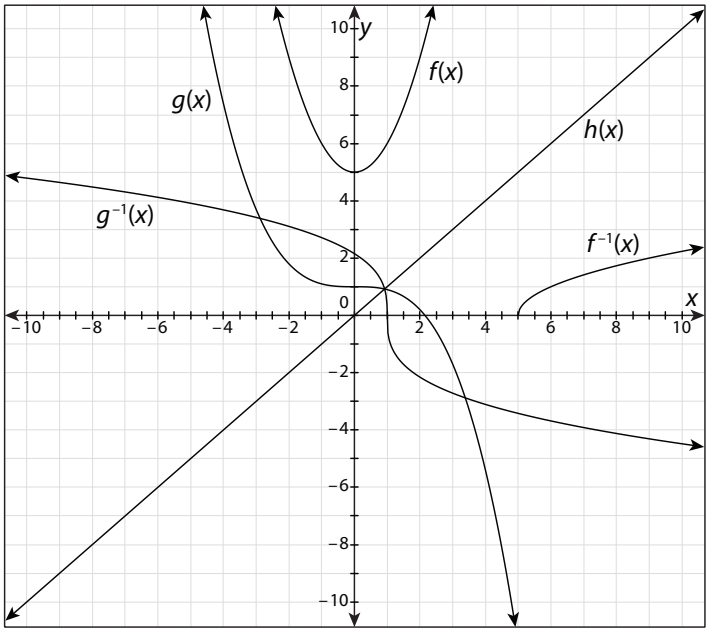
10. How do the values of both functions change as the intensity changes from about –100 to about –1 and from about –1 to about 100?

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Lesson 4B.4.3: Square Root and Cube Root Functions

Warm-Up 4B.4.3

The following graph shows two functions, $f(x)$ and $g(x)$, used to represent the optimal dimension, x, of a shipping container's surface area and volume. The surface area function, $f(x) = x^2 + 5$, is a quadratic function, and its inverse is $f^{-1}(x) = \sqrt{x-5}$. The volume function, $g(x) = -0.1x^3 + 1$, is a cubic function, and its inverse is $g^{-1}(x) = \sqrt[3]{10-10x}$. The function $h(x) = x$ is the line across which each function and its inverse are reflected.



1. Is the inverse of $f(x)$ a function? Explain.

2. What is the domain of $f(x)$?

3. What is the domain of the inverse of $f(x)$?

4. Is the inverse of $g(x)$ a function? Explain.

5. What is the domain of $g(x)$?

6. What is the domain of the inverse of $g(x)$?

7. Why do the domains of the inverses of $f(x)$ and $g(x)$ differ?

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Instruction

Lesson 4B.4.3: Square Root and Cube Root Functions

Common Core State Standards

F–IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F–IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

F–BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Warm-Up 4B.4.3 Debrief

1. Is the inverse of $f(x)$ a function? Explain.

   If both the positive and negative square roots of an inverse function are shown on a graph, then the inverse is not a function. Showing both means that each domain value will be associated with two function values, which violates one-to-one correspondence for functions. If one or the other of the positive and negative square roots is shown, it will be a function because it will show one-to-one correspondence between its domain and function values.

   On this graph, only the positive half of the inverse, $f^{-1}(x) = \sqrt{x-5}$, is shown. Therefore, $f^{-1}(x)$ is a function because it shows one-to-one correspondence between its domain and function values.

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Instruction

2. What is the domain of $f(x)$?

   It can be seen from the graph that the domain of $f(x)$ is all real numbers, or $(-\infty, \infty)$.

3. What is the domain of the inverse of $f(x)$?

   The domain of the inverse of $f(x)$, $f^{-1}(x)$, is $[5, \infty)$ because its graph has an endpoint at 5.

4. Is the inverse of $g(x)$ a function? Explain.

   $g^{-1}(x)$ is a function because its graph shows one-to-one correspondence between its domain and function values.

5. What is the domain of $g(x)$?

   It can be seen from the graph that the domain of $g(x)$ is all real numbers, or $(-\infty, \infty)$.

6. What is the domain of the inverse of $g(x)$?

   The domain of the inverse of $g(x)$, $g^{-1}(x)$, is also all real numbers, or $(-\infty, \infty)$.

7. Why do the domains of the inverses of $f(x)$ and $g(x)$ differ?

   The positive and negative halves of $f^{-1}(x)$ are functions individually, but not when both are graphed together, since this violates the criteria for one-to-one correspondence. Also, $f^{-1}(x)$ is undefined for values of x that are less than or equal to 5 because it is a square root function.

   $g^{-1}(x)$ is a function because it meets the one-to-one correspondence criteria for functions. It is also defined for all values of x since x can be less than, equal to, or greater than 0 for a cube root function like $g(x)$.

Connection to the Lesson

- Students will recognize the domain and other limitations of cube root functions and square root functions.
- Students will use the fact that quadratic and square root functions are inverses of each other to solve real-world problems.
- Students will use the fact that special cases of cubic functions and cube root functions are inverses of each other to solve real-world problems.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- writing the inverse of a quadratic function and its domain (F-BF.4)
- writing the inverse of a cubic function and its domain (F-BF.4)
- identifying and using the features of the inverses in solving real-world problems (F-BF.4)

Introduction

Cubic functions and quadratic functions have many applications in real-world problems. The inverses of those functions can be just as useful in modeling situations in which the independent variable in a function model is inaccessible, immeasurable, or otherwise not available for use in a calculation. While the domain of a cubic function and its inverse function are often the same, the domain of a quadratic function is often not the same as the domain of its inverse because of restrictions on the value of variables under the radical sign of the inverse. And, as is always the case with real-world problems, the domains of both types of functions must reflect the realities of real-world conditions and factors.

Key Concepts

- A **cube root function** is a function that contains the cube root of a variable. The general form is $f(x)=a\sqrt[3]{(x-h)-k}$, where a, h, and k are real numbers.
- A **square root function** contains a square root of a variable. Square root functions have the general form $f(x)=\sqrt{ax^2+bx+c}$, where a, b, and c are real numbers.
- Cube root and square root functions, also known as radical functions, are the inverses of cubic and quadratic functions, respectively.
- In both functions, any of the constants a, b, c, and d can be equal to 0, so many such functions are simpler than these general forms.
- The highest power of x does not exceed the reciprocal of the root in this general form because it is assumed that any higher powers of x have been factored out of the terms that are under the radical sign.
- Just as with absolute value, step, and other piecewise functions, domain and range variables for real-world problems involving cube root or square root functions may require adjusting the scale of their graphs to suit the situation.

SRB p. U4B-178

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Common Errors/Misconceptions

- applying a function model to a real-world problem without checking to see if the model's features and solutions actually explain the problem's data
- confusing cubic functions with cube root functions, or quadratic functions with square root functions, and vice versa
- choosing the domain instead of the range and vice versa in applying cube root and square root function models to real-world problems

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Guided Practice 4B.4.3

Example 1

A marine biologist measures the volume of a roughly spherical piece of coral by submerging it in a container of seawater of known volume. The coral displaces a volume of $\frac{4}{375}\pi$ cubic meters of seawater. What is the radius of the coral? The function for the volume of the coral is given by $V(r)=\frac{4}{3}\pi r^3$, where r is the radius of the sphere.

1. Determine the value of r.

   Let $V(r)$ be $\frac{4}{375}\pi$ in the formula for the volume of a sphere. Solve for r.

$V(r)=\frac{4}{3}\pi r^3$	Formula for the volume of a sphere
$\left(\frac{4}{375}\pi\right)=\frac{4}{3}\pi r^3$	Substitute $\frac{4}{375}\pi$ for $V(r)$.
$\frac{4}{125}\pi=4\pi r^3$	Multiply both sides by 3.
$\frac{1}{125}=r^3$	Divide both sides by 4π.
$r=\sqrt[3]{\frac{1}{125}}$	Take the cube root of both sides.
$r=\frac{1}{5}=0.2$	Simplify.

   The radius of a sphere with a volume of $\frac{4}{375}\pi$ cubic meters is 0.2 meter.

2. Rewrite the result of step 1 in the context of the problem.

   The radius of the piece of coral is roughly 0.2 meter, or about 20 centimeters.

SWB p. 131; SRB p. U4B-179

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Example 2

An ecologist drops a data-collection buoy into a river from a bridge. With what velocity does the buoy hit the river surface if the acceleration due to gravity has a magnitude of about 32 feet per second squared and the bridge is 100 feet above the surface of the river? The formula relating an object's acceleration, distance, and velocity is $v(h)^2 = 2ah$, where a is the acceleration, h is the height, and v is the final velocity in feet per second.

1. Rewrite the given formula as a radical function.

$v(h)^2 = 2ah$	Original function
$v(h) = \pm\sqrt{2ah}$	Take the square root of both sides.

The related radical function is $v(h) = \pm\sqrt{2ah}$.

2. Assign signs to the constants given in the problem, and explain the choice of signs so that the real-world conditions of the problem are met.

If the acceleration due to gravity is defined as a negative value ($a = -32$ feet per second squared) because the object is falling toward Earth, then the height h also has to be negative so that the expression under the radical, $2ah$, will be defined. However, both a and h can be defined as positive as long as all other conditions of the problem (in this case, v, the velocity) are similarly defined.

3. Use the radical function and the defined constants to find the value of v.

Use the negative square root option because the quantities describing the journey of the buoy from the bridge to the river are defined as negative.

$v(h) = -\sqrt{2ah}$	Negative radical function
$v(-100) = -\sqrt{2(-32)(-100)}$	Substitute –100 for h and –32 for a.
$v(-100) = -\sqrt{6400}$	Simplify.
$v(-100) = -80$	Take the square root.

The velocity of the buoy when it hits the river is –80 feet per second.

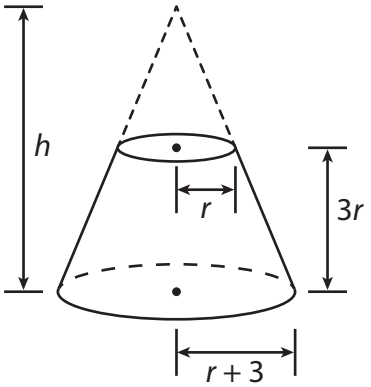
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Example 3

A 3-D printer is programmed to produce a plastic game piece. From top to bottom, the game piece has parallel circular bases with radii of r centimeters and $r + 3$ centimeters, respectively, as shown in the diagram. The distance between the circular bases is $3r$ centimeters. The dashed lines on the diagram represent the imaginary tip of the cone that would result if the sides of the game piece continued to that point.

The volume of plastic in cubic centimeters used to produce the game piece is known, but the parts engineer would like to have a function model for the volume that can be solved for r. Use the diagram and the general formula for the volume of a cone, $V_{cone} = \frac{1}{3}\pi r^2 h$, to write a function for the volume of the game piece as a function of r. Determine the domain over which the volume exists. Then, write the volume function of the game piece in such a way that it matches the general form of a cube root function, $f(x) = \sqrt[3]{ax^3 + bx^2 + cx + d}$. Finally, write the cube root function that has to be solved for a game piece volume of 27π cubic centimeters, and use it to calculate the actual height of the game piece and the radii of its bases.



1. Write an expression for the height of the imaginary small cone, shown with dashed lines atop the smaller circular base of the game piece.

   The height of the combined game piece and the small cone is h. The height of the game piece is $3r$. Therefore, the height of the small cone is $h - 3r$.

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2. Write an equation for finding h in terms of r.

   Let $r + 3$ be the radius of the base of the large cone formed by the game piece and the small cone, let r be the radius of the base of the small cone, let h be the height of the large cone, and let $h - 3r$ be the height of the small cone.

   Recall that sides of similar triangles are proportional. Use this fact to set up a proportion comparing the large cone to the small cone, and solve this proportion for h: $\frac{r}{r+3}=\frac{h-3r}{h}$.

$\frac{r}{r+3}=\frac{h-3r}{h}$	Sides of similar triangles are proportional
$rh=(r+3)(h-3r)$	Cross multiply.
$rh=rh-3r^2+3h-9r$	Simplify.
$3r^2+9r=3h$	Isolate h.
$r^2+3r=h$	Simplify.
$r(r+3)=h$	Factor out r.
$h=r(r+3)$	Apply the Symmetric Property of Equality.

   The equation for finding h in terms of r is $h=r(r+3)$.

3. Write an equation for the volume of the game piece in terms of the volumes of both cones in the diagram.

   Let V_{GP} represent the volume of the game piece, and V_{LC} and V_{SC} represent the volumes of the large and small cones, respectively.

   The volume of the game piece is equal to the difference in volumes of the large and small cones, so the function is given by $V_{GP} = V_{LC} - V_{SC}$.

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4. Use the known values for r and h of the large cone and the general equation for the volume of a cone to write an equation for the volume of the large cone.

Recall that the general formula for the volume of a cone is $V_{\text{cone}} = \frac{1}{3}\pi r^2 h$.

The diagram shows that the radius of the large cone is $r + 3$ and the height is h.

From step 2, we know that the equation for h in terms of r is $h = r(r + 3)$.

$V_{\text{cone}} = \frac{1}{3}\pi r^2 h$	Formula for the volume of a cone
$(V_{LC}) = \frac{1}{3}\pi (r+3)^2 h$	Substitute V_{LC} for V_{cone} and $r + 3$ for r.
$V_{LC} = \frac{1}{3}\pi (r+3)^2 [r(r+3)]$	Substitute $r(r + 3)$ for h.
$V_{LC} = \frac{1}{3}\pi r(r+3)^3$	Rearrange and combine terms.

The formula for the volume of the large cone in terms of r is $V_{LC} = \frac{1}{3}\pi r(r+3)^3$.

Write an equation for the volume of the small cone.

From step 1, we know that the height of the small cone is $h - 3r$.

Recall that the equation for h in terms of r is $h = r(r + 3)$.

Therefore, the height of the small cone can also be written as $[r(r + 3)] - 3r$, which simplifies to $r^2 + 3r - 3r$.

Substitute this value into the general volume formula and simplify.

$V_{\text{cone}} = \frac{1}{3}\pi r^2 h$	Formula for the volume of a cone
$(V_{SC}) = \frac{1}{3}\pi r^2 (r^2 + 3r - 3r)$	Substitute V_{SC} for V_{cone} and $r^2 + 3r - 3r$ for h.
$V_{SC} = \frac{1}{3}\pi r^4$	Simplify and combine terms.

The formula for the volume of the small cone in terms of r is $V_{SC} = \frac{1}{3}\pi r^4$.

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5. Write a specific equation for the volume of the game piece.

   Use the volume formulas for the large and small cones to write an equation for the volume of the game piece.

   Recall the general equation of the game piece from step 3: $V_{GP} = V_{LC} - V_{SC}$.

$V_{GP} = V_{LC} - V_{SC}$	Formula for the volume of the game piece
$(V_{GP}) = \left[\frac{1}{3}\pi r(r+3)^3\right] - \left[\frac{1}{3}\pi r^4\right]$	Substitute values for the large and small cones.
$V_{GP} = \frac{1}{3}\pi r(r^3+9r^2+27r+27) - \frac{1}{3}\pi r^4$	Expand the cubic term.
$V_{GP} = \cancel{\frac{1}{3}\pi r^4} + 3\pi r^3 + 9\pi r^2 + 9\pi r - \cancel{\frac{1}{3}\pi r^4}$	Cancel and combine terms.
$V_{GP} = 3\pi r(r^2+3r+3)$	Simplify.

   Therefore, the specific equation for the volume of the game piece is $V_{GP} = 3\pi r(r^2+3r+3)$.

6. Determine the domain for r such that the volume of the game piece exists and is a positive number.

   The radius r is a real-world quantity, so it is positive; therefore, $r > 0$. Thus, the minimum volume of the game piece occurs as r approaches 0.

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7. Use the specific equation for V_{GP} to write the cubic equation that has to be solved for r if the volume of the game piece is 27π cubic centimeters.

   Substitute 27π into the specific equation for V_{GP}, then simplify.

$V_{GP} = 3\pi r(r^2 + 3r + 3)$	Formula for the volume of the game piece
$(27\pi) = 3\pi r(r^2 + 3r + 3)$	Substitute 27π for V_{GP}.
$9 = r(r^2 + 3r + 3)$	Divide both sides by 3π.
$9 = r^3 + 3r^2 + 3r$	Distribute.
$0 = r^3 + 3r^2 + 3r - 9$	Subtract 9 from both sides.

   The cubic equation to be solved for r is $0 = r^3 + 3r^2 + 3r - 9$.

8. Use a graphing calculator to find the value(s) of r that can result in a volume of 27π cubic centimeters.

   The equation found in the previous step represents the difference between the function V_{GP} and the desired volume, 27π. To estimate a value for r, plot the function $r^3 + 3r^2 + 3r - 9$ and estimate where it crosses the x-axis. *Note*: Results may vary depending on your viewing window and calculator model.

   - On a TI-83/84, tracing the graph shows that the solution lies between $x = 1.06$ and $x = 1.28$ in the standard window.
   - On a TI-Nspire, tracing the graph yields an x-value of 1.15444 for a y-value of 0.

   The value of r that gives a game piece volume of 27π cubic centimeters is about 1.15 centimeters. This means that the game piece has bases with radii of about 1.15 and 4.15 centimeters, respectively, and a height of about 3.45 centimeters.

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Example 4

Exoplanets are planets that exist outside of Earth's solar system. As astronomers discover new exoplanets orbiting other stars in our galaxy, they find that some of the same relationships governing the motion of planets in our own solar system apply to those exoplanets, too. One of these is the function $T(r)=\left(\frac{2\pi}{\sqrt{GM}}\right)\cdot r^{\frac{3}{2}}$. It relates the length of a "year," $T(r)$, of a planet orbiting a star with an orbit that has an average radius of r, with "year" referring to the amount of time the exoplanet takes to complete one orbit. Compare the average orbital radii of two exoplanets orbiting the same star if one of the exoplanets has a year that is twice as long as the other. (*Note*: The quantities G and M are constants—the universal gravitation constant and the mass of the star about which the planet is orbiting—and do not figure into the calculations for this example.)

1. Write the relationship of the year of the first exoplanet to that of the exoplanet whose year is twice as long.

   If the year of one exoplanet is T, then the year of the other exoplanet is $2T$.

2. Write the function for the exoplanet with the shorter year, and label its average orbital radius r_1.

   Let T be the length of a year. Substitute T for $T(r)$ and r_1 for r in the given function, $T(r)=\left(\frac{2\pi}{\sqrt{GM}}\right)\cdot r^{\frac{3}{2}}$:

   $$T=\left(\frac{2\pi}{\sqrt{GM}}\right)\cdot r_1^{\frac{3}{2}}$$

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Instruction

3. Rearrange the resulting function to solve for r_1.

$T=\left(\frac{2\pi}{\sqrt{GM}}\right)\bullet r_1^{\frac{3}{2}}$	Function from the previous step
$T^2=\left(\frac{2\pi}{\sqrt{GM}}\right)^2\bullet r_1^{\left(\frac{3}{2}\right)^2}$	Square both sides.
$T^2=\left(\frac{4\pi^2}{GM}\right)\bullet r_1^3$	Simplify.
$\left(\frac{GM}{4\pi^2}\right)T^2=\left(\frac{\cancel{GM}}{\cancel{4\pi^2}}\right)\bullet\left(\frac{\cancel{4\pi^2}}{\cancel{GM}}\right)\bullet r_1^3$	Multiply both sides by the reciprocal of the constant term in parentheses; simplify.
$\frac{GMT^2}{4\pi^2}=r_1^3$	Multiply the terms on the left side.
$\sqrt[3]{\frac{GMT^2}{4\pi^2}}=r_1$	Take the cube root of both sides.

The average orbital radius of the exoplanet with the shorter year is $r_1=\sqrt[3]{\frac{GMT^2}{4\pi^2}}$.

4. Write the function for the exoplanet with the longer year, and label its average radius r_2.

Let $2T$ be the length of a year. Substitute $2T$ for $T(r)$ and r_2 for r in the given function, $T(r)=\left(\frac{2\pi}{\sqrt{GM}}\right)\bullet r^{\frac{3}{2}}$:

$$2T=\left(\frac{2\pi}{\sqrt{GM}}\right)\bullet r_2^{\frac{3}{2}}$$

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5. Rearrange the resulting function to solve for r_2.

$2T = \left(\frac{2\pi}{\sqrt{GM}}\right) \bullet r_2^{\frac{3}{2}}$	Function from the previous step
$4T^2 = \left(\frac{2\pi}{\sqrt{GM}}\right)^2 \bullet r_2^{\left(\frac{3}{2}\right)^2}$	Square both sides.
$4T^2 = \left(\frac{4\pi^2}{GM}\right) \bullet r_2^3$	Simplify.
$\left(\frac{GM}{4\pi^2}\right) \bullet 4T^2 = \left(\frac{\cancel{GM}}{\cancel{4\pi^2}}\right) \bullet \left(\frac{\cancel{4\pi^2}}{\cancel{GM}}\right) \bullet r_2^3$	Multiply both sides by the reciprocal of the constant term; simplify.
$\frac{GMT^2}{\pi^2} = r_2^3$	Multiply the terms on the left side.
$\sqrt[3]{\frac{GMT^2}{\pi^2}} = r_2$	Take the cube root of both sides.

The average orbital radius of the exoplanet with the longer year is $r_2 = \sqrt[3]{\frac{GMT^2}{\pi^2}}$.

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6. Compare the average orbital radii of the two exoplanets.

Start by simplifying the formula for r_1.

$r_1 = \sqrt[3]{\frac{GMT^2}{4\pi^2}}$ Orbital radius of the exoplanet with the shorter year

$r_1 = \sqrt[3]{\frac{1}{4} \bullet \frac{GMT^2}{\pi^2}}$ Factor $\frac{1}{4}$ from the argument.

$r_1 = \sqrt[3]{\frac{1}{4}} \bullet \sqrt[3]{\frac{GMT^2}{\pi^2}}$ Rewrite the cube root as a product.

Notice that one of the factors of r_1 is equal to the equation for the larger exoplanet: $r_2 = \sqrt[3]{\frac{GMT^2}{\pi^2}}$. Therefore, the equation for r_1 can be rewritten in terms of r_2 by substituting r_2 into $r_1 = \sqrt[3]{\frac{1}{4}} \bullet \sqrt[3]{\frac{GMT^2}{\pi^2}}$.

$r_1 = \sqrt[3]{\frac{1}{4}} \bullet \sqrt[3]{\frac{GMT^2}{\pi^2}}$ Simplified equation for r_1

$r_1 = \sqrt[3]{\frac{1}{4}} \bullet (r_2)$ Substitute r_2 for $\sqrt[3]{\frac{GMT^2}{\pi^2}}$.

The resulting equation for r_1 is $r_1 = \sqrt[3]{\frac{1}{4}} \bullet r_2$.

The average orbital radius of the exoplanet with the shorter year is $\sqrt[3]{\frac{1}{4}}$ (or about 0.63) times the average orbital radius of the exoplanet with the longer year.

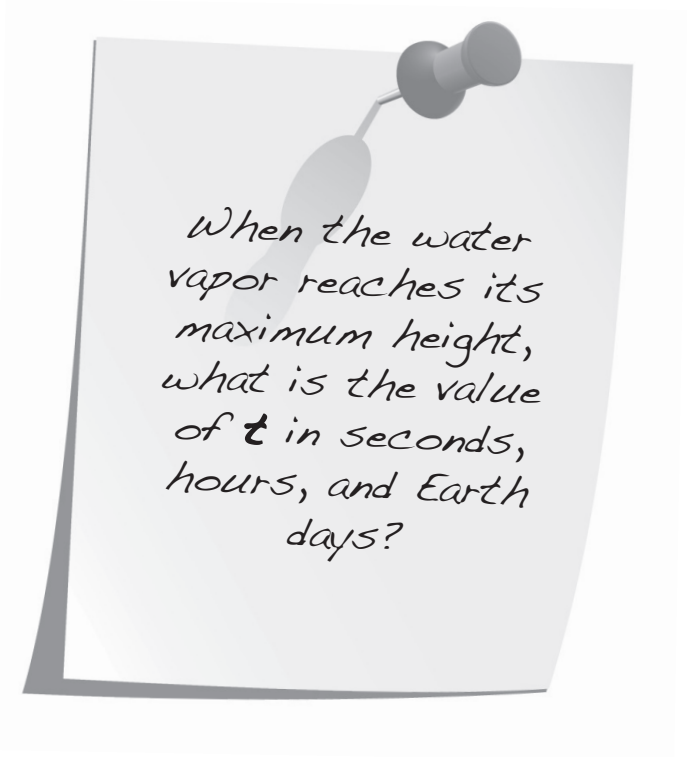


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Problem-Based Task 4B.4.3: Europa's Water-Vapor Geysers

SMP	
1 ✓	2 ✓
3	4
5	6
7 ✓	8

Water is one of the necessary conditions for life as it is defined on Earth. Photos taken of Jupiter's moon, Europa, in 2012 by the Hubble Space Telescope show plumes of water vapor reaching 200 kilometers above the moon's surface. The function describing the height reached by the water vapor can be modeled by the function $h(t)=v_0t+\frac{1}{2}at^2$. The variable $h(t)$ is the height of the plume in meters, v_0 is the initial velocity of the water vapor molecules in meters per second as they leave the surface of Europa, t is the time in seconds, and a is Europa's gravitational acceleration, which is –0.01 meter per second squared if $v_0 > 0$ and $h(t) > 0$. Assume the water molecules are ejected from Europa's surface at a speed of 500 meters per second. When the water vapor reaches its maximum height, what is the value of t in seconds, hours, and Earth days?



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Problem-Based Task 4B.4.3: Europa's Water-Vapor Geysers

Coaching

a. What are the given values in the problem? State your answer in terms of numerical values and units of measurement.

b. Which one of these values' units should be changed in order for all of the units to be of the same scale? Perform the unit conversion for this value.

c. Rewrite the given function for $h(t)$ using the values found in parts a and b, then simplify the result.

d. What is the value of t?

e. What are the restrictions on the domain of t? How do they affect the choice of answers in part d?

f. Approximate the value of t in seconds, hours, and Earth days.

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Instruction

Problem-Based Task 4B.4.3: Europa's Water-Vapor Geysers

Coaching Sample Responses

a. What are the given values in the problem? State your answer in terms of numerical values and units of measurement.

The height, h, reached by the water molecules is 200 kilometers, the acceleration of gravity is –0.01 meter per second squared, and the initial velocity of the molecules is 500 meters per second.

b. Which one of these values' units should be changed in order for all of the units to be of the same scale? Perform the unit conversion for this value.

Convert the height of 200 kilometers to meters to match the other units that include meters.

$$200 \text{ kilometers} \cdot \frac{1000 \text{ meters}}{1 \text{ kilometer}} = 200{,}000 \text{ meters}$$

Therefore, 200 kilometers is equal to 200,000 or $2 \cdot 10^5$ meters.

c. Rewrite the given function for $h(t)$ using the values found in parts a and b, then simplify the result.

Substitute –0.01 for a and $2 \cdot 10^5$ for $h(t)$ in the given function $h(t) = v_0 t + \frac{1}{2}at^2$.

$$h(t) = v_0 t + \frac{1}{2}at^2$$
$$\left(2 \cdot 10^5\right) = 500 \cdot t + \frac{1}{2}(-0.01)t^2$$
$$4 \cdot 10^5 = 1000 \cdot t - 0.01 \cdot t^2$$
$$0.01 \cdot t^2 - 1000 \cdot t + 4 \cdot 10^5 = 0$$
$$t^2 - 10^5 \cdot t + 4 \cdot 10^7 = 0$$

d. What is the value of t?

Use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, which will result in a square-root equation for t.

For $t^2 - 10^5 \cdot t + 4 \cdot 10^7 = 0$, $x = t$, $a = 1$, $b = -10^5$, and $c = 4 \cdot 10^7$. Substitute and simplify.

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$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$(t)=\frac{-(-10^5)\pm\sqrt{(-10^5)^2-4(1)(4\bullet10^7)}}{2(1)}$$

$$t=\frac{10^5\pm\sqrt{10^{10}-16\bullet10^7}}{2}$$

$$t=5\bullet10^4\pm\frac{\sqrt{1\bullet10^{10}-0.016\bullet10^{10}}}{2}$$

$$t=5\bullet10^4\pm5\bullet10^4\sqrt{0.984}$$

e. What are the restrictions on the domain of t? How do they affect the choice of answers in part d?

The domain of t is $[0,\infty)$, so the positive value of the square root has to be used. This gives $t=5\bullet10^4+5\bullet10^4\sqrt{0.984}$ seconds.

f. Approximate the value of t in seconds, hours, and Earth days.

Recall that t is time in seconds. Simplify the value of t from part e.

$$t=5\bullet10^4+5\bullet10^4\sqrt{0.984}$$

$$t\approx99{,}600 \text{ seconds}$$

Convert the result to hours, and then convert that result to Earth days using the ratios $\frac{1 \text{ hour}}{3600 \text{ seconds}}$ and $\frac{1 \text{ Earth day}}{24 \text{ hours}}$.

$$t\approx99{,}600 \text{ seconds}\bullet\frac{1 \text{ hour}}{3600 \text{ seconds}}\approx28 \text{ hours}$$

$$t\approx28 \text{ hours}\bullet\frac{1 \text{ Earth day}}{24 \text{ hours}}\approx1.15 \text{ days}$$

It takes a little longer than one Earth day for a water-vapor plume to reach 200 kilometers above Europa if it is ejected from Europa's surface at a speed of 500 meters per second.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4B.4.3: Square Root and Cube Root Functions

For problems 1–3, write the equation of each square root function described.

1. the area, $A(x)$, of a right triangle with a side of length x and a hypotenuse of length $2x - 1$

2. the function value, $f(x)$, in an ordered pair, $(x, f(x))$, on a circle of radius 5 with its center at the origin of a coordinate system. (*Hint*: The equation for a circle of radius r and with its center at (0, 0) is $x^2 + y^2 = r^2$.)

3. the average velocity, v, of an object of mass m and known kinetic energy K if kinetic energy is given by $K(m,v)=\frac{1}{2}mv^2$. (*Note*: The notation "$K(m, v)$" means that the kinetic energy is a function of mass and velocity.)

For problems 4–6, write the equation of each cube root function described.

4. the radius, r, of a cylindrical container of known height $2r$ and volume $V(r)$. (*Hint*: The volume formula for a cylinder is $V = \pi r^2 h$.)

5. the time, t, it takes for the concentration of a chemical in an industrial process to reach a known value if the concentration is modeled by the function $c(t)=\left(\frac{t}{t+1}\right)^3$

6. the basic territorial volume unit, b, of the habitat of a population of steelhead trout if the population, $p(b)$, is modeled by the function $p(b)=\frac{250}{b^3}$

continued

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Use the information given in each scenario to complete problems 7–10.

7. The speed at which a softball is released can be modeled by the function $s(x)^2 = 2ax$. One pitcher, Daisy, has an acceleration of a meters per second squared, and accelerates the softball over a distance of x meters. A second pitcher, Gemma, has the same acceleration as the first but she has longer arms, so the softball is accelerated over a distance of $1.25x$ meters. How much faster is Gemma's pitch than Daisy's?

8. At what value of r is the known surface area of a sphere, given by the function $S(r) = 4\pi r^2$, numerically equal to the known volume of the sphere given by the function $V(r) = \frac{4}{3}\pi r^3$?

9. The time, $T(r)$, required for a small exoplanet to complete one orbit around its star is given by the function $T(r) = \left(\frac{2\pi}{\sqrt{GM}}\right) \bullet r^{\frac{3}{2}}$, where G and M are constants. Write an equation comparing the length of one orbit for this small planet (i.e., one "year") to that of another exoplanet orbiting the same star with an average orbital radius that is twice as large.

10. The volume of a rectangular solid with dimensions of x, $2x$, and $3x$ feet is 12 cubic feet. How does the value of x for the same relationship among the dimensions compare if the volume is 24 cubic feet?

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Progress Assessment

Circle the letter of the best answer.

1. A study by a fishing industry trade organization found that the California sardine catch each year during a 50-year period varied according to the data in the table. The value $t = 1$ corresponds to the decade of the 1930's. The catch $C(t)$ is in hundreds of millions of pounds.

t	1	2	3	4	5	6
$C(t)$	1.2	2.9	4.0	4.3	4.1	3.6

Which function form models this data if the domain of t is a positive integer and a, b, c, h, and k are all greater than 0?

a. $f(x) = ax^2$

b. $f(x) = -ax^2$

c. $f(x) = ax^2 + bx + c$

d. $f(x) = -a(x - h)^2 + k$

2. The table shows the percentage, p, of retail sales revenues from online retailers from 2002 through 2012, with the year 2002 corresponding to $t = 1$. Which type of function *best* models this data?

t	1	2	3	4	5	6	7	8	9	10	11
p	1.4	1.8	2.1	2.5	2.9	3.3	3.6	3.8	4.2	4.6	5.1

a. absolute value

b. exponential

c. linear

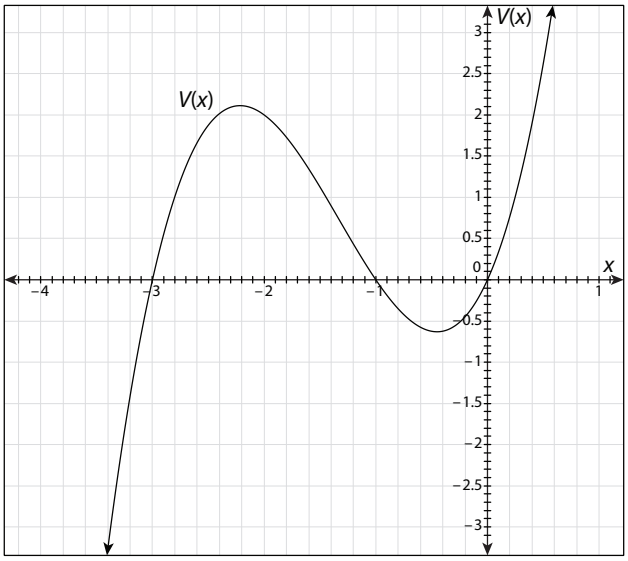
d. logarithmic

continued

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Assessment

3. The graph shows the volume of soil excavated from an environmental clean-up site. For what value of x does the volume of excavated soil reach a maximum value for the cubic function shown?



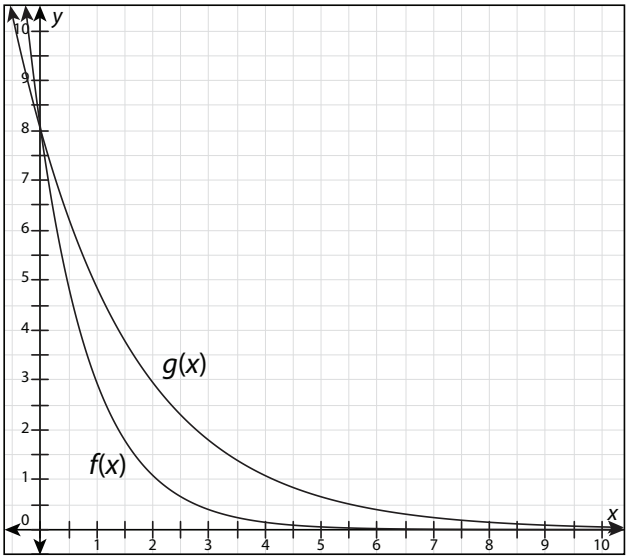
a. $x \approx -3.0$

b. $x \approx -2.2$

c. $x \approx -1.0$

d. $x \approx 0.0$

4. The graph shows the decay of radioactive elements in two rock samples represented by the exponential functions $f(x)$ and $g(x)$. Which equation forms fit the graphs of $f(x)$ and $g(x)$, assuming a, b, and c are all greater than 0?



a. $f(x) = ae^{-bx}$ and $g(x) = ae^{-cx}$

b. $f(x) = ae^{bx}$ and $g(x) = ae^{cx}$

c. $f(x) = ae^{-bx}$ and $g(x) = ce^{-bx}$

d. $f(x) = ae^{bx}$ and $g(x) = ce^{bx}$

continued

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Assessment

5. Manatees often graze for food in Florida's Crystal River. The abundance of vegetation available for manatee grazing varies with the deviation of the monthly mean temperature from the mean annual temperature. For temperatures that fall below the mean (represented by negative x-values), the abundance varies according to the quadratic function model $A_1(d) = -kd^2$. For temperatures that fall above the mean (represented by positive x-values), the abundance varies according to the exponential function model $A_2(d) = d \bullet e^{-kd}$. What is this an example of?

   a. a composition function

   b. a composition function and its inverse

   c. a combination function

   d. a piecewise function

6. A variable-volume cylindrical gas tank at a fuel storage facility is filled weekly via a pipeline. The pipeline's rate of flow is modeled by the linear function $R = 2400t$, in which t is time in minutes and the rate, R, is in kiloliters per minute. The variable-volume tank has a radius of 50 meters and an adjustable height. The height changes according to the linear function $h = 0.8R$. The time-dependent volume-function model $V(t) = \pi r^2 h$ incorporates both of these functions. What type of function is $V(t)$?

   a. a combination function of the form $(h \bullet R)(t)$

   b. a combination function of the form $(h + R)(t)$

   c. a composition function of the form $(R \circ h)(t)$

   d. a composition function of the form $(h \circ R)(t)$

7. An absolute-value function has the general form $f(x) = \left|\frac{x+a}{x+b}\right|$. What is the specific function if $f(1) = 1$ and $f(2) = 3$?

   a. $f(x) = \left|\frac{x-1}{x-3}\right|$

   b. $f(x) = \left|\frac{x+1}{x+3}\right|$

   c. $f(x) = \left|\frac{x-1}{x+3}\right|$

   d. $f(x) = \left|\frac{x+1}{x-3}\right|$

continued

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Assessment

8. The piecewise function $f(x)$ is given below. Over which restricted domain is $f(x)$ continuous?

$$f(x)=\begin{cases}4; x\le -3\\ 3x-2; -3<x<3\\ x^2-2; x\ge 3\end{cases}$$

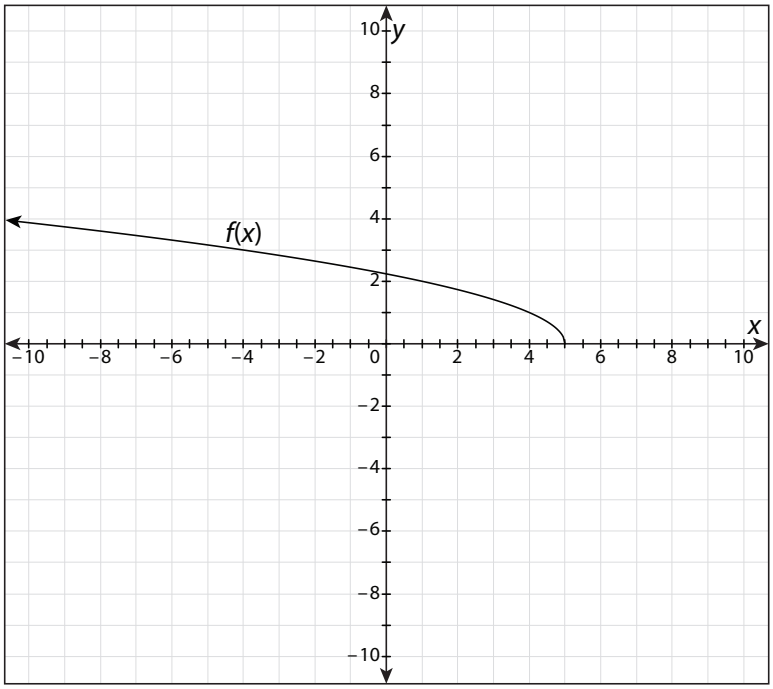
a. $(3,\infty)$

b. $(-\infty,-3]\cap[3,\infty)$

c. $(-3,\infty)$

d. $(-\infty,\infty)$

9. What is the degree of the inverse of $f(x)$?



a. 3

b. 2

c. 1

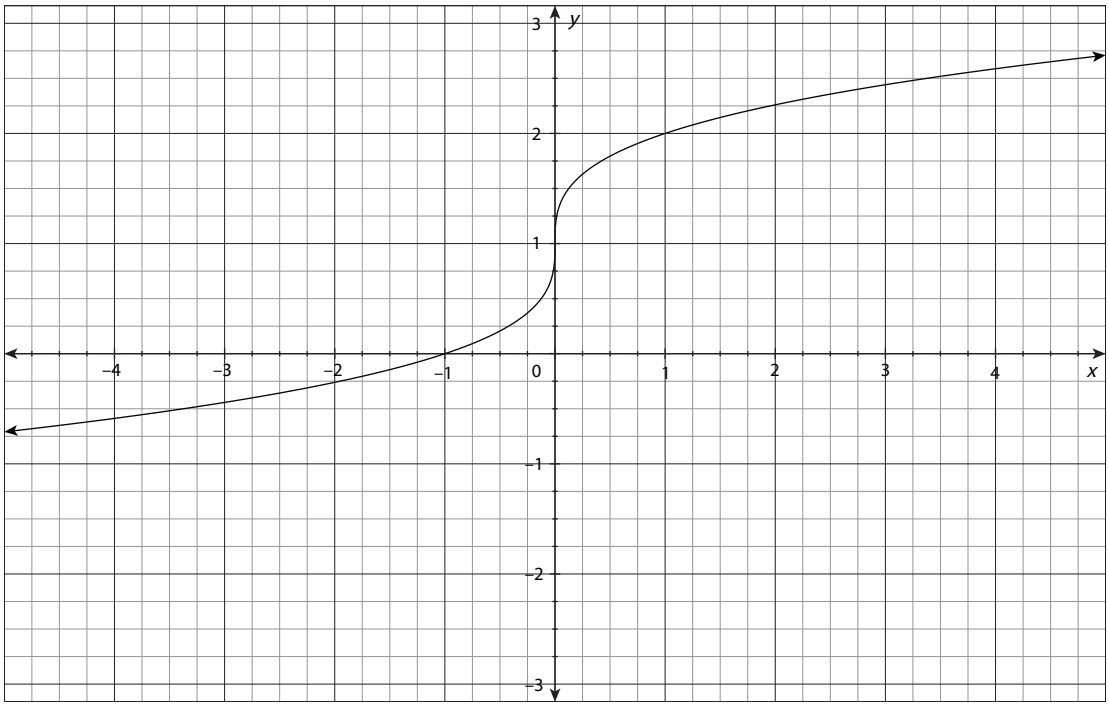
d. $\frac{1}{2}$

continued

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Assessment

10. What is the degree of the inverse of $f(x)$? Assume that the function has no other solutions.



a. 3

b. 2

c. 1

d. $\frac{1}{3}$

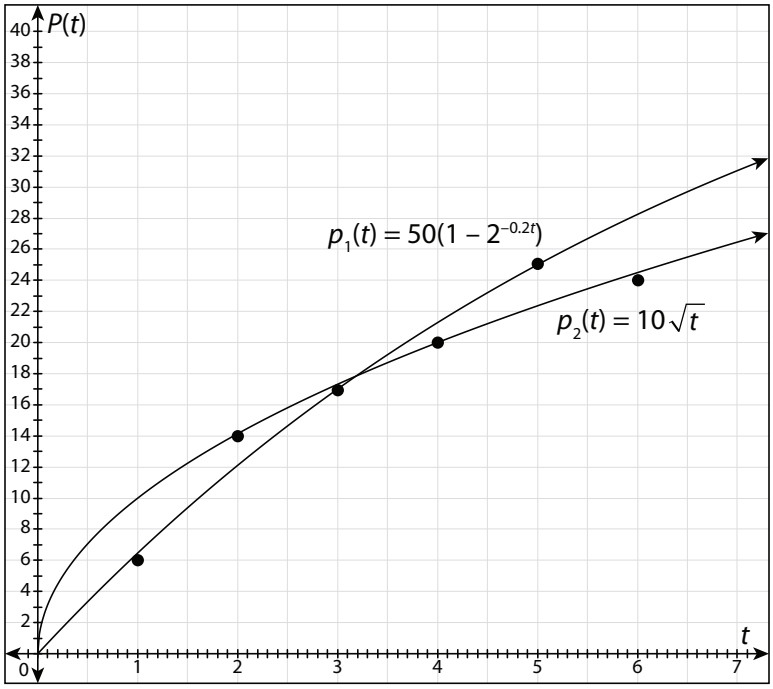
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Assessment

Use the given information, the graph, and the table to complete all parts of problem 11.

11. A wildlife biologist and her assistants have been studying the behavior of wild turkeys in a long-needle pine forest over several years. The graph represents two function models used by the biologists that relate the elapsed time, t, in years and the net population, $p(t)$, of turkeys (in tens) in the same habitat from year to year. The table contains the values of the graphed data points.



Year	1	2	3	4	5	6
Turkeys (in tens)	6	14	17	20	25	24

a. What advantage(s) does one model have over the other?

b. Based on the table data and the combination functions you have seen in this lesson, what change(s) could be made to the exponential function to make it a better fit for the data points?

c. How is the function model in part b like the square root function?

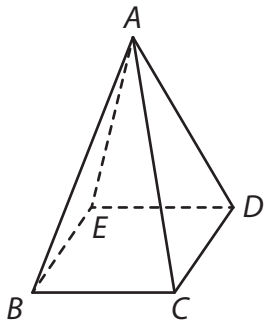
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Assessment

Pre-Assessment

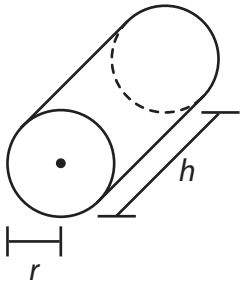
Circle the letter of the best answer.

1. The sides of the base of the following solid figure are congruent, and $\angle BCD$ is a right angle. What plane figures, and how many total figures, make up the solid?



   a. 4 triangles; 4 total figures
   b. 4 triangles and 1 square; 5 total figures
   c. 1 triangle and 1 square; 2 total figures
   d. 2 triangles; 2 total figures

2. What figures make up the cylinder shown, and which of those figures are three-dimensional in the cylinder?



   a. 2 circles of radius r and 1 rectangle of dimensions h by $2\pi r$; 1 circle of radius r is three-dimensional
   b. 1 circle and 1 rectangle; the rectangular surface of the cylinder is three-dimensional
   c. 1 rectangle of dimensions h by $2\pi r$; 1 circle of radius r is three-dimensional
   d. 1 rectangle of dimensions h by $2\pi r$ and 2 circles of radius r; the rectangular surface of the cylinder is three-dimensional

continued

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Assessment

3. The state wildlife service found that 4 breeding pairs of cougars live on 4,800 acres of a state park. When not breeding, cougars are solitary animals. What is the average territory for each cougar if the cougars do not cross into one another's territory?

   a. 38,400 acres per cougar
   b. 19,200 acres per cougar
   c. 600 acres per cougar
   d. 1,200 acres per cougar

4. A small calorie is the amount of energy needed to raise the temperature of 1 gram of water by 1°C. A large or "food" calorie is equal to 1,000 small calories. There are 100 food calories in a 250-gram can of light clam chowder. How many food calories does each gram of clam chowder contain?

   a. 25,000 food calories per gram
   b. 2.5 food calories per gram
   c. 0.4 food calorie per gram
   d. 4 food calories per gram

5. As a dancer spins, her voluminous skirt of length L inches flares out to form a cone shape. The widest the skirt is at the bottom is D inches when she is spinning. Which statement describes the shape of the skirt?

   a. a cone of base diameter $\frac{1}{2}D$ and height $\sqrt{L^2-(0.5D)^2}$
   b. a cone of base diameter $\frac{1}{2}D$ and height $\sqrt{L^2-D^2}$
   c. a cone of base diameter D and height $\sqrt{L^2-(0.5D)^2}$
   d. a cone of base diameter D and height $\sqrt{L^2-D^2}$

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Lesson 5: Geometric Modeling

Instruction

SMP
1 ✓ 2 ✓
3 ✓ 4 ✓
5 ✓ 6
7 8 ✓

Common Core State Standards

G–GMD.4 Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

G–MG.1 Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).★

G–MG.2 Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★

G–MG.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★

Essential Questions

1. What plane geometric figure is formed when a plane intersects a solid figure?
2. What solid figure is formed when a plane figure is rotated a specific number of degrees or radians?
3. How can the density of a quantity be computed for the area or volume in which the quantity is located?
4. How can density be used to compute the total amount of a quantity in an area or a volume?
5. How can the properties of geometric figures and their measurements be used to design solutions to real-world problems?

WORDS TO KNOW

axis of rotation	a line about which a plane figure can be rotated in three-dimensional space to create a solid figure, such as a diameter or a symmetry line
congruent	having the same shape, size, lines, and angles; the symbol for representing congruency between figures is ≅
cross section	the plane figure formed by the intersection of a plane with a solid figure, where the plane is at a right angle to the surface of the solid figure

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Instruction

density	the amount, number, or other quantity per unit of area or volume of some substance or population being studied
plane	a flat, two-dimensional figure without depth that has at least three non-collinear points and extends infinitely in all directions
plane figure	a two-dimensional shape on a plane
polyhedron	a three-dimensional object that has faces made of polygons
regular polyhedron	a polyhedron with faces that are all congruent regular polygons; the angles created by the intersecting faces are congruent, and the cross sections are similar figures
***rho* (ρ)**	a lowercase Greek letter commonly used to represent density
rotation	in three dimensions, a transformation in which a plane figure is moved about one of its sides, a fixed point, or a line that is not located in the plane of the figure, such that a solid figure is produced
similar figures	two figures that are the same shape but not necessarily the same size; the symbol for representing similarity between figures is $\sim$.
solid figure	a three-dimensional object that has length, width, and height (depth)
translation	in three dimensions, the horizontal or vertical movement of a plane figure in a direction that is not in the plane of the figure, such that a solid figure is produced

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Instruction

Recommended Resources

- Annenberg Learner. "Cross Sections."

  http://www.walch.com/rr/00249

  This site provides an overview of cross sections, along with an interactive cube that can be rotated and cut in various locations to show the resulting cross section. Requires Flash.

- LearnZillion.com. "Predict 3D Results of Rotating Simple Figures."

  http://www.walch.com/rr/00250

  This video tutorial reviews how rotations of a plane figure about an axis can be used to create three-dimensional solids.

- MathIsFun.com. "Solid Geometry."

  http://www.walch.com/rr/00251

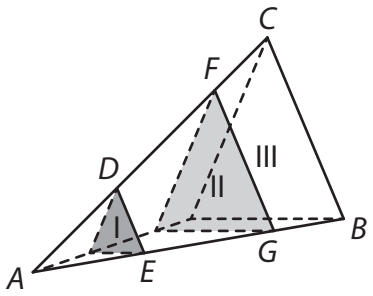
  This site reviews the geometry of three-dimensional objects; in particular, it covers properties of solids and compares polyhedra to non-polyhedra, with examples of each.

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Lesson 4B.5.1: Two-Dimensional Cross Sections of Three-Dimensional Objects

Warm-Up 4B.5.1

The blueprints for a planned science and nature museum include an aviary that will be in the shape of a triangular pyramid, as shown in the diagram. The aviary will be divided into three sections—I, II, and III—in order to accommodate the different environmental needs of different species of birds. The sections will be formed by two parallel planes that intersect the pyramid to form the shaded triangles I and II. Use the diagram to answer the questions.



1. How are triangles I and II different from the pyramid base, triangle III?

2. How are all three triangles alike?

3. How does the ratio of length $\overline{AD}$ to length $\overline{DF}$ compare to the ratio of length $\overline{AE}$ to length $\overline{EG}$?

4. Which ratio is the same as the ratio of length $\overline{DF}$ to length $\overline{DC}$?

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Instruction

Lesson 4B.5.1: Two-Dimensional Cross Sections of Three-Dimensional Objects

Common Core State Standards

G–GMD.4 Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

G–MG.1 Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).★

Warm-Up 4B.5.1 Debrief

1. How are triangles I and II different from the pyramid base, triangle III?

   The areas and lengths of the corresponding sides of the triangles are unequal.

2. How are all three triangles alike?

   The corresponding angles of the triangles are congruent, which means that the three triangles are similar.

3. How does the ratio of length $\overline{AD}$ to length $\overline{DF}$ compare to the ratio of length $\overline{AE}$ to length $\overline{EG}$?

   The ratio of length $\overline{AD}$ to length $\overline{DF}$ equals the ratio of length $\overline{AE}$ to length $\overline{EG}$ because they are corresponding sides of a plane figure (the pyramid face, $\triangle ABC$) that is intersected by two parallel line segments, $\overline{DE}$ and $\overline{FG}$.

   Symbolically, the ratios can be represented by a proportion: $\frac{AD}{DF}=\frac{AE}{EG}$.

4. Which ratio is the same as the ratio of length $\overline{DF}$ to length $\overline{DC}$?

   The line segments $\overline{DF}$ and $\overline{DC}$ correspond to the line segments $\overline{EG}$ and $\overline{EB}$. Therefore, the ratio of length $\overline{DF}$ to length $\overline{DC}$ equals the ratio of length $\overline{EG}$ to length $\overline{EB}$.

   Symbolically, the ratios can be represented by a proportion: $\frac{DF}{DC}=\frac{EG}{EB}$.

Connection to the Lesson

- Students will apply basic axioms, rules, and theorems of plane geometry to problems involving solid figures.
- Students will recognize plane geometric figures formed by the intersection of a plane with a solid figure.
- Students will calculate areas, distances, ratios, and other derived quantities in problems involving plane and solid geometric figures.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- applying knowledge of ratio and proportion (7.RP.3)
- recognizing and applying the properties of basic plane geometry figures, such as circles and polygons (G–C.2, G–CO.11)
- recognizing and applying the properties of basic solid geometry figures, such as cones, pyramids, and spheres (8.G.9)

Introduction

A chef takes a knife and slices a carrot in half. What shape results? Depending on the direction of the cut, the resulting shape may resemble a circle (for a widthwise cut), a triangle (for a lengthwise cut from the tip of the carrot to its base), or an oval (for a diagonal cut). The circle, triangle, and oval are each a type of **plane figure**, a two-dimensional shape on a plane. Recall that a **plane** is a flat, two-dimensional figure without depth that is defined by three non-collinear points and extends infinitely in all directions. When planes intersect with **solid figures**, or three-dimensional objects that have length, width, and height (or depth), plane figures are created. In this example, the carrot can be thought of as the solid figure, the knife as the intersecting plane, and the resulting shape as the plane figure.

Plane figures have many applications in real-world problems. Technical drawings of solid figures, such as the parts of a redesigned spacecraft, often rely on back, bottom, front, and rear views of a figure that exists in three dimensions or in three-dimensional coordinate systems. These applications can be as complex as the real-world circumstances in which they are found, but many rely on relatively basic properties of plane and solid geometry for their description and use in calculations.

Similarly, two-dimensional figures can be used to generate solid figures with transformations in three-dimensional space. Solids can be generated by the horizontal or vertical movement of a plane figure through a third dimension, or by the rotation of a plane figure about an axis of rotation or line in a third dimension.

Key Concepts

- In its basic form, a **cross section** is the plane figure formed by the intersection of a plane and a solid figure. In this lesson we will focus on cross sections where the plane is at a right angle to the surface of the solid figure or, in the cases of rotated figures such as a cone or sphere, at a right angle to the axis of the figure.
- An **axis of rotation** is a line about which a plane figure can be rotated in three-dimensional space to create a solid figure, such as a diameter or a symmetry line. For example, a line that connects the centers of the circular bases of a cylinder is an axis of rotation.

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Instruction

- The shape of a cross section depends on the solid figure and the location of the cross section.
- A **polyhedron** is a three-dimensional solid that has faces that are polygons. Such polygonal plane figures include squares, rectangles, and trapezoids. **Regular polyhedra** are solids that have faces that are all congruent regular polygons. (Recall that **congruent** means having the same shape, size, lines, and angles; the symbol for representing congruency between figures is $\cong$.)
- Cross sections of some polyhedra such as pyramids form similar figures. **Similar figures** are two figures that are the same shape but not necessarily the same size. In other words, corresponding sides have equal ratios. The symbol for representing similarity between figures is $\sim$.
- A **right prism** is a three-dimensional solid that has two congruent bases and rectangular faces that join the two bases at 90° angles. In a prism, cross sections that are parallel to the bases are congruent figures.
- An infinite number of plane figures can be formed by the intersection of a plane with a solid figure, but the cross section is a special case due to its perpendicularity with the surface of the solid.
- A solid figure can be generated by translating a plane figure along one of its two dimensions. In three dimensions, **translation** refers to the horizontal or vertical movement of a plane figure in a direction that is not in the plane of the figure, such that a solid figure is produced. Imagine a pop-up cylindrical clothes hamper—it can be compressed into a flat circle for storage, but when you need to use it you lift up the top part to "open" it up into its cylindrical form.
- A solid figure can also be generated by the rotation of a plane figure in three dimensions. A **rotation** in three dimensions is a transformation in which a plane figure is revolved about one of its sides or a line that isn't in the plane of the figure, such that a solid figure is produced. Similarly, imagine a toothpick with a small paper flag attached. Twirling the toothpick quickly back and forth between your fingers causes the paper flag to form the illusion of a cylinder.

Common Errors/Misconceptions

- thinking that a cross section can be formed by *any* intersection of a plane and a solid figure
- labeling cross sections formed by planes and solids as having linear or cubic units rather than square units
- thinking that the axis of rotation of a plane figure is a single point instead of a line

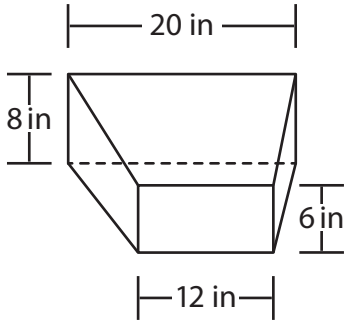
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Guided Practice 4B.5.1

Example 1

Compare the areas of two rectangular cross sections of an air-conditioning duct along a length of the duct that shrinks in width from 20 inches to 12 inches and in height from 8 inches to 6 inches. Find the difference of the areas, and determine the percentage that each cross section is of the other. Then, name the solid figure from which the two rectangular cross sections are formed.



1. Calculate the area of the larger cross section.

   The larger cross section is a rectangle, so use the formula for the area of a rectangle, *area* = *length* • *width*. In this case, the "length" is actually the height, so use the formula *area* = *height* • *width*, or $A = hw$.

   The larger cross section has a height of 8 inches and a width of 20 inches. Substitute these measurements into the formula $A = hw$ and then solve to determine the area of the larger cross section.

$A = hw$	Rectangular area formula
$A = (8)(20)$	Substitute 8 for h and 20 for w.
$A = 160$	Simplify.

   Recall that area is expressed in square units.

   Thus, the area of the larger cross section is 160 square inches.

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Instruction

2. Calculate the area of the smaller cross section.

The smaller cross section is also a rectangle, so use the formula $A = hw$. This cross section has a height of 6 inches and a width of 12 inches. Substitute these measurements into the formula $A = hw$ and then solve to determine the area of the smaller cross section.

$A = hw$	Rectangular area formula
$A = (6)(12)$	Substitute 6 for h and 12 for w.
$A = 72$	Simplify.

The area of the smaller cross section is 72 square inches.

3. Find the difference of the cross section areas.

Subtract the area of the smaller cross section from the area of the larger cross section.

$160 - 72 = 88$

The difference of the areas is 88 square inches.

4. Calculate the percentage that the smaller area is of the larger area.

Divide the smaller area by the larger area and then multiply by 100 to find the percentage.

$\frac{\text{smaller area}}{\text{larger area}} \cdot 100$	Percentage formula
$= \frac{(72)}{(160)} \cdot 100$	Substitute 72 for the smaller area and 160 for the larger area.
$= 45$	Simplify.

The smaller area is 45% of the size of the larger area.

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5. Calculate the percentage that the larger area is of the smaller area.

Divide the larger area by the smaller area and then multiply by 100 to find the percentage.

$$\frac{\text{larger area}}{\text{smaller area}} \cdot 100$$ Percentage formula

$$= \frac{(160)}{(72)} \cdot 100$$ Substitute 160 for the larger area and 72 for the smaller area.

$$\approx 222.2$$ Simplify.

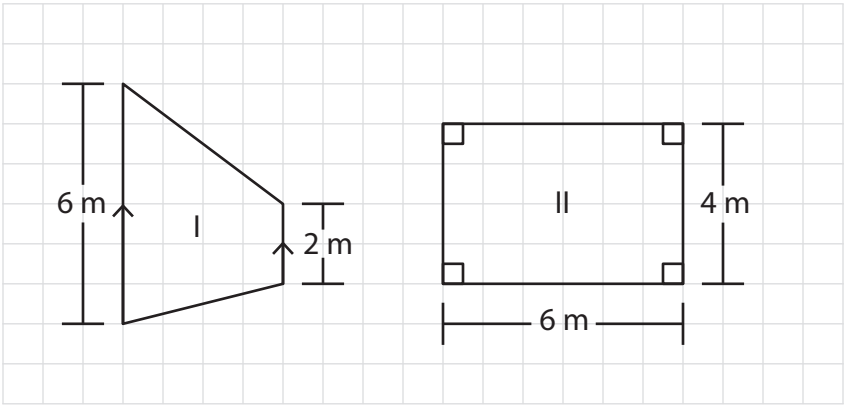
The larger area is about 222% of the size of the smaller area.

6. Determine the solid figure from which the cross sections are formed.

The areas of the rectangular cross sections are unequal, and the rectangular cross sections are not similar. Given this information, the only solid figure from which these cross sections could have been formed is like a pyramid with a rectangular base, except that instead of coming to a point at the top, it comes to a line like the ridge of a roof.

Example 2

The diagram shows the two cross sections, I and II, of a solid figure. What is the geometric solid from which the cross sections are formed?



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1. Name the plane figure represented by cross section I.

   Cross section I is a trapezoid because it has one pair of parallel sides.

2. Name the plane figure represented by cross section II.

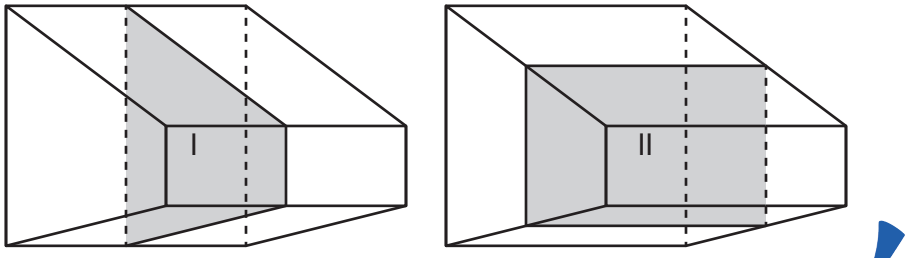
   Cross section II is a rectangle because it has four right angles and two pairs of congruent and parallel sides.

3. Describe how the cross sections are oriented with respect to each other and why.

   The cross sections represent two of the three dimensions of the solid. By definition, a cross section intersects a solid figure at a right angle to the surface of the solid figure. Therefore, since both cross sections must intersect the figure at a right angle and there are two different cross sections, the cross sections are perpendicular to each other.

4. Name the solid figure implied by the cross sections.

   The two cross sections are the only ones for this solid figure. Therefore, the trapezoid is the base of a prism. The rectangle is a cross section of the prism that is parallel to the faces of the prism formed by the parallel sides of the trapezoid, because it has a width that is less than the longer parallel side of the trapezoid and greater than the shorter parallel side of the trapezoid. The diagram that follows shows the solid and its cross sections.

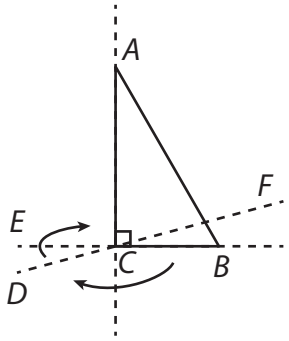


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Example 3

A right triangle is rotated 270° about its longer leg, $\overline{AC}$, as shown. Describe the solid figure produced by this rotation as well as what this partial revolution means in terms of the solid figure.



1. Name the sides of the triangle that will move during the rotation.

   Sides $\overline{AB}$ and $\overline{BC}$ will move during the rotation. Side $\overline{AC}$ will remain in place, since it represents an axis of rotation.

2. Name the plane figure that the line $\overline{BC}$ will create if it is rotated in space one complete revolution about point C.

   If it is rotated in space one complete revolution, $\overline{BC}$ will create a circle with its center at point C; $\overline{BC}$ will be the radius of the circle.

3. Describe the solid figure that $\overline{AB}$ will create if it is rotated in space one complete revolution about point A.

   Rotating $\overline{AB}$ one revolution around point A will create a cone that has a slant height of $\overline{AB}$ and a radius that is equal to the circumference of circle C; this circumference will be equal to 2π times the length of $\overline{BC}$, or $2\pi \cdot BC$.

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4. Describe the solid figure produced by the rotation.

   The solid figure is a cone with a highest point at point A, a height of $\overline{AC}$, and a circular base with a radius of length $\overline{BC}$.

5. Determine what fraction of a complete revolution 270° represents.

   Recall that 270° is the number of degrees that the lines $\overline{AB}$ and $\overline{BC}$ rotate. Multiply 270° by the conversion factor $\frac{1 \text{ revolution}}{360°}$ to convert the degree measure to revolutions.

   $$270° \bullet \frac{1 \text{ revolution}}{360°} = \frac{3}{4} \text{ revolution}$$

   270° is equivalent to $\frac{3}{4}$ of a complete revolution.

6. Describe what this partial revolution means in terms of the solid figure.

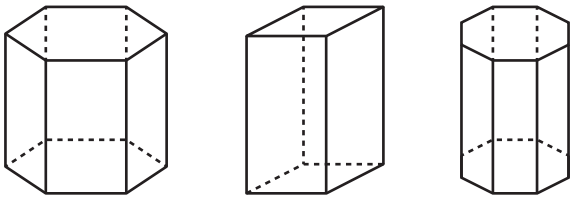
   A 270° revolution of the right triangle results in a figure that is $\frac{3}{4}$ of the cone that would be produced if the right triangle were rotated by one complete revolution.

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Example 4

In regular prisms, equilateral polygons form the bases, and the rectangular faces of the prism make up the sides, which are perpendicular to the bases. Planes can intersect regular prisms in such a way as to divide the prism into two congruent halves; these planes are said to bisect the prisms. The diagram shows a regular hexagonal prism (I), a square prism (II), and an octagonal prism (III). Identify the number of such cross sections for each regular prism, and describe how this number relates to the number of faces of the prism.



I II III

1. Determine the number of faces of each prism.

   Faces include the rectangular sides and the polygonal bases.

   Organize the information in a table.

Figure	Rectangular sides	Bases	Total faces
I	6	2	8
II	4	2	6
III	8	2	10

2. Describe the number of ways that each base could be bisected by a plane to produce two congruent halves.

   - Figure I: The 3 pairs of parallel sides of the hexagonal bases can be bisected to produce 3 pairs of congruent prisms, and the 3 pairs of opposite angles can be bisected to produce 3 more pairs of congruent prisms, so there are 6 ways.
   - Figure II: The 2 pairs of parallel opposite sides of the square bases can be bisected to produce 2 pairs of congruent prisms, and the 2 pairs of opposite angles can be bisected to produce 2 more pairs of congruent prisms, so there are 4 ways.
   - Figure III: The 4 pairs of parallel sides of the hexagonal bases can be bisected to produce 4 pairs of congruent prisms, and the 4 pairs of opposite angles can be bisected to produce 4 more pairs of congruent prisms, so there are 8 ways.

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3. Describe the number of ways that each prism could be bisected by a plane that is parallel to the bases to produce two congruent halves.

   To produce two congruent prisms, each figure could be intersected by 1 plane located halfway between and parallel to the bases. Therefore, there is only 1 way this is possible for each prism.

4. Use the results of steps 2 and 3 to determine the total number of cross sections that produce congruent halves of each prism.

   Add the number of ways each prism's base could be bisected (found in step 2) to the number of ways each prism could be bisected (step 3). This gives the total number of cross sections that produce congruent halves.

   - Figure I: 6 + 1 = 7 cross sections
   - Figure II: 4 + 1 = 5 cross sections
   - Figure III: 8 + 1 = 9 cross sections

5. Use the number of faces and cross sections for each figure to suggest a rule relating the total number of faces to the number of cross sections that produce congruent prisms.

   Refer to the table from step 1 and the results of step 4 to determine the number of faces and cross sections.

   - Figure I has 8 faces and 7 cross sections.
   - Figure II has 6 faces and 5 cross sections.
   - Figure III has 10 faces and 9 cross sections

   Based on these numbers, the rule appears to be that the number of cross sections producing congruent prisms is one less than the number of faces of the "parent" prism.

6. Use the number of faces and cross sections result for Figure II to predict how many cross sections would be needed to divide a regular decagonal (10-sided) prism into congruent halves.

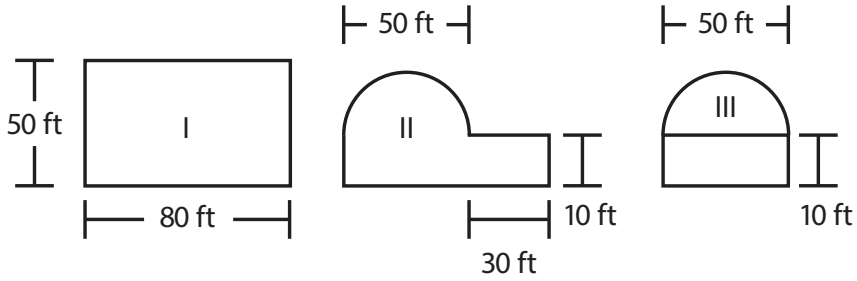
   Based on these results, the number of cross sections would be one less than the number of faces of the decagonal prism. A decagonal prism has 10 rectangular faces and 2 bases, for a total of 12 faces. Therefore, there are 11 cross sections that can divide the regular decagonal prism into congruent halves.

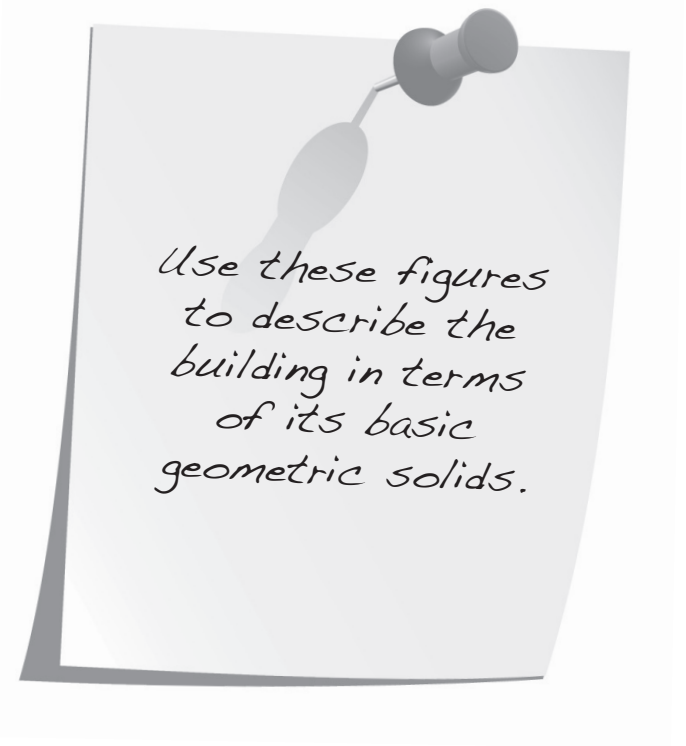
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Problem-Based Task 4B.5.1: The School Observatory and Its Footprint

SMP	
1	2 ✓
3 ✓	4
5	6
7	8 ✓

The figures in the following diagram show the parts of a new observatory planned for a local college. Figure I represents the "footprint" or base of the observatory building. Figures II and III show the other cross sections of the observatory. Use these figures to describe the building in terms of its basic geometric solids. Then, determine the volume of the dome and the volume of the other building shape(s).





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Problem-Based Task 4B.5.1: The School Observatory and Its Footprint

Coaching

a. What kind of plane figure is the footprint (Figure I) of the building, and what are its dimensions?

b. What kind of plane figures make up Figure II?

c. What are the dimensions of each part of Figure II?

d. What kind of plane figures make up Figure III?

e. What are the dimensions of each part of Figure III?

f. What are the geometric solids that produced the cross sections in Figures I, II, and III, and what are the dimensions of each?

g. What are the general formulas for each of the volumes of the solid figures?

h. What are the volumes of each of the solid figures?

i. How do the volumes of the dome and the other solid(s) compare?

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Instruction

Problem-Based Task 4B.5.1: The School Observatory and Its Footprint

Coaching Sample Responses

a. What kind of plane figure is the footprint (Figure I) of the building, and what are its dimensions?

Figure I is a rectangle of dimensions 50 feet by 80 feet.

b. What kind of plane figures make up Figure II?

Figure II is made up of a rectangle and a semicircle.

c. What are the dimensions of each part of Figure II?

Figure II is made up of a rectangle of dimensions 10 feet by 80 feet and a semicircle of diameter 50 feet.

d. What kind of plane figures make up Figure III?

Figure III is made up of a rectangle and a semicircle.

e. What are the dimensions of each part of Figure III?

Figure III is made up of a rectangle of dimensions 10 feet by 50 feet and a semicircle of diameter 50 feet.

f. What are the geometric solids that produced the cross sections in Figures I, II, and III, and what are the dimensions of each?

The rectangles in Figures I, II, and III are formed by the cross sections of a rectangular prism of dimensions 10 feet by 50 feet by 80 feet. The semicircles in Figures II and III have the same diameters, so they are formed by the cross sections of a hemisphere of diameter 50 feet.

g. What are the general formulas for each of the volumes of the solid figures?

The formulas are as follows:

- Rectangular prism: $V = lwh$
- Hemisphere: $V = \frac{2}{3}\pi r^3$

Since we are given a value for the diameter, d, determine the length of the radius by dividing the diameter by 2.

$$r = \frac{d}{2} = \frac{50}{2} = 25$$

The radius is 25 feet.

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h. What are the volumes of each of the solid figures?

Calculate the volume of the rectangular solid for $l = 80$ feet, $w = 50$ feet, and $h = 10$ feet.

$V = lwh$

$V = (10)(80)(50)$

$V = 40{,}000$

The volume of the rectangular-solid part of the observatory is 40,000 cubic feet.

Calculate the volume of the hemisphere for a diameter of 50 feet.

$$V = \frac{2}{3}\pi r^3$$

$$V = \frac{2}{3}\pi(25)^3$$

$$V = \frac{31{,}250}{3}\pi$$

The volume of the hemisphere is $V = \frac{31{,}250}{3}\pi$ cubic feet.

i. How do the volumes of the dome and the other solid(s) compare?

To compare the volumes of the two figures, approximate the volume of the hemisphere.

$$V = \frac{31{,}250}{3}\pi \approx 32{,}725$$

Rounding to the nearest ten thousand, the volume of the hemisphere is approximately 30,000 cubic feet. By comparison, the rectangular-solid "base" building on which the dome sits has a volume of 40,000 cubic feet. Therefore, the volume of the dome is about $\frac{3}{4}$ of the volume of the base building.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4B.5.1: Two-Dimensional Cross Sections of Three-Dimensional Objects

For problems 1–4, calculate the area of the cross section for each object described.

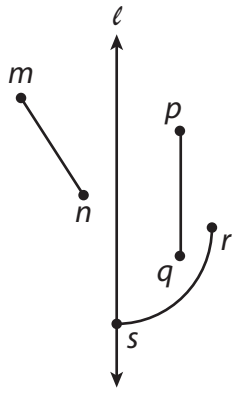
1. a cross section that is located $\frac{2}{3}$ of the distance from the highest point to the base of a pyramid; the pyramid has a height of 12 inches and a square base with 4-inch sides

2. the three cross sections of a rectangular prism with dimensions of 4 mm, 5 mm, and 6 mm

3. the cross section of maximum area for a sphere whose surface area is 36π ft^2. (*Hint*: The surface area of a sphere of radius r is given by the expression $4\pi r^2$.)

4. the cross section of a cone of radius 10 meters and height 25 meters that divides the cone into two congruent halves

For problems 5–7, use the diagram to describe the object formed by rotating each given figure one complete revolution about line ℓ.



5. segment *mn*

6. figure *rs*

7. segment *pq*

continued

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For problems 8–10, calculate the quantity indicated.

8. A parallelepiped is a three-dimensional figure with six faces that are parallelograms. Find the areas of the three cleavage planes (cross sections) of a mineral crystal that is in the shape of a parallelepiped and whose faces have the dimensions 2 microns by 3 microns, 2 microns by 5 microns, and 3 microns by 5 microns. (*Hint*: The area of a parallelogram is the product of its height and the side to which the height is perpendicular.)

9. Find the surface area of a plastic socket of a ball-and-socket joint if the socket is in the shape of a part of a sphere generated by a circular arc that measures 60° and has a length of 1.5π millimeters.

10. Find the maximum radius of a cross section of a spherical gumball that has a volume of 32π cubic millimeters.

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Lesson 4B.5.2: Density

Warm-Up 4B.5.2

The circular pool at the city park has a diameter of 10 meters and a depth of 3 meters. The city keeps the pool stocked with koi all year. One particular morning, 20 koi swim to the surface of the pool right away for feeding time, while the others linger below. The attendant knows that there are currently 48 koi in the pool.

1. Considering only the koi on the surface, what is the average number of koi per square meter of the pool's surface at feeding time?

2. What is the average number of koi per cubic meter in the pool?

3. Describe the domain for the number of koi in the pool. In light of this domain, how could you revise your answers to problems 1 and 2 to make the results more meaningful?

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Lesson 4B.5.2: Density

Common Core State Standard

G–MG.2 Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★

Warm-Up 4B.5.2 Debrief

1. Considering only the koi on the surface, what is the average number of koi per square meter of the pool's surface at feeding time?

   Recall that the formula for the area of a circle is $A = \pi r^2$.

   Since we are given the diameter (10 meters), divide the diameter by 2 to find the radius; the radius is 5 meters. Substitute this value of r into the formula for area.

$A = \pi r^2$	Area formula
$A = \pi(5)^2$	Substitute 5 for r.
$A = 25\pi$	Simplify.

   The area of the surface of the pool is 25π square meters.

   To find the number of koi per square meter, divide the given number of koi at the surface of the pool (20) by this area.

$\frac{\text{number of koi at the surface}}{\text{area of the pool's surface}}$	Formula for koi per square meter
$= \frac{(20)}{(25\pi)}$	Substitute 20 for the number of koi and 25π for the total area of the pool's surface.
$= \frac{4}{5\pi}$	Simplify.

   There is an average of $\frac{4}{5\pi}$ or about 0.26 koi per square meter of the pool's surface.

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Instruction

2. What is the average number of koi per cubic meter in the pool?

   "Cubic meter" indicates volume, so multiply the known area, $A = 25\pi$, by the given depth (height) of the pool, 3 meters, to find the pool's volume, V.

$V = Ah$	Formula for the volume of a cylinder
$V = (25\pi)(3)$	Substitute 25π for A and 3 for h.
$V = 75\pi$	Multiply.

   Thus, the volume of the pool is 75π cubic meters.

   Next, divide the total number of koi in the pool, 48, by the pool's volume, 75π cubic meters.

$\frac{\text{total number of koi}}{\text{volume of the pool}}$	Formula for koi per cubic meter
$= \frac{(48)}{(75\pi)}$	Substitute 48 for the number of koi and 75π for the total pool volume.
$= \frac{16}{25\pi}$	Simplify.

   There is an average of $\frac{16}{25\pi}$ or about 0.2 koi per cubic meter of the pool's volume.

3. Describe the domain for the number of koi in the pool. In light of this domain, how could you revise your answers to problems 1 and 2 to make the results more meaningful?

   The domain of koi in the pool is the set of whole numbers. The answers to problems 1 and 2 are both fractional amounts, so they are not in the domain. To yield more meaningful results that fall within the domain, create proportions to apply the calculated averages for koi per 1 square meter and koi per 1 cubic meter to different amounts for area and volume.

   For example, we know that there is an average of approximately 0.26 koi per 1 square meter of the pool's surface. Write this as a ratio:

   $$\frac{0.26 \text{ koi}}{1 \text{ square meter}}$$

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Instruction

Use this ratio to create a proportion that can then be used to determine the number of square meters of the pool's surface, x, per 1 koi.

$\frac{0.26 \text{ koi}}{1 \text{ square meter}} = \frac{1 \text{ koi}}{x \text{ square meters}}$ Create a proportion.

$0.26 \bullet x = 1 \bullet 1$ Cross multiply.

$x = \frac{1 \bullet 1}{0.26}$ Divide both sides by 0.26.

$x \approx 3.85$ Simplify.

There is 1 koi for about every 3.85 square meters of the pool's surface.

Follow a similar process to determine the number of cubic meters of the pool's volume, y, per 1 koi.

Recall that in problem 2, we found that the average number of koi per 1 cubic meter of pool volume was approximately 0.2.

$\frac{0.2 \text{ koi}}{1 \text{ cubic meter}} = \frac{1 \text{ koi}}{y \text{ cubic meters}}$ Create a proportion.

$0.2 \bullet y = 1 \bullet 1$ Cross multiply.

$y = \frac{1 \bullet 1}{0.2}$ Divide both sides by 0.2.

$y = 5$ Simplify.

There is 1 koi for every 5 cubic meters of the pool's volume.

Connection to the Lesson

- Students will distinguish between density per unit of area and density per unit of volume.
- Students will use proportions to find the average number of a quantity or variable per unit of area or volume.
- Students will interpret the meaning of the density values they calculate in the context of the problem, and compute alternate measures when they might be more useful in real-world situations.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- using proportions to scale up or scale down the relationship between two quantities (7.RP.3)
- assigning appropriate units to calculations involving two or more units (N–Q.1)
- converting units of measurement (6.RP.3d)
- identifying the domain in real-world problems (F–IF.5)
- checking calculations and solutions to make sure that they reflect real-world factors (7.EE.3)

Introduction

Density is used in a variety of fields to quantify the amount, number, or other quantity per unit of area, volume, or some other basic variable. In some situations, density is used to compare the total number of a quantity with the *average* value of the quantity when it is associated with a second variable. In real-world problems, it is important to carefully define the quantity being averaged, especially with regard to what kind(s) of numbers can be used for quantification. In this sub-lesson, the density of a quantity will be most often compared to a basic unit of area or volume, but these are by no means the only way of averaging a quantity.

Key Concepts

- **Density** is the amount, number, or other quantity per unit of area or volume of some substance or population being studied.
- Density is often symbolized with the lowercase Greek letter ***rho*** **(ρ)**.
- In many real-world problems, density is used to calculate an *average* value of some quantity in terms of a basic unit of area or volume.

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Instruction

- The most basic units of density for these two types of calculations are given by the following formulas:

Equation in words	Equation in symbols
$\text{Density}_{\text{Area}} = \dfrac{\text{mass or quantity}}{\text{number of } \textit{square} \text{ units}}$	$\rho_A = \dfrac{m}{A}$
$\text{Density}_{\text{Volume}} = \dfrac{\text{mass or quantity}}{\text{number of } \textit{cubic} \text{ units}}$	$\rho_V = \dfrac{m}{V}$

- Note that these formulas are quite similar, but the area density formula divides by the area, A, (square units) and the volume density formula divides by the volume, V (cubic units). Keeping this distinction in mind can be a useful guide in writing equations for many different density situations.
- Density can vary considerably from one place to another in an area or volume of interest, so it is important to determine if the density calculation will be meaningful for a situation as a whole or if the calculation cannot be reliably used to describe variations in a quantity per unit throughout a problem setting.
- Sometimes the domain of a quantity can determine the most appropriate basic unit to use in order for a density calculation to be meaningful. Fractions of a counting number may be unrealistic for a given situation, such as determining the population density of people living in a certain square mile of a city. Density calculations in such situations make more sense if the contexts in which the quantity is averaged or in which the density is calculated are changed to accommodate the smallest meaningful unit of a quantity of interest.

Common Errors/Misconceptions

- switching the quantities that belong in the denominator to the numerator and vice versa
- forgetting to confirm that a density calculation reflects the real-world conditions of a problem or its solution
- generalizing a density calculation to an entire population or problem context that includes wide variation in the quantity of interest across that context

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Guided Practice 4B.5.2

Example 1

A laser-jet printer uses 7,500 droplets of soy-based printer ink to deposit a 6-inch long line of type that is an average of 0.125 of an inch high. How many droplets of ink are deposited per square inch?

1. Calculate the area represented by the line of type.

   The area is the product of the height, 0.125 inch, and the length of the line of type, 6 inches.

   0.125 inch • 6 inches = 0.75 square inch

2. Use the area to calculate the density of ink drops per square inch.

   Recall that density is equal to some quantity divided by the area unit.

   Here, we're dealing with the two-dimensional area of a piece of paper, so use the formula $\text{Density}_{\text{Area}} = \frac{\text{quantity}}{\text{number of square units}}$.

   The number of drops, or quantity, is given as 7,500.

   The area, or number of square units, is 0.75.

   Substitute these values into the formula to calculate the density.

$\text{Density}_{\text{Area}} = \frac{\text{quantity}}{\text{number of square units}}$	Area density formula
$\text{Density}_{\text{Area}} = \frac{7500}{0.75}$	Substitute 7,500 for the quantity and 0.75 for the number of square units.
$\text{Density}_{\text{Area}} = 10{,}000$	Simplify.

   The ink-jet printer has a print density of 10,000 droplets per square inch.

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Example 2

There are 35,000 bats roosting in a pyramid-shaped vaulted cave ceiling that has a height of 30 meters and a rectangular floor that measures 75 by 100 meters. What is the average density of the bat population in the cave? Make sure that the density is a number that reflects the fact that the bat population is measured in counting numbers.

1. Calculate the volume of the cave.

   The cave is pyramid-shaped, with a rectangular floor representing the base of the pyramid.

   Recall that the volume of a pyramid is found using the formula $V = \frac{1}{3}Ah$, where A is the area of the base of the pyramid and h is the pyramid's height.

   The rectangular floor's area can be found using the formula $A = lw$, so use lw for the area, A, substitute the known values, and solve for the volume.

$V = \frac{1}{3}Ah$	Pyramid volume formula
$V = \frac{1}{3}(lw)h$	Substitute lw for A.
$V = \frac{1}{3}(75)(100)(30)$	Substitute 75 for l, 100 for w, and 30 for h.
$V = 75{,}000$	Simplify.

   The pyramid-shaped cave has a volume of 75,000 cubic meters.

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Instruction

2. Calculate the volume density of the bats in the cave.

Use the formula $\text{Density}_{\text{Volume}} = \dfrac{\text{quantity}}{\text{number of cubic units}}$.

The number of bats, or the quantity, is given as 35,000.

The volume of the cave is 75,000 cubic meters.

Substitute these values into the formula to calculate the density.

$\text{Density}_{\text{Volume}} = \dfrac{\text{quantity}}{\text{number of cubic units}}$	Volume density formula
$\text{Density}_{\text{Volume}} = \dfrac{(35{,}000)}{(75{,}000)}$	Substitute 35,000 for the quantity and 75,000 for the number of cubic units.
$\text{Density}_{\text{Volume}} = 0.4\overline{6}$	Simplify.

The average density of the bat population in the cave is $0.4\overline{6}$ bats per cubic meter.

3. Determine the domain of the bat population.

The bat population consists of individual bats that are counted using positive integers. Therefore, the domain of the bat population is [1, 35,000].

4. Suggest a different basic volume unit that will result in a bat density measured in positive integers.

One option for the basic volume unit could be a $2 \times 2 \times 2$ meter space within the cave (8 m^3) rather than 1 cubic meter. We can write this conversion factor as a ratio, $\dfrac{1 \text{ unit of volume}}{8 \text{ cubic meters}}$.

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5. Use the response to step 4 to calculate a more realistic value for the bat density.

   Start by converting the total volume of the cave to the new unit of volume measurement.

   $$75{,}000 \text{ cubic meters} \cdot \frac{1 \text{ volume unit}}{8 \text{ cubic meters}}$$ Multiply the volume of the cave by the conversion ratio from step 4.

   $9{,}375$ volume units — Simplify.

   Next, substitute the new volume units into the volume formula for the density and recalculate the density. Since the number of bats in the total volume of the cave is unchanged, use 35,000 for the bat quantity.

   $$\text{Density}_{\text{Volume}} = \frac{\text{quantity}}{\text{number of cubic units}}$$ Volume density formula

   $$\text{Density}_{\text{Volume}} = \frac{(35{,}000)}{(9{,}375)}$$ Substitute 35,000 for the quantity of bats and 9,375 for the volume units.

   $\text{Density}_{\text{Volume}} = 3.7\overline{3}$ — Simplify.

   Therefore, there are approximately $3.7\overline{3}$ bats per 1 volume unit, which was defined as 8 m^3.

   Rounded up to the nearest whole integer, the new calculation gives a density of about 4 bats for each 8 cubic meter space of the cave.

6. Compare the number of bats predicted by the result of step 5 to the number of bats given in the problem.

   Multiply the total number of volume units in the cave (9,375 volume units) by the estimated number of bats per volume unit from step 5 (4 bats) to determine the total predicted number of bats in the cave.

   $$9375 \text{ volume units} \cdot \frac{4 \text{ bats}}{1 \text{ volume unit}} = 37{,}500$$

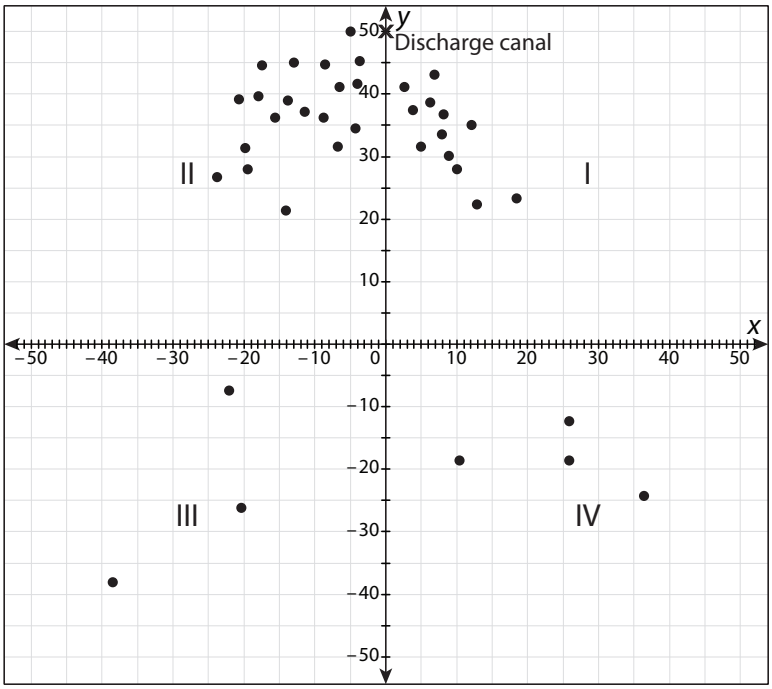
   Based on our estimate, there are about 37,500 bats in the cave. The original problem gave a bat population of 35,000. Therefore, the estimated population of 37,500 bats is about 7% greater than the given population of 35,000 bats. This is a realistic measurement.

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Example 3

Manatees are known to gather in the warmer waters surrounding power plants, which dispose of heated wastewater using discharge canals. The diagram shows the distribution of manatees on a cool winter day in the hot-water discharge basin surrounding a Gulf Coast power plant, as seen on an aerial security photograph. The plant's discharge canal is marked by an X at the end of the positive *y*-axis. Each data point represents 10 manatees, and each quadrant represents an area measuring 50 yards by 50 yards.



Calculate the density of manatees in each of the four quadrants. Then, calculate the average density of the manatees for all four combined quadrants that make up the discharge basin. Contrast and compare the densities, explain any variations, and make note of any other factors that should be included in relation to the reasonableness of the results and units.

1. Calculate the area of each quadrant.

   Each quadrant has an area of 50 yards • 50 yards, or 2,500 square yards.

2. Write the area density formula using the variable names for the problem.

   The area density formula is $\text{Density}_{\text{Area}} = \dfrac{\text{number of manatees}}{\text{number of square yards}}$.

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Instruction

3. Calculate the density in manatees per square yard for each quadrant.

   Divide the number of manatees in each quadrant by the quadrant's area in square yards. Recall that each data point represents 10 manatees.

   Organize the calculations in a table.

Quadrant	Manatees	Quadrant area (yd^2)	Density formula	Manatee density per yd^2
I	120	2,500	$\frac{120}{2500}$	0.048
II	190	2,500	$\frac{190}{2500}$	0.076
III	30	2,500	$\frac{30}{2500}$	0.012
IV	40	2,500	$\frac{40}{2500}$	0.016

4. Calculate the total area for all four quadrants of the discharge basin.

   The area of the discharge basin is found by multiplying the area of each quadrant (2,500 square yards) by the number of quadrants (4).

   $2500 \cdot 4 = 10{,}000$

   The discharge basin has a total area of 10,000 square yards.

5. Calculate the total number of manatees in the discharge basin.

   The number of manatees in the discharge basin is found by summing the number of manatees in each quadrant.

   $120 + 190 + 30 + 40 = 380$

   There are a total of 380 manatees in all four quadrants of the discharge basin.

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6. Calculate the average density for all four quadrants.

The total number of manatees is 380 and the area is 10,000 square yards.

Substitute these values into the formula and calculate the area density.

$$\text{Density}_{\text{Area}} = \frac{\text{total number of manatees}}{\text{total square yards}}$$ Density formula

$$\text{Density}_{\text{Area}} = \frac{(380)}{(10{,}000)}$$ Substitute 380 for the total number of manatees and 10,000 for the total square yards.

$$\text{Density}_{\text{Area}} = 0.038$$ Simplify.

The average density of manatees for the whole discharge basin is approximately 0.038 manatees per square yard.

7. Compare each of the quadrant densities with the average discharge basin density and explain any variations between the individual quadrant densities and the density for the discharge basin as a whole.

Organize the information in a table.

Quadrant(s)	Manatee density
I	0.048
II	0.076
III	0.012
IV	0.016
I–IV	0.038

The large numbers of manatees in quadrants I and II near the hot-water discharge canal and the small numbers of manatees in quadrants III and IV skew the average density for the whole discharge basin.

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8. Explain how to change the density formulas so that the densities can be expressed in "whole" manatees.

   The area of the discharge basin could be re-scaled so that the basic area unit is small enough to yield manatee densities that are greater than 1. For example, by defining the basic unit of area as a 10-yard by 10-yard area (100 square yards), the resulting density numbers would be greater than 1. This would give a density of about 5 manatees per 100 square yards in Quadrant I, about 8 manatees in Quadrant II, about 1 manatee in Quadrant III, and about 2 manatees in Quadrant IV. The overall density would be about 4 manatees in each 100-square-yard area.

Example 4

The table shows five different volume densities of water taken from an inland salt lake, in kilograms per cubic meter. The samples were all taken at locations that are equidistant and representative of the overall dimensions of the lake. Determine the average saltwater density for the lake.

Density (kg/m^3)	Number of samples	Weighted samples (density • samples) (kg/m^3)
1,020	12	12,240
1,022	20	20,440
1,025	15	15,375
1,030	22	22,660
1,032	16	16,512

1. Describe how the number of samples for each density affects the average density.

   The numbers of samples "weight" the densities they represent, such that a large number of samples for any one density will have a greater effect on the average density. Thus, the 22 samples with a density of 1,030 kilograms per cubic meter will have a greater effect on the average density than the smaller samples collected for the other four densities. However, a combination of any two other densities will have a "weight" greater than that of 1,030 kilograms per cubic meter.

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Instruction

2. Calculate the average saltwater density for the lake.

First, determine the total number of samples by summing them.

$12 + 20 + 15 + 22 + 16 = 85$

The total number of samples is 85.

Next, find the sum of the weighted samples. All samples are in kg/m^3.

$12{,}240 + 20{,}440 + 15{,}375 + 22{,}660 + 16{,}512 = 87{,}227$

The sum of the weighted samples is 87,227 kg/m^3.

Calculate the average by dividing sum of the weighted samples by the total number of samples.

$$\frac{87{,}227}{85} = 1026.2 \text{ kg/m}^3$$

The average saltwater density of the lake is 1,026.2 kilograms per cubic meter, or 1,026 kilograms per cubic meter. Interpret the average saltwater density of the samples in terms of the overall saltwater density of the lake.

Based on these five densities taken in a total of 85 representative samples, the average saltwater density of the lake is about 1,026 kilograms per cubic meter.

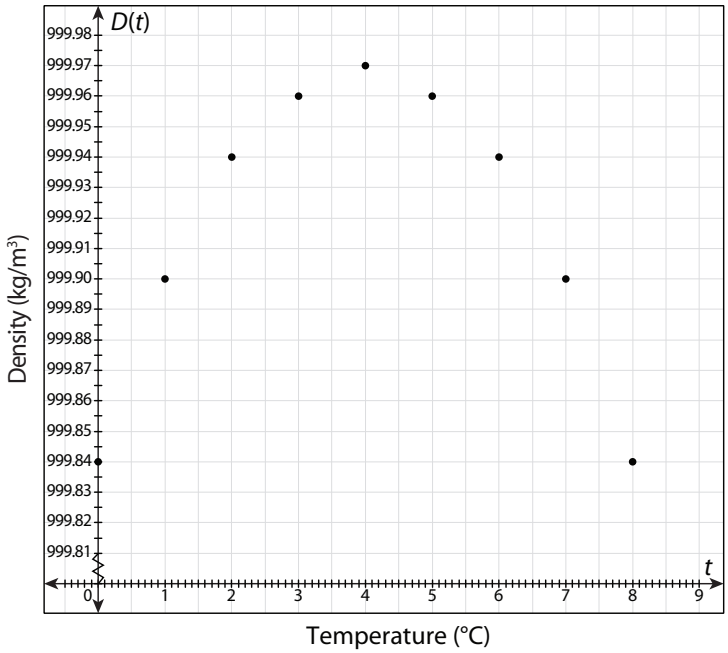


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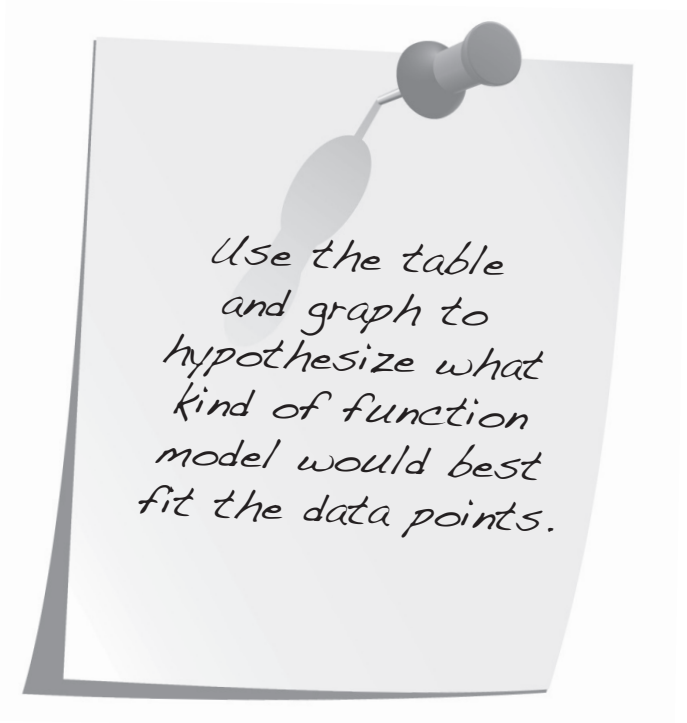
Problem-Based Task 4B.5.2: Ice-Water Density Deviations

SMP	
1	2 ✓
3	4 ✓
5 ✓	6
7	8

The table gives the changes in the density of water as it melts from solid ice to liquid water over the temperature range from 0°C to 8°C. The graph shows the data points. Use the table and graph to hypothesize what kind of function model would best fit the data points. Then, use a graphing calculator to derive an equation relating the temperature to the density.



Temperature (°C)	Density (kg/m³)
0	999.84
1	999.90
2	999.94
3	999.96
4	999.97
5	999.96
6	999.94
7	999.90
8	999.84



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Problem-Based Task 4B.5.2: Ice-Water Density Deviations

Coaching

a. Which variable is the independent variable and what is its domain?

b. Which variable is the dependent variable?

c. Based on the graph, what type of function model would best fit the data?

d. Using a graphing calculator, determine the equation of the function model for the temperature and density data.

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Instruction

Problem-Based Task 4B.5.2: Ice-Water Density Deviations

Coaching Sample Responses

a. Which variable is the independent variable and what is its domain?

Temperature is the independent variable, and its domain is [0, 8].

b. Which variable is the dependent variable?

Density is the dependent variable.

c. Based on the graph, what type of function model would best fit the data?

The graph suggests that a quadratic function would model the data best.

d. Using a graphing calculator, determine the equation of the function model for the temperature and density data.

Enter the given data into the graphing calculator, then generate the quadratic equation that best fits the data.

The calculator will generate the following values for a, b, and c for a quadratic equation of the form $y = ax^2 + bx + c$: $a = -0.008$, $b = 0.0644$, and $c = 999.84$.

The function model is $D(T) = -0.008T^2 + 0.0644T + 999.84$.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4B.5.2: Density

Use the information given in each scenario to complete problems 1–4.

1. A bracelet made of pure silver has a density of about 10,500 kilograms per cubic meter. A pendant with a mass of 25 grams that is supposed to be pure silver has a volume of 2 cubic centimeters. Is the pendant pure silver? Explain.

2. Adrian recently bought an office building at a rate of $10 per square foot, and spent $750,000 on renovations. If the building has 30,000 square feet of office space, what is the price per square foot that Adrian would need to sell the building for to break even on his investment?

3. A student group conducted a sample of plants in local meadows. The students dropped a $1.25\text{ m} \times 1.75\text{ m}$ rectangular plastic frame onto a section of a meadow, then they recorded and counted every type of plant found within the rectangle. They did this 6 times in each meadow, at evenly spaced locations throughout the meadows. For one meadow measuring $100\text{ m} \times 200\text{ m}$, the counts of black-eyed Susan plants within the rectangle were 2, 10, 7, 19, 5, and 13. What is a reasonable estimate of density of this plant throughout the entire meadow? How should a person looking for black-eyed Susans interpret this result?

4. The next day, the same student group sampled a different meadow that measured $75\text{ m} \times 125\text{ m}$. One of the rectangular plastic frames had broken on the way home the previous day, so one member of the group brought a circular hoop with a diameter of 1.2 m to replace the broken rectangle. The students recorded and counted 0, 14, 3, 9, and 11 black-eyed Susans in the rectangles, and 6 black-eyed Susans in the hoop.

   a. Estimate the density of this plant throughout the entire meadow. What is a reasonable estimate of the total black-eyed Susan population in this meadow?

   b. Compare the estimated density for this meadow with the estimated density in the meadow from problem 3. In which meadow would you expect to have the best luck finding a black-eyed Susan quickly, and why?

continued

SWB p. 161; SRB Practice 4B.5.2 p. U4B-220

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Use the information that follows to complete problems 5–10.

> The average density of human bone and muscle is 1,100 kilograms per cubic meter; the other material in the human body has an average density of 900 kilograms per cubic meter.

5. Write a formula for the density of the whole human body (ρ) in terms of its mass m and volume V.

6. Let the mass of a human body be 50 kilograms. Let x represent the mass of bone and muscle ($m_{\text{bone and muscle}}$). Write an expression for the mass of other material in the body (m_{other}) in terms of the whole body mass of 50 kilograms.

7. Write a formula for the volume of bone and muscle ($V_{\text{bone and muscle}}$) and the volume of the other material (V_{other}) in terms of the whole body volume, V.

8. Write formulas for the density of bone and muscle ($\rho_{\text{bone and muscle}}$) and the density of the other material (ρ_{other}) in the human body in terms of its mass and volume from problems 6 and 7.

9. Rewrite the formula from problem 5 in terms of the density and mass of the other two types of body material from problems 6 and 8.

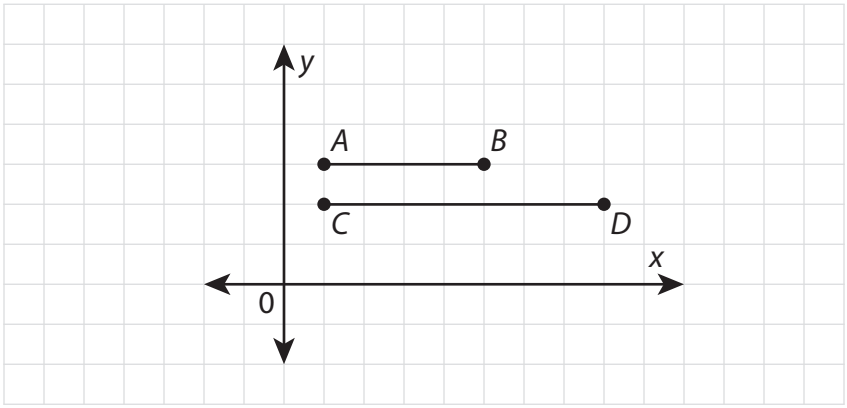
10. Substitute the original two densities given by the problem statement into the result of problem 9. Use the resulting equation to solve for the mass of bone and muscle if the density of a whole body is 1,000 kilograms per cubic meter.

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Lesson 4B.5.3: Design

Warm-Up 4B.5.3

Cedar fencing boards come in 4-foot and 7-foot lengths. Use the line segments in the diagram to represent the boards and answer the questions that follow.



1. How can the two fencing line segments shown be used to create a fence pattern that is in the shape of a parallelogram?

2. How can the fencing line segments be used to create a fence pattern that is in the shape of a rectangle?

3. How can the fencing line segments be used to create a fence pattern that is in the shape of a right triangle?

4. What plane figure is formed if the line segments serve as perpendicular bisectors of each other? (*Hint*: Imagine "connecting the dots" of the segments' endpoints so that the endpoints represent the vertices of the figure.)

5. What plane figure is formed if one of the line segments is perpendicular to the other but neither is a bisector of the other?

6. What plane figure is formed if only one of the line segments is the perpendicular bisector of the other?

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Lesson 4B.5.3: Design

Common Core State Standard

G–MG.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★

Warm-Up 4B.5.3 Debrief

1. How can the two fencing line segments shown be used to create a fence pattern that is in the shape of a parallelogram?

   The line segments can be used as the diagonals of a parallelogram or sides of a parallelogram.

2. How can the fencing line segments be used to create a fence pattern that is in the shape of a rectangle?

   The line segments can be used as the sides of a rectangle.

3. How can the fencing line segments be used to create a fence pattern that is in the shape of a right triangle?

   The line segments can be used as the legs of a right triangle. One line segment can be set perpendicular to and anywhere along the length of the other line segment to create a right triangle.

4. What plane figure is formed if the line segments serve as perpendicular bisectors of each other? (*Hint*: Imagine "connecting the dots" of the segments' endpoints so that the endpoints represent the vertices of the figure.)

   The line segments form a rhombus.

5. What plane figure is formed if one of the line segments is perpendicular to the other but neither is a bisector of the other?

   The line segments form a quadrilateral.

6. What plane figure is formed if only one of the line segments is the perpendicular bisector of the other?

   The line segments form a kite.

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Instruction

Connection to the Lesson

- Students will use given geometric figures, numerical conditions, or other mathematical constraints to solve real-world design problems.
- Students will choose among alternative mathematical models and solutions to meet the conditions of a real-world design problem.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- using function models to represent real-world problems and their solutions (F–BF.1)
- using numerical representations to model solutions of real-world problems (N–Q.2)
- representing data using algebraic, graphic, or tabular formats (F–IF.9)

Introduction

In the real world of designing products and services for business, education, government, health care, national security, and other societal institutions, financial and physical constraints regulate the designs that are created and used. Often, the *purpose* of the design dictates the boundaries of these constraints, resulting in a complex interplay among the applied and theoretical factors that go into creating a design. Mathematics can play an essential role in this process.

Key Concepts

- Mathematical concepts that identify the domains, limits, extrema (maximum and minimum values), and solutions of equations and function models are often the first used after the purpose of a design has been identified.
- Concepts in coordinate and plane geometry help designers visualize prototypes. Often, the basic principles of geometry dictate the kinds of mathematical models that are used, too.
- Many real-world objects can be modeled by mathematical figures. For example, a tree trunk can be modeled by a cylinder, or a park may be modeled by a regular trapezoid.
- The cost of development of a new design and its ultimate utility value (i.e., value based on expected performance) can also play into the choice of geometric and mathematical models used in the design.
- The number-crunching and visualizing capabilities of calculators and computers can be used to prototype and/or simulate design options.

Common Errors/Misconceptions

- applying particular mathematical concepts and models to real-world design situations for which they are not as relevant as other options
- misinterpreting information from a graph or data table and inappropriately applying that information to a real-world design situation
- using financial or purposive factors of a design with geometric, mathematical, or statistical models that do not effectively accommodate those factors

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Guided Practice 4B.5.3

Example 1

Caleb has fourteen 3-foot fence sections and would like to use them all to create a rectangular garden. If he uses only whole fence sections, how should Caleb use the fence sections to create a garden with the most area? Use the formula for the area of a rectangle, $A = lw$.

1. Assign a variable to the number of fence sections needed to create the maximum area.

   Let x represent the number of sections for the length of the rectangle. There are a total of 14 sections in all, so the length plus the width must be made up of 7 sections. The width can be represented by $7 - x$.

2. Write a function model for the area of the garden.

   Substitute the values for the length and width into the formula $A = lw$ to calculate the area.

$A = lw$	Area formula for a rectangle
$A = (x)(7 - x)$	Substitute x for l and $7 - x$ for w.
$A = 7x - x^2$	Distribute.

   The area of the garden can be modeled by the function $A = 7x - x^2$.

3. Use a graphing calculator to generate a table of data for different values of x.

   Follow the steps appropriate to your calculator model.

   On a TI-83/84:

   Step 1: Press [Y=]. At Y_1, use your keypad to enter the function. Use [X, T, θ, n] for x and [x^2] for any exponents.

   Step 2: Press [GRAPH]. Press [WINDOW] to adjust the graph's axes.

   Step 3: Press [2ND][GRAPH] to display a table of values.

(*continued*)

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On a TI-Nspire:

Step 1: Press [home]. Arrow down to the graphing icon, the second icon from the left, and press [enter].

Step 2: Enter the function to the right of "*f1*(*x*) =" and press [enter].

Step 3: To adjust the *x*- and *y*-axis scales on the window, press [menu] and select 4: Window and then 1: Window Settings. Enter each setting as needed. Tab to "OK" and press [enter].

Step 4: To see a table of values, press [menu] and scroll down to 2: View and 9: Show Table. (For some models, press [menu] and select 7: Table, then 1: Split-screen Table.)

4. Use the table of values to determine the maximum value(s) of *x*.

   The maximum value of the function occurs when x = 3.5, but since we must use complete fence sections, use the integers on either side of 3.5: $x = 3$ and $x = 4$.

5. Use the results of step 4 to determine the maximum area of the garden.

   $A = 12$ at both $x = 3$ and $x = 4$. Multiply 12 by the squared length of the individual 3-foot fence sections ($3^2 = 9$) to find the maximum area.

   The maximum area at $x = 3$ or $x = 4$ is 12 fence sections squared, or 108 square feet of garden area ($108 = 12 \cdot 9$).

6. Explain how Caleb should use the fence sections to create a garden with the most area. Refer to the domain of *x* in your explanation.

   The domain of *x* is whole fence sections, so it can be represented by the interval [0, 7]. Parts of a fence section cannot be used, so either $x = 3$ or $x = 4$ will give the largest area possible given the constraint of the domain. Both values of *x* give the same dimensions for the garden. That is, either 3 fence sections by 4 fence sections or 4 fence sections by 3 fence sections gives 108 square feet.

   $108 = 9 \cdot 12 = 3^2 \cdot 12 = 3^2 \cdot 4 \cdot 3 = 3^2 \cdot 3 \cdot 4$

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Example 2

The city council is comparing two tank options for a new water tower: a cylindrical tank and a spherical tank. Both designs have a diameter of 20 meters. The cylindrical tank has a height of 12 meters. Both designs would cost $300 per square meter of surface area. The average monthly water bill is $100 per cubic meter of water, and either tank would hold enough water for 1 month. The city council decides to go with whichever tank will produce a net profit in its first month, assuming the tank's full volume of water will be emptied completely (i.e., sold to customers) by the end of that month. Which tank will yield the desired profit?

1. Calculate the surface area of the spherical water tank using the formula $A(r) = 4\pi r^2$.

   The radius r of the spherical tank is 10 meters. Substitute this value into the formula to calculate the surface area.

$A(r) = 4\pi r^2$	Formula for the surface area of a sphere
$A(10) = 4\pi(10)^2$	Substitute 10 for r.
$A(10) = 400\pi$	Simplify.

   The surface area of the spherical water tank is 400π or about 1,257 square meters.

2. Use the surface area to calculate the cost of the spherical water tank.

   The cost of the spherical water tank is $300 per square meter of surface area. Therefore, the cost to build the tank is $300 \cdot 400\pi = 120{,}000\pi$, or about $376,991.

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Instruction

3. Calculate the volume of the spherical water tank using the formula $V(r)=\frac{4}{3}\pi r^3$.

   The radius r of the spherical tank is 10 meters. Use this to calculate the volume.

   $V(r)=\frac{4}{3}\pi r^3$ Formula for the volume of a sphere

   $V(10)=\frac{4}{3}\pi(10)^3$ Substitute 10 for r.

   $V(10)=\frac{4000}{3}\pi$ Simplify.

   The volume of the spherical water tank is $\frac{4000}{3}\pi$ or about 4,189 cubic meters.

4. Use the volume to calculate the expected revenue from emptying the spherical tank.

   The average monthly water bill is \$100 per cubic meter. The tank holds enough water for 1 month. Therefore, the revenue produced by emptying the spherical water tank is $100 \bullet \frac{4000}{3}\pi = \frac{400{,}000}{3}\pi$, or about \$418,879.

5. Compare the cost of the spherical tank to its expected revenue to determine the net profit or loss for the first month.

   The cost of constructing the spherical water tank is about \$376,991. The expected revenue from the tank is about \$418,879.

   $418{,}879 - 376{,}991 = 41{,}888$

   Therefore, the spherical water tank would produce a net profit of about \$41,888 for the first month.

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6. Calculate the surface area of the cylindrical water tank using the formula $A(h, r) = 2\pi r^2 + 2\pi rh$.

   The height of the tank is 12 meters and the radius is 10 meters. Use these values to calculate the surface area.

$A(h, r) = 2\pi r^2 + 2\pi rh$	Formula for the surface area of a cylinder with bases
$A[(12), (10)] = 2\pi(10)^2 + 2\pi(10)(12)$	Substitute 12 for h and 10 for r.
$A(12, 10) = 440\pi$	Simplify.

   The surface area of the cylindrical water tank is 440π or about 1,382 square meters.

7. Use the surface area to calculate the cost of the cylindrical water tank.

   The cost of the cylindrical water tank is $300 per square meter. Therefore, the cost to build the tank is $300 \cdot 440\pi = 132{,}000\pi$, or about $414,690.

8. Calculate the volume of the cylindrical water tank using the formula $V(h, r) = \pi rh$.

   The height of the tank is 12 meters and the radius is 10 meters. Use these values to calculate the volume.

$V(h, r) = \pi r^2 h$	Formula for the volume of a cylinder
$V[(12), (10)] = \pi(10)^2(12)$	Substitute 12 for h and 10 for r.
$V(12, 10) = 1200\pi$	Simplify.

   The volume of the cylindrical water tank is 1200π or about 3,770 cubic meters.

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Instruction

9. Use the volume to calculate the expected revenue from emptying the cylindrical tank.

   The average water bill is $100 per cubic meter. Therefore, the revenue produced by emptying the cylindrical water tank is $100 \cdot 1200\pi = 120{,}000\pi$, or about $376,991.

10. Compare the cost of building the cylindrical water tank to its expected revenue to determine the net profit or loss for the first month.

    The cost of constructing the cylindrical water tank is about $414,690. The expected revenue is about $376,991.

    $376{,}991 - 414{,}690 = -37{,}699$

    Therefore, the cylindrical water tank would produce a net loss of about $37,699 for the first month.

11. Compare the results for the two water tanks to determine which tank the city council will likely choose.

    Based on the criteria used by the city council to compare the water tanks, the spherical water tank will be a more cost-effective tank to build because the spherical tank is expected to generate a profit in the first month (if completely emptied of water). The cylindrical tank, on the other hand, would generate a net loss.

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Instruction

Example 3

Autumn is an artist who specializes in decorative silk-screen panels. She recently bought several 60-centimeter and 80-centimeter wooden dowels to use as frames for the panels. Autumn decides to make several panels that are in the shape of quadrilaterals with two perpendicular diagonals made from the dowels. One diagonal will be 60 cm, and the other will be 80 cm. Autumn wants these panels to have the largest screen area possible, but she also needs to ensure the design makes the best use of her resources. What configuration of the two sizes of dowels should she use to realize this goal?

1. List all the possible configurations of the dowels in a table.

   The diagonals can be perpendicular bisectors of each other, or one of the dowels can be a perpendicular bisector of the other dowel.

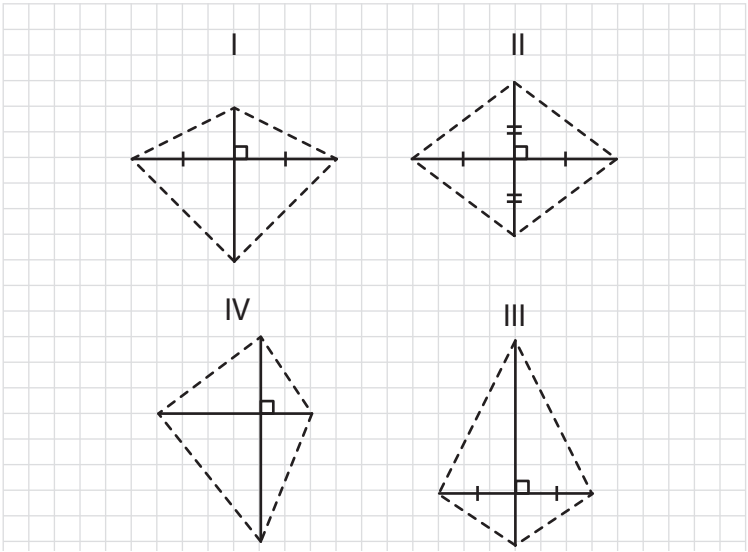
Case	Description
I	The 60-cm dowel is a perpendicular bisector of the 80-cm dowel.
II	Both dowels are perpendicular bisectors of each other.
II	The 80-cm dowel is a perpendicular bisector of the 60-cm dowel.
IV	Neither dowel is a perpendicular bisector of the other.

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Instruction

2. Sketch the four cases from step 1 and find the areas of the resulting figures.

   The sketched figures are shown as follows, with each grid box representing 10 cm. Notice that cases I–IV are each comprised of four right triangles. Use this fact to find each area by adding the areas of the triangles contained within each figure. (Recall that the formula for the area of triangle is $A = \frac{1}{2}bh$.)



- Case I area: 2,400 cm^2
- Case II area: 2,400 cm^2
- Case III area: 2,400 cm^2
- Case IV area: 2,400 cm^2

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Instruction

3. Explain why the figures have equal areas.

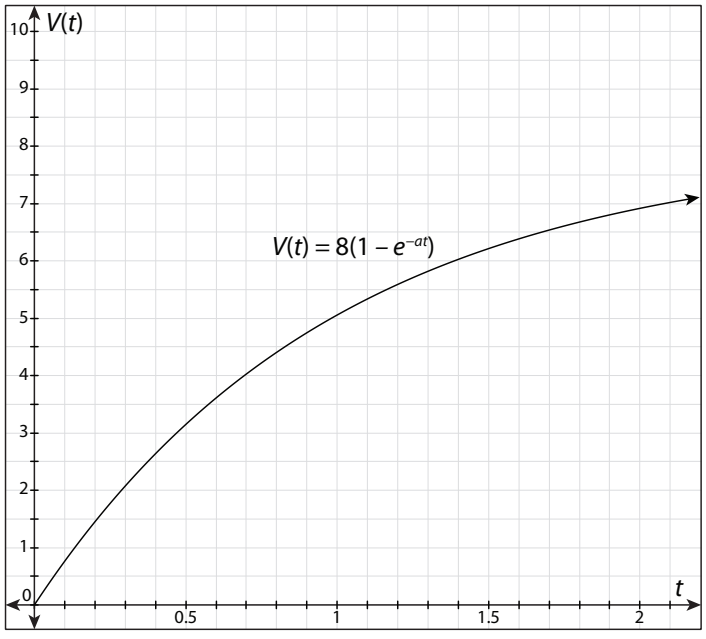
   Each panel has diagonal lengths of 60 cm and 80 cm. Visualize sliding the dowels within a panel's frame, while still maintaining perpendicularity. Sliding the dowels changes the sizes of individual triangles within the panel, but does not change the total area enclosed within the panel.

4. Determine which configuration of dowels Autumn should use.

   Since cases I–IV each use the same number of dowels and have the same area, Autumn could use any one of these configurations for her design.

Example 4

A mobile phone manufacturer has to upgrade the circuitry in one of its products. Currently, the voltage in the signal-capture circuit rises according to the exponential function model $V(t) = 8(1 - e^{-at})$, in which a is a positive constant and the coefficient of t, the time in seconds, and V is the circuit voltage in volts. The graph shows how the voltage increases when a cell-phone tower signal is detected for $a = 1$.



How will the function model have to be modified if the upgraded phone circuit has a voltage "rise time" that results in a circuit voltage of at least 7.9 volts in less than 1 second when a cell-phone tower signal is detected?

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Instruction

1. Write an inequality using the function that reflects the necessary conditions for a faster rise time.

   The voltage has to reach at least 7.9 volts in less than 1 second. Substituting $V = 7.9$ volts and $t = 1$ second in the function model gives $7.9 \geq 8(1 - e^{-a})$.

2. Solve the resulting inequality.

$7.9 \geq 8(1 - e^{-a})$	Inequality from the previous step
$7.9 \geq 8 - 8e^{-a}$	Distribute.
$-0.1 \geq -8e^{-a}$	Isolate the exponential term.
$0.0125 \geq e^{-a}$	Simplify.
$-a \leq \ln(0.0125)$	Rewrite as a natural logarithm.
$a \geq 4.4$	Use a calculator to simplify.

   A value of $a \geq 4.4$ will give a function that models a circuit that will reach a voltage of at least 7.9 volts in a rise time of 1 second.

3. Check the result of the previous step by substituting it into the original exponential function.

$V(t) = 8(1 - e^{-at})$	General exponential function model
$V(1) = 8[1 - e^{-(4.4)(1)}]$	Substitute 1 for t and 4.4 for a.
$V(1) = 8 - 8e^{-4.4}$	Distribute.
$V(1) \approx 8 - 8(0.012)$	Substitute the calculated value of $e^{-4.4}$.
$V(1) \approx 7.9$	Simplify.

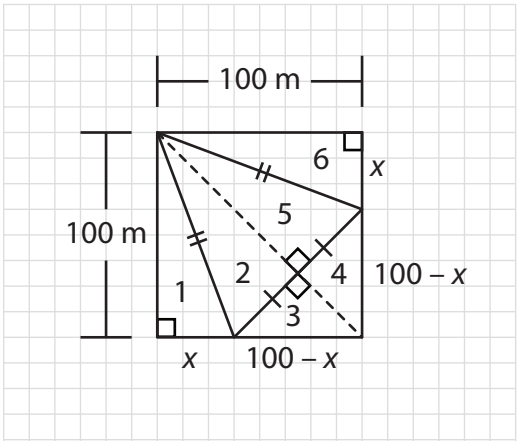
   The value $a = 4.4$ meets the conditions of the problem. Therefore, by modifying the function $V(t) = 8(1 - e^{-at})$ so that $a = 4.4$, the upgraded phone circuit will achieve a voltage of at least 7.9 volts in less than 1 second.

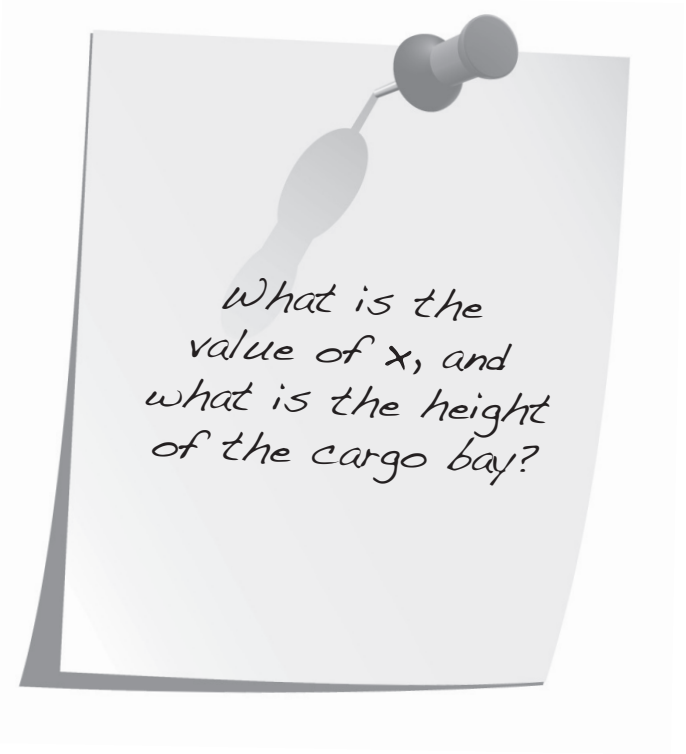
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Problem-Based Task 4B.5.3: A Solar Sail

SMP	
1 ✓	2
3 ✓	4
5	6
7	8 ✓

The design for an experimental spacecraft includes a square solar sail with sides measuring 100 meters. The sail will use photons from the sun and other forces to propel the craft. The sail will be comprised of 6 triangular panels arranged in the configuration shown in the diagram and numbered 1–6. When not in use, the sail will be folded up for storage in a cylindrical cargo bay, which will have a radius of 6 meters. The cargo bay's height will be equal to the length of the common side of triangles 2 and 5. The shortest sides of triangles 2 and 5 are each equal to the bay's radius. What is the value of x, and what is the height of the cargo bay? (*Note*: The diagram is not drawn to scale.)





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Problem-Based Task 4B.5.3: A Solar Sail

Coaching

a. What is the equation for the total area of the sail in terms of its triangular parts?

b. How many pairs of congruent triangles make up the sail?

c. What is the area of triangle 1?

d. What is the combined area of triangles 3 and 4?

e. What is the area of triangle 6?

f. What do the results of parts b–e mean for the area equation determined in part a and for the areas of triangles 2 and 5?

g. How are triangles 2 and 5 related?

h. How are triangles 3 and 4 related?

i. What is the length of the hypotenuse of the triangle that is made up of triangles 3 and 4?

j. How does the result of part i relate to the shortest sides of triangles 2 and 5?

k. Find the lengths of the shortest sides of triangles 2 and 5.

l. How do the shortest sides of triangles 2 and 5 relate to the radius of the cylindrical cargo bay?

m. Find x using the results of parts k and l.

n. How can you find the length of the common side of triangles 2 and 5, which is also the height of the cylindrical cargo bay?

o. Calculate the height of the cargo bay.

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Instruction

Problem-Based Task 4B.5.3: A Solar Sail

Coaching Sample Responses

a. What is the equation for the total area of the sail in terms of its triangular parts?

Let the areas of triangles 1–6 be represented by A_1 through A_6.

100 meters • 100 meters = 10,000 square meters

Total area of the sail = $A_1 + A_2 + A_3 + A_4 + A_5 + A_6$

$10{,}000 = A_1 + A_2 + A_3 + A_4 + A_5 + A_6$

b. How many pairs of congruent triangles make up the sail?

Three pairs: A_1 and A_6; A_2 and A_5; and A_3 and A_4.

c. What is the area of triangle 1?

Substitute values into the formula for the area of a triangle and solve. From the diagram, $b = x$ and $h = 100$.

$$A = \frac{1}{2}bh$$

$$A_1 = \frac{1}{2}(x)(100)$$

$$A_1 = 50x$$

The area of triangle 1 is $50x$ square meters.

d. What is the combined area of triangles 3 and 4?

The diagram shows that the base and height of the combined triangle are equal: $b = h = 100 - x$.

$$A = \frac{1}{2}bh$$

$$A_3 + A_4 = \frac{1}{2}(100-x)^2$$

The combined area of triangles 3 and 4 is $A_3 + A_4 = \frac{1}{2}(100-x)^2$ square meters.

e. What is the area of triangle 6?

Since triangles 1 and 6 are congruent, they have the same area.

$A_1 = A_6 = 50x$

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f. What do the results of parts b–e mean for the area equation determined in part a and for the areas of triangles 2 and 5?

The area equation for the entire sail is $10{,}000 = A_1 + A_2 + A_3 + A_4 + A_5 + A_6$.

We know that $A_1 = A_6 = 50x$ and $A_3 + A_4 = \frac{1}{2}(100 - x)^2$ square meters.

Substitute the areas for triangles 1, 3, 4, and 6 into the area equation for the sail.

$$10{,}000 = A_1 + A_2 + A_3 + A_4 + A_5 + A_6$$

$$10{,}000 = (50x) + A_2 + \left[\frac{1}{2}(100 - x)^2\right] + A_5 + (50x)$$

$$10{,}000 - 100x - \frac{1}{2}(100 - x)^2 = A_2 + A_5$$

The sum of the areas of triangles 2 and 5 is $10{,}000 - 100x - \frac{1}{2}(100 - x)^2$ square meters.

g. How are triangles 2 and 5 related?

It can be determined from the diagram that they are congruent according to the Side-Side-Side Congruence Postulate.

h. How are triangles 3 and 4 related?

According to the diagram, they are congruent by the Side-Side-Side Congruence Postulate.

i. What is the length of the hypotenuse of the triangle that is made up of triangles 3 and 4?

Since it is known that each leg of the triangle equals $100 - x$, substitute this value for a and b into the Pythagorean Theorem and solve for the hypotenuse.

$$a^2 + b^2 = c^2$$

$$(100 - x)^2 + (100 - x)^2 = c^2$$

$$2(100 - x)^2 = c^2$$

$$\sqrt{2(100 - x)^2} = c$$

$$c = \sqrt{2}(100 - x)$$

The hypotenuse of the triangle made up of triangles 3 and 4 is $\sqrt{2}(100 - x)$ meters.

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Instruction

j. How does the result of part i relate to the shortest sides of triangles 2 and 5?

The sum of the lengths of the shortest sides of triangles 2 and 5 is the same length as the hypotenuse of the triangle made up of triangles 3 and 4 in part i.

k. Find the lengths of the shortest sides of triangles 2 and 5.

The shortest sides of triangles 2 and 5 are each equal in length to half of the result of part i, or $\frac{\sqrt{2}(100-x)}{2}$.

l. How do the shortest sides of triangles 2 and 5 relate to the radius of the cylindrical cargo bay?

The shortest sides of triangles 2 and 5 are each the same length as the radius of the cargo bay, which is 6 meters.

m. Find x using the results of parts k and l.

Since both expressions are equal to r, set the two expressions equal and solve for x.

$$6=\frac{\sqrt{2}(100-x)}{2}$$
$$12=100\sqrt{2}-x\sqrt{2}$$
$$x\sqrt{2}=100\sqrt{2}-12$$
$$x=100-\frac{12}{\sqrt{2}}$$
$$x\approx 91.5$$

The value of x is approximately 91.5 meters.

n. How can you find the length of the common side of triangles 2 and 5, which is also the height of the cylindrical cargo bay?

The shortest sides of triangles 2 and 5 are already known from the problem statement. The hypotenuses of triangles 2 and 5 are also the hypotenuses of triangles 1 and 6, respectively; these lengths can be calculated with the Pythagorean Theorem. The common side of triangles 2 and 5 can then be computed with the Pythagorean Theorem, too.

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o. Calculate the height of the cargo bay.

First find the hypotenuse of triangle 1 using the Pythagorean Theorem. Let $a = 100$ and $b = 91.5$.

$$a^2 + b^2 = c^2$$

$$(100)^2 + (91.5)^2 = c^2$$

$$c = \sqrt{100^2 + 91.5^2} = 135.5$$

Use the result to find the unknown longer side of triangle 2, which is equal to the height of the cargo bay. Use the variables h and r in the equation for the Pythagorean Theorem: $h^2 + r^2 = c^2$.

$$h^2 + r^2 = c^2$$

$$h^2 + (6)^2 = (135.5)^2$$

$$h^2 = 135.5^2 - 6^2$$

$$h^2 = 18{,}324.3$$

$$h = \sqrt{18{,}324.3}$$

$$h \approx 135.4 \text{ meters}$$

The height of the cylindrical cargo bay is about 135.5 meters.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4B.5.3: Design

Use the information given in each problem to answer the questions.

1. A honey farmer has two options for plastic honey jars: a cylindrical one with a radius of 3.5 cm and a height of 6 cm, and a square prism with a base length of 6.5 cm per side and a height of 5.5 cm. Find the volume of each jar, rounded to the nearest tenth. Which jar has the greater volume? If the jars are packed in a box that measures 52 cm × 52 cm × 22 cm, could the farmer pack more cylindrical jars or more square prism jars in the box? Assume that a thin layer of cardboard of negligible thickness separates the layers of jars so that they can be stacked.

2. Consider a circle with radius of 3 cm. If the radius is increased by 1 cm, by what percentage does the circumference increase? By what percentage does the area increase? Why does one measurement grow so much faster than the other?

3. An equilateral triangle with sides measuring 10 m contains a circle with radius r whose center lies on the incenter of the triangle. A circle centered on the incenter that lies within the triangle will become tangent to all three sides simultaneously as it expands. The vertices of the triangle are at (0, 0), (10, 0), and $\left(5,5\sqrt{3}\right)$. The incenter is at $\left(5,\frac{5\sqrt{3}}{3}\right)$. Define a function $A(r)$ that calculates the area between the outside of the circle and the inside of the triangle as a function of the circle's radius r. Describe $A(r)$. What type of curve is it, and what are its key characteristics? What are the domain and range of $A(r)$?

continued

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4. A 100-meter-diameter circular ring of stones is being designed as part of an arboretum project. The stones used to build the ring measure 90 cm × 80 cm × 70 cm. Estimate the minimum and maximum number of stones that would be needed to build the ring, depending on whether their long or short dimension is used along the edge of the ring.

5. A building in a major city has a roofline that starts out on the left side with a height of 785 ft, then rises continuously along a diagonal for 184 ft until it reaches a maximum height of 915 ft on the right side of the building, giving it a distinctive look. For safety reasons, an aircraft warning light must be installed exactly halfway up the roofline of this building. What is the width of the building below the level of the roof? What coordinates can be assigned to the left and right edges of the roof if the building's left side at ground level is considered the origin? What are the coordinates of the new light?

6. A barbecue grill manufacturer makes grills by cutting 55-gallon steel drums in half along their longest dimension, then fastening them together to make even longer grills, up to a maximum of four drum halves long. The drums are 34.5 inches long with a diameter of 22.5 inches before being cut. The resulting half-cylinder is covered with iron grates to use as a cooking surface, and has a bracket welded to one end to connect it to another drum. A 55-gallon drum costs $100, a 17.25 in × 22.5 in cooking grate costs $45, and a bracket costs $5.45. Write and simplify a function that describes the cost of a grill (drum, grill, and bracket) as a function of its length l (measured in drum halves). Specify the domain and range of this function. In terms of l, what volume of charcoal (in cubic feet) could a grill of length l hold?

7. The Colemans want to reline the interior wall and floor of their 48-inch-tall circular aboveground pool, which has a diameter of 18 ft. Two thicknesses of liners in the shape of the pool are available: 20 gauge, which has a 15-year warranty and costs $100; and 30 gauge, which has a 25-year warranty and costs $160. What is the cost of each liner per square foot of the pool's interior wall and floor? What is the cost of each liner per year of warranty (*not* per square foot)? Which liner represents a better value and why?

continued

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8. A sculptor is creating a hollow square plastic prism with external dimensions of 4 ft × 4 ft × 6 ft. The square ends of the prism will be open, allowing the same thickness of plastic to extend for the full 6 feet on all four sides. Write an equation that describes the total volume of plastic as a function of its thickness, w. How thick must the plastic be so that the total volume of plastic in the finished piece is exactly 30 cubic feet? Express the thickness in inches.

9. A cabinetmaker has been commissioned to build a round table out of mahogany. The design is to have bare mahogany in the center of the table top, with a circle of maple veneer (a very thin layer of wood glued on top) covering the outer edge of the tabletop. The table must have a diameter of 3 ft, but the width of the maple veneer circle is the cabinetmaker's choice. The total cost of materials for the table must not exceed $300. The mahogany boards cost $10 per square foot, the maple veneer costs $16.25 per square foot, and the base that the table will rest on costs $125. Write and simplify an equation that describes the total cost of materials as a function of the width w of the maple veneer. Specify the domain and range of the function. What is the widest veneer that the cabinetmaker could use and still remain within the materials budget (not considering any waste material)? Express your answer in inches.

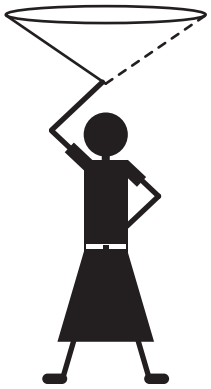
10. A mason must provide two quotes for a 30 ft × 2 ft × 2 ft stone wall. The first quote is for a "dry" wall, which is held together by its weight and the construction technique instead of mortar. The second quote is for a "wet" wall, for which mortar is used to hold the stones together. A dry wall requires a 3-foot-deep foundation to ensure it does not shift too much as the ground freezes and thaws during the winter, whereas a wet wall only requires a 1-foot-deep foundation. The foundation material costs $20 per cubic yard for either wall. Stone costs $10 per cubic foot for either wall. Compared with a wet wall, a dry wall requires 15% more stone, and has labor costs that are 20% higher. Labor costs to build the wet wall would be $1,500. Find the total cost of the wet wall and the total cost of the dry wall. On a percentage basis, how much more expensive would the dry wall be?

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Progress Assessment

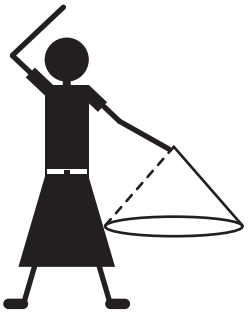
Circle the letter of the best answer.

1. Kennedy does a trick with a rope lasso as shown in the diagram. The rope moves up and down as Kennedy raises and lowers her arm. Kennedy stands still throughout the trick. What solid figures are formed by the motion of the rope?



a. cones
b. cones and cylinders
c. cylinders
d. cylinders and spheres

2. Kennedy does another rope trick in which she spins the rope in a circle while she hops in and out of the area created by the spinning rope. What solid figure is formed by the motion of the rope?



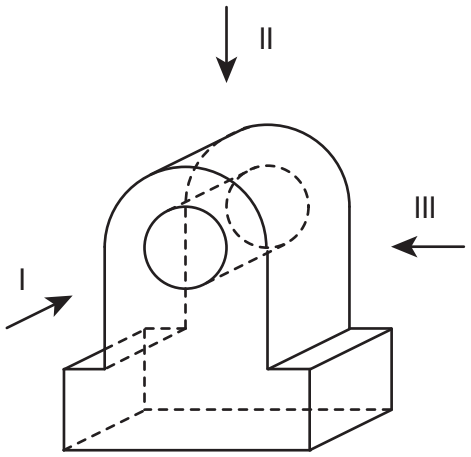
a. a triangle
b. a cylinder
c. a cone
d. a circle

continued

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Assessment

The arrows in the following diagram refer to three different directional views of a machine part. The perspectives are labeled I (the front), II (the top), and III (the side). Use the diagram for problems 3 and 4.



3. Which perspectives include the same kind of plane figure?

   a. I and II
   b. I, II, and III
   c. I and III
   d. II and III

4. List the plane figure(s) seen from perspective I in the diagram.

   a. a circle of radius a, a semicircle of radius b ($b > a$), and two rectangles
   b. a circle of radius a, a semicircle of radius b ($a > b$), and two rectangles
   c. a cylinder of radius a, a half cylinder of radius b ($b > a$), and two rectangles
   d. a cylinder of radius a, a half cylinder of radius b ($a > b$), and two rectangles

5. An ecology student in southeast Texas determines that 1 square yard of roadside wildflowers contains an average of 650 bluebonnet blossoms. A random sample of the bluebonnet plants in the square-yard area contains 125 plants, each with multiple blossoms. A nearby roadside embankment is shaped like a trapezoid and is covered with bluebonnets. The embankment is 75 yards long and has a plateau that is 50 yards long; the distance from the roadside to the plateau is 30 yards. What is the density of bluebonnet *blossoms*, and how many bluebonnet *plants* cover the embankment?

   a. about 5 blossoms per plant; 468,750 plants
   b. about 5 blossoms per plant; 234,375 plants
   c. about 25 blossoms per plant; 468,750 plants
   d. about 27 blossoms per plant; 234,375 plants

continued

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6. The temperature generated by a bacteria population that has a maximum potential size of 10^8 bacteria due to resource restrictions is given by the function model $N(T) = N_0(1 - 2^{-T})$, in which T is the temperature change in degrees Celsius and N_0, the original number of bacteria (in millions of bacteria per liter), is 100. What is the density of bacteria in a 10-liter incubator in millions of bacteria per liter after the temperature has risen 3°C?

   a. 87.5
   b. 875
   c. 12.5
   d. 125

7. Three 1-gram sugar cubes are dissolved in 250 milliliters of coffee that has a mass 325 grams. Which variable does not change?

   a. the density of the coffee before and after the sugar cubes are added
   b. the mass of the coffee before and after the sugar cubes are added
   c. the volume of the coffee before and after the sugar cubes are added
   d. the weight of the coffee before and after the sugar cubes are added

8. One of the two main types of waves produced by an earthquake is the *P* wave, which is described as a compressional wave because it moves horizontally in magnitude as well as in time. The *P* wave can be modeled by the function $\rho_{average}(t) = \rho_{maximum} \bullet \sin(2\pi nt)$, in which the two ρ variables are the density of the material through which the wave is moving, n is a correction factor that is equal to 1 in the simplest version of the model, and t is the time in seconds. Which of the following qualities of a rock would not change as the *P* wave moves through the rock with a period of 1 second?

   a. the density of the rock
   b. the volume of the rock
   c. the rate of change of the density of the rock
   d. the mass of the rock

continued

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Assessment

9. A toy designer needs to create a part shaped like a right triangular prism such that the angle between one of the surfaces and the surface with the hypotenuse as an edge is 30°. The ratio of the sides should be such that their ratio is no more than 0.001 greater or less than the sine of 30°. Which compound inequality models these specifications if one side of the angle is represented by x and the other side is represented by $x + 3$?

a. $30.001 > \frac{x}{x+3} > 29.999$

b. $30.001 > \frac{x^2}{(x+3)^2} > 29.999$

c. $0.501 > \frac{x}{x+3} > 0.499$

d. $0.501 > \frac{x^2}{(x+3)^2} > 0.499$

10. A team of researchers is developing a prototype "bio battery" that uses common energy-storing organic chemicals. The prototype is in the shape of a rectangular prism. Because of the need to have an interface of a certain size between the positive and negative electrode chambers in the battery, the battery needs to have an outside surface area A of no more than 4 square centimeters and rectangular dimensions of x, $x + 2$, and $x + 4$ centimeters. The volume V of the battery should be no more than 9 cubic centimeters. Which equation represents the function model for the volume of the prototype battery in terms of these dimensions?

a. $A(x) = V(x)$

b. $5 \geq x(x+2)(x+4) - 2x(x+2) + 2(x+2)(x+4) + 2x(x+4)$

c. $\frac{V(x)}{A(x)} = \frac{9}{4}$

d. $9 \geq x(x+2)(x+4)$

continued

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Assessment

Use the given information to complete all parts of problem 11.

11. A manufacturer of GPS-equipped buoys for use in rivers is designing a new model that will be cone-shaped, with a base radius of 0.5 meter and a height of 1.5 meters. The densities of the component materials in the new model result in a buoy density (ρ) of 750 kilograms per cubic meter.

    a. Calculate the volume V of the new buoy.

    b. Calculate the mass m of the new buoy.

    c. What condition must be met for the buoy to float? (*Hint*: Look at the unit of density for the new buoy.)

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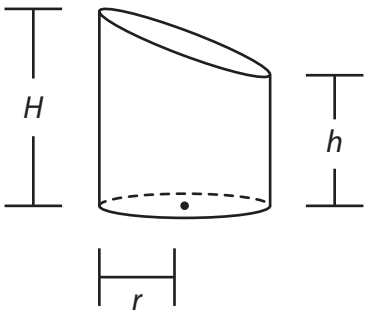
Unit 4B Assessment

Circle the letter of the best answer.

1. Dry-cleaning machines use solvents to clean clothing. At the end of the rinse cycle, the solvent is extracted from the machine for reuse. The effectiveness of a dry-cleaning solvent decreases by $\frac{1}{3}$ each time it is used. The solvent is given a numerical rating after each use, and when the rating drops below 1 the solvent is replaced. The numerical rating of unused solvent is 18. If a batch of solvent has a rating of 2, which of the following equations models this situation? Let n represent the number of times the solvent has been used.

   a. $2 = 18 \cdot 3^{-n}$
   b. $2 = 3 \cdot 18^{-n}$
   c. $2 = 18(1 + 3^{-n})$
   d. $2 = 18 - 3^{n}$

2. A portion of a regular cylinder with heights H and h and a base radius of r is shown in the diagram that follows. What is the correct formula for the volume V of this portion in terms of the lateral area A (the surface area of the shape excluding the top and bottom) if A and V are given by $A = \pi r(H + h)$ and $V = \pi r^2\left(\frac{H+h}{2}\right)$?



   a. $V = \frac{1}{2rA}$
   b. $V = \frac{1}{2}rA$
   c. $V = \frac{r}{2A}$
   d. $V = \frac{2r}{A}$

continued

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3. Which function represents a transformation of the parent quadratic function 4 units to the right and 3 units down on a coordinate plane?

   a. $f(x) = x^2 - 8x + 13$
   b. $f(x) = x^2 + 8x + 13$
   c. $f(x) = x^2 - 8x + 19$
   d. $f(x) = x^2 + 8x + 19$

4. Which function is neither odd nor even?

   a. $a(x) = \cos x$
   b. $b(x) = \sin x$
   c. $c(x) = |x|$
   d. $d(x) = 2^x$

5. A sailboat is propelled at a constant rate by wind according to the function $w(t) = 30t$. A shear turbulence force slows down the motion of the boat according to the function $r(t) = -0.05t^2 - 0.5t$. Which function combination represents the net motion of the sailboat relative to the direction of the wind?

   a. $(w + r)(t) = -0.05t^2 + 29.5t$
   b. $(w - r)(t) = 0.05t^2 + 29.5t$
   c. $(w \bullet r)(t) = -1.5t^2 - 15t$
   d. $(w \div r)(t) = -0.001\overline{6}t - 0.01\overline{6}$

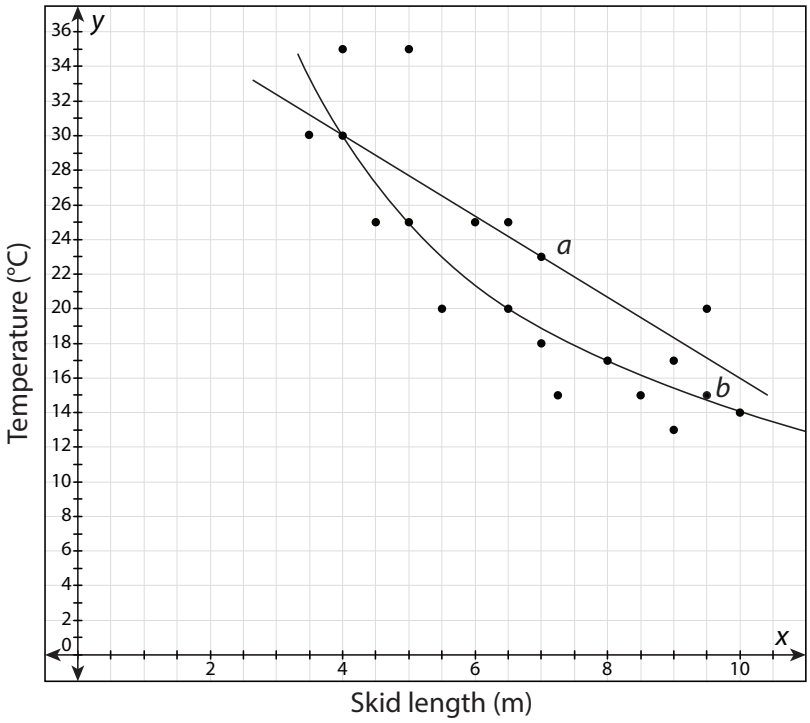
6. What is the average rate of change of the function $f(x) = \dfrac{x}{x-1}$ between $x = 0.99$ and $x = 1.01$?

   a. It is infinite because the function is undefined at $x = 1$.
   b. It is undefined because the function is undefined at $x = 1$.
   c. It is 100 regardless of whether the function is undefined at $x = 1$.
   d. It is 0 because the function is undefined at $x = 1$.

continued

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7. Skid marks on roads are caused by the heat of friction of a braking car's rubber tires against the road's surface—the heat melts the tires slightly, leaving behind streaks of rubber. The outside temperature can heat or cool the asphalt, which in turn affects the length of a skid mark at the time the vehicle's brakes are applied. The graph shows a linear function *a* and a quadratic function *b* that serve as models for the relationship between data points representing the temperatures that correlate to an increase in skid length. Temperatures are in degrees Celsius and skid lengths are in meters. What do the rates of change of the functions have in common over the interval [6, 8]?



a. Both are equal to 5°C per meter.

b. Both are equal to –5°C per meter.

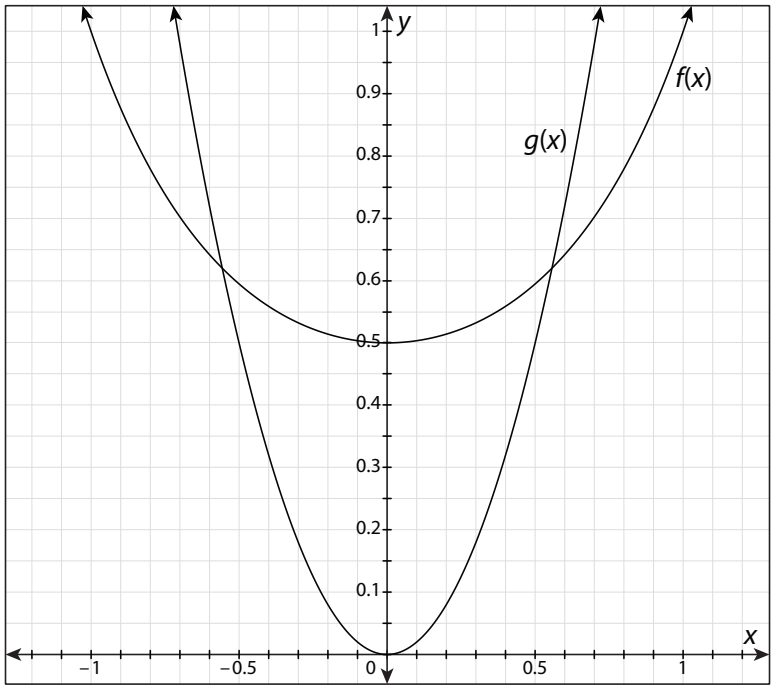
c. Both are negative.

d. Both are positive.

continued

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8. The graphs of two functions $f(x)=2^{x^2-1}$ and $g(x) = 2x^2$ are shown. If $(\pm a, f(\pm a))$ and $(\pm a, g(\pm a))$ represent the points at which the functions intersect, which of the following is true?



a. over the interval $(-\infty,-a)\cap(a,\infty)$, $f(x) = g(x)$

b. over the interval $(-a,0)\cap(0,a)$, $f(x) = 0.5$ and $g(x) = 0$

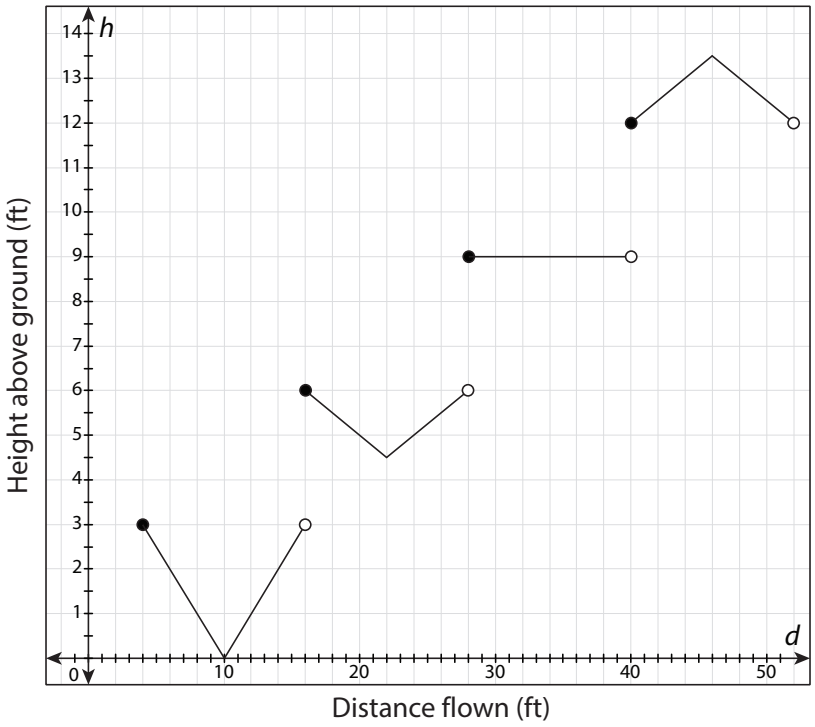
c. over the interval $[-a, a]$, $g(x) < f(x)$

d. over the interval $(-a, a)$, $g(x) < f(x)$

continued

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9. Some members of a school's STEM club came up with a way to model the wing movements of a heron as it takes off from a flooded area where it has been foraging. The model is shown in the following graph, with the *d* and *h* axes representing the bird's distance from its takeoff point and height above ground, respectively, in feet for each of four segments of its flight. Which function model(s) were used to form the mathematical model?



a. absolute value; linear; step

b. absolute value; linear

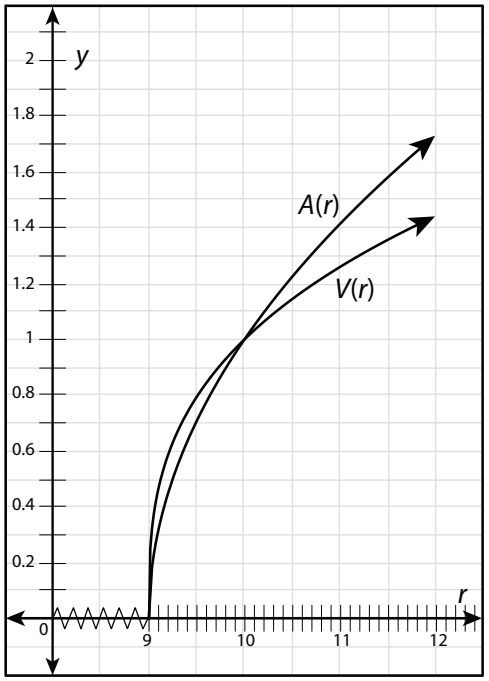
c. absolute value; step

d. absolute value

continued

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10. An astronomer found that the cross-sectional radius in microns of carbon-oxygen-silicon space dust can be modeled by the two functions shown in the following graph. The cube root function $V(r)$ models the volume of a dust particle based on the estimated mass of the dust cloud of which it is a part. The square root function $A(r)$ models the particle's surface area as indicated by light reflectivity. Over what interval of r do the two models give approximately equal results?



a. [0, 3]

b. [9.5, 10.5]

c. $[11, \infty)$

d. $(-\infty, 9]$

11. A solid figure has two cross sections: an isosceles triangle with a base of 8 inches and a height of 5 inches, and a circle with a diameter of 8 inches. Which of the following best describes the solid?

a. It is a cone with a height of 8 inches and a base radius of 5 inches.

b. It is a cone with a height of 4 inches and a base radius of 5 inches.

c. It is a cone with a height of 5 inches and a base radius of 8 inches.

d. It is a cone with a height of 5 inches and a base radius of 4 inches.

continued

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12. Which density measurement would best reflect the density of finches in an aviary if there are 75 finches in an enclosure that measures 5 by 10 by 15 meters? (*Note*: 100 meters = 1 hectometer = 10 decameters)

    a. 0.1 finch per cubic meter
    b. 10 finches per cubic meter
    c. 100 finches per cubic decameter
    d. 100,000 finches per cubic hectometer

Use what you have learned about mathematical modeling to solve the following problems.

13. Consider the functions $f(x) = \sin(x + 4)$ and $g(x) = x + 4$. Let $h(x) = f(x) - g(x)$.

    a. Is $h(x)$ a combination, a composition, or both a combination and a composition? Explain your reasoning.

    b. What is the function that represents $h(x)$?

    c. What is the domain of $h(x)$?

14. An absolute value function has the general form $f(x) = \left|\frac{x+a}{x+b}\right|$.

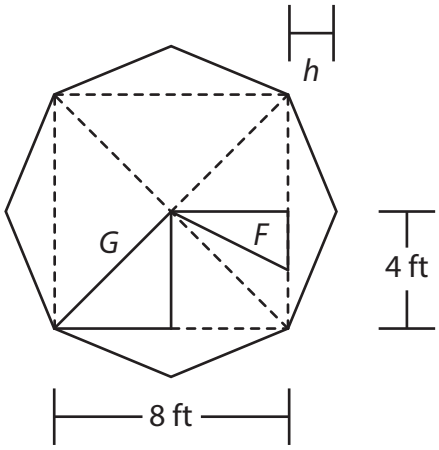
    a. What is the specific function if $f(4) = 6$, $f(6) = 8$, and $a = 2$?

    b. What is the domain of the specific function?

continued

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15. A wilderness-supply manufacturer is designing a simple pyramid-shaped tent with a square base. The central tent pole has a height of 4 feet. The waterproof sheeting that will be used for the tent is in the shape shown in the following overhead view.



To form a pyramid, the sheeting is staked at the outermost end of each edge, *G*. A flap of area *F* is cut into the sheeting as shown, but not removed from it; the bottom edge of the flap has a length of 2 feet. The square base of the pyramid is 8 feet by 8 feet. The four isosceles triangles attached to each side of the square base are needed so that the sheeting fits snugly to the ground around all sides once the sheeting is raised with the tent pole.

a. Find the exact length of the edge, *G*.

b. Find the exact area, *F*, of the flap.

c. If the height *h* of each of the four isosceles triangles is $\sqrt{2}$, what is the exact total area of the waterproof tent sheeting?

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Answer Key

Lesson 1: Creating Equations

Pre-Assessment, p. U4B-1

1. a
2. b
3. a
4. b
5. d

Warm-Up 4B.1.1, p. U4B-7

1. The dependent variable decreases as the independent variable increases. The independent variable increases by 1 each time, but the dependent variable does not decrease by this same rate.
2. The graph does not appear to represent a linear equation. The slope between pairs of consecutive points on the graph changes, so the points do not form a straight line. The graph could represent an exponential, quadratic, or logarithmic equation.
3. The domain of the time variable is positive because it represents a real-world quantity and, as noted, time cannot be negative. The data also limit the maximum of time to 7 years. The range of the percentage values is theoretically [0, 100], but the data limit the range to the maximum and minimum values shown on the graph for the points at which $t = 1$ year and $t = 7$ years, respectively.

Practice 4B.1.1: Creating Equations in One Variable, p. U4B-34

1. $y = x - 1$
2. $y = 2x^2 + 2x$
3. $y = \left(\frac{1}{2}\right)x^2 + 2x - 4$
4. $y = 3^x$
5. There is no linear equation that fits the three given data points.
6. $y = -\log_{\left(\frac{1}{3}\right)} x$
7. $y = 6x^2 - 12x + 9$
8. $d_1(t) = 20t_1$
9. $d_2(t) = 10t_2$
10. $d_{\text{Gale}}(t) = -\frac{40}{3}t^2 + \frac{80}{3}t$

Warm-Up 4B.1.2, p. U4B-36

1. $2t + 1 > 0$ or $t > -\frac{1}{2}$
2. [0, 1]
3. $t > 0$
4. $t \geq \frac{1}{2}(e-1)$
5. The two quantities are related by the inequality $\frac{1}{2}(e-1) > -\frac{1}{2}$. This means that $t \geq \frac{1}{2}(e-1)$ satisfies the mathematical and real-world constraints on the independent variable in the problem.
6. horizontal lines: $P(t) \geq 0$ and $P(t) \leq 1$; vertical line: $t \geq \frac{1}{2}(e-1)$

Practice 4B.1.2: Representing and Interpreting Constraints, p. U4B-55

1. $x < 2$ or $(-\infty, 2)$
2. $x \geq -\sqrt{7}$ and $x \leq \sqrt{7}$, or $\left[-\sqrt{7}, \sqrt{7}\right]$
3. $x < \log_3 2$ or $(-\infty, \log_3 2)$
4. $\{1 < x\}, \left\{\frac{1}{2} < y\right\}$
5. $\{x \leq -2\}, \left\{y \leq -\frac{5}{2}\right\}$
6. There is no system solution since the x constraint ($x < 0$) does not correspond to a common solution for y with a constraint of $y > 1$.
7. $\{1 < x < 2\}, \{2 < y < 4\}$
8. $h(t) < 0$; $t > 0$
9. They must sell at least 63 bags.
10. $[6.91, \infty)$

Warm-Up 4B.1.3, p. U4B-56

1. $h = \frac{3V}{\pi r^2}$
2. $h = 4$
3. $h^2 = s^2 - r^2$
4. $h = 4$
5. Yes, the results are the same. Either method can be used to determine that the cone's height is 4 meters.
6. The lobby is 1 meter higher than the cone, so Mahomet's cone will fit in the lobby.

Practice 4B.1.3: Rearranging Formulas, p. U4B-80

1. $P = P^2 \bullet R$
2. $E_a = RT(\ln A - \ln k)$
3. $[H^+] = 10^{-\frac{28}{3}} = 10^{-10} \bullet \sqrt[3]{10^2}$; $[OH^-] = 10^{-\frac{14}{3}} = 10^{-5} \bullet \sqrt[3]{10}$
4. $s_{\text{wire}} > s_{\text{ring}}$

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5. $L_i = \dfrac{c \bullet L_v}{\sqrt{c^2 - v^2}}$

6. $B(-1) = \dfrac{-\mu_0}{4\pi r}; B(1) = \dfrac{\mu_0}{4\pi(r-5)}$; $B(-1) < 0$ and $B(1) > 0$ since $r > 5$.

7. $\dfrac{RT}{N_A} \bullet 10^{-5} < f_c < \dfrac{RT}{N_A} \bullet 10^{-3}$

8. $V_m = \dfrac{V}{n} = \dfrac{RT}{P}$

9. $a < \dfrac{c\log C + d\log D - b\log B}{\log A}$ or $a < c\log_A C + d\log_A D - b\log_A B$

10. $h = \dfrac{9}{2r} - r$

Progress Assessment, p. U4B-82

1. b
2. a
3. d
4. a
5. c
6. d
7. b
8. c
9. a
10. b
11. The best type of equation to model this situation is an exponential equation with a positive power for the exponential base because the data is increasing and the dependent variable is not 0.

Lesson 2: Transforming a Model and Combining Functions

Pre-Assessment, p. U4B-86

1. b
2. d
3. c
4. d
5. a

Warm-Up 4B.2.1, p. U4B-90

1. $b(x)$ and $g(x)$
2. $b(x)$ and $g(x)$
3. $c(x)$ and $d(x)$
4. $c(x)$ and $d(x)$

Practice 4B.2.1: Transformations of Parent Graphs, p. U4B-113

1. $f(x) = (x + 2)^2$; $g(x) = (x + 3)^2$
2. $f(x) = e^x - 1$; $g(x) = e^x - 3$
3. $f(x) = \log(x + 2)$; $g(x) = \log(x + 4)$
4. domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$
5. domain: $(-\infty, \infty)$; range: $(-\infty, 2)$
6. domain: $(-2, \infty)$; range: $(-\infty, \infty)$
7. If the area is doubled, then $x = 1 + \sqrt{1 + 2A}$, in which A is the area of the original rectangle.
8. If the rate r is tripled and the time t increases by 3 hours, then $d = 3rt + 9r$, in which r and t are the original rate and time.
9. If $g(t)$ is cut in half, then t is doubled: $t = 2 \bullet \dfrac{\ln 2}{g(t)}$.
10. If the range of $p(n)$ is (0, 5000), the domain is (300, 320). If the upper bound of the range of $p(n)$ is decreased by 2,500, then the domain is reduced: (300, 310).

Warm-Up 4B.2.2, p. U4B-116

1. $f(x) = -f(-x)$
2. $g(x) \neq g(-x)$
3. $h(x) = h(-x)$

Practice 4B.2.2: Recognizing Odd and Even Functions, p. U4B-129

1. For an even function: Remove $-2x$ to give $a(x) = 5 + x^2$.
   For an odd function: Remove 5 and x^2 to give $y = -2x$.
2. For an even function: Remove 3 to give $b(x) = -2 \bullet |-4x|$.
   For an odd function: Remove 3 and change $|-4x|$ to $-4x$ to give $b(x) = -2 \bullet (-4x)$.
3. For an even function: Remove x from the logarithm argument to give $c(x) = x^2 \bullet \log x^2$.
   For an odd function: Remove x from the logarithm argument and remove an x from the term that is multiplied by the logarithm term to give $c(x) = x \bullet \log x^2$.
4. For an even function: Remove the sin x term to give $d(x) = \cos x$.
   For an odd function: Remove the cos x term to give $d(x) = \sin x$.
5. $f(x) = 2 \bullet \sin 3x$; odd
6. $g(x) = \sqrt{x^2 - 1}$; even for $\{x \leq 1 \cap x \geq 1\}$
7. $h(x) = 5 - |x + 1|$; neither
8. $I(t)$ is even; $V(t)$ is odd.
9. $P(t)$ is odd because $P(t) = -P(-t)$. $P(t)$ is a sine function because it is an odd function.
10. The period of $P(t)$ is half that of the current and voltage functions, or $\dfrac{\pi}{a}$.

Warm-Up 4B.2.3, p. U4B-131

1. \$1,243.88
2. $c(n) = 24.875n + 0.125$
3. $T(c) = 1500.125 + 24.875n$

Practice 4B.2.3: Combining Functions, p. U4B-143

1. both
2. composition function
3. both
4. $\left(\dfrac{f}{g}\right)(x) = \dfrac{\sin(x+1)}{x+1}$; domain: $\{x \neq -1\}$
   $(g)(x) = \dfrac{x+1}{\sin(x+1)}$; domain: $\{x \neq -1 \pm 2\pi n, n = 0, 1, 2 \ldots\}$
5. $(f - g)(x) = \sin(x + 1) - x - 1$; domain: $(-\infty, \infty)$
   $(g - f)(x) = x + 1 - \sin(x + 1)$; domain: $(-\infty, \infty)$
6. $(f \circ g)(x) = \sin(x+2)$; domain: $(-\infty, \infty)$
   $(g \circ f)(x) = \sin(x+1) + 1$; domain: $(-\infty, \infty)$
7. $(f \circ f)(x) = \sin[\sin(x+1) + 1]$; domain: $(-\infty, \infty)$
   $(g \circ g)(x) = x + 2$; domain: $(-\infty, \infty)$
8. $V(x) = x(x+20)(3x-1)$
9. $V(t) = (50t)(50t+20)(150t-1)$
10. $V(5) = (250)(270)(749) \approx 5.06 \times 10^7$;
    $V(10) = (500)(570)(1499) \approx 4.3 \times 10^8$

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Progress Assessment, p. U4B-145

1. b
2. c
3. a
4. d
5. c
6. b
7. d
8. a
9. b
10. c
11. a. $\Delta T = \dfrac{dW}{kA}$
    b. $\Delta T \approx 255$ Kelvin. (*Note*: This change is approximately 459°F.)

Lesson 3: Comparing Properties Within and Between Functions

Pre-Assessment, p. U4B-148

1. c
2. c
3. d
4. b
5. c

Warm-Up 4B.3.1, p. U4B-155

1. The mathematical domain of t is $(-\infty,\infty)$. The real-world domain of $N(t)$ is approximately $[0, \infty)$.
2. The function values increase continuously as time increases. However, the rate at which the function values increase is decreasing as time goes on. The function also appears to approach a maximum value as time increases.
3. $N(t) = 5\left(1 - \dfrac{1}{2^t}\right)$
4. As t increases, the fraction $\dfrac{1}{2^t}$ becomes smaller and approaches 0. This causes the function value to approach $5 \bullet 1 - 0$, or a value of 5.
5. For the real-world domain found in problem 1, approximately $[0, \infty)$, the minimum function value would be 0, which corresponds to one breeding pair of cougars at a time of 0 years. The maximum function value is close to 5, but since the function is limited to whole numbers, it will have to be rounded down to 4.

Practice 4B.3.1: Reading and Identifying Key Features of Real-World Situation Graphs, p. U4B-185

1. The rate of new designs completed between $t = 0$ and $t = 1$ is proportional to the number of designers in the group; the rate is greatest for the group having 16 designers and least for the group of 2 designers.
2. Between $t = 2$ and $t = 3$, the greatest rate of new designs completed comes from the group of 2 designers. The rate of design creation for the other three groups has decreased as the number of designs approaches the limit of 20 new designs.
3. The graph suggests that the larger groups have a greater initial production of new designs, but the rate of new design production decreases with group size over time. All groups are working toward a boundary condition of 20 new designs, as stated in the problem.
4. A linear function can model the increasing consumption of coal by Asian nations and Oceania; a linear function with a rate of change of approximately 0 can model the consumption of coal in North America because the annual consumption is almost constant.
5. The consumption of coal in Asian nations and Oceania is an increasing function with an approximately constant rate of increase of coal consumption; the consumption of coal in North America is approximately constant from year to year with an approximately 0 rate of consumption from one year to the next, except for the exceptions shown when the rate of increasing consumption is negative.
6. The coal consumption by Asian nations and Oceania was about three times that of North America between 2004 (about 3.1 billion tons) and 2005 (about 3.5 billion tons).
7. The trend of the data suggests that there would be more than 16,300 NEOs with a diameter less than 100 meters.
8. The trend of the data suggests that there would be fewer than 900 NEOs with a diameter greater than 1,000 meters.
9. A linear function fits the data points for $c = 1$ and $c = 2$, but a different linear function fits the data points for $c = 2$, 3, and 4. An exponential function of the form $f(x) = ab^x$ (in which a is a constant greater than n for $c = 1$ and x is a negative exponent) fits the data because n is decreasing—but not constantly—as c increases.
10. A combination of two linear functions for different restricted domains can be derived to exactly fit the four data points. An exponential function would not fit the data exactly, so an exponential function would be an approximation of the relationship between c and n.

Warm-Up 4B.3.2, p. U4B-188

1. $g(5) - g(15) = -[g(15) - g(5)]$
2. $d(5) - d(15) = -[d(15) - d(5)]$
3. $g(x) - d(x) = -[d(x) - g(x)]$
4. As x increases by 1, $g(x)$ increases by 3.5 and $d(x)$ increases by 3.3.

Practice 4B.3.2: Calculating Average Rates of Change, p. U4B-208

1. 0.1
2. 0.2
3. –0.09
4. 0.09
5. For [0.01, 0.1], $r = -0.01$; for [0.1, 1], $r = -0.01$.
6. For [0.01, 0.1], $r = -1.1 \bullet 10^{-4}$; for [0.1, 1], $r = -1.11 \bullet 10^{-2}$.
7. For [0.01, 0.1], $r = \dfrac{\sqrt{5} - 10\sqrt{4.01}}{9} \approx 0.260$; for [0.1, 1], $r = \dfrac{\sqrt{5} - 10\sqrt{1.04}}{9} \approx 0.885$.
8. The average rate of change is approximately constant and positive.
9. The average rate of change is approximately constant and positive.

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10. The average rate of change of $f(t)$ over the interval [1, 10] is approximately 24; the average rate of change of $g(t)$ over the interval [1, 10] is approximately 20. The number of followers of the social network represented by $f(t)$ is increasing faster than the number of followers of the social network represented by $g(t)$.

Warm-Up 4B.3.3, p. U4B-210

1. $r = \frac{18}{\pi}$
2. $r = \frac{12\sqrt{2}}{\pi}$
3. The average rates of change for the two functions over the interval $\left[0, \frac{\pi}{12}\right]$ are very close in value, but not equal.

Practice 4B.3.3: Comparing Functions, p. U4B-231

1. three different maximum values
2. three different minimum values
3. The maximum values appear just before the game, at the half, and toward the end of the game or during overtime if one is played.
4. –33.6 feet per second
5. –32.16 feet per second
6. –32.016 feet per second
7. Answers may vary, but should be reflective of the given data. Sample answer: The speed at $t = 1$ second appears to be –32 feet per second. The speed function appears to be $s(t) = -32t$. The "–32" is the product of the exponent 2 of the variable t in the distance function multiplied by the coefficient "–16".
8. For $a(x)$, $r \approx 2.86$; for $b(x)$, $r \approx 0.497$.
9. For $c(x)$, $r \approx 2.1$; for $d(x)$, $r = 1$.
10. For $e(x)$, $r \approx 0.41$; for $f(x)$, $r \approx 0.488$.

Progress Assessment, p. U4B-233

1. d
2. c
3. b
4. b
5. a
6. d
7. b
8. b
9. a
10. b
11. a. Company B has a greater incentive rate compared to its project amount than company A: 1.5% versus 1%.
    b. There is no time at which the two companies will have earned the same amount, because the solution to the two earnings equations is $t < 0$.

Lesson 4: Choosing a Model

Pre-Assessment, p. U4B-241

1. d
2. a
3. b
4. a
5. b

Warm-Up 4B.4.1, p. U4B-246

1. For $f(x) = 5^x$, $r = 4$; for $g(x) = 5x^2$, $r = 5$; for $h(x) = 5x$, $r = 5$.
2. The y-intercept is the function value at $x = 0$. In the context of this problem, it means that at the beginning of the study ($x = 0$), two of the models show a population of 0 prairie dogs and one model shows a population of 100 prairie dogs. Therefore, the y-intercept of $f(x)$ is at $y = 1$, and the y-intercepts of $g(x)$ and $h(x)$ are at $y = 0$.
3. (1, 5); after 1 year, the prairie dog population will be 500 for each of the three models.

Practice 4B.4.1: Linear, Exponential, and Quadratic Functions, p. U4B-268

1. Wi-Fi signal speed in megabits per second per meter
2. slope of (0, 450) and (18, 0): –25
   slope of (0, 110) and (55, 0): –2
   slope of (0, 50) and (91, 0): approximately –0.5
   The slopes show that Wi-Fi speed initially decreases very quickly as the distance from the router increases. However, though the Wi-Fi speed continues to decrease as the distance increases, the decline becomes more gradual when the receiving device is farther from the router.
3. An exponential model would best describe the inverse relationship between the distance between a Wi-Fi transmitter (a computer) and a receiver (another computer), because the rate of change of Wi-Fi speed is decreasing very quickly near the signal and very slowly farther from the signal.
4. exponential; [1, 365]
5. linear; $\left(0, \frac{50}{t_{\min}}\right)$
6. exponential; $y = 168$
7. quadratic; the ejection velocity of the magma bomb and the time it is in the air
8. linear; domain: [1, 10]; range: [0.200, 0.250]
9. linear; domain: [–5, 30]; average temperature change per quarter hour: $-2.91\overline{6}\,^{\circ}\text{F}$
10. quadratic; $A(x) = x(2x - 1)$

Warm-Up 4B.4.2, p. U4B-270

1. $(2, \infty)$
2. $f(x) = 4$
3. $(-\infty, -3)$
4. $f(x) = \frac{4}{5}x + \frac{17}{5}$
5. [–3, 2]

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Practice 4B.4.2: Piecewise, Step, and Absolute Value Functions, p. U4B-290

1. The function pieces over the interval $(0, 10^5)$ and $(7 \bullet 10^5, 8 \bullet 10^5)$ are linear functions; the function piece over the interval $[10^5, 7 \bullet 10^5]$ is a constant function, $T(H) = 100$.
2. $T(H) = 0.0006 \bullet H + 40$ over the interval $(0, 10^5)$.
3. $T(H) = 100$ over the interval $[10^5, 7 \bullet 10^5]$.
4. $T(H) = 0.0006 \bullet H - 320$ over the interval $(7 \bullet 10^5, 8 \bullet 10^5)$.
5. Between the values of t at which the function is undefined, the steps of $V(t) = 4$ and $V(t) = -4$ indicate a constant value over their respective intervals.
6. If the pattern of the function values continues for values of t greater than π, then the function's period is π.
7. The function $V(t) = \dfrac{4 \bullet |\sin 2t|}{\sin 2t}$ is equal to either 4 or –4 for all t, except when $t = \dfrac{\pi}{2}$, π, or integer multiples of those values.
8. The efficiency increases as the intensity approaches –1.
9. The efficiency decreases as the intensity approaches –1.
10. $A(i)$ approaches 1 as i becomes more negative. $B(i)$ increases as i becomes more positive.

Warm-Up 4B.4.3, p. U4B-293

1. $f^{-1}(x)$ is a function because its graph shows one-to-one correspondence between its domain and function values.
2. $(-\infty, \infty)$
3. $[5, \infty)$
4. $g^{-1}(x)$ is a function because its graph shows one-to-one correspondence between its domain and function values.
5. $(-\infty, \infty)$
6. $(-\infty, \infty)$
7. The positive and negative halves of $f^{-1}(x)$ are functions individually, but not when both are graphed together, since this violates the criteria for one-to-one correspondence. Also, $f^{-1}(x)$ is undefined for values of x that are less than or equal to 5 because it is a square root function.
   $g^{-1}(x)$ is a function because it meets the one-to-one correspondence criteria for functions. It is also defined for all values of x since x can be less than, equal to, or greater than 0 for a cube root function like $g(x)$.

Practice 4B.4.3: Square Root and Cube Root Functions, p. U4B-313

1. $A(x) = \dfrac{x}{2}\sqrt{3x^2 - 4x + 1}$
2. $f(x) = \pm\sqrt{25 - x^2}$
3. $v = \pm\sqrt{\dfrac{2K}{m}}$
4. $r = \sqrt[3]{\dfrac{V}{2\pi}}$
5. $t = \dfrac{\sqrt[3]{c}}{1 - \sqrt[3]{c}}$
6. $b = \sqrt[3]{\dfrac{250}{p}}$
7. Gemma's pitch is approximately 12% faster than Daisy's.
8. $r = 3$
9. $r_{\text{larger orbit}} = 2\sqrt{2} \bullet r_{\text{smaller orbit}}$
10. $x_{24} - x_{12} = \sqrt[3]{4} - \sqrt[3]{2} \approx 0.33$ cubic unit greater for a volume of 24 cubic feet

Progress Assessment, p. U4B-315

1. d
2. c
3. b
4. a
5. d
6. d
7. d
8. c
9. b
10. a

11. a. The square root function, $p_2(t) = 10\sqrt{t}$, appears to be a slightly better fit. However, populations are usually best modeled by exponential functions.
    b. Creating a combination exponential-linear function of the form $(f \bullet g)(x)$ better fits the shape of the data. A function of this type that fits the data well is $p(t) = 9.5t \bullet 2^{-0.2t}$.
    c. The part b combination function is like the square root function because both depend only on time, and neither depends on any parameters of the wild turkey population.

Lesson 5: Geometric Modeling

Pre-Assessment, p. U4B-321

1. b
2. d
3. c
4. c
5. c

Warm-Up 4B.5.1, p. U4B-326

1. The areas and lengths of the corresponding sides of the triangles are unequal.
2. The corresponding angles of the triangles are congruent, which means that the three triangles are similar.
3. The ratio of length $\overline{AD}$ to length $\overline{DF}$ equals the ratio of length $\overline{AE}$ to length $\overline{EG}$ because they are corresponding sides of a plane figure (the pyramid face, $\triangle ABC$) that is intersected by two parallel line segments, $\overline{DE}$ and $\overline{FG}$. Symbolically, the ratios can be represented by a proportion: $\dfrac{AD}{DF} = \dfrac{AE}{EG}$.
4. The line segments $\overline{DF}$ and $\overline{DC}$ correspond to the line segments $\overline{EG}$ and $\overline{EB}$. Therefore, the ratio of length $\overline{DF}$ to length $\overline{DC}$ equals the ratio of length $\overline{EG}$ to length $\overline{EB}$. Symbolically, the ratios can be represented by a proportion: $\dfrac{DF}{DC} = \dfrac{EG}{EB}$.

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Practice 4B.5.1: Two-Dimensional Cross Sections of Three-Dimensional Objects, p. U4B-342

1. $1.\overline{7}$ in^2
2. 20 mm^2, 24 mm^2, and 30 mm^2
3. 9π ft^2
4. 250 m^2
5. The rotation forms the surface of a portion of a regular cone, not including its bases.
6. The rotation forms a hemisphere.
7. The rotation forms a right regular cylinder.
8. $3\sqrt{3}$ square microns, $5\sqrt{3}$ square microns, and $10\sqrt{2}$ square microns
9. $\frac{27}{2}\pi$ mm^2
10. approximately 2.88 mm

Warm-Up 4B.5.2, p. U4B-344

1. There is an average of $\frac{4}{5\pi}$ or about 0.26 koi per square meter of the pool's surface.
2. There is an average of $\frac{16}{25\pi}$ or about 0.2 koi per cubic meter of the pool's volume.
3. The domain of koi in the pool is the set of whole numbers. The answers to problems 1 and 2 are both in fractional amounts, so they are not in the domain. To yield more meaningful results, use proportions to determine that there is 1 koi about every 3.85 square meters of the pool's surface, and 1 koi for every 5 cubic meters of the pool's volume.

Practice 4B.5.2: Density, p. U4B-362

1. No, because the density of the pendant is 12.5 grams per cubic centimeter. This is greater than the density of pure silver, which is 10.5 grams per cubic centimeter.
2. Adrian would need to sell the building for $35 per square foot in order to recoup the $750,000 he spent on renovations.
3. density estimate: 4.27 plants per m^2. Based on this result, on average a person should expect to find 4 black-eyed Susans within 1 square meter of the meadow—however, actual counts may vary widely from one portion of the meadow to another.
4. a. density estimate: 2.78 plants per m^2; estimated total population: 26,063 plants
   b. Based on the results, you would expect to find a black-eyed Susan more quickly in the meadow from problem 3, because the density of these plants is higher there (4.27 plants/m^2 versus 2.78 plants/m^2).
5. $\rho = \frac{m}{V}$
6. $m_{\text{other}} = 50 - x$
7. $V = V_{\text{bone and muscle}} + V_{\text{other}}$
8. $\rho_{\text{bone and muscle}} = \frac{m_{\text{bone and muscle}}}{V_{\text{bone and muscle}}} = \frac{x}{V_{\text{bone and muscle}}}$ and $\rho_{\text{other}} = \frac{m_{\text{other}}}{V_{\text{other}}} = \frac{50-x}{V_{\text{other}}}$
9. $\rho = \frac{50 \bullet \rho_{\text{bone and muscle}} \bullet \rho_{\text{other}}}{x \bullet \rho_{\text{other}} + \rho_{\text{bone and muscle}} \bullet (50-x)}$
10. about 27.5 kilograms of bone and muscle

Warm-Up 4B.5.3, p. U4B-364

1. The line segments can be used as the diagonals of a parallelogram.
2. The line segments can be used as the sides of a rectangle.
3. The line segments can be used as the legs of a right triangle. One line segment can be set perpendicular to and anywhere along the length of the other line segment to create a right triangle.
4. The line segments form a rhombus.
5. The line segments form a quadrilateral.
6. The line segments form a kite.

Practice 4B.5.3: Design, p. U4B-384

1. $V_{\text{square prism}} = 232.4$ cm^3, and $V_{\text{cylinder}} = 230.9$ cm^3; the square prism jar has the greater volume. Although the square prism jars have the greater volume, 256 square prism jars can fit in the box, whereas only 147 cylindrical jars fit.
2. circumference increase: 33%; area increase: 78%. As the radius increases, the area grows faster than the circumference because the radius is squared to find the area, but it is just multiplied by 2r to find the circumference.
3. $A(r) = 43.30 - \pi r^2$; the graph is a parabola that opens down with a vertex of (0, 43.30) and roots of ±3.71. Domain: (0, 2.89]; range: [17.12, 43.30]
4. minimum: 349; maximum: 449
5. width ≈ 130.22 ft; left roof edge: (0, 785); right roof edge: (130.22, 915); light coordinates: (65.11, 850)
6. $C(l) = 145.45l$; domain: {1, 2, 3, 4}; range: {145.45, 290.90, 436.35, 581.80}; $V_{\text{charcoal}} = 3.97l$ ft^3
7. Costs per square foot: The 20-gauge liner is $0.21 per square foot, and the 30-gauge liner is $0.33 per square foot. Costs per warranty year: The 20-gauge liner is $6.67 per year, and the 30-gauge liner is $6.40 per year. The 30-gauge liner is a better value because it costs less per likely year of use.
8. $V_{\text{total}} = 96w - 24w^2$ ft^3; $w \approx 0.34$ ft. The plastic must be approximately 0.34 ft, or 4.08 in thick.
9. $C(w) = 195.69 + 153.15w - 51.05w^2$; domain: [0, 1.5]; range: [195.69, 310.55]; widest possible width of veneer: approximately 1.045 ft or 12.54 in
10. wet wall cost: $2,744.44; dry wall cost: $3,313.33. The dry wall would be approximately 20.7% more expensive to build.

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Progress Assessment, p. U4B-387

1. b
2. c
3. d
4. a
5. b
6. a
7. c
8. d
9. c
10. d
11. a. $\frac{\pi}{8}$ or about 0.393 cubic meters
    b. $\frac{3\pi}{32} \cdot 10^3$ or about 295 kilograms
    c. The mass of the buoy has to equal the mass of the water displaced when the buoy is put in the river.

Unit Assessment
p. U4B-392

1. a
2. b
3. a
4. d
5. a
6. b
7. c
8. d
9. a
10. b
11. d
12. b
13. a. $h(x)$ is a combination because the functions are subtracted.
    b. $h(x) = \sin(x + 4) - x - 4$
    c. $(-\infty, \infty)$
14. a. $f(x) = \left|\frac{x+2}{x-5}\right|$
    b. all real numbers except 5
15. a. $4\sqrt{3}$ feet
    b. $4\sqrt{2}$ square feet
    c. $80\sqrt{2}$ square feet

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Student Book Answer Key

Lesson 1: Creating Equations

Practice 4B.1.1: Creating Equations in One Variable, p. U4B-23

1. $y = 5x - 2$
2. $y = 2x^2 - 2x$
3. $y = x^2 + 2x - 3$
4. An exponential equation cannot be written for all three points because one of the points has $y < 0$ for $x > 0$, which is not allowed for an exponential equation of the form $y = ab^x$. However, an exponential equation of the form $y = a + bc^x$ can be written for the three points: $y = 4 - 3\left(\frac{3}{2}\right)^x$.
5. There is no linear equation that fits the three given data points.
6. There is no logarithmic equation in one variable that fits the three given data points.
7. $y = -\frac{1}{4}x^2 - \frac{5}{4}x + 1$
8. $d = 30t$
9. $d = 20t + 10$
10. 20 miles per hour

Practice 4B.1.2: Representing and Interpreting Constraints, p. U4B-36

1. $x \le \frac{7}{4}$ or $\left(-\infty, \frac{7}{4}\right]$
2. $x < 10^{-20} - 10$ or $\left(-\infty, 10^{-20} - 10\right)$
3. $x \ge \frac{5}{2}$ and $x \le -\frac{5}{2}$ or $\left(-\infty, -\frac{5}{2}\right] \cup \left[\frac{5}{2}, +\infty\right)$
4. $\left\{\frac{1}{2} < x < 1\right\}, \{0 < y < 1\}$
5. $\{x \le 0\}, \left\{y \le -\frac{1}{2}\right\}$
6. $\{x < 3\}, \{\varnothing\}$. There are no common y-values for the two inequalities over the common domain.
7. $\{\varnothing\}$
8. acceleration: constant; distance: [0, 0.25 mile]; speed: [0, s); time: [0, 2.5 seconds]
9. The constraint on the capacity of the trough is given by the interval [0, 900]; the constraint on the rate at which water is removed from the trough is a constant, $r = -60$ gallons per day; however, the automatic refilling of the trough, 30 gallons every two days, is added to the daily removal rate: $r = -60$ gallons per day + 15 gallons per day = -45 gallons per day; the equation $900 - 60d + 15d = 0$ gives the number of days it will take to empty the trough, or $d = 20$ days; if the trough is refilled by the rancher before 20 days is up, it will not go empty.
10. $N(t) < 30$ pairs or the interval [2, 30); $N_0 = 30$ pairs and is the defined carrying capacity of the ecosystem; t is time in years and $t \ge 0$.

Practice 4B.1.3: Rearranging Formulas, p. U4B-53

1. $k = \frac{P}{2L^2}$
2. $E_a = \frac{2.3RT}{A}\left(\log A - \log k\right)$
3. $[\mathrm{A}^-] = \frac{[\mathrm{HA}] \bullet 10^{\mathrm{pH}}}{10^{\mathrm{p}K_a}}$ or $[\mathrm{A}^-] = [\mathrm{HA}] \bullet 10^{\mathrm{pH} - \mathrm{p}K_a}$
4. $i = \frac{f \bullet o}{o - f}$. If $i < 0$, $\frac{f \bullet o}{o - f} < 0$ or $\frac{f \bullet o}{f - o} > 0$; this means one of two things: $f \bullet o > 0$ and $f > o$, which means the object is located between the focal point and the lens; or $f \bullet o < 0$ and $f < o$, which means that the object is located farther from the lens than the focal point.
5. The escape velocity $v_{\text{escape}} = \sqrt{\frac{2GM_E}{r_{\text{actual}} + 7500}}$ is less than the escape velocity $v_{\text{escape}} = \sqrt{\frac{2GM_E}{r_{\text{actual}} - 7500}}$ because the denominators differ based on the radius inequality.
6. $\chi_L = \chi_C + \sqrt{Z^2 - R^2}$
7. $\mu = \frac{\log 2}{t_{\text{gen}} \bullet \log e}$
8. $\frac{P}{T} = \frac{nR}{V}$. If the number of moles, n, stays the same and V increases, the ratio of pressure to temperature will decrease. If the volume stays the same and the number of moles increases, the ratio of pressure to temperature will increase.

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9. [A] • [B] < [AB]; the product of the concentrations of the elements A and B is less than the concentration of the compound AB.
10. $p = -q \pm 1$ or for $p > 0$ and $q > 0$, $p = 1 - q$; $p = \frac{1}{3}$

Lesson 2: Transforming a Model and Combining Functions

Practice 4B.2.1: Transformations of Parent Graphs, p. U4B-71

1. $f(x) = 4(x - 1)$; $g(x) = 4(x + 1)$
2. $f(x) = -2x^2$; $g(x) = 3x^2$
3. $f(x) = 2(4^x)$; $g(x) = -3(4^x)$
4. domain: $(-\infty, \infty)$; range: $[0, \infty)$
5. domain: $(0, \infty)$; range: $(-\infty, \infty)$
6. domain: $(-\infty, \infty)$; range: $(-\infty, 1)$
7. If the value of a is doubled and b is increased by 2, then $A(a, b) = \pi(2a - b - 2)^2$.
8. If the volume is doubled and r is decreased by a factor of $\frac{2}{3}$, then $t = \frac{1}{1200} \bullet \frac{V}{r}$.
9. For a pH range of (0, 14), the domain of c is (10^{-14}); for a pH range of (7, 10), the domain of c is $(10^{-10}, 10^{-7})$.
10. If the range of $p(n)$ is (0, 5000), the domain of n is [67, 80). If the constant term –25,000 changes to –40,000 and the coefficient of n increases to 400, the domain of p changes to (100, 113].

Practice 4B.2.2: Recognizing Odd and Even Functions, p. U4B-79

1. For an even function: Remove x^3 and x to give $a(x) = x^2 + 1$.
   For an odd function: Remove x^2 and 1 to give $a(x) = x^3 + x$.
2. For an even function: Remove –5 from the absolute term to give $b(x) = 5 - |x|$.
   For an odd function: Change $|x - 5|$ to $(x + 5)$ to give $b(x) = 5 - (x + 5) = -x$.
3. For an even function: $c(x) = \frac{x^2}{1 + x^2}$.
   For an odd function: $c(x) = \frac{x}{1 + x^2}$.
4. For an even function: No changes are required because the given function is already even: since sin (90° – x) is equal to cos x, then $d(x) = 2 \bullet \cos x$, which is even.
   For an odd function: Remove cos x and "90° –" from the sine term, which gives $d(x) = \sin x$.
5. $f(x) = 0.5 \bullet \cos 0.2x$; even
6. $g(x) = x \bullet e^{|x|}$; odd
7. $h(x) = \frac{x}{x - 1}$; neither
8. Both are even.
9. $I_{\text{average}} = \frac{1}{2}(I_1 + I_2)$
10. The function $f(x)$ is even and the function $g(x)$ is odd.

Practice 4B.2.3: Combining Functions, p. U4B-85

1. composition
2. both
3. combination
4. $(f \circ g)(x) = 2^{\log_2 x} = x$; domain: $(0, \infty)$
   $(g \circ f)(x) = \log_2 2^x = x$; domain: $(-\infty, \infty)$
5. $\frac{(f \circ g)(x)}{(g \circ f)(x)} = \frac{2^{\log_2 x}}{\log_2 2^x} = 1$; domain: $(0, \infty)$
   $\frac{(g \circ f)(x)}{(f \circ g)(x)} = \frac{\log_2 2^x}{2^{\log_2 x}} = 1$; domain: $(0, \infty)$
6. $(f \circ f)(x) = 2^{2^x}$; domain: $(-\infty, \infty)$
   $(g \circ g)(x) = \log_2(\log_2 x)$; domain: $(0, \infty)$
7. $(f^2 \circ g)(x) = 2^{2 \bullet \log_2 x} = x^2$; domain: $(0, \infty)$
   $(g \circ f^2)(x) = \log_2(2^{2x}) = 2x$; domain: $(-\infty, \infty)$
8. $S = \pi t^4$; 1,963 in^2
9. $V(t) = \frac{r \bullet Q(t)}{t}$
10. 30 meters per second

Lesson 3: Comparing Properties Within and Between Functions

Practice 4B.3.1: Reading and Identifying Key Features of Real-World Situation Graphs, p. U4B-112

1. The robot represented by $f(t)$ takes longer to complete the initial steps than the robots represented by $g(t)$ and $h(t)$; the robot represented by $h(t)$ completes the initial steps fastest, but its rate of completion of the 10 steps is lower than that of the other two robots.
2. The intersection point is the time at which $f(t) = g(t) = h(t)$; for these functions, that value of t is $t = 1$. At $t = 1$, all three robots have completed the same step, so $f(t) = g(t) = h(t)$.
3. In the real-world of manufacturing, there would be a finite time t at which all three robots would complete the 10 CPU-fabrication steps since a "step" might be assumed to be a positive whole number; mathematically, none of the three robots would ever complete the tenth step if a "step" could be any positive number; all three would get arbitrarily close to 10, depending on the value of t selected.
4. All of the data points could be fitted to a linear function; however, two or more linear functions over restricted domains that make up the whole domain of the data points might give a better model.
5. If the function $N(t)$ given in the problem is used, the y-axis scale would have to be 0 to at least 15. The graph represents the exponential function given in the problem, so values of $N(t)$ would increase quickly for initial values of time but would decline over time.

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6. The corn-yield linear-function model results in higher yields as the nitrogen initially increases. The corn yield continues to grow at a constant rate as the rate of nitrogen use decreases.
7. The domain for this function model is a restricted domain of 13 years; the domain of the mathematical sine function is unrestricted. The range of $H(t)$ is defined by the constant 3.2 and the amplitude of 2.2, which gives a range of [1, 5.4].
8. $H(0) = 1\%$. This represents the initial market share of the hybrid cars when the market-share tracking started in 2007.
9. approximately [0, 1.9]
10. The hybrid function model shows a decreasing rate of market share, whereas the battery function model shows a rate of market share that is decreasing less than that of the hybrid model. The periods of the two function models differ, which is one reason the hybrid function model (with a period of $t = 8\pi$) reaches a maximum before the battery function model (with a period of $t = 10\pi$) does.

Practice 4B.3.2: Calculating Average Rates of Change, p. U4B-129

1. 0
2. –1
3. 0.841
4. –0.46
5. For [–2, –1], $r = -\frac{1}{6}$; for [–1, 0], $r = -\frac{1}{2}$.
6. For [–2, –1], $r = \sin 4 - \sin 2$; for [–1, 0], $r = \sin 2$.
7. For [–2, –1], $r = \log 2$; for [–1, 0], $r = \log\left(\frac{3}{2}\right)$.
8. The yield of the variety represented by $f(x)$ is greater than that of the variety represented by $g(x)$ when the plants use the nitrogen mostly for forming leaves.
9. The amount of fertilizer needed to produce a yield of mostly stems increases at the same rate for both varieties after the leaves are formed, because the lines are parallel, indicating the same rate of nitrogen usage.
10. Biofuel producers would most likely prefer the variety represented by $g(x)$ for economic reasons; this variety requires less nitrogen fertilizer per metric ton of yield per hectare, meaning reduced fertilizer expenses.

Practice 4B.3.3: Comparing Functions, p. U4B-142

1. There are four maximum function values.
2. There are four minimum function values.
3. Either a quadratic or sine function could be used; in either case, the domain would differ for three or four different functions. If a sine function is used, the function value would be an absolute value of the sine function value.
4. –27.16 feet per second
5. –28.16 feet per second
6. –29.16 feet per second
7. The average change of Josh's function, $J(t) = -16t^2 + 4t$, can be used to find the speed over any interval. For example, the speed between 1 and 2 seconds is $s = \frac{J(2) - J(1)}{(2) - (1)} = -44$ feet per second.
8. $[x, x + \ln 2 - \ln 3]$
9. Two intervals are possible: $\left[x, \frac{x + \sqrt{x^2 - 40x}}{2}\right]$ and $\left[x, \frac{x - \sqrt{x^2 - 40x}}{2}\right]$.
10. $[x, 2 - x]$

Lesson 4: Choosing a Model

Practice 4B.4.1: Linear, Exponential, and Quadratic Functions, p. U4B-161

1. average rates of change in degrees per second (°/s): –1°/s; –0.5°/s; –0.25°/s; –0.125°/s; –0.0625°/s
2. exponential
3. average rates of change in meters per second per second (m/s/s): 2.5 m/s/s; –1.5 m/s/s; –1 m/s/s; –0.5 m/s/s; 0 m/s/s; an exponential model could be used for the speed changes.
4. exponential; [1, 30]
5. linear; $\left(0, 33.\overline{3}\right)$
6. quadratic; –2 strokes/game; –1 strokes/game
7. exponential; asymptote: $y = 0$
8. quadratic; average rates of change: 2 points/game, 3 points/game, and –1 point/game; predicted score for the next game: about 78 points
9. linear; $\frac{10}{7}$ miles per hour per day
10. quadratic; $V(x) = 0.12(0.3x)(0.4x - 0.3) = 0.044x^2 - 0.0108x$

Practice 4B.4.2: Piecewise, Step, and Absolute Value Functions, p. U4B-175

1. The total piecewise relation does not associate each domain value with a unique function value; e.g., the functions represented by curves using the series of points *A*, *B*, and *C* and *C*, *E*, and *F* do not show this one-to-one correspondence.
2. Points *A*, *B*, and *C* and points *C* and *D* could form a piecewise function; points *C* and *D* and points *C*, *E*, and *F* could also form a piecewise function.
3. (80, 250)
4. One function could consist of *ABC* and *CD*: *ABC* could define a quadratic function, and *CD* could define an exponential or a linear function. Another function could consist of *CD* and *CEF*: *CD* could define an exponential or a linear function, and *CEF* could define a quadratic function.
5. The data represents a piecewise function because for each interval, [–3, –1], [–1, 1], and [1, 3], the slopes of the linear pieces are different.

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6. $a = 1$
7. $a = 1$ and $b = 2$ for all three restricted domains
8. points C_1, C_2, and C_3 could form a linear function over the domain [0, 6] because of its constant slope; C_3, C_4, and C_5 could form a quadratic function over the domain [6, 19.5] because of its decreasing rate of change over that interval; C_5 and C_6 could form a linear function over the domain [19.5, 27] because of its constant slope.
9. $S_1S_2S_3S_4$ could form a linear function over the domain [0, 18] because of its constant slope; $S_4S_5S_6$ could form a quadratic function over the domain [18, 27] because of its increasing rate of change over that interval.
10. Profit occurs when $S(t) > C(t)$, which is between $t = 25.5$ and $t = 27$.

Practice 4B.4.3: Square Root and Cube Root Functions, p. U4B-190

1. $A(x) = (x-3)\sqrt{6x-9}$
2. $f(x) = \frac{3}{2}\sqrt{4-x^2}$
3. $L = \frac{1}{2\pi}\sqrt{gT}$
4. $x = \sqrt[3]{V}$
5. $g = \frac{1}{5} \bullet \sqrt[3]{\frac{1}{P}}$
6. $\sqrt[3]{-bt^3 + 10^2 \bullet t^2 - 10^4 \bullet t} = 5 \bullet \sqrt[3]{4v}$
7. The arrow from the first bow is approximately 8% faster than the arrow from the second bow.
8. $r = \sqrt[3]{\frac{3 \bullet V_{\text{difference}}}{4(6-\pi)}}$
9. $T_{\text{larger sun}} = \frac{1}{\sqrt{2}} \bullet T_{\text{smaller sun}} \approx 0.7 \bullet T_{\text{smaller sun}}$
10. Yes; for a volume of 5 cubic feet, $x \approx 0.457$.

Lesson 5: Geometric Modeling

Practice 4B.5.1: Two-Dimensional Cross Sections of Three-Dimensional Objects, p. U4B-207

1. 12 in^2
2. 4 m^2; 14 m^2
3. $\frac{(3\pi-1)^2}{\pi}$ yd^2
4. $300\sqrt{3}$ cm^2
5. The rotation forms a sphere.
6. The rotation forms a hemisphere.
7. The rotation forms a portion of a cylinder reduced by the volume of the cone created by the triangle.
8. 2 m^2; 6 m^2
9. $2 \bullet \sqrt[3]{\frac{75}{4}} \approx 5.3$ microns
10. $\frac{\sqrt{3}}{3} \bullet 10^{-9}$ m and $\sqrt[3]{3} \bullet 10^{-9}$ m

Practice 4B.5.2: Density, p. U4B-220

1. 0.098 or 9.8% of the iceberg is above the water line.
2. a. 113.2 PSI
   b. 256.1 pounds
3. The dust-free surface area of the electric eye was reduced to 11.67 square centimeters.
4. The nuggets are about 77.2% gold.
5. $\text{mass}_{\text{displaced oil}} = \text{mass}_{\text{doughnut}}$
6. $\text{mass}_{\text{displaced oil}} = \frac{925\text{ kg}}{\text{m}^3} \bullet \frac{\text{m}^3}{100^3\text{ cm}^3} \bullet \frac{1000\text{ g}}{\text{kg}} \bullet$
   $x\text{ cm} \bullet 6\text{ cm} \bullet 10\text{ cm} = 55x\text{ grams}$
7. $\text{mass}_{\text{doughnut}} = \frac{400\text{ kg}}{\text{m}^3} \bullet \frac{\text{m}^3}{100^3\text{ cm}^3} \bullet \frac{1000\text{ g}}{\text{kg}} \bullet$
   $4\text{ cm} \bullet 6\text{ cm} \bullet 10\text{ cm} = 96\text{ grams}$
8. $55.5x = 96$; $x \approx 1.73$ cm
9. $\frac{1.73}{4} \approx 0.43$
10. $\frac{400}{925} \approx 0.43$; the values of this ratio and the ratio from problem 9 are equal because the relative density of the oil and dough determines the proportion of the doughnut bar that will float above the oil. The denser the oil is relative to the dough, the higher the doughnut bar will float, and the smaller x will be.

Practice 4B.5.3: Design, p. U4B-234

1. triangular chocolate: 11.25 cm^3; square chocolate: 10.8 cm^3. The triangular bar contains more chocolate.
2. perimeter increase: 25%; area increase: 56.25%. The area increases at a greater rate than the perimeter because the perimeter is based on the side length, whereas the area is based on the *square* of the side length.
3. $A(r) = 100 - \pi r^2$; the graph is a parabola that opens down with its vertex at (0, 100) and roots at $r \approx \pm 5.64$; domain: (0, 5]; range: [21.46, 100)
4. 2,373,718 blocks
5. 16 ft
6. $C(l) = 29.6l + 258.40$
7. Brand A paint: \$254.80; Brand B paint: \$313.60; the homeowner should buy Brand A.
8. The hollow interior must have a radius of 21.7 cm, and the remaining concrete will be 28.3 cm thick.
9. 1.13 in
10. The smaller stones cost \$593.42 to install, and the larger stones cost \$503.11 to install. Clive should choose the larger stones.

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Instruction

Goal: To provide opportunities for students to interpret and analyze functions using different representations

Common Core State Standards

F–IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F–IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

F–BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Student Activities Overview and Answer Key

Station 1

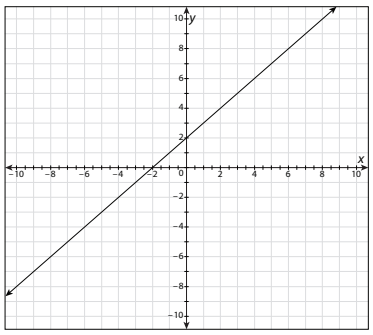
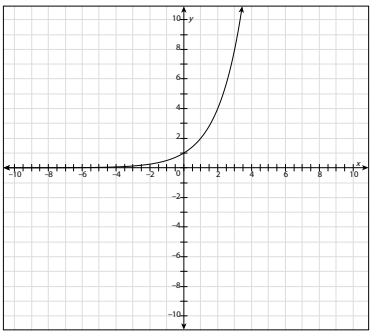
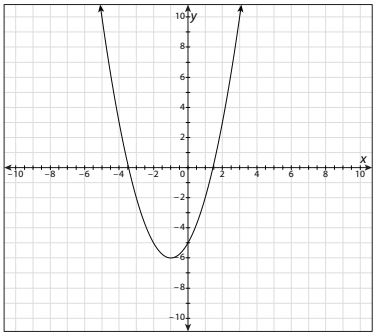
At this station, students will find four sets of index cards, each depicting a function in one of four formats (equations, tables, graphs, or real-world situations). Students will use their knowledge of the key features of functions to identify the functions as linear, exponential, or quadratic, and sort the cards accordingly. Students will then explain how they used the key features of the three types of functions to sort the cards.

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Instruction

Answers

1. Completed table (including brief descriptions of the cards):

	Linear functions	Exponential functions	Quadratic functions
Equation	E3 $f(x) = -2x + 7$	E2 $f(x) = 4(2^x)$	E1 $f(x) = 2x^2 + 5x + 10$
Table	T1 (y-values: 1, 3, 5, 7, 9)	T2 (y-values: 1, 2, 4, 8, 16)	T3 (y-values: –1, –2, –1, 2, 7)
Graph	G3 	G1 	G2 
Situation	S1 Bryson has \$20 and saves \$10 each week.	S2 Camila took in a stray dog with fleas.	S3 Noah tosses a ball to his younger sister.

2. Answers may vary, but should reflect an application of the key features of each type of function to sort the cards. Sample answers:

   Linear functions: The graph of a linear function is a straight line and the average rate of change is the same for every interval. In the equation, the variable does not have an exponent other than 1.

   Exponential functions: The graph of an exponential function is curved and the overall shape indicates growth. As x increases, y also increases, but at a rapid rate; the average rate of change increases as intervals move to the right. The table of values shows the y-value doubling for each x-value, and in the equation the variable is an exponent.

   Quadratic functions: The graph of a quadratic function opens up and the table of values shows the second difference to be the same. In the equation, the variable is raised to a power of 2.

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Instruction

Station 2

Given a graph or table, students will identify the key features of the function depicted and use these key features to answer questions.

Answers

1. a. domain: [–4, 6]; range: [–6, 6]
   b. x-intercept: (–3.5, 0); y-intercept: (0, 3)
   c. maximum: $y = 6$, (–2, 6); minimum: $y = –6$, (–4, –6)
   d. increasing: (–4, –2); decreasing: (–2, 0)∪(1, 3)
   e. As x approaches $–\infty$, y approaches –6; as x approaches $+\infty$, y approaches 1.
2. a. domain: (–2, –1, 0, 1, 2); range: (1, 2, 5)
   b. x-intercept: none; y-intercept: (0, 1)
   c. maximum: none; minimum: (0, 1)
   d. As x approaches $–\infty$, y approaches $+\infty$; as x approaches $+\infty$, y approaches $+\infty$.
   e. even function
3. a. domain: [–4, 4]; range: [–1, 3]
   b. x-intercept: (–1, 0), (2, 0); y-intercept: (0, –1)
   c. maximum: none; minimum: $y = –1$, (0, –1)
   d. increasing: (–4, 3)∪(0, 4); decreasing: (–3, 0)
   e. neither
4. a. $T(8) = 40$
   b. $T(12) = 6$ P.M.
   c. 50 degrees
   d. after $T(12)$ or 6 P.M.
   e. Answers may vary. Sample answer: between 11 A.M. ($T(5)$) and 10 P.M. ($T(16)$).
5. a. 0
   b. 4
   c. 1 and –1
   d. –3, 5

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Instruction

Station 3

Students will explore the relationships between square root and cube root functions by completing tables of values for related functions using a graphing calculator, then observing the differences between the functions.

Answers

1.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
$f(x)$	—	—	—	—	—	0	1	1.41	1.73	2	2.24

a. domain: $(0, \infty)$; range: $(0, \infty)$

b. The graph increases toward the right; the table shows only positive values for $f(x)$.

c. There is nothing to the left of the y-axis because there are no real values for the square root of a negative number.

2.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
$g(x)$	—	—	—	—	0	1	1.41	1.73	2	2.24	2.45

a. domain: $(–1, \infty)$; range: $(0, \infty)$

b. Transformation: the graph of $g(x)$ is shifted 1 unit to the left of $f(x)$.

c. The graph increases toward the right (in a positive direction).

d. Since there is a shift of 1 unit to the left, there is nothing to the left of –1 on the graph.

3.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
$h(x)$	—	—	—	—	—	1	3	3.83	4.46	5	5.47

a. domain: $(0, \infty)$; range: $(1, \infty)$

b. Transformation: the graph of $h(x)$ is vertically stretched by a factor of 2 and shifted up 1 unit from $f(x)$.

c. The graph increases toward the right; the table shows only positive values for $h(x)$.

4. The domain is limited to any value of x that will make the expression underneath the radical positive because there are no real values for the square root of a negative number.

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5.

x	–5	–4	–3	–2	–1	0	1	2	3
$j(x)$	–1.71	–1.59	–1.44	–1.26	–1	0	1	1.26	1.44

a. domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

b. The table includes positive and negative values that are opposites of each other.

c. The x- and y-values keep decreasing to the left and increasing to the right; there is no asymptote.

6.

x	–5	–4	–3	–2	–1	0	1	2	3
$k(x)$	–0.44	–0.26	0	1	2	2.26	2.44	2.59	2.71

a. The table includes positive and negative values.

b. Transformation: $k(x)$ is shifted 2 units to the left and up 1 unit from $j(x)$.

Station 4

Students will write function equations to reflect given transformations of parent functions, and use graphing calculators to confirm their results.

Answers

1. $g(x)=2\sqrt{x-3}+1$
2. $g(x)=\sqrt[3]{x-1}$
3. $g(x)=-\frac{1}{2}(2)^{x+2}$
4. $g(x)=(x-1)^2+2$
5. $g(x)=|x+2|-1$
6. $g(x)=\begin{cases}-(x+3), x<-3\\(x+3), x\geq -3\end{cases}$

Materials List/Setup

Station 1 sheets of card stock or 15 index cards, for printing out the Station 1 cards from the Card Master Sheets that follow. There are 15 cards in all.

Station 2 graphing calculator

Station 3 graphing calculator

Station 4 graphing calculator

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Instruction

Card Master Sheets: Station 1

There are five sets of cards for this station: one set of Function Category cards, and four sets of Function cards (Equations, Tables, Graphs, and Real-world Situations). You may wish to copy or print these cards onto card stock, or make copies that you can cut out and paste onto index cards.

Function Category Cards

Linear	Exponential	Quadratic

Equation Cards

E1. $f(x)=2x^2+5x+10$	E2. $f(x)=4\left(2^x\right)$	E3. $f(x)=-2x+7$

(*continued*)

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Instruction

Table Cards

T1.

x	–2	–1	0	1	2
y	1	3	5	7	9

T2.

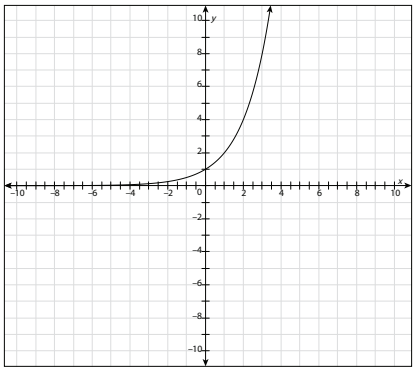
x	–2	–1	0	1	2
y	1	2	4	8	16

T3.

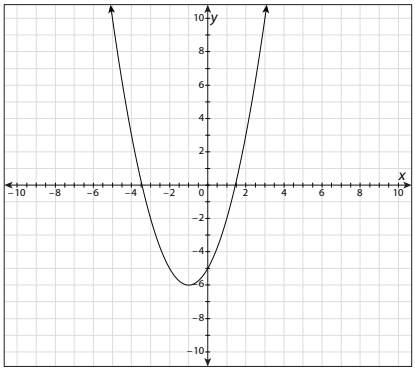
x	–2	–1	0	1	2
y	–1	–2	–1	2	7

Graph Cards

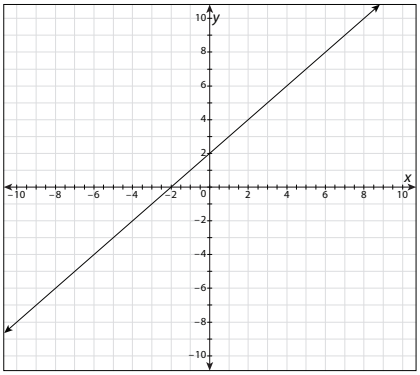
G1.



G2.



G3.



(*continued*)

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Instruction

Real-world Situation Cards

S1.	S2.	S3.
Bryson needs to save up for a new guitar case. He starts with $20 and saves $10 each week.	Camila took in a stray dog. As she was leaving for work the next morning, she noticed 10 fleas on his neck. After she came home, she found that the amount of fleas appeared to have doubled.	Noah tosses a ball to his younger sister. The height, in feet, of the ball above the ground is given by $-16t^2 + 164t + 190$, where $t \geq 0$ and represents the time in seconds.

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Instruction

Discussion Guide

To support students in reflecting on the activities and synthesizing their ideas, and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. When given a graph, what is the most noticeable feature that will help you identify the type of function in the graph?
2. How can you distinguish a linear function, an exponential function, and a quadratic function from one another using a table of data?
3. What are the parent functions for each of the following functions: linear, exponential, quadratic, absolute value, square root, and cube root?
4. What type of function is similar to the function $f(x)=|x|$?
5. Consider the functions $f(x)=\sqrt[3]{x}$ and $g(x)=\sqrt{x}$. Which function has a restriction on its domain, and why?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. The most important feature is the overall shape of the graph. A straight line would indicate a linear function, a graph that shows growth or decay would indicate an exponential function, and a graph that opens upward or downward (i.e., a parabola) would indicate a quadratic function.
2. A linear function has a constant rate of change, an exponential function has a constant ratio between *y*-values, and in quadratic functions, the second difference is the same.
3. Parent linear function: $f(x) = x$; parent exponential function: $f(x) = b^x$; parent quadratic function: $f(x) = x^2$; parent absolute value function: $f(x)=|x|$; parent square root function: $f(x)=\sqrt{x}$; and parent cube root function: $f(x)=\sqrt[3]{x}$.
4. A piecewise function is similar to the function $f(x)=|x|$.
5. $g(x)=\sqrt{x}$ has a restricted domain because there are no real values for the square root of a negative number.

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Instruction

Possible Misunderstandings/Mistakes

- Confusing *x*- and *y*-intercepts
- Forgetting to look at the *y*-values when analyzing end behavior
- Forgetting to look at the *x*-values when analyzing intervals of increase/decrease
- Incorrectly reading a graph or table
- Making arithmetical errors when completing a table of values

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Station 1

At this station, you will find three cards labeled with the following categories of functions: Linear, Exponential, and Quadratic. You will also find four sets of cards depicting functions:

- 3 Equation cards, numbered E1–E3
- 3 Table cards, lettered T1–T3
- 3 Graph cards, lettered G1–G3
- 3 Real-world Situation cards, lettered S1–S3

Using your knowledge of the key features of functions, work with your group to complete the problems that follow.

1. Spread out the three category cards. Identify each of the twelve function cards as a linear, exponential, or quadratic function. Arrange the function cards beneath the correct category card. Record the code for each card (e.g., S1) in the appropriate cell of the table that follows.

	Linear functions	Exponential functions	Quadratic functions
Equation			
Table			
Graph			
Situation			

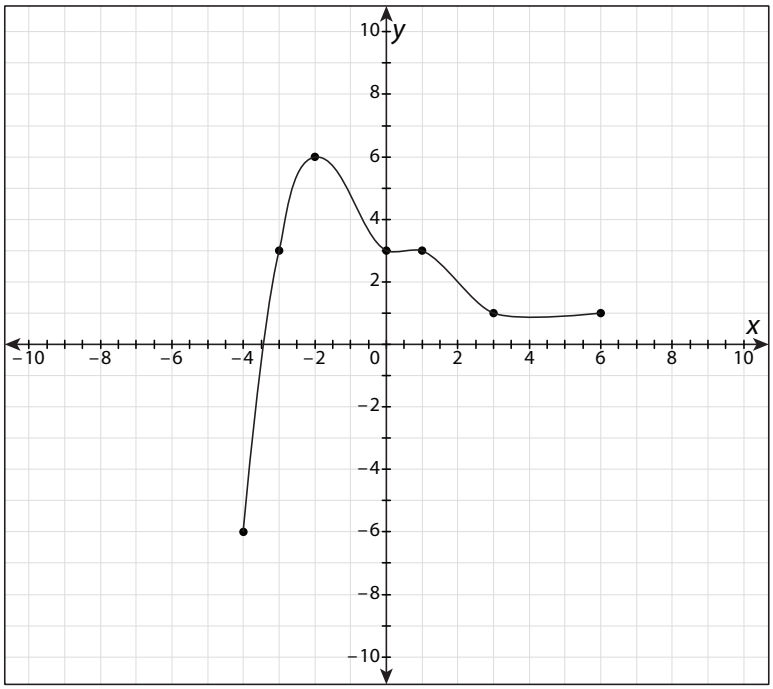
2. Describe the key features of each type of function (linear, exponential, and quadratic), and explain how you used these features to sort the cards.

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Station 2

Work with your group to complete all parts of the problems involving the key features of the function described in each table or graph.

1.



a. Identify the domain and range.

b. Determine the *x*- and *y*-intercepts.

c. Identify the maximum and minimum values.

d. Determine the intervals where the function is increasing and/or decreasing.

e. Describe the end behavior of the graph.

2.

x	–2	–1	0	1	2
y	5	2	1	2	5

a. Identify the domain and range.

b. Determine the *x*- and *y*-intercepts.

c. Identify the maximum and minimum values.

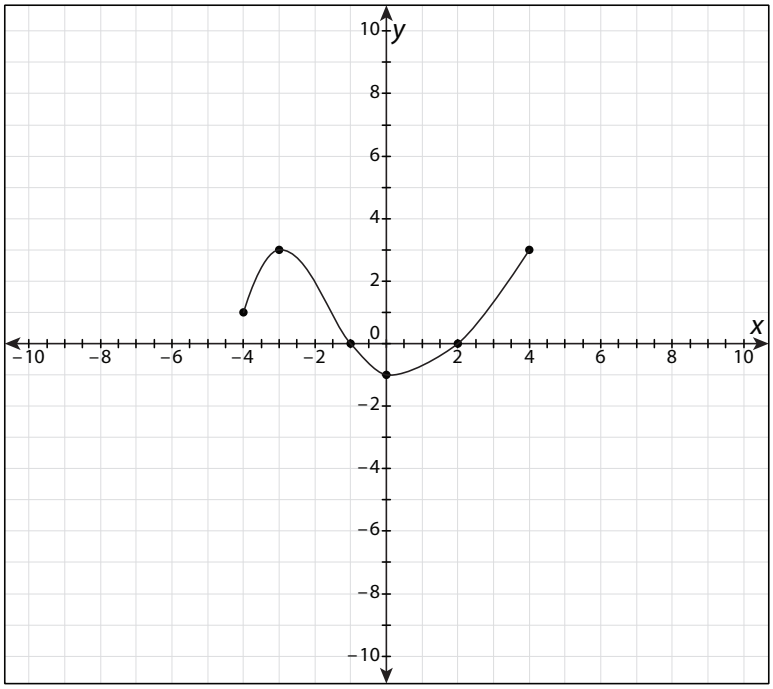
d. Describe the end behavior of the graph.

e. State whether the function is even or odd.

continued

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3.



a. Identify the domain and range.

b. Determine the *x*- and *y*-intercepts.

c. Identify the maximum and minimum values.

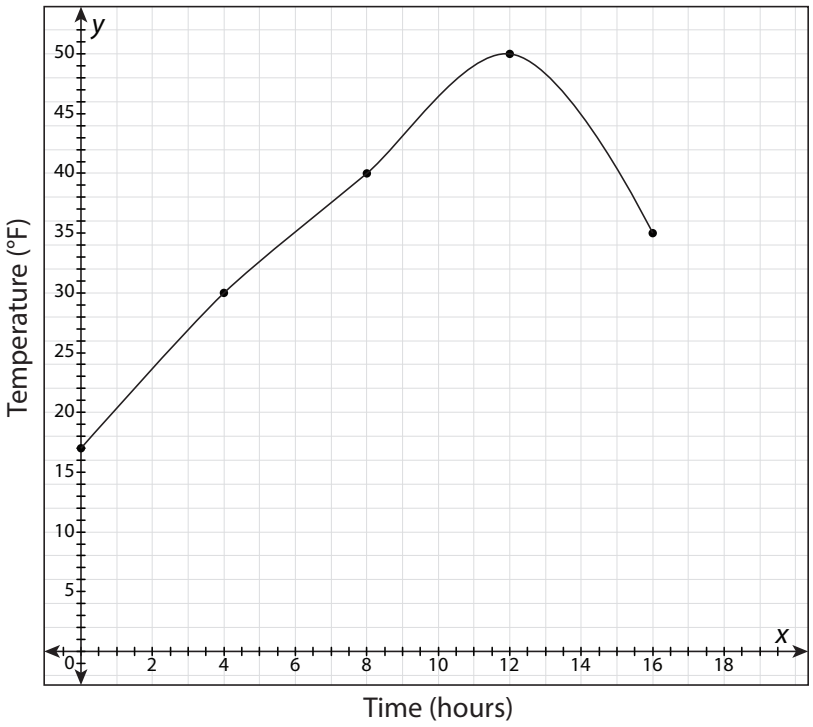
d. Determine the intervals where the function is increasing and/or decreasing.

e. State whether the function is even or odd.

continued

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4. The following graph shows the Temperature (T) in degrees Fahrenheit over a 16-hour period in New York as a function of time (t). Use the graph to complete all parts of the problem.



a. Use the graph to estimate the value of $T(8)$.

b. If $T(0)$ corresponds to 6 A.M. and $T(2)$ corresponds to 8 A.M., what time corresponds to $T(12)$?

c. Estimate the time or time frame when the temperature is the highest.

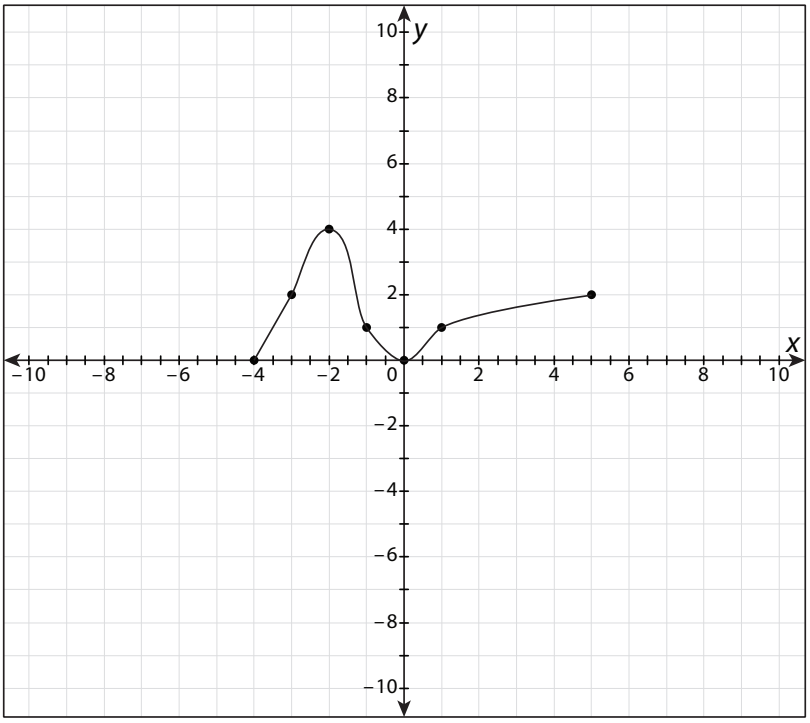
d. Estimate the time or time frame when the temperature is decreasing.

e. If you wanted to go for a walk and return before the temperature drops below freezing (32°F), when should you go?

continued

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5. Use the following graph to determine each of the values as indicated.



a. $f(0)$

b. $f(-2)$

c. $f(x) = 1$

d. $f(x) = 2$

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Station 3

Work as a group to explore the given square root and cube root functions. Complete each table and graph each function using a graphing calculator. Then, use the table and graph to complete the problems.

1. Complete the table of values for the square root function $f(x)=\sqrt{x}$.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
$f(x)$											

   a. Determine the domain and range of the function.

   b. What do you notice as you complete the table?

   c. What do you notice about the graph on the calculator?

2. The square root function $g(x)=\sqrt{x+1}$ is a transformation of the graph of $f(x)$ from problem 1. Complete the table of values for $g(x)$.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
$g(x)$											

   a. Determine the domain and range of the function.

   b. Describe the transformation from the graph of $f(x)$ to $g(x)$.

   c. What do you notice as you complete the table?

   d. What do you notice about the graph on the calculator?

continued

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3. The square root function $h(x)=2\sqrt{x}+1$ is a transformation of the graph of $f(x)$ from problem 1. Complete the table of values for $h(x)$.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
$h(x)$											

   a. Determine the domain and range of this function.

   b. Describe the transformation from the graph of $f(x)$ to $h(x)$.

   c. What do you notice as you complete the table?

4. Based on your answers to problems 1–3, what can you conclude about the domain of a square root function?

5. Complete the table of values for the cube root function $j(x)=\sqrt[3]{x}$.

x	–5	–4	–3	–2	–1	0	1	2	3
$j(x)$									

   a. Determine the domain and range of this function.

   b. What do you notice as you complete the table?

   c. What do you notice about the graph on the calculator?

6. The cube root function $k(x)=\sqrt[3]{x+2}+1$ is a transformation of the graph of $j(x)$ from problem 5. Complete the table of values for $k(x)$.

x	–5	–4	–3	–2	–1	0	1	2	3
$k(x)$									

   a. What do you notice as you complete the table?

   b. Describe the transformation from the graph of $j(x)$ to $k(x)$.

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Station 4

For each problem, the equation of a parent function is given. Work with your group to create a new equation that depicts the transformation(s) described. Use a graphing calculator to verify your equations.

1. Parent function: $f(x)=\sqrt{x}$

   Transformations: vertical stretch by a factor of 2 units; shift 3 units to the right and 1 unit up

2. Parent function: $f(x)=\sqrt[3]{x}$

   Transformation: shift 1 unit to the right

3. Parent function: $f(x) = 2^x$

   Transformations: shrink by a factor of $\frac{1}{2}$; reflect over the x-axis; shift 2 units to the left

4. Parent function: $f(x) = x^2$

   Transformations: shift 1 unit to the right and 2 units up

5. Parent function: $f(x)=|x|$

   Transformations: shift 2 units to the left and 1 unit down

6. Parent function: $f(x)=|x|$

   Transformations: write as a piecewise function with a shift of 3 units to the left

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Instruction

Goal: To use geometric methods to design and solve problems

Common Core State Standards

G–GMD.4 Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

G–MG.1 Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).★

G–MG.2 Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★

G–MG.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★

Student Activities Overview and Answer Key

Station 1

Students will sculpt various shapes from modeling clay, and predict the shapes of possible cross sections of their sculptures. Then, students will use given information to first find the area of a cross section of a three-dimensional object and then find the object's volume.

Answers

1. cylinder: circle, rectangle; cube: square, triangle; cone: circle, triangle; sphere: circle
2. 113.1 in^2
3. 17,157.3 cm^3
4. 1,000 mm^2
5. 6,283.19 m^3

Station 2

Students will apply the concept of density based on volume to determine the substance from which a given canister is constructed.

Answers

1. 261.5 in^3
2. 4285.22 cm^3
3. 8.9 g/cm^3
4. copper

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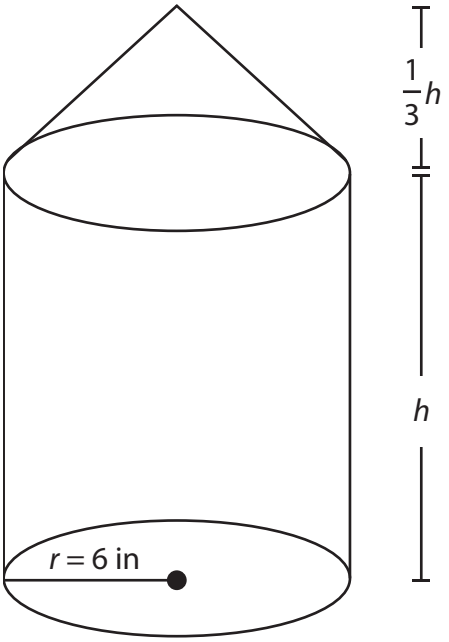
Instruction

Station 3

Students will apply geometric methods to solve problems.

Answers

1. Answers will vary. Sample sketch:



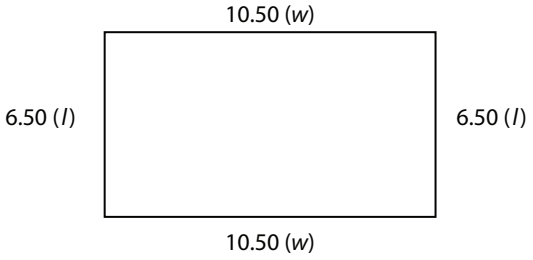
2. $V = 40\pi h$
3. cylinder: $h = 5.5$ in; cone: $h \approx 1.83$ in; entire container: $h \approx 7.33$ in
4. approximately 5 bags

Station 4

Students will apply geometric methods to area problems.

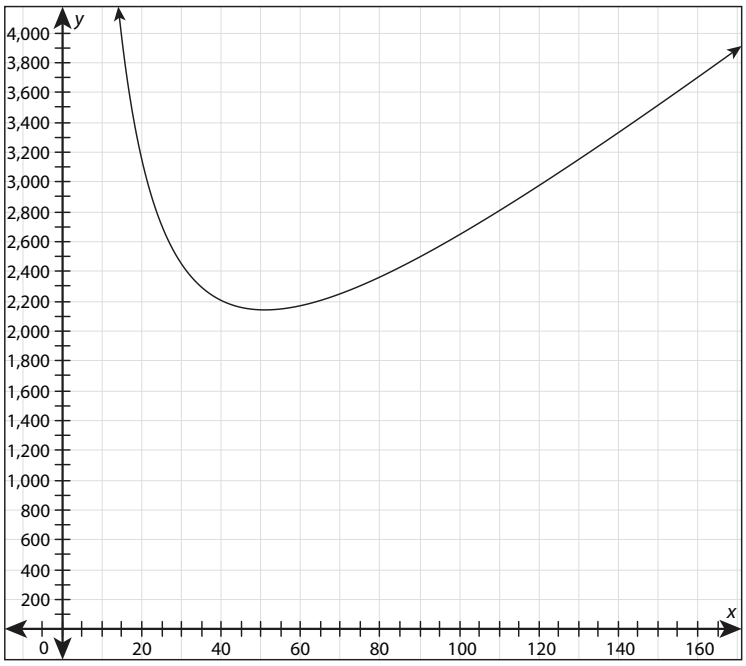
Answers

1. Answers will vary. Sample sketch:



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2. $l = \frac{A}{w} = \frac{4200}{w}$
3. $C = 13\left(\frac{4200}{w}\right) + 21w$
4. $w \approx 51$ ft; $l \approx 82$ ft. Graph:



Materials List/Setup

Station 1 modeling clay

Station 2 none

Station 3 none

Station 4 graphing calculator

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Instruction

Discussion Guide

To support students in reflecting on the activities and synthesizing their ideas, and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. What is geometric modeling? Describe some real-world situations in which geometric shapes can be used to solve problems.
2. What is the difference between a two-dimensional figure and a three-dimensional object? Provide examples of each.
3. How can you relate two-dimensional figures to three-dimensional objects?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. Answers may vary. Sample answer: Geometric modeling is the use of appropriate mathematical tools to analyze situations and make decisions; examples of such tools include spreadsheets and geometry software. Sample real-world situations requiring geometric modeling include: finding the maximum and minimum costs of a construction project; modeling the growth or decay of a population; and determining the amount of water needed for water stations at a marathon and how far apart the water stations should be placed.
2. Two-dimensional figures have length and width but lack depth. Examples of two-dimensional figures include circles and squares. Three-dimensional objects have length, width, and depth; examples include cubes and cones.
3. Cross sections are one way to relate two-dimensional figures to three-dimensional objects. A cube or cylinder can be sliced in different ways that would result in different two-dimensional cross sections of the three-dimensional figure.

Possible Misunderstandings/Mistakes

- Using incorrect formulas for area and volume
- Using inappropriate units for area and volume
- Making arithmetical errors

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Station 1

At this station, you will find modeling clay. Use the clay to sculpt a cylinder, a cube, a cone, and a sphere.

1. Predict the shapes of all possible cross sections for each object you sculpted. Once a prediction has been made, cut the objects to verify their cross sections' shapes.

Use the given information to complete problems 2–5. Round answers to the nearest tenth.

2. Given a cylinder with a radius of 6 in and a height of 8 in, find the area of a cross section that is parallel to the cylinder's base.

3. A circle has a radius of 16 cm. What is the volume of a sphere that has the same radius?

4. Given a cylinder with a height of 50 mm and a radius of 10 mm, find the area of the rectangle formed by cutting a perpendicular cross section right down the cylinder's center.

5. A square with an area of 400 m^2 is rotated to form a cylinder. What is the volume of the cylinder?

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Station 2

Use the given information to complete the problems that follow. Round answers to the nearest tenth.

A canister is approximately 9.25 inches tall. The top and bottom of the canister each have a diameter of 6 inches.

1. Find the approximate volume of the canister.

2. Convert the volume of the canister from cubic inches (in^3) to cubic centimeters (cm^3) using the conversion 2.54 cm = 1 in.

3. If the mass of the canister is 38,250 grams, what is the density of the canister in grams per cubic centimeter (g/cm^3)?

4. Based on this information and the table that follows, what material was likely used to make the canister?

Substance	Density (g/cm^3)
Wood (oak)	0.85
Aluminum	2.7
Copper	8.93
Iron	7.8
Gold	19.3

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Station 3

You have decided to start your own business selling homemade candy corn. The candy is packed in a uniquely shaped container, which consists of a cylinder and a top that is in the shape of a cone. The radius of the cylinder is 6 in, and the height of the cone-shaped top is $\frac{1}{3}$ the height of the cylinder.

1. Sketch the container and label its dimensions.

2. What is the equation for the volume of the container in its simplest form?

3. If the volume of the container is 220π in^3, find the heights of the cylinder, the cone, and the entire container.

4. Suppose you have spherical packages of candy corn that each have a radius of 3 in. How many of these spherical packages can you fit in the cylindrical part of the container?

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Station 4

Luke wants to create a fenced-in garden with an area of 4,200 ft^2. He wants to use wooden fencing on two sides, and wire fencing on the other two sides; each pair of opposite sides will be made of matching material. The sides with wire fencing will cost \$6.50/ft and the two sides with wooden fencing will cost \$10.50/ft. Find the dimensions of the length and width of the garden that will minimize the cost of the fence.

1. Draw a sketch of the garden and define the variables.

2. If $A = lw$, write the formula for the length in terms of the area and width.

3. The cost C of the fence can be expressed as $C = 6.5(2l) + 10.5(2w)$. Use the expression for l found in problem 2 to simplify the cost equation.

4. Use a graphing calculator to determine the smallest width (or minimum point) in order to find the dimensions (length and width) of the garden.

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APPENDIX

Problem-Based Task Implementation Guides

Introduction

This addendum contains a collection of Implementation Guides (IGs) designed to support the use of the Problem-Based Tasks found throughout this unit.

Each Implementation Guide includes an overview and suggestions for introducing and facilitating the task. These contain specific recommendations for selected Standards for Mathematical Practice (SMPs) to highlight, guidance on addressing common errors and misconceptions associated with the task, and ideas for monitoring and coaching students as they work through the task. Finally, there is guidance for debriefing the task, with special attention to the SMPs and relevant Key Concepts from the instruction, as well as information on suitable extensions and alternate strategies.

Throughout the Implementation Guides, suggestions are provided in a question/prompt format to support your efforts to elicit student thinking and facilitate classroom discourse.

We developed these Implementation Guides for our Problem-Based Tasks at the request of math educators, and with advice and feedback from mathematics supervisors and teachers. Please let us know if they are helpful. Visit us at www.walch.com, follow us on Twitter (@WalchEd), or e-mail suggestions to customerservice@walch.com.

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Implementation Guide

Problem-Based Task 4B.1.1: How Long Will the Fertilizer Last?

Common Core State Standard

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*⋆

Task Overview

Focus

How can equations be derived given data points? What types of equations best represent a data set? In this task, students will analyze two data sets and use graphing technology to determine the most appropriate type of equation to represent the given scenario. They will also determine which of two teams from the state agricultural department made a better prediction about how long nitrogen-based fertilizer stays in soil, based on the results of the equations.

This activity will provide practice with:

- identifying independent and dependent variables in a real-world scenario
- using technology to determine equations that represent a data set
- analyzing predictions given in the context of a real-world scenario
- comparing graphing technology results of equations for two data sets
- substituting domain values into equations and calculating the corresponding range values
- performing error analysis on results based on prior knowledge of exponent rules

Introduction

This task should be used to explore or to apply the skill of creating equations in one variable and using them to solve problems, including equations arising from linear and quadratic functions, and simple rational and exponential functions. Students should already be familiar with identifying independent and dependent variables and recognizing real-world constraints on them, and understanding the basic conditions under which a nonlinear relationship can be described as exponential, logarithmic, or quadratic.

Begin by reading the problem and clarifying the meaning of the following terms:

dependent variable	labeled on the *y*-axis; the quantity that is based on the input values of the independent variable; the output variable of a function
domain	the set of all input values (*x*-values) that satisfy the given function
exponential equation	an equation that has a variable in the exponent

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independent variable	labeled on the x-axis; the quantity for which values are chosen; the input variable of a function
quadratic equation	an equation that can be written in the form $y = ax^2 + bx + c$, where x is the independent variable, y is the dependent variable, a, b, and c are constants, and $a \neq 0$
range	the set of all outputs of a function; the set of y-values that are valid for the function

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to use graphing technology to derive exponential and quadratic equations for two sets of data to determine which set provides the most accurate model for predicting future amounts of nitrogen-based fertilizer (N-15) remaining in the soil over time. They will accomplish this goal by first identifying the exponential and quadratic equations for each data set, given the set of ordered pairs for each team of members of the state agricultural department. Then they will discuss the advantages and limitations of each equation for each data set, including any restrictions on the domain and range in the context of the scenario. Finally, they will choose either Team A's or Team B's exponential or quadratic equation to best represent the model that can be used to make predictions about the amount of the nitrogen-based fertilizer in the soil after a certain number of years.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for determining the best equation model for making predictions for amounts of nitrogen-based fertilizer in soil from two sets of data. They will base their arguments about whether Team A's or Team B's data lends to the most appropriate exponential or quadratic equation on the closest fit of each equation to the data set and the advantages and limitations of each equation. They will also use comparisons of the results from the calculator to identify domain restrictions in the context of the scenario. Then, they will present their arguments to others and explain their decision based on their analysis of the types of equations for each team.

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- **SMP 7:** Look for and make use of structure.

  Students will apply the structure of the general formats of exponential and quadratic equations by first using graphing technology and deriving each type of equation for two data sets. They will recognize the structure of an exponential function with the variable in the exponent, and they will recognize the structure of a quadratic equation in the form $y = ax^2 + bx + c$. They will also recognize the structure of an ordered pair as data points displayed in a table, noting that the x-coordinate represents the independent variable, and the y-coordinate represents the dependent variable for the scenario. They will apply the structures of the exponential and quadratic equations by comparing the constants and coefficients in each equation. Once they have compared all aspects of the two types of equations, they will analyze the results to determine which type of equation is the better model for making predictions about the amount of nitrogen-based fertilizer in the soil.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- misidentifying the dependent and independent variables

  Before students begin the task, review the definitions of dependent and independent variables with them. Provide examples of scenarios in which the variables need to be defined, based on which one depends on the other.

- switching the dependent and independent variables in data points or ordered pairs

  Remind students that the dependent variable is represented by the second data point in an ordered pair, or the y-coordinate of the point, as its value depends on the independent variable, which is represented by the first point in the ordered pair, or the x-coordinate.

- using too few data points to determine a line of best fit in the cases of linear or quadratic equations

  Discuss the importance of using at least three or more data points to determine the line of best fit. Remind students that using less than three points can lead to the incorrect equation for the data, since any two points will make a straight line.

- forgetting the domain and range restrictions on the equations

  Discuss the given scenario and determine the restrictions on the domain based on the context of the dependent and independent variables. Remind students that time is always a positive quantity, and the amount of nitrogen remaining in the soil must be greater than or equal to zero.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, review the shapes of the graphs of quadratic and exponential equations. Ask them, "How do the graphs of quadratic equations and exponential equations compare? Explain the similarities and differences." (**Answer:** Both quadratic and exponential equations have curve-shaped graphs. A quadratic equation has a specific parabolic shape, whereas an exponential equation does not have a graph that is parabolic-shaped.)
- Before students begin the task, discuss the given data table. Ask them, "Why is the year 2006 represented as year 25, and why is the year 2056 represented as year 75? Explain the calculations behind the designations." (**Answer:** The first year that a data sample was obtained, 1982, is designated Year 1, since it is the first data point, or first x-coordinate in the data table. Each year from that point is designated as the number of years from 1982. So, the year 2006 is 25 years past 1982, including 1982, and is designated Year 25. Since 2056 is 75 years past 1982, including 1982, it is designated as Year 75 in the data table.)
- Before students begin the task, discuss the idea of sketching a graph of the data without technology. Ask them, "Why is it not a very practical idea to sketch a graph of the given data by hand to try and determine the type of equation that would be the most accurate model?" (**Answer:** One reason that sketching graphs by hand is not very practical is because the range values for the last two data points are very small, and it would be difficult to accurately draw the coordinate point on a graph. Another reason is because deriving equations using technology will provide more accurate results.)
- Before students begin the task, discuss domain and range restrictions in the context of the scenario. Ask them, "Are there any restrictions on the domain and range for the data? Explain." (**Answer:** Since the domain represents the amount of time passing, in years, the domain is positive, and it is greater than or equal to one (since 1 was designated as the first year a sample was taken). Since the range represents the amount of N-15 remaining in the soil, the range is also positive, and it is greater than or equal to 0. There cannot be negative amounts of a chemical in the soil.)
- As students are beginning the task, discuss the process of identifying the dependent and independent variables for the scenario. Ask them, "Why is it important to first identify the dependent and independent variables before deriving the equations for the data sets?" (**Answer:** It is important to identify the dependent and independent variables first before deriving the equations because a set of ordered pairs is necessary for the graphing calculator to determine the equations. To have a set of ordered pairs, the variables must be defined so that the proper numerical values represent each x- and y-coordinate.)

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- As students are beginning the task, ask them to analyze the differences in the predictions in the teams for the remaining amount of the nitrogen isotope in the soil in the year 2056. Ask them, "What is the main difference in what Team A is predicting for the amount of N-15 left in the soil in the year 2056 compared to the prediction that Team B is making for the same year?" (**Answer:** Team A predicts that there will be 10^{-5}, or 0.00001 kg of N-15 left in the soil in the year 2056. Although this is a very small amount, it indicates that Team A predicts trace amounts of N-15 will still be in the soil after 75 years. Team B predicts that there will not be any N-15 left in the soil after 75 years.)
- As an extension of the previous question, ask students, "What is one method that could be used to determine which team is more accurate with their prediction for the year 2056?" (**Answer:** You could find which team has a more accurate prediction by making more measurements of the N-15 in the soil and comparing the actual measurements to the predicted measurements in the model. More specifically, you could measure the amount of N-15 in the soil in 2056 and see which team's prediction was closer.)
- After students have generated the exponential and quadratic equations from their calculators, ask them, "What do you notice about the place values for the constants given for each equation in comparison with the other numerical values in the equations? Why is this difference a potential source of error?" (**Answer:** The values of the constants in the equations are given as values in the tenths place. The other numerical values in the equations are given as values in the hundredths and thousandths place values. The constants could have been rounded, which leads to accuracy errors. If more digits were used to represent the constants, then the equations might be more accurate.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss the given information and why it was important to identify the dependent and independent variables first. Ask students to explain their initial predictions about which team's prediction for the year 2056 seemed more accurate. Discuss any difficulties students may have encountered when deriving the exponential and quadratic equations for each data set using their graphing calculators. Also discuss the importance of identifying the real-world constraints on the domain and range values in the context of the scenario. Ask students to explain the features of each equation that were used when comparing the equations representing each team's data set. Also discuss the process of substituting values into the equations for further analysis. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.

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- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and thoughts about the different methods of comparing the calculator-generated results for the equations. Discuss the errors that can occur when working with data in a real-world scenario such as the one in the task, specifically rounding the constant values. Ask students to explain how an error occurred in the graphing calculator because of the domain values for Team B's exponential equation. Focus on the use of precise mathematical language and clarity, specifically when discussing the terms and general format of an exponential equation and a quadratic equation, as well as when discussing the identification of the dependent and independent variables and the domain and range.

Connecting to Key Concepts

Make explicit connections to key concepts:

- An independent variable, labeled on the x-axis, is the quantity that changes based on values chosen. It is also referred to as the input variable of an equation or function.

  In this task, students will identify the independent variable as the time, in years, that the soil was tested for the nitrogen isotope, N-15.

- A dependent variable, labeled on the y-axis, is the quantity that changes based on the input values of the independent variable. The dependent variable is often referred to as the output variable of an equation or function.

  Students will identify the dependent variable as the amount of N-15 that is found in the soil. They will recognize that this entity is the dependent variable because the value depends on the time that has passed.

- Recollect that a quadratic equation is an equation that can be written in the form $y = ax^2 + bx + c$, where x is the independent variable, y is the dependent variable, a, b, and c are constants, and $a \neq 0$. The graph of a quadratic equation is a parabola on a coordinate plane.

  Students will use graphing technology to create quadratic equations for the data given in the problem. They will recognize the general form of a quadratic equation.

- Also recall that an exponential equation is an equation that has a variable in the exponent.

  Students will recognize that the exponential equations derived from the data are in an exponential equation format, based on the exponent in the equation.

- A graphing calculator can be used to compare the different kinds of equations in one variable that can be written based on a finite number of data points or ordered pairs.

  Students will use the four given data points from the table for each team and follow the specific instructions for their calculator and determine both quadratic and exponential equations to represent the data. They will compare the results for each equation and analyze the best fit of each.

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Extending the Task

To extend the task, ask students to work together to create a third data set, which represents another team from the state agricultural department, Team C. Ask them to create data values based on Team C predicting that in the year 2056, there will be 10^{-3} kilograms of the nitrogen-15 isotope from the fertilizer in the soil. Ask students to create a chart displaying the data points, types of equations, and comparisons of the equation features for all three teams. When students have completed the task, discuss how the new data set changes the results from the original task. Encourage a class discussion by asking questions such as, "Does the new data point for Team C result in better fitting equations?", and "Can you determine which data value for the nitrogen-based fertilizer remaining in the soil in the year 2056 would create an equation with an almost perfect fit?", and "Why do you think it is difficult to get a "perfect fit equation"?"

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of this problem by first recognizing that the goal of the task is to use graphing technology to derive exponential and quadratic equations for two sets of data to determine which set provides the most accurate model for predicting future amounts of nitrogen-based fertilizer (N-15) remaining in the soil over time. I accomplished this goal by first identifying the exponential and quadratic equations for each data set, given the set of ordered pairs for each team of members of the state agricultural department. Then I discussed the advantages and limitations of each equation for each data set, including any restrictions on the domain and range in the context of the scenario. Finally, I decided which team's exponential or quadratic equation best represented the model that can be used to make predictions about the amount of nitrogen-based fertilizer in the soil after a certain number of years.)
- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by determining the best equation model for making predictions for amounts of nitrogen-based fertilizer in soil from two sets of data. I based my arguments about whether Team A's or Team B's data lent to the most appropriate exponential or quadratic equation on the closest fit of each equation to the data set and the advantages and limitations of each equation. I also used comparisons of the results from the calculator to identify domain restrictions in the context of the scenario. Then, I presented my arguments to others and explained my decision based on my analysis of the types of equations for each team.)

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- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I looked for and made use of structure when solving this problem by applying the structure of the general formats of exponential and quadratic equations by first using graphing technology and deriving each type of equation for two data sets. I recognized the structure of an exponential function with the variable in the exponent, and I recognized the structure of a quadratic equation in the form $y = ax^2 + bx + c$. I also recognized the structure of an ordered pair as data points displayed in a table, and I noted that the x-coordinate represented the independent variable, and the y-coordinate represented the dependent variable for the scenario. I applied the structures of the exponential and quadratic equations by comparing the constants and coefficients in each equation. Once I had compared all aspects of the two types of equations, I analyzed the results and determined which type of equation was the better model for making predictions about the amount of nitrogen-based fertilizer in the soil.)

Alternate Strategies or Solutions

Some students may choose to sketch graphs of each data set on the same coordinate plane to get a visual method of making a prediction about which type of equation would best fit the data. Remind students that although three points is technically the minimum requirement for graphing exponential and quadratic equations, deriving equations using technology will provide a more accurate method of making predictions. Also remind them of the accuracy limitations when sketching small values, such as 10^{-5}, without the use of technology.

Technology

Students can use calculators or other graphing technology for this task.

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Implementation Guide

Problem-Based Task 4B.1.3: A Math Melody Mash-Up

Common Core State Standard

A–CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.*★

Task Overview

Focus

How can mathematical formulas be arranged to solve for a specific unknown variable? If multiple formulas are involved, how can parts of formulas be substituted into a main formula in order to solve for one variable? In this task, students will derive a formula for the tension in a stringed musical instrument in terms of all other variables given from related formulas. They will also determine how the tension stays the same for a guitar string if the frequency changes, as well as what the corresponding wavelengths are with these different frequencies.

This activity will provide practice with:

- isolating a variable in a formula using algebraic operations (inverse operations)
- analyzing the relationship between variables in different formulas
- substituting numerical values into formulas for known quantities
- simplifying a formula and evaluating numerical values
- rewriting equations in order to isolate a variable
- drawing conclusions about missing information in a formula

Introduction

This task should be used to explore or to apply the skill of rearranging formulas to highlight a quantity of interest, using the same reasoning as in solving equations. Students should be familiar with using the mathematical operation(s) of addition, exponentiation, finding roots, multiplying, and/or using inverses to solve equations.

Begin by reading the problem and clarifying the meaning of the following terms:

formula	a mathematical statement of the relationship among two or more variables
frequency	the number of wave cycles that occur in one second; measured in units of Hertz (Hz)
tension	the amount of force pulling a string tight
wavelength	the length of the sound wave, along with the speed of the wave, that determines the pitch of a musical note

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goals of the task are to derive a formula for the tension in a stringed musical instrument in terms of all other variables given from other musically-related formulas, and also to determine how the tension stays the same for a guitar string if the frequency changes, as well as what the corresponding wavelengths are with these different frequencies.

  To accomplish these goals, they will first examine the three given formulas and determine which one has the desired unknown variable. Then they will perform mathematical operations necessary for isolating the desired variable. Once they have isolated the one unknown variable, they will rewrite the other formulas and substituted in the necessary quantities in order to create a formula containing all the variables from the three given formulas. Finally, they will substitute the given numerical values for specific variables, simplify the formula, and then analyze the results to explain how the tension stays the same for a guitar string with different frequencies, and also determine if additional information is needed in order to find the last unknown variable, the wavelength.

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they recognize that they will need to rewrite, simplify, and substitute known numerical values into the given formulas in order to derive a formula to represent the tension in a stringed musical instrument in terms of all other variables given in three musically-related formulas, as well as determining the wavelength of the given frequencies. They will recognize the need to analyze each formula and choose the one which has the desired unknown variable, the tension. They will also reason that they will need to perform mathematical operations on the formulas in order to isolate the desired variable. Finally, they will reason abstractly as they analyze the final formula and draw conclusions about whether or not additional information is needed to find the value of the remaining unknown variable, the wavelength. They will reason quantitatively as they perform the necessary mathematical operations in order to isolate the unknown variable, as well as when they rewrite the formulas. They will also reason quantitatively as they substitute the given values into the formulas and then simplify them.

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- **SMP 8:** Look for and express regularity in repeated reasoning.

  Students will express regularity in repeated reasoning as they explain and justify their steps involved in choosing the appropriate formulas for the goal of isolating the desired unknown variable. They will also use repeated reasoning as they determine how the variables are related to one another, and they will apply this reasoning as they substitute in the appropriate components of each given formula into a new formula. Finally, they will use repeated reasoning as they perform the necessary mathematical operations in order to isolate a variable, as well as to simplify formulas.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- using the wrong operation (e.g., dividing instead of multiplying) in the process of isolating a variable in a formula

  Review inverse operations with students. Remind them that the method for cancelling out a term or variable is to perform the opposite/inverse operation to both sides of an equation. Have them practice by providing samples of multi-step equations for them to solve.

- incorrectly substituting parts of a formula into a new formula

  As students are working through the task, encourage them to keep an organized list of all formulas that they create as they rewrite them by substituting in parts of other formulas. Also remind them to use parentheses when substituting a part containing an expression into a single variable.

- misidentifying variables when rewriting a formula or creating a new one

  Before students begin the task, have them create a list at the top of their papers of what each variable represents. Encourage them to refer to this list while working through the parts of the task, so that they can relate the variables to their meaning in the context of the scenario.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, review inverse operations with them. Ask them, "When isolating a variable in an equation or formula, inverse operations are used. What are inverse operations, and how are they used to accomplish this task?" (**Answer:** Inverse operations are opposites. For example, addition and subtraction are inverse operations, multiplication and division are inverses, and taking the square root and squaring a root are inverse operations. They are used to eliminate variables or numerical values when simplifying an equation or a formula.)

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- Before students begin the task, review the variables which represent the different musical terms. Ask students, "What do the lowercase variable *f* and the uppercase variable *F* represent in musical terminology?" (**Answer:** The lowercase variable *f* represents the frequency of the sound, and the uppercase variable *F* represents the tension in a string on an instrument.)
- As students begin working on the task, ask them, "What are the two steps, or inverse operations, that are used to isolate the variable *F* in the formula $v = \sqrt{\frac{F}{\mu}}$? Explain the steps and the purpose of each one." (**Answer:** The first step in isolating the variable *F* is to square both sides so that the square root is eliminated on the right side of the equation. Squaring a square root eliminates the radical. Next, since the variable *F* is divided by μ, then the inverse operation of division, which is multiplication, is performed. Both sides of the formula are multiplied by μ, which eliminates μ on the right side, and then the remaining variable is *F* on the right side, which was the goal.)
- As students are working through the task, ensure their understanding of the tension of the string and different frequencies. Ask them, "How can the tension stay the same for a guitar string if the frequency is changed from 275 cycles per second to 400 cycles per second?" (**Answer:** The tension is the amount of force pulling the string tight. The mass and length of a string on an instrument do not change when the frequency changes. Therefore, even though the frequency changes, the amount of tension stays the same since the mass and length of the string did not change.)
- Ask students, "The task states that the fraction, $\frac{m}{L}$, is a constant. What is another way to explain what this means in the context of the scenario?" (**Answer:** The constant $\frac{m}{L}$ is the ratio of the mass per unit length of a string to its length. The ratio of these two quantities stays the same, as the mass and length change proportionally to each other.)
- Ask students, "Based on the information given in the task about the musical terms, what is the relationship between wavelength and frequency?" (**Answer:** The higher the frequency of a wave means more cycles of waves that are shorter, so therefore the wavelength is shorter. So, the higher the frequency, the shorter the wavelength, and the lower the frequency, the longer the wavelength.)

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- As students are completing the task, ask them, "Why is more information needed to determine an actual value for the wavelengths of the two frequencies? Explain in terms of what is known or unknown in the formulas." (**Answer:** To find the values for the wavelengths using the formula $v = f \bullet \lambda$, the frequency and the velocity must be known. The values for the frequencies are given in the original problem, but the value for the velocity is found from the formula $v = \sqrt{\frac{F}{\mu}}$. The value for F is unknown, and it is only represented in a formula containing other variables. There is still more than one unknown variable in the formulas.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss what the frequency, mass, wavelength, tension, velocity, and length are in the context of the scenario of stringed musical instruments. Review the formulas with students and discuss the components of each. Ask students to explain how to determine which formula to choose in order to isolate the variable representing the tension. Also ask them to explain what mathematical operations were used to rearrange the formulas and how mathematical operations were used to simplify them throughout the task. Discuss the relationship between frequency and wavelength and how the given values for the frequencies can be substituted into the formulas to create new formulas that can be used to represent the unknown wavelengths of the two frequencies. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process as they determine which formulas are needed for each part of the task. Ask them to explain how they algebraically manipulated the formulas in order to create new formulas. Discuss the relationship of the musical terms to one another and how these relationships were applied when rewriting and creating new formulas. Also discuss which components of which formulas were substituted into the new formulas in each step. Discuss how the tension can stay the same for a guitar string if the frequency is changed from 275 cycles per second to 400 cycles per second. Focus on the use of precise mathematical language and clarity, specifically when discussing each variable in the given formulas and their meanings in terms of how they affect notes played on stringed instruments. Also focus on the correct terminology when discussing the mathematical operations used to isolate the variables in the formulas.

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Connecting to Key Concepts

Make explicit connections to key concepts:

- A formula relates two or more variables in a mathematical or a real-world problem context.

  In this task, students will examine the relationships of the variables in several formulas given which represent musical terms relating to stringed instruments.

- The values of the variables in a formula can be constrained or limited by mathematical or real-world conditions.

  Students will recognize that the values of the variables in the musically-related formulas are positive values.

- Formulas can be rearranged to express one or more variables in terms of another variable. A rearranged formula can reduce the number of variables to be calculated.

  Students will rearrange and rewrite several of the given formulas in order to accomplish one of the goals of the task, which is to derive one formula for a specific variable which encompasses all other variables in the formulas.

Extending the Task

- To extend the task, provide students with a value for the wavelength corresponding to a frequency of 275 cycles per second, or a value for the wavelength corresponding to a frequency of 400 cycles per second. Ask them to find the other missing wavelength. Ask for volunteers to discuss their calculations and share their results. Remind them to use the correct units of measurement for wavelength.

- Another option for extending the task is to ask students to practice rearranging formulas to solve for an unknown variable by providing an example of a mathematical formula. For example, provide the formula for the volume of a cone: $V = \frac{1}{3}\pi r^2 h$. Ask students to rearrange the formula in order to solve for the radius. Ask them to create an organized list of their steps and indicate what operations were done in each step in order to accomplish the goal of isolating the variable *r*. Ask for volunteers to share their results and explain their steps.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of this problem by first recognizing that the goals of the task are to derive a formula for the tension in a stringed musical instrument in terms of all other variables given from other musically-related formulas, and also to determine how the tension stays the same for a guitar string if the frequency changes, as well as what the corresponding wavelengths are with these different frequencies. To accomplish these goals, first I examined the three given formulas and determined which one has the desired unknown variable. Then I performed mathematical operations necessary for isolating the desired variable. Once I isolated the one unknown variable, I rewrote the other formulas and substituted in the necessary quantities in order to create a formula containing all the variables from the three given formulas. Finally, I substituted the given numerical values for specific variables, simplified the formula, and then analyzed the results to explain how the tension stays the same for a guitar string with different frequencies, and also determine if additional information was needed in order to find the last unknown variable, the wavelength.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first recognized that I needed to rewrite, simplify, and substitute known numerical values into the given formulas in order to derive a formula to represent the tension in a stringed musical instrument in terms of all other variables given in three musically-related formulas. I recognized the need to analyze each formula and choose the one which has the desired unknown variable, the tension. I also reasoned that I needed to perform mathematical operations on the formulas in order to isolate the desired variable. Finally, I reasoned abstractly as I analyzed the final formula and drew conclusions about whether or not additional information was needed to find the value of the remaining unknown variable, the wavelength.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I reasoned quantitatively as I performed the necessary mathematical operations in order to isolate the unknown variable, as well as when I rewrote the formulas. I also reasoned quantitatively as I substituted the given values into the formulas and then simplified them.)
- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** I looked for and expressed regularity in repeated reasoning as I explained and justified my steps involved in choosing the appropriate formulas for the goal of isolating the desired unknown variable. I also used repeated reasoning as I determined how the variables were related to one another, and I applied this reasoning as I substituted in the appropriate components of each given formula into a new formula. Finally, I used repeated reasoning as I performed the necessary mathematical operations in order to isolate a variable, as well as to simplify formulas.)

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Alternate Strategies or Solutions

Some students may have chosen to substitute the known numerical values given in the original problem into each of the three formulas before they rearranged any of them to isolate an unknown variable. Remind them that although this method is acceptable, it may be easier to perform the operations necessary for isolating the variable first without numerical values.

Technology

Students can use scientific calculators for this task.

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Problem-Based Task 4B.2.1: An Absolute Gas?

Common Core State Standard

F–BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Task Overview

Focus

How can functions be changed, or transformed, by performing algebraic operations on them? How will these changes affect the graphs of the functions? How can a family of functions be derived from a parent function and with data given in a table? In this task, students will create linear functions for three different natural gases in gas tanks where the temperature and pressure on each gas tank is measured, and they will describe the transformations of the parent linear function represented by the family of functions. They will also apply the functions to determine the temperature that each gas must be in order for the pressure of the tank to equal zero.

This activity will provide practice with:

- graphing linear functions given data represented in a table
- comparing and contrasting linear functions represented on a graph
- calculating the slope of a line given points on a graph
- writing linear functions using point-slope form
- describing transformations of a parent function given a family of functions
- identifying x-intercepts for linear functions

Introduction

This task should be used to explore or to apply the skill of identifying the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative), and finding the value of k given the graphs. Students should already be familiar with identifying the parent functions of linear functions, calculating the slope of a line, creating a function equation using point-slope form, and calculating the change in a function value when a constant is added to or multiplied by the independent variable in the function equation.

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Begin by reading the problem and clarifying the meaning of the following terms:

family of functions	a set of functions whose graphs have the same general shape as their parent function
parent function	a function with a simple algebraic rule that represents a family of functions. The graphs of the functions in the family have the same general shape as the parent function.
point-slope form of a linear function	formula which states $y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) is a point on the line

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled with a family of linear functions which represent three different natural gases in a fracking well. They will recognize that the graphs of these three functions can be used to determine the amount of pressure in each gas tank given changing temperatures. They will identify the temperature and pressure as independent and dependent variables in the scenario, and they will identify the types of transformations that occur with each variable for each function. They will also apply the slope formula and the point-slope formula in order to determine the necessary values for analyzing and describing the transformations. They will also recognize that the functions created to represent each natural gas can be applied to determine the temperature necessary so that each gas has a pressure of zero in its tank.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure of a linear function to identify and apply its key features when accomplishing the goal of the task. First, they will recognize and apply the structure for graphing the three functions, as they will graph each one based on the data presented in a table. Then they will create linear functions for each gas by first using the structure of the slope formula, and then the point-slope formula. They will recognize that the slope and the y-intercept for each linear function represent different types of transformations performed on the independent and dependent variables. They will also use the structure of each function equation to solve for the temperature of each gas when the pressure is at zero in the tank.

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- **SMP 8:** Look for and express regularity in repeated reasoning.

  Students will express regularity in repeated reasoning as they perform the same procedures and apply the same formulas for three different functions, which represent the three natural gases. They will apply the same reasoning for each function as they first graph each one given data points in a table. Then they will calculate the slope for each function and use these values to create linear functions for each using the point-slope formula. They will also use repeated reasoning as they analyze the structure of the functions and describe the transformations. Finally, they will use repeated reasoning and procedures as they determine the temperature of each gas when the pressure in the tank has a value of zero.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- misidentifying the direction in which a transformation of the form $x + k$ or $f(x) + k$ shifts the graph of a function

  Have students create a list of the general format of the types of transformations that can be done to a parent function. Ask students to write a brief description of each transformation. Review sample transformations with students before they begin the task. Discuss the connection between the constant that is added to each function in the task to the type of transformation it relates to in the list of transformations.

- incorrectly determining the effect (e.g., dilation or shrinkage) that a transformation of the form kx or $k \cdot f(x)$ has on the graph of a function

  As students are creating the family of functions in the task, discuss what the slope of each function represents in terms of transformations to the parent linear function. Remind students that the independent variable is multiplied by the constant in this task.

- miscalculating the slope of each graph by switching the x- and y-variables

  Have students write the formula for slope at the top of their papers. Remind them to label each x- and y-coordinate carefully before substituting the values into the slope formula.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, review independent and dependent variables with them. Ask them, "Explain what is meant by the independent and dependent variables in a function. What are the independent and dependent variables in this task? Explain them in the context of the scenario." (**Answer:** In a function, the independent variable is the input value that is entered into the function. Its value does not depend on the outcome of the dependent variable. The dependent variable is the output value that depends on the results of the independent variable in the function. In this task, the independent variable is the temperature of the gas, and the dependent variable is the pressure in the gas tank, because the tank pressure depends on what the temperature of the gas is.)

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- As an extension to the previous question, ask students to explain what a linear function means in the context of the scenario. Ask them, "The problem states that the relationship between the pressure (P) and the temperature (t) can be represented by a linear function. What does this mean in the context of the scenario?" (**Answer:** If a function is linear, then there is a constant rate of change between the independent and dependent variables. Or, the slope between any two points on the graph is the same. In the context of the scenario, as the temperature of the gas increases, the pressure in the tank increases at a constant rate.)
- After students have determined the linear functions for the three gases, ask them, "What are the two types of transformations that have occurred for each function? Explain in terms of the variables and the context of the scenario." (**Answer:** For each function, the independent variable representing the temperature (t) of each gas was multiplied by a constant. Also for each function, a constant was added to the dependent variable, or the pressure (P).)
- As an extension to the previous question, ask students, "In terms of the format of each function, what do the values for the two types of transformations represent in geometrical terminology on the graph?" (**Answer:** The constant that is multiplied by the temperature represents the slope of the function, and the constant that is added represents the y-intercept of the function.)
- As students are working to find the temperature at which the pressure of the gas tanks is at zero, ask them, "In algebraic terms, what is the point on a graph where the dependent variable of a function is 0 for a linear function? For a quadratic function?" (**Answer:** For a linear function, the point on the graph where the dependent variable has a value of 0 is called the x-intercept, because it is the point where the graph intersects, or crosses, the x-axis. For a quadratic function, this point, or points, is also called the x-intercept(s), but another common term is the "zeros of the function".)
- Ask students, "After analyzing the functions representing the three gases, which gas has the greatest change in pressure when the temperature rises? Explain your answer." (**Answer:** Gas C has the greatest change in pressure when the temperature rises because it has the greatest slope of the three functions. In other words, the pressure in the tank for Gas C is greater than the pressure in the tanks for Gas A and Gas B, given the same temperature change.)
- After students have completed the task, ask them, "What are some practical applications for a technician working in a fracking well, in terms of applying the linear functions representing the three natural gases in the three tanks?" (**Answer:** By having linear functions which represent the effects of changing temperature on the pressure inside of the tanks, a technician can predict and plan for a gas tank that becomes too highly pressurized if the temperature reaches a certain degree. A technician might also want to know what the pressure will be if the temperature of a gas gets below a certain temperature. These values might affect the uses and extraction of a natural gas.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss how linear functions and the features of their graphs can be applied to real-world scenarios such as the one presented in the task. Discuss the parent function of a linear function, and discuss how the family of functions representing the three different natural gases displayed different types of transformations, in comparison with the parent function. Ask students to explain how they calculated the slope of each function, as well as how the equation for each function was created using the point-slope formula. Discuss the connection of the slope and *y*-intercept of each function in the context of the variables representing the temperature and the pressure of the gases. Also discuss the slopes and *y*-intercepts in the context of the transformations of the functions. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and their steps for graphing each of the three functions, as well as their steps for calculating the slope of each graph and then writing the function for each. Discuss the meaning of the slope for each gas in the context of the scenario. Ask them to explain how they calculated the temperature of each gas if the pressure in the tanks was at a value of zero. Focus on the use of precise mathematical language and clarity, specifically when discussing the parent function and family of functions, as well as when discussing the components of each linear function in terms of the slope and *y*-intercept, as well as in the context of the types of transformations that occurred.

Connecting to Key Concepts

Make explicit connections to key concepts:

- For a linear function, the parent function can be defined as $f(x) = x$.

  In this task, students will recognize that the data given in the table represents linear functions, and they will identify the parent function of a linear function as $f(x) = x$ and compare the family of functions to it.

- A **family of functions** is a set of functions whose graphs have the same general shape as their parent function.

  Students will graph three linear functions which represent three different natural gases contained in gas tanks. They will recognize that the graphs have the same general shape as the parent linear function, $f(x) = x$.

- Transformations of the dependent or independent variables of a function can shift the location of the function on the coordinate plane.

  Students will observe the different locations of the three functions on the coordinate plane, and they will identify the types of transformations that cause the location of the graph of each function.

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- The transformation $g(x) = f(x) + k$ adds a constant, k, to the value of the dependent variable, $f(x)$.

  Students will identify that there is a constant value added to each of the three linear functions in the function family. They will also identify that this constant represents the pressure in each gas tank when the temperature reading for each tank was 0° C.

- The transformation $g(x) = f(kx)$ multiplies the independent variable, x, by a constant, k.

  Students will identify that the variable t, which represents the independent variable in the task, is multiplied by a constant in each of the three functions in the family of functions.

Extending the Task

To extend the task, ask students to work with a partner and create a new scenario in which a family of linear functions will be created, but this time ask them to devise a scenario in which the slope of the three functions will be negative. Encourage them to brainstorm and think of real-world scenarios in which the independent and dependent variables have a negative correlation with each other, or in other words, a scenario where the values of the dependent variable decrease as the values of the independent variable increase, at a constant rate. Allow students to create their own fictional data for their scenarios. Ask for volunteers to share their scenarios and discuss how their family of linear functions was created, as well as what types of transformations occurred. Ask them to explain the differences in the constants in their functions and the constants in the functions in the original task.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by recognizing that this scenario could be modeled with a family of linear functions which represent three different natural gases in a fracking well. I recognized that the graphs of these three functions could be used to determine the amount of pressure in each gas tank given changing temperatures. I identified the temperature and pressure as independent and dependent variables in the scenario, and I also identified the types of transformations that occurred with each variable for each function. I applied the slope formula and the point-slope formula in order to determine the necessary values for analyzing and describing the transformations. I also recognized that the functions created to represent each natural gas could be applied to determine the temperature necessary so that each gas has a pressure of zero in its tank.)

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- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I looked for and made use of structure when solving this problem by applying the structure of a linear function in order to identify and apply its key features when accomplishing the goal of the task. First, I recognized and applied the structure for graphing the three functions, as I graphed each one based on the data presented in a table. Then I created linear functions for each gas by first using the structure of the slope formula, and then the point-slope formula. I recognized that the slope and the *y*-intercept for each linear function represented different types of transformations performed on the independent and dependent variables. I also used the structure of each function equation to solve for the temperature of each gas when the pressure was at zero in the tank.)
- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** I looked for and expressed regularity in repeated reasoning as I performed the same procedures and applied the same formulas for three different functions, which represented three natural gases contained in tanks in a fracking well. I applied the same reasoning for each function as I first graphed each one given data points in a table. Then I calculated the slope for each function and used these values to create linear functions for each using the point-slope formula. I also used repeated reasoning as I analyzed the structure of the functions and described the transformations. Finally, I used repeated reasoning and procedures as I determined the temperature of each gas when the pressure in the tank had a value of zero.)

Alternate Strategies or Solutions

Since the task did not specify the method used to graph the three linear functions, some students may have chosen to graph them by hand, and some may have chosen to graph them with graphing technology. Remind students to carefully choose the scales on each axis for either method. Encourage them to check their graphs with a calculator, if they chose to initially graph them by hand.

Technology

Students can use graphing calculators for this task.

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Problem-Based Task 4B.2.3: Blender Compositions

Common Core State Standard

F–BF.1 Write a function that describes a relationship between two quantities.★

b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*

Task Overview

Focus

How can the composition of functions be applied to real-world scenarios? How are the conditions of a real-world problem applied to the parts of an equation or function that models the problem? In this task, students will write a composition function for a sedimentation constant that is a function of the rotational time of the blending apparatus used in an industrial-strength high-speed blender for the purposes of mixing various flavorings or toppings with milk and ice cream to create milkshakes in an ice cream shop.

This activity will provide practice with:

- interpreting written information given in a problem
- determining the appropriate function to use in a function composition
- performing composition of functions by substituting one function into another
- simplifying function compositions
- identifying constants and variables in a function modeling a real-world scenario
- analyzing a function composition in the context of a real-world scenario

Introduction

This task should be used to explore or to apply the skills of writing a function that describes a relationship between two quantities and combining standard function types using arithmetic operations. Students should already be familiar with combining the mathematical terms in a problem into an equation or function that models the problem, substituting one expression for another in a mathematical statement, and simplifying expressions containing multiple variables and operations.

Begin by reading the problem and clarifying the meaning of the following term:

composition of functions the process of substituting one function for the independent variable of another function to create a new function

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goals of the task are to first write a composition function for a sedimentation constant that is a function of the rotational time of the blending apparatus used in an industrial-strength high-speed blender in an ice cream shop. Then, they will use the relationship for the distance that a sediment particle travels in the blending apparatus to the edge of the mixing container to simplify the composition function. Finally, they will analyze the simplified composition function to determine the units of the final function, based on the variables in the function. They will also interpret the results in the context of the scenario and discuss other factors regarding the components of the ingredients used in an ice cream shop that can affect the sedimentation constant.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled with composition of functions, as they recognize that the real-world context of the functions represent the blending apparatus of an industrial-strength high-speed blender used in an ice cream shop. They will apply the given function representing the given sedimentation constant c, which is a function that quantifies how fast a blender can mix a customer's request, and substitute the function representing the rotational speed of the mixture into the function representing the sedimentation constant. Once they have composed the functions and simplified the resulting function, they will determine the units of the sedimentation constant, and they will also interpret the composed function in the context of the scenario.

- **SMP 6:** Attend to precision.

  Students will use precision as they perform the operations necessary to write a composition function for the sedimentation constant c that is a function of the rotational time t. Specifically, they will use precision as they substitute one function into another and simplify the resulting expression using algebraic operations. They will also use precision as they identify and define the variables that are contained in the resulting composed function, and then they will interpret the results and discuss factors that might affect the sedimentation constant that are not directly represented mathematically in its function. Encourage them to use the correct terminology throughout the task, specifically when identifying the variables used in the function composition, as well as the terminology used when simplifying the components of each function. Also ensure that they use correct function composition notation throughout the task.

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Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- substituting the wrong function into a second function during a composition of functions

  Before students begin the task, review the process of composing functions with them. Have them practice the process with simple functions.

- incorrectly simplifying the complex fraction in the first function composition

  Provide students with a sample problem containing a complex fraction similar to the one given in the task. Remind students how to raise a fraction to an exponential value, as well as how to simplify an expression containing a complex fraction.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "In your own words, what does it mean to compose functions?" (**Answer:** Answers may vary. One response is: "Composing functions is another method of combining functions. It means to substitute one function into the independent variable of another function and simplify to create a new function.)
- As students are reading through the information given in the problem, discuss the sedimentation constant with them. Ask them, "What does the sedimentation constant mean in the context of a blender? What do the variables represent?" (**Answer:** The sedimentation constant is a function that gives a numerical value of how fast a blender can mix a customer's request for a milkshake. It also measures how creamy or thick the blend of the milkshake is. The variables in the sedimentation constant are: v; the speed (in cm per second) at which the solid topping particle travels from the blending apparatus to the edge of the mixing container, s; the distance in centimeters the solid topping particle travels from the blending apparatus to the edge of the mixing container, and ω (*omega*); the angular or rotational speed of the mixture.)

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- As students are working through the substitution of the functions, ensure their understanding of the simplification process. Ask, "What expression are the numerator and denominator multiplied by in order to eliminate the complex fraction in the substitution step $(c \circ \omega)(t) = \frac{v}{\left(\frac{2\pi}{t}\right)^2 s}$? Explain the steps that are missing in the task." (**Answer:** The first step is to simplify the fraction raised to the second power in the denominator: $(c \circ \omega)(t) = \frac{v}{\left(\frac{2\pi}{t}\right)^2 s}$; $(c \circ \omega)(t) = \frac{v}{\left(\frac{4\pi^2 s}{t^2}\right)}$. Then, the numerator and denominator of the entire fraction are multiplied by the reciprocal of the denominator, which is $\frac{t^2}{4\pi^2 s}$, in order to eliminate the complex fraction: $(c \circ \omega)(t) = \frac{v \bullet \frac{t^2}{4\pi^2 s}}{\left(\frac{4\pi^2 s}{t^2}\right) \bullet \frac{t^2}{4\pi^2 s}} = \frac{vt^2}{4\pi^2 s}$.)
- As students are working through the task, ask them, "There are two variables which represent units of time in the task. What are the differences between the variable *t* and the variable *m*?" (**Answer:** The variable *t* represents the rotational time of the blending apparatus. The variable *m* is the "topping migration time", which is the time it takes a topping particle to travel from the blending apparatus to the edge of the mixing container.)
- Ask students to explain the simplified function in terms of which values are constants and which are variables. Ask them, "Once the expression $v \bullet m$ is substituted into the function $(c \circ \omega)(t) = \frac{vt^2}{4\pi^2 s}$, and the resulting function is $(c \circ \omega)(t) = \frac{t^2}{4\pi^2 m}$, which components of the function are variables, and which components are constants? Explain." (**Answer:** In the simplified function $(c \circ \omega)(t) = \frac{t^2}{4\pi^2 m}$, $4\pi^2$ is a constant, because π is a constant with a value of approximately 3.14. The variables are *t* and *m*, since those values are unknown in the problem.)

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- As an extension to the previous question, ask students, "How could you rewrite the function $(c \circ \omega)(t) = \frac{t^2}{4\pi^2 m}$ in a more simplified form? Explain." (**Answer:** Since the term $4\pi^2$ is a constant, the value for the term can be found. Since π is a constant with a value of approximately 3.14, then 3.14 is squared and then multiplied by 4: $4 \bullet (3.14^2) = 4 \bullet 9.86 \approx 39.44$. Therefore, the function in more simplified form is $(c \circ \omega)(t) = \frac{t^2}{39.44m}$.)
- As students are completing the task, ask them, "When an ice cream shop owner is looking to buy an industrial-strength high-speed blender, what factors do you think will influence the owner to buy one brand over another?" (**Answer:** Some factors that might influence an ice cream shop owner to buy one brand of a blender over another are the cost, the speed of the blender, and the quality of the mixture, in terms of thickness and smoothness.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss how composition functions can be applied to real-world scenarios such as the one presented in the task. Discuss the variables that are involved in the formula for the sedimentation constant (c) of the blending apparatus. Ask students to explain their decision regarding which composition led to an expression for c as a function of t. Discuss the steps involved in the composition process, such as the substitution of one function into another, and then the simplification of the composition. Ask students to explain how they reduced variables, simplified the exponents, and eliminated the complex fraction in the simplification process. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and their procedures for composing the appropriate functions. Discuss the components of the simplified composition, in terms of which parts are variables and which parts are constant terms. Ask students to explain why the units of the simplified composed function are units of time. Ask students to discuss and explain factors that might affect the sedimentation constant that are not directly represented mathematically in its function. Focus on the use of precise mathematical language and clarity, specifically when discussing the variables related to the blending apparatus, as well as the function notation related to the function composition.

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Connecting to Key Concepts

Make explicit connections to key concepts:

- Functions can be combined in many ways, such as by being added, subtracted, multiplied, divided, and substituted into each other and used to model real-world problems.

  In this task, students will combine functions representing the components of the apparatus used to blend ingredients in an ice cream shop by substituting them into each other.

- The **composition of functions** is the process of substituting one function for the independent variable of another function to create a new function.

  Students will write a composition function for the sedimentation constant c that is a function of the rotational time t of the blending apparatus that is used to quantify how fast a blender can mix a customer's order in an ice cream shop.

- The notation $g(f(x))$ or $(g \circ f)(x)$ means that the function $f(x)$ is substituted for the independent variable x in the function $g(x)$. Similarly, the notation $f(g(x))$ or $(f \circ g)(x)$ means that the function $g(x)$ is substituted for the independent variable x in the function $f(x)$.

  When working on writing and simplifying the composition functions, students will apply the appropriate function notation, as they substitute a function into the independent variable for another function.

Extending the Task

To extend the task, ask students to work with a partner to create a scenario in which two blender companies are competing for the business of an ice cream shop owner. Applying the simplified function given in the task, $(c \circ \omega)(t) = \frac{t^2}{4\pi^2 m}$, ask them to create values for the two unknown variables for Blender Company A and Blender Company B. Ask them to create a table of values, and substitute these values into the function, to determine which blender company ultimately has the better blender, based on the rotational speed of the blending apparatus (t), and the "topping migration time" (m). Encourage them to examine the relationship between the two variables in the function, since one is located in the numerator, and one is located in the denominator. Ask for volunteers to share their scenarios, and specifically discuss how they chose values for t and m which resulted in one company producing a better blender than the other company. Discuss other factors that could be involved in creating a high-quality industrial-strength blender.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I recognized that the first goal of the task was to write a composition function for a sedimentation constant that is a function of the rotational time of the blending apparatus. The second goal was to use the relationship for the distance that a sediment particle travels in the blending apparatus to the edge of the mixing container to simplify the composition function. The final goal was to analyze the simplified composition function to determine the units of the final function, based on the variables in the function. I interpreted the results of the function in the context of the scenario, and I determined and discussed other factors regarding the components of the ingredients used in an ice cream shop that can affect the sedimentation constant.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by recognizing that the scenarios could be modeled with composition of functions. I recognized that that the real-world context of the functions represented the blending apparatus of an industrial-strength high-speed blender used in an ice cream shop. I applied the given function representing the given sedimentation constant c, which is a function that quantifies how fast a blender can mix a customer's request, and I substituted the function representing the rotational speed of the mixture into the function representing the sedimentation constant. After I composed the functions and simplified the resulting function, I determined the units of the sedimentation constant, and I also interpreted the composed function in the context of the scenario.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I performed the operations necessary to write a composition function for the sedimentation constant c that is a function of the rotational time t. Specifically, I used precision as I substituted one function into another and simplified the resulting expression using algebraic operations. I also used precision as I identified and defined the variables that were contained in the resulting composed function, and then I interpreted the results and discussed factors that might affect the sedimentation constant that were not directly represented mathematically in its function. I used the correct terminology throughout the task, specifically when I identified the variables used in the function composition, as well as the terminology I used when simplifying the components of each function. I also made sure I used correct function composition notation throughout the task.)

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Alternate Strategies or Solutions

Some students may have chosen to substitute $v \bullet m$ into the function composition originally, before the initial steps for the composition were performed. Remind them to carefully check their work for simplifying the function to make sure it results in the same simplified function composition as given in the task. Some students may also have chosen to represent the constant term $4\pi^2$ as a numerical value when writing the final function composition.

Technology

Students can use scientific calculators for this task.

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Problem-Based Task 4B.3.1: Tracking Carbon Dioxide

Common Core State Standards

F–IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F–IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F–IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

Task Overview

Focus

How can linear and trigonometric functions be applied to model real-world scenarios? How can graphing calculators be used to identify the key features of a graph of a function and create an equation to fit the graph? In this task, students will use graphing technology to graph a given function representing the amount of carbon dioxide in the atmosphere over a period of decades in Hawaii, and they will compare the given function to the graph of a function from data given in a table. They will also combine the functions and use the newly created function to predict the amount of carbon dioxide in the atmosphere in April of the year 2019.

This activity will provide practice with:

- graphing points given in a table using a graphing calculator
- determining the appropriate function type to represent data shown on a graph
- comparing key features of two functions
- writing a combined function given two separate functions
- identifying the amplitude and period of a sine function
- substituting values into a function
- calculating an output value of a function
- analyzing results in the context of a scenario

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Implementation Guide

Introduction

This task should be used to explore or to apply the skills of modeling a relationship between two quantities of a given function, interpreting key features of graphs and tables in terms of the quantities, and sketching graphs showing key features of the relationship between the quantities. Students should already be familiar with identifying features of a sine function, graphing functions using a graphing calculator, and comparing how function values change for two or more functions with the same domain values.

Begin by reading the problem and clarifying the meaning of the following terms:

amplitude	the coefficient a of the sine term in a function of the form $f(x) = a \sin bx$; on a graph of the sine function, the vertical distance from the y-coordinate of the maximum point on the graph to the midline of the sine curve
maximum value	the greatest value of a function for a particular interval of the function
minimum value	the smallest value of a function for a particular interval of the function
period	in a sine function graph, the horizontal distance from a maximum to a maximum or from a minimum to a minimum; one repetition of the period of a function is called a *cycle*
sine function	a trigonometric function of the form $f(x) = a \sin bx$, in which a and b are constants and x is a variable defined in radians over the domain $(-\infty,\infty)$

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they recognize that they will need to graph and compare two different functions representing the amount of carbon dioxide in the atmosphere over a time period of six decades. They will reason abstractly as they compare the features of the given sine function to the linear function created from the graph of the data given in a table. They will also reason that the two functions can be combined by modifying one of the functions, and they will apply the newly created function in order to predict the amount of carbon dioxide in the atmosphere in April of 2019. They will reason quantitatively as they substitute the coefficients which were generated by the calculator into the combined function model, as well as when they solve the equation in order to determine the period of the sine function. They will also reason quantitatively as they substitute the value representing April of 2019 into the combined function model and calculate the predicted amount of carbon dioxide in the atmosphere in April of 2019.

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- **SMP 5:** Use appropriate tools strategically.

  Students will use a graphing calculator, or other graphing technology, in order to first graph data given in a table in the problem. They will analyze the graph in order to determine the type of function represented by the data. Then, they will create a function on the calculator based on its graph, and they will round the calculator results for the coefficients of the constants of the linear function given for the variables a and b to three significant digits. Once they have combined the given sine function and the linear function to create a new function, they will use their calculators to graph the new function and compare the values given on the graph to the original values given in the table.

- **SMP 7:** Look for and make use of structure.

  Students will apply the basic structure of a linear function in order to determine values of the coefficients which represent the function describing the graph of the data given in a table. Students will also apply the structure of a sine function in order to identify the key features of the function representing the concentration of carbon dioxide present in the atmosphere. They will determine the amplitude and the period of the sine function and use these values when writing the combined function, along with the values given for the linear function. Finally, they will apply the structure of the combined function in order to calculate the amount of carbon dioxide that is predicted to be present in the atmosphere in April of 2019.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- applying function characteristics in mathematical models to real-world problems and solutions that cannot be supported by the real-world conditions found in the problems

  Before students begin the task, review the scenario presented. Discuss how the context of the scenario relates to the use of functions to describe the scenario. Specifically discuss the amplitude and period of a sine function and how these features relate to the scenario.

- using mathematical models that inaccurately reflect the real-world conditions of the problems; e.g., using rational functions when exponential or logarithmic functions would be more appropriate

  Review with students the general format and shapes of the different types of functions. Ensure their understanding of fitting data points to the closest type of function that can represent the data.

- miscalculating the value for t representing April of 2019

  Review with students what the values for t represent in the table and in the context of the scenario. Have them draw a visual representation of which decades April 2019 falls between, as well as what fraction of a year is represented by the month of April. Carefully walk students through the process of converting the month of April and the year 2019 into decimals in order to find the numerical value of t.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Once students have found the linear function that approximately fits the data from the table, ask them. "The linear function that approximately fits the data is $C(t) = 14.6 \bullet t + 296$. What do the two coefficients represent in terms of the graph of the function? How do these coefficients relate to the context of the scenario?" (**Answer:** The value of the coefficient of the variable t, 14.6, represents the slope of the graph of the linear function. The constant term 296 represents the y-intercept of the graph. In the context of the scenario, the value of the slope represents the approximate amount of increase of carbon dioxide in the atmosphere every ten years. The y-intercept represents the amount of carbon dioxide in the atmosphere when the scientists began tracking the carbon dioxide levels, in the 1950s.)
- As an extension to the previous question, ask students, "If you were to calculate the slope of each consecutive pair of points given in the table, each slope does not have a value of 14.6. How can you explain the differing slopes, given that this function is linear?" (**Answer:** When the data points from the table are graphed using a calculator, the points do not form a perfectly straight line. A linear function is the closest representation of a function fitting the points. Therefore, the slope values between any two points are not going to be exactly the same. The value generated by the calculator to represent the slope, 14.6, is the average of the slopes found for each consecutive pair of points.)
- As students are comparing the format of the calculator-generated linear function to the given sine function, ask them, "What does the value c represent in the given sine function?" (**Answer:** The value c represents the amount that the carbon dioxide concentration varies during each year of the decade. In the problem, this value is given as 3 parts per million.)
- As an extension to the previous question, ask students, "If the value of c is given in the problem as 3 parts per million, then why is the value of 1.5 used in the combined function instead of 3? Explain in terms of which feature this value represents in a sine function." (**Answer:** The value of c, which is given as 3 parts per million, is the difference between the maximum and minimum value of the sine terms. In a sine function, this value is the amplitude, and the value of the amplitude is half of the distance between the maximum and minimum values. Therefore, the value of the amplitude in this problem is 1.5, so that is the value that is used in the combined function.)
- As students are comparing the graph of the given sine function to the graph of the linear function from the data given in the table, ask them, "The basic sine function has a graph which displays maximum and minimum values. How can the given sine function and the linear function found when graphing the data from the table be similar when the basic look of each graph is different?" (**Answer:** The data from the table is graphed using an interval of ten years. The behavior of the graph of each year is not displayed in this linear function. The given sine function represents the annual variation in the amount of carbon dioxide, which is shown as fluctuations between maximum and minimum values for each year.)

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- After students have graphed the newly combined function on their calculators, ask them, "Why is the graph of the combined function not completely linear? In other words, why does it show small, squiggly lines?" (**Answer:** The combined function represents the graph of the linear function which displays the amount of carbon dioxide present every ten years. The sine function represents the amount of carbon dioxide annually, and this graph displays how the amounts of carbon dioxide fluctuate from maximum to minimum values in a given from year to year, within a decade. The combined graph displays the results of the data given every ten years, as well as the data given annually.)
- Ask students, "How can the value for April 2019 be converted into an equivalent value of t, so that the amount of carbon dioxide predicted to be present in the atmosphere in April of 2019 can be calculated?" (**Answer:** The value for t that represents the year 2010 is $t = 6$. The value that would represent the year 2020 is $t = 7$. Therefore, the value of t that will represent the month of April in the year 2019 is a decimal between 6 and 7. April of 2019 is nine years and four months (one-third of a year) into the decade of 2010. The nine out of ten years into a decade is represented by 0.9, and four months into a year of a decade is represented by $\frac{1}{3} \bullet 0.1 = 0.0\overline{3}$. The sum of these two values is $0.9\overline{3}$, so the decimal representing April of 2019 is $6.9\overline{3}$.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss the general format of the given sine function and ask students to identify what the components of the function represent in the context of the scenario. Ask students to explain how they used their calculators to first graph and then create the linear function. Discuss the values of the coefficients given by the calculator for the linear function. Ask students to explain how the given sine function compares to the calculator-generated linear function. Specifically ask them to explain what component of the sine function represents the annual fluctuation of carbon dioxide in the atmosphere, and why the linear function only represents data given every ten years. Discuss the process of combining the functions, as well as graphing the combined function on a calculator. Ask students to explain how they calculated the predicted amount of carbon dioxide for April in the year 2019. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.

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- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process as they determined that the graphed data points represented a linear function. Ask them to explain their choices for substituting the specific coefficients from the calculator-generated linear function into the combined function model, with the given sine function. Also ask them to explain what the amplitude and the period of the combined function represent in the context of the scenario, and how those values were used in order to create the final combined function model. Discuss the "squiggly line" graph resulting from the combined function model in terms of how its shape represents the original sine function and the linear function. Focus on the use of precise mathematical language and clarity, specifically when discussing the features of the sine function, such as the amplitude and period, as well as the components of the linear function. Also focus on the correct terminology when discussing the units of the variables in the functions, specifically the time and the amount of carbon dioxide.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The graphs and the accompanying data tables of functions produced by graphing calculators can be invaluable in interpreting function features in real-world applications.

  In this task, students will graph a given sine function on a calculator and then compare it to the graph of data from a table which can be represented by a linear function. They will examine the features on both graphs and relate them to the context of the scenario.

- Function values in mathematical and real-world models of problems can be described as increasing, decreasing, or constant.

  Students will examine the graphs of the two functions in the problem and observe the increasing values in each function, which represent the increasing amounts of carbon dioxide in the atmosphere over a time period of decades.

- Mathematical and real-world function models are sometimes restricted to certain unchanging constants. Examples include the values of e and π, the speed of light in a vacuum (often denoted c), and the temperature value of absolute zero.

  Students will recognize that the value of π is a constant contained in the sine function in the problem.

- The domain value(s) at which a real-world model has a maximum or a minimum function value are of importance in a wide variety of physical applications.

  Students will recognize that in the given sine function, the argument portion of the function ($20\pi t$) shows that the amount of carbon dioxide in the atmosphere fluctuates from a maximum value to a minimum value annually.

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- With some complicated functions, maximum and minimum function values occur locally; that is, over a restricted domain interval of the function, which may or may not have physical meaning in a real-world model.

  Students will note that while the data in the table represents domain intervals of a decade, the maximum and minimum values of the sine function occur within the ten-year interval, specifically every year.

Extending the Task

To extend the task, ask students to predict in what year the amount of carbon dioxide present in the atmosphere will reach 450 ppm. Ask them to create a detailed list of their steps and calculations that were performed in order to solve for the value of the year, and ask them to explain how they converted the value to a month and a year. Ask for volunteers to share their results, and discuss the process of solving for the unknown variable of t and how this is accomplished when the variable is represented in two places in the combined function. Discuss any difficulty that students may have encountered while working through this extended task, and allow students to help each other correct any errors.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first recognized that I needed to graph and compare two different functions representing the amount of carbon dioxide in the atmosphere over a time period of six decades. I also reasoned abstractly as I compared the features of the given sine function to the linear function created from the graph of the data given in a table. I reasoned that the two functions could be combined by modifying one of the functions, and I applied the newly created function in order to predict the amount of carbon dioxide in the atmosphere in April of 2019.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I reasoned quantitatively as I substituted the coefficients which were generated by the calculator into the combined function model, as well as when I solved the equation in order to determine the period of the sine function. I also reasoned quantitatively as I substituted the value representing April 2019 into the combined function model and calculated the predicted amount of carbon dioxide in the atmosphere then.)
- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** I used appropriate tools strategically as I first used a graphing calculator in order to graph data given in a table in the problem. I analyzed the graph in order to determine the type of function represented by the data. Then, I created a function on the calculator based on its graph, and I rounded the calculator results for the coefficients of the constants of the linear function given for the variables a and b to three significant digits. Once I combined the given sine function and the linear function to create a new function, I used my calculator to graph the new function and compare the values given on the graph to the original values in the table.)

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- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I looked for and made use of structure when solving this problem by applying the basic structure of a linear function in order to determine values of the coefficients which represented the function describing the graph of the data given in a table. I also applied the structure of a sine function in order to identify the key features of the function representing the concentration of carbon dioxide present in the atmosphere. I determined the amplitude and the period of the sine function and used these values when writing the combined function, along with the values given for the linear function. Finally, I applied the structure of the combined function in order to calculate the amount of carbon dioxide that is predicted to be present in the atmosphere in April of 2019.)

Alternate Strategies or Solutions

Some students may choose to graph the data given in the table by hand instead of using a calculator. They may also choose to derive the equation for the function by using the point-slope formula. Remind students that the data given in the table does not represent a perfectly linear function, so they would need to find the average of the slope values between each pair of consecutive points in order to find the same slope value generated by the calculator. Remind them that the graphing calculator will graph the points in a more efficient and accurate manner, and it will also generate a function to represent the data more quickly.

Technology

Students can use graphing calculators or other graphing technology for this task.

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Problem-Based Task 4B.3.3: Comparing Energy Sources

Common Core State Standards

F–IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

F–IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Task Overview

Focus

How can linear functions be applied to model real-world scenarios? How is the rate of change of a function calculated, and how can rates of change be used to compare functions? What types of combinations of functions can be applied to draw conclusions about and make comparisons of pairs of functions to a singular function? In this task, students will calculate and compare the rates of change of five linear functions representing types of energy sources. They will also create combinations of different energy sources and evaluate the usage of these combined energy sources and compare them to a singular source.

This activity will provide practice with:

- interpreting data given in an table and a graph in the context of the scenario
- calculating rate of change of a function
- comparing rates of change
- combining functions using addition
- determining the domain of a function
- solving an inequality with the variable on both sides
- analyzing the domain of a function in the context of a real-world scenario

Introduction

This task should be used to explore or to apply the skill of calculating and interpreting the average rate of change of a function (presented symbolically or as a table) over an interval. Students should be familiar with calculating the rate of change, evaluating a function at a specific domain value, and combining functions.

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Begin by reading the problem and clarifying the meaning of the following terms:

domain	the set of all input values (x-values) that satisfy the given function without restriction
rate of change	a ratio that describes how much one quantity changes with respect to the change in another quantity; also known as the slope of a line

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goals of the task are to determine which of five different energy sources has the greatest rate of change of usage and which has the least, and also to determine when any one of the fuel types will overtake the usage of any one of the other fuel types. They will also compare the combined use of coal and natural gas to the use of renewables as an energy source. To accomplish these goals, first they will calculate and compare the rates of change for each energy source. To do this, they will use the rate of change formula, and they will compare the results to the value given for the slope of each linear function. They will also determine if any one of the fuel types will overtake the usage of other fuel types by setting any two of the functions equal to each other to see if there is a value for the variable t that will cause the functions to equal each other. Finally, they will combine the functions representing coal and natural gas by adding the functions, and they will compare this result to the function representing renewables by creating and solving an inequality. The results of the inequality will yield the domain interval for which the combined coal-and-natural gas function will have greater usage than renewables as a source of energy.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled with linear functions representing five different energy sources which are different types of fuel. They will observe that the functions are represented with a graph and a table displaying the function equation for each energy source. They will recognize that the graph displays the usage of each energy source beginning with 1990, and that the graph also projects future usage based on the rate of change given for each function. They will compare the rates of change for each function in order to draw conclusions about which energy source that has the greatest usage and which has the least usage. They will also combine functions by adding them in order to compare the usage of two energy sources to the usage of only one, and they will find the domain values which make the combined energy sources greater than the usage of one energy source. They will analyze the graphs of the functions and determine if the usage of one type of fuel of an energy source will ever become greater than the usage of another type of fuel.

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- **SMP 7:** Look for and make use of structure.

  Students will apply the structure of a linear function as one method of identifying the rate of change for each of the five linear functions representing different energy sources. Specifically, they will recognize that the coefficient of the *t* term in each function represents the rate of change. They will also apply the structure of a linear inequality, as they create an inequality to compare the combined coal-and-natural gas function to the renewables function in order to determine the domain value at which the value of the combination of the coal and natural gas functions is greater than the value of the renewables function.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- misunderstanding the variable or variable value(s) that are relevant to a comparison of like or unlike functions

  Review the meaning of the variables in the problem. Have students write at the top of their papers that *t* represents the time in years from 1990 to 2020, and $E(t)$ represents the amount of energy that is used or projected in quadrillions of BTUs each year.

- confusing mathematical features with real-world features and values of functions in comparing functions

  Discuss the real-world constraints of the domain in the given scenario. Review with students that the time in years is a positive quantity, as well as the amount of energy usage by the five different sources.

- incorrectly calculating the rate of change of a function

  Have students write the formula for calculating the rate of change at the top of their papers. Review the components in the numerator and the denominator with them. Work through some sample rate of change problems with them before they begin the task.

- misinterpreting the resulting domain when the combined coal-and-natural gas function is substituted into the inequality comparing this combined function to the renewables function

  Remind students of the real-world constraints on the domain, specifically that the value of *t* cannot be negative. Remind them to focus only on the interval of the domain that is greater than zero.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "What is another term for 'rate of change' of a linear function? What is needed in order to calculate the rate of change for a linear function?" (**Answer:** Another term for "rate of change" is "slope". The rate of change, or slope, of a linear function is calculated using two points from the line representing the function.)

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- As students begin working on the task and calculating the rates of change, ask them, "How is it possible that the rates of change found from using the formula could be different from the value of the coefficient of t for each function?" (**Answer:** When the rate of change formula is used, the points are found from the graph of the function. It is possible that the values of the coordinates of the points on the graph are not exact and have been approximated. Therefore, the resulting rate of change value might differ from the value of the coefficient of t.)
- Ensure students' understanding of the components of a linear function and how they are represented on a graph. Ask students, "What do the y-intercepts of each function represent graphically and also in the context of the scenario?" (**Answer:** The y-intercept of a linear function represents the point where the graph of the function crosses, or intercepts, the y-axis. In the context of the scenario, it represents the amount of usage for each fuel type when the data collection began, in the year 1990. The year 1990 is represented as $t = 0$.)
- Ask students, "Based on the rates of change for the five different energy sources, which energy source has the greatest rate of change, and which has the least rate of change? Explain what these values mean in the context of the scenario." (**Answer:** The renewables energy source has the greatest rate of change with 13 quadrillion BTUs per year, and liquid hydrocarbons has the least rate of change with 3 quadrillion BTUs per year. In the context of the scenario, the usage of renewables as an energy source is growing at a faster rate each year than any other energy source, and the usage of liquid hydrocarbons as an energy source is growing at the slowest rate compared to the other sources.)
- Ensure students' understanding of the behavior of the five linear functions representing the different energy sources. Ask them, "What does it mean that the functions are diverging across the domain of [0, 30]? Explain the term 'diverging'." (**Answer:** The term "diverging" means to move away from, or go in a different direction. In the given domain shown on the graph of [0, 30], the five functions are increasing and going off in all different directions, which means they will never meet, or cross each other on a graph.)
- As an extension to the previous question, ask students, "How can you mathematically prove that the five functions will not meet or cross each other beyond the domain of $t = 30$, or beyond the year 2020?" (**Answer:** Two functions will meet, or intersect at a point, if there is a value within the given domain for which the functions equal each other, i.e. $f(t) = g(t)$. If all possible pairs of functions for the five energy sources are set equal to each other, the value found for t is a negative value each time. Since t can only be a positive value representing time in years, then there is not a value of t greater than 30 that will allow any pair of the five functions to equal each other.)
- As students are combining the coal and natural gas functions, ask them, "Why can't you subtract the natural gas function from the coal function in order to combine them? Explain your answer in the context of the scenario." (**Answer:** The amount of energy usage represented in the functions and graphs of the functions is positive and increasing. Although the natural gas function has a smaller rate of change than the coal function, subtracting the functions would not be meaningful or practical in the context of the scenario, because the functions represent two different energy sources, and the usage is a positive amount.)

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- After students have completed the task, discuss the practical applications of this real-world scenario. Ask them, "How do you think the results of this data collection could be applied in terms of government funding for energy resources?" (**Answer:** The graphs of the functions representing the five different energy sources show the increasing trend of the usage of the types of fuel for each energy source. A person lobbying for funding for resources used for renewable energy, such as solar energy, wind energy, etc., could cite the fact that renewable energy sources are the fastest growing sources of energy. Therefore, there could be a case made for the necessity of funding to support the growth.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss how the linear functions given algebraically for each energy source correspond to the graphical representations on the coordinate plane. Specifically discuss how the slopes and *y*-intercepts are represented based on their values in the function equations. Also discuss the possible difference between calculating the rate of change for each function using two points and using the value of the slope for the rate of change. Ask students to explain how they added the coal function to the natural gas function to yield a combined function. Also ask them to explain how they applied this combined function to compare it with the renewables function. Discuss how the inequality was created and solved to find the domain value that allowed the combined function of coal and natural gas to have greater usage than the usage of the renewables as an energy source. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process for why at no time will the usage for one fuel type overtake the usage of any one of the other fuel types. Ask them to explain this reasoning in terms of the rates of change of the five functions representing the different energy sources. Discuss how the solution of the inequality necessitated restricting the domain over a specific interval. Ask students to explain the domain restrictions in the context of the scenario. Also discuss the real-world applications of comparing different energy sources and how their graphs and functions can be used to predict and plan for future usages. Focus on the use of precise mathematical language and clarity, specifically when discussing the variables contained in the linear functions and how these variables correspond to the labels on the axes on the graphs of the functions. Also focus on the correct use of terminology and notation when calculating the rate of change for each function.

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Connecting to Key Concepts

Make explicit connections to key concepts:

- The function characteristics that are used to compare and contrast function types can depend on the purpose of the comparison.

  In this task, students will compare the function characteristic of rate of change. They will calculate the rates of change of five linear functions representing different energy sources. They will compare the rates of change to determine which energy source has the greatest amount of growth of usage each year and which has the least amount of growth of usage.

- The rate of change of a function value can be calculated to determine which of several different functions (or functions of the same form) best represent a condition of decreasing or increasing function values.

  Students will calculate the rate of change for five linear functions which represent different energy sources. They will compare these rates of change in order to determine the current and future usage of the different sources.

- Domains and ranges can also be used to compare the validity of a function serving as a model for real-world phenomena. In some cases, only a restricted domain or range of a mathematical model is relevant to a specific problem.

  Students will calculate the domain interval when comparing the combined coal-and-natural gas function to the renewables function in order to determine at which value the combination of the coal and natural gas functions is greater than the value of the renewables function. They will evaluate the result and conclude that the domain is undefined at values of t that are less than zero; based on the real-world constraint that t is a positive quantity.

Extending the Task

- Ask students to create a new graph of the five functions by hand on graph paper and include the second quadrant on their graph. Ask them to suspend the real-world constraints on the domain values and extend the lines on the five functions to the point where the graph of each function intercepts the x-axis. Remind students to use the same scale on the x-axis for the second quadrant as was used in the first quadrant, where the intervals represent one year. Ask students to calculate the x-intercepts for each of the five functions representing the different energy sources. Have them round their values for t to the nearest whole number. Ask them to create a table with their results, indicating what year the value of t represents. (Remind them that the negative values of t represent the years counting backwards from 1990. For example, $t = -5$ would represent the year 1985.)

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- Ask for volunteers to share their results. Discuss what real-world meanings the *x*-intercepts for the functions have in regards to what year data collection began for the usage of each energy source. Encourage a group discussion about whether or not the different energy sources were available (in the sense of being officially used as an energy source) in the year indicated by the *x*-intercepts. (For example, the *x*-intercept for the renewables function is at $t = -26$, which represents the year 1964. Ask students if they think solar power, wind power, hydroelectric power, etc. was being monitored and used as energy sources back in the year 1964.)

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by recognizing that the goals of the task are to determine which of five different energy sources has the greatest rate of change of usage and which has the least, and also to determine when any one of the fuel types will overtake the usage of any one of the other fuel types. Another goal is to compare the combined use of coal and natural gas to the use of renewables as an energy source. To accomplish these goals, first I calculated and compared the rates of change for each energy source. To do this, I used the rate of change formula and compared the results to the value given for the slope of each linear function. I also determined if any one of the fuel types will overtake the usage of other fuel types by setting any two of the functions equal to each other to see if there was a value for the variable *t* that would cause the functions to equal each other. Finally, I combined the functions representing coal and natural gas by adding the functions, and I compared this result to the function representing renewables by creating and solving an inequality. The results of the inequality yielded the domain interval for which the combined coal-and-natural gas function would have greater usage than renewables as a source of energy.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by recognizing that this scenario could be modeled with linear functions representing five different energy sources which are different types of fuel. I observed that the functions were represented with a graph and a table displaying the function equation for each energy source. I recognized that the graph displayed the usage of each energy source beginning with 1990, and that the graph also projected future usage based on the rate of change given for each function. I compared the rates of change for each function in order to draw conclusions about which energy source that had the greatest usage and which had the least usage. I also combined functions by adding them to compare the usage of two energy sources to the usage of only one, and I found the domain values which made the combined energy sources greater than the usage of one energy source. I analyzed the graphs of the functions and determined if the usage of one type of fuel of an energy source will ever become greater than the usage of another type of fuel.)

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- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I looked for and made use of structure when solving this problem by applying the structure of a linear function as one method of identifying the rate of change for each of the five linear functions representing different energy sources. Specifically, I recognized that the coefficient of the *t* term in each function represents the rate of change. I also applied the structure of a linear inequality, as I created an inequality to compare the combined coal-and-natural gas function to the renewables function in order to determine the domain value at which the value of the combination of the coal and natural gas functions was greater than the value of the renewables function.)

Alternate Strategies or Solutions

- Some students may choose not to use the rate of change formula, but instead demonstrate understanding that the slope of the linear functions is the same as the rate of change. Encourage them to calculate the rate of change also in order to compare with the slopes given in the equations for the linear functions.
- Some students may choose to test all pairs of linear functions in order to see if there is a positive value for the domain in which the pair of functions is equal. Remind them to check their work and calculations to verify that all combinations will result in values of *t* that are less than zero.

Technology

Students can use scientific calculators for this task.

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Implementation Guide

Problem-Based Task 4B.4.1: Comparing Social Media Growth

Common Core State Standards

A–CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

F–IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F–IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F–BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Task Overview

Focus

What types of functions can be used to model a real-world scenario? How can graphing technology be applied to determine the appropriate function type, as well as identify key features of a data set? In this task, students will use graphing technology to create exponential, linear, and quadratic functions for data sets representing the growth in users of a popular social media website in the United States and also in the rest of the world. They will also graph the three types of functions for each data set and determine which function type best represents each set of data.

This activity will provide practice with:

- interpreting written information about a real-world scenario
- entering data points into a graphing calculator
- writing exponential, linear, and quadratic functions based on data from a calculator
- graphing exponential, linear, and quadratic functions using a graphing calculator
- analyzing graphs of exponential, linear, and quadratic functions
- determining the most appropriate function type for a data set

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Implementation Guide

Introduction

This task should be used to explore or to apply the skills of interpreting key features of graphs and tables of functions that model a relationship between two quantities and using graphing technology to display key features of functions, and also choosing an appropriate function to model a real-world scenario. Students should already be familiar with using graphing technology to create functions and graph them, and identifying exponential, linear, and quadratic function models that can be used to represent tables of data points, data graphs, or symbolic expressions of data given a verbal description of the relationship.

Begin by reading the problem and clarifying the meaning of the following terms:

exponential function	a function of the form $f(x) = ab^{cx}$, in which a, b, and c are constants; a function that has a variable in the exponent, such as $f(x) = 5^x$
linear function	a function that can be written in the form $ax + by = c$, where a, b, and c are constants; can also be written as $f(x) = mx + b$, in which m is the slope and b is the y-intercept
quadratic function	a function defined by a second-degree expression of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine which function would be the best model for each of two sets of data representing the growth in users of a popular social media website in the United States and the growth in users of the website in the rest of the world. To accomplish this goal, they will use graphing calculators to first input data points for each data set. Then they will create exponential, linear, and quadratic functions based on the coefficients generated by the calculator for each set of points. Then, they will input the function equations into the calculator and generate graphs representing an exponential, a linear, and a quadratic function for each data set. They will analyze the graphs in order to determine which function type best models each data set, based on the best fitting function for the data points.

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- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for choosing the best function type to model each data set representing the growth in users of a popular social media website in the United States and in the rest of the world. They will base their arguments on their analysis of each exponential, linear, and quadratic function graphed for each data set. They will examine how closely each of the function graphs passes to the data points, and then they will determine which type of function provides the best fit for the data points. They will present their arguments to others and explain their conclusions for each data set based on their results from the calculator and their analysis of the graphs.

- **SMP 5:** Use appropriate tools strategically.

  Students will use a graphing calculator, or other graphing technology, in order to first input the data points given in the tables which represent the growth in users of a popular social media website in the United States and the growth in users of the same site in the rest of the world. Then they will apply the calculator-generated coefficients in order to write exponential, linear, and quadratic functions for each data set. After the functions have been created, they will graph all three functions for each data set with the calculator, and then they will analyze the resulting graphs in order to determine which function type best fits each set of data.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly inputting data points and/or functions into the calculator

  Remind students to enter the data points carefully, paying attention to each set of coordinates. Have them double check their results with another student's to ensure accuracy. Also remind them to pay close attention when inputting intervals for the scale of their graphs. Encourage them to recognize possible data input errors if their graphs are not anywhere close to the data points on the graphs given in the original problem.

- reversing the coefficients for a and b when writing the functions

  Have students write the general formats for an exponential, linear, and quadratic function at the top of their papers. Remind them to write down each a and b value for each function before substituting the coefficients into the function equation.

- incorrectly choosing the function type to best fit the data points for each data set

  Before students begin the task, discuss the general shapes of the graphs for an exponential, linear, and quadratic function. Remind students that most of the given data points should fall close to, or on, the line, or curve, for one of the three function types to conclude it is the best fit for the data. Also remind them that it is unlikely that any of the three functions will be a perfect fit for the data points.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- After students have read the information presented in the task, but before they begin working on the task, ask them, "After examining the graph representing the amount of active users for the social media site in the United States and also the rest of the world, what type of function seems to best represent each data set?" (**Answer:** Answers may vary. For the non-United States users, some students may think the graph represents an exponential function, and some may think it represents a quadratic function. For the United States users, students may think the graph represents a linear function.)
- As an extension to the previous question, ask students, "Based on the data in the table, as well as the graphs, what observations can you make about the number of active users on the social media site in the United States compared to the number of active users on the social media site in the rest of the world?" (**Answer:** Based on the data, the number of active users on the social media site in non-United States areas is greater than the number of active users on the site in the United States. Also, based on the data in the table as well as the graphs, the number of active users in non-United States areas is growing at a faster rate than the number of active users in the United States for the years 2010–2013.)
- Before students begin the task, ask them, "Given the data, what could you calculate to mathematically verify that the number of active users on the site is growing faster in non-United States areas than it is in the United States?" (**Answer:** To verify that the growth rate is greater in the non-United States than the United States, you could calculate the rate of change for each data set.)
- As students are working on inputting the data into their calculators and then writing the functions for each function type, ask them, "What do the results of the linear functions representing the United States users and the non-United States users verify for you in terms of the growth rate? Explain using features/values from the functions." (**Answer:** For United States users, the linear function is $y = 3.354x + 3.4$. For non-United States users, the linear function is $y = 15x + 1.3$. The slope of the function for United States users is 3.354, and the slope for the non-United States users is 15. The slope is greater for non-United States users, by almost five times, so the growth rate is much higher for non-United States users.)
- As students are analyzing the fit of the graphs for each data set, ask them, "Why is an exponential function not a good fit for either data set?" (**Answer:** For the data set representing the United States users, the exponential function only passes close to the first four data points. It does not even come close to the other eleven data points. For the data set representing non-United States users, the exponential function only passes close to about four or five of the fifteen data points. For both data sets, the exponential functions diverge away from the data points after passing close to only a few points. As a result, the exponential function is not a good function model for either data set.)

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- Ask students, "After examining the original given graph and then the graphs created by the three function types for the number of United States users, which function type would you choose to best represent the data set—linear or quadratic? Explain and defend your choice." (**Answer:** Answers may vary. One possible response is: The initial graph of the data for the number of active United States users seems to display a linear function. Once the three function types were created and graphed, both the linear and quadratic functions seemed to fit the data well, although the linear function passes close to but higher than some of the data points. However, since the initial graph of the data points seemed to have a more linear shape, the linear function would best represent the data set of United States users.)
- Ask students, "After examining the original given graph and then the graphs created by the three function types for the number of non-United States users, which function type would you choose to best represent the data set – linear or quadratic? Explain and defend your choice." (**Answer:** Answers may vary. One possible response is: The initial graph of the data for the number of active non-United States users seems to display either a quadratic or an exponential function, based on the slight curve of the graph. Once the three function types were created and graphed, both the linear and quadratic functions seemed to fit the data well, as both functions pass closely to most of the data points. Since the initial graph seems to display more of a curve, and the quadratic function that was created passes through most of the data points, the quadratic function would best represent the data set of United States users.)
- After students have completed the task, ask them to analyze the results in terms of real-world applications. Ask students, "What are some possible reasons for the fact that the number of active users for the site is much greater for the rest of the world in comparison to the United States? Why do you think the growth rate is greater in the rest of the world?" (**Answer:** Answers may vary. Some possible reasons include the fact that the population of the rest of the world compared to the population of the United States is much greater. Also, the technology and internet might be better and faster in some of the highly-populated countries, which might lend to the social media site having more users. A possible reason for the greater growth rate in the rest of the world is that the level of penetration of the internet itself may be lower, and growing faster, over the time period.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Ask them to explain their analysis of the given graphs representing the number of active site users for the four-year time period in terms of what type of function appears to best represent the data for both the United States and the non-United States. Discuss the process of inputting the data given in the table in order to obtain the values for a and b such that an exponential, a linear, and a quadratic function can be written. Ask students to explain how they input these functions into their calculators such that all three types of graphs were displayed for both United States users and non-United States users. Discuss how well each graph fits the given data points and what conclusions can be drawn about the best fitting function for each data set. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their analysis and reasoning process for choosing the type of function that best represents each set of data. Specifically, ask them to explain how the data points fell close to or away from each function. Discuss why one function type may be a slightly better fit than another based on the quantity of data points that are closest to the function graph. Ask students to compare their conclusions with their initial thoughts about the graphs given for each data set in the original problem. Focus on the use of precise mathematical language and clarity, specifically when discussing the variables contained in the three function types. Also focus on the correct use of terminology and notation when writing the functions, as well as when determining which function type best fits each data set.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The key to selecting a function model is an accurate analysis of the data in a problem or its visual representation.

  In this task, students will graph an exponential, a linear, and a quadratic function for two different sets of data, and they will analyze the graphs in order to determine which of the three graphs best represents each data set, based on how closely the given data points fit each graph.

- The choice of exponential, linear, or quadratic functions as models for real-world problems can sometimes be justified using graphing calculator techniques. The data-fitting tools of graphing calculators offer a variety of options for comparing function models.

  Students will use the features of their graphing calculators in order to first input data points in order to create an exponential, a linear, and a quadratic function for each set of data. They will also use the graphing feature of their calculators to generate graphs of each function type for both sets of data.

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- Recall that **exponential functions** are functions that have a variable in the exponent. Exponential functions have the form $f(x) = ab^{cx}$, in which a, b, and c are constants.

  Students will create an exponential function based on the calculator-generated values for the constants a and b given when the data points from the given table are inputted in the calculator.

- A linear function can also be written as $f(x) = mx + b$, in which m is the slope and b is the y-intercept. The graph of a linear function is a straight line.

  Students will create a linear function based on the calculator-generated values for the constants a and b given when the data points from the given table are inputted. They will recognize that the value for a is the same value for m given in the slope-intercept form of a linear function.

- Recall that the general form of a **quadratic function** is $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants.

  Students will create a quadratic function based on the calculator-generated values for the constants a and b given when the data points from the given table are inputted in the calculator.

Extending the Task

To extend the task, ask students to apply the functions that they created for each data set and make future projections for the social media website for both United States users and non-United States users. Ask them to predict the number of users for both in the fourth quarter of 2020. Have them assign domain values to represent the specific quarters of the years given in the original table. For example, Q1 in the year 2010 would be $x = 1$; Q2 in the year 2010 would be $x = 2$, and so on. Based on this system, have them calculate what Q4 in the year 2020 would be for the value of x. Ask them to use both the linear and quadratic functions that were created to represent the number of active users for both the United States and non-United States. Have them write down their analysis of the resulting values. Ask for volunteers to share their results, and ask them to share their comparisons of their values for each function type for both United States users and non-United States users. Discuss the difference in values for the two functions and their analysis of whether or not they think the predicted values are valid.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the task was to determine which function would be the best model to represent each of two sets of data representing the growth in users of a popular social media website in the United States and the growth in users of the website in the rest of the world. To accomplish this goal, I used my graphing calculator to first input data points for each data set. Then I created exponential,

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linear, and quadratic functions based on the coefficients generated by the calculator for each set of points. I inputted the function equations into the calculator and generated graphs representing an exponential, a linear, and a quadratic function for each data set. Finally, I analyzed the graphs in order to determine which function type best modeled each data set, based on the best fitting function for the data points.)

- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining my choice for the best function type to model each data set representing the growth in users of a popular social media website in the United States and the growth in users of the same site in the rest of the world. I based my arguments on my analysis of each exponential, linear, and quadratic function graphed for each data set. I examined how closely each of the function graphs passed to the data points, and then I determined which type of function provided the best fit for the data points. I presented my arguments to others and explained my conclusions for each data set based on my results from the calculator and my analysis of the graphs.)
- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** I used appropriate tools strategically as I used my graphing calculator in order to first input the data points given in the tables which represented the growth in users of a popular social media website in the United States and the growth in users of the same site in the rest of the world. Then I applied the calculator-generated coefficients in order to write exponential, linear, and quadratic functions for each data set. After I created the three functions, I graphed all three functions for each data set with the calculator, and then I analyzed the resulting graphs in order to determine which function type best fits each set of data.)

Alternate Strategies or Solutions

Some students may choose to only create the functions for the number of active users in the United States and non-United States based on their analysis of the graphs given in the original problem. They may conclude that for the number of United States users, the linear function is the one that needs to be created, and for the number of non-United States users, the quadratic function or the exponential function is the one that needs to be created. Remind them that it is necessary to create and graph functions for all three types so that they will be able to make an accurate analysis based on all possibilities.

Technology

Students can use graphing calculators or other graphing technology for this task.

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Problem-Based Task 4B.4.3: Europa's Water-Vapor Geysers

Common Core State Standards

F–IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F–IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

F–BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Task Overview

Focus

How can a quadratic function and a square root function be applied to model a real-world scenario? How is the domain of the functions affected by real-world restrictions? In this task, students will apply quadratic and square root functions in order to determine the amount of time it will take for the water vapor of Jupiter's moon, Europa, to reach its maximum height when it is ejected from the surface of the moon at a certain velocity.

This activity will provide practice with:

- identifying values and units of measurement for variables
- performing units of measurement conversions
- substituting numerical values into functions
- applying the quadratic formula
- simplifying square root and quadratic function equations
- determining domain restrictions in the context of the scenario
- converting a solution to different units of measurement

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Implementation Guide

Introduction

This task should be used to explore or to apply the skill of relating the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. Students should already be familiar with writing the inverse of a quadratic function and its domain, applying the quadratic formula, and identifying restrictions on the domain of a function in the context of a real-world scenario.

Begin by reading the problem and clarifying the meaning of the following terms:

quadratic formula — a formula that states the solutions of a quadratic equation of the form $ax^2 + bx + c = 0$ are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

quadratic function — a function defined by a second-degree expression of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants. The graph of any quadratic function is a parabola.

square root function — a function that contains a square root of a variable. The general form is $f(x) = \sqrt{ax^2 + bx + c}$, where a, b, and c are real numbers.

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine the time in seconds, hours, and Earth days that it will take for the water vapor from the plumes of the surface of Jupiter's moon, Europa, to reach its maximum height. To accomplish this task, first they will rewrite the given function for the height of the water vapor by substituting in known values from the given problem and then simplify the result. Next, they will use the quadratic formula in order to solve for the value of t. Once they have simplified the operations used in the quadratic formula, they will analyze the result and determine the appropriate domain for t. Finally, they will use conversion factors in order to determine the numerical value of t in seconds, hours, and Earth days.

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they recognize that the values given in the original problem can be substituted into the given quadratic function, in order to simplify the function and reduce the number of unknown variables to one. They will also reason abstractly as they recognize that the domain for the unknown variable t is positive, and as a result, they will

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apply only the positive result from applying the quadratic formula when determining the numerical value of *t*. They will reason quantitatively as they substitute in the known values and perform the operations necessary to simplify the equation. They will also reason quantitatively as they substitute values into the quadratic formula and simplify the result. Finally, they will use quantitative reasoning as they identify and apply the appropriate conversion factors necessary for converting the value of *t* into hours and Earth days.

- **SMP 7:** Look for and make use of structure.

  Students will first apply the structure of the given quadratic function as they substitute in given numerical values for the height and the velocity, in order to simplify the function to the form where only one variable, *t*, is unknown. Then they will use this simplified quadratic function and identify the coefficients and constants of the function and apply them to the quadratic formula in order to solve for the unknown variable, *t*, which represents the time it will take for the water vapor from Jupiter's moon, Europa, to reach its maximum height. They will also use the structure of conversion factors, as they will identify the necessary conversion factors and perform the operations necessary to label the variable *t* with the requested units of measurement.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- applying a function model to a real-world problem without checking to see if the model's features and solutions actually explain the problem's data

  Before students begin the task, review the components of the given function representing the height reached by the water vapor from Jupiter's moon Europa. Have students explain what each variable in the function represents and ensure their understanding of the connections of the variables in the function to the given scenario.

- confusing quadratic functions with square root functions, and vice versa

  Review the general formats of quadratic functions and square root functions with students. Have them write these formats at the top of their papers to refer to when working through the task.

- misidentifying the domain or not considering the restrictions on the domain based on the given real-world scenario

  Remind students that the variable *t* represents time, so the domain of *t* is positive, and therefore the positive value for the simplified square root has to be used. Remind them that the negative value will not be considered in the problem.

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Implementation Guide

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "The information given in the task states that the gravitational acceleration for Europa is –0.01 meter per second squared. How does this compare with the Earth's gravitational acceleration?" (**Answer:** The Earth's gravitational acceleration is 9.8 meters per second squared. Therefore, the Earth's gravity is greater than the gravity on Europa.)
- Before students begin the task, ensure their understanding of the components of the quadratic function representing the maximum height reached by the water vapor. Ask them, "In the function $h(t) = v_0 t + \frac{1}{2}at^2$, which variables have values that are constants, and which variables have values that are changing? Explain the variables in the function." (**Answer:** In the given function, the variable *a* represents the gravitational acceleration of Europa. This value is a constant value. The other constant value is the maximum height that the water vapor plumes reach, which is represented by the variable $h(t)$. The initial velocity, v_0, represents the given speed of the water vapor molecules in meters per second as they leave the surface of Europa. The variable *t* represents the time in seconds it will take the water vapor to reach the given maximum height.)
- Ask students, "Why is it necessary to change the units of the given height value of 200 kilometers? What conversion is used to accomplish this?" (**Answer:** It is necessary to change the units of the height because it is the only value that is not represented in units of meters. The conversion used to change it to meters is 1 kilometer = 1,000 meters.)
- As students are rewriting the given function and simplifying it after substituting in the known values, ask them, "How can you eliminate the fraction, $\frac{1}{2}$, on the right side of the function equation? What do you need to be careful of when performing this operation?" (**Answer:** To eliminate the fraction on the right side, multiply both sides of the equation by the reciprocal of $\frac{1}{2}$, which is 2. Make sure all terms on each side are multiplied by 2 in order to maintain equality on both sides.)

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- As students are working on applying the quadratic formula in order to solve for the value of t, ask them, "Why is the quadratic formula the most practical method for solving for t as opposed to factoring or completing the square?" (**Answer:** The constants given for each term of the quadratic equation do not allow for an easily factorable quadratic function. The method of completing the square would also provide "messy" values. The quadratic formula is most likely the most efficient and less cumbersome method for this particular function.)
- Ask students, "Why do you think the values representing the variables for b and c in the quadratic formula were left as powers of 10?" (**Answer:** Since the values for the variables b and c are large numbers, it may be easier to perform calculations when simplifying if the numbers are represented as powers of 10, so that exponent rules can be applied.)
- Ensure students' understanding of the domain restrictions on t. Ask them, "Why is only the positive value of the square root used for the resulting value of t?" (**Answer:** In the context of the scenario, the variable t represents the time in seconds that it takes for the water vapor to reach its maximum height. Therefore, the value of t must be positive. The quadratic formula resulted in a positive and negative value of the square root. To ensure that the value of t stays in the positive domain, only the positive value of the square root is considered.)
- As students are converting the resulting value for t into hours and Earth days, ask them, "Could you have converted the value of t in seconds to Earth days using conversions that when straight from seconds to Earth days? If so, what conversion would you have used?" (**Answer:** In order to convert t from seconds to Earth days, it would be necessary to determine how many seconds are in one Earth day. The conversion is 1 Earth day = 86,400 seconds.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss the given information and what each variable in the given quadratic function represented in the context of the scenario. Ask students to explain the operations they used to simplify the original function when they substituted in the known values for the height and the velocity. Discuss how the coefficients that represent the variables a, b, and c were chosen for the quadratic formula. Ask students to explain the substitution and the simplification that they performed in order to solve for the values of t. Also discuss the domain restrictions on t and how these restrictions affected the application of the results of the quadratic formula. Also ask students to explain how they converted the results of the value of t to the requested units of hours and Earth days. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.

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- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and thoughts about how they used order of operations to simplify the original quadratic function and also when they simplified the expressions in the quadratic formula. Ask them to explain how they identified the proper terms to use in the quadratic formula. Discuss their choices for the appropriate conversion factors used in order to represent the value of t in hours and Earth days, as well as their choice for converting the given height from kilometers to meters. Ensure their understanding of how the restrictions on the domain of t resulted in eliminating the negative value for t found from the quadratic formula. Focus on the use of precise mathematical language and clarity, specifically when discussing the component of the variables in the given quadratic function, as well as when discussing the components of the quadratic formula. Also focus on the correct terminology when referring to the domain of the function, as well as the conversion factors used throughout the task.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Quadratic functions have many applications in real-world problems.

  In this task, students will recognize that the function given to represent the height reached by the water vapor from Jupiter's moon Europa is a quadratic function.

- A **square root function** contains a square root of a variable. Square root functions have the general form $f(x) = \sqrt{ax^2 + bx + c}$, where a, b, and c are real numbers.

  Students will simplify the given quadratic function, and then they will use the quadratic formula in order to solve for the value of the variable t. They will recognize that the resulting function once they have simplified the quadratic formula is a square root function.

- Domain and range variables for real-world problems involving square root functions may require adjusting based on the context of the situation.

  After students have simplified the quadratic formula and are left with a square root function, they will recognize that the domain must be a positive value, so the negative value will be eliminated.

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Extending the Task

To extend the task, ask students to work through the task again, but this time with a different given value for the initial velocity. Ask them to use the value of 325 meters per second for the initial velocity and determine the amount of time it will take the water vapor to reach its maximum height of 200 kilometers. Encourage them to organize their steps showing their calculations, and when they have completed the task, ask them to verify their results with those of another classmate. Ask for volunteers to share their results. Discuss the comparison of the results of the extended task with the results of the original task, and encourage a discussion about the relationship between the initial velocity and time. Discuss any difficulties that students may have experienced with the extended task, and allow students to help each other with error analysis.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the task was to determine the time in seconds, hours, and Earth days that it would take for the water vapor from the plumes of the surface of Jupiter's moon, Europa, to reach its maximum height. To accomplish this task, first I rewrote the given function for the height of the water vapor by substituting in known values from the given problem, and then I simplified the result. Next, I used the quadratic formula in order to solve for the value of t. Once I simplified the operations used in the quadratic formula, I analyzed the result and determined the appropriate domain for t. Finally, I used conversion factors in order to determine the numerical value of t in seconds, hours, and Earth days.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first recognized that the values given in the original problem could be substituted into the given quadratic function in order to simplify the function and reduce the number of unknown variables to one. I also reasoned abstractly as I recognized that the domain for the unknown variable t was positive, and as a result, I used only the positive result from applying the quadratic formula when determining the numerical value of t.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning as I substituted in the known values and performed the operations necessary to simplify the equation. I also reason quantitatively as I substituted values into the quadratic formula and simplified the result. Finally, I used quantitative reasoning as I identified and applied the appropriate conversion factors necessary for converting the value of t into hours and Earth days.)

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- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I looked for and made use of structure when solving this problem as I first applied the structure of the given quadratic function by substituting in given numerical values for the height and the velocity, in order to simplify the function to the form where only one variable, *t*, was unknown. Then I used this simplified quadratic function and identified the coefficients and constants of the function and applied them to the quadratic formula in order to solve for the unknown variable, *t*, which represented the time it would take for the water vapor from Jupiter's moon, Europa, to reach its maximum height. I also used the structure of conversion factors, as I identified the necessary conversion factors and performed the operations necessary to label the variable *t* with the requested units of measurement.)

Alternate Strategies or Solutions

- When working through the parts of the task that involved rewriting the given quadratic function by substituting in known values, as well as when working through simplifying the quadratic formula, some students may have chosen to rewrite the powers of ten with standard notation. Some students may have an easier time with the calculations without the exponential representation of the numerical values. Encourage them to check their results with those given in the task.
- Some students may have chosen to convert the value of *t* in second to the number of Earth days using one conversion (1 Earth day = 86,400 seconds). Remind them that although this is a correct alternate conversion, they still need to determine the value of *t* in hours as well.

Technology

Students can use scientific calculators for this task.

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Implementation Guide

Problem-Based Task 4B.5.1: The School Observatory and Its Footprint

Common Core State Standards

G–GMD.4 Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

G–MG.1 Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).★

Task Overview

Focus

What are the different types of plane geometric figures formed by the intersection of a plane with a solid figure? How can cross sections of figures be used to describe objects in terms of their shapes and dimensions? How can the volume of a building be calculated from knowing the dimensions of its measurements? In this task, students will describe the building of a new observatory planned for a local college in terms of its basic geometric solids, and they will also determine the volume of the dome of the building as well as the other shapes of the building.

This activity will provide practice with:

- identifying plane figures and their dimensions
- classifying cross sections of geometric solids
- applying volume formulas for solid figures
- calculating volumes of rectangular prisms and hemispheres
- comparing volumes of solid figures
- analyzing results in the context of the scenario

Introduction

This task should be used to explore or to apply the skills of identifying the shapes of two-dimensional cross-sections of three-dimensional objects and using geometric shapes, their measures, and their properties to describe objects. Students should already be familiar with recognizing and applying the properties of basic plane geometry figures, such as circles and polygons, recognizing and applying the properties of basic solid geometry figures, such as cones, pyramids, and spheres, and calculating the volume of three-dimensional figures.

Begin by reading the problem and clarifying the meanings of *cross section*, *plane figure*, and *solid figure.*

cross section the plane figure formed by the intersection of a plane with a solid figure, where the plane is at a right angle to the surface of the solid figure

plane figure a two-dimensional shape on a plane

solid figure a three-dimensional object that has length, width, and height (depth)

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they recognize that they will need to analyze and then identify the three plane figures that represent the parts of a new observatory building planned for a local college. They will recognize that the first figure is a rectangle, and the second and third figures are cross sections of the observatory. They will examine the features of the cross sections and determine which solid figures produced each cross section. They will also reason abstractly as they compare the volumes of the two solid figures that comprise the entire observatory building. They will reason quantitatively as they calculate the radius of the hemisphere given the diameter. They will also reason quantitatively as they substitute the dimensions of each figure into the appropriate volume formulas and calculate the volumes of each solid figure.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for choosing the specific plane figures that represent the parts of the observatory. They will also justify their choices of which geometric solids produced the cross sections that are given in the original information. They will base their arguments on the features and characteristics of the types of plane and solid figures. They will also use their knowledge to choose the appropriate volume formulas, as they will identify the specific dimensions necessary for computing the volume of each figure. They will also justify their choices for making a determination about how much greater the volume of one figure is than the other.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  Students will express regularity in repeated reasoning as they explain and justify their steps involved in choosing the appropriate plane figures that represent the parts of the observatory. They will also use repeated reasoning as they identify the appropriate solid figures that created the cross sections of the given figures. Specifically, they will identify that the solid figure that has a cross section of a rectangle is a rectangular prism, and the solid figure that has a cross section of a semicircle is a hemisphere.

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Implementation Guide

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- thinking that a cross section can be formed by any intersection of a plane and a solid figure

  Have students draw a sketch of each figure representing the observatory on their paper. Have them list the types of figures as well as the possible types of figures formed by the cross sections of a plane. Remind them that the direction of the cross section makes a difference in the type of plane figure that is created.

- labeling cross sections formed by planes and solids as having linear or cubic units rather than square units

  Remind students that the area of a two-dimensional figure will have square units, and that the volume of a three-dimensional figure will have cubic units. Also, remind them that a plane is a two-dimensional figure, and a solid is a three-dimensional figure.

- mistakenly using the value for the diameter of the hemisphere in the volume formula instead of the radius

  Have students write the formula for the volume of a hemisphere at the top of their papers. Point out that the variable in the formula is the radius. Remind students that the dimension given by the figure in the problem represents the diameter of the semicircle, so the value of the radius must be found and substituted into the volume formula.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- As students are beginning the task and examining the three figures, ask them, "How else could you sketch Figure II in order to possibly get a better representation of the cross section?" (**Answer:** Figure II could be decomposed into two separate two-dimensional shapes—a rectangle, and a semicircle.)
- As an extension to the previous question, ask students, "What are the dimensions of the two figures, if Figure II is decomposed?" (**Answer:** The semicircle has a diameter of 50 feet, which means that the radius is 25 feet. The rectangle has a length of 80 feet, and a width of 10 feet.)
- Ask students, "What solid figure does a hemisphere relate to, and what are the possible cross sections of this solid figure?" (**Answer:** A hemisphere is exactly half of a sphere. Any cross section of a sphere will be a circle, no matter where the plane intersects the sphere.)
- As students are calculating the volume of the rectangular prism, ask them, "What is an alternate formula for the volume of a rectangular prism, other than the formula $V = lwh$ that is used in the task? Explain the alternate formula." (**Answer:** An alternate volume formula for a rectangular prism is $V = Bh$, where B represents the area of the base of the prism. Since the area of a rectangle is the product of the length and the width, the alternate formula can be used if the area of the base is known instead of the individual dimensions.)

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- As students are calculating the volume of the hemisphere and comparing it to the volume of the rectangular base of the building, ask them, "Why do you think the volume of the hemisphere was rounded from approximately 32,725 cubic feet to 30,000 cubic feet?" (**Answer:** When the volume of the rectangular solid was calculated, the dimensions were values such that the volume was exactly 40,000 cubic feet. In order to properly compare the volume of the rectangular solid to the volume of the rectangular solid, the volumes should be compared close to the same place value. Therefore, the volume of the hemisphere was rounded to the nearest ten-thousandths place so it could be compared to the rectangular solid with a volume represented in the ten-thousandths place.)
- As an extension to the previous question, ask students, "Why is the value for the volume of the rectangular prism given as exactly 40,000 cubic feet, but the value for the volume of the hemisphere is given as approximately 32,725 cubic feet?" (**Answer:** The variables given in the formula for the volume of a rectangular prism, $V = lwh$, have exact values. Therefore, when they are multiplied together, the result is an exact value. However, the constant π is in the formula for the volume of a hemisphere, $V = \frac{2}{3}\pi r^3$. Since the value of π is approximated, then when it is used in calculations, the resulting values will also be approximated.)
- Ask students, "When comparing the volume of the rectangular prism to the volume of the hemisphere, the task states that the volume of the dome, represented by the hemisphere, is about three-fourths that of the rectangular solid shaped building, represented by the rectangular prism. How can you compare these volumes in terms of percents? Explain." (**Answer:** The volumes can be compared in terms of percents by recognizing that 30,000 is 75% of 40,000. Therefore, 40,000 is 25% greater of an amount than 30,000. The volume of the rectangular prism, which represents the rectangular solid-shaped part of the building, is 25% greater than the volume of the hemisphere, which represents the dome-shaped part of the building.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Ask them to explain their analysis of the three given figures in terms of what two-dimensional plane figures they identified for each. Discuss the two different plane figures represented in each of the cross sections. Ask students to explain how they identified a rectangular prism as the solid figure that produced the cross section of the rectangular base of the building, and also how they identified a hemisphere as the solid figure that produced the semicircles shown

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in the cross sections. Discuss how the given dimensions for each figure corresponded to the respective variables in the formulas for the volume of a rectangular prism and a hemisphere. Ask them to explain how they calculated the volumes for the solid figures and used the results to compare the values. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.

- Compare students' strategies and ways of justifying responses. Ask students to share their analysis and reasoning process for choosing the specific plane figures representing the base of the observatory building and the cross sections. Discuss why a rectangular prism and a hemisphere were chosen as the solid figures that created the given cross sections. Also discuss the specific dimensions of each solid figure and which variables each dimension represented in the volume formulas for each figure. Ask students to explain their justification for rounding the volume of the hemisphere when comparing its value to the volume of the rectangular prism. Focus on the use of precise mathematical language and clarity, specifically when referring to plane figures, cross sections, and solid figures. Focus on the correct use of terminology and notation when substituting in the values for the dimensions of each figure into the volume formulas.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Plane figures have many applications in real-world problems.

  In this task, students will recognize that the observatory that is planned for a local college is a three-dimensional figure that can be broken down into two-dimensional plane figures. They will notice that the floor of the observatory is a rectangle, and that the cross sections are also plane figures.

- When planes intersect with **s**olid figures, or three-dimensional objects that have length, width, and height (or depth), plane figures are created.

  Students will recognize that parts of the building are represented by cross sections, and that these cross sections were created from a two-dimensional plane intersecting a three-dimensional figure, or solid figure.

- In its basic form, a cross section is the plane figure formed by the intersection of a plane and a solid figure.

  Students will examine and analyze the given cross sections representing parts of the building. They will identify the solid figures that created the specific cross sections and use the dimensions of these figures to calculate the volume of the observatory building.

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Extending the Task

To extend the task, ask students to work with a partner and create a new task with a building that represents a grain silo found on a farm. (If they are unfamiliar with the shape of a grain silo, let them know that it is a right circular cylinder with a hemisphere/dome on the top. Draw a sketch if necessary.) Ask each pair of students to sketch the base of the silo, as well as the cross sections of the dome of the silo. Remind them to use a cross section found when slicing a plane at a right angle to the base of the figure. Have them devise their own dimensions for the diameter of the silo, as well as the height of the silo. Ask them to calculate the entire volume of the silo. Ask for volunteers to share their results. Discuss the plane figures that represent the cross sections of the silo. Also discuss the dimensions used for the formulas for each solid figure. Discuss any difficulties students may have encountered while working through this extended task, and allow students to help each other with error analysis.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first recognized that I needed to analyze and then identify the three plane figures that represented the parts of a new observatory building planned for a local college. I recognized that the first figure was a rectangle, and the second and third figures were cross sections of the observatory. I examined the features of the cross sections and determined which solid figures produced each cross section. I also reasoned abstractly as I compared the volumes of the two solid figures that comprised the entire observatory building.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I reasoned quantitatively as I calculated the radius of the hemisphere given the diameter. I also reasoned quantitatively as I substituted the dimensions of each figure into the appropriate volume formulas and calculated the volumes of each solid figure.)
- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining my choices for choosing the specific plane figures that represent the parts of the observatory. I also justified my choices of which geometric solids produced the cross sections that were given in the original information. I based my arguments on the features and characteristics of the types of plane and solid figures. I also used my knowledge to choose the appropriate volume formulas, as I identified the specific dimensions necessary for computing the volume of each figure. I also justified my choices for making a determination about how much greater the volume of one figure is than the other.)

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- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** I looked for and expressed regularity in repeated reasoning as I explained and justified my steps involved in choosing the appropriate plane figures that represented the parts of the observatory. I also used repeated reasoning as I identified the appropriate solid figures that created the cross sections of the given figures. Specifically, I identified that the solid figure that had a cross section of a rectangle is a rectangular prism, and the solid figure that had a cross section of a semicircle is a hemisphere.)

Alternate Strategies or Solutions

- Some students may choose to draw alternate sketches of the cross sections. They may choose to decompose the cross sections into two figures in order to get a better representation of the dimensions of each figure. Encourage them to carefully label the appropriate dimensions on each figure.
- Some students may choose to initially simplify the result of the volume of the hemisphere, as opposed to leaving it in terms of π, as presented in the task. They may also choose to leave the volume with decimal place values. Remind them that since the volume of the rectangular prism does not have any decimal places, then the volume of the hemisphere should also be represented without decimal places.

Technology

Students can use scientific calculators for this task.

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Implementation Guide

Problem-Based Task 4B.5.3: A Solar Sail

Common Core State Standard

G–MG.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★

Task Overview

Focus

How can geometric methods be applied to solve for missing lengths in a real-world scenario? In this task, students will apply area formulas, congruence postulates, and the Pythagorean Theorem in order to determine missing lengths of triangular-shaped panels of a solar sail that is part of the design of an experimental spacecraft. They will also use these geometric methods to calculate the height of a cylindrical-shaped cargo bay that will house the solar sail when it is not in use.

This activity will provide practice with:

- creating an equation for the total area of a combination of triangles
- identifying pairs of congruent triangles
- calculating the area of a triangle
- classifying congruent triangles by identifying triangle congruence postulates
- applying the Pythagorean Theorem to solve for a missing side
- interpreting solutions of equations in the context of the scenario

Introduction

This task should be used to explore or to apply the skill of applying geometric methods to solve design problems. Students should already be familiar with using function models to represent real-world problems and their solutions, representing data using algebraic, graphic, or tabular formats, and applying the Pythagorean Theorem and congruence postulates.

Begin by reading the problem and reviewing the meaning of *Pythagorean Theorem and Side-Side-Side Congruence Postulate.*

Pythagorean Theorem	a theorem that relates the length of the hypotenuse of a right triangle (c) to the lengths of its legs (a and b). The theorem states that $a^2 + b^2 = c^2$.
Side-Side-Side Congruence Postulate	a postulate that states that if three sides of one triangle are congruent to three sides of another triangle, then these two triangles are congruent.

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goals of the task are to determine the value of the unknown variable (x), which represents an expression for a side length of a solar sail of the design of an experimental spacecraft, and also to find the height of a cylindrical-shaped cargo bay which will be used to house the solar sail when it is not in use. To accomplish these goals, students will analyze the six triangles that represent the panels that make up the square sail. More specifically, they will determine which pairs of triangles are congruent by identifying congruent markings on the given diagram, as well as by applying the Side-Side-Side Congruence Postulate. They will also determine the area of each triangle by applying the Pythagorean Theorem to find missing side lengths. They will create equations relating the area of the sail to its triangular parts, and they will identify the necessary information from these results in order to find the solution of the task.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for choosing the pairs of congruent triangles that represent the panels in the square-shaped solar sail. They will identify the appropriate congruence postulate and explain their choice by noting the corresponding congruent sides of the triangles. They will also justify their choices for determining the side lengths to use for each triangle when finding the area of the triangles by applying the Pythagorean Theorem. They will base their arguments for the lengths of the sides by applying their knowledge of the relationship of sides of congruent triangles, as well as their knowledge of creating equivalent expressions and solving equations with radical expressions.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  Students will express regularity in repeated reasoning as they explain and justify their steps involved in choosing the pairs of congruent triangles that represent the panels that comprise the solar sail. They will also use repeated reasoning as they determine which triangles have the same area based on congruency. They will identify the congruent sides of the pairs of triangles by using the structure of the Side-Side-Side Congruence Postulate, and they will apply this structure to the pairs of triangles. They will also determine the common side lengths of the triangle pairs, and they will substitute these values into the appropriate component of the Pythagorean Theorem, as they apply this theorem to solve for the unknown lengths.

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Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- applying particular mathematical concepts and models to real-world design situations for which they are not as relevant as other options

  Remind students that the given design problem is modeled with a square with a given value for its area, and the triangles that comprise the square are labeled with specific expressions for the side lengths. Encourage students to apply their knowledge of area, congruent triangles, and the Pythagorean Theorem in order to work through the steps in this task that will lead to the solutions.

- misinterpreting information from a graph or data table and inappropriately applying that information to a real-world design situation

  Remind students to carefully choose the appropriate values from the given diagram when working through specific parts of the task. Encourage them to double check their calculations each time to ensure accuracy of their results.

- incorrectly identifying pairs of congruent triangles

  Have students make a table to display their pairs of congruent triangles that they can refer to while working through the task. Encourage them to create a column in the table that lists the values for the side lengths of each pair of congruent triangles.

- incorrectly identifying the relationship between the sides of the triangles to the radius of the cylindrical-shaped cargo bay

  Have students sketch a separate diagram of the solar sail and the cargo bay. Remind them to carefully label each component of the solar sail and the cargo bay. Have them write down the comparisons of the parts of the triangles of the solar sail to the radius of the cargo bay.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, review congruence postulates with them. Ask them, "What are some postulates that can be used to verify that two triangles are congruent? (**Answer:** Some triangle congruence postulates that are used for verifying congruency are the Angle-Side-Angle Postulate, Side-Side-Side Postulate, Side-Angle-Side Postulate, and Angle-Angle-Side Postulate.)
- As an extension to the previous question, ask students, "Based on the given information and the diagram of the triangles that comprise the sail, which postulates do you think will be most applicable in verifying that two triangles are congruent? Explain your answer using the three pairs of congruent triangles." (**Answer:** For triangles 1 and 6, the diagram shows that the triangles have two congruent side lengths, 100 meters and x, as well as an included right angle. Therefore, triangles 1 and 6 are congruent by the Side-Angle-Side Postulate.

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For triangles 2 and 5, the diagram shows that they share a common side, and the congruent markings indicate that their other two corresponding sides are congruent as well. Therefore, triangles 2 and 5 are congruent by the Side-Side-Side Postulate. For triangles 3 and 4, they share a common side, and there is a congruent marking indicating a congruent corresponding side. They also each have a side length of $(100 - x)$. Therefore, triangles 3 and 4 are congruent by the Side-Side-Side Postulate.)

- As students are examining the diagram ask them, "Is it possible to determine the length of the diagonal of the solar sail? If so, how would you find this value?" (**Answer:** Since the solar sail is a square, and the side lengths are 100 m, the Pythagorean Theorem could be used to determine the length of the diagonal: $(100)^2 + (100)^2 = (\text{diagonal})^2$; diagonal $\approx$ 141 m.)
- Also ask students, "How could the value of the length of the diagonal be helpful in determining lengths of any of the triangles?" (**Answer:** The diagonal of the entire sail is also one of the common sides of triangles 2 and 5, as well as one of the common sides of triangles 3 and 4. If one of the lengths was known for either of these common sides, the length could be subtracted from the total length of the diagonal in order to find the length of the other common side.)
- As students are working ask them to explain the components of the Pythagorean Theorem. Ask them, "How does the Pythagorean Theorem work in terms of which side lengths in a triangle are substituted into which parts of the theorem?" (**Answer:** The Pythagorean Theorem states that the sum of the squares of the two legs of a right triangle is equal to the square of the length of the hypotenuse.)
- As students are working through the task and determining the side lengths of the triangles, ask them, "Why do you set the expressions representing the shortest sides of triangles 2 and 5 equal to the length of the radius of the cargo bay?" (**Answer:** In the information given in the stem of the problem, it was given that the radius of the cargo bay is the same as each of the shortest sides of triangles 2 and 5. Therefore, the expressions representing these sides can be set equal to the radius of the cargo bay.)
- As students are determining the numerical values for the height of the cargo bay, ask them, "Why do you think the values are given in terms of approximations?" (**Answer:** Since the calculations for the height of the cargo bay involve radical values that contain decimals listed out to many place values, it is more practical to round these values for the scenario. Therefore, since the values given are not the exact values, the word "approximately" or "approximate" must be used.)
- Ask students, "Based on the values of the radius and height of the cylindrical-shaped cargo bay, what other information about the cargo bay could be calculated?" (**Answer:** Since the radius and height of the cargo bay are known, the volume of the cargo bay, as well as the surface area of the bay, could be calculated.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goals of the task. Ask them to explain their analysis of the geometric figures that represent the solar sail. Discuss how it was determined which pairs of triangles were congruent, as well as how the Side-Side-Side Congruence Postulate, or the Side-Angle-Side Congruence Postulate, was used to verify the congruent pairs. Ask students how the sides of each adjacent triangle related to each other. Discuss how the expressions for the specific sides were substituted into the area formula, and ask students to explain how the expression for the sum of the areas for triangles 2 and 5 was derived. Also discuss how the Pythagorean Theorem was applied, once the length of the hypotenuse was determined, and then how this value was used to find the height of the cylindrical-shaped cargo bay that housed the solar sail when it was not in use. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their analysis and reasoning process for determining the pairs of congruent triangles. Discuss the relationship of the sides of the triangles that were chosen as the corresponding sides for the pairs. Ask students to explain how the expressions representing side lengths of triangles were substituted into the area formulas, as well as into the formula for the Pythagorean Theorem. Discuss the relationship between the radius of the cylindrical-shaped cargo bay and the shortest sides of triangles 2 and 5. Focus on the use of precise mathematical language and clarity, specifically when referring to the types of geometric shapes that represent the solar sail. Also focus on the correct use of terminology and notation when substituting expressions into the area formulas, as well as the formula for the Pythagorean Theorem. Ensure that students distinguish between the radius and height of the cargo bay and the side lengths of the triangles.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Many real-world objects can be modeled by mathematical figures.

  In this task, students will recognize that the solar sail, which is a part of an experimental design of a spacecraft, is modeled as a square containing six triangular-shaped panels.

- Real-world design problems can be solved using given geometric figures, numerical conditions, or other mathematical constraints.

  Students will recognize that the given task is modeled after a real-world design problem that presents itself with a square-shaped component that is comprised of six triangles. They will also be given numerical values, as well as algebraic expressions, in order to solve for missing components of the solar sail, represented by the square.

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- Concepts in coordinate and plane geometry help designers visualize prototypes.

  Students will apply concepts of plane geometry, as well as concepts of trigonometry, in order to accomplish the goals of the task. They will analyze congruent triangles and use their properties to determine unknown values, and they will also use the Pythagorean Theorem in order to determine unknown side lengths.

Extending the Task

To extend the task, ask students to decompose the given solar sail and sketch each of the six triangles that represent the solar panels separately. Ask them to create a table of rows for each of the six triangles, and columns that represent the three sides of each triangle. Ask them to find the numerical values for each side length, to the nearest tenth of a meter. Then ask them to record their results in the table. Once they have completed this task, ask for volunteers to share their results. Discuss the calculation methods for determining the values of the side lengths of all triangles, and specifically discuss any "short-cuts" and/or patterns which students discovered, based on the pairs of congruent triangles. Discuss any difficulties students may have encountered while working through this extended task, and allow students to help each other with error analysis.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of this problem by first recognizing that the goals of the task were to find the value of the variable (x), which is the side length of a solar sail, and also to find the height of a cylindrical-shaped cargo bay. I analyzed the six triangles that represented the sail's panels, and determined which pairs of triangles were congruent by identifying congruent markings on the diagram, and by applying the Side-Side-Side Congruence Postulate. I also found the area of each triangle by using the Pythagorean Theorem to find missing side lengths. I created equations relating the area of the sail to its triangular parts, and I identified the necessary information from these results to find the solution of the task.")
- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** "I constructed viable arguments by explaining my choices for choosing the pairs of congruent triangles that represented the panels in the square-shaped solar sail. I identified the appropriate congruence postulate and explained my choice by noting the corresponding congruent sides of the triangles. I also justified my choices for determining the side lengths to use for each triangle when finding the area of the triangles by applying the Pythagorean Theorem. I based my arguments for the lengths of the sides by applying my knowledge of the relationship of sides of congruent triangles, as well as my knowledge of creating equivalent expressions and solving equations with radical expressions.)

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- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** I explained and justified my steps involved in choosing the pairs of congruent triangles that represented the panels that comprised the solar sail. I used repeated reasoning as I determined which triangles had the same area based on congruency. I identified the congruent sides of the pairs of triangles by using the structure of the Side-Side-Side Congruence Postulate, and I applied this structure to the pairs of triangles. I also determined the common side lengths of the triangle pairs, and substituted these values into the appropriate component of the Pythagorean Theorem. I applied this theorem to solve for the unknown lengths.)

Alternate Strategies or Solutions

Some students may have chosen to calculate the lengths of the sides of the triangles in a different order than what was presented in the task. For example, some students may have chosen to find the length of the diagonal of the entire solar sail, and then used this value to find the missing common side of congruent triangles 3 and 4. If students choose different methods of calculating the missing side lengths, using the Pythagorean Theorem, encourage them to check their final results with those of another classmate, as well as the results given in the task.

Technology

Students can use scientific calculators for this task.

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Unit 4 Comprehensive Assessment

Circle the letter of the best answer.

1. If p is the inverse of q and $p(-2) = 7$, what is $p(-2) - q(7)$?

   a. -14
   c. 9
   b. 5
   d. 14

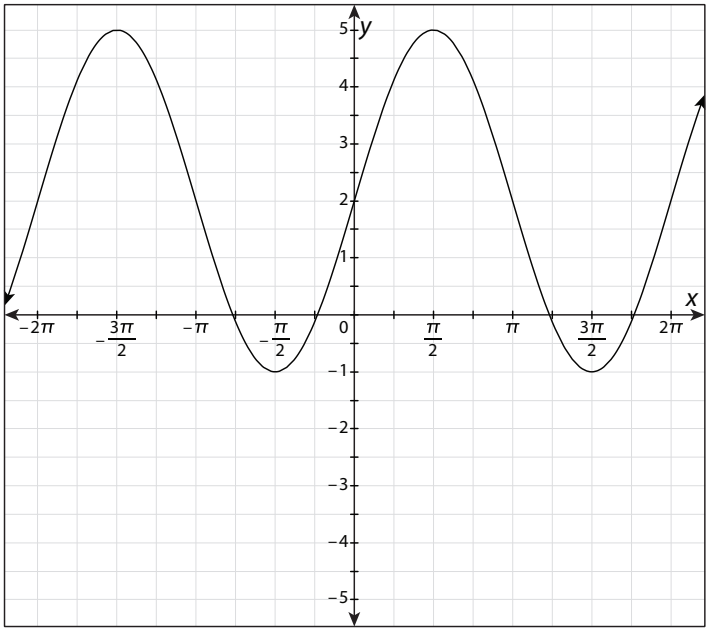
2. What is the inverse function of the exponential function $g(x) = 3 + 8^x$?

   a. $g^{-1}(x) = \log_8 (x - 3)$
   c. $g^{-1}(x) = \log_8 x$
   b. $g^{-1}(x) = \log_8 x + 3$
   d. $g^{-1}(x) = 3 - \log_8 x$

3. What is the inverse of the exponential function $f(x) = 4(5)^{2x}$?

   a. $\frac{1}{2}\log_5 x - \frac{1}{2}\log_5 4$
   c. $\frac{\log_5 x}{2\log_5 4}$
   b. $2\log_5 x - 2\log_5 4$
   d. $\frac{2\log_5 x}{\log_5 4}$

4. Which function does the following graph represent?



   a. $h(x) = 3 \sin (x + 2)$
   c. $h(x) = \sin (x + 5)$
   b. $h(x) = 5 \sin x$
   d. $h(x) = 3 \sin (x) + 2$

continued

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5. Shaina, a personal trainer, is mixing workout shakes for some of her clients. She would like to mix a total of 2 gallons of workout shakes using two different protein supplements: one that is 35% protein and another that is 55% protein. Shaina would like the resulting shake to be 47% protein. Which equation describes how to find the amount of each of the two types of protein supplements that Shaina will need?

   a. $0.35x + 0.55(2 - x) = 0.47$
   b. $0.35x + 0.55(2 - x) = 0.94$
   c. $0.35x - 0.55(2 - x) = 0.47$
   d. $35x + 55(2 - x) = 47$

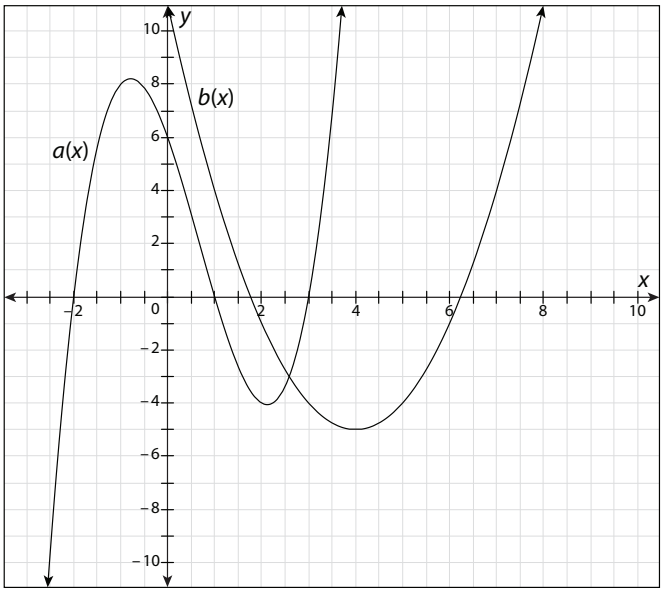
6. Which function represents a transformation of the parent quadratic function 6 units to the left and 2 units down?

   a. $f(x) = x^2 + 12x - 34$
   b. $f(x) = x^2 + 12x + 34$
   c. $f(x) = x^2 + 12x - 38$
   d. $f(x) = x^2 - 12x + 38$

7. If $g(x) = \ln(x + 4)$ and $h(x) = \ln(x - 7)$, which choice represents $g(x) + h(x)$?

   a. $g(x) + h(x) = \ln(2x - 3)$
   b. $g(x) + h(x) = \ln(x^2 + 3x - 28)$
   c. $g(x) + h(x) = \ln(x^2 - 3x - 28)$
   d. $g(x) + h(x) = \ln(2x^2 + 28)$

8. Over which interval are the graphed functions $a(x)$ and $b(x)$ both increasing?



   a. $(-2, 2)$
   b. $(-1, 4)$
   c. $(4, \infty)$
   d. $(-\infty, \infty)$

continued

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9. What is the average rate of change of the function $h(x)=\frac{2x}{x-4}$ between $x = 3.99$ and $x = 4.01$?

   a. It is infinite because the function is undefined at $x = 4$.

   b. It is undefined because the function is undefined at $x = 4$.

   c. It is 4 regardless of whether the function is undefined at $x = 4$.

   d. It is 0 because the function is undefined at $x = 4$.

10. The following table shows the percentage of students from a state university who passed the national nursing exam from 2004 to 2012. (The year 2004 corresponds to $t = 1$.) Which type of function best models the data?

t	1	2	3	4	5	6	7	8	9
p	63	65.4	67.8	70.1	72.6	75	77.3	79.7	82.2

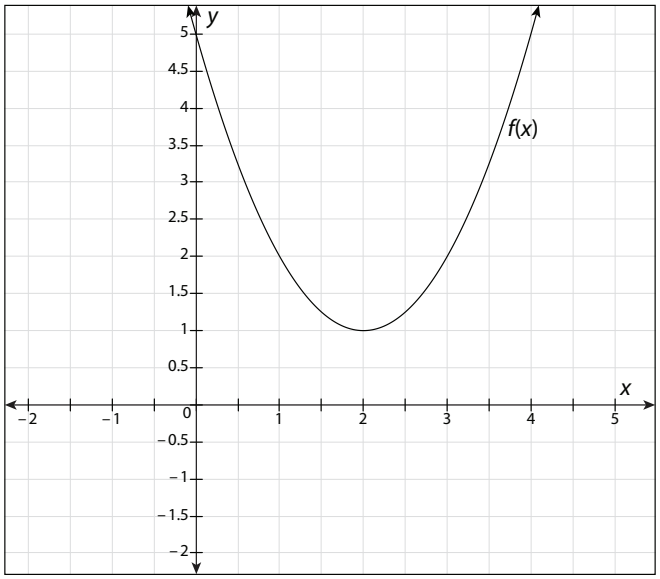
   a. absolute value function

   b. exponential function

   c. quadratic function

   d. linear function

11. What is the degree of the inverse of the function shown?



   a. 0

   b. $\frac{1}{2}$

   c. 1

   d. 2

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12. The size of a piston in a car engine is greater when the piston is heated than when the piston is cold. During manufacture, engineers must allow for this difference in size. The ratio of the piston size can be no more than 0.008 greater or less than the tangent of 45°. Which compound inequality models these specifications if a piston is represented by p when it is cold and by $p + 0.2$ when it is heated?

    a. $44.992 < \frac{p}{p+0.2} < 45.008$

    b. $44.992 < \frac{p}{(p+0.2)^2} < 45.008$

    c. $0.992 < \frac{p}{p+0.2} < 1.008$

    d. $0.992 < \frac{p}{(p+0.2)^2} < 1.008$

Use what you have learned about creating equations and modeling real-world situations to complete all parts of the following problems.

13. The amplitude of a radio wave y can be written as a function of x, the distance the sound wave travels, using the sine function $y(x) = A\sin\left(\frac{2\pi}{\beta} \bullet x\right)$. The wavelength β represents one complete cycle of the function and x is the total distance the wave travels. Show that $x + 4\beta$ gives the same result as x in the argument of the function.

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14. The volume of a cube with sides of length x is given by the formula $V = x^3$. The surface area of the cube is given by $A = 6x^2$.

    a. Rewrite the formula for A in terms of V.

    b. Rewrite the formula for V in terms of A.

    c. Calculate the volume of a cube with a surface area of 24 cm^2.

15. A university veterinarian has been tracking the number of cases of canine asthma at his clinic. He has recorded the number of dogs seen in his office that tested positive for canine asthma over the past 8 months. The increase in positive test results is given by the following data. What type of equation best models this data? Explain your reasoning.

Month	1	2	3	4	5	6	7	8
Number of positive results	1	2	3	5	7	11	17	27

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Answer Key

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1. c
2. a
3. a
4. d
5. b
6. b
7. c
8. c
9. b
10. d
11. b
12. c
13. Substitute $x + 4\beta$ for x in the given function to result in $y(x) = A\sin\left[\frac{2\pi}{\beta}(x+4\beta)\right] = A\sin\left(\frac{2\pi x}{\beta}+\frac{8\pi\beta}{\beta}\right) = A\sin\left(\frac{2\pi x}{\beta}+8\pi\right)$. This shows a phase shift of 8π, which would have the same value as y at $x = 0$.
14. a. $A = 6v^{\frac{2}{3}}$
    b. $V = \left(\frac{A}{6}\right)^{\frac{3}{2}}$
    c. $V = 8\text{ cm}^3$
15. The best type of equation to model this situation is an exponential equation with a positive exponent because the y-values are increasing at a faster rate than the x-values.

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