Common Core State Standards

Mathematics II

Integrated Pathway

Program Overview

WALCH
INTEGRATED MATH

This program was developed and reviewed by experienced math educators who have both academic and professional backgrounds in mathematics. This ensures: freedom from mathematical errors, grade level appropriateness, freedom from bias, and freedom from unnecessary language complexity.

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PROGRAM OVERVIEW

Introduction to the Program

Introduction

The *Common Core State Standards Integrated Pathway: Mathematics II Program* is a complete set of materials developed around the Common Core State Standards (CCSS), the overview of the Integrated Pathway for the Common Core State Mathematics Standards, and the Mathematics II content map found in Appendix A of the Common Core State Standards. Topics are built around accessible core curricula, ensuring that the *CCSS Integrated Pathway: Mathematics II Program* is useful for striving students and diverse classrooms.

This program realizes the benefits of exploratory and investigative learning and employs a variety of instructional models to meet the learning needs of students with a range of abilities.

The *CCSS Integrated Pathway: Mathematics II Program* includes components that support problem-based learning, instruct and coach as needed, provide practice, and assess students' skills. Instructional tools and strategies are embedded throughout.

The program includes:

- More than 150 hours of lessons, addressing the six units of CCSS IP: Mathematics II
- Essential Questions for each instructional topic
- Vocabulary
- Instruction and Guided Practice
- Problem-based Tasks and Coaching questions
- Step-by-step graphing calculator instructions for the TI-Nspire and the TI-83/84
- Station activities to promote collaborative learning and problem-solving skills

Purpose of Materials

The *CCSS Integrated Pathway: Mathematics II Program* has been organized to coordinate with the CCSS Integrated Pathway: Mathematics II content map and specifications from Appendix A of the Common Core State Standards.

Each lesson includes activities that offer opportunities for exploration and investigation. These activities incorporate concept and skill development and guided practice, then move on to the application of new skills and concepts in problem-solving situations. Throughout the lessons and activities, problems are contextualized to enhance rigor and relevance.

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PROGRAM OVERVIEW
Introduction to the Program

This program includes all the topics addressed in the CCSS Integrated Pathway: Mathematics II content map. These include:

- Extending the Number System
- Quadratic Functions and Modeling
- Expressions and Equations
- Applications of Probability
- Similarity, Right Triangle Trigonometry, and Proof
- Circles With and Without Coordinates

The eight Mathematical Practices described in the Common Core are infused throughout:

- CCSS.MP.1: Make sense of problems and persevere in solving them.
- CCSS.MP.2: Reason abstractly and quantitatively.
- CCSS.MP.3: Construct viable arguments and critique the reasoning of others.
- CCSS.MP.4: Model with mathematics.
- CCSS.MP.5: Use appropriate tools strategically.
- CCSS.MP.6: Attend to precision.
- CCSS.MP.7: Look for and make use of structure.
- CCSS.MP.8: Look for and express regularity in repeated reasoning.

Structure of the Teacher Resource

The *CCSS Integrated Pathway: Mathematics II Program* is provided in an online, digital format, and/or in hard copy (Program Overview and six unit volumes). The materials, either online or hard copy, are completely reproducible. Online materials can be provided in your Learning Management System (such as Canvas or Schoolology) or in Walch's proprietary system, WalchConnect. The nested folder organization in WalchConnect allows you to access the materials quickly and easily. The digital format also facilitates printing and copying student pages and/or making assignments online.

The Program Overview is the first section. This section helps you to navigate the materials, offers a collection of graphic organizers and suggested strategies for their use, and shows the correlation between the Common Core State Standards and the CCSS Integrated Pathway: Mathematics II content map found in Appendix A of the Common Core State Standards.

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The remaining books focus on content, knowledge, and application of the six units in the CCSS Integrated Pathway Mathematics II curriculum: Extending the Number System; Quadratic Functions and Modeling; Expressions and Equations; Applications of Probability; Similarity, Right Triangle Trigonometry, and Proof; and Circles With and Without Coordinates. The units in the *CCSS Integrated Pathway: Mathematics II Program* are designed to be flexible so that you can mix and match activities as the needs of your students and your instructional style dictate.

The Station Activities correspond to the content in the units and provide students with the opportunity to apply concepts and skills, while you have a chance to circulate, observe, speak to individuals and small groups, and informally assess and plan.

Each lesson begins with a pre-assessment and ends with a progress assessment. These allow you to assess students' progress as you move from lesson to lesson, enabling you to gauge how well students have understood the material and to differentiate as appropriate.

***Please note*:** Throughout the Teacher Resource, page references are provided on the lower, inner corner of some pages. These references indicate the corresponding page(s) in the Student Workbook (SWB) and/or Student Resource Book (SRB) as appropriate. These page references have been included here to facilitate assigning materials to students.

Glossary

The Glossary contains vocabulary terms and formulas from throughout the program, organized alphabetically. Each listing provides the term and the definition in both English and Spanish. The listings include the lesson number(s) where the terms can be found in the Words to Know.

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Correspondence to Standards for Mathematical Practice

How Do Walch Integrated Mathematics Resources Address the CCSS 8 Standards for Mathematical Practice?

Walch's programs for the *CCSS Integrated Pathway Mathematics I, II,* and *III* courses employ a problem-based model of instruction that supports and reinforces the 8 Standards for Mathematical Practice. Although the following table focuses on Problem-Based Tasks, Walch's full programs also include hundreds of additional problems in warm-ups and practices. The Implementation Guides for selected PBTs highlight SMPs to focus on during implementation and discussion.

CCSS Standards for Mathematical Practice		Relevant Attributes of Walch Integrated Math Resources
1	Make sense of problems and persevere in solving them.	Each sub-lesson is built around a Problem-Based Task (PBT) that requires students to "make sense of problems and persevere in solving them."
2	Reason abstractly and quantitatively.	Each PBT uses a meaningful real-world context that requires students to reason both abstractly about the situation/relationships and quantitatively about the values representing the elements and relationships.
3	Construct viable arguments and critique the reasoning of others.	Since the PBT provides opportunities for multiple problem-solving approaches and varied solutions, students are required to construct viable arguments to support their approach and answer. This, in turn, provides other students the opportunity to analyze and critique their classmates' reasoning.
4	Model with mathematics.	Each PBT represents a real-world situation and requires students to model it with mathematics.
5	Use appropriate tools strategically.	PBTs require students to make choices about using appropriate tools, such as calculators, spreadsheets, graph paper, manipulatives, protractors, and compasses. The tasks do not prescribe specific tools, but instead provide opportunities for their use.
6	Attend to precision.	The real-world contexts of the PBTs require students to be precise in their solutions, both in the ways that the solutions are stated, labeled, and explained, and in the degree of precision necessary given the context (e.g., tripling chili for a crowd vs. machining a part for an airplane engine).
7	Look for and make use of structure.	The PBTs present students with complicated scenarios that must be analyzed to discern patterns and significant mathematical features.
8	Look for and express regularity in repeated reasoning.	PBTs require multiple steps, providing opportunities for students to note repeated calculations, monitor their process, and continually evaluate reasonableness of intermediate results before arriving at a solution.

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Correspondence to CCSS Publishers' Criteria

How Do Walch Integrated Mathematics Resources Address the High School Publishers' Criteria for CCSS Mathematics?

Walch's programs for the *CCSS Integrated Pathway Mathematics I, II,* and *III* courses were designed to the specifications of Appendix A and the Publishers' Criteria. Focus, Coherence, and Rigor represent the fundamental assumptions of the resources.

Publishers' Criteria	Relevant Attributes of Walch Integrated Math Resources
FOCUS Focus strongly where the standards focus.	Each course is built around the units specified in the CCSS for Mathematics Appendix A. Units are organized around important topics in mathematics and reflect the focus of the standards. Units range from 22 sub-lessons for Math I Unit 2, "Linear and Exponential Relationships," which address 21 Common Core standards, to three sub-lessons for Math I Unit 6, "Connecting Algebra and Geometry Through Coordinates," which address only three Common Core standards.
COHERENCE Think across grades/courses, and link to major topics in each course.	Walch Integrated Math programs include two features contributing to coherence. Each sub-lesson identifies necessary prerequisite skills (our forthcoming Support Supplement will point to where they were introduced, including in elementary or middle school, and provide instructional materials for remediation). Warm-Ups tie to these prerequisites in order to activate prior knowledge. Also, the major topics of each course are mapped forward to show where they are revisited and expanded upon in future courses.
RIGOR In major topics, pursue with equal intensity • conceptual understanding, • procedural skill and fluency, and • applications.	The instructional components of Walch's Integrated Math programs guide students from conceptual understanding and procedural skill developed through modeling and guided practice to the application opportunities of the PBT. The resources move students on to additional practice to refine procedural skill and fluency, and later to station activities to crystalize conceptual understanding.

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Correspondence to NCTM *Principles to Actions* Teaching Practices

How Do Walch Integrated Mathematics Resources Address the NCTM *Principles to Actions* Mathematics Teaching Practices?

Walch's programs for the *CCSS Integrated Pathway Mathematics I, II,* and *III* courses were designed by experienced educators and curriculum developers, informed by best-practice research, and refined through an iterative process of implementation and feedback. Together with professional development, these materials support and sustain good teaching practices.

NCTM Mathematics Teaching Practices	Relevant Attributes of Walch Integrated Math Resources
Establish mathematics goals to focus learning.	Each sub-lesson in Walch's programs addresses specified standards which can be used as goals to focus learning. Essential Questions offer further focus.
Implement tasks that promote reasoning and problem solving.	Each sub-lesson in Walch's programs is built around a Problem-Based Task (PBT), set in a meaningful real-world context and designed to promote reasoning and problem solving. The courses include dozens of PBTs as well as warm-up and practice problems.
Use and connect mathematical representations.	Walch's High School Math programs make frequent use of, and connections among and between, equations, tables, and graphs. PBTs often require students to use and connect two or more of these representations, and the representations are modeled through guided practice.
Facilitate meaningful mathematical discourse.	Several features of the programs support mathematical discourse, including warm-up debriefs with connections to upcoming lessons, implementation guides and optional coaching questions for the PBTs, and discussion guides for Station Activities. Explanations of PBT solutions are another opportunity for discourse. Please note: Mathematical discourse is an important topic for professional development, in conjunction with implementation of these materials.
Pose purposeful questions.	The implementation guides, coaching questions and discussion guides provide samples of purposeful questions. Note that this is another important topic for professional development.
Build procedural fluency from conceptual understanding.	The programs develop conceptual understanding through modeling, guided practice, and application, and then provide additional opportunities to practice and develop fluency.
Support productive struggle in learning mathematics.	The PBTs require "productive struggle;" implementation guides include suggestions for facilitation and monitoring, and coaching questions provide an option for additional support as appropriate, allowing students to proceed through the task and ensuring that the struggle remains productive rather than too frustrating.
Elicit and use evidence of student thinking.	Various discussions and PBTs require students to display their thinking. Implementation guides offer specific prompts and suggestions for eliciting and responding to student thinking. Professional development supports teachers in using that evidence to respond in instructionally appropriate ways.

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Unit Structure

All of the instructional units have common features. Each unit begins with a list of all the standards addressed in the lessons, and a list of one or more conceptual activities. Each lesson begins with a pre-assessment, followed by the list of standards addressed in the lesson; Essential Questions; vocabulary (titled "Words to Know"); and a list of recommended websites to be used as additional resources.

Each sub-lesson begins with a list of identified prerequisite skills that students need to have mastered in order to be successful with the new material in the upcoming sub-lesson. This is followed by an introduction, key concepts, common errors/misconceptions, guided practice examples, a problem-based task with coaching questions and sample responses, a closure activity, and practice. Each lesson ends with a progress assessment to evaluate students' learning.

All of the components are described below and on the following pages for your reference.

Standards Correlations and Connections to Future Courses

In this section, you'll find a comprehensive list of the Common Core State Standard(s) addressed in each sub-lesson, followed by a map detailing where and how future courses will build upon the topics introduced in the unit.

Conceptual Activities

Conceptual understanding serves as the foundation on which to build deeper understanding of mathematics. In an effort to build conceptual understanding of mathematical ideas and to provide more than procedural fluency and application, links to interactive open education and Desmos resources are included. (*Note*: These website links will be monitored and repaired or replaced as necessary.)

Pre-Assessment

This can be used to gauge students' prior knowledge and to inform instructional planning.

Common Core State Standards for the Lesson

All standards that are addressed in the entire lesson are listed.

Essential Questions

These are intended to guide students' thinking as they proceed through the lesson. By the end of each lesson, students should be able to respond to the questions.

Words to Know

Vocabulary terms and formulas are provided as background information for instruction or to review key concepts that are addressed in the lesson.

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Recommended Resources

This is a list of websites that can be used as additional resources. Some websites are games; others provide additional examples and/or explanations. The links for these resources are live in the PDF version of the Teacher Resource. (*Note*: These website links will be monitored and repaired or replaced as necessary.)

Common Core State Standards for the Sub-Lesson

When lessons are broken down into sub-lessons, the specific standard or standards that are addressed are presented at the beginning of the instructional portion of the lesson.

Warm-Up

Each warm-up takes approximately 5 minutes and addresses either prerequisite and critical-thinking skills or previously taught math concepts.

Warm-Up Debrief

Each debrief provides the answers to the warm-up questions, and offers suggestions for situations in which students might have difficulties. A section titled Connection to the Lesson is also included in the debrief to help answer students' questions about the relevance of the particular warm-up activity to the upcoming instruction. Warm-Ups with debriefs are also provided in PPT presentations with short video clips to enhance engagement.

Identified Prerequisite Skills

Presented at the beginning of each sub-lesson, this list cites the skills necessary to be successful with the new material.

Introduction

This brief paragraph gives a description of the concepts about to be presented and often contains some Words to Know.

Key Concepts

Provided in bulleted form, this instruction highlights the important ideas and/or processes for meeting the standard.

Common Errors/Misconceptions

This is a list of the common errors students make when applying Key Concepts. This list suggests what to watch for when students arrive at an incorrect answer or are struggling with solving the problems.

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Guided Practice

This section provides step-by-step examples of applying the Key Concepts. The three to five examples are intended to aid during initial instruction, but are also for individuals needing additional instruction and/or for use during review and test preparation. GeoGebra applets that model selected Guided Practice examples are provided on the flash drive that accompanies this program.

Problem-Based Task

This activity can serve as the centerpiece of a problem-based lesson, or it can be used to walk students through the application of the standard, prior to traditional instruction or at the end of instruction. The task makes use of critical-thinking skills.

Problem-Based Task Implementation Guides (Appendix)

This instructional overview, found in an appendix at the end of each unit, highlights connections between the task and the sub-lesson's key concepts and SMPs. It also offers suggestions for facilitating and monitoring, and provides alternative solutions.

Optional Problem-Based Task Coaching Questions

These questions scaffold the task and guide students to solving the problem(s) presented in the task. They should be used at the discretion of the teacher for students requiring additional support.

Problem-Based Task Coaching Questions Sample Responses

These are the answers and suggested appropriate responses to the coaching questions. In some cases answers may vary, but a sample answer is given for each question.

Recommended Closure Activity

Students are given the opportunity to synthesize and reflect on the lesson through a journal entry or discussion of one or more of the Essential Questions.

Practice

Each sub-lesson includes practice problems to support students' achievement of the learning objectives. These sheets are written for the student. They can be used in any combination of teacher-led instruction, cooperative learning, or independent application of knowledge.

Progress Assessment

Each lesson ends with 10 multiple-choice questions, as well as one extended-response question that incorporates critical thinking and writing components. This can be used to document the extent to which students grasp the concepts and skills addressed during instruction.

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Unit Assessment

Each unit ends with 12 multiple-choice questions and three extended-response questions that incorporate critical thinking and writing components. This can be used to document the extent to which students grasped the concepts and skills of each unit.

Answer Key

Answers for all of the Warm-Ups, Assessments, and Practice problems from the Teacher Resource and all of the problems from the Student Resource Book are provided at the end of each unit. (Student editions include odd answers for the exercises in the student book.)

Graphing Calculators

Step-by-step instructions for using a TI-Nspire and a TI-83/84 are provided whenever graphing calculators are referenced.

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Standards Correlations

Each lesson in this Integrated Pathway: Mathematics II program was written specifically to address the Common Core State Standards. Each lesson lists the standards covered in all the sub-lessons, and each sub-lesson lists the standards addressed in that particular section. In this section, you'll find a comprehensive list mapping the sub-lessons to the CCSS.

Guide to Common Core State Standards Annotation

As you use this program, you will come across a symbol included with the Common Core standards for some of the lessons and activities. These symbols are explained below.

Symbol: ⋆

Denotes: Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (⋆).

From http://www.walch.com/CCSS/00003

Symbol: (+)

Denotes: College and Career Readiness Standards

Advanced mathematics standards that are required in higher-level courses such as advanced statistics may also be included in lower-level courses. These additional standards are denoted by (+). According to the Common Core State Standards Initiative, "the evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6–8."

From http://www.walch.com/CCSS/00004

Connections to Future Courses

This section provides a map between topics introduced in each unit of this course and subsequent courses where each topic is revisited and built upon.

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CCSS INTEGRATED PATHWAY: MATHEMATICS II STANDARDS CORRELATIONS

Unit 1: Extending the Number System				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Working with the Number System**			
	1.1.1	Defining, Rewriting, and Evaluating Rational Exponents	N–RN.1 N–RN.2	U1-4
	1.1.2	Rational and Irrational Numbers and Their Properties	N–RN.2 N–RN.3	U1-17
Lesson 2	**Operating with Polynomials**			
	1.2.1	Adding and Subtracting Polynomials	A–APR.1	U1-36
	1.2.2	Multiplying Polynomials	A–APR.1	U1-48
Lesson 3	**Operating with Complex Numbers**			
	1.3.1	Defining Complex Numbers, i, and i^2	N–CN.1	U1-66
	1.3.2	Adding and Subtracting Complex Numbers	N–CN.2	U1-78
	1.3.3	Multiplying Complex Numbers	N–CN.2	U1-90

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Unit 2: Quadratic Functions and Modeling				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Analyzing Quadratic Functions**			
	2.1.1	Graphing Quadratic Functions	F–IF.7a★	U2-4
	2.1.2	Interpreting Various Forms of Quadratic Functions	F–IF.7a★ F–IF.8a	U2-27
Lesson 2	**Interpreting Quadratic Functions**			
	2.2.1	Interpreting Key Features of Quadratic Functions	F–IF.4★	U2-55
	2.2.2	Identifying the Domain of a Quadratic Function	F–IF.5★	U2-77
	2.2.3	Identifying the Average Rate of Change	F–IF.6★	U2-91
Lesson 3	**Building Functions**			
	2.3.1	Building Functions from Context	F–BF.1a★	U2-113
	2.3.2	Operating on Functions	F–BF.1b★	U2-133
Lesson 4	**Graphing Other Functions**			
	2.4.1	Square Root and Cube Root Functions	F–IF.7b★	U2-155
	2.4.2	Absolute Value and Step Functions	F–IF.7b★	U2-190
	2.4.3	Piecewise Functions	F–IF.7b★	U2-219
Lesson 5	**Analyzing Functions**			
	2.5.1	Analyzing Exponential Functions	F–IF.8b	U2-254
	2.5.2	Comparing Properties of Functions Given in Different Forms	F–IF.9 F–LE.3★	U2-268
Lesson 6	**Transforming Functions**			
	2.6.1	Replacing $f(x)$ with $f(x) + k$ and $f(x + k)$	F–BF.3	U2-295
	2.6.2	Replacing $f(x)$ with $k \cdot f(x)$ and $f(k \cdot x)$	F–BF.3	U2-315
Lesson 7	**Finding Inverse Functions**			
	2.7.1	Finding Inverse Functions	F–BF.4a	U2-346

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Unit 3: Expressions and Equations				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Interpreting Structure in Expressions**			
	3.1.1	Identifying Terms, Factors, and Coefficients	A–SSE.1a★	U3-4
	3.1.2	Interpreting Complicated Expressions	A–SSE.1b★	U3-17
Lesson 2	**Creating and Solving Quadratic Equations in One Variable**			
	3.2.1	Taking the Square Root of Both Sides	A–CED.1★ A–REI.4b	U3-37
	3.2.2	Factoring Expressions by the Greatest Common Factor	A–SSE.2	U3-48
	3.2.3	Factoring Expressions with $a = 1$	A–SSE.2	U3-64
	3.2.4	Factoring Expressions with $a > 1$	A–SSE.2	U3-85
	3.2.5	Solving Quadratic Equations by Factoring	A–SSE.2 A–CED.1★ A–REI.4b	U3-107
	3.2.6	Completing the Square	A–SSE.2 A–CED.1★ A–REI.4a A–REI.4b	U3-124
	3.2.7	Applying the Quadratic Formula	A–CED.1★ A–REI.4a A–REI.4b	U3-136
	3.2.8	Solving Quadratic Inequalities	A–SSE.2 A–CED.1★ A–REI.4b	U3-150

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Lesson	Sub-lesson number	Title	Standard(s)	Pages
Lesson 3	**Creating Quadratic Equations in Two or More Variables**			
	3.3.1	Creating and Graphing Equations Using Standard Form	A–SSE.3a★ A–CED.2★	U3-171
	3.3.2	Creating and Graphing Equations Using the *x*-intercepts	A–SSE.3a★ A–CED.2★	U3-199
	3.3.3	Creating and Graphing Equations Using Vertex Form	A–SSE.3b★ A–CED.2★	U3-216
	3.3.4	Rearranging Formulas	A–CED.4★	U3-230
Lesson 4	**Fundamental Theorem of Algebra**			
	3.4.1	Extending Polynomial Identities to Include Complex Numbers	N–CN.8 (+)	U3-249
	3.4.2	Solving Quadratic Equations with Complex Solutions	N–CN.7 N–CN.9 (+)	U3-271
Lesson 5	**Rational Equations**			
	3.5.1	Creating Rational Equations	A–CED.1★	U3-306
	3.5.2	Graphing Rational Equations	A–CED.2★	U3-328
	3.5.3	Creating Rational Inequalities	A–CED.1★	U3-361
Lesson 6	**Writing Exponential Expressions in Equivalent Forms**			
	3.6.1	Writing Exponential Expressions in Equivalent Forms	A–SSE.3c★	U3-410
Lesson 7	**Solving Systems of Equations**			
	3.7.1	Solving Systems Graphically	A–REI.7	U3-442
	3.7.2	Solving Systems Algebraically	A–REI.7	U3-462

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Unit 4: Applications of Probability				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Events**			
	4.1.1	Describing Events	S–CP.1★	U4-6
	4.1.2	The Addition Rule	S–CP.7★	U4-33
	4.1.3	Understanding Independent Events	S–CP.2★	U4-48
Lesson 2	**Conditional Probability**			
	4.2.1	Introducing Conditional Probability	S–CP.3★ S–CP.5★ S–CP.6★	U4-78
	4.2.2	Using Two-Way Frequency Tables	S–CP.4★ S–CP.5★ S–CP.6★	U4-102
	4.2.3	The Multiplication Rule	S–CP.8★ (+)	U4-125
Lesson 3	**Combinatorics**			
	4.3.1	Combinations and Permutations	S–CP.9★ (+)	U4-153
	4.3.2	Probability with Combinatorics	S–CP.9★ (+)	U4-173
Lesson 4	**Making and Analyzing Decisions**			
	4.4.1	Making Decisions	S–MD.6★ (+)	U4-196
	4.4.2	Analyzing Decisions	S–MD.7★ (+)	U4-216

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Unit 5: Similarity, Right Triangle Trigonometry, and Proof				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Line Segments**			
	5.1.1	Midpoints and Other Points on Line Segments	G–GPE.6	U5-3
Lesson 2	**Investigating Properties of Dilations**			
	5.2.1	Investigating Properties of Parallelism and the Center	G–SRT.1a	U5-34
	5.2.2	Investigating Scale Factors	G–SRT.1b	U5-56
Lesson 3	**Defining and Applying Similarity**			
	5.3.1	Defining Similarity	G–SRT.2	U5-80
	5.3.2	Applying Similarity Using the Angle-Angle (AA) Criterion	G–SRT.3	U5-103
Lesson 4	**Proving Similarity**			
	5.4.1	Proving Triangle Similarity Using Side-Angle-Side (SAS) and Side-Side-Side (SSS) Similarity	G–SRT.4	U5-130
	5.4.2	Working with Ratio Segments	G–SRT.4	U5-147
	5.4.3	Proving the Pythagorean Theorem Using Similarity	G–SRT.4	U5-167
	5.4.4	Solving Problems Using Similarity and Congruence	G–SRT.5	U5-187
Lesson 5	**Proving Theorems About Lines and Angles**			
	5.5.1	Proving the Vertical Angles Theorem	G–CO.9	U5-222
	5.5.2	Proving Theorems About Angles in Parallel Lines Cut by a Transversal	G–CO.9	U5-251
Lesson 6	**Proving Theorems About Triangles**			
	5.6.1	Proving the Interior Angle Sum Theorem	G–CO.10	U5-291
	5.6.2	Proving Theorems About Isosceles Triangles	G–CO.10	U5-313
	5.6.3	Proving the Midsegment of a Triangle	G–CO.10	U5-334
	5.6.4	Proving Centers of Triangles	G–CO.10	U5-363

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Lesson	Sub-lesson number	Title	Standard(s)	Pages
Lesson 7	**Proving Theorems About Parallelograms**			
	5.7.1	Proving Properties of Parallelograms	G–CO.11	U5-411
	5.7.2	Proving Properties of Special Quadrilaterals	G–CO.11	U5-438
Lesson 8	**Exploring Trigonometric Ratios**			
	5.8.1	Defining Trigonometric Ratios	G–SRT.6	U5-481
	5.8.2	Exploring Sine and Cosine As Complements	G–SRT.7	U5-508
Lesson 9	**Applying Trigonometric Ratios**			
	5.9.1	Calculating Sine, Cosine, and Tangent	G–SRT.8★	U5-534
	5.9.2	Calculating Cosecant, Secant, and Cotangent	G–SRT.8★	U5-557
	5.9.3	Problem Solving with the Pythagorean Theorem and Trigonometry	G–SRT.8★	U5-576
	5.9.4	Proving the Fundamental Pythagorean Identity	F–TF.8	U5-596

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Unit 6: Circles With and Without Coordinates				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Introducing Circles**			
	6.1.1	Similar Circles and Central and Inscribed Angles	G–C.1 G–C.2	U6-5
	6.1.2	Chord Central Angles Conjecture	G–C.2	U6-27
	6.1.3	Properties of Tangents of a Circle	G–C.2	U6-41
Lesson 2	**Inscribed Polygons and Circumscribed Triangles**			
	6.2.1	Constructing Inscribed Circles	G–C.3	U6-69
	6.2.2	Constructing Circumscribed Circles	G–C.3	U6-94
	6.2.3	Proving Properties of Inscribed Quadrilaterals	G–C.3	U6-111
Lesson 3	**Constructing Tangent Lines**			
	6.3.1	Constructing Tangent Lines	G–C.4 (+)	U6-134
Lesson 4	**Finding Arc Lengths and Areas of Sectors**			
	6.4.1	Defining Radians	G–C.5	U6-167
	6.4.2	Deriving the Formula for the Area of a Sector	G–C.5	U6-180
Lesson 5	**Explaining and Applying Area and Volume Formulas**			
	6.5.1	Circumference and Area of a Circle	G–GMD.1	U6-198
	6.5.2	Volumes of Cylinders, Pyramids, Cones, and Spheres	G–GMD.1 G–GMD.3★	U6-220
Lesson 6	**Deriving Equations**			
	6.6.1	Deriving the Equation of a Circle	G–GPE.1	U6-249
	6.6.2	Deriving the Equation of a Parabola	G–GPE.2	U6-275
Lesson 7	**Using Coordinates to Prove Geometric Theorems About Circles and Parabolas**			
	6.7.1	Using Coordinates to Prove Geometric Theorems About Circles and Parabolas	G–GPE.4	U6-310

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TOPICS FROM FUTURE COURSES: CCSS INTEGRATED PATHWAY MATHEMATICS II

Math II	Topic introduced	Course/unit where addressed	How addressed
Unit 1	Extending the number system to include rational exponents	Math III Unit 4A	applied to properties of logarithms
		Math III Unit 4B	extended to real-world modeling
	Operating with polynomials	Math III Unit 4B	combining functions
	Complex numbers	Math III Unit 4B	using complex numbers in graphing rational functions
		Math IV	dividing complex numbers, operating and representing complex numbers geometrically in the coordinate plane, and representing and operating with vectors
Unit 2	Quadratic functions for analyzing, solving, and transforming square root, cube root, and absolute value functions	Math III Units 3, 4A, and 4B	extended to rational, logarithmic, periodic, and polynomial functions
	Inverse functions	Math III Unit 4A	finding inverses for other function families
		Math IV	composing functions and verifying inverses

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TOPICS FROM FUTURE COURSES: CCSS INTEGRATED PATHWAY MATHEMATICS II

Math II	Topic introduced	Course/unit where addressed	How addressed
Unit 3	Interpreting the structures of quadratic expressions	Math III Unit 2A	interpreting higher-order polynomials
		Math III Unit 2B	interpreting rational expressions
	Quadratic and rational functions in standard form	Math III Unit 2A	applied to polynomial equations in standard form
	Using the Fundamental Theorem of Algebra in solving quadratic equations	Math III Unit 2A	using the Fundamental Theorem of Algebra to solve polynomials of higher degree
	Graphing rational equations	Math III Unit 4B	modeling real-world situations
	Exponential equations including rational exponents and more complex equations in equivalent form	Math IV	graphing logarithmic functions
	Linear-quadratic systems	Math III Unit 2A	applied to linear-rational, linear-polynomial, polynomial-polynomial, and absolute value-polynomial systems
		Math III Unit 2B	extended to linear-rational and linear-radical systems
		Math IV	applied to 3×3 and larger systems of equations using vectors and matrices
	Graphing simple rational functions	Math III Unit 2B	solving rational equations
		Math IV	graphing more complex rational functions and exploring features of the graphs

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TOPICS FROM FUTURE COURSES: CCSS INTEGRATED PATHWAY MATHEMATICS II

Math II	Topic introduced	Course/unit where addressed	How addressed
Unit 4	Probability	Math III Unit 1	applied to inferences and decision-making in statistics
Unit 5	Similarity, special angle pairs formed by parallel lines cut by a transversal and other angle pairs, the properties of triangles and special triangles, and parallelograms	Math III Unit 4B	using planes to create two-dimensional figures and slice three-dimensional objects
	Trigonometric ratios and proofs of trigonometric identities	Math III Unit 3	proving and applying the laws of sines and cosines, studying periodic phenomena, and modeling using trigonometric functions
		Math IV	graphing the sine and cosine functions and representing and operating with vectors; extended to complex trigonometric identities
Unit 6	Circles, their properties, and special segments	Math IV	using formulas for the volumes of spheres and other solids
	Deriving the formulas for a circle and a parabola	Math IV	expanded to working with equations of other conic sections

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Conceptual Activities

Use these interactive open education and/or Desmos resources to build conceptual understanding of mathematical ideas. (*Note*: Activity links will be monitored and repaired or replaced as necessary.)

Unit 1

- Illustrative Mathematics. "Computations with Complex Numbers."

  http://www.walch.com/ca/10004

  Students will practice operations on complex numbers using the fact that $i^2 = -1$. Encourage students to examine the structure of each expression and look for shortcuts (SMP 7), as this task allows for the shortening of some tedious calculations. This task is also an excellent candidate for comparison of different approaches to the same problem.

Unit 2

- Desmos. "Card Sort: Parabolas."

  http://www.walch.com/ca/01022

  Find the shape of a parabola by using its form to reveal its characteristics. The activity begins with a review of both the characteristics and forms of a parabola, then moves on to determine characteristics of the graph of a parabola given in standard form, vertex form, or intercept form.

- Desmos. "Domain and Range Introduction."

  http://www.walch.com/ca/01049

  In this activity, students practice finding the domain and range of piecewise functions. Students begin with an informal exploration of domain and range using a graph, and build up to representing the domain and range of piecewise functions using inequalities.

- Desmos. "Free-Range Functions."

  http://www.walch.com/ca/01028

  This activity challenges students to strengthen their ideas about the range of quadratic functions.

- Desmos. "Polygraph: Absolute Value."

  http://www.walch.com/ca/01050

  This activity is designed to spark vocabulary-rich conversations about transformations of the absolute value parent function. Key vocabulary terms that may appear in student questions include *translation*, *shift*, *slide*, *dilation*, *stretch*, *horizontal*, *vertical*, and *reflect*.

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- Desmos. "Polygraph: Piecewise Functions."

  http://www.walch.com/ca/01052

  This activity is designed to spark vocabulary-rich conversations about piecewise functions. Key vocabulary terms that may appear in student questions include *piecewise*, *continuous*, and *interval*.

- Desmos. "Polygraph: Square Root Functions."

  http://www.walch.com/ca/01032

  This activity is designed to spark vocabulary-rich conversations about transforming square root functions. Key vocabulary terms that may appear in student questions include *translation*, *reflection*, *intercept*, and *quadrant*.

Unit 3

- Desmos. "Build a Bigger Field."

  http://www.walch.com/ca/01021

  Use quadratic models to optimize the area of a field for a given perimeter.

- Desmos. "Match My Parabola."

  http://www.walch.com/ca/01017

  A series of graphing challenges builds understanding of quadratic functions in various forms and graphing transformations of quadratic functions.

- Desmos. "Penny Circle."

  http://www.walch.com/ca/01023

  Gather data, build a model, and use that model to answer the question, "How many pennies fit in a large circle?"

Unit 4

- Illustrative Mathematics. "The Titanic 3."

  http://www.walch.com/ca/10005

  This task poses an open-ended question that forces students to think about how they can answer it. Students will analyze data given in the two-way table to find meaningful probabilities and increase their understanding of conditional probability and independence.

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- Mathematics Assessment Resource Service, University of Nottingham. "Representing Conditional Probabilities 1."

  http://www.walch.com/ca/10007

  In this resource, students will use tables and tree diagrams to represent events as a subset of a sample space and answer questions involving conditional probabilities. This resource includes a diagnostic assessment, a learning task, and sample solutions that students will analyze and evaluate. Some extra materials are required.

Unit 5

- Desmos. "Lines, Transversals, and Angles."

  http://www.walch.com/ca/01033

  In this activity, students explore the relationship of angles formed by a transversal and a system of two lines. In particular, students consider what happens when the two lines are parallel versus when they are not.

- Desmos. "Polygraph: Angle Relationships."

  http://www.walch.com/ca/01034

  This activity is designed to spark vocabulary-rich conversations about angle relationships. Key vocabulary terms that may appear in student questions include *parallel*, *transversal*, *adjacent*, *opposite*, *alternate interior*, *corresponding*, *alternate exterior*, *vertical*, and *right*.

- Desmos. "Polygraph: Figure It Out."

  http://www.walch.com/ca/01035

  This activity is intended as an introduction to geometric notation and vocabulary. Depending on prior knowledge, students could use the following to distinguish figures: points, lines, rays, segments, parallel, perpendicular, angles, congruence, midpoints, bisectors, betweenness, collinearity, and more.

Unit 6

- Desmos. "Card Sort: Parabolas."

  http://www.walch.com/ca/01022

  Find the shape of a parabola by using its form to reveal its characteristics. The activity begins with a review of both the characteristics and forms of a parabola, then moves on to determine characteristics of the graph of a parabola given in standard form, vertex form, or intercept form.

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- Desmos. "Equations of Circles."

  http://www.walch.com/ca/01040

  In this activity, students write equations of circles with different given information. The activity involves writing equations in both standard and general form.

- Desmos. "Marbleslides: Parabolas."

  http://www.walch.com/ca/01029

  In this activity, students will rapidly become fluent in transforming, translating, and restricting the domains of parabolas by manipulating their symbolic form to cause marbles to slide down the parabola through stars.

- Desmos. "Polygraph: Parabolas."

  http://www.walch.com/ca/01030

  In this paired game, one student picks a single parabola from a collection and the other student asks yes/no questions in an attempt to guess which parabola was chosen. The game provides students with a reason for noticing important features of parabolas, which in turn provides a need for words to name the features. "Does your graph cross the x-axis twice?" is a common question during play that leads to the naming roots, for example.

- Desmos. "Sector Area."

  http://www.walch.com/ca/01041

  In this proportional reasoning activity, students explore the relationship between circle area, sector area, and sector angle.

- Desmos. "Will It Hit the Hoop?"

  http://www.walch.com/ca/01031

  In this activity, students predict whether various basketball shots will go through the hoop, and then model these shots with parabolas to check their predictions.

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Station Activities Guide

Introduction

Each unit includes a collection of station-based activities to provide students with opportunities to practice and apply the mathematical skills and concepts they are learning. You may use these activities in addition to the instructional lessons, or, especially if the pre-test or other formative assessment results suggest it, instead of direct instruction in areas where students have the basic concepts but need practice. The debriefing discussions after each set of activities provide an important opportunity to help students reflect on their experiences and synthesize their thinking. Debriefing also provides an additional opportunity for ongoing, informal assessment to guide instructional planning.

Implementation Guide

The following guidelines will help you prepare for and use the activity sets in this section.

Setting Up the Stations

Each activity set consists of four or five stations. Set up each station at a desk, or at several desks pushed together, with enough chairs for a small group of students. Place a card with the number of the station on the desk. Each station should also contain the materials specified in the teacher's notes, and a stack of student activity sheets (one copy per student). Place the required materials (as listed) at each station.

When a group of students arrives at a station, each student should take one of the activity sheets to record the group's work. Although students should work together to develop one set of answers for the entire group, each student should record the answers on his or her own activity sheet. This helps keep students engaged in the activity and gives each student a record of the activity for future reference.

Forming Groups of Students

All activity sets consist of four or five stations. You might divide the class into four or five groups by having students count off from 1 to 4 or 5. If you have a large class and want to have students working in small groups, you might set up two identical sets of stations, labeled A and B. In this way, the class can be divided into eight groups, with each group of students rotating through the "A" stations or "B" stations.

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Assigning Roles to Students

Students often work most productively in groups when each student has an assigned role. You may want to assign roles to students when they are assigned to groups and change the roles occasionally. Some possible roles are as follows:

- Reader—reads the steps of the activity aloud
- Facilitator—makes sure that each student in the group has a chance to speak and pose questions; also makes sure that each student agrees on each answer before it is written down
- Materials Manager—handles the materials at the station and makes sure the materials are put back in place at the end of the activity
- Timekeeper—tracks the group's progress to ensure that the activity is completed in the allotted time
- Spokesperson—speaks for the group during the debriefing session after the activities

Timing the Activities

The activities in this section are designed to take approximately 10 minutes per station. Therefore, you might plan on having groups change stations every 10 minutes, with a two-minute interval for moving from one station to the next. It is helpful to give students a "5-minute warning" before it is time to change stations.

Since each activity set consists of four or five stations, the above time frame means that it will take about 50 to 60 minutes for groups to work through all stations.

Guidelines for Students

Before starting the first activity set, you may want to review the following "ground rules" with students. You might also post the rules in the classroom.

- All students in a group should agree on each answer before it is written down. If there is a disagreement within the group, discuss it with one another.
- You can ask your teacher a question only if everyone in the group has the same question.
- If you finish early, work together to write problems of your own that are similar to the ones on the activity sheet.
- Leave the station exactly as you found it. All materials should be in the same place and in the same condition as when you arrived.

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Debriefing the Activities

After each group has rotated through every station, bring students together for a brief class discussion. At this time, you might have the groups' spokespersons pose any questions they had about the activities. Before responding, ask if students in other groups encountered the same difficulty or if they have a response to the question. The class discussion is also a good time to reinforce the essential ideas of the activities. The questions that are provided in the teacher's notes for each activity set can serve as a guide to initiating this type of discussion.

You may want to collect the student activity sheets before beginning the class discussion. However, it can be beneficial to collect the sheets afterward so that students can refer to them during the discussion. This also gives students a chance to revisit and refine their work based on the debriefing session. If you run out of time to hold class discussions, you might want to have students journal about their experiences and follow up with a class discussion the next day.

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Digital Enhancements Guide

Introduction

With the *Common Core State Standards Integrated Pathway: Mathematics II Program*, you have access to the following digital components, described here with guidelines and suggestions for implementation.

Digital Warm-Ups

Each of the warm-ups in this collection includes a video clip to enhance student engagement. Hard copies of all warm-ups are also included in the Teacher Resource. These optional versions of the warm-ups and debriefs run on PowerPoint. (*Please note*: Computers may render PowerPoint images differently. For best viewing and display, use a PowerPoint Viewer and adjust your settings to optimize images and text.)

We suggest that the video clip be played as students enter the class. The green "play" button on the first PowerPoint slide will link your computer to a browser and then to the video, so you will need an Internet connection. Make sure that you use the "full screen" option for display and adjust the volume appropriately. The video clips have been selected to catch students' attention by addressing the context (not the content) of the warm-up problem in an engaging way.

After the video, each warm-up takes approximately 5 minutes and addresses critical-thinking skills and/or previously taught math concepts that are prerequisites for the new content in the lesson.

The warm-ups include debriefs that reveal the answers to the warm-up questions. The answers slide into view and can be used to facilitate discussion.

In the notes section, you will find the URL for the video and "Connections to the Lesson," which describes concepts students will glean or skills they will need in the upcoming lesson. The "Connections" help transition from the warm-up to instruction.

GeoGebra Applets

In addition to the hard copy materials found in the Teacher Resource, two interactive GeoGebra applets are provided for most lessons. The applets model the mathematics in two of the Guided Practice examples. Links to all of the applets can be found on the flash drive for this program. With an Internet connection, simply click on the link identified for the lesson and example.

Please adjust your view to maximize the image. Each applet illustrates the specific problem addressed in the Guided Practice example. The applets allow you to walk through the solution by visually demonstrating the steps, such as defining points and drawing lines. Variable components of the applets (usually fill-in boxes or sliders) allow you to substitute different values in order to explore

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the mathematics. For example, "What happens to the line when we increase the amount of time?" or "What if we cut the number of students in half?" This experimentation and discussion supports development of conceptual understanding.

GeoGebra for PC/MAC

GeoGebra is not required for using the applets, but can be downloaded for free for further exploration at the following link:

http://www.geogebra.org/cms/en/download

GeoGebra Applet Troubleshooting

If you are experiencing any difficulty in using the applets in your browser, please visit the following link for our troubleshooting document.

http://www.walch.com/applethelp

ExamView Assessment Banks

These are available with sufficient items to compile up to 5 versions of each pre-, progress, and unit assessment in the program. ExamView items may also be used to generate additional worksheets for practice.

For information about ExamView, please contact Customer Service at (800) 341-6094 or customerservice@walch.com.

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Standards for Mathematical Practice Implementation Guide

Introduction

The eight Standards for Mathematical Practice describe features of lesson design, teaching pedagogy, and student actions that will lead to a true conceptual understanding of the mathematics standards. The Walch lessons, practice problems, and Problem-Based Tasks lend themselves to teaching through this framework. When the Walch resources are combined with high-level questioning and engaging teacher decisions in the classroom, it will lead to high-level math instruction and student achievement.

Here is a brief description of the SMPs and how they can be applied in the classroom:

CCSS.MP.1: Make sense of problems and persevere in solving them.

Students will read, interpret, and understand complicated mathematical and real-world problems, and they will be willing to try multiple methods with the ultimate goal of determining the correct answer. Strategies such as annotation and student discourse can lead to improvement on this standard. Presenting students with higher-level problems is essential to ensuring students achieve maximum understanding. Teacher prompts that can enhance this standard include:

- What is the problem asking you to solve?
- What are some (other) strategies you could use to solve this problem?
- Compare your answer with a classmate's answer. Who is correct? Why?

CCSS.MP.2: Reason abstractly and quantitatively.

Mathematical reasoning with numbers and variables is essential to understanding the connections among the standards. Students must be able to discover and formalize general rules using numbers and variables, and apply them to determine numerical quantities in other situations. Teacher prompts that can enhance this standard include:

- Substitute realistic numbers into the situation.
- What operation/strategy would you use?
- Will your strategy work for any number?
- For which categories of numbers (negative integers, all real numbers, etc.) will your strategy work?

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CCSS.MP.3: Construct viable arguments and critique the reasoning of others.

Many students are most concerned with the "what" aspects of mathematics, i.e. "what" do we do or "what" is the answer. However, math educators must develop the "why" of mathematics. Students must learn to question algorithms, challenge answers, and justify their reasoning in order to truly understand the concepts behind their answers. Teacher prompts that can enhance this standard include:

- How did you determine your answer?
- Why did you choose that strategy?
- Defend your answer based on a real-world situation.

CCSS.MP.4: Model with mathematics.

An important goal of mathematics instruction is for students to be able to apply mathematics to the world around them. Students should be able to link a real problem to a mathematical concept, identify quantities that are modeled well with mathematics, and use mathematics to find a solution. Emphasizing this standard will help students represent and interpret information using physical, visual, and abstract models. Encourage students to use any or all of their learning experiences to gain a deep and flexible understanding of mathematics. Teacher prompts that can enhance this standard include:

- Can you represent this situation with a visual model?
- How will it help you solve the problem?
- What information is needed to solve this problem?
- Is there another way to solve this problem?
- While working to solve this problem, what do you notice/wonder?

CCSS.MP.5: Use appropriate tools strategically.

There are many available tools suitable for mathematics, such as calculators, manipulatives, formulas, rulers, computers, and developed mathematical strategies. Choosing and using the correct tool to work through a problem is an important skill for mathematicians. Teacher prompts that can enhance this standard include:

- Can you graph this equation in the calculator to see a relationship?
- What formula or strategy might help you determine the answer to this question?
- How can you represent the situation using handheld tools (rulers, protractors, etc.) to determine an answer?

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CCSS.MP.6: Attend to precision.

When using mathematics to solve problems, an answer can be considered correct only if it is sufficiently precise and accurate for the situation to which it pertains. When applying mathematics, it is vital to clearly define the question, the reasoning, the answer, and the explanation. Vocabulary, units, numerical responses, and pictures must be represented precisely in questions and answers to ensure that the mathematical solutions represent the true answer to a question. Teacher prompts that can enhance this standard include:

- What does your answer represent in a real-world context?
- Is your answer reasonable based on your initial estimate?
- What units of measure help describe your numerical answer?

CCSS.MP.7: Look for and make use of structure.

Structure, whether geometric, algebraic, statistical, or numerical, is an important aspect of mathematical reasoning that students often overlook. Teachers often explicitly refer to geometric and other visual structures as explanations of mathematical concepts, but algebraic and numerical structures can often be just as important in analyzing and interpreting mathematical situations. These structures yield clues as to the meaning of expressions, equations, graphs, and other representations. As students interpret these structures, they will gain a greater understanding of the mathematical concepts. Teacher prompts that can enhance this standard include:

- What do the characteristics of the graph tell us about the situation?
- What do each of the variables and numbers in the equation/formula represent?
- How are these situations the same and different based on their representations?

CCSS.MP.8: Look for and express regularity in repeated reasoning.

Just as patterns appear in real life, patterns appear throughout the subject of mathematics. Recognizing and applying these patterns, and applying the reasoning contained within, is one of the most important skills teachers can instill in their students. Rather than teaching isolated algorithms to determine answers, have students discover relationships, create their own algorithms, and apply the reasoning to other situations. These skills can be applied throughout their education and will enrich their lives after high school. Teacher prompts that can enhance this standard include:

- What relationship do you notice in the graph/table/numbers?
- Why did you choose to use this process to solve this word problem/equation?
- How can you apply this process in other situations?

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Suggested Pacing Guide

Overview

The sub-lessons in Walch Integrated Math are designed to be covered over one, two, or three days, depending on the depth and complexity of the topic. The pacing guide that follows provides *suggested* pacing, but this should be viewed as a flexible guideline only. If students need an extra day to solidify understanding, a one-day lesson may be spread over two days, or a two-day lesson may be spread over three days. Conversely, if students grasp a concept more quickly than suggested by the pacing guide, it is fine to shorten the duration of the lesson and move on to the next topic.

Each sub-lesson features several components. The estimated time for each component is given in parentheses. Components include: a Warm-Up, Instruction (including an Introduction, Key Concepts, and Guided Practice), a Problem-Based Task, and Practice.

Please note: PowerPoint versions of each sub-lesson's Warm-Up and Instruction are provided. Warm-Up PowerPoints include a video clip and the Warm-Up Debrief. Instructional PowerPoints feature the Introduction, Key Concepts, and selected Guided Practice Examples with links to corresponding GeoGebra applets. Estimated times reflect presentation of these PowerPoint versions and applets.

Guided Practice, Scaffolded Practice, and Practice

For each Guided Practice in the Teacher Resource (TRB), corresponding Scaffolded Practice worksheets are provided in the Student Workbook (SWB). Practice problems may be assigned from the SWB and/or the parallel Student Resource (SRB) worksheet for in-class work or homework assignments. The SRB worksheets are designed to closely resemble the format of the SWB worksheets, but are composed of different problems for additional practice.

Please note: When a sub-lesson is spread across more than one class period, the Guided Practice Examples will also be spread across the same number of class periods. Assign Practice problems that correspond to the Guided Practice Example(s) you select for each day, to ensure appropriate prior instruction.

Suggested Progression of One-, Two-, and Three-Day Sub-Lessons

One Day

- Warm-Up PowerPoint and Debrief (10 minutes)
- Introduction and Key Concepts (10 minutes)
- One or two Guided/Scaffolded Practice Examples, including PowerPoint applet presentation as appropriate (15 minutes)
- Problem-Based Task and discussion (15 minutes)
- Homework: Practice from the SWB or SRB

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Two Days

Day 1

- Warm-Up PowerPoint and Debrief (10 minutes)
- Introduction and Key Concepts (15 minutes)
- Two or three Guided/Scaffolded Practice Examples, including PowerPoint applet presentation as appropriate (15 minutes)
- In-class practice: selected problems from the SWB Practice (10 minutes)
- Homework: selected problems from the SRB Practice

Day 2

- Review Practice problems (10 minutes)
- One or two remaining Guided/Scaffolded Practice Examples, including PowerPoint applet presentation as appropriate (10 minutes)
- Problem-Based Task and discussion (20 minutes)
- In-class practice: the remainder of the SWB Practice (10 minutes)
- Homework: the remainder of the SRB Practice

Three Days

Day 1

- Warm-Up PowerPoint and Debrief (10 minutes)
- Introduction and Key Concepts (25 minutes)
- One or two Guided/Scaffolded Practice Examples, including PowerPoint applet presentation as appropriate (15 minutes)
- Homework: selected problems from the SRB Practice

Day 2

- Review Practice problems (15 minutes)
- Two or three remaining Guided/Scaffolded Practice Examples, including PowerPoint applet presentation as appropriate (15 minutes)
- Problem-Based Task and discussion (20 minutes)
- Homework: the remainder of the SRB Practice

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Day 3

- Review Practice problems (15 minutes)
- In-class practice: selected problems from the SWB Practice (35 minutes)
- Homework: the remainder of the SWB Practice

Assessments and Station Activities

Each lesson includes a Pre-Assessment and a Progress Assessment, and each unit concludes with a Unit Assessment. Units also feature sets of Station Activities to complement instruction.

Pre-Assessments are short, multiple-choice assessments with five problems, designed to evaluate prior knowledge of the upcoming lesson. These are brief and should not factor into a student's grade. Pre-Assessments are often given on the same day as the preceding lesson's Progress Assessment, to prepare for the upcoming lesson.

Progress Assessments include 10 multiple-choice problems and one extended-response problem. Progress Assessments may not take an entire class period. The additional time may be used to review before the assessment, or to begin the next lesson after the assessment.

Unit Assessments include 12 multiple-choice problems and three extended-response problems, and generally require a full class period.

Station Activities generally require a full class period for the students to rotate through each station and then to engage in a class discussion at the end.

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Unit 1: Extending the Number System (15 Days)

Unit Overview: This introductory unit covers many of the foundational skills needed for the rest of the course. Students learn about rational exponents and the properties of rational and irrational numbers. This is followed by operating with polynomials. Finally, students define an imaginary number and learn to operate with complex numbers.

Lesson 1: Working with the Number System

Students are introduced to the properties of exponents, including rational exponents. They use these properties to rewrite and evaluate expressions that contain rational exponents. Next, students learn the properties of rational and irrational numbers.

Day	Area of study/content	Pages	Standard(s)
Day 1 Day 2	Lesson 1 Pre-Assessment 1.1.1: Defining, Rewriting, and Evaluating Rational Exponents	**Pre-Assessment** TRB: U1-1 **Instruction/Practice** TRB: U1-4 SRB: U1-4 SWB: 7	N–RN.1 N–RN.2
Day 3	1.1.2: Rational and Irrational Numbers and Their Properties	TRB: U1-17 SRB: U1-11 SWB: 17	N–RN.2 N–RN.3
Day 4	Lesson 1 Progress Assessment Lesson 2 Pre-Assessment	TRB: U1-32 TRB: U1-34	

Lesson 2: Operating with Polynomials

Students learn how to add, subtract, and multiply polynomials. These skills help students simplify expressions, and are foundational skills that are used in later units and in future math courses.

Day	Area of study/content	Pages	Standard(s)
Day 5	1.2.1: Adding and Subtracting Polynomials	TRB: U1-36 SRB: U1-22 SWB: 27	A–APR.1
Day 6	1.2.2: Multiplying Polynomials	TRB: U1-48 SRB: U1-29 SWB: 37	A–APR.1
Day 7	Station Activities Set 2: Operations with Polynomials	TRB: U1-118 SWB: 81	A–APR.1
Day 8	Lesson 2 Progress Assessment Lesson 3 Pre-Assessment	TRB: U1-61 TRB: U1-63	

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Lesson 3: Operating with Complex Numbers

Students are introduced to the imaginary unit *i* and to the concept of a complex number. Then, students learn to add, subtract, and multiply complex numbers. Students explore how the imaginary unit enables them to solve problems that they would not otherwise be able to solve.

Day	Area of study/content	Pages	Standard(s)
Day 9 Day 10	1.3.1: Defining Complex Numbers, *i*, and i^2	TRB: U1-66 SRB: U1-37 SWB: 47	N–CN.1
Day 11	1.3.2: Adding and Subtracting Complex Numbers	TRB: U1-78 SRB: U1-43 SWB: 57	N–CN.2
Day 12	1.3.3: Multiplying Complex Numbers	TRB: U1-90 SRB: U1-49 SWB: 67	N–CN.2
Day 13	Station Activities Set 1: Operations with Complex Numbers	TRB: U1-111 SWB: 77	N–CN.1 N–CN.2 N–CN.3
Day 14	Lesson 3 Progress Assessment	TRB: U1-102	
Unit Assessment			
Day 15	Unit Assessment	TRB: U1-104	

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Unit 2: Quadratic Functions and Modeling (29 Days)

Unit Overview: Students begin by exploring and interpreting the graphs of quadratic functions. Next, they learn how to build quadratic functions from a context and how to carry out operations with functions. This gives way to the exploration of other types of functions, including square root, cube root, absolute value, step, and piecewise functions. The unit progresses to analyzing exponential functions and comparing linear, quadratic, and exponential models given in different forms. The unit ends with transforming functions and finding the inverse of functions.

Lesson 1: Analyzing Quadratic Functions

Students learn what a quadratic function is and how to graph one. They learn how to find the vertex and the intercepts of the graph of a quadratic function, and they discover the various forms of a quadratic function and how each form helps identify key features of the graph. This lesson lays the foundation for the more in-depth examination of quadratic functions in the next lesson.

Day	Area of study/content	Pages	Standard(s)
Day 1 Day 2	Lesson 1 Pre-Assessment 2.1.1: Graphing Quadratic Functions	**Pre-Assessment** TRB: U2-1 **Instruction/Practice** TRB: U2-4 SRB: U2-4 SWB: 7	F–IF.7a★
Day 3	2.1.2: Interpreting Various Forms of Quadratic Functions	TRB: U2-27 SRB: U2-21 SWB: 17	F–IF.7a★ F–IF.8a
Day 4	Lesson 1 Progress Assessment Lesson 2 Pre-Assessment	TRB: U2-49 TRB: U2-51	

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Lesson 2: Interpreting Quadratic Functions

This lesson delves deeper into the analysis of quadratic functions, giving students an opportunity to use equations, graphs, tables, and verbal descriptions to identify and apply the key features of the function's graph. Students also learn how to identify the domain and range of a quadratic function and how to calculate the average rate of change. The skills learned in this lesson prepare students to apply this knowledge to real contexts in the next lesson.

Day	Area of study/content	Pages	Standard(s)
Day 5 Day 6	2.2.1: Interpreting Key Features of Quadratic Functions	TRB: U2-55 SRB: U2-38 SWB: 27	F–IF.4★
Day 7	2.2.2: Identifying the Domain of a Quadratic Function	TRB: U2-77 SRB: U2-53 SWB: 37	F–IF.5★
Day 8	Station Activities Set 1: Graphing Quadratic Equations	TRB: U2-399 SWB: 161	F–IF.7a★
Day 9	2.2.3: Identifying the Average Rate of Change	TRB: U2-91 SRB: U2-61 SWB: 49	F–IF.6★
Day 10	Lesson 2 Progress Assessment Lesson 3 Pre-Assessment	TRB: U2-106 TRB: U2-110	

Lesson 3: Building Functions

This lesson provides students with the opportunity to apply their knowledge of quadratic functions to real-world contexts. By modeling scenarios with quadratic equations and graphs, students discover the usefulness of being able to identify key features such as the maximum, minimum, and intercepts to solve problems.

Day	Area of study/content	Pages	Standard(s)
Day 11 Day 12	2.3.1: Building Functions from Context	TRB: U2-113 SRB: U2-71 SWB: 59	F–BF.1a★
Day 13	2.3.2: Operating on Functions	TRB: U2-133 SRB: U2-84 SWB: 69	F–BF.1b★
Day 14	Lesson 3 Progress Assessment Lesson 4 Pre-Assessment	TRB: U2-148 TRB: U2-150	

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Lesson 4: Graphing Other Functions

Students are introduced to several new types of functions that they probably have not seen before. They explore the graphs of square root functions, cube root functions, absolute value functions, step functions, and piecewise functions. The material in this lesson allows students to move beyond the three types of functions they have already studied (linear, exponential, and quadratic) to see that there are many more types of functions.

Day	Area of study/content	Pages	Standard(s)
Day 15 Day 16	2.4.1: Square Root and Cube Root Functions	TRB: U2-155 SRB: U2-95 SWB: 79	F–IF.7b★
Day 17	2.4.2: Absolute Value and Step Functions	TRB: U2-190 SRB: U2-119 SWB: 89	F–IF.7b★
Day 18	2.4.3: Piecewise Functions	TRB: U2-219 SRB: U2-139 SWB: 99	F–IF.7b★
Day 19	Lesson 4 Progress Assessment Lesson 5 Pre-Assessment	TRB: U2-244 TRB: U2-248	

Lesson 5: Analyzing Functions

This lesson begins with an in-depth look at exponential functions. Then, students learn to compare linear, quadratic, and exponential functions that are presented in various forms—tables, equations, graphs, and words. This allows students to see the similarities and differences between these different categories of functions, and prepares them for the more in-depth study of functions that they will encounter in future math courses.

Day	Area of study/content	Pages	Standard(s)
Day 20 Day 21	2.5.1: Analyzing Exponential Functions	TRB: U2-254 SRB: U2-157 SWB: 109	F–IF.8b
Day 22	2.5.2: Comparing Properties of Functions Given in Different Forms	TRB: U2-268 SRB: U2-164 SWB: 119	F–IF.9 F–LE.3★
Day 23	Lesson 5 Progress Assessment Lesson 6 Pre-Assessment	TRB: U2-287 TRB: U2-291	

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Lesson 6: Transforming Functions

Students learn how to vary the shape and position of a function by multiplying by a number or adding a constant to various parts of the function. These operations are called transformations. The study of transformations helps students see the connection between the algebraic equation for a function and the graphical representation of the function.

Day	Area of study/content	Pages	Standard(s)
Day 24	2.6.1: Replacing $f(x)$ with $f(x) + k$ and $f(x + k)$	TRB: U2-295 SRB: U2-177 SWB: 129	F–BF.3
Day 25	2.6.2: Replacing $f(x)$ with $k \bullet f(x)$ and $f(k \bullet x)$	TRB: U2-315 SRB: U2-189 SWB: 139	F–BF.3
Day 26	Lesson 6 Progress Assessment Lesson 7 Pre-Assessment	TRB: U2-338 TRB: U2-343	

Lesson 7: Finding Inverse Functions

Students are introduced to the concept of an inverse and how to determine the equation for the inverse of a given function. Students also learn how to use inverse functions to solve real-world problems.

Day	Area of study/content	Pages	Standard(s)
Day 27	2.7.1: Finding Inverse Functions	TRB: U2-346 SRB: U2-206 SWB: 151	F–BF.4a
Day 28	Lesson 7 Progress Assessment	TRB: U2-362	
Unit Assessment			
Day 29	Unit Assessment	TRB: U2-364	

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Unit 3: Expressions and Equations (38 Days)

Unit Overview: Students reexamine the basic structures of expressions, but this time these structures are applied to quadratic expressions. Then, students learn to solve quadratic equations using various methods. Next, students create quadratic equations in various forms and learn how to rearrange formulas to solve for a variable that is being squared. The unit builds on previous units by introducing the Fundamental Theorem of Algebra and showing students how complex numbers are solutions to some quadratic equations. At this point, students are introduced to rational functions. Again, students learn to write exponentially structured expressions in equivalent forms. The unit ends by returning to a familiar topic—solving systems of equations—but now complex solutions can be determined.

Lesson 1: Interpreting Structure in Expressions

Students begin by reviewing algebraic vocabulary, such as *term*, *factor*, and *coefficient*. Next, they analyze algebraic expressions to see how each term of an expression affects the value of the expression. Understanding how all the parts of algebraic expressions and equations work together will make the process of creating equations easier when encountered in Lesson 3.

Day	Area of study/content	Pages	Standard(s)
Day 1	Lesson 1 Pre-Assessment 3.1.1: Identifying Terms, Factors, and Coefficients	**Pre-Assessment** TRB: U3-1 **Instruction/Practice** TRB: U3-4 SRB: U3-3 SWB: 7	A–SSE.1a★
Day 2 Day 3	3.1.2: Interpreting Complicated Expressions	TRB: U3-17 SRB: U3-11 SWB: 17	A–SSE.1b★
Day 4	Lesson 1 Progress Assessment Lesson 2 Pre-Assessment	TRB: U3-30 TRB: U3-32	

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Lesson 2: Creating and Solving Quadratic Equations in One Variable

Students learn to create quadratic equations based on real-world scenarios, and to solve quadratic equations by taking the square root of both sides. Then, students learn how to factor quadratic equations and use the factored form to solve equations. The lesson includes factoring by grouping and completing the square. Next, students learn how to apply the quadratic formula to solve equations, and finally they solve quadratic inequalities.

Day	Area of study/content	Pages	Standard(s)
Day 5	3.2.1: Taking the Square Root of Both Sides	TRB: U3-36 SRB: U3-21 SWB: 27	A–CED.1★ A–REI.4b
Day 6 Day 7	3.2.2: Factoring	TRB: U3-47 SRB: U3-26 SWB: 37	A–SSE.2 A–CED.1★ A–REI.4b
Day 8	Station Activities Set 1: Factoring	TRB: U3-449 SWB: 205	A–SSE.3a★
Day 9	3.2.3: Completing the Square	TRB: U3-63 SRB: U3-35 SWB: 47	A–SSE.2 A–CED.1★ A–REI.4a A–REI.4b
Day 10	3.2.4: Applying the Quadratic Formula	TRB: U3-75 SRB: U3-41 SWB: 57	A–CED.1★ A–REI.4a A–REI.4b
Day 11 Day 12	3.2.5: Solving Quadratic Inequalities	TRB: U3-89 SRB: U3-49 SWB: 67	A–SSE.2 A–CED.1★ A–REI.4b
Day 13	Lesson 2 Progress Assessment Lesson 3 Pre-Assessment	TRB: U3-104 TRB: U3-106	

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Lesson 3: Creating Quadratic Equations in Two or More Variables

Students solidify previously learned skills by applying them to problem solving. Students create and graph quadratic equations, both with and without a real-world context, and practice using standard form, factored form, and vertex form. Students also learn how to rearrange formulas in which the variable that is being solved for is squared.

Day	Area of study/content	Pages	Standard(s)
Day 14 Day 15	3.3.1: Creating and Graphing Equations Using Standard Form	TRB: U3-110 SRB: U3-60 SWB: 77	A–SSE.3a★ A–CED.2★
Day 16	3.3.2: Creating and Graphing Equations Using the *x*-intercepts	TRB: U3-138 SRB: U3-82 SWB: 87	A–SSE.3a★ A–CED.2★
Day 17	3.3.3: Creating and Graphing Equations Using Vertex Form	TRB: U3-155 SRB: U3-92 SWB: 97	A–SSE.3b★ A–CED.2★
Day 18	Station Activities Set 3: Quadratic Transformations in Vertex Form	TRB: U3-466 SWB: 213	A–SSE.3b★ F–IF.7a★ F–IF.8a F–BF.3
Day 19	3.3.4: Rearranging Formulas	TRB: U3-169 SRB: U3-100 SWB: 107	A–CED.4★
Day 20	Lesson 3 Progress Assessment Lesson 4 Pre-Assessment	TRB: U3-182 TRB: U3-185	

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Lesson 4: Fundamental Theorem of Algebra

Students gain more practice with polynomial identities by exploring those that contain complex numbers. Next, students learn one of the most useful algebraic theorems, the Fundamental Theorem of Algebra, which tells the number of solutions, both real and complex, that a polynomial can have. Students apply this theorem when solving quadratic equations that have complex solutions.

Day	Area of study/content	Pages	Standard(s)
Day 21 Day 22	3.4.1: Extending Polynomial Identities to Include Complex Numbers	TRB: U3-188 SRB: U3-110 SWB: 117	N–CN.8 (+)
Day 23	3.4.2: Solving Quadratic Equations with Complex Solutions	TRB: U3-210 SRB: U3-125 SWB: 127	N–CN.7 N–CN.9 (+)
Day 24	Station Activities Set 2: Solving Quadratics	TRB: U3-457 SWB: 209	N–CN.7 A–SSE.3a★ A–REI.4b
Day 25	Lesson 4 Progress Assessment Lesson 5 Pre-Assessment	TRB: U3-236 TRB: U3-238	

Lesson 5: Rational Equations

Students further develop their problem-solving skills by learning how to create and solve rational equations from context. Next, they learn how to graph rational functions and use these graphs to solve real-world problems.

Day	Area of study/content	Pages	Standard(s)
Day 26 Day 27	3.5.1: Creating Rational Equations	TRB: U3-245 SRB: U3-148 SWB: 137	A–CED.1★
Day 28	3.5.2: Graphing Rational Equations	TRB: U3-267 SRB: U3-163 SWB: 147	A–CED.2★
Day 29	3.5.3: Creating Rational Inequalities	TRB: U3-300 SRB: U3-188 SWB: 159	A–CED.1★
Day 30	Lesson 5 Progress Assessment Lesson 6 Pre-Assessment	TRB: U3-339 TRB: U3-344	

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Lesson 6: Writing Exponential Expressions in Equivalent Forms

Students explore the usefulness of rewriting exponential expressions in equivalent forms in order to solve real-world problems involving exponential growth and decay.

Day	Area of study/content	Pages	Standard(s)
Day 31 Day 32	3.6.1: Writing Exponential Expressions in Equivalent Forms	TRB: U3-349 SRB: U3-224 SWB: 171	A–SSE.3c★
Day 33	Lesson 6 Progress Assessment Lesson 7 Pre-Assessment	TRB: U3-374 TRB: U3-377	

Lesson 7: Solving Systems of Equations

Students build on their prior knowledge of systems of linear equations from previous courses by extending the concept to include quadratic equations. Students learn to solve systems made up of a linear and a quadratic equation by graphing, and then by using algebraic methods. The quadratic equations involved are for both parabolas and circles.

Day	Area of study/content	Pages	Standard(s)
Day 34	3.7.1: Solving Systems Graphically	TRB: U3-381 SRB: U3-244 SWB: 181	A–REI.7
Day 35 Day 36	3.7.2: Solving Systems Algebraically	TRB: U3-401 SRB: U3-258 SWB: 193	A–REI.7
Day 37	Lesson 7 Progress Assessment	TRB: U3-419	
Unit Assessment			
Day 38	Unit Assessment	TRB: U3-425	

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Unit 4: Applications of Probability (21 Days)

Unit Overview: Students begin by defining events, applying the Addition Rule, and learning about independence. Then, they progress toward conditional probabilities and the Multiplication Rule. This builds into using combinatorics to count and calculate probabilities. Finally, students learn to make and analyze decisions based on probability.

Lesson 1: Events

This introductory lesson defines probability and provides students with the opportunity to learn the many vocabulary words associated with probability, including the concepts of experiments, outcomes, sample spaces, events, and set notation. Then, students learn how to solve probability problems with the Addition Rule. Finally, students learn the difference between independent and dependent events.

Day	Area of study/content	Pages	Standard(s)
Day 1 Day 2	Lesson 1 Pre-Assessment 4.1.1: Describing Events	**Pre-Assessment** TRB: U4-7 **Instruction/Practice** TRB: U4-12 SRB: U4-4 SWB: 7	S–CP.1★
Day 3	4.1.2: The Addition Rule	TRB: U4-39 SRB: U4-24 SWB: 21	S–CP.7★
Day 4	4.1.3: Understanding Independent Events	TRB: U4-54 SRB: U4-34 SWB: 31	S–CP.2★
Day 5	Lesson 1 Progress Assessment Lesson 2 Pre-Assessment	TRB: U4-75 TRB: U4-80	

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Lesson 2: Conditional Probability

Students learn what conditional probability is, and how to solve problems in which conditional probability exists. Then, the concept is expanded to using two-way frequency tables to organize information and solve problems. Lastly, students learn how to solve conditional probability problems using the Multiplication Rule.

Day	Area of study/content	Pages	Standard(s)
Day 6 Day 7	4.2.1: Introducing Conditional Probability	TRB: U4-84 SRB: U4-50 SWB: 45	S–CP.3★ S–CP.5★ S–CP.6★
Day 8 Day 9	4.2.2: Using Two-Way Frequency Tables	TRB: U4-108 SRB: U4-69 SWB: 55	S–CP.4★ S–CP.5★ S–CP.6★
Day 10	4.2.3: The Multiplication Rule	TRB: U4-131 SRB: U4-85 SWB: 69	S–CP.8★ (+)
Day 11	Station Activities Set 1: Probability	TRB: U4-261 SWB: 121	S–CP.1★ S–CP.2★ S–CP.3★
Day 12	Lesson 2 Progress Assessment Lesson 3 Pre-Assessment	TRB: U4-152 TRB: U4-156	

Lesson 3: Combinatorics

Students are introduced to combinations and permutations and how to calculate both. They use these powerful concepts to solve a variety of problems. Then, students combine the probability skills developed earlier in this unit with combinatorics to solve more real-world problems.

Day	Area of study/content	Pages	Standard(s)
Day 13 Day 14	4.3.1: Combinations and Permutations	TRB: U4-159 SRB: U4-101 SWB: 79	S–CP.9★ (+)
Day 15	4.3.2: Probability with Combinatorics	TRB: U4-179 SRB: U4-113 SWB: 89	S–CP.9★ (+)
Day 16	Lesson 3 Progress Assessment Lesson 4 Pre-Assessment	TRB: U4-195 TRB: U4-198	

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Lesson 4: Making and Analyzing Decisions

Students apply skills gained from previous lessons in this unit to decide whether a situation is fair by analyzing the probabilities of the possible outcomes. Then, they analyze the probabilities associated with various situations in order to make decisions, calculate estimates, and determine which choice is most likely to result in the desired outcome.

Day	Area of study/content	Pages	Standard(s)
Day 17 Day 18	4.4.1: Making Decisions	TRB: U4-202 SRB: U4-122 SWB: 99	S–MD.6★ (+)
Day 19	4.4.2: Analyzing Decisions	TRB: U4-222 SRB: U4-134 SWB: 109	S–MD.7★ (+)
Day 20	Lesson 4 Progress Assessment	TRB: U4-242	
Unit Assessment			
Day 21	Unit Assessment	TRB: U4-247	

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Unit 5: Similarity, Right Triangle Trigonometry, and Proof (46 Days)

Unit Overview: Students begin by learning about midpoints and other points of interest in a line segment. Then, they work with dilations and similarity, which builds into learning about and proving the various similarity statements. Next, students learn about special angles in intersecting lines and about relationships among the angles formed by a set of parallel lines intersected by a transversal. Students then return to working with triangles and proving theorems about them, including the Interior Angle Sum Theorem, as well as theorems about isosceles triangles, midsegments, and centers of triangles. Next, students will explore proofs involving parallelograms and other quadrilaterals. The unit ends with an introduction to trigonometric ratios and solving problems using those ratios and the Pythagorean Theorem.

Lesson 1: Line Segments

Students learn how to find the length of a segment in the coordinate plane by using the distance formula, and they calculate the coordinates of the midpoints of segments. The distance and midpoint formulas are then used to solve problems. These formulas are used for several proofs later in the unit.

Day	Area of study/content	Pages	Standard(s)
Day 1	Lesson 1 Pre-Assessment 5.1.1: Midpoints and Other Points on Line Segments	**Pre-Assessment** TRB: U5-1 **Instruction/Practice** TRB: U5-3 SRB: U5-2 SWB: 7	G–GPE.6
Day 2	Lesson 1 Progress Assessment	TRB: U5-26	

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Lesson 2: Investigating Properties of Dilations

Students explore the properties of dilations, the difference between rigid and non-rigid motions, and the meaning of a scale factor. These skills are first explored in isolation and then through the context of applications.

Day	Area of study/content	Pages	Standard(s)
Day 3 Day 4	Lesson 2 Pre-Assessment 5.2.1: Investigating Properties of Parallelism and the Center	**Pre-Assessment** TRB: U5-28 **Instruction/Practice** TRB: U5-34 SRB: U5-20 SWB: 17	G–SRT.1a
Day 5	5.2.2: Investigating Scale Factors	TRB: U5-56 SRB: U5-35 SWB: 31	G–SRT.1b
Day 6	Lesson 2 Progress Assessment Lesson 3 Pre-Assessment	TRB: U5-70 TRB: U5-75	

Lesson 3: Defining and Applying Similarity

Students study the properties of similarity, and they explore the Angle-Angle Similarity Statement as a way to prove that two triangles are similar. The lesson focuses on similar triangles, while the Station Activities set extends the concept to other figures.

Day	Area of study/content	Pages	Standard(s)
Day 7 Day 8	5.3.1: Defining Similarity	TRB: U5-80 SRB: U5-43 SWB: 41	G–SRT.2
Day 9	5.3.2: Applying Similarity Using the Angle-Angle (AA) Criterion	TRB: U5-103 SRB: U5-58 SWB: 57	G–SRT.3
Day 10	Station Activities Set 1: Similarity and Scale Factor	TRB: U5-643 SWB: 299	G–SRT.1b G–SRT.2
Day 11	Lesson 3 Progress Assessment Lesson 4 Pre-Assessment	TRB: U5-119 TRB: U5-126	

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Lesson 4: Proving Similarity

Students extend their knowledge of triangle similarity statements to include Side-Angle-Side (SAS) and Side-Side-Side (SSS). Note that these statements, which rely on the ratios of the sides and not the lengths of the sides, should not be confused with the SAS and SSS methods of proving congruence. These similarity statements are then used to prove the Pythagorean Theorem.

Day	Area of study/content	Pages	Standard(s)
Day 12 Day 13	5.4.1: Proving Triangle Similarity Using Side-Angle-Side (SAS) and Side-Side-Side (SSS) Similarity	TRB: U5-130 SRB: U5-72 SWB: 69	G–SRT.4
Day 14	5.4.2: Working with Ratio Segments	TRB: U5-147 SRB: U5-83 SWB: 81	G–SRT.4
Day 15	5.4.3: Proving the Pythagorean Theorem Using Similarity	TRB: U5-167 SRB: U5-96 SWB: 95	G–SRT.4
Day 16	5.4.4: Solving Problems Using Similarity and Congruence	TRB: U5-187 SRB: U5-110 SWB: 109	G–SRT.5
Day 17	Lesson 4 Progress Assessment Lesson 5 Pre-Assessment	TRB: U5-210 TRB: U5-216	

Lesson 5: Proving Theorems About Lines and Angles

This lesson continues the exploration of proofs. Students discover a proof for the Vertical Angles Theorem, as well as proofs of theorems concerning the angles that are formed when parallel lines are cut by a transversal.

Day	Area of study/content	Pages	Standard(s)
Day 18 Day 19	5.5.1: Proving the Vertical Angles Theorem	TRB: U5-222 SRB: U5-128 SWB: 125	G–CO.9
Day 20 Day 21	5.5.2: Proving Theorems About Angles in Parallel Lines Cut by a Transversal	TRB: U5-251 SRB: U5-149 SWB: 139	G–CO.9
Day 22	Station Activities Set 2: Parallel Lines and Transversals	TRB: U5-656 SWB: 307	G–CO.9
Day 23	Lesson 5 Progress Assessment Lesson 6 Pre-Assessment	TRB: U5-278 TRB: U5-286	

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Lesson 6: Proving Theorems About Triangles

Students explored proofs involving similarity of triangles in previous lessons in this unit. In this lesson, they return to triangles to discover several more proofs of some of the properties of triangles.

Day	Area of study/content	Pages	Standard(s)
Day 24 Day 25	5.6.1: Proving the Interior Angle Sum Theorem	TRB: U5-291 SRB: U5-173 SWB: 153	G–CO.10
Day 26	5.6.2: Proving Theorems About Isosceles Triangles	TRB: U5-313 SRB: U5-188 SWB: 167	G–CO.10
Day 27	5.6.3: Proving the Midsegment of a Triangle	TRB: U5-334 SRB: U5-203 SWB: 179	G–CO.10
Day 28 Day 29	5.6.4: Proving Centers of Triangles	TRB: U5-363 SRB: U5-226 SWB: 193	G–CO.10
Day 30	Lesson 6 Progress Assessment Lesson 7 Pre-Assessment	TRB: U5-402 TRB: U5-408	

Lesson 7: Proving Theorems About Parallelograms

Many of the geometrical proofs that students have seen are brought together in this lesson, which explores the theorems for parallelograms and other special quadrilaterals.

Day	Area of study/content	Pages	Standard(s)
Day 31 Day 32	5.7.1: Proving Properties of Parallelograms	TRB: U5-411 SRB: U5-256 SWB: 207	G–CO.11
Day 33	5.7.2: Proving Properties of Special Quadrilaterals	TRB: U5-438 SRB: U5-276 SWB: 219	G–CO.11
Day 34	Station Activities Set 3: Rhombi, Squares, Kites, and Trapezoids	TRB: U5-668 SWB: 315	G–CO.11
Day 35	Lesson 7 Progress Assessment Lesson 8 Pre-Assessment	TRB: U5-473 TRB: U5-476	

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Lesson 8: Exploring Trigonometric Ratios

Students are introduced to trigonometric ratios. Students learn how to calculate these ratios, and explore sine and cosine as complements. This introduction to trigonometry prepares students for the next lesson, in which trigonometric ratios are used to solve problems, as well as for the much deeper study of trigonometry in later courses.

Day	Area of study/content	Pages	Standard(s)
Day 36 Day 37	5.8.1: Defining Trigonometric Ratios	TRB: U5-481 SRB: U5-303 SWB: 229	G–SRT.6
Day 38	5.8.2: Exploring Sine and Cosine As Complements	TRB: U5-508 SRB: U5-322 SWB: 243	G–SRT.7
Day 39	Lesson 8 Progress Assessment Lesson 9 Pre-Assessment	TRB: U5-525 TRB: U5-529	

Lesson 9: Applying Trigonometric Ratios

Students use the trigonometric ratios learned in the previous lesson and the Pythagorean Theorem to solve problems.

Day	Area of study/content	Pages	Standard(s)
Day 40	5.9.1: Calculating Sine, Cosine, and Tangent	TRB: U5-534 SRB: U5-335 SWB: 255	G–SRT.8★
Day 41	5.9.2: Calculating Cosecant, Secant, and Cotangent	TRB: U5-557 SRB: U5-351 SWB: 267	G–SRT.8★
Day 42	5.9.3: Problem Solving with the Pythagorean Theorem and Trigonometry	TRB: U5-576 SRB: U5-364 SWB: 279	G–SRT.8★
Day 43	5.9.4: Proving the Pythagorean Identity	TRB: U5-596 SRB: U5-378 SWB: 289	F–TF.8
Day 44	Station Activities Set 4: Sine, Cosine, and Tangent Ratios, and Angles of Elevation and Depression	TRB: U5-679 SWB: 323	G–SRT.6 G–SRT.7 G–SRT.8★
Day 45	Lesson 9 Progress Assessment	TRB: U5-613	
Unit Assessment			
Day 46	Unit Assessment	TRB: U5-617	

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Unit 6: Circles With and Without Coordinates (30 Days)

Unit Overview: Students study the properties of circles, including central and inscribed angles, chords of a circle, and tangents to a circle. Then, they build on this knowledge to explore polygons circumscribed by or inscribed in a circle. Next, students move on to the properties of tangent lines and their construction. Students are introduced to radians, and they use radians to find the area of a sector and the circumference and area of a circle. Students build from one- and two-dimensional arenas to a three-dimensional one by exploring more deeply the volume formulas for cylinders, pyramids, cones, and spheres. The links between algebra and geometry are explored through deriving the equations for circles and parabolas. Finally, students use coordinates to prove geometric theorems about circles and parabolas.

Lesson 1: Introducing Circles

This lesson begins by introducing circles and vocabulary related to circles, including central and inscribed angles. Next, students explore the relationship between chords and the angles and arcs they create. Then, students learn the definitions of a tangent line and a secant line, and the properties associated with each.

Day	Area of study/content	Pages	Standard(s)
Day 1 Day 2	Lesson 1 Pre-Assessment 6.1.1: Similar Circles and Central and Inscribed Angles	**Pre-Assessment** TRB: U6-1 **Instruction/Practice** TRB: U6-5 SRB: U6-3 SWB: 7	G–C.1 G–C.2
Day 3	6.1.2: Chord Central Angles Conjecture	TRB: U6-27 SRB: U6-21 SWB: 19	G–C.2
Day 4	6.1.3: Properties of Tangents of a Circle	TRB: U6-41 SRB: U6-30 SWB: 31	G–C.2
Day 5	Lesson 1 Progress Assessment Lesson 2 Pre-Assessment	TRB: U6-61 TRB: U6-66	

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Lesson 2: Inscribed Polygons and Circumscribed Triangles

Students learn how to construct a circle that is inscribed in a triangle by first finding the point of concurrency of the angle bisectors (the incenter). Then, students move on to the construction of circumscribed circles, which requires them to first find the point of concurrency for the perpendicular bisectors (the circumcenter). Finally, students explore the properties of inscribed quadrilaterals.

Day	Area of study/content	Pages	Standard(s)
Day 6	6.2.1: Constructing Inscribed Circles	TRB: U6-69 SRB: U6-44 SWB: 45	G–C.3
Day 7	6.2.2: Constructing Circumscribed Circles	TRB: U6-94 SRB: U6-62 SWB: 57	G–C.3
Day 8	6.2.3: Proving Properties of Inscribed Quadrilaterals	TRB: U6-111 SRB: U6-73 SWB: 69	G–C.3
Day 9	Station Activities Set 3: Circumcenter, Incenter, Orthocenter, and Centroid	TRB: U6-384 SWB: 185	G–CO.12 G–CO.13 G–C.3
Day 10	Lesson 2 Progress Assessment Lesson 3 Pre-Assessment	TRB: U6-126 TRB: U6-130	

Lesson 3: Constructing Tangent Lines

This brief lesson shows how to construct a tangent line through a given point on a circle, and how to construct a tangent line through a point that is not on the circle.

Day	Area of study/content	Pages	Standard(s)
Day 11	6.3.1: Constructing Tangent Lines	TRB: U6-134 SRB: U6-86 SWB: 81	G–C.4 (+)
Day 12	Lesson 3 Progress Assessment	TRB: U6-160	

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Lesson 4: Finding Arc Lengths and Areas of Sectors

Students continue their study of circles by learning how to calculate arc length based on the relationship between the arc length, the radian measure of the central angle, and the length of the radius. Then, students learn how to calculate the area of a sector, again using the length of the radius and the measure of the central angle in radians.

Day	Area of study/content	Pages	Standard(s)
Day 13	Lesson 4 Pre-Assessment 6.4.1: Defining Radians	**Pre-Assessment** TRB: U6-164 **Instruction/Practice** TRB: U6-167 SRB: U6-107 SWB: 93	G–C.5
Day 14 Day 15	6.4.2: Deriving the Formula for the Area of a Sector	TRB: U6-180 SRB: U6-114 SWB: 103	G–C.5
Day 16	Station Activities Set 1: Circumference, Angles, Arcs, Chords, and Inscribed Angles	TRB: U6-359 SWB: 167	G–C.2 G–C.5
Day 17	Lesson 4 Progress Assessment Lesson 5 Pre-Assessment	TRB: U6-192 TRB: U6-194	

Lesson 5: Explaining and Applying Area and Volume Formulas

Students have seen the formulas for circumference and area of a circle in previous math courses. In this lesson, students prove why these formulas work, and they also see a proof of the ratio π. Then, students move on to explore the formulas for calculating the volumes of cylinders, pyramids, cones, and spheres. Through the study of volume, students are exposed to Cavalieri's Principle.

Day	Area of study/content	Pages	Standard(s)
Day 18	6.5.1: Circumference and Area of a Circle	TRB: U6-198 SRB: U6-122 SWB: 113	G–GMD.1
Day 19 Day 20	6.5.2: Volumes of Cylinders, Pyramids, Cones, and Spheres	TRB: U6-220 SRB: U6-137 SWB: 123	G–GMD.1 G–GMD.3★
Day 21	Lesson 5 Progress Assessment Lesson 6 Pre-Assessment	TRB: U6-242 TRB: U6-245	

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Lesson 6: Deriving Equations

Students discover two of the four conic sections, the circle and the parabola. The study of the circle continues, this time combining circles with algebra as students discover the algebraic equation for a circle. Then, students learn how to create equations for parabolas using the geometric definition of a parabola.

Day	Area of study/content	Pages	Standard(s)
Day 22 Day 23	6.6.1: Deriving the Equation of a Circle	TRB: U6-249 SRB: U6-155 SWB: 133	G–GPE.1
Day 24	Station Activities Set 2: Special Segments, Angle Measurements, and Equations of Circles	TRB: U6-372 SWB: 177	G–C.1 G–C.2 G–C.4 G–C.5 G–GPE.1
Day 25	6.6.2: Deriving the Equation of a Parabola	TRB: U6-275 SRB: U6-173 SWB: 143	G–GPE.2
Day 26	Lesson 6 Progress Assessment Lesson 7 Pre-Assessment	TRB: U6-303 TRB: U6-306	

Lesson 7: Using Coordinates to Prove Geometric Theorems About Circles and Parabolas

Students further expand their understanding of the connection between geometric definition of circles and parabolas and the algebraic equations for these shapes. Students use figures in the coordinate plane to illustrate and prove theorems about circles and parabolas algebraically.

Day	Area of study/content	Pages	Standard(s)
Day 27 Day 28	6.7.1: Using Coordinates to Prove Geometric Theorems About Circles and Parabolas	TRB: U6-310 SRB: U6-197 SWB: 155	G–GPE.4
Day 29	Lesson 7 Progress Assessment	TRB: U6-336	
Unit Assessment			
Day 30	Unit Assessment	TRB: U6-339	

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Graphic Organizers

Overview

Graphic organizers can be a versatile tool in your classroom. Organizers offer an easy, straightforward way to visually present a wide range of material. Research suggests that graphic organizers support learning in the classroom for all levels of learners. Gifted students, students on grade level, and students with learning difficulties all benefit from the use of graphic organizers. They reduce the cognitive demand on students by helping them access information quickly and easily. Using graphic organizers, learners can understand content more clearly and can take concise notes. Ultimately, learners find it easier to retain and apply what they've learned.

Graphic organizers help foster higher-level thinking skills. They help students identify main ideas and details in their reading. They make it easier for students to see patterns such as cause and effect, comparing and contrasting, and chronological order. Organizers also help students master critical-thinking skills by asking them to recall, evaluate, synthesize, analyze, and apply what they've learned. Research suggests that graphic organizers contribute to better test scores because they help students understand relationships between key ideas, and enable them to be more focused as they study.

Types of Graphic Organizers

There are four main purposes for using graphic organizers in mathematics and a variety of tools within each category:

Purpose 1: Organizing, Categorizing, and Classifying	**Purpose 2: Problem Solving**	**Purpose 3: Understanding Mathematical Information**	**Purpose 4: Communicating Mathematical Information**
Tables Flowcharts Webs Venn Diagrams	Number Lines Geometric Drawings Factor Trees Attribute Tables Cause and Effect Maps Coordinate Plane Probability Trees	Frayer Model Semantic Map/ Concept Map Compare-and-Contrast Diagram	Line Graphs Bar Charts

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Tables

A table is simply a grid with rows and columns. Tables are useful because information stored in a table is easy to find—much easier than the same information embedded in text.

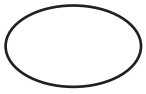
Usually, a table has a row (horizontal) for each item being listed. The columns (vertical) provide places for details about the listed items—the things they have in common. The places where the rows and columns meet are called cells. In each cell, we write information that fits both the topic of the row (the thing being listed) and the topic of the column (the aspect being examined). To create a table, we make rows and columns to fit the number of items and attributes.

Flowcharts

Flowcharts are graphic organizers that show the steps in a process. Flowcharts can be very simple—just a series of boxes with one step in each box. However, there is also a more formal type of flowchart. These flowcharts use special symbols to show different things, such as starting and stopping points, or points where decisions must be made. These symbols make flowcharts especially useful for showing complicated processes.

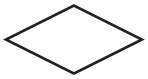
Each step in a flowchart is written in a box. The boxes are connected by arrows to show the sequence of steps. The boxes aren't all rectangular; different shapes are used to indicate different actions. The shapes and symbols are a kind of visual shorthand. Whenever a certain symbol is used, it always has the same meaning.

- Circles and ovals show starting and stopping points. They often contain the words start or stop. The "start" circle or oval has no arrows in and one arrow out. The "stop" circle or oval has one arrow in and no arrows out.



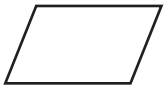
- Arrows show the direction in which the process is moving.

- Diamonds show points where a decision must be made or a question must be answered. The question can usually be answered either "yes" or "no."



- Rectangles and squares show steps where a process or an operation takes place.

- Parallelograms show input or output, such as writing or printing a result or solution.



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Webs

Webs are graphic organizers that help take notes, identify important ideas, and show relationships between and among pieces of information. In a web, the main idea is written in the center circle. Details are recorded in other circles with lines to connect related topics. Circles or lines can be added or deleted as necessary.

Number Lines

In its simplest form, a number line is any line that uses equally spaced marks to show numbers. Number lines are used to visualize equalities and inequalities, positive and negative numbers, and measurements of all kinds. They can "map" math problems, especially ones that involve negative numbers or distances.

Geometric Drawings

A geometric drawing is a representation on paper (or some other surface) of a geometric figure. The geometric drawings we make can never be as perfect as the geometric figures they represent, but as long as they are reasonably accurate, they can help us visualize the figures. In fact, it's often impossible to solve a geometry problem without making a drawing.

Factor Trees

There are several ways to find factors. One that helps to visually keep track of all the factors is called a factor tree. This is a diagram with a tree-like shape. It uses "branches" to show the factors of a number.

All whole numbers other than 1 can be written as the product of factors. A prime number is a number that has only two factors, itself and 1. An example of a prime number is 13. Its only factors are 13 and 1. A composite number is a number that has more than two factors. An example of a composite number is 6. Its factors include 6, 3, 2, and 1. Prime factors are factors that are also prime numbers. The greatest common factor (GCF) of two numbers is the largest number that is a factor of both numbers.

Coordinate Plane

This is the plane determined by a horizontal number line, called the x-axis, and a vertical number line, called the y-axis, intersecting at a point called the origin. A coordinate plane can be used to illustrate locations and relationships using ordered pairs of numbers.

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Venn Diagrams

A set is a list of objects in no particular order. Items in a set can be numbers, but they can also be letters or words. Venn diagrams are a visual way of showing how sets of things can include one another, overlap, or be distinct from one another.

Venn diagrams are often used to compare and contrast things. But they are also a useful tool to sort and classify information. You can use Venn diagrams to take notes on material that shows relationships between things or ideas. You can also use them to solve certain types of word problems. When a word problem names two or three different categories and asks you how many items fall into each category, a Venn diagram can be a useful problem-solving tool.

A Venn diagram begins with a rectangle representing the universal set. Then each set in the problem is represented by a circle. Circles can be separate, overlapping, or one within another. When two circles overlap, it means that the two sets intersect. Some members of one set are also members of the other set.

Venn Diagrams AND Compare-and-Contrast Diagrams

The Venn diagram is an organizing device for planning comparisons and contrasts. A completed Venn diagram helps students categorize and organize similarities and differences, and provides a blueprint for a comparison-and-contrast exercise. The compare-and-contrast diagram provides a structure to identify or list similarities and differences between two objects.

Attribute Tables

To solve logic problems, you need a way to keep track of the subjects and which attributes they have or don't have. An attribute table can help. This is a table with a row for each subject in the problem, and a column for each attribute. The rows and columns meet to form cells. Because the attributes in logic problems are usually exclusive, you can use Xs or check marks (✔) to show which attribute belongs to which subject.

Cause and Effect Maps

Cause and effect maps help you work through information to make sense of it. Write each cause in the oval. Write all its effects in the boxes. Add or delete ovals and boxes as needed.

Frayer Model

The Frayer Model is a word categorization activity that helps learners to develop their understanding of concepts. Using this model, students provide a definition, list characteristics, and provide examples and non-examples of the concept.

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Semantic Map

A semantic word map allows students to conceptually explore their knowledge of a new term or concept by mapping it with other related words, concepts, or phrases that are similar in meaning. Semantic maps portray the schematic relations that compose a concept. It assumes that there are multiple relations between a concept and the knowledge that is associated with the concept.

Line Graphs

Line graphs are often used to show how things change over time. They clearly show trends in data and can let you make predictions about future trends, too. Line graphs use two number lines, one horizontal and one vertical. The horizontal number line is called the *x*-axis. The vertical line is called the *y*-axis. The *x*-axis often shows the passage of time. The *y*-axis often shows a quantity of some kind, such as height, speed, cost, and so forth.

Bar Charts

Bar charts are useful when you want to compare things or to show how one thing changes over time. They are a good way to show overall trends. Bar charts use horizontal or vertical bars to represent data. Longer bars represent higher values. Different colors can be used to show different variables. When you look at a bar chart, it's easy to see which element has the greatest value—the one with the longest bar.

Bar charts have an *x*-axis (horizontal) and a *y*-axis (vertical). If the graph is being used to show how something changes over time, the *x*-axis has numbers for the time period. If the graph is being used to compare things, the *x*-axis shows which things are being compared. The *y*-axis has numbers that show how much of each thing there is.

Probability Trees

When we have probability problems with many possible outcomes, or events that depend on one another, probability trees can help. Probability trees show all the possible outcomes of an event. Whenever a problem calls for figuring out how many possible outcomes there are, and the probability that any one of them will happen, a probability tree can be useful.

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Table

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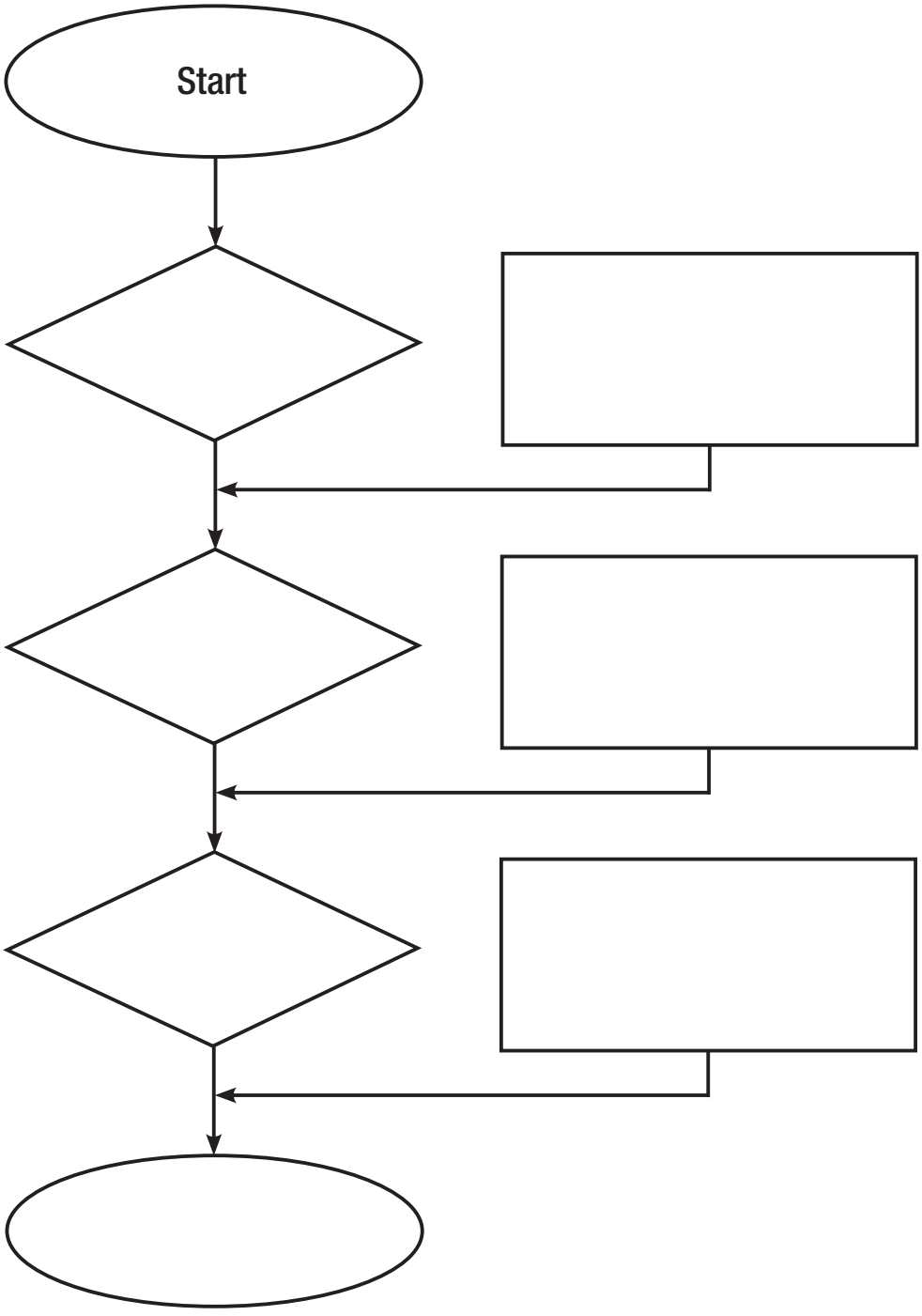
Flowchart



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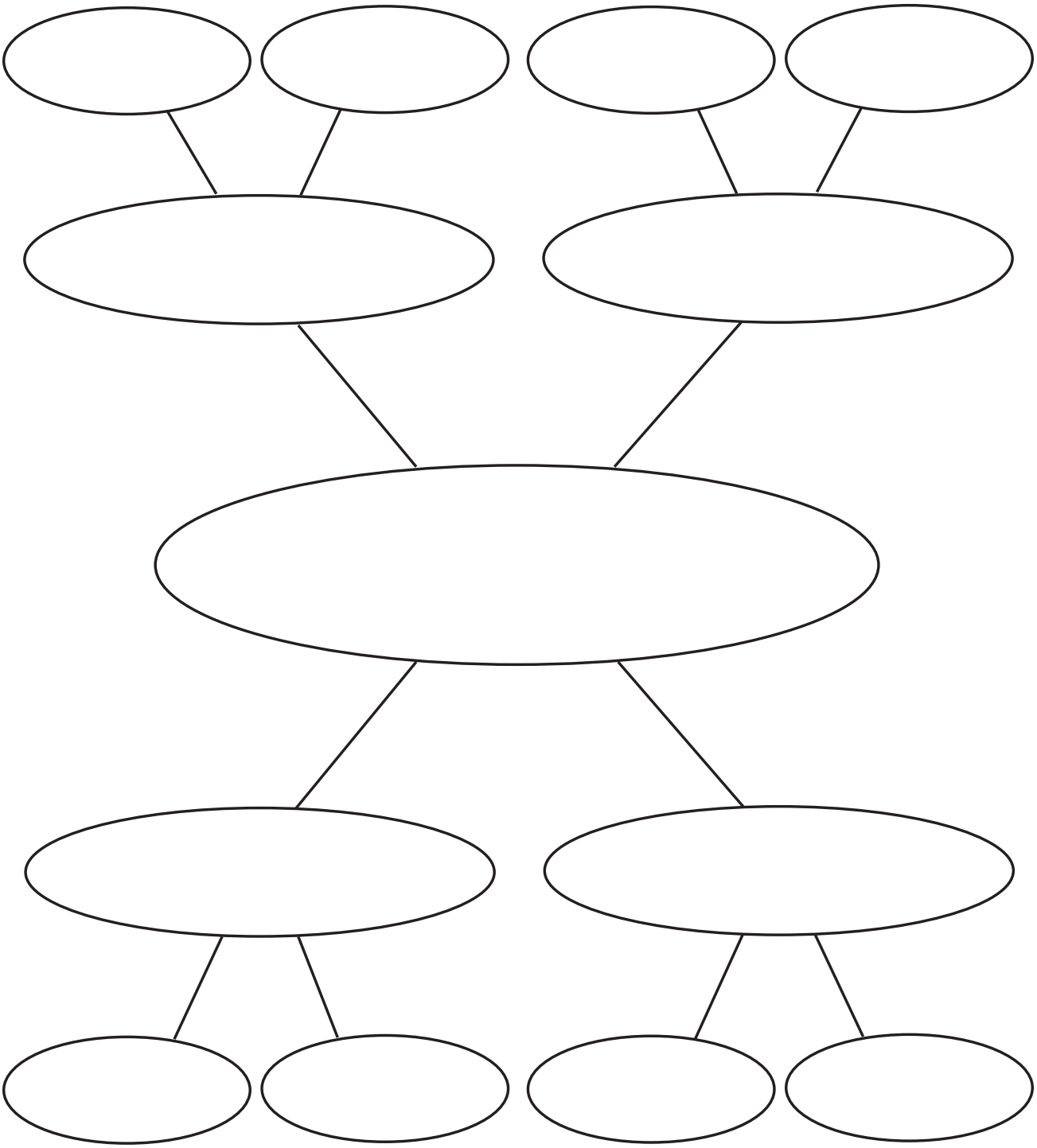
Web



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Number Line

0

0

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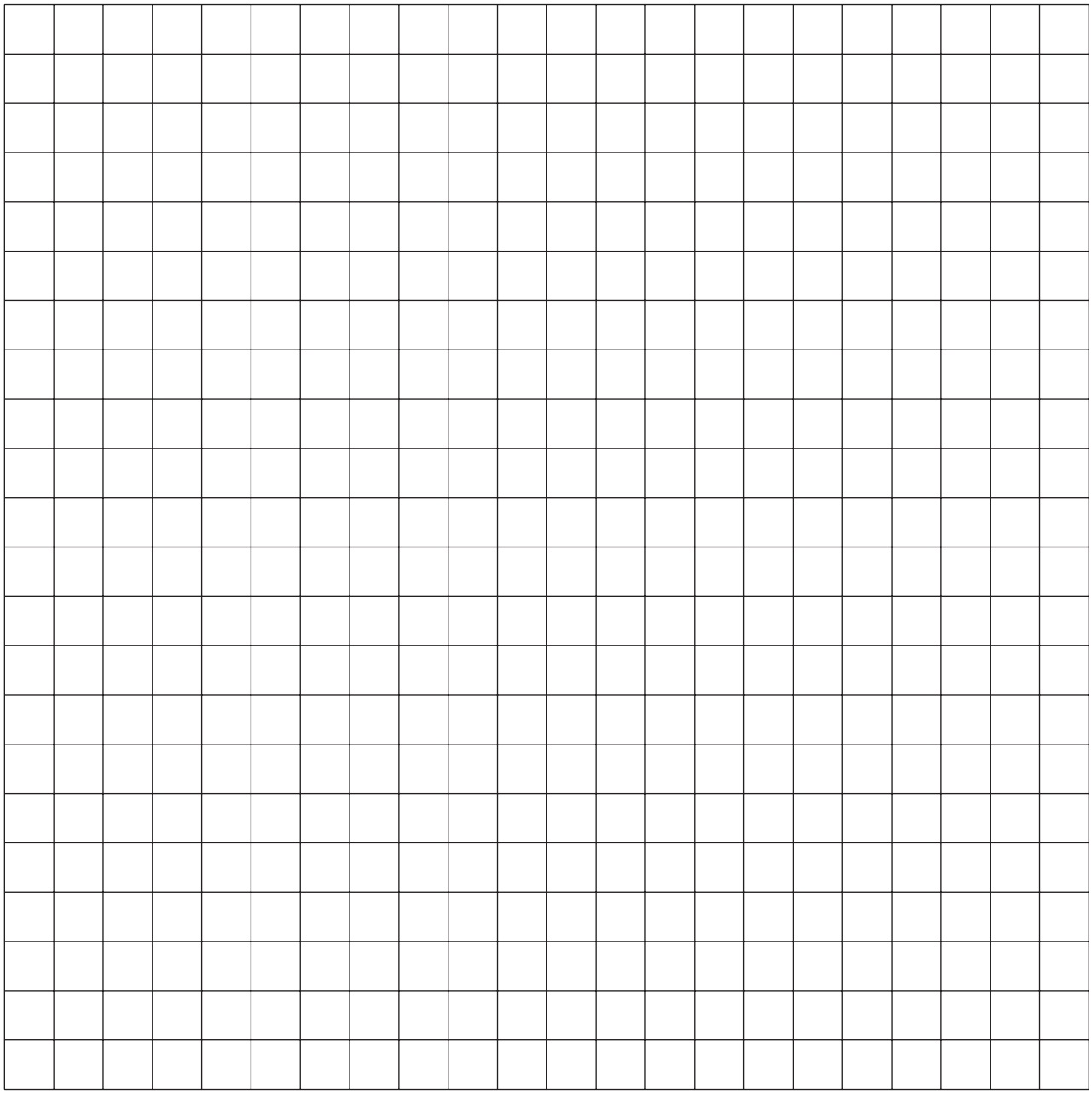
Geometric Drawing



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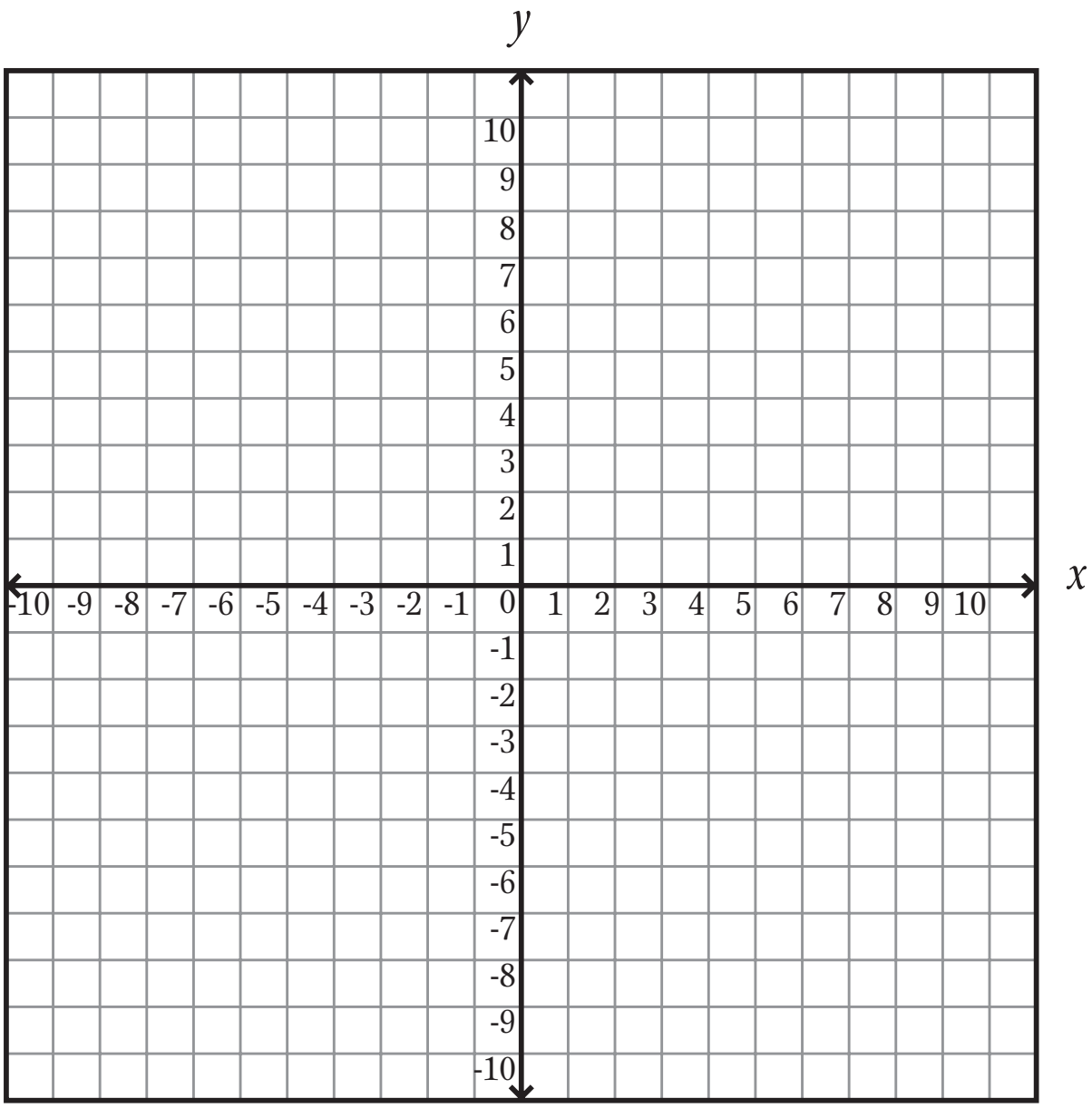
Coordinate Plane



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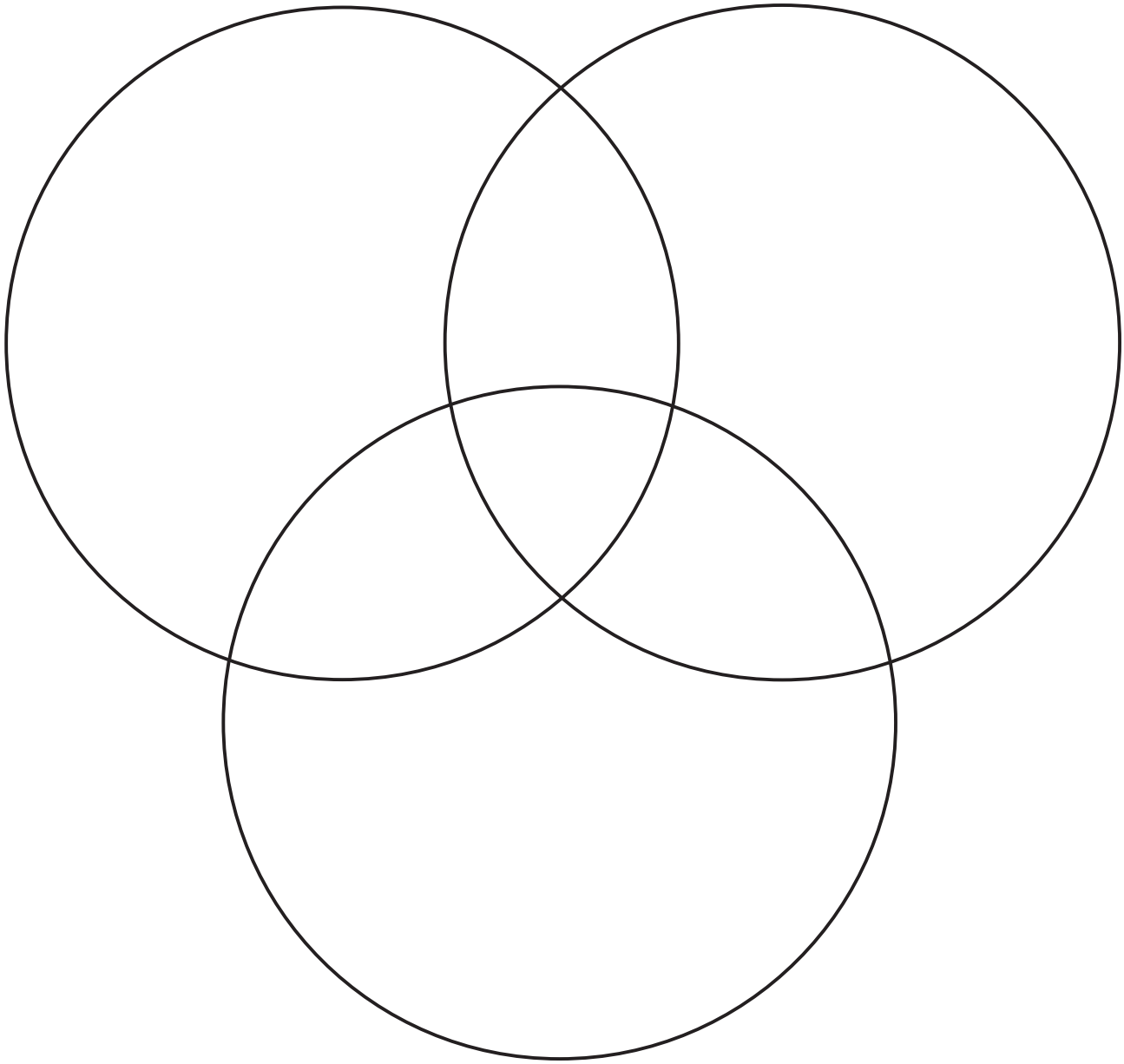
Venn Diagram



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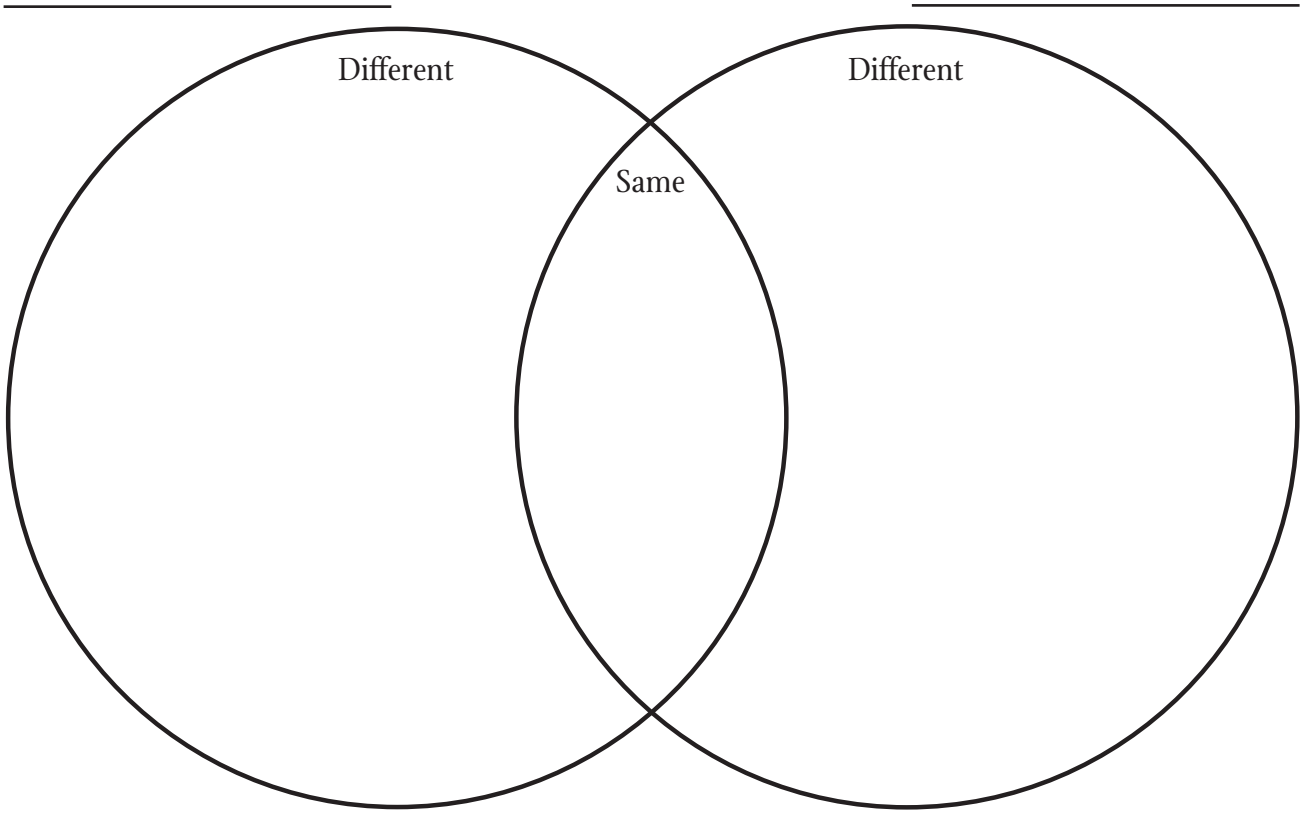
Venn Diagram



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Compare-and-Contrast Diagram

Item 1 _____ Item 2 _____

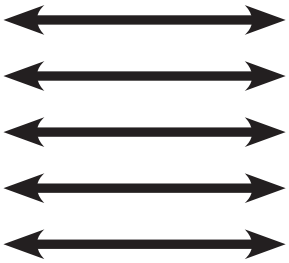
How Alike?

_____
_____
_____
_____
_____
_____

How Different?

_____
_____
_____
_____
_____
_____
_____
_____

With Regard To



_____
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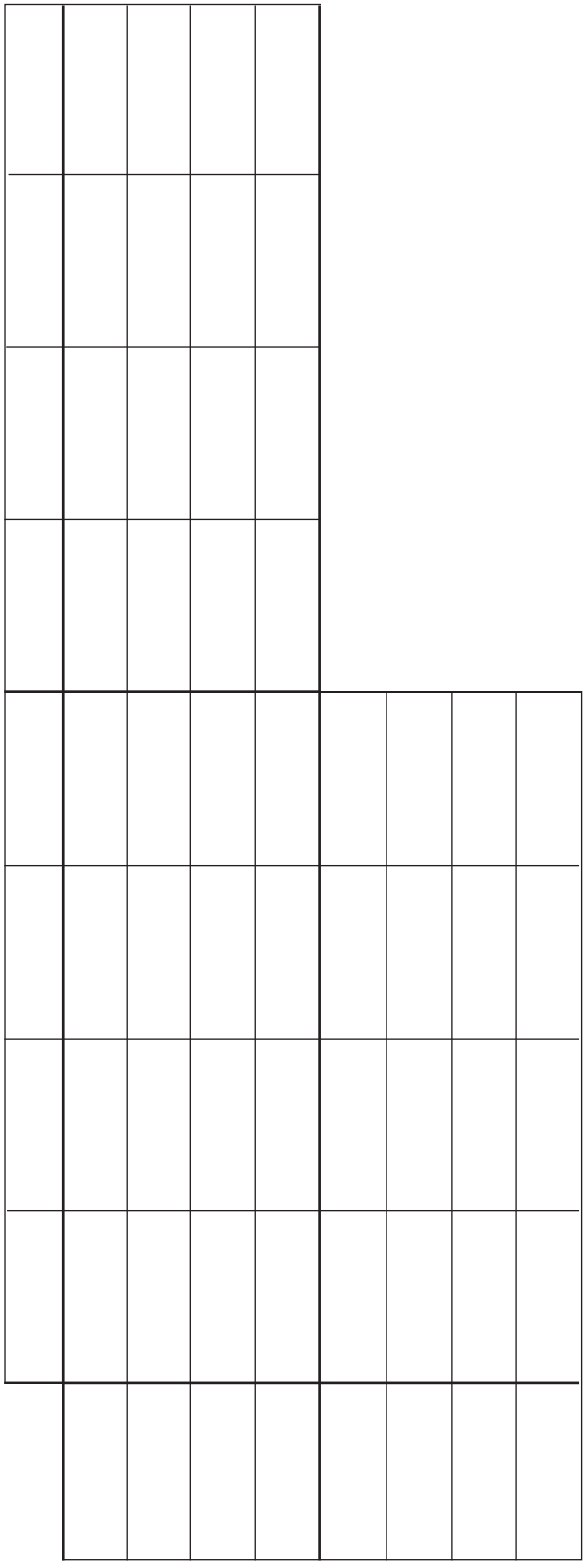
Attribute Table



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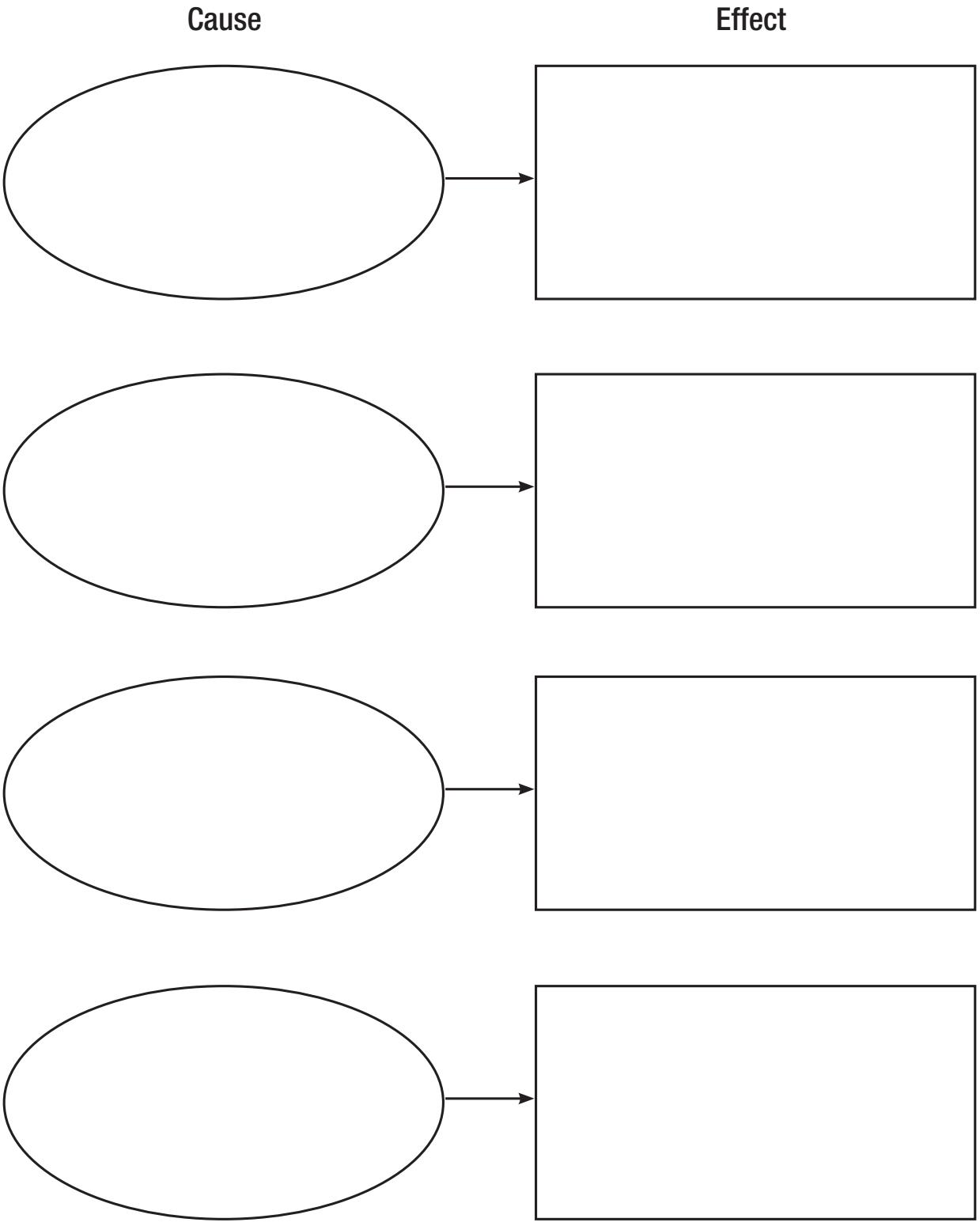
Cause and Effect Map



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Frayer Model

Definition	Characteristics
WORD	
Examples from Life	Non-Examples

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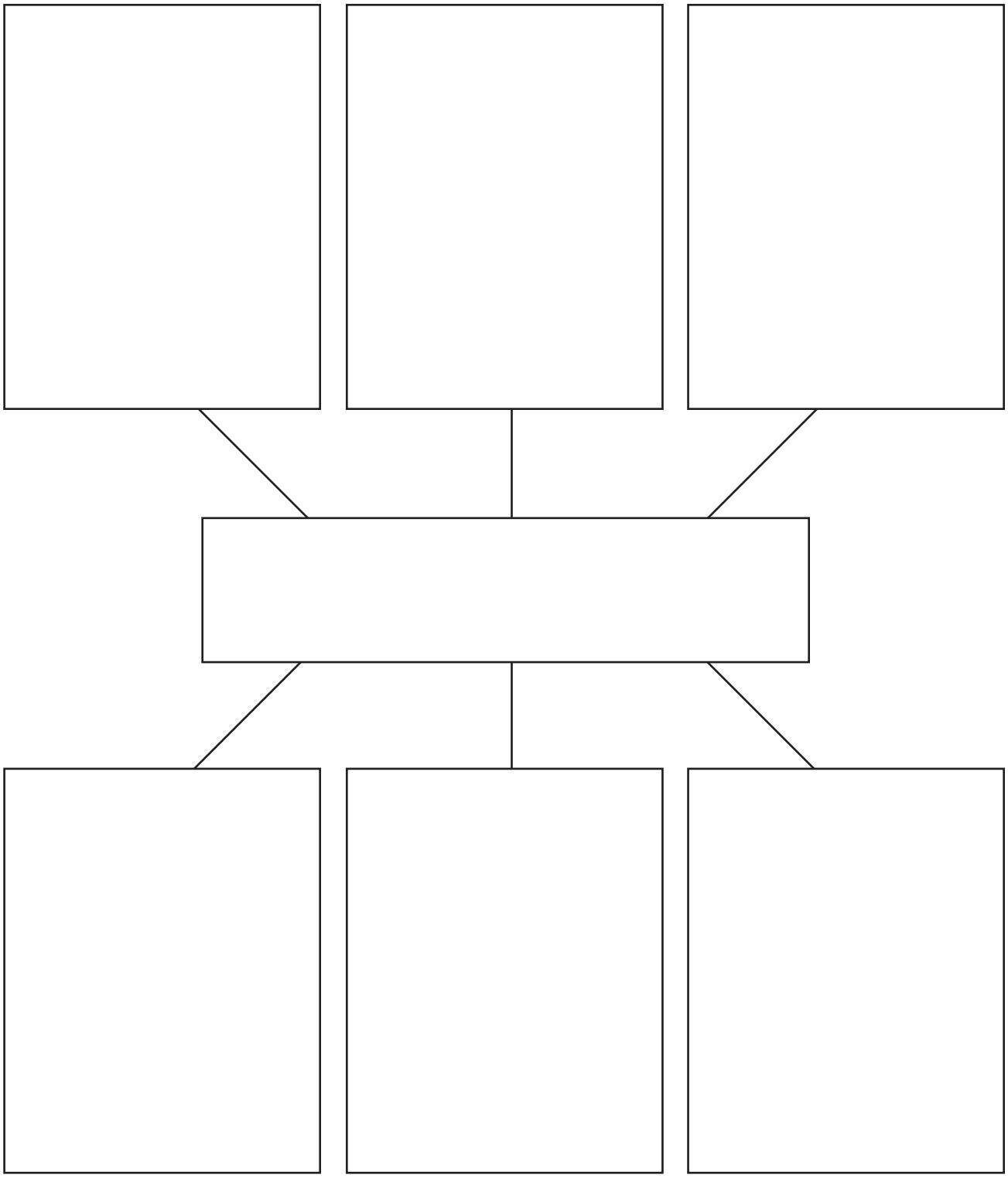
Semantic Map/Concept Map



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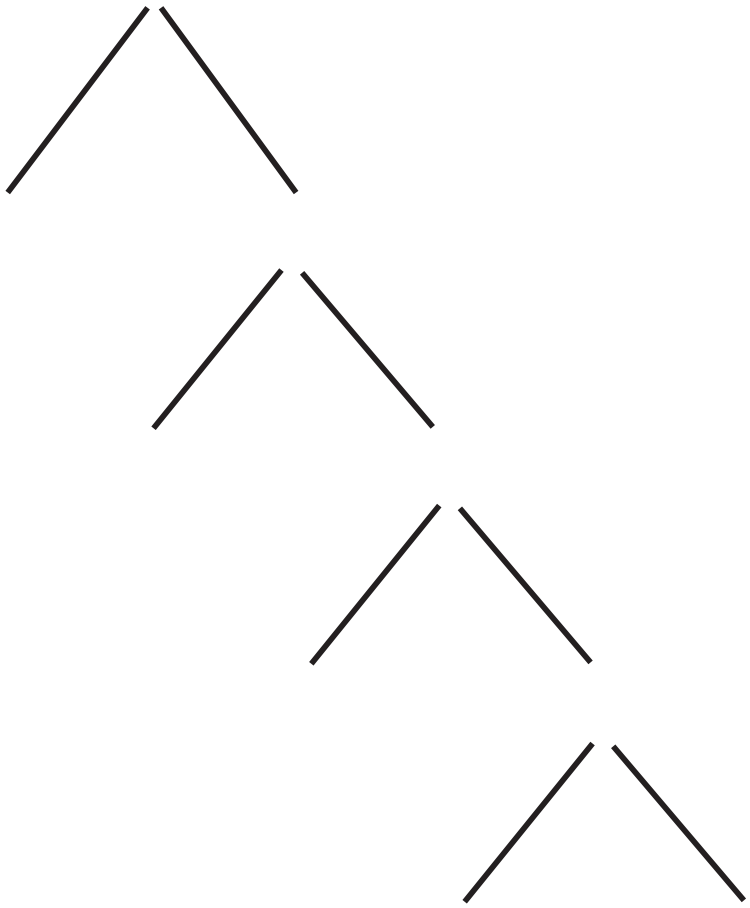
Factor Tree



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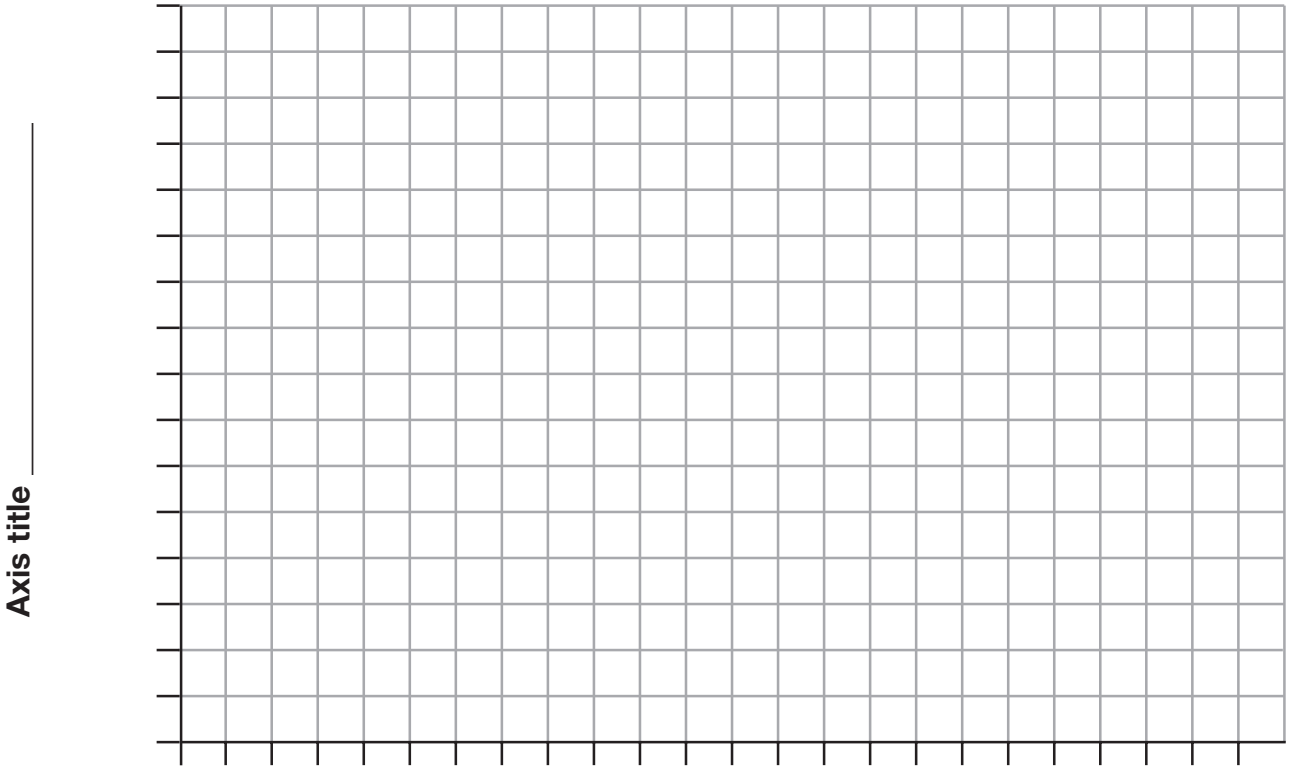
Line Graph

Graph title _____



Axis title _____

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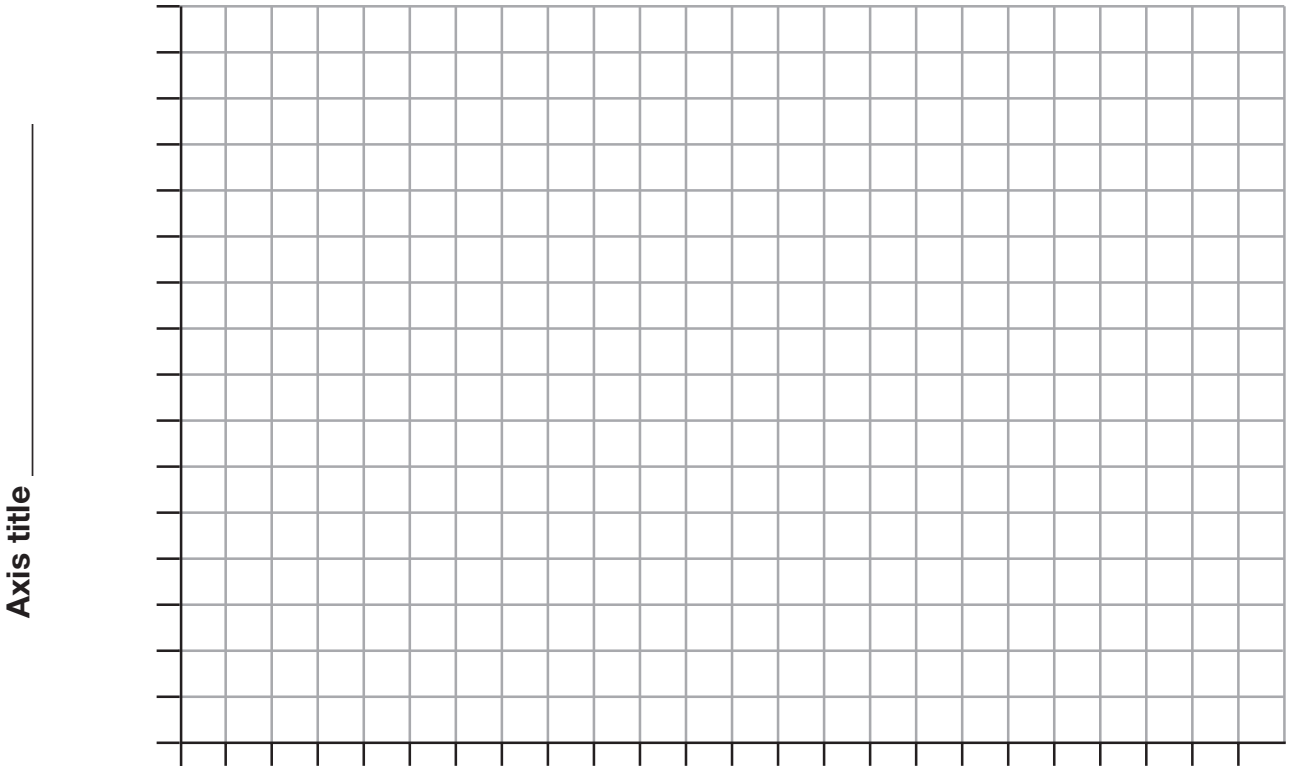
Bar Chart/Histogram

Graph title _____



KEY

- ☐
- ☐
- ☐
- ☐

Axis title _____

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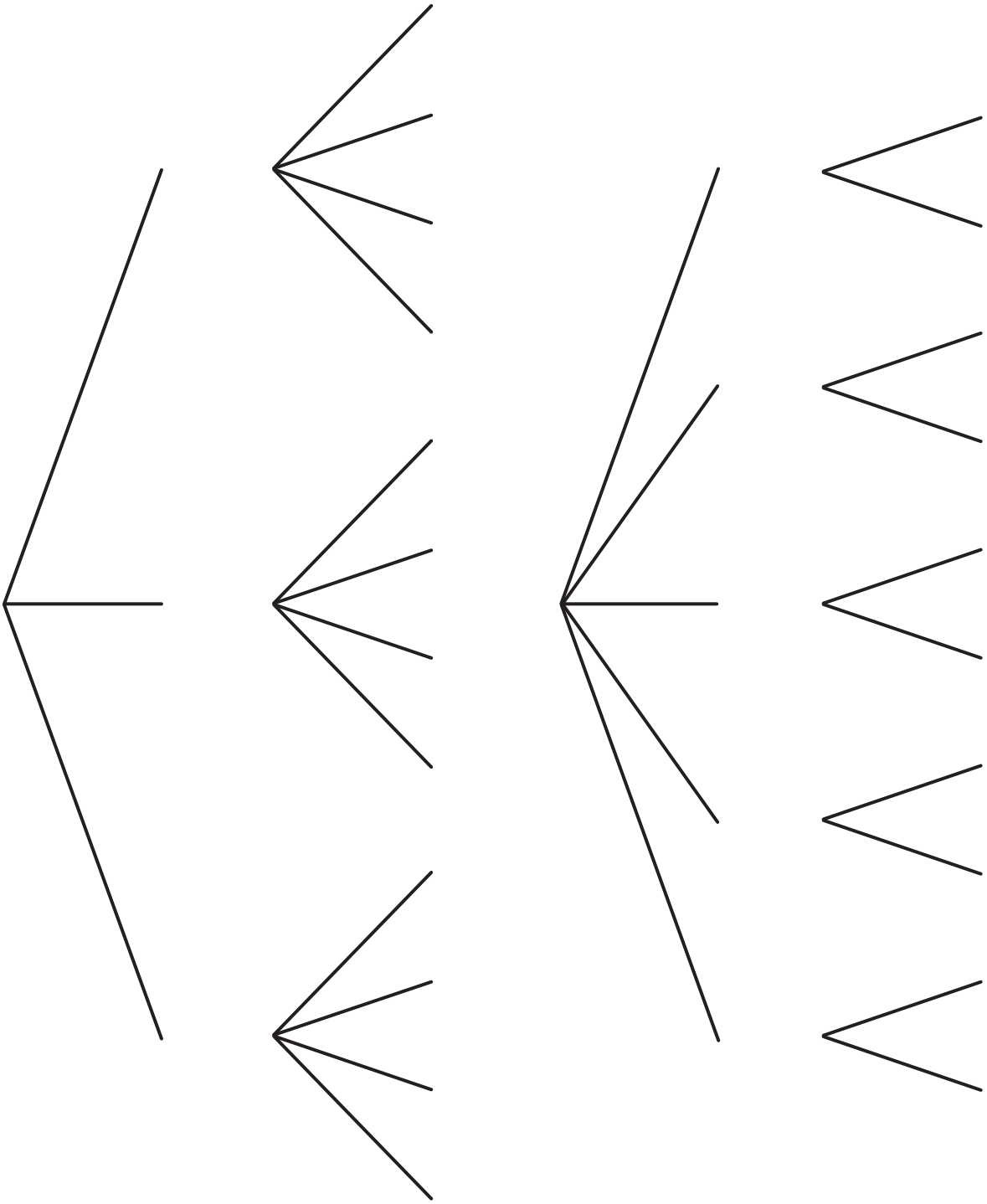
Probability Trees



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ALGEBRA

Functions	
$f(x)$	Function notation, "f of x"
$f^{-1}(x)$	Inverse function notation
$f(x) = mx + b$	Linear function
$f(x) = b^x + k$	Exponential function
$(f+g)(x) = f(x) + g(x)$	Addition
$(f-g)(x) = f(x) - g(x)$	Subtraction
$(f \bullet g)(x) = f(x) \bullet g(x)$	Multiplication
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	Division
$\frac{f(b)-f(a)}{b-a}$	Average rate of change
$f(-x) = -f(x)$	Odd function
$f(-x) = f(x)$	Even function
$f(x) = \lfloor x \rfloor$	Floor/greatest integer function
$f(x) = \lceil x \rceil$	Ceiling/least integer function
$f(x) = a\sqrt[3]{(x-h)} + k$	Cube root function
$f(x) = a\sqrt[n]{(x-h)} + k$	Radical function
$f(x) = a\lvert x-h \rvert + k$	Absolute value function
$f(x) = \frac{p(x)}{q(x)};\ \ q(x) \neq 0$	Rational function

Symbols	
$\approx$	Approximately equal to
$\neq$	Is not equal to
$\lvert a \rvert$	Absolute value of a
$\sqrt{a}$	Square root of a
∞	Infinity
[	Inclusive on the lower bound
]	Inclusive on the upper bound
(	Non-inclusive on the lower bound
)	Non-inclusive on the upper bound

Linear Equations	
$m = \frac{y_2 - y_1}{x_2 - x_1}$	Slope
$y = mx + b$	Slope-intercept form
$ax + by = c$	General form
$y - y_1 = m(x - x_1)$	Point-slope form

Exponential Equations	
$A = P\left(1 + \frac{r}{n}\right)^{nt}$	Compounded interest formula
Compounded...	n (number of times per year)
Yearly/annually	1
Semi-annually	2
Quarterly	4
Monthly	12
Weekly	52
Daily	365

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Quadratic Functions and Equations	
$x = \frac{-b}{2a}$	Axis of symmetry
$x = \frac{p+q}{2}$	Axis of symmetry using the midpoint of the x-intercepts
$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$	Vertex
$f(x) = ax^2 + bx + c$	General form
$f(x) = a(x - h)^2 + k$	Vertex form
$f(x) = a(x - p)(x - q)$	Factored/intercept form
$b^2 - 4ac$	Discriminant
$x^2 + bx + \left(\frac{b}{2}\right)^2$	Perfect square trinomial
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Quadratic formula
$(ax)^2 - b^2 = (ax + b)(ax - b)$	Difference of squares
$(x - h)^2 = 4p(y - k)$	Standard form for a parabola that opens up or down
$(y - k)^2 = 4p(x - h)$	Standard form for a parabola that opens right or left
$F(h, k + p)$	Focus for a parabola that opens up or down
$F(h + p, k)$	Focus for a parabola that opens right or left
$y = k - p$	Directrix for a parabola that opens up or down
$x = h - p$	Directrix for a parabola that opens right or left

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Exponential Functions	
$1 + r$	Growth factor
$1 - r$	Decay factor
$f(t) = a(1+r)^t$	Exponential growth function
$f(t) = a(1-r)^t$	Exponential decay function
$f(x) = ab^x$	Exponential function in general form

General	
(x, y)	Ordered pair
$(x, 0)$	x-intercept
$(0, y)$	y-intercept

Equations of Circles	
$(x-h)^2 + (y-k)^2 = r^2$	Standard form
$x^2 + y^2 = r^2$	Center at (0, 0)
$Ax^2 + By^2 + Cx + Dy + E = 0$	General form

Properties of Radicals
$\sqrt{ab} = \sqrt{a} \bullet \sqrt{b}$
$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Imaginary Numbers
$i = \sqrt{-1}$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$

Radicals to Rational Exponents
$\sqrt[n]{a} = a^{\frac{1}{n}}$
$\sqrt[n]{x^m} = x^{\frac{m}{n}}$

Multiplication of Complex Conjugates
$(a + bi)(a - bi) = a^2 + b^2$

Properties of Exponents	
Property	General rule
Zero Exponent	$a^0 = 1$
Negative Exponent	$b^{-\frac{m}{n}} = \frac{1}{b^{\frac{m}{n}}}$
Product of Powers	$a^m \bullet a^n = a^{m+n}$
Quotient of Powers	$\frac{a^m}{a^n} = a^{m-n}$
Power of a Power	$\left(b^m\right)^n = b^{mn}$
Power of a Product	$(bc)^n = b^n c^n$
Power of a Quotient	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

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DATA ANALYSIS

Rules and Equations	
$P(E)=\dfrac{\#\text{ of outcomes in } E}{\#\text{ of outcomes in sample space}}$	Probability of event E
$P(A\cup B)=P(A)+P(B)-P(A\cap B)$	Addition rule
$P(\overline{A})=1-P(A)$	Complement rule
$P\left(B\middle\vert A\right)=\dfrac{P(A\cap B)}{P(A)}$	Conditional probability
$P(A\cap B)=P(A)\bullet P\left(B\middle\vert A\right)$	Multiplication rule
$P(A\cap B)=P(A)\bullet P(B)$	Multiplication rule if A and B are independent
${}_nC_r=\dfrac{n!}{(n-r)!r!}$	Combination
${}_nP_r=\dfrac{n!}{(n-r)!}$	Permutation
$n!=n\bullet(n-1)\bullet(n-2)\bullet\cdots\bullet 1$	Factorial

Symbols	
$\varnothing$	Empty/null set
$\cap$	Intersection, “and”
$\cup$	Union, “or”
$\subset$	Subset
$\overline{A}$	Complement of Set A
!	Factorial
${}_nC_r$	Combination
${}_nP_r$	Permutation

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GEOMETRY

Symbols	
$\overset{\frown}{ABC}$	Major arc length
$\overset{\frown}{AB}$	Minor arc length
$\angle$	Angle
$\odot$	Circle
$\cong$	Congruent
$\overleftrightarrow{PQ}$	Line
$\overline{PQ}$	Line segment
$\overrightarrow{PQ}$	Ray
$\parallel$	Parallel
$\perp$	Perpendicular
$\bullet$	Point
$\triangle$	Triangle
$\square$	Parallelogram
A'	Prime
$^\circ$	Degrees
θ	Theta
ϕ	Phi
π	Pi

Area	
$A = lw$	Rectangle
$A = \frac{1}{2}bh$	Triangle
$A = \pi r^2$	Circle
$A = \frac{1}{2}(b_1 + b_2)h$	Trapezoid

Trigonometric Ratios		
$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$	$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$	$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$
$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$	$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$	$\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$

Trigonometric Identities
$\sin\theta = \cos(90° - \theta)$
$\cos\theta = \sin(90° - \theta)$
$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$\csc\theta = \frac{1}{\sin\theta}$
$\sec\theta = \frac{1}{\cos\theta}$
$\cot\theta = \frac{1}{\tan\theta}$
$\cot\theta = \frac{\cos\theta}{\sin\theta}$
$\sin^2\theta + \cos^2\theta = 1$

Distance Formula
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Pi Defined
$\pi = \frac{\text{circumference}}{\text{diameter}} = \frac{\text{circumference}}{2 \bullet \text{radius}}$

Pythagorean Theorem
$a^2 + b^2 = c^2$

Volume	
$V = lwh$	Rectangular prism
$V = Bh$	Prism
$V = \frac{1}{3}\pi r^2 h$	Cone
$V = \frac{1}{3}Bh$	Pyramid
$V = \pi r^2 h$	Cylinder
$V = \frac{4}{3}\pi r^3$	Sphere

Dilation
$D_k(x, y) = (kx, ky)$

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Circumference of a Circle	
$C = 2\pi r$	Circumference given the radius
$C = \pi d$	Circumference given the diameter

Converting Between Degrees and Radians
$\frac{\text{radian measure}}{\pi} = \frac{\text{degree measure}}{180}$

Inverse Trigonometric Functions
Arcsin $\theta = \sin^{-1}\theta$
Arccos $\theta = \cos^{-1}\theta$
Arctan $\theta = \tan^{-1}\theta$

Arc Length	
$s = \theta r$	Arc length (θ in radians)

Midpoint Formula
$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

MEASUREMENTS

Length
Metric
1 kilometer (km) = 1000 meters (m)
1 meter (m) = 100 centimeters (cm)
1 centimeter (cm) = 10 millimeters (mm)
Customary
1 mile (mi) = 1760 yards (yd)
1 mile (mi) = 5280 feet (ft)
1 yard (yd) = 3 feet (ft)
1 foot (ft) = 12 inches (in)

Volume and Capacity
Metric
1 liter (L) = 1000 milliliters (mL)
Customary
1 gallon (gal) = 4 quarts (qt)
1 quart (qt) = 2 pints (pt)
1 pint (pt) = 2 cups (c)
1 cup (c) = 8 fluid ounces (fl oz)

Weight and Mass
Metric
1 kilogram (kg) = 1000 grams (g)
1 gram (g) = 1000 milligrams (mg)
1 metric ton (MT) = 1000 kilograms
Customary
1 ton (T) = 2000 pounds (lb)
1 pound (lb) = 16 ounces (oz)

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English	Unit/Lesson	Español
	A	
absolute value a number's distance from 0 on a number line; the positive value of a quantity	2.4	**valor absoluto** distancia de un número a partir del 0 en una recta numérica; valor positivo de una cantidad
absolute value function a function with a variable inside an absolute value	2.4	**función de valor absoluto** función con una variable dentro de un valor absoluto
acute triangle a triangle in which all of the angles are acute (less than 90°)	5.6	**triángulo agudo** triángulo en el que todos los ángulos son agudos (menos de 90°)
Addition Rule If A and B are any two events, then the probability of A or B, denoted $P(A \text{ or } B)$, is given by: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$. Using set notation, the rule is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.	4.1	**Regla de la suma** Si A y B son dos eventos cualquiera, entonces la probabilidad de A o B, que se indicacon $P(A \text{ o } B)$, está dada por: $P(A \text{ o } B) = P(A) + P(B) - P(A \text{ y } B)$. Con el uso de notación de conjuntos, la regla es $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
adjacent angles angles that lie in the same plane and share a vertex and a common side. They have no common interior points.	5.5	**ángulos adyacentes** ángulos en el mismo plano que comparten un vértice y un lado común. No tienen puntos interiores comunes.
adjacent side the leg next to an acute angle in a right triangle that is not the hypotenuse	5.8	**lado adyacente** el cateto junto a un ángulo agudo en un triángulo rectángulo que no es la hipotenusa
alternate exterior angles angles that are on opposite sides of the transversal and lie on the exterior of the two lines that the transversal intersects	5.5	**ángulos exteriores alternos** ángulos en lados opuestos de la transversal que se sitúan en el exterior de las dos líneas que corta la transversal
alternate interior angles angles that are on opposite sides of the transversal and lie within the interior of the two lines that the transversal intersects	5.5	**ángulos interiores alternos** ángulos que están en los lados opuestos de la transversal y se ubican en el interior de las dos líneas que corta la transversal
altitude the perpendicular line from a vertex of a figure to its opposite side; height	5.4 5.9	**altitud** línea perpendicular desde el vértice de una figura hasta su lado opuesto; altura

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English	Unit/Lesson	Español
Angle-Angle (AA) Similarity Statement If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.	5.3	**Criterio de semejanza ángulo-ángulo (AA)** Si dos ángulos de un triángulo son congruentes con dos ángulos de otro triángulo, entonces los triángulos son similares.
angle bisector a ray that divides an angle into two congruent angles	5.4 6.2	**bisectriz del ángulo** semirrecta que divide un ángulo en dos ángulos congruentes
angle of depression the angle created by a horizontal line and a downward line of sight to an object that is below the observer	5.9	**ángulo de depresión** ángulo creado por una línea horizontal y una línea de mira descendente en relación a un objeto que se encuentra por debajo del observador
angle of elevation the angle created by a horizontal line and an upward line of sight to an object that is above the observer	5.9	**ángulo de elevación** ángulo creado por una línea horizontal y una línea de mira ascendente en relación a un objeto que se encuentra por encima del observador
arc part of a circle's circumference	6.1	**arco** parte de la circunferencia de un círculo
arc length the distance between the endpoints of an arc; written as $m\overset{\frown}{AB}$	6.4	**longitud de arco** distancia entre los extremos de un arco; se expresa como $m\overset{\frown}{AB}$
arccosine the inverse of the cosine function, written $\cos^{-1}\theta$ or $\arccos\theta$	5.9	**arcocoseno** inversa de la función coseno; se expresa $\cos^{-1}\theta$ o $\arccos\theta$
Archimedes a Greek mathematician, physician, engineer, and inventor who lived from 287–212 B.C.; considered to be one of the greatest mathematicians of all time	6.5	**Arquímedes** fue un matemático, físico, ingeniero e inventor griego que vivió entre 287 y 212 A.C.; se lo considera uno de los matemáticos más importantes de todos los tiempos
arcsine the inverse of the sine function, written $\sin^{-1}\theta$ or $\arcsin\theta$	5.9	**arcoseno** inversa de la función seno; se expresa $\text{sen}^{-1}\theta$ o $\text{arcsen}\,\theta$
arctangent the inverse of the tangent function, written $\tan^{-1}\theta$ or $\arctan\theta$	5.9	**arcotangente** inversa de la función tangente; se expresa $\tan^{-1}\theta$ o $\arctan\theta$
asymptote a line that a function gets closer and closer to as one of the variables increases or decreases without bound	3.5	**asíntota** una línea que una función se acerca cada vez más cerca de una de las variables aumenta o disminuye sin límite

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English	Unit/Lesson	Español
average rate of change the ratio of the difference of output values to the difference of the corresponding input values: $\frac{f(b)-f(a)}{b-a}$; a measure of how a quantity changes over some interval	2.2	**tasa de cambio promedio** proporción de la diferencia de valores de salida a la diferencia de valores correspondientes de entrada: $\frac{f(b)-f(a)}{b-a}$; medida de cuánto cambia una cantidad en cierto intervalo
axis of symmetry of a parabola the line through the vertex of a parabola about which the parabola is symmetric. The equation of the axis of symmetry is $x=\frac{-b}{2a}$.	2.1 3.3 6.7	**eje de simetría de una parábola** línea que atraviesa el vértice de una parábola sobre la que la parábola es simétrica. La ecuación del eje de simetría es $x=\frac{-b}{2a}$.
	B	
base the quantity that is being raised to an exponent in an exponential expression; in a^x, a is the base; or, the quantity that is raised to an exponent which is the value of the logarithm, such as 2 in the equation $\log_2 g(x) = 3 - x$	1.1 5.6	**base** cantidad que es elevada a un exponente en una expresión exponencial; en a^x, a es la base; o, la cantidad que se eleva a un exponente que es el valor del logaritmo, tal que 2 en la ecuación $\log_2 g(x) = 3 - x$
base angle an angle formed by the base and one congruent side of an isosceles triangle	5.6	**ángulo base** ángulo formado por la base y un lado congruente de un triángulo isósceles
binomial a polynomial with two terms	3.1	**binomio** polinomio con dos términos
bisect to cut in half	6.5	**bisecar** cortar por la mitad
	C	
Cavalieri's Principle The volumes of two objects of equal height are equal if the areas of their corresponding cross sections are in all cases equal.	6.5	**Principio de Cavalieri** Los volúmenes de dos objetos de igual altura son iguales si las superficies de sus correspondientes secciones transversales son en todos los casos iguales.

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English	Unit/Lesson	Español
ceiling function also known as the least integer function; a function represented as $y = \lceil x \rceil$. For any input x, the output is the smallest integer greater than or equal to x; for example, $\lceil -3 \rceil = -3$, $\lceil 2.1 \rceil = 3$, and $\lceil -2.1 \rceil = -2$.	2.4	**función techo** también conocida como función del mínimo entero; función representada como $y = \lceil x \rceil$. Para cualquier entrada x, la salida es el entero más pequeño mayor que o igual a x; por ejemplo, $\lceil -3 \rceil = -3$, $\lceil 2.1 \rceil = 3$, y $\lceil -2.1 \rceil = -2$.
center of a circle the point in the plane of the circle from which all points on the circle are equidistant. The center is not part of the circle; it is in the interior of the circle.	6.6	**centro de un círculo** punto en el plano del círculo desde el cual son equidistantes todos los puntos del círculo. El centro no es parte del círculo: se encuentra en el interior del círculo.
center of dilation a point through which a dilation takes place; all the points of a dilated figure are stretched or compressed through this point	5.2	**centro de dilatación** punto a través del cual se produce una dilatación; todos los puntos de una figura dilatada se alargan o comprimen a través de este punto
central angle an angle with its vertex at the center of a circle	6.1 6.4	**ángulo central** ángulo con su vértice en el centro de un círculo
centroid the intersection of the medians of a triangle	5.6	**centroide** intersección de las medianas de un triángulo
chord a segment whose endpoints lie on the circumference of the circle	6.1	**cuerda** segmento cuyos extremos se ubican en la circunferencia del círculo
circle the set of all points in a plane that are equidistant from a reference point in that plane, called the center. The set of points forms a two-dimensional curve that measures 360°.	3.7 6.1 6.6 6.7	**círculo** conjunto de todos los puntos de un plano equidistantes desde un punto de referencia en ese plano, denominado centro. El conjunto de puntos forma una curva bidimensional que mide 360°.
circumcenter the intersection of the perpendicular bisectors of a triangle	5.6 6.2	**circuncentro** intersección de las bisectrices perpendiculares de un triángulo
circumference the distance around a circle; $C = 2\pi r$ or $C = \pi d$, for which C represents circumference, r represents the circle's radius, and d represents the circle's diameter.	6.1 6.4	**circunferencia** distancia alrededor de un círculo; $C = 2\pi r$ o $C = \pi d$, en donde C representa la circunferencia, r representa el radio del círculo y d, su diámetro.

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circumscribed angle the angle formed by two tangent lines whose vertex is outside of the circle	6.1	**ángulo circunscrito** ángulo formado por dos líneas tangentes cuyo vértice está fuera del círculo
circumscribed circle a circle that contains all vertices of a polygon	5.6 6.2	**círculo circunscrito** círculo que contiene todos los vértices de un polígono
circumscribed triangle triangle whose sides are tangent to an interior circle	6.2	**triángulo circunscrito** triángulo cuyos lados son tangentes a un círculo interior
closed interval an interval that includes its endpoints	3.5	**intervalo cerrado** intervalo que incluye sus extremos
closure a system is closed, or shows closure, under an operation if the result of the operation is within the system	1.2	**cierre** un sistema es cerrado, o tiene cierre, en una operación si el resultado de la misma está dentro del sistema
coefficient the number multiplied by a variable in an algebraic expression	3.1	**coeficiente** número multiplicado por una variable en una expresión algebraica
cofunction a trigonometric function whose ratios have the same values when applied to the two acute angles in the same right triangle. The sine of one acute angle is the cofunction of the cosine of the other acute angle.	5.8	**cofunción** función trigonométrica cuyas proporciones tienen los mismos valores cuando se aplican a los dos ángulos agudos en el mismo triángulo rectángulo. El seno de un ángulo agudo es la cofunción del coseno del otro ángulo agudo.
collinear points points that lie on the same line	5.2	**puntos colineales** puntos que se ubican en la misma línea
combination a subset of a group of objects taken from a larger group of objects; the order of the objects does not matter, and objects may be repeated. A combination of size r from a group of n objects can be represented using the notation ${}_nC_r$, where ${}_nC_r = \frac{n!}{(n-r)!r!}$.	4.3	**combinación** subconjunto de un grupo de objetos tomado de un grupo de objetos más grande; el orden de los objetos no importa y los objetos pueden repetirse. Una combinación de tamaño r de un grupo de n objetos puede representarse con la notación ${}_nC_r$, donde ${}_nC_r = \frac{n!}{(n-r)!r!}$.

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common external tangent a tangent that is common to two circles and does not intersect the segment joining the radii of the circles	6.3	**tangente común externa** tangente común a dos círculos que no corta el segmento que une los radios de los círculos
common internal tangent a tangent that is common to two circles and intersects the segment joining the radii of the circles	6.3	**tangente común interna** tangente común a dos círculos que corta el segmento que une los radios de los círculos
common tangent a line tangent to two circles	6.3	**tangente común** recta tangente a dos círculos
complement a set whose elements are not in another set, but are in some universal set being considered. The complement of set A, denoted by $\overline{A}$, is the set of elements that are in the universal set, but not in A. The event does not occur. The probability of an event not occurring is 1 minus the probability of the event occurring, $P\left(\overline{A}\right)=1-P(A)$.	4.1	**complemento** conjunto cuyos elementos no se encuentran en otro conjunto, pero están en algún conjunto universal que se considera. El complemento del conjunto A, que se indica con $\overline{A}$, es el conjunto de elementos que se encuentran en el conjunto universal, pero no en A. El evento no se produce. La probabilidad de que un evento no se produzca es 1 menos la probabilidad de que se produzca, $P\left(\overline{A}\right)=1-P(A)$.
complementary angles two angles whose sum is 90°	5.5 5.8	**ángulos complementarios** dos ángulos cuya suma es 90°
complex conjugate the complex number that when multiplied by another complex number produces a value that is wholly real; the complex conjugate of $a + bi$ is $a - bi$	1.3	**conjugado de número complejo** número complejo que cuando se multiplica por otro número complejo produce un valor totalmente real; el conjugado complejo de $a + bi$ es $a - bi$
complex conjugates two complex numbers of the form $a + bi$ and $a - bi$	3.4	**conjugados de números complejos** dos números complejos de la forma $a + bi$ y $a - bi$
complex number a number in the form $a + bi$, where a and b are real numbers, and i is the imaginary unit	1.3 3.4	**número complejo** número en la forma $a + bi$, donde a y b son números reales e i es la unidad imaginaria
complex number system all numbers of the form $a + bi$, where a and b are real numbers, including complex numbers (neither a nor b equal 0), real numbers ($b = 0$), and imaginary numbers ($a = 0$)	1.3	**sistema de números complejos** todos los números de la forma $a + bi$, donde a y b son números reales, incluidos los números complejos (ni a ni b son iguales a 0), reales ($b = 0$) e imaginarios ($a = 0$)

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compound event the combination of two or more simple events	4.2	**evento compuesto** combinación de dos o más eventos simples
compound interest interest earned on both the initial amount and on previously earned interest	3.6	**interés compuesto** interés devengado tanto de la cantidad inicial como del interés previamente devengado
compound probability the probability of compound events	4.2	**probabilidad compuesta** probabilidad de eventos compuestos
compression a transformation in which a figure becomes smaller; compressions may be horizontal (affecting only horizontal lengths), vertical (affecting only vertical lengths), or both	5.2	**compresión** transformación en la que una figura se hace más pequeña; las compresiones pueden ser horizontales (cuando afectan sólo la longitud horizontal), verticales (cuando afectan sólo la longitud vertical), o en ambos sentidos
concave down a graph of a curve that is bent downward, such as a quadratic function with a maximum value	2.2	**cóncavo hacia abajo** gráfico de una curva que se inclina hacia abajo, tal como una función cuadrática con un valor máximo
concave polygon a polygon with at least one interior angle greater than 180° and at least one diagonal that does not lie entirely inside the polygon	5.7	**polígono cóncavo** polígono con al menos un ángulo interior de más de 180° y con al menos una diagonal que no se ubica por completo dentro de él
concave up a graph of a curve that is bent upward, such as a quadratic function with a minimum value	2.2	**cóncavo hacia arriba** gráfico de una curva que se inclina hacia arriba, tal como una función cuadrática con un valor mínimo
concavity with respect to a curve, the property of being arched upward or downward. A quadratic with positive concavity will increase on either side of the vertex, meaning that the vertex is the minimum or lowest point of the curve. A quadratic with negative concavity will decrease on either side of the vertex, meaning that the vertex is the maximum or highest point of the curve.	2.2 2.3	**concavidad** con respecto a una curva, la propiedad de ser arqueado hacia arriba o hacia abajo. Una función cuadrática con concavidad positiva se incrementará en ambos lados del vértice, lo que significa que el vértice es el punto mínimo o más bajo de la curva. Una función cuadrática con concavidad negativa disminuirá a cada lado del vértice, lo que significa que el vértice es el punto máximo o más alto de la curva.
concentric circles coplanar circles that have the same center	6.1	**círculos concéntricos** círculos coplanares que tienen el mismo centro

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concurrent lines lines that intersect at one point	5.6	**rectas concurrentes** rectas con intersección en un punto
conditional probability of *B* given *A* the probability that event *B* occurs, given that event *A* has already occurred. If *A* and *B* are two events from a sample space with $P(A) \neq 0$, then the conditional probability of *B* given *A*, denoted $P(B\|A)$, has two equivalent expressions: $P(B\|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\text{number of outcomes in } (A \text{ and } B)}{\text{number of outcomes in } A}$.	4.2	**probabilidad condicional de *B* dado *A*** la probabilidad de que el evento *B* se produzca, dado que el evento *A* ya se ha producido. Si *A* y *B* son dos eventos de un espacio muestral con $P(A) \neq 0$, entonces la probabilidad condicional de *B* dado *A*, indicado $P(B\|A)$ tiene dos expresiones equivalentes: $P(B\|A) = \frac{P(A \text{ y } B)}{P(A)} = \frac{\text{numero de resultados en } (A \text{ y } B)}{\text{numero de resultados en } A}$.
cone a solid or hollow object that tapers from a circular or oval base to a point	6.5	**cono** objeto sólido o hueco que se estrecha desde una base circular u ovalada hasta un punto
congruency transformation a transformation in which a geometric figure moves but keeps the same size and shape; a dilation where the scale factor is equal to 1	5.2	**transformación de congruencia** transformación en la cual una figura geométrica se mueve pero mantiene el mismo tamaño y la misma forma; dilatación en la que el factor de escala es igual a 1
congruent arcs two arcs that have the same measure and are either of the same circle or of congruent circles	6.1	**arcos congruentes** dos arcos que tienen la misma medida y son parte del mismo círculo o de círculos congruentes
consecutive angles angles that lie on the same side of a figure	5.7	**ángulos consecutivos** ángulos ubicados en el mismo lado de una figura
constant term a term whose value does not change	3.1	**término constante** término cuyo valor no cambia
converse of the Pythagorean Theorem If the sum of the squares of the measures of two sides of a triangle equals the square of the measure of the longest side, then the triangle is a right triangle.	5.4	**conversa del teorema de Pitágoras** Si la suma de los cuadrados de las medidas de dos lados de un triángulo equivale al cuadrado de la medida del lado más largo, entonces el triángulo es rectángulo.

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convex polygon a polygon with no interior angle greater than 180°; all diagonals lie inside the polygon	5.7	**polígono convexo** polígono sin ángulo interior de más de 180°; todas las diagonales están dentro del polígono
coordinate proof a proof that involves calculations and makes reference to the coordinate plane	5.6	**prueba de coordenadas** prueba que involucra cálculos y hace referencia al plano de coordenadas
corollary a theorem that accompanies another theorem and is usually easily deduced from the other theorem	3.4	**corolario** teorema que acompaña a otro teorema y por lo general se deduce con facilidad del primero
Corollary to the Fundamental Theorem of Algebra If $P(x)$ is a polynomial function of degree $n \geq 1$ with complex coefficients, then the related equation $P(x) = 0$ has exactly n complex solutions (roots), if a double solution is counted as two separate solutions.	3.4	**Corolario del teorema fundamental del álgebra** Si $P(x)$ es una función polinómica de grado $n \geq 1$ con coeficientes complejos, entonces la ecuación relacionada $P(x) = 0$ tiene exactamente n soluciones complejas (raíces), si una solución doble se cuenta como dos soluciones individuales.
corresponding angles angles in the same relative position with respect to the transversal and the intersecting lines	5.5	**ángulos correspondientes** ángulos en la misma posición relativa con respecto a las líneas transversal y de intersección
corresponding sides sides of two figures that lie in the same position relative to the figure. In transformations, the corresponding sides are the preimage and image sides, so $\overline{AB}$ and $\overline{A'B'}$ are corresponding sides and so on.	5.2	**lados correspondientes** lados de dos figuras que están en la misma posición relativa a la figura. En las transformaciones, los lados correspondientes son los de preimagen e imagen, entonces $\overline{AB}$ y $\overline{A'B'}$ son los lados correspondientes, etc.
cosecant the reciprocal of the sine ratio, $\csc\theta = \frac{1}{\sin\theta}$; the cosecant of $\theta = \csc\theta = \frac{\text{length of hypotenuse}}{\text{length of opposite side}}$	5.8 5.9	**cosecante** razón inversa del seno, $\csc\theta = \frac{1}{\operatorname{sen}\theta}$; la cosecante de $\theta = \csc\theta = \frac{\text{longitud de la hipotenusa}}{\text{longitud del lado opuesto}}$

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cosine a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the side adjacent to the length of the hypotenuse; the cosine of $\theta = \cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$	5.8	**coseno** función trigonométrica de un ángulo agudo en un triángulo rectángulo que es la proporción de la longitud de lado adyacente a la longitud de la hipotenusa; el coseno de $\theta = \cos\theta = \frac{\text{longitud del lado adyacente}}{\text{longitud de la hipotenusa}}$
cotangent the reciprocal of tangent, $\cot\theta = \frac{1}{\tan\theta}$; the cotangent of $\theta = \cot\theta = \frac{\text{length of adjacent side}}{\text{length of opposite side}}$	5.8 5.9	**cotangente** recíproco de la tangente, $\cot\theta = \frac{1}{\tan\theta}$; la cotangente de $\theta = \cot\theta = \frac{\text{longitud del lado adyacente}}{\text{longitud del lado opuesto}}$
critical number of a polynomial inequality an x-value that makes $f(x) = 0$, where $f(x)$ is a polynomial function and the inequality is written in any of these forms: $f(x) < 0$, $f(x) \le 0$, $f(x) > 0$, or $f(x) \ge 0$	3.5	**número crítico de una desigualdad polinómica** valor de x que hace $f(x) = 0$, donde $f(x)$ es una función polinómica y la desigualdad se expresa en cualquiera de estas formas: $f(x) < 0$, $f(x) \le 0$, $f(x) > 0$, o $f(x) \ge 0$
critical number of a rational inequality an x-value that makes $f(x) = 0$ or makes $f(x)$ undefined, where $f(x)$ is a rational function and the inequality is written in any of these forms: $f(x) < 0$, $f(x) \le 0$, $f(x) > 0$, or $f(x) \ge 0$	3.5	**número crítico de una desigualdad racional** valor de x que hace $f(x) = 0$ o $f(x)$ indefinido, donde $f(x)$ es una función racional y la desigualdad se expresa en cualquiera de estas formas: $f(x) < 0$, $f(x) \le 0$, $f(x) > 0$, o $f(x) \ge 0$
cube root For any real numbers a and b, if $a^3 = b$, then a is a cube root of b. The cube root of b is written using a radical: $\sqrt[3]{b}$.	2.4	**raíz cúbica** para cualquiera de los números reales a y b, si $a^3 = b$, entonces a es la raíz cúbica de b. La raíz cúbica de b se escribe con un radical: $\sqrt[3]{b}$.
cube root function a function that contains the cube root of a variable. The general form is $y = a\sqrt[3]{(x-h)} + k$, where a, h, and k are real numbers.	2.4	**función raíz cúbica** función que contiene la raíz cúbica de una variable. La forma general es $y = a\sqrt[3]{(x-h)} + k$, donde a, h, y k son números reales.

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curve the graphical representation of the solution set for $y = f(x)$. In the special case of a linear equation, the curve will be a line.	2.3	**curva** representación gráfica del conjunto de soluciones para $y = f(x)$. En el caso especial de una ecuación lineal, la curva será una recta.
cylinder a solid or hollow object that has two parallel bases connected by a curved surface; the bases are usually circular	6.5	**cilindro** objeto sólido o hueco que tiene dos bases paralelas conectadas por medio de una superficie curva; las bases por lo general son circulares
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decay factor $1 - r$ in the exponential decay model $f(t) = a(1 - r)^t$, or b in the exponential function $f(t) = ab^t$ if $0 < b < 1$; the multiple by which a quantity decreases over time. The general form of an exponential function modeling decay is $f(t) = a(1 - r)^t$.	2.5 3.6	**factor de decaimiento** $1 - r$ en el modelo de decaimiento exponencial $f(t) = a(1 - r)^t$, o b en la función exponencial $f(t) = ab^t$ si $0 < b < 1$; el múltiplo por el que una cantidad disminuye con el tiempo. La forma general de una función exponencial que determina decaimiento es $f(t) = a(1 - r)^t$.
decay rate r in the exponential decay model $f(t) = a(1 - r)^t$	2.5 3.6	**tasa de decaimiento** r en el modelo de decaimiento exponencial $f(t) = a(1 - r)^t$
decreasing the interval of a function for which the output values are becoming smaller as the input values are becoming larger	2.2	**decreciente** intervalo de una función por el que los valores de salida se hacen más pequeños a medida que los valores de entrada se hacen más grandes
decreasing function a function such that as the independent values increase, the dependent values decrease	2.4	**función decreciente** función en la que a medida que aumentan los valores independientes, disminuyen los dependientes
degree of a one-variable polynomial the greatest exponent attached to the variable in the polynomial	3.4	**grado de un polinomio de una variable** el mayor exponente anexado a la variable en el polinomio
dependent events events that are not independent. The outcome of one event affects the probability of the outcome of another event.	4.1	**eventos dependientes** eventos que no son independientes. El resultado de un evento afecta la probabilidad del resultado de otro.

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dependent variable labeled on the *y*-axis; the quantity that is based on the input values of the independent variable; the output variable of a function	3.5	**variable dependiente** designada en el eje de *y*; cantidad que se basa en los valores de entrada de la variable independiente; variable de salida de una función
diagonal a line that connects nonconsecutive vertices	5.7	**diagonal** línea que conecta vértices no consecutivos
diameter a straight line passing through the center of a circle connecting two points on the circle; equal to twice the radius	6.1	**diámetro** línea recta que atraviesa el centro de un círculo y conecta dos puntos en él; equivale a dos veces del radio
difference of two squares a squared number that is subtracted from another squared number	3.2	**diferencia de dos cuadrados** un número cuadrado que se resta de otro número cuadrado
dilation a transformation in which a figure is either enlarged or reduced by a scale factor in relation to a center point	5.2	**dilatación** transformación en la que una figura se amplía o se reduce por un factor de escala en relación con un punto central
directrix of a parabola a line that is perpendicular to the axis of symmetry of a parabola and that is in the same plane as both the parabola and the focus of the parabola; the fixed line referenced in the definition of a parabola	6.6 6.7	**directriz de una parábola** línea perpendicular al eje de simetría de una parábola que está en el mismo plano tanto de la parábola como de su foco; línea fija mencionada en la definición de parábola
discriminant an expression whose solved value indicates the number and types of solutions for a quadratic. For a quadratic equation in standard form ($ax^2 + bx + c = 0$), the discriminant is $b^2 - 4ac$.	3.2	**discriminante** expresión cuyo valor resuelto indica la cantidad y los tipos de soluciones para una ecuación cuadrática. En una ecuación cuadrática en forma estándar ($ax^2 + bx + c = 0$), el discriminante es $b^2 - 4ac$.
disjoint events events that have no outcomes in common. If A and B are disjoint events, then they cannot both occur. Disjoint events are also called mutually exclusive events.	4.1	**eventos disjuntos** eventos que no tienen resultados en común. Si A y B son eventos disjuntos, entonces no pueden producirse ambos. También se denominan eventos mutuamente excluyentes.
dissection breaking a figure down into its components	6.5	**disección** desglose de una figura en sus componentes

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distance formula a formula that states the distance between points (x_1, y_1) and (x_2, y_2) is equal to $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	5.1 6.6 6.7	**fórmula de distancia** fórmula que señala la distancia entre puntos (x_1, y_1) y (x_2, y_2) es igual a $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
dodecagon a 12-sided polygon	6.5	**dodecágono** polígono de 12 lados
domain the set of all input values (x-values) that satisfy the given function without restriction	2.2 2.4 3.5	**dominio** conjunto de todos los valores de entrada (valores de x) que satisfacen la función dada sin restricciones
double root two roots that are equal	3.4	**raíz doble** dos raíces que son iguales
double solution two solutions that are equal	3.4	**solución doble** dos soluciones que son iguales
	E	
element an item in a set; also called a member	4.1	**elemento** ítem en un conjunto; también se denomina miembro
empty set a set that has no elements, denoted by $\varnothing$. The empty set is also called the null set.	4.1	**conjunto vacío** conjunto que no contiene elementos, indicado con $\varnothing$. También se denomina conjunto nulo.
end behavior the behavior of the graph as x approaches positive infinity and as x approaches negative infinity	2.2 3.5	**comportamiento final** el comportamiento de la gráfica al aproximarse x a infinito positivo o a infinito negativo
enlargement a dilation of a figure where the scale factor is greater than 1	5.2	**ampliación** dilatación de una figura en la que el factor de escala es mayor que 1
equal sets sets with all the same elements	4.1	**conjuntos iguales** conjuntos con todos los mismos elementos
equiangular having equal angles	5.6	**equiangular** que tiene ángulos iguales
equidistant a point or points that lie the same distance away from a given object	5.5 6.2	**equidistante** punto o puntos que están a la misma distancia de un determinado objeto
equilateral triangle a triangle with all three sides equal in length	5.6	**triángulo equilátero** triángulo que tiene los tres lados de la misma longitud
even function a function that, when evaluated for $-x$, results in a function that is the same as the original function; $f(-x) = f(x)$	2.2	**función par** función que, cuando se la evalúa para $-x$, tiene como resultado una función que es igual a la original; $f(-x) = f(x)$

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event an outcome or set of outcomes of an experiment. An event is a subset of the sample space.	4.1	**evento** resultado o conjunto de resultados de un experimento. Un evento es un subconjunto del espacio de muestral.
expected value an estimate of value that is determined by finding the product of a total value and a probability of a given event	4.4	**valor esperado** estimación de valor que se determina al encontrar el producto de un valor total y una probabilidad de un evento determinado
experiment a process or action that has observable results. The results are called outcomes.	4.1	**experimento** proceso o acción con consecuencias observables. Las consecuencias se denominan resultados.
exponent the number which tells how many times a number is multiplied by itself; in the expression a^x, the exponent is x	1.1	**exponente** el número que indica cuántas veces un número se multiplica por sí mismo; en la expresión a^x, x es el exponente
exponential decay an exponential equation with a base, b, that is between 0 and 1 ($0 < b < 1$); can be represented by the formula $y = a(1 - r)^t$, where a is the initial value, $(1 - r)$ is the decay rate, t is time, and y is the final value	2.5 3.6	**decaimiento exponencial** ecuación exponencial con una base, b, que está entre 0 y 1 ($0 < b < 1$); puede representarse con la fórmula $y = a(1 - r)^t$, en la que a es el valor inicial, $(1 - r)$ es la tasa de decaimiento, t es el tiempo y y es el valor final
exponential decay model an exponential function, $f(t) = a(1 - r)^t$, where $f(t)$ is the final output value at the end of t time periods, a is the initial value, r is the percent decrease per time period (expressed as a decimal), and t is the number of time periods	2.5 3.6	**modelo de decaimiento exponencial** función exponencial, $f(t) = a(1 - r)^t$, en la que $f(t)$ es el valor de salida final despues de t períodos de tiempo, a es el valor inicial, r es el porcentaje de disminución por período (expresado como decimal), y t es la cantidad de períodos
exponential equation an equation that has a variable in the exponent	1.1	**ecuación exponencial** una ecuación que tiene una variable en el exponente
exponential expression an expression that contains a base and a power/exponent	1.1 3.6	**expresión exponencial** expresión que incluye una base y una potencia o exponente
exponential function a function of the form $f(x) = ab^{cx}$, in which a, b, and c are constants; a function that has a variable in the exponent, such as $f(x) = 5^x$	2.5 3.6	**función exponencial** función de la fórmula $f(x) = ab^{cx}$ en la cual a, b y c son constantes; una función que tiene una variable en el exponente, tal como $f(x) = 5^x$

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exponential growth an exponential function with a base, b, greater than 1 ($b > 1$); can be represented by the formula $f(t) = a(1 + r)^t$, where a is the initial value, $(1 + r)$ is the growth rate, t is time, and $f(t)$ is the final value	2.5 3.6	**crecimiento exponencial** función exponencial con una base, b, mayor que 1 ($b > 1$); puede representarse la fórmula $f(t) = a(1 + r)^t$, en la que a es el valor inicial, $(1 + r)$ es la tasa de crecimiento, t es el tiempo, y $f(t)$ es el valor final
exponential growth model an exponential function, $f(t) = a(1 + r)^t$, where $f(t)$ is the final output value at the end of t time periods, a is the initial value, r is the percent increase per time period (expressed as a whole number or decimal), and t is the number of time periods	2.5 3.6	**modelo de crecimiento exponencial** función exponencial, $f(t) = a(1 – r)^t$, en la que $f(t)$ es el valor de salida final despues de t períodos de tiempo, a es el valor inicial, r es el porcentaje de aumento por período (expresado como entero o decimal), y t es la cantidad de períodos
exterior angle of a polygon an angle formed by one side of a polygon and the extension of another side	5.6	**ángulo exterior de un polígono** ángulo formado por un lado de un polígono y la extensión de otro lado
exterior angles angles that lie outside a pair of parallel lines	5.5	**ángulos exteriores** ángulos que están fuera de un par de líneas paralelas
extraneous solution (extraneous root) of an equation a solution of an equation that arises during the solving process, but which is not a solution of the original equation	3.5	**solución extraña (raíz extraña) de una ecuación** solución de una ecuación que surge durante el proceso de resolución pero que no es una solución de la ecuación original
extrema the minima or maxima of a function	2.1 2.2 2.4	**extremos** los mínimos o máximos de una función
	F	
factor (noun) one of two or more numbers or expressions that when multiplied produce a given product	3.1 3.2	**factor** uno de dos o más números o expresiones que al multiplicarse dan un producto determinado
factor (verb) to write an expression as the product of its factors	3.2	**factorizar** escribir una expresión como el producto de sus factores
factored form of a quadratic function the intercept form of a quadratic equation, written as $f(x) = a(x – p)(x – q)$, where p and q are the x-intercepts of the function; also known as the *intercept form of a quadratic function*	2.1	**forma factorizada de una función cuadrática** forma de intercepto de una ecuación cuadrática, se expresa como $f(x) = a(x – p)(x – q)$, en la que p y q son los interceptos de x de la función; también se conoce como la *forma de intercepto de una función cuadrática*

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factorial the product of an integer and all preceding positive integers, represented using a ! symbol; $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. By definition, $0! = 1$.	4.3	**factorial** producto de un entero y todos los enteros positivos anteriores, que se representa con el símbolo !; $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$. Por ejemplo, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. Por definición, $0! = 1$.
family of functions a set of functions whose graphs have the same general shape as their parent function. The parent function is the function with a simple algebraic rule that represents the family of functions.	3.5	**familia de funciones** conjunto de funciones cuyos gráficos tienen la misma forma general que su función principal. La función principal es la función con una regla algebraica simple que representa la familia de funciones.
first difference in a set of data, the change in the y-value when the x-value is increased by 1	2.5	**primera diferencia** en un conjunto de datos, el cambio en el valor y cuando el valor x aumenta por 1
floor function also known as the greatest integer function; a function represented as $y = \lfloor x \rfloor$. For any input x, the output is the largest integer less than or equal to x; for example, $\lfloor -3 \rfloor = -3$, $\lfloor 2.1 \rfloor = 2$, and $\lfloor -2.1 \rfloor = -3$.	2.4	**función piso** también conocida como la función del mayor entero; función representada como $y = \lfloor x \rfloor$. Para cualquier entrada x, la salida es el entero más grande que es menor que o igual a x; por ejemplo, $\lfloor -3 \rfloor = -3$, $\lfloor 2.1 \rfloor = 2$, y $\lfloor -2.1 \rfloor = -3$.
flow proof a graphical method of presenting the logical steps used to show an argument. In a flow proof, the logical statements are written in boxes and the reason for each statement is written below the box.	5.4	**prueba de flujo** método gráfico para presentar los pasos lógicos utilizados para mostrar un argumento. En una prueba de flujo, las declaraciones lógicas se expresan en casillas y la razón de cada declaración se escribe debajo de la casilla.
focus of a parabola a fixed point on the interior of a parabola that is not on the directrix of the parabola but is on the same plane as both the parabola and the directrix; the fixed point referenced in the definition of a parabola	6.6 6.7	**foco de una parábola** punto fijo en el interior de una parábola que no está en la directriz de la parábola sino en el mismo plano que la parábola y la directriz; punto fijo mencionado en la definición de parábola
function a relation in which every element of the domain is paired with exactly one element of the range; that is, for every value of x, there is exactly one value of y.	2.3 2.7	**función** relación en la que cada elemento del dominio se empareja con un único elemento del rango; es decir, para cada valor de x, existe exactamente un valor de y.

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function notation the use of $f(x)$, which means "function of x," instead of y or another dependent variable in an equation of a function; $f(x) = 2x + 1$ and $y = 2x + 1$ are equivalent functions	2.7	**notación de funciones** el uso de $f(x)$, que significa "función de x", en lugar de y u otra variable dependiente en la ecuación de una función; $f(x) = 2x + 1$ e $y = 2x + 1$ son funciones equivalentes
Fundamental Theorem of Algebra If $P(x)$ is a polynomial function of degree $n \geq 1$ with complex coefficients, then the related equation $P(x) = 0$ has at least one complex solution (root).	3.4	**Teorema fundamental del álgebra** Si $P(x)$ es una función polinómica de grado $n \geq 1$ con coeficientes complejos, entonces la ecuación relacionada $P(x) = 0$ tiene al menos una solución compleja (raíz).
	G	
general form of an equation of a circle $Ax^2 + By^2 + Cx + Dy + E = 0$, where $A = B$, $A \neq 0$, and $B \neq 0$	6.6	**forma general de ecuación de un círculo** $Ax^2 + By^2 + Cx + Dy + E = 0$, en la que $A = B$, $A \neq 0$, y $B \neq 0$
greatest common factor (GCF) the largest factor that two or more terms share	3.2	**máximo común divisor (GCF)** el factor más grande que comparten dos o más términos
greatest integer function also known as the floor function; a function represented as $y = \lfloor x \rfloor$. For any input x, the output is the largest integer less than or equal to x; for example, $\lfloor -3 \rfloor = -3$, $\lfloor 2.1 \rfloor = 2$, and $\lfloor -2.1 \rfloor = -3$.	2.4	**función del mayor entero** también conocida como función piso; función que se representa como $y = \lfloor x \rfloor$. Para cualquier entrada x, la salida es el entero más grande que es menor que o igual a x; por ejemplo, $\lfloor -3 \rfloor = -3$, $\lfloor 2.1 \rfloor = 2$, y $\lfloor -2.1 \rfloor = -3$.
growth factor the multiple by which a quantity increases over time	2.5 3.6	**factor de crecimiento** múltiplo por el que una cantidad aumenta con el tiempo
growth rate the rate of increase in size per unit of time; r in the exponential growth model $f(t) = a(1 + r)^t$	2.5 3.6	**tasa de crecimiento** tasa de aumento de tamaño por unidad de tiempo; r en el modelo de crecimiento exponencial $f(t) = a(1 + r)^t$
	H	
half-closed interval an interval that includes one endpoint but not the other; also called a half-open interval	3.5	**intervalo medio cerrado** intervalo que incluye un punto final pero no el otro; también denominado intervalo medio abierto

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half-open interval an interval that includes one endpoint but not the other; also called a half-closed interval	3.5	**intervalo medio abierto** intervalo que incluye un punto final pero no el otro; también denominado intervalo medio cerrado
horizontal asymptote a line defined as follows: The line $y = b$ is a horizontal asymptote of the graph of a function f if $f(x)$ gets closer to b as x either increases or decreases without bound.	3.5	**asíntota horizontal** línea recta que se define de la siguiente manera: La línea $y = b$ es una asíntota horizontal del gráfico de una función f si $f(x)$ se acerca a b a medida que x aumenta o disminuye sin límites.
horizontal compression squeezing of the parabola toward the y-axis	2.6	**compresión horizontal** contracción de la parábola hacia el eje y
horizontal stretch pulling of the parabola and stretching it away from the y-axis	2.6	**estiramiento horizontal** jalar de la parábola y estirarla lejos del eje y
hypotenuse the side opposite the vertex of the 90° angle in a right triangle	5.8	**hipotenusa** lado opuesto al vértice del ángulo de 90° en un triángulo rectángulo
	I	
identity an equation that is true regardless of what values are chosen for the variables	3.4 5.8 5.9	**identidad** ecuación verdadera independientemente de los valores elegidos para las variables
imaginary number any number of the form bi, where b is a real number, $i = \sqrt{-1}$, and $b \neq 0$	1.3 3.4	**número imaginario** cualquier número de la forma bi, en el que b es un número real, $i = \sqrt{-1}$, y $b \neq 0$
imaginary unit, *i* the letter i, used to represent the non-real value, $i = \sqrt{-1}$	1.3 3.4	**unidad imaginaria, *i*** la letra i, utilizada para representar el valor no real $i = \sqrt{-1}$
incenter the intersection of the angle bisectors of a triangle	5.6 6.2	**incentro** intersección de las bisectrices del ángulo de un triángulo
increasing the interval of a function for which the output values are becoming larger as the input values are becoming larger	2.2	**creciente** intervalo de una función para el que los valores de salida se hacen más grandes a medida que los valores de entrada también se vuelven más grandes
increasing function a function such that as the independent values increase, the dependent values also increase	2.4	**función creciente** función en la que a medida que aumentan los valores independientes, también aumentan los valores dependientes

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independent events events such that the outcome of one event does not affect the probability of the outcome of another event	4.1	**eventos independientes** eventos en los que el resultado de un evento no afecta la probabilidad del resultado de otro evento
independent variable labeled on the x-axis; the quantity that changes based on values chosen; the input variable of a function	3.5	**variable independiente** designada en el eje x; cantidad que cambia según los valores seleccionados; variable de entrada de una función
infinity going on without bound; represented by the symbol ∞	3.5	**infinito** continuación sin límites; se representa con el símbolo ∞
inflection point a point on a curve at which the sign of the curvature (i.e., the concavity) changes	2.2	**punto de inflexión** punto en una curva en el que cambia el signo de la curvatura (es decir, la concavidad)
inscribed angle an angle formed by two chords whose vertex is on the circle	6.1	**ángulo inscrito** ángulo formado por dos cuerdas cuyo vértice está en el círculo
inscribed circle a circle whose tangents form a triangle	5.6 6.2	**círculo inscrito** círculo cuyos tangentes forman un triángulo
inscribed quadrilateral a quadrilateral whose vertices are on a circle	6.2	**cuadrilátero inscrito** cuadrilátero cuyos vértices están en un círculo
inscribed triangle a triangle whose vertices are on a circle	6.2	**triángulo inscrito** triangulo cuyos vértices están en un círculo
integer a number that is not a fraction or a decimal	1.1	**entero** un número que no es una fracción ni un decimal
intercept the point at which a line intercepts the x- or y-axis	2.1	**intercepto** punto en el que una línea intercepta el eje x o y
intercept form of a quadratic function the factored form of a quadratic equation, written as $f(x) = a(x - p)(x - q)$, where p and q are the x-intercepts of the function; also known as the *factored form of a quadratic function*	2.1 3.3	**forma de intercepto de una función cuadrática** forma factorizada de una ecuación cuadrática, expresada como $f(x) = a(x - p)(x - q)$, donde p y q son los interceptos de x de la función; también se conoce como la *forma factorizada de una ecuación cuadrática*
intercepted arc an arc whose endpoints intersect the sides of an inscribed angle and whose other points are in the interior of the angle	6.1	**arco interceptado** arco cuyos extremos intersecan los lados de un ángulo inscrito y cuyos otros puntos se sitúan en el interior del ángulo

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interior angle of a polygon an angle formed by two sides of a polygon	5.6	**ángulo interior de un polígono** ángulo formado por dos lados de un polígono
interior angles angles that lie between a pair of parallel lines	5.5	**ángulos interiores** ángulos ubicados entre un par de líneas paralelas
intersection a set whose elements are each in both of two other sets. The intersection of sets A and B, denoted by $A \cap B$, is the set of elements that are in both A and B.	4.1	**intersección** conjunto cuyos elementos están todos en otros dos conjuntos. La intersección de los conjuntos A y B, indicada por $A \cap B$, es el conjunto de elementos que se encuentran tanto en A como en B.
interval the set of all real numbers between two given numbers. The two numbers on the ends are the endpoints. The endpoints might or might not be included in the interval depending on whether the interval is open, closed, or half-open/half-closed.	2.5 3.2 3.5	**intervalo** conjunto de todos los números reales entre dos números dados. Los dos números en los finales son los extremos. Los extremos podrían o no estar incluidos en el intervalo, según si el intervalo está abierto, cerrado, o medio abierto o medio cerrado.
interval notation a way of representing an interval using a pair of parentheses, a pair of brackets, or a parenthesis and a bracket	3.5	**notación de intervalos** modo de representar un intervalo con un par de paréntesis, un par de corchetes, o un paréntesis y un corchete
inverse function the function that results from switching the x- and y-variables in a given function; the inverse of $f(x)$ is written as $f^{-1}(x)$	2.7	**función inversa** función que se produce como resultado de cambiar las variables x y y en una función determinada; la inversa de $f(x)$ se expresa como $f^{-1}(x)$
inverse operation the operation that reverses the effect of another operation	2.7	**operación inversa** operación que revierte el efecto de otra
irrational number a number that cannot be written as $\frac{m}{n}$, where m and n are integers and $n \neq 0$; any number that cannot be written as a decimal that ends or repeats	1.1 3.2 6.5	**números irracionales** un número que no pueden expresarse como $\frac{m}{n}$, en los que m y n son enteros y $n \neq 0$; cualquier número que no puede expresarse como decimal finito o periódico
isosceles trapezoid a trapezoid with one pair of opposite parallel lines and congruent legs	5.7	**trapezoide isósceles** trapezoide con un par de líneas paralelas opuestas y catetos congruentes
isosceles triangle a triangle with at least two congruent sides	5.6	**triángulo isósceles** triángulo con al menos dos lados congruentes

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	K	
key features of a quadratic the *x*-intercepts, *y*-intercept, where the function is increasing and decreasing, where the function is positive and negative, relative minimums and maximums, symmetries, and end behavior of the function used to describe, draw, and compare quadratic functions	2.2 3.3	**características clave de una función cuadrática** interceptos de *x*, intercepto de *y*, donde la función aumenta y disminuye, donde la función es positiva y negativa, máximos y mínimos relativos, simetrías y comportamiento final de la función utilizado para describir, dibujar y comparar las funciones cuadráticas
kite a quadrilateral with two distinct pairs of congruent sides that are adjacent	5.7	**cometa** cuadrilátero con dos pares distintos de lados congruentes que son adyacentes
	L	
leading coefficient the coefficient of the term with the highest power. For a quadratic equation in standard form ($y = ax^2 + bx + c$), the leading coefficient is a.	2.3 3.2	**coeficiente líder** coeficiente del término con la mayor potencia. En una ecuación cuadrática en forma estándar ($y = ax^2 + bx + c$), el coeficiente líder es a.
least common denominator (LCD) of fractions the least common multiple of the denominators of the fractions	3.5	**mínimo común denominador (LCD) de fracciones** múltiplo mínimo común de los denominadores de las fracciones
least common multiple (LCM) of polynomials with two or more polynomials, the common multiple of the polynomials that has the least degree and the least positive constant factor	3.5	**mínimo común múltiplo (LCM) de polinomios** con dos o más polinomios, el múltiplo común de los polinomios que tiene el menor grado y el menor factor constante positivo
least integer function also known as the ceiling function; a function represented as $y = \lceil x \rceil$. For any input x, the output is the smallest integer greater than or equal to x; for example, $\lceil -3 \rceil = -3$, $\lceil 2.1 \rceil = 3$, and $\lceil -2.1 \rceil = -2$.	2.4	**función de mínimo entero** también conocida como función techo; función representada como $y = \lceil x \rceil$. Para cualquier entrada x, la salida es el entero más pequeño mayor que o igual a x; por ejemplo, $\lceil -3 \rceil = -3$, $\lceil 2.1 \rceil = 3$, y $\lceil -2.1 \rceil = -2$.
legs congruent sides of an isosceles triangle	5.6	**catetos** lados congruentes de un triángulo isósceles
like terms terms that contain the same variables raised to the same power	1.2 3.1	**términos semejantes** términos que contienen las mismas variables elevadas a la misma potencia

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limit the value that a sequence approaches as a calculation becomes more and more accurate	6.5	**límite** valor al que se aproxima una secuencia cuando un cálculo se vuelve cada vez más exacto
line segment a part of a line that is noted by two endpoints, (x_1, y_1) and (x_2, y_2)	5.1	**segmento de recta** parte de una línea comprendida entre dos extremos, (x_1, y_1) y (x_2, y_2)
linear function a function that can be written in the form $f(x) = mx + b$, in which m is the slope, b is the y-intercept, and the graph is a straight line	2.5 2.7	**función lineal** función que puede expresarse en la forma $f(x) = mx + b$, en la que m es la pendiente, b es el intercepto de y, y la gráfica es una línea recta
linear pair a pair of adjacent angles whose non-shared sides form a straight angle	5.5	**par lineal** par de ángulos adyacentes cuyos lados no compartidos forman un ángulo recto
literal equation an equation that involves two or more variables	3.3	**ecuación literal** ecuación que incluye dos o más variables
	M	
major arc part of a circle's circumference that is larger than its semicircle	6.1	**arco mayor** parte de la circunferencia de un círculo que es mayor que su semicírculo
maximum the largest y-value of a quadratic equation	2.1 3.3	**máximo** el mayor valor de y de una ecuación cuadrática
median of a triangle the segment joining the vertex to the midpoint of the opposite side	5.6	**mediana de un triángulo** segmento que une el vértice con el punto medio del lado opuesto
member an item in a set; also called an element	4.1	**miembro** ítem en un conjunto; también se denomina elemento
midpoint a point on a line segment that divides the segment into two equal parts	5.1 5.6	**punto medio** punto en un segmento de recta que lo divide en dos partes iguales
midpoint formula formula that states the midpoint of a segment created by connecting (x_1, y_1) and (x_2, y_2) is given by the formula $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$	5.1 5.6 6.7	**fórmula de punto medio** fórmula que establece el punto medio de un segmento creado al conectar (x_1, y_1) con (x_2, y_2) está dado por la fórmula $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

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midsegment a line segment joining the midpoints of two sides of a figure	5.6	**segmento medio** segmento de recta que une los puntos medios de dos lados de una figura
midsegment triangle the triangle formed when all three of the midsegments of a triangle are connected	5.6	**segmento medio de un triángulo** triángulo que se forma cuando los tres segmentos medios de un triángulo están conectados
minimum the smallest *y*-value of a quadratic equation	2.1 3.3	**mínimo** el menor valor de *y* en una ecuación cuadrática
minor arc part of a circle's circumference that is smaller than its semicircle	6.1	**arco menor** parte de la circunferencia de un círculo que es menor que su semicírculo
monomial an expression with one term, consisting of a number, a variable, or the product of a number and variable(s)	1.2 3.1	**monomio** expresión con un solo término, que consiste en un número, una variable, o el producto de un número y una o más variables
Multiplication Rule the probability of two events, A and B, is $P(A \text{ and } B) = P(A) \bullet P(B\|A) = P(B) \bullet P(A\|B)$; for independent events A and B, the rule is $P(A \text{ and } B) = P(A) \bullet P(B)$.	4.2	**Regla de multiplicación** probabilidad de que dos eventos, A y B, sea $P(A \text{ y } B) = P(A) \bullet P(B\|A) = P(B) \bullet P(A\|B)$; para eventos independientes A y B, la regla es $P(A \text{ y } B) = P(A) \bullet P(B)$.
mutually exclusive events events that have no outcomes in common. If A and B are mutually exclusive events, then they cannot both occur. Mutually exclusive events are also called disjoint events.	4.1	**eventos mutuamente excluyentes** eventos que no tienen resultados en común. Si A y B son eventos mutuamente excluyentes, entonces no pueden producirse ambos. También se denominan eventos disjuntos.
	N	
neither describes a function that, when evaluated for $-x$, does not result in the opposite of the original function (odd) or the original function (even)	2.2	**ni** describe una función que, cuando se evalúa para $-x$, no tiene como resultado lo opuesto de la función original (impar) ni la función original (par)
non-rigid motion a transformation done to a figure that changes the figure's shape and/or size	5.2	**movimiento no rígido** transformación hecha a una figura que cambia su forma o tamaño

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nonadjacent angles angles that have no common vertex or common side, or have shared interior points	5.5	**ángulos no adyacentes** ángulos que no tienen vértices ni lados comunes, o que tienen puntos interiores compartidos
null set a set that has no elements, denoted by ∅. The null set is also called the empty set.	4.1	**conjunto nulo** conjunto que no tiene elementos, indicado con ∅. También se denomina conjunto vacío.
	O	
obtuse triangle a triangle with one angle that is obtuse (greater than 90°)	5.6	**triángulo obtuso** triángulo con un ángulo que es obtuso (de más de 90°)
odd function a function that, when evaluated for $-x$, results in a function that is the opposite of the original function; $f(-x) = -f(x)$	2.2	**función impar** función que, cuando se evalúa para $-x$, tiene como resultado una función que es lo opuesto a la función original; $f(-x) = -f(x)$
one-to-one a relationship wherein each point in a set of points is mapped to exactly one other point	2.7	**unívoca** relación en la que cada punto de un conjunto de puntos se corresponde con otro con exactitud
open interval an interval that does not include its endpoints	3.5	**intervalo abierto** intervalo que no incluye sus extremos
opposite side the side across from an angle	5.8	**lado opuesto** lado al otro lado de un ángulo
orthocenter the intersection of the altitudes of a triangle	5.6	**ortocentro** intersección de las alturas de un triángulo
outcome a result of an experiment	4.1	**resultado** consecuencia de un experimento
	P	
parabola the U-shaped graph of a quadratic equation; the set of all points that are equidistant from a fixed line, called the directrix, and a fixed point not on that line, called the focus. The parabola, directrix, and focus are all in the same plane. The vertex of the parabola is the point on the parabola that is closest to the directrix.	2.1 3.3 6.6 6.7	**parábola** gráfico de una ecuación cuadrática en forma de U; conjunto de todos los puntos equidistantes de una línea fija denominada directriz y un punto fijo que no está en esa línea, llamado foco. La parábola, la directriz y el foco están todos en el mismo plano. El vértice de la parábola es el punto más cercano a la directriz.
paragraph proof statements written out in complete sentences in a logical order to show an argument	5.4	**prueba de párrafo** declaraciones redactadas en oraciones completas en orden lógico para demostrar un argumento

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parallel lines lines in a plane that either do not share any points and never intersect, or share all points; written as $\overleftrightarrow{AB} \parallel \overleftrightarrow{PQ}$	5.4	**líneas paralelas** líneas en un plano que no comparten ningún punto y nunca se cortan, o que comparten todos los puntos; se expresan como $\overleftrightarrow{AB} \parallel \overleftrightarrow{PQ}$
parallelogram a special type of quadrilateral with two pairs of opposite sides that are parallel; denoted by the symbol $\square$	5.7	**paralelogramo** un tipo especial de cuadrilátero con dos pares de lados opuestos paralelos; se expresa con el símbolo $\square$
parent function a function with a simple algebraic rule that represents a family of functions. The graphs of the functions in the family have the same general shape as the parent function.	3.5	**función principal** función con una regla algebraica simple que representa una familia de funciones. Los gráficos de las funciones en la familia tienen la misma forma general que la función principal.
percent of change $\frac{\text{amount of change}}{\text{original amount}}$, written as a percent	3.6	**porcentaje de cambio** se expresa como porcentaje $\frac{\text{porcentaje de cambio}}{\text{cantidad original}}$
perfect square the product of an integer and itself	3.2	**cuadrado perfecto** el producto de un número entero multiplicado por sí mismo
perfect square trinomial a trinomial of the form $x^2 + bx + \left(\frac{b}{2}\right)^2$ that can be written as the square of a binomial	3.2 6.6 6.7	**trinomio cuadrado perfecto** trinomio de la forma $x^2 + bx + \left(\frac{b}{2}\right)^2$ que puede expresarse como el cuadrado de un binomio
permutation a selection of objects where the order matters and is found either using n^r, if repetitions are allowed, or by using ${}_nP_r = \frac{n!}{(n-r)!}$, where n is the number of objects to select from and r is the number of objects being selected and ordered.	4.3	**permutación** selección de objetos en la que el orden importa y se encuentra con el uso de n^r, si se permiten las repeticiones, o con ${}_nP_r = \frac{n!}{(n-r)!}$, donde n es la cantidad de objetos de donde seleccionar y r es la cantidad de objetos seleccionados y ordenados.
perpendicular bisector a line that intersects a segment at its midpoint at a right angle	5.5 6.2	**bisectriz perpendicular** línea que corta un segmento en su punto medio en ángulo recto

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perpendicular lines two lines that intersect at a right angle (90°). The lines form four adjacent and congruent right angles.	5.5	**líneas perpendiculares** dos líneas que se cortan en un ángulo recto (90°). Las líneas forman cuatro ángulos rectos adyacentes y congruentes.
***phi* (ϕ)** a Greek letter sometimes used to refer to an unknown angle measure	5.8	***fi* (ϕ)** letra del alfabeto griego que se utiliza a veces para referirse a la medida desconocida de un ángulo
***pi* (π)** the ratio of circumference of a circle to the diameter; equal to approximately 3.14	6.1	***pi* (π)** proporción de la circunferencia de un círculo al diámetro; equivale aproximadamente a 3.14
piecewise function a function that is defined by two or more expressions on separate portions of the domain	2.4	**función por partes** función definida por dos o más expresiones en porciones separadas del dominio
plane a flat, two-dimensional figure without depth that has at least three non-collinear points and extends infinitely in all directions	5.5	**plano** figura plana, bidimensional, sin profundidad, que tiene al menos tres puntos no colineales y se extiende infinitamente en todas direcciones
point of concurrency a single point of intersection of three or more lines	5.6 6.2	**punto de concurrencia** punto único de intersección de tres o más líneas
point of tangency the only point at which a line and a circle intersect	6.3	**punto de tangencia** punto único de intersección entre una línea y un círculo
point(s) of intersection the ordered pair(s) where graphed functions intersect on a coordinate plane; these are also the solutions to systems of equations	3.7	**puntos de intersección** pares ordenados en los que se intersecan funciones representadas en gráficos en un plano de coordenadas; son también las soluciones a sistemas de ecuaciones
polyhedron a three-dimensional object that has faces made of polygons	6.5	**poliedro** objeto tridimensional que tiene caras compuestas por polígonos
polynomial a monomial or the sum of monomials	1.2 3.1	**polinomio** monomio o suma de monomios
polynomial function a function whose rule is a one-variable polynomial; $P(x)$ is a polynomial function if $P(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1 x+a_0$, where n is a nonnegative integer and $a_n \neq 0$	3.4	**función polinómica** función cuya regla es un polinomio de una variable; $P(x)$ es una función polinómica si $P(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1 x+a_0$, donde n es un entero no negativo y $a_n \neq 0$

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postulate a true statement that does not require a proof	5.5	**postulado** declaración verdadera que no requiere prueba
power the result of raising a base to an exponent; 32 is a power of 2 since $2^5 = 32$	1.1	**potencia** el resultado de elevar una base a un exponente; 32 es una potencia de 2 desde $2^5 = 32$
prime an expression that cannot be factored	3.2	**número primo** expresión que no puede ser factorizada
prime factor a factor that is a prime number	3.2	**factor primario** un factor que es un número primo
prime number a whole number greater than 1, whose only two whole-number factors are 1 and itself	3.2	**número primo** un número entero mayor que 1, cuyos únicos dos factores de número entero son 1 y sí mismo
probability a number from 0 to 1 inclusive or a percent from 0% to 100% inclusive that indicates how likely an event is to occur	4.1	**probabilidad** número de 0 a 1 inclusivo o porcentaje de 0% a 100% inclusivo que indica cuán probable es que se produzca un evento
probability model a mathematical model for observable facts or occurrences that are assumed to be random; a representation of a random phenomenon	4.1	**modelo de probabilidad** modelo matemático para hechos o sucesos observables que se presumen aleatorios; representación de un fenómeno aleatorio
probability of an event *E* denoted $P(E)$, and is given by $P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}$ in a uniform probability model	4.1	**probabilidad de un evento *E*** se expresa como $P(E)$, y está dado por $P(E) = \frac{\text{número de resultados en } E}{\text{número de resultados en el espacio de muestreo}}$ en un modelo de probabilidad uniforme
proof a set of justified statements organized to form a convincing argument that a given statement is true	5.4 5.5	**prueba** conjunto de declaraciones justificadas y organizadas para formar un argumento convincente de que determinada declaraciónes verdadera
proportional having a constant ratio to another quantity	5.3	**proporcional** que tiene una proporción constante con otra cantidad
pyramid a solid or hollow polyhedron object that has three or more triangular faces that converge at a single vertex at the top; the base may be any polygon	6.5	**pirámide** objeto poliedro sólido o hueco con tres o más caras triangulares que convergen en un único vértice en la parte superior; la base puede ser cualquier polígono

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Pythagorean identities trigonometric identities that are derived from the Pythagorean Theorem: $\sin^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$, and $1 + \cot^2\theta = \csc^2\theta$	5.9	**identidades Pitagóricas** identidades trigonométricas que se derivan de el teorema de Pitágoras: $\text{sen}^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$, and $1 + \cot^2\theta = \csc^2\theta$
Pythagorean Theorem a theorem that relates the length of the hypotenuse of a right triangle (c) to the lengths of its legs (a and b). The theorem states that $a^2 + b^2 = c^2$.	5.9 6.6	**Teorema de Pitágoras** teorema que relaciona la longitud de la hipotenusa de un triángulo rectángulo (c) con las longitudes de sus catetos (a y b). El teorema establece que $a^2 + b^2 = c^2$.
	Q	
quadratic equation an equation that can be written in the form $ax^2 + bx + c = 0$, where x is the variable, a, b, and c are constants, and $a \neq 0$	3.1 3.2	**ecuación cuadrática** ecuación que se puede expresar en la forma $ax^2 + bx + c = 0$, donde x es la variable, a, b, y c son constantes, y $a \neq 0$
quadratic expression an algebraic expression that can be written in the form $ax^2 + bx + c$, where x is the variable, a, b, and c are constants, and $a \neq 0$	3.1	**expresión cuadrática** expresión algebraica que se puede expresar en la forma $ax^2 + bx + c$, donde x es la variable, a, b, y c son constantes, y $a \neq 0$
quadratic formula a formula that states the solutions of a quadratic equation of the form $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. A quadratic equation in this form can have no real solutions, one real solution, or two real solutions.	3.2 3.5 3.7	**fórmula cuadrática** fórmula que establece que las soluciones de una ecuación cuadrática de la forma $ax^2 + bx + c = 0$ están dadas por $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Una ecuación cuadrática en esta forma tener ningún solución real, o tener una solución real, o dos soluciones reales.
quadratic function a function that can be written in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$. The graph of any quadratic function is a parabola.	2.1 2.5 2.7 3.3 6.6 6.7	**función cuadrática** función que puede expresarse en la forma $f(x) = ax^2 + bx + c$, donde $a \neq 0$. El gráfico de cualquier función cuadrática es una parábola.

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quadratic inequality an inequality that can be written in the form $ax^2 + bx + c < 0$, $ax^2 + bx + c \leq 0$, $ax^2 + bx + c > 0$, or $ax^2 + bx + c \geq 0$	3.2	**desigualdad cuadrática** desigualdad que puede expresarse en la forma $ax^2 + bx + c < 0$, $ax^2 + bx + c \leq 0$, $ax^2 + bx + c > 0$, o $ax^2 + bx + c \geq 0$
quadratic-linear system a system of equations in which one equation is quadratic and one is linear	3.7	**sistema lineal cuadrático** sistema de ecuaciones en el que una ecuación es cuadrática y una es lineal
quadratic polynomial in one variable a one-variable polynomial of degree 2; it can be written in the form $ax^2 + bx + c$, where $a \neq 0$	3.4	**polinomio cuadrático en una variable** polinomio de una variable de grado 2; se puede expresar en la forma $ax^2 + bx + c$, donde $a \neq 0$
quadrilateral a polygon with four sides	5.7	**cuadrilátero** polígono con cuatro lados
	R	
radian the measure of the central angle that intercepts an arc equal in length to the radius of the circle; π radians = 180°	6.4	**radián** medida del ángulo central que intercepta un arco de longitud igual al radio del círculo; π radianes = 180°
radian measure the ratio of the arc intercepted by the central angle to the radius of the circle	6.4	**medida de radián** proporción del arco interceptado por el ángulo central al radio del círculo
radical expression an expression containing a root, such as $\sqrt[5]{9}$	1.1	**expresión radical** expresión que contiene una raíz, tal como $\sqrt[5]{9}$
radical function a function with the independent variable under a root. The general form is $y = a\sqrt[n]{(x-h)} + k$, where n is a positive integer root and a, h, and k are real numbers.	2.4	**función radical** función con la variable independiente bajo una raíz. La forma general es $y = a\sqrt[n]{(x-h)} + k$, donde n es una raíz de entero positivo y a, h, y k son números reales.
radius the distance from the center to a point on the circle; equal to one-half the diameter	6.1 6.6	**radio** distancia desde el centro a un punto en el círculo; equivale a la mitad del diámetro
random number generator a tool to select a number without following a pattern, where the probability of any number in the set being generated is equal	4.4	**generador de números aleatorios** herramienta para seleccionar un número sin seguir un patrón, por lo que la probabilidad de generar cualquier número del conjunto es igual
range the set of all outputs of a function; the set of y-values that are valid for the function	2.4 3.5	**rango** conjunto de todas las salidas de una función; conjunto de valores de y que son válidos para la función

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rate a ratio that compares measurements with different kinds of units	3.5	**tasa** proporción que compara medidas con distintos tipos de unidades
ratio the relation between two quantities; can be expressed in words, fractions, decimals, or as a percentage	3.5 5.8	**proporción** relación entre dos cantidades; puede expresarse en palabras, fracciones, decimales o como porcentaje
ratio identities identities that define tangent and cotangent in terms of sine and cosine; the following two identities are ratio identities: $\tan\theta=\frac{\sin\theta}{\cos\theta}$ **and** $\cot\theta=\frac{\cos\theta}{\sin\theta}$	5.9	**identidades de proporciones** identidades que definen tangente y cotangente en términos de seno y el coseno; las dos identidades siguientes son identidades de proporciones: $\tan\theta=\frac{\text{sen}\,\theta}{\cos\theta}$ y $\cot\theta=\frac{\cos\theta}{\text{sen}\,\theta}$
ratio of similitude a ratio of corresponding sides; also known as the scale factor	5.3	**proporción de similitud** proporción de lados correspondientes; se conoce también como factor de escala
rational equation an algebraic equation that contains at least one rational expression	3.5	**ecuación racional** una ecuación algebraica que contiene al menos una expresión racional
rational exponent an exponent of the form $\frac{m}{n}$, where m and n are integers. If m and n are positive integers and a is a real number, then $a^{\frac{m}{n}}=\left(\sqrt[n]{a}\right)^m=\sqrt[n]{a^m}$.	3.6	**exponente racional** exponente de la forma $\frac{m}{n}$, donde m y n son enteros. Si m y n son enteros positivos y a es un número real, entonces $a^{\frac{m}{n}}=\left(\sqrt[n]{a}\right)^m=\sqrt[n]{a^m}$.
rational expression an expression made of the ratio of two polynomials, in which a variable appears in the denominator of a polynomial	3.5	**expresión racional** expresión formada por la proporción de dos polinomios, en la que aparece una variable en el denominador de un polinomio
rational function a function that can be written in the form $f(x)=\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x)\neq 0$	3.5	**función racional** función que puede expresarse en la forma $f(x)=\frac{p(x)}{q(x)}$, donde $p(x)$ y $q(x)$ son polinomios y $q(x)\neq 0$

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rational inequality an inequality that includes the ratio of two rational expressions, in which a variable appears in the denominator of at least one rational expression	3.5	**desigualdad racional** desigualdad que incluye la proporción de dos expresiones racionales, en la que aparece una variable en el denominador de al menos una expresión racional
rational number any number that can be written as $\frac{m}{n}$, where both m and n are integers and $n \neq 0$; any number that can be written as a decimal that ends or repeats	1.1 3.2	**números racionales** números que pueden expresarse como $\frac{m}{n}$, en los que m y n son enteros y $n \neq 0$; cualquier número que puede escribirse como decimal finito o periódico
real numbers the set of all rational and irrational numbers	1.1 1.3 3.2	**números reales** conjunto de todos los números racionales e irracionales
reciprocal a number that, when multiplied by the original number, has a product of 1	5.8	**recíproco** número que multiplicado por el número original tiene producto 1
reciprocal identities trigonometric identities that define cosecant, secant, and cotangent in terms of sine, cosine, and tangent: $\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\cot\theta = \frac{1}{\tan\theta}$, $\sin\theta = \frac{1}{\csc\theta}$, $\cos\theta = \frac{1}{\sec\theta}$, and $\tan\theta = \frac{1}{\cot\theta}$	5.9	**identidades recíprocas** identidades trigonométricas que definen cosecante, secante y cotangente en términos de seno, coseno y tangente: $\csc\theta = \frac{1}{\text{sen}\,\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\cot\theta = \frac{1}{\tan\theta}$, $\text{sen}\,\theta = \frac{1}{\csc\theta}$, $\cos\theta = \frac{1}{\sec\theta}$, y $\tan\theta = \frac{1}{\cot\theta}$
rectangle a special parallelogram with four right angles	5.7	**rectángulo** paralelogramo especial con cuatro ángulos rectos
reduction a dilation where the scale factor is between 0 and 1	5.2	**reducción** dilatación en la que el factor de escala está entre 0 y 1
Reflexive Property of Congruent Segments a segment is congruent to itself; $\overline{AB} \cong \overline{AB}$	5.4	**Propiedad reflexiva de congruencia de segmentos** un segmento es congruente con él mismo; $\overline{AB} \cong \overline{AB}$
relative frequency (of an event) the number of times an event occurs divided by the number of times an experiment is performed	4.1	**frecuencia relativa (de un evento)** cantidad de veces que un evento se produce dividido por la cantidad de veces que se realiza el experimento

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remote interior angles interior angles that are not adjacent to the exterior angle	5.6	**ángulos interiores remotos** ángulos interiores que no son adyacentes al ángulo exterior
restricted domain a subset of a function's defined domain	2.4	**dominio restringido** subconjunto del dominio definido de una función
restricted range a subset of a function's defined range	2.4	**rango restringido** subconjunto del rango definido de una función
rhombus a special parallelogram with all four sides congruent	5.7	**rombo** paralelogramo especial con sus cuatro lados congruentes
right angle an angle measuring 90°	5.5	**ángulo recto** ángulo que mide 90°
right triangle a triangle with one angle that measures 90°	5.6 5.8	**triángulo rectángulo** triángulo con un ángulo que mide 90°
rigid motion a transformation done to a figure that maintains the figure's shape and size or its segment lengths and angle measures	5.2	**movimiento rígido** transformación que se realiza a una figura que mantiene su forma y tamaño o las longitudes de sus segmentos y las medidas de ángulos
root the inverse of a power/exponent; the root of a number x is a number that, when multiplied by itself a given number of times, equals x	1.1	**raíz** inversa de una potencia o exponente; la raíz de un número x es un número que, multiplicado por sí mismo una cantidad determinada de veces, equivale a x
root(s) solution(s) of a quadratic equation	3.2	**raíces** soluciones de una ecuación cuadrática
	S	
same-side exterior angles angles that lie on the same side of the transversal and are outside the lines that the transversal intersects; sometimes called consecutive exterior angles	5.5	**ángulos exteriores del mismo lado** ángulos que se ubican en el mismo lado de la transversal y están fuera de las líneas que corta la transversal; a veces se denominan ángulos exteriores consecutivos
same-side interior angles angles that lie on the same side of the transversal and are in between the lines that the transversal intersects; sometimes called consecutive interior angles	5.5	**ángulos interiores del mismo lado** ángulos que se ubican en el mismo lado de la transversal y están en medio de las líneas que corta la transversal; a veces se los denomina ángulos interiores consecutivos
sample space the set of all possible outcomes of an experiment	4.1	**espacio de muestreo** conjunto de todos los resultados posibles de un experimento

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scale factor a multiple of the lengths of the sides from one figure to the transformed figure. If the scale factor is larger than 1, then the figure is enlarged. If the scale factor is between 0 and 1, then the figure is reduced.	5.2 5.8	**factor de escala** múltiplo de las longitudes de los lados de una figura a la figura transformada. Si el factor de escala es mayor que 1, entonces la figura se agranda. Si el factor de escala se encuentra entre 0 y 1, entonces la figura se reduce.
scalene triangle a triangle with no congruent sides	5.6	**triángulo escaleno** triángulo sin lados congruentes
secant the reciprocal of cosine, $\sec\theta = \frac{1}{\cos\theta}$; the secant of $\theta = \sec\theta = \frac{\text{length of hypotenuse}}{\text{length of adjacent side}}$	5.8 5.9	**secante** recíproco del coseno, $\sec\theta = \frac{1}{\cos\theta}$; secante de $\theta = \sec\theta = \frac{\text{longitud de la hipotenusa}}{\text{longitud del lado adyacente}}$
secant line a line that intersects a circle at two points	6.1	**línea secante** recta que corta un círculo en dos puntos
second difference in a set of data, the change in successive first differences	2.5	**segunda diferencia** en un conjunto de datos, el cambio en sucesivas primeras diferencias
sector a portion of a circle bounded by two radii and their intercepted arc	6.4	**sector** porción de un círculo limitado por dos radios y el arco que cortan
Segment Addition Postulate If B is between A and C, then $AB + BC = AC$. Conversely, if $AB + BC = AC$, then B is between A and C.	5.4	**Postulado de la suma de segmentos** Si B está entre A y C, entonces $AB + BC = AC$. A la inversa, si $AB + BC = AC$, entonces B se encuentra entre A y C.
semicircle an arc that is half of a circle	6.1	**semicírculo** arco que es la mitad de un círculo
set a collection or list of items	4.1	**conjunto** colección o lista de elementos
Side-Angle-Side (SAS) Similarity Statement If the measures of two sides of a triangle are proportional to the measures of two corresponding sides of another triangle and the included angles are congruent, then the triangles are similar.	5.4	**Criterio de semejanza lado-ángulo-lado (SAS)** Si las medidas de dos lados de un triángulo son proporcionales a las medidas de dos lados correspondientes de otro triángulo y los ángulos incluidos son congruentes, entonces los triángulos son similares.

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Side-Side-Side (SSS) Similarity Statement If the measures of the corresponding sides of two triangles are proportional, then the triangles are similar.	5.4	**Criterio de semejanza lado-lado-lado (SSS)** Si las medidas de los lados correspondientes de dos triángulos son proporcionales, entonces los triángulos son similares.
similar two figures that are the same shape but not necessarily the same size; the symbol for representing similarity between figures is $\sim$	5.3 5.8	**similar** dos figuras que tienen la misma forma pero no necesariamente el mismo tamaño; el símbolo para representar similitud entre figuras es $\sim$
similarity transformation a rigid motion followed by a dilation; a transformation that results in the position and size of a figure changing, but not the shape	5.3	**transformación de similitud** movimiento rígido seguido por una dilatación; transformación que tiene como resultado el cambio de posición y tamaño, pero no la forma, de una figura
simple event an event that has only one outcome; sometimes called a single event	4.2	**evento simple** evento que sólo tiene un resultado; a veces se denomina evento único
sine a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the hypotenuse; the sine of θ = $\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$	5.8	**seno** función trigonométrica de un ángulo agudo en un triángulo rectángulo que es la proporción de la longitud del lado opuesto a la longitud de la hipotenusa; sen de θ = sen $\theta = \frac{\text{longitud del lado opuesto}}{\text{longitud de la hipotenusa}}$
slope the measure of the rate of change of one variable with respect to another variable; slope = $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$; the slope in the equation $y = mx + b$ is m	2.2	**pendiente** medida de la tasa de cambio de una variable con respecto a otra; pendiente = $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$; la pendiente en la ecuación $y = mx + b$ es m
slope formula a formula that states the slope of the line through (or the line segment connecting) $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$ is $\frac{y_2 - y_1}{x_2 - x_1}$	6.7	**fórmula de pendiente** fórmula que determina la pendiente de la línea que atraviesa (o el segmento de recta que conecta) $A\ (x_1, y_1)$ y $B\ (x_2, y_2)$ es $\frac{y_2 - y_1}{x_2 - x_1}$

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sphere a three-dimensional surface that has all its points the same distance from its center	6.5	**esfera** superficie tridimensional que tiene todos sus puntos a la misma distancia de su centro
square a special parallelogram with four congruent sides and four right angles	5.7	**cuadrado** paralelogramo especial con cuatro lados congruentes y cuatro ángulos rectos
square root For any real numbers a and b, if $a^2 = b$, then a is a square root of b. The square root of b is written using a radical: $\sqrt{b}$.	2.4	**raíz cuadrada** para cualquier número real a y b, si $a^2 = b$, entonces a es la raíz cuadrada de b. La raíz cuadrada de b se expresa con un radical: $\sqrt{b}$.
square root function a function that contains a square root of a variable	2.4	**función raíz cuadrada** función que contiene una raíz cuadrada de una variable
square root of a negative number a number defined such that for any positive real number a, $\sqrt{-a} = i\sqrt{a}$.	3.4	**raíz cuadrada de un número negativo** número definido de forma tal que para cualquier número real positivo a, $\sqrt{-a} = i\sqrt{a}$.
standard form of a quadratic function a quadratic function written as $f(x) = ax^2 + bx + c$, where a is the coefficient of the quadratic term, b is the coefficient of the linear term, and c is the constant term	2.1 3.3	**forma estándar de función cuadrática** función cuadrática expresada como $f(x) = ax^2 + bx + c$, donde a es el coeficiente del término cuadrático, b es el coeficiente del término lineal, y c es el término constante
standard form of an equation of a circle $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the center and r is the radius	3.7 6.6 6.7	**forma estándar de ecuación de un círculo** $(x - h)^2 + (y - k)^2 = r^2$, donde (h, k) es el centro y r es el radio
standard form of an equation of a parabola $(x - h)^2 = 4p(y - k)$ for parabolas that open up or down; $(y - k)^2 = 4p(x - h)$ for parabolas that open right or left. For all parabolas, $p \neq 0$ and the vertex is (h, k).	6.6 6.7	**forma estándar de ecuación de una parábola** $(x - h)^2 = 4p(y - k)$ para parábolas que abren hacia arriba o hacia abajo; $(y - k)^2 = 4p(x - h)$ para parábolas que abren a la derecha o a la izquierda. Para todas las parábolas, $p \neq 0$ y el vértice es (h, k).
step function a function that is a series of disconnected constant functions	2.4	**función escalonada** función que es una serie de funciones constantes desconectadas
straight angle an angle with rays in opposite directions; i.e., a straight line	5.5	**ángulo recto** ángulo con semirrectas en direcciones opuestas; es decir, línea recta

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English	Unit/Lesson	Español
stretch a transformation in which a figure becomes larger; stretches may be horizontal (affecting only horizontal lengths), vertical (affecting only vertical lengths), or both	5.2	**ampliación** transformación en la que una figura se hace más grande; las ampliaciones pueden ser horizontales (cuando afectan sólo las longitudes horizontales), verticales (cuando afectan sólo las longitudes verticales), o en ambos sentidos
subset a set whose elements are in another set. Set A is a subset of set B, denoted by $A \subset B$, if all the elements of A are also in B.	4.1	**subconjunto** conjunto cuyos elementos están en otro conjunto. El conjunto A es un subconjunto del conjunto B, indicado por $A \subset B$, si todos los elementos de A se encuentran también en B.
substitution the replacement of a term of an equation by another term that is known to have the same value	3.7	**sustitución** reemplazo de un término de una ecuación por otro que se sabe que tiene el mismo valor
supplementary angles two angles whose sum is 180°	5.5 5.6	**ángulos suplementarios** dos ángulos cuya suma es 180°
Symmetric Property of Congruent Segments If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$.	5.4	**Propiedad simétrica de congruencia de segmentos** Si $\overline{AB} \cong \overline{CD}$, entonces $\overline{CD} \cong \overline{AB}$.
system of equations a set of equations with the same unknowns	3.7	**sistema de ecuaciones** conjunto de ecuaciones con las mismas incógnitas
	T	
tangent a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the adjacent side; the tangent of $\theta = \tan \theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$	5.8	**tangente** función trigonométrica de un ángulo agudo en un triángulo rectángulo que es la proporción de la longitud del lado opuesto a la longitud del lado adyacente; tangente de $\theta = \tan \theta = \frac{\text{longitud del lado opuesto}}{\text{longitud del lado adyacente}}$
tangent line a line that intersects a circle at exactly one point and is perpendicular to the radius of the circle	6.1 6.3	**recta tangente** línea que corta un círculo en exactamente un punto y es perpendicular al radio del círculo

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English	Unit/Lesson	Español
term a number, a variable, or the product of a number and variable(s)	1.2 3.1 3.4	**término** número, variable, o producto de un número y una o más variables
test interval for a polynomial or rational inequality in x, an interval on the x-axis formed by one or more critical numbers. The sign of the function on the test interval is the same as the sign of the function value at any x-value in the interval.	3.5	**intervalo de prueba** para una desigualdad polinómica o racional en x, intervalo en el eje x formado por uno o más números críticos. El signo de la función del intervalo de prueba es el mismo que el del valor de la función en cualquier valor de x en el intervalo.
theorem a statement that is shown to be true	5.4 6.7	**teorema** declaración que se demuestra que es verdadera
theta (θ) a Greek letter commonly used to refer to unknown angle measures	5.8	***teta* (θ)** letra griega que se utiliza por lo general para referirse a medidas de ángulos desconocidas
transformation adding or multiplying a constant to a function that changes the function's position and/or shape	2.6	**transformación** suma o multiplicación de una constante con una función que cambia la posición y/o forma de la función
Transitive Property of Congruent Segments If $\overline{AB} \cong \overline{CD}$, and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.	5.4	**Propiedad transitiva de congruencia de segmentos** Si $\overline{AB} \cong \overline{CD}$, y $\overline{CD} \cong \overline{EF}$, entonces $\overline{AB} \cong \overline{EF}$.
translation transforming a function where the shape and size of the function remain the same but the function moves horizontally and/or vertically; adding a constant to the independent or dependent variable	2.6	**traslación** transformación de una función en la que la forma y el tamaño de la función permanecen iguales pero la función se traslada en sentido horizontal y/o vertical; suma de una constante a la variable independiente o dependiente
transversal a line that intersects a system of two or more lines	5.5	**transversal** línea que corta un sistema de dos o más líneas
trapezoid a quadrilateral with exactly one pair of opposite parallel lines	5.7	**trapezoide** cuadrilátero con exactamente un par de líneas paralelas opuestas
trigonometry the study of triangles and the relationships between their sides and the angles between these sides	5.8	**trigonometría** estudio de los triángulos y las relaciones entre sus lados y los ángulos entre ellos
trinomial a polynomial with three terms	3.1	**trinomio** polinomio con tres términos

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English	Unit/Lesson	Español
two-column proof numbered statements and corresponding reasons that show the argument in a logical order	5.4	**prueba de dos columnas** declaraciones numeradas y las razones correspondientes que muestran el argumento en orden lógico
two-way frequency table a frequency table that shows two categories of characteristics, one in rows and the other in columns. Each cell value is a frequency that shows how many times two different characteristics appear together, or how often characteristics are associated with a person, object, or type of item that is being studied.	4.2	**tabla de frecuencia de dos vías** tabla de frecuencia que muestra dos categorías de características, una en filas y la otra en columnas. Cada valor de celda es una frecuencia que demuestra cuántas veces dos características diferentes aparecen juntas, o con qué frecuencia las características se asocian con una persona, objeto, o tipo de elemento que se está analizando.
	U	
uniform probability model a probability model in which all the outcomes of an experiment are assumed to be equally likely	4.1	**modelo de probabilidad uniforme** modelo de probabilidad en el que se presume que todos los resultados de un experimento son igualmente probables
union a set whose elements are in at least one of two other sets. The union of sets A and B, denoted by $A \cup B$, is the set of elements that are in either A or B or both A and B.	4.1	**unión** conjunto cuyos elementos están al menos en uno de otros dos conjuntos. La unión de los conjuntos A y B, indicada por $A \cup B$, es el conjunto de elementos que están en A o en B, o a la vez en A y B.
universal set a set of all elements that are being considered in a particular situation. In a probability experiment, the universal set is the sample space.	4.1	**conjunto universal** conjunto de todos los elementos que se consideran en una situación particular. En un experimento de probabilidad, el conjunto universal es el espacio de muestreo.
	V	
variable a letter used to represent a value or unknown quantity that can change or vary	3.1	**variable** letra que se utiliza para representar un valor o cantidad desconocida que puede cambiar o variar
Venn diagram a diagram that shows how two or more sets in a universal set are related	4.1	**diagrama de Venn** diagrama que muestra cómo se relacionan dos o más conjuntos en un conjunto universal

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English	Unit/Lesson	Español
vertex angle angle formed by the legs of an isosceles triangle	5.6	**ángulo vértice** ángulo formado por los catetos de un triángulo isósceles
vertex form a quadratic function written as $f(x) = a(x - h)^2 + k$, where the vertex of the parabola is the point (h, k); the form of a quadratic equation where the vertex can be read directly from the equation	2.1 3.3	**fórmula de vértice** función cuadrática que se expresa como $f(x) = a(x - h)^2 + k$, donde el vértice de la parábola es el punto (h, k); forma de una ecuación cuadrática en la que el vértice se puede leer directamente de la ecuación
vertex of a parabola the point on a parabola that is closest to the directrix and lies on the axis of symmetry; the point at which the curve changes direction; the maximum or minimum	2.1 2.3 3.3 6.6 6.7	**vértice de una parábola** punto en una parábola que está más cercano a la directriz y se ubica sobre el eje de simetría; punto en el que la curva cambia de dirección; el máximo o mínimo
vertical angles nonadjacent angles formed by two pairs of opposite rays	5.5	**ángulos verticales** ángulos no adyacentes formados por dos pares de semirrectas opuestas
vertical asymptote a line defined as follows: The line $x = a$ is a vertical asymptote of the graph of a function f if $f(x)$ either increases or decreases without bound as x gets closer to a.	3.5	**asíntota vertical** recta definida de la siguiente manera: La línea $x = a$ es una asíntota vertical del gráfico de una función f si $f(x)$ aumenta o disminuye sin límites a medida que x se acerca a a.
vertical compression squeezing of the parabola toward the x-axis	2.6	**compresión vertical** contracción de la parábola hacia el eje x
vertical stretch pulling of the parabola and stretching it away from the x-axis	2.6	**estiramiento vertical** jalar y estirar la parábola lejos del eje x
	W	
wholly imaginary a complex number that has a real part equal to 0; written in the form $a + bi$, where a and b are real numbers, i is the imaginary unit, $a = 0$, and $b \neq 0$: $0 + bi$	1.3	**totalmente imaginario** número complejo que tiene una parte real igual a 0; se expresa en la forma $a + bi$, donde a y b son números reales, i es la unidad imaginaria, $a = 0$, y $b \neq 0$: $0 + bi$
wholly real a complex number that has an imaginary part equal to 0; written in the form $a + bi$, where a and b are real numbers, i is the imaginary unit, $b = 0$, and $a \neq 0$: $a + 0i$	1.3	**totalmente real** número complejo que tiene una parte imaginaria igual a 0; se expresa en la forma $a + bi$, donde a y b son números reales, i es la unidad imaginaria, $b = 0$, y $a \neq 0$: $a + 0i$

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English	Unit/Lesson	Español
	X	
***x*-intercept** the point at which the graph crosses the *x*-axis; written as (*x*, 0)	2.1 3.3	**intercepto de *x*** punto en el que el gráfico cruza el eje *x*; se expresa como (*x*, 0)
	Y	
***y*-intercept** the point at which the graph crosses the *y*-axis; written as (0, *y*)	2.1 3.3	**intercepto de *y*** punto en el que el gráfico cruza el eje *y*; se expresa como (0, *y*)
	Z	
Zero Product Property If the product of two factors is 0, then at least one of the factors is 0.	3.2	**Propiedad de producto cero** Si el producto de dos factores es 0, entonces al menos uno de los factores es 0.
zeros the *x*-values of a function for which the function value is 0	3.4	**ceros** valores de *x* de una función para la que el valor de la función es 0

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Common Core State Standards

Mathematics II

Integrated Pathway

Teacher Resource

Unit 1

WALCH
INTEGRATED MATH

This program was developed and reviewed by experienced math educators who have both academic and professional backgrounds in mathematics. This ensures: freedom from mathematical errors, grade level appropriateness, freedom from bias, and freedom from unnecessary language complexity.

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Standards Correlations

Each lesson in this Integrated Pathway: Mathematics II program was written specifically to address the Common Core State Standards. Each lesson lists the standards covered in all the sub-lessons, and each sub-lesson lists the standards addressed in that particular section. In this section, you'll find a comprehensive list mapping the sub-lessons to the CCSS.

Guide to Common Core State Standards Annotation

As you use this program, you will come across a symbol included with the Common Core standards for some of the lessons and activities. These symbols are explained below.

Symbol: ⋆

Denotes: Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (⋆).

From http://www.walch.com/CCSS/00003

Symbol: (+)

Denotes: College and Career Readiness Standards

Advanced mathematics standards that are required in higher-level courses such as advanced statistics may also be included in lower-level courses. These additional standards are denoted by (+). According to the Common Core State Standards Initiative, "the evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6–8."

From http://www.walch.com/CCSS/00004

Connections to Future Courses

This section provides a map between topics introduced in each unit of this course and subsequent courses where each topic is revisited and built upon.

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CCSS INTEGRATED PATHWAY: MATHEMATICS II STANDARDS CORRELATIONS

Unit 1: Extending the Number System				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Working with the Number System**			
	1.1.1	Defining, Rewriting, and Evaluating Rational Exponents	N–RN.1 N–RN.2	U1-4
	1.1.2	Rational and Irrational Numbers and Their Properties	N–RN.2 N–RN.3	U1-17
Lesson 2	**Operating with Polynomials**			
	1.2.1	Adding and Subtracting Polynomials	A–APR.1	U1-36
	1.2.2	Multiplying Polynomials	A–APR.1	U1-48
Lesson 3	**Operating with Complex Numbers**			
	1.3.1	Defining Complex Numbers, i, and i^2	N–CN.1	U1-66
	1.3.2	Adding and Subtracting Complex Numbers	N–CN.2	U1-78
	1.3.3	Multiplying Complex Numbers	N–CN.2	U1-90

TOPICS FROM FUTURE COURSES: CCSS INTEGRATED PATHWAY MATHEMATICS II

Math II	Topic introduced	Course/unit where addressed	How addressed
Unit 1	Extending the number system to include rational exponents	Math III Unit 4A	applied to properties of logarithms
		Math III Unit 4B	extended to real-world modeling
	Operating with polynomials	Math III Unit 4B	combining functions
	Complex numbers	Math III Unit 4B	using complex numbers in graphing rational functions
		Math IV	dividing complex numbers, operating and representing complex numbers geometrically in the coordinate plane, and representing and operating with vectors

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Conceptual Activities

Use this interactive open education resource to build conceptual understanding of mathematical ideas. (*Note*: The link will be monitored and repaired or replaced as necessary.)

- Illustrative Mathematics. "Computations with Complex Numbers."

  http://www.walch.com/ca/10004

  Students will practice operations on complex numbers using the fact that $i^2 = -1$. Encourage students to examine the structure of each expression and look for shortcuts (SMP 7), as this task allows for the shortening of some tedious calculations. This task is also an excellent candidate for comparison of different approaches to the same problem.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Which expression is equivalent to $(-2)^{\frac{12}{14}}$?

   a. $\sqrt{(-2)^{14}}$
   b. $\sqrt[14]{(-2)^{12}}$
   c. $\sqrt[12]{(-2)}$
   d. $\sqrt[12]{(-2)^{14}}$

2. What is the approximate value of $\sqrt[8]{12^3}$?

   a. 2.54
   b. 36.00
   c. 754.77
   d. 1,728.00

3. What is the value of $16^{\frac{5}{4}}$?

   a. 2
   b. 4
   c. 32
   d. 1,024

4. Which sum is rational?

   a. $\sqrt[2]{25}+4^{\frac{2}{3}}$
   b. $\frac{3}{5}+6^{\frac{1}{9}}$
   c. $\sqrt[4]{81^3}+\frac{5}{9}$
   d. $\sqrt{7}+10$

5. Let $\sqrt[5]{x^2}=11$. What is the approximate value of x?

   a. 3.32
   b. 2.61
   c. 121.00
   d. 401.31

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Lesson 1: Working with the Number System

Instruction

Common Core State Standards

SMP	
1	2
3	4
5 ✓	6 ✓
7 ✓	8

N–RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.*

N–RN.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents.

N–RN.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Essential Questions

1. What is the relationship between a power and a root?
2. How can the Power of a Power Property be used to rewrite an expression with a rational exponent?
3. How can an equation of the form $x^{\frac{m}{n}} = b$ be solved?
4. How can you determine whether the sum or product of two real numbers will be rational or irrational?

WORDS TO KNOW

base the factor being multiplied together in an exponential expression; in the expression a^b, a is the base

exponent the number of times a factor is being multiplied together in an exponential expression; in the expression a^b, b is the exponent

exponential equation an equation whose independent variable is in the exponent; the general form of its equation is $f(x) = ab^x + k$, where a is the initial value, b is the base, x is the input value, k is the vertical shift, and $f(x)$ is the output. Another form is $y = ab^{\frac{x}{t}}$, where t is the interval over which y changes by a factor of b, and x is measured in the same units as t.

SRB p. U1-1

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Instruction

exponential expression	an expression containing a base and an exponent
integer	the set of positive and negative whole numbers and 0; the set {... –3, –2, –1, 0, 1, 2, 3, ...}
irrational number	a real number that cannot be written as $\frac{m}{n}$, where m and n are integers and $n \neq 0$; a non-terminating or non-repeating decimal
power	the result of raising a base to an exponent; 32 is a power of 2 since $2^5 = 32$
radical expression	an expression containing a root, such as $\sqrt[5]{9}$
rational number	a real number that can be written as $\frac{m}{n}$, where both m and n are integers and $n \neq 0$; a terminating or repeating decimal
real numbers	the set of all rational and irrational numbers
root	the inverse of a power/exponent; the root of a number x is a number that, when multiplied by itself a given number of times, equals x
whole numbers	the set of positive integers and 0: {0, 1, 2, 3, ...}

Recommended Resources

- Math-Play.com. "Rational and Irrational Numbers Game."

  http://www.walch.com/rr/00080

  Users classify numbers as rational or irrational in this fast-paced game.

- Math Warehouse. "Simplify Fraction Exponents."

  http://www.walch.com/rr/00081

  This site provides a brief explanation of properties of rational exponents as well as a few interactive practice problems.

- Oswego City School District Regents Exam Prep Center. "Rational (Fractional) Exponents."

  http://www.walch.com/rr/00082

  This site provides an overview of the rules for an alternate way to express roots and includes worked examples.

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Lesson 1.1.1: Defining, Rewriting, and Evaluating Rational Exponents

Warm-Up 1.1.1

A population at any time t can be estimated using the equation $p_t = p_0 \bullet (1 + r)^t$, where p_0 is the initial population, r is the annual growth rate, and t is the time in years from now. Today, Tinyville and Littletown both have a population of 10,000 people. Tinyville's yearly growth rate is 2.5% and Littletown's yearly growth rate is 1.8%.

1. Which town will have the larger population after 5 years?

2. What equation can be used to find the approximate population of Littletown after t years?

3. What will be the approximate population of Tinyville in 10 years?

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Instruction

Lesson 1.1.1: Defining, Rewriting, and Evaluating Rational Exponents

Common Core State Standards

N–RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.*

N–RN.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Warm-Up 1.1.1 Debrief

1. Which town will have the larger population after 5 years?

   Since both towns have the same population today, each town's population growth rate will determine which town has the larger population in 5 years. Tinyville has a higher yearly growth rate, and will have a greater population in 5 years.

2. What equation can be used to find the approximate population of Littletown after t years?

   Write an equation using Littletown's starting population and yearly growth rate. Replace p_0 with 10,000 and r with 0.018 in the equation $p_t = p_0 \bullet (1 + r)^t$. Simplify the equation.

   $p_t = 10{,}000 \bullet (1 + 0.018)^t$

   $p_t = 10{,}000 \bullet (1.018)^t$

3. What will be the approximate population of Tinyville in 10 years?

   Write an equation using Tinyville's starting population, its yearly growth rate, and the time in years. Replace p_0 with 10,000, r with 0.025, and t with 10 in the equation $p_t = p_0 \bullet (1 + r)^t$ such that $p_{10} = 10{,}000 \bullet (1 + 0.025)^{10}$. Evaluate the expression to find p_{10}.

   $p_{10} = 10{,}000 \bullet (1 + 0.025)^{10}$

   $p_{10} = 10{,}000 \bullet (1.025)^{10}$

   $p_{10} \approx 10{,}000 \bullet (1.28008)$

   $p_{10} \approx 12{,}800.8$

   The approximate population of Tinyville in 10 years will be 12,801 people.

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Instruction

Connection to the Lesson

- Students will develop a greater understanding of evaluating and simplifying expressions, including expressions with rational exponents.
- Students will apply the order of operations to evaluate exponential expressions that include other operations.
- Students will write equations that represent real-world scenarios.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- evaluating expressions using the order of operations (5.OA.1)
- evaluating exponential expressions involving integer exponents (8.EE.1)
- rewriting fractions in the simplest form (4.NF.1)
- rewriting mixed fractions as improper fractions (4.NF.3c)

Introduction

An **exponent** is a quantity that shows the number of times a factor is being multiplied together in an exponential expression. In an expression written in the form a^b, b is the exponent. So far, the exponents you have worked with have been **whole numbers**, or the set of positive integers and 0: {0, 1, 2, 3, ...}. **Integers** are positive and negative whole numbers and 0. Exponents can be negative numbers. They can also be **rational numbers**, which are real numbers that can be expressed as the ratio of two integers. Rational exponents are simply another way to write radical expressions. For example, $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[3]{x} = x^{\frac{1}{3}}$.

As we will see in this lesson, the rules and properties that apply to integer exponents also apply to rational exponents.

Key Concepts

- An **exponential expression** contains a base and a power/exponent. A **base** is the factor that is being multiplied together in an exponential expression. In other words, it is the quantity that is being raised to a power, and the power is determined by the exponent.
- A **power** is the result of raising a base to an exponent; 32 is a power of 2 since $2^5 = 32$. In the expression 2^5, 2 is the base, 5 is the exponent, and the entire expression, 2^5, is a power.
- A **radical expression** contains a root, which can be shown using the radical symbol, $\sqrt{\ }$. The **root** of a number x is a number that, when multiplied by itself a given number of times, equals x.
- The root of a function is also referred to as the inverse of an exponent, and "undoes" the exponent. For example, $\sqrt[3]{8} = 2$ and $2^3 = 8$.
- In the radical expression $\sqrt[n]{a^n}$, the nth root of the nth power of a is a, if $a \geq 0$. For example, $\sqrt[3]{2^3} = 2$.
- Roots can be expressed using rational exponents instead of radical symbols. For example, $\sqrt[n]{a} = a^{\frac{1}{n}}$ and $\sqrt[n]{x^m} = x^{\frac{m}{n}}$.

SRB p. U1-4

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Instruction

- A rational exponent is an exponent that is a rational number.
- A rational number is a real number that can be written as $\frac{m}{n}$, where both m and n are integers and $n \neq 0$.
- The denominator of the rational exponent is the root, and the numerator indicates the power. For example, $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.
- An **exponential equation** is an equation whose independent variable is in the exponent. An exponential equation can be written as $y = ab^x$, where x is the independent variable, y is the dependent variable, and a and b are real numbers. **Real numbers** are the set of all rational and irrational numbers.
- The properties of exponents apply to both integer and rational exponents.

Properties of Exponents

Words	Symbols	Numbers
Zero Exponent Property A base raised to the power of 0 is equal to 1.	$a^0 = 1$	$12^0 = 1$
Negative Exponent Property A number raised to a negative exponent is equal to the reciprocal of the number raised to a positive power.	$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}, a \neq 0, n \neq 0$	$64^{-\frac{2}{3}} = \frac{1}{64^{\frac{2}{3}}} = \frac{1}{\left(\sqrt[3]{64}\right)^2} = \frac{1}{16}$
Product of Powers Property To multiply powers with the same base, add the exponents.	$a^m \bullet a^n = a^{m+n}$	$3^{\frac{1}{4}} \bullet 3^{\frac{7}{4}} = 3^{\frac{1}{4}+\frac{7}{4}} = 3^2 = 9$
Quotient of Powers Property To divide powers with the same base, subtract the exponents.	$\frac{a^m}{a^n} = a^{m-n}$	$\frac{8^{\frac{4}{9}}}{8^{\frac{1}{9}}} = 8^{\frac{4}{9}-\frac{1}{9}} = 8^{\frac{1}{3}} = 2$

(*continued*)

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Power of a Power Property To raise one power to another power, multiply the exponents.	$(a^m)^n = a^{m \bullet n}$	$\left(5^{\frac{2}{3}}\right)^3 = 5^{\frac{2}{3} \bullet 3} = 5^2 = 25$
Power of a Product Property To find the power of a product, raise each factor in the product to the given power.	$(ab)^m = a^m b^m$	$(25 \bullet 36)^{\frac{1}{2}} = 25^{\frac{1}{2}} \bullet 36^{\frac{1}{2}} = 5 \bullet 6 = 30$
Power of a Quotient Property To find the power of a quotient, raise each term in the quotient to the given power.	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{25}{49}\right)^{\frac{1}{2}} = \frac{25^{\frac{1}{2}}}{49^{\frac{1}{2}}} = \frac{5}{7}$

- Either the power or root can be determined first when evaluating an exponential expression with a rational exponent.
- Rational exponents can be reduced to simplest form before evaluating a radical expression.
- Use absolute value when simplifying expressions with an even root or variable roots, if the base contains a variable or negative number. For example, the square root of x^2 can be written as $(x^2)^{\frac{1}{2}}$, which is equal to $|x|$.
- An even root is always positive. For example, although 2^2 and $(-2)^2$ are both equal to 4, $\sqrt{4}$ is equal to 2, not –2.
- Sometimes rational exponents appear as decimals. For example, $x^{0.25}$ is equal to $x^{\frac{1}{4}}$ or $\sqrt[4]{x}$.

Common Errors/Misconceptions

- not identifying the denominator of a rational exponent as being a root
- incorrectly evaluating an exponential expression with multiple operations

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Instruction

Guided Practice 1.1.1

Example 1

How can the exponential expression $3^{\frac{6}{5}}$ be rewritten as a radical expression?

1. Identify the power.

   The power is the numerator of the rational exponent: 6.

2. Identify the root.

   The root is the denominator of the rational exponent: 5. Since the root is odd, it is not necessary to take the absolute value of the expression.

3. Rewrite the expression in either of the following forms: $\sqrt[\text{root}]{\text{base}^{\text{power}}}$ or $\left(\sqrt[\text{root}]{\text{base}}\right)^{\text{power}}$, where the base is the quantity being raised to the rational exponent.

   $3^{\frac{6}{5}} = \sqrt[5]{3^6} = \left(\sqrt[5]{3}\right)^6$

Example 2

How can the radical expression $\sqrt[8]{a^c}$ be rewritten as an exponential expression?

1. Identify the numerator of the rational exponent.

   The numerator is the power: c.

2. Identify the denominator of the rational exponent.

   The denominator is the root: 8.

3. Rewrite the expression in the form $\text{base}^{\frac{\text{power}}{\text{root}}}$, where the base is the quantity raised to a power and of which the root is being taken.

   $\sqrt[8]{a^c} = a^{\frac{c}{8}}$

SWB p. 9; SRB p. U1-6

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Instruction

Example 3

Evaluate the expression $\sqrt[8]{4^{10}}$. Round your answer to the nearest thousandth.

1. Evaluate the power.

   $4^{10} = 1{,}048{,}576$

2. Find an exact root or approximate root using a calculator.

   Use a calculator to approximate the eighth root of 1,048,576. The root, 8, is even, so we only want the positive value of the root.

   $\sqrt[8]{1{,}048{,}576} \approx 5.657$

Example 4

A town's population is decreasing. The population in the year 2000 was 4,000, and the population t years after 2000 can be found by using the function $f(t) = 4000(0.96)^t$. What was the town's approximate population 2.5 years after the year 2000?

1. Replace the variable in the equation with the known value.

   The variable, t, is the number of years after 2000. To find the approximate population 2.5 years after 2000, replace t with 2.5.

   $f(2.5) = 4000(0.96)^{2.5}$

2. Evaluate the expression by first calculating the power.

   The base of the exponential expression is a decimal. In this case, use a calculator to approximate the population for $t = 2.5$. Since the evaluated function is a population, round to the nearest whole number.

   $f(2.5) = 4000(0.96)^{2.5} \approx 4000(0.9030) \approx 3612$

   2.5 years after the year 2000, the town's approximate population was 3,612 people.

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Problem-Based Task 1.1.1: Population Growth

SMP	
1	2
3	4
5 ✓	6 ✓
7 ✓	8

A town takes a census, or a count of its population, every 10 years. The town uses the census to estimate the population's growth rate. The town's population today is 42,000 people, and the 10-year growth rate is approximately 35%. The town's population, y, can be estimated at any year t using the equation $y = y_0 \bullet (1+r)^{\frac{t}{10}}$, where y_0 is the current population and r is the 10-year growth rate. What will be the town's approximate population 8 years from today?

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Problem-Based Task 1.1.1: Population Growth

Coaching

a. What is the general equation that you can use to estimate this population?

b. What do each of the variables in the equation represent, and what are their values?

c. What is the specific equation that models this situation?

d. How can you further simplify this equation before you estimate the population?

e. What will be the town's approximate population 8 years from today?

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Instruction

Problem-Based Task 1.1.1: Population Growth

Coaching Sample Responses

a. What is the general equation that you can use to estimate this population?

$$y = y_0 \bullet (1+r)^{\frac{t}{10}}$$

b. What do each of the variables in the equation represent, and what are their values?

y is the population in 8 years, y_0 is the population today, r is the 10-year growth rate, and t is the time in years. The value of y is unknown, y_0 is 42,000, r is 0.35, and t is 8.

c. What is the specific equation that models this situation?

$$y = 42{,}000 \bullet (1+0.35)^{\frac{8}{10}}$$

d. How can you further simplify this equation before you estimate the population?

The rational exponent can be simplified. Note that the original root is even; therefore, the solution must be positive.

$$y = 42{,}000 \bullet (1+0.35)^{\frac{8}{10}}$$

$$y = 42{,}000 \bullet (1+0.35)^{\frac{4}{5}}$$

e. What will be the town's approximate population 8 years from today?

Use a calculator to find the town's approximate population 8 years from today.

$$y = 42{,}000 \bullet (1+0.35)^{\frac{4}{5}}$$

$$y = 42{,}000 \bullet (1.35)^{\frac{4}{5}}$$

$$y \approx 42{,}000 \bullet (1.27136)$$

$$y \approx 53{,}397$$

The town's approximate population will be 53,397 people 8 years from today.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.1.1: Defining, Rewriting, and Evaluating Rational Exponents

Rewrite each exponential expression as a radical expression. Do not evaluate.

1. $5^{\frac{1}{4}}$

2. $g^{\frac{2}{9}}$

3. $-10^{\frac{3}{7}}$

Rewrite each radical expression as an exponential expression. Do not evaluate.

4. $\sqrt[2]{20^3}$

5. $\sqrt[6]{r^s}$

Evaluate each expression.

6. $5^{\frac{5}{2}}$

7. $\sqrt[4]{5^3}$

continued

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Use the information given in each scenario to solve the problems.

8. A population of bacteria is growing rapidly. The population at any hour, h, can be represented by the function $f(h) = 2 \cdot 4^h$. What is the population of bacteria after $4\frac{1}{2}$ hours?

9. A car loses value each year. The value of the car t years from today can be modeled using the function $f(t) = 15{,}000(0.85)^t$. If Elizabeth wants to sell her car in $2\frac{1}{3}$ years, what will the car's value be when she sells it?

10. Isaac deposits \$2,000 in a savings account with an annually compounded interest rate of 3%. The amount of money in the account in any year t after opening the account can be represented by the function $f(t) = 2000(1.03)^t$. Isaac plans to take all of his savings out of the account in $6\frac{3}{4}$ years. How much money will he have in savings at that time?

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Lesson 1.1.2: Rational and Irrational Numbers and Their Properties

Warm-Up 1.1.2

Ahmed started a savings account in 2000. The balance of the account is modeled by the equation $f(t) = 225(1.04)^t$, where $t = 0$ represents the year 2012.

1. What was the balance in the account in 2010?

2. What was the balance in the account in 2012?

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Instruction

Lesson 1.1.2: Rational and Irrational Numbers and Their Properties

Common Core State Standards

N–RN.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents.

N–RN.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Warm-Up 1.1.2 Debrief

1. What was the balance in the account in 2010?

   For the year 2010, the value of t is –2, since 2010 is two years before 2012.

   $f(-2) = 225(1.04)^{-2}$

   $f(-2) = 225\left(\frac{1}{1.04^2}\right)$

   $f(-2) \approx 208.03$

   In 2010, the balance was approximately \$208.03.

2. What was the balance in the account in 2012?

   For the year 2012, the value of t is 0.

   $f(0) = 225(1.04)^0$

   $f(0) = 225(1)$

   $f(0) = 225$

   In 2012, the balance in the account was \$225.00.

Connection to the Lesson

- Students will apply previously explored properties of exponents to irrational numbers.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- applying the properties of integer exponents (8.EE.1)
- understanding the real number system, including rational numbers (N–RN.1)
- differentiating between roots and powers (6.EE.1)

Introduction

The properties of exponents apply even when the result is an irrational number. In this section, we will see how the properties of exponents can be used to write equivalent exponential expressions involving irrational numbers. We will also learn how to tell if a sum or product will be rational or irrational, and we will solve equations involving irrational expressions.

Key Concepts

- An **irrational number** is a real number that cannot be expressed as the ratio of two integers. In other words, it cannot be written as a fraction that has integers for both the numerator and denominator.
- The decimal representations of irrational numbers neither end nor repeat.
- The root of a number for which there is no rational root is an irrational number, such as $\sqrt{2}$ or $\sqrt[3]{5}$.
- The set of integers is closed under addition and multiplication, meaning that the sum of any integers is an integer, and the product of any integers is also an integer.
- Equations involving a power can be solved by taking a root, and equations involving a root can be solved by raising to a power.
- In order to solve an equation involving a power or a root, isolate the variable.
- To solve an equation that has a rational exponent, raise both sides to the multiplicative inverse (reciprocal) of the exponent.
- If an expression is written using a power and a root, it can be rewritten using a rational exponent and then solved using the reciprocal of the rational exponent. Or, each operation, the root and the power, can be "undone" in separate steps.
- The same operation must be performed on both sides of the equation.

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Instruction

- In the following equations, x is the variable being solved for and a, b, m, and n represent integers, with $m \neq 0$ and $n \neq 0$.

Power:

$$x^a = b$$
$$\sqrt[a]{x^a} = \sqrt[a]{b}$$
$$x = \sqrt[a]{b}$$

Root:

$$\sqrt[a]{x} = b$$
$$\left(\sqrt[a]{x}\right)^a = b^a$$
$$x = b^a$$

Rational exponent:

$$x^{\frac{m}{n}} = b$$
$$\left(x^{\frac{m}{n}}\right)^{\frac{n}{m}} = b^{\frac{n}{m}}$$
$$x^{\frac{m}{n} \bullet \frac{n}{m}} = b^{\frac{n}{m}}$$
$$x = b^{\frac{n}{m}}$$

Rational root:

$$\sqrt[n]{x^m} = b$$
$$\left(\sqrt[n]{x^m}\right)^n = b^n$$
$$x^m = b^n$$
$$\sqrt[m]{x^m} = \sqrt[m]{b^n}$$
$$x = \sqrt[m]{b^n}$$

- The set of rational numbers is closed under addition and multiplication, so the sum of two rational numbers is rational and the product of two rational numbers is rational.
- On the other hand, the sum of a rational number and an irrational number is an irrational number, as is the product of a rational number and an irrational number. An irrational number has a decimal that never ends or repeats. So, if a rational number is added to or multiplied by an irrational number, the result will always be irrational because its decimal will never end or repeat. For example, $2 + \sqrt{2} \approx 2 + 1.41421... \approx 3.41421...$, and $2 \bullet \sqrt{2} \approx 2 \bullet 1.41421... \approx 2.82842...$.
- The sum or product of two irrational numbers can be either rational or irrational. To determine which it is, simplify the expression.

Common Errors/Misconceptions

- incorrectly categorizing a root as irrational when an exact root exists
- incorrectly categorizing a repeating or terminating decimal as irrational
- incorrectly raising a power to a power when solving an exponential equation
- trying to solve an exponential equation without first isolating the variable
- confusing the rules for the Power of a Power Property, the Power of a Product Property, and the Product of Powers Property

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Instruction

Guided Practice 1.1.2

Example 1

Simplify the expression $a^{\frac{6}{5}} \bullet a^{\frac{3}{2}}$.

1. Identify which property can be used to simplify the expression.

   This is the product of two exponential expressions with the same base. Use the Product of Powers Property to simplify.

2. Apply the property to simplify the expression.

   The Product of Powers Property states that if the bases are the same, the expression can be written as the single base raised to the sum of the powers.

   $$a^{\frac{6}{5}} \bullet a^{\frac{3}{2}} = a^{\frac{6}{5}+\frac{3}{2}} = a^{\frac{27}{10}}$$

Example 2

Simplify the expression $\dfrac{b^{\frac{7}{9}}}{b^{\frac{8}{3}}}$.

1. Identify which property can be used to simplify the expression.

   This is the quotient of two exponential expressions with the same base. Use the Quotient of Powers Property to simplify.

2. Apply the property to simplify the expression.

   The Quotient of Powers Property states that since the bases are the same, the expression can be written as the base raised to the difference between the power of the numerator and the power of the denominator. If the difference is negative, the Negative Exponent Property can be used to rewrite the expression so that the exponent is positive.

   $$\frac{b^{\frac{7}{9}}}{b^{\frac{8}{3}}} = b^{\frac{7}{9}-\frac{8}{3}} = b^{-\frac{17}{9}} \text{ or } \frac{1}{b^{\frac{17}{9}}}$$

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Instruction

Example 3

Evaluate the exponential expression $\left(3^{\frac{1}{2}}\right)^{\frac{4}{3}}$. Round your answer to the nearest thousandth, if necessary. Then, determine whether the answer is rational or irrational.

1. Simplify the expression using properties of exponents.

   An expression with a power of a power can be rewritten using the product of the powers.

   $$\left(3^{\frac{1}{2}}\right)^{\frac{4}{3}} = 3^{\frac{1}{2}\cdot\frac{4}{3}} = 3^{\frac{4}{6}}$$

2. Write the rational exponent in simplest form. Be sure to include absolute value if the original expression involved finding an even root.

   The exponent, $\frac{4}{6}$, can be reduced to $\frac{2}{3}$. The original root is even, so the sixth root is positive.

   $$3^{\frac{4}{6}} = 3^{\frac{2}{3}}$$

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Instruction

3. Evaluate the power and root of the function, using a calculator if needed.

   Note that the power of a power exponent property can be used to rewrite the expression $x^{\frac{a}{b}}$ as $\left(x^{\frac{1}{b}}\right)^{a}$ or $\left(x^{a}\right)^{\frac{1}{b}}$, so either the root or power can be evaluated first.

   The cube root of 3 is not an integer, so a calculator will be needed to approximate the root.

   The expression 3^2 can be evaluated first without using a calculator: $3^2 = 9$.

   $$3^{\frac{2}{3}} = \left(3^{2}\right)^{\frac{1}{3}} = 9^{\frac{1}{3}} \approx 2.080$$

4. Determine whether the answer is rational or irrational.

   When evaluated, the expression $\left(3^{\frac{1}{2}}\right)^{\frac{4}{3}}$ is equal to $9^{\frac{1}{3}}$, which cannot be expressed as a ratio of integers. Therefore, this expression is irrational.

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Example 4

Lochlan has a savings account. The total account balance, y, after any number of years t can be found using the equation $y = 5000(x)^t$, where x is equal to 1 plus the annual interest rate. The total balance in the account is currently \$6,203.74, and Lochlan has had the account for $5\frac{1}{2}$ years. What is the annual interest rate?

1. Replace any variables with known quantities.

   The equation showing the total account balance y after t years is $y = 5000(x)^t$. If the total account balance is \$6,203.74, then $y = \$6{,}203.74$, $t = 5\frac{1}{2} = \frac{11}{2}$, and x is unknown.

   $$6203.74 = 5000 \bullet x^{\frac{11}{2}}$$

2. Use inverse operations to find an equivalent equation in the form $x^a = b$, where x is the variable.

   Multiply both sides by the reciprocal of 5,000 to isolate the exponential expression, $x^{\frac{11}{2}}$.

   $$\left(\frac{1}{5000}\right) \bullet 6203.74 = \left(\frac{1}{5000}\right) \bullet 5000 \bullet x^{\frac{11}{2}}$$

   $$1.2407 = x^{\frac{11}{2}}$$

3. Solve by raising both sides to the reciprocal of the rational exponent.

   $$(1.2407)^{\frac{2}{11}} = \left(x^{\frac{11}{2}}\right)^{\frac{2}{11}}$$

   $$1.040 = x$$

   Since x is equal to 1 plus the interest rate, the interest rate must be 1.04 – 1 = 0.04, or 4%.

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Example 5

Solve the equation $\sqrt[4]{x^3} = 125$.

1. Rewrite the equation using a rational exponent.

   The variable x is being raised to the third power with a fourth root, so the rational exponent is $\frac{3}{4}$.

   $\sqrt[4]{x^3} = 125$

   $x^{\frac{3}{4}} = 125$

2. Determine the reciprocal of the rational exponent.

   The rational exponent is $\frac{3}{4}$. The reciprocal is $\frac{4}{3}$.

3. Raise both sides to the reciprocal of the rational exponent.

   $\left(x^{\frac{3}{4}}\right)^{\frac{4}{3}} = (125)^{\frac{4}{3}}$

   $x = 125^{\frac{4}{3}}$

4. If possible, find the root of the quantity before raising to the power to find x.

   $125 = 5^3$, so the third root of 125 is 5.

   $x = 125^{\frac{4}{3}} = \left(125^{\frac{1}{3}}\right)^4 = 5^4$

   $x = 625$

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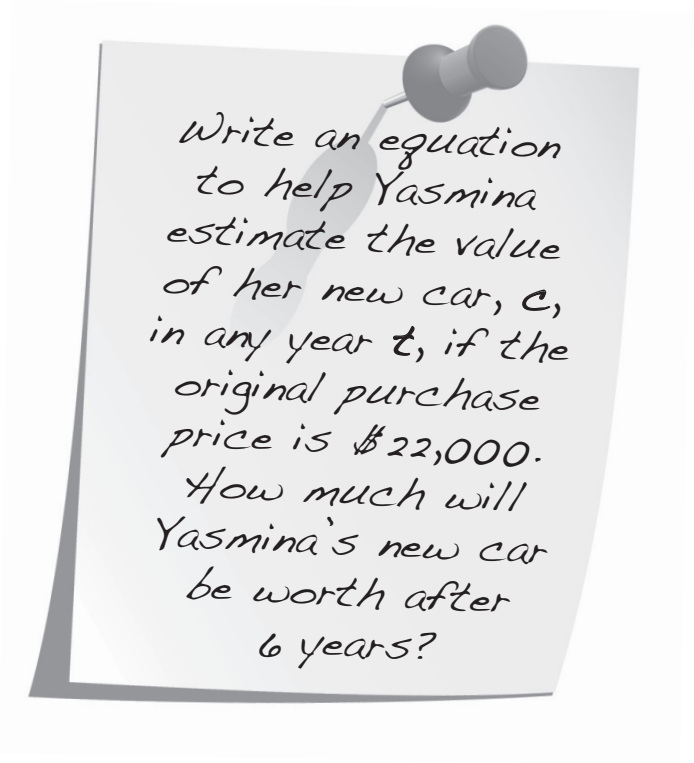
Problem-Based Task 1.1.2: Estimating Depreciation

SMP	
1	2
3	4
5 ✓	6 ✓
7 ✓	8

Yasmina is buying a new car for $22,000. To estimate how her car will decrease in value, or depreciate, she looks at the price of older versions of the same car. She finds a similar car that is 2.5 years old. The original price of the older car was also $22,000, and its current selling price is $16,905. She knows the equation $c = 22{,}000 \cdot d^t$ can be used to estimate the value of either car, c, in any year t since the car was originally purchased for $22,000. The value d is used to calculate the new value of the car each year. Write an equation to help Yasmina estimate the value of her new car, c, in any year t, if the original purchase price is $22,000. How much will Yasmina's new car be worth after 6 years?



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Problem-Based Task 1.1.2: Estimating Depreciation

Coaching

a. Which variable needs to be determined in the equation $c = 22{,}000 \cdot d^t$ so that Yasmina can estimate the value of a car in year t?

b. Use the data about the older car to help Yasmina complete the equation. Which variables are known for this equation?

c. Substitute the known quantities for the variables in the equation.

d. Use inverse operations to find an equivalent equation in the form $x^a = b$.

e. Solve for the variable.

f. What is the equation Yasmina can use to estimate the value of a car purchased for $22,000 in any year t?

g. How much will Yasmina's new car be worth after 6 years?

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Instruction

Problem-Based Task 1.1.2: Estimating Depreciation

Coaching Sample Responses

a. Which variable needs to be determined in the equation $c = 22{,}000 \cdot d^t$ so that Yasmina can estimate the value of a car in year t?

The variable c represents the value of the car in year t; t is the independent variable and c is the dependent variable. The only variable that needs to be determined is d.

b. Use the data about the older car to help Yasmina complete the equation. Which variables are known for this equation?

For the older car, Yasmina knows the current value, c, and the time, t, in years.

c. Substitute the known quantities for the variables in the equation.

$c = \$16{,}905$

$t = 2.5$ years

$16{,}905 = 22{,}000 \cdot d^{2.5}$

d. Use inverse operations to find an equivalent equation in the form $x^a = b$.

$16{,}905 = 22{,}000 \cdot d^{2.5}$

$0.768 \approx d^{2.5}$

e. Solve for the variable.

$0.768 \approx d^{2.5}$

$0.768 \approx d^{\frac{5}{2}}$

$(0.768)^{\frac{2}{5}} \approx \left(d^{\frac{5}{2}}\right)^{\frac{2}{5}}$

$0.90 \approx d$

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f. What is the equation Yasmina can use to estimate the value of a car purchased for $22,000 in any year t?

Replace d with the calculated value of 0.90.

$c = 22{,}000(0.90)^t$

g. How much will Yasmina's new car be worth after 6 years?

Replace t with 6 in the equation found in part f.

$c = 22{,}000(0.90)^6$

$c = 11{,}691.70$

After 6 years, the car will be worth approximately $11,692, or about half of its original value.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.1.2: Rational and Irrational Numbers and Their Properties

Use the properties of exponents to simplify the expressions. Do not evaluate.

1. $g^{-\frac{4}{9}}$

2. $8^{\frac{3}{10}} \bullet 8^{\frac{2}{7}}$

3. $\left(19^{\frac{4}{15}}\right)^{\frac{5}{2}}$

Simplify each expression, and then determine whether each answer is rational or irrational.

4. $\sqrt{4}+8$

5. $1+\sqrt[3]{10^2}$

6. $\left(\sqrt[4]{4^2}\right) \bullet \sqrt{25}$

Solve each equation for the given variable.

7. $\sqrt[3]{x^4}=1296$

8. $d^{\frac{4}{6}}=18$

continued

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Use the information given in each scenario to solve the problems.

9. Mia is tracking her savings account balance. She knows the equation $y = 8000p^t$ can be used to find her balance y in any year t, but she can't remember what p represents. Her balance today, $3\frac{2}{3}$ years after opening her account, is \$9,905.54. What is the value of p?

10. A new fashion trend is catching on at a high school. Five students came to school after the holidays wearing new Palioxis-brand sneakers, and 6 months later, 36 total students were wearing Palioxis sneakers. In the equation $y = 5(r^t)$, y is the number of students wearing the sneakers after time t in years. Find r, and write an equation to estimate the number of students in Palioxis sneakers after t years.

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Progress Assessment

Circle the letter of the best answer.

1. Which expression is equivalent to $5^{\frac{7}{13}}$?

   a. $\sqrt{5^{13}}$
   b. $\sqrt[7]{5^{13}}$
   c. $\sqrt[13]{5}$
   d. $\sqrt[13]{5^7}$

2. What is the value of $\sqrt[3]{27^4}$?

   a. 3
   b. 9
   c. 27
   d. 81

3. What is the approximate value of $6^{\frac{7}{2}}$?

   a. 1.67
   b. 2.45
   c. 529.09
   d. 279,936

4. Which product is irrational?

   a. $0.21 \bullet \sqrt[4]{4}$
   b. $\frac{3}{11} \bullet 100^{\frac{5}{2}}$
   c. $\sqrt[3]{512^2} \bullet 13$
   d. $121^{\frac{9}{2}} \bullet 3.3$

5. Let $x^{\frac{3}{5}} = 8$. What is the value of x?

   a. 1.6
   b. 2
   c. 32
   d. 64

6. Which expression is equivalent to $\sqrt[w]{g^{14}}$?

   a. $g^{\frac{14}{w}}$
   b. $14^{\frac{g}{w}}$
   c. $g^{\frac{w}{14}}$
   d. $g^{\frac{1}{w}}$

continued

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Assessment

7. What is the value of $1024^{\frac{4}{5}}$?

   a. 4
   b. 16
   c. 256
   d. 1,024

8. What is the approximate value of $\sqrt[9]{7^{10}}$?

   a. 5.76
   b. 8.69
   c. 23.08
   d. 16,807.00

9. Which sum is rational?

   a. $\sqrt{10}+15$
   b. $6^{\frac{4}{2}}+\sqrt[3]{1000}$
   c. $8^3+8^{\frac{1}{2}}$
   d. $\sqrt{144^3}+4^{\frac{1}{4}}$

10. Let $x^3 = 220$. What is the approximate value of x?

    a. 6.04
    b. 7.00
    c. 14.83
    d. 73.33

Read the scenario and use the information in it to answer the questions that follow.

11. A town counts its population every 10 years, and uses the change in the population to determine the town's growth rate. The equation $y = 52{,}000(1.03)^d$ can be used to estimate the town's population d decades after the year 2000.

    a. What will be the approximate population in 2020?

    b. What will be the approximate population in 2035?

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. What is the result of $(4x - 8) + (x + 16)$?
   a. $4x + 8$
   b. $5x + 8$
   c. $4x + 24$
   d. $5x + 16$

2. What is the result of $(-2x^3 + x) + (3x - 6)$?
   a. $-6x - 6$
   b. $-2x^3 - 6$
   c. $-2x^3 + 4x$
   d. $-2x^3 + 4x - 6$

3. What is the result of $(x^2 - 10x) - (-5x^2 + x)$?
   a. $-4x^2 - 9x$
   b. $-4x^2 - 11x$
   c. $6x^2 + 9x$
   d. $6x^2 - 11x$

4. What is the result of $(-x + 2)(x + 3)$?
   a. $-x^2 - x + 6$
   b. $x^2 + 5x + 6$
   c. $-x^2 + 6$
   d. $x^2 - 6$

5. What is the result of $(x^2 + 1)(-x^3 - 4x + 2)$?
   a. $x^5 - 5x^3 + 2x^2 + 4x + 2$
   b. $-x^5 - 5x^3 + 2x^2 - 4x + 2$
   c. $-x^4 - 4x^3 + 2x^2$
   d. $-x^6 - 4x^4 + 2x^2 + 2$

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Lesson 2: Operating with Polynomials

Instruction

Common Core State Standard

A–APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

SMP	
1 ✓	2
3	4 ✓
5	6
7 ✓	8 ✓

Essential Questions

1. How can a variable and its power be used to determine which terms are like terms?
2. What is the relationship between addition of polynomials and subtraction of polynomials?
3. How can we determine if polynomials are closed under addition, subtraction, or multiplication?

WORDS TO KNOW

binomial	a polynomial with two terms
closure	a system is closed, or shows closure, under an operation if the result of the operation is within the system
like terms	terms that contain the same variables raised to the same power
monomial	an expression with one term, consisting of a number, a variable, or the product of a number and variable(s)
polynomial	a monomial or a sum of monomials
term	a number, a variable, or the product of a number and variable(s)

Recommended Resources

- Khan Academy. "Adding Polynomials."

  http://www.walch.com/rr/00083

  This video tutorial explains how to add polynomials.
- MathIsFun.com. "Multiplying Polynomials."

  http://www.walch.com/rr/00084

  This website shows what a polynomial is, and explains how to multiply them.
- MathIsFun.com. "Polynomials."

  http://www.walch.com/rr/00085

  This website defines polynomials and their components, and provides examples of expressions that are polynomials as well as expressions that are not.

SRB p. U1-20

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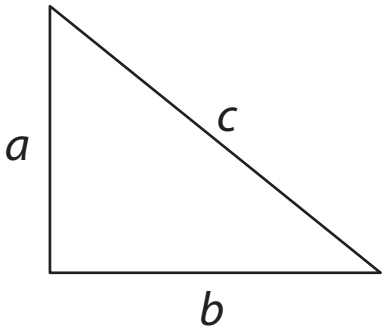
Lesson 1.2.1: Adding and Subtracting Polynomials

Warm-Up 1.2.1

Penelope is a playground designer. She's considering different sizes of a triangular climbing wall for her latest project. Penelope has drawn up three potential designs for the climbing wall, each with different side lengths. For each design, she needs to determine the perimeter of the climbing wall in order to know how much material will be needed to build it. The perimeter of a triangle is the sum of the lengths of the three sides. Help Penelope by finding the perimeter of a climbing wall with each of the given side lengths. Write the perimeter in the simplest expression possible. All side lengths are in feet.



1. $a = 5$, $b = 12$, and $c = 20$

2. $a = 8$, $b = x$, and $c = 15$

3. $a = x$, $b = 1$, and $c = 6$

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Instruction

Lesson 1.2.1: Adding and Subtracting Polynomials

Common Core State Standard

A–APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Warm-Up 1.2.1 Debrief

1. $a = 5$, $b = 12$, and $c = 20$

   Recall that the perimeter of a triangle = $a + b + c$.

   Replace the values of a, b, and c in the formula for perimeter.

   perimeter = $a + b + c$

   perimeter = $5 + 12 + 20$

   perimeter = 37 Simplify.

   The perimeter is 37 feet.

2. $a = 8$, $b = x$, and $c = 15$

   Replace the values of a, b, and c in the formula for perimeter.

   perimeter = $8 + x + 15$

   perimeter = $23 + x$ Combine like terms.

   The perimeter is $(23 + x)$ feet.

3. $a = x$, $b = 1$, and $c = 6$

   Replace the values of a, b, and c in the formula for perimeter.

   perimeter = $x + 1 + 6$

   perimeter = $x + 7$ Combine like terms.

   The perimeter is $(x + 7)$ feet.

Connection to the Lesson

- Students will extend their understanding of finding sums and differences to finding sums and differences of polynomial expressions.
- Students will substitute given values for the variables in formulas.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- replacing variables in a formula with given quantities (6.EE.2c)
- using variables to express unknown quantities (6.EE.6)
- using the Distributive Property to find equivalent expressions (6.EE.3)
- combining like terms in polynomials (8.EE.7b)

Introduction

Polynomials can be added and subtracted like real numbers. Adding and subtracting polynomials is a way to simplify expressions. It can also allow us to find a shorter way to represent a sum or difference.

Key Concepts

- A **monomial** is an expression with one term, consisting of a number, a variable, or the product of a number and variable(s).
- A **polynomial** is a monomial or a sum of monomials that contains variables, numeric quantities, or both. The variables of polynomials are raised to integer powers greater than 0. For example, $mn^4 - 6n$ and 12 are both polynomials.
- A **binomial** is a polynomial with two terms.
- Each part of a polynomial is called a term. A **term** is a number, a variable, or the product of a number and variable(s). For example, the second term in the polynomial $8p + 4r^3 + 9$ is $4r^3$.
- **Like terms** are terms that contain the same variables raised to the same power. Numeric quantities are like terms; for example, 1 and 3.4 are like terms. The terms $2x^3$ and $-4x^3$ are also like terms.
- Polynomials are typically written in descending order of the exponents; that is, from left to right, the term with the highest exponent is written first, followed by the term with the next highest exponent, and so on down to the term with the lowest exponent or no exponent.
- When there are two or more variables in a polynomial, the terms are written in descending order alphabetically. For example, $x^2 + x + y^2 + y + 12$ is written in descending order alphabetically.
- To add or subtract like terms containing a variable, use the Distributive Property to add or subtract the variable's coefficients. If a and b are coefficients and n is a positive integer, then $ax^n + bx^n = (a + b)x^n$.

SRB p. U1-22

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Instruction

- Subtraction can be represented in a similar way: $ax^n - bx^n = (a - b)x^n$.
- To add polynomials, combine like terms.
- Before subtracting one polynomial from another, rewrite the difference as a sum. For example, $(n + ax) - (m + bx)$ can be rewritten as the sum $(n + ax) + (-m - bx)$:

$(n + ax) - (m + bx)$	Difference of two polynomials
$= (n + ax) + [-(m + bx)]$	Rewrite the difference as a sum.
$= (n + ax) + [-1(m + bx)]$	Rewrite $-(m + bx)$ as $-1(m + bx)$.
$= (n + ax) + [(-1)m + (-1)bx]$	Distribute -1 over $(m + bx)$.
$= (n + ax) + (-m - bx)$	Simplify.

- Therefore, subtracting two polynomials is the same as adding the opposite of the second polynomial.
- A system shows **closure**, or is closed, if the result of the operation is also in the system.
- For example, when an integer is added to an integer, the result is also an integer, so the set of integers is closed under addition.
- In the same way, the result of adding two polynomials is also a polynomial, so the system of polynomials is closed under addition.
- The result of subtracting one polynomial from another polynomial is also a polynomial; therefore, the system of polynomials is closed under subtraction.

Common Errors/Misconceptions

- finding a sum or difference of two terms with the same variable raised to different powers ($2x^3 + 5x^4 \neq 7x^3$)
- incorrectly adding or subtracting like terms by adding the exponents ($x^3 + x^3 \neq x^6$)
- incorrectly subtracting two polynomials by not rewriting the expression as a sum

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Instruction

Guided Practice 1.2.1

Example 1

Find the sum of $(4 + 3x) + (2 + x)$.

1. Rewrite the sum so that like terms are together.

   There are two numeric quantities, 4 and 2, and two terms that contain a variable, $3x$ and x. All the terms are positive.

   $(4 + 3x) + (2 + x)$

   $= 4 + 2 + 3x + x$

2. Find the sum of any numeric quantities.

   The numeric quantities in this example are 4 and 2.

   $4 + 2 + 3x + x$

   $= 6 + 3x + x$

3. Find the sum of any terms with the same variable raised to the same power.

   The two terms $3x$ and x both contain only the variable x raised to the first power.

   $6 + 3x + x$

   $= 6 + 4x$

   The result of $(4 + 3x) + (2 + x)$ is $6 + 4x$.

SWB p. 29; SRB p. U1-24

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Instruction

Example 2

Find the sum of $(7x^2 - x + 15) + (6x + 12)$.

1. Rewrite the sum so that like terms are together.

   Be sure to keep any negatives with the expression that follows, such as $-x$.

   $(7x^2 - x + 15) + (6x + 12)$

   $= 7x^2 - x + 6x + 15 + 12$

2. Find the sum of any numeric quantities.

   $7x^2 - x + 6x + 15 + 12$

   $= 7x^2 - x + 6x + 27$

3. Find the sum of any terms with the same variable raised to the same power.

   There is only one term with the variable x raised to the second power.

   There are two terms with the variable x raised to the first power, $-x$ and $6x$, so these can be combined.

   Add the coefficients of the variable.

   $7x^2 - x + 6x + 27$

   $= 7x^2 + 5x + 27$

   The result of $(7x^2 - x + 15) + (6x + 12)$ is $7x^2 + 5x + 27$.

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Instruction

Example 3

Find the difference of $(x^5 + 8) - (3x^5 + 5x)$.

1. Rewrite the difference as a sum.

   A difference can be written as a sum by adding the opposite of the second expression.

   Simplify "$-(3x^5 + 5x)$" by distributing -1 and writing the polynomial as $(-3x^5 - 5x)$.

   $(x^5 + 8) - (3x^5 + 5x)$

   $= (x^5 + 8) + [-1(3x^5 + 5x)]$

   $= (x^5 + 8) + (-3x^5 - 5x)$

2. Rewrite the sum so that any like terms are together.

   Be sure to keep any negatives with the expression that follows, such as $-3x^5$.

   $(x^5 + 8) + (-3x^5 - 5x)$

   $= x^5 + (-3x^5) + (-5x) + 8$

3. Find the sum of any terms with the same variable raised to the same power.

   There are two terms with the variable x raised to the fifth power.

   There is only one term with x raised to the first power, and only one numeric quantity.

   The sum of the two terms with x^5 can be combined by adding their coefficients.

   $x^5 + (-3x^5) + (-5x) + 8$

   $= -2x^5 - 5x + 8$

   The result of $(x^5 + 8) - (3x^5 + 5x)$ is $-2x^5 - 5x + 8$.

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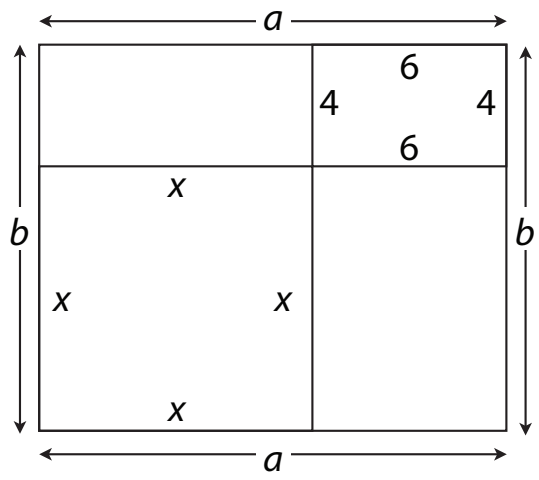
Problem-Based Task 1.2.1: Cabin Perimeter

SMP	
1 ✓	2
3	4
5	6
7 ✓	8 ✓

Soren has been hired to design a small cabin. He has drawn a blueprint for the cabin, as shown. His client is still determining the overall size of the cabin, but Soren has labeled the known lengths in feet. He wants to find an expression to represent the perimeter of the entire space. The perimeter of the cabin is the sum of all four sides and can be written as $P = 2a + 2b$. Find an expression in terms of x that shows the total perimeter.



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Problem-Based Task 1.2.1: Cabin Perimeter

Coaching

a. What is an expression for a in terms of x?

b. What is an expression for b in terms of x?

c. Using the formula for perimeter and the expressions for a and b, write a formula for the cabin's perimeter in terms of x.

d. What is the simplified expression for the perimeter of the cabin in terms of x?

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Instruction

Problem-Based Task 1.2.1: Cabin Perimeter

Coaching Sample Responses

a. What is an expression for a in terms of x?

The length of a contains two labeled lengths: x and 6.

The sum of these two lengths, in feet, equals a: $(x + 6)$.

b. What is an expression for b in terms of x?

The length of b contains two labeled lengths: x and 4.

The sum of these two lengths, in feet, equals b: $(x + 4)$.

c. Using the formula for perimeter and the expressions for a and b, write a formula for the cabin's perimeter in terms of x.

Replace a and b in the perimeter formula with the expressions found in parts a and b.

$P = 2a + 2b$

$= 2(x + 6) + 2(x + 4)$

d. What is the simplified expression for the perimeter of the cabin in terms of x?

Apply the Distributive Property, then simplify by combining like terms.

$P = 2(x + 6) + 2(x + 4)$

$= 2x + 12 + 2(x + 4)$

$= 2x + 12 + 2x + 8$

$= 2x + 2x + 12 + 8$

$= 4x + 20$

The simplified expression for the perimeter of the cabin in terms of x is $(4x + 20)$ feet.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.2.1: Adding and Subtracting Polynomials

Find each sum or difference.

1. $(x^3 - 5) + (6x^3 + 2)$

2. $(x^3 - 4x + 2) + (x^4 + 12x)$

3. $(-3x^2 + 16) - (x^2 - 22x - 4)$

4. $(5x^5 - 2x) - (4x^4 + 3x^2)$

5. $(10x - 9) - (-x^2 + 22x)$

6. $(6x^4 + 8) + (x^4 - 2x^3 + 1)$

continued

SWB p. 33; SRB Practice 1.2.1 p. U1-27

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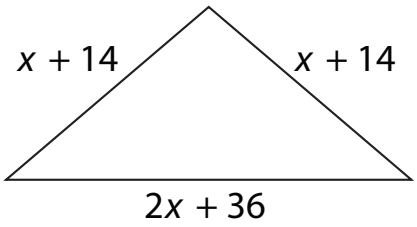
The perimeter of a polygon is the sum of the lengths of the sides of the polygon. For problems 7–10, find the perimeter of each shape in terms of x. All lengths are given in centimeters.

7.



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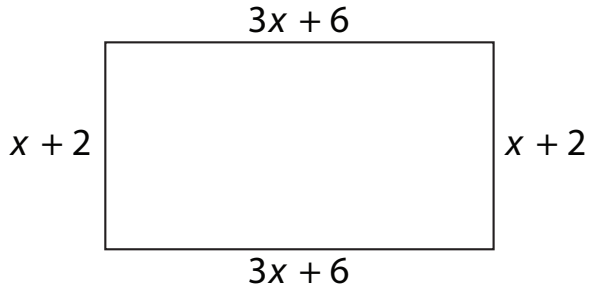

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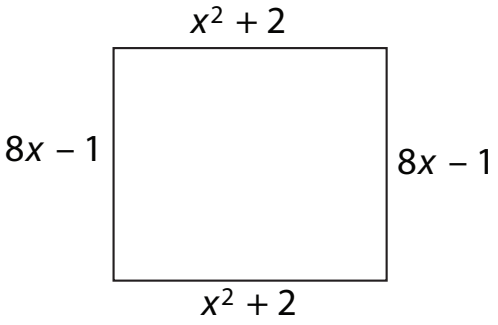

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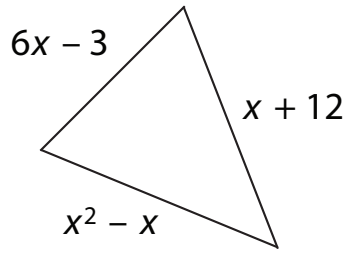

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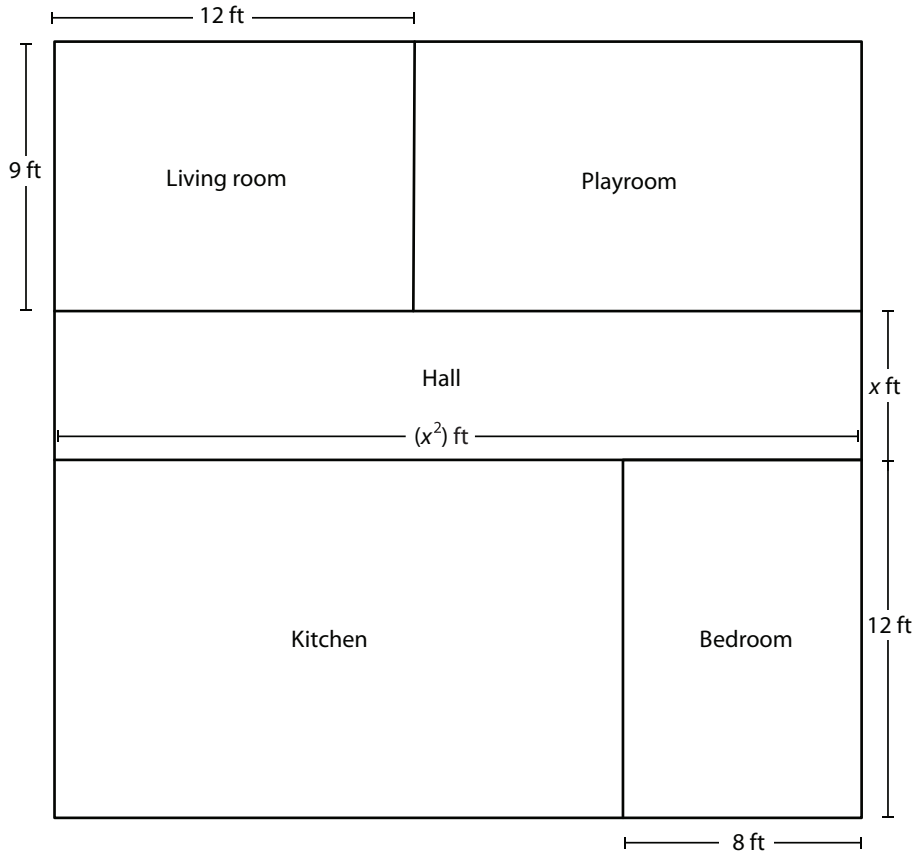
Lesson 1.2.2: Multiplying Polynomials

Warm-Up 1.2.2

A carpet installer charges different prices based on the size of the room where the carpet is being installed. Iskra wants to have the same carpet installed in her bedroom, living room, and hall. To determine the cost, she first needs to determine the area of each rectangular room. The area of a rectangle is the product of the rectangle's length, l, and width, w: $A = lw$. Find the area in simplest form for each of the three rooms Iskra wants to have carpeted.



1. The bedroom has a length of 12 feet and a width of 8 feet.

2. The living room has a length of 12 feet and a width of 9 feet.

3. The hall has a length of x^2 feet and a width of x feet.

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Instruction

Lesson 1.2.2: Multiplying Polynomials

Common Core State Standard

A–APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Warm-Up 1.2.2 Debrief

1. The bedroom has a length of 12 feet and a width of 8 feet.

   Replace l and w in the formula for area with the given values of l and w.

   $A = lw$

   $A = 12 \cdot 8$

   $A = 96$

   The area of the bedroom is 96 ft^2.

2. The living room has a length of 12 feet and a width of 9 feet.

   Replace l and w in the formula for area with the given values of l and w.

   $A = lw$

   $A = 12 \cdot 9$

   $A = 108$

   The area of the living room is 108 ft^2.

3. The hall has a length of x^2 feet and a width of x feet.

   Replace l and w in the formula for area with the given values of l and w.

   $A = lw$

   $A = x^2 \cdot x$

   Use the properties of exponents to simplify the answer. Since the two factors have the same base, x, add the exponents.

   $A = x^2 \cdot x$

   $A = x^{2+1}$

   $A = x^3$

   The area of the hall is x^3 ft^2.

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Instruction

Connection to the Lesson

- Students will extend their understanding of finding products to finding products of polynomials.
- Students will need to replace variables in a formula with given quantities.
- Students will simplify expressions by using properties of exponents and multiplication.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- replacing variables in a formula with given quantities (6.EE.2c)
- using variables to express unknown quantities (6.EE.6)
- using properties of exponents to simplify expressions (8.EE.1)
- combining like terms in polynomials (8.EE.7b)

Introduction

To simplify an expression, such as $(a + bx)(c + dx)$, polynomials can be multiplied. Unlike addition and subtraction of polynomial terms, any two terms can be multiplied, even if the variables or powers are different. Laws of exponents and combining like terms can be used to simplify products of polynomials.

Key Concepts

- To multiply two polynomials, multiply each term in the first polynomial by each term in the second polynomial.
- The Distributive Property can be used to simplify the product of two or more polynomials. For example, if each polynomial has two terms, the product can be simplified as follows.

$(a + b)(c + d)$	Product of two binomials
$= (a + b)c + (a + b)d$	Distribute.
$= ac + bc + ad + bd$	Simplify.

- If each polynomial has three terms, then the product $(a + b + c)(d + e + f)$ can be found in the same manner.

$(a + b + c)(d + e + f)$	Product of two polynomials
$= (a + b + c)d + (a + b + c)e + (a + b + c)f$	Distribute.
$= ad + bd + cd + ae + be + ce + af + bf + cf$	Simplify.

- This procedure will work for multiplying any number of polynomials with any number of terms.
- To find the product of two like variables raised to powers, use the properties of exponents. If the bases are the same, add the exponents: $x^n \bullet x^m = x^{n+m}$.
- If the bases are not the same, then the exponents cannot be added. Example: $x^n \bullet y^m = x^n y^m$.

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Instruction

- To find the product of a variable with a coefficient and a number, multiply the coefficient by the number. If a and b are real numbers, then $ax \cdot b = abx$.
- After multiplying all terms, simplify the expression by combining like terms.
- The product of two polynomials is a polynomial, so the system of polynomials is closed under multiplication.

Common Errors/Misconceptions

- forgetting to multiply all terms in each polynomial
- incorrectly simplifying a product
- when multiplying two terms with the same variable, multiplying the exponents instead of adding them

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Guided Practice 1.2.2

Example 1

Find the product of $(2x - 1)(x + 18)$.

1. Distribute the first polynomial over the second.

   Ensure that any negative signs are included in the products where appropriate.

   $(2x - 1)(x + 18)$

   $= 2x \bullet x + 2x \bullet 18 + (-1) \bullet x + (-1) \bullet 18$

2. Use properties of exponents to simplify any expressions.

   x is x to the first power, or x^1.

   $2x \bullet x$

   $= 2x^1 \bullet x^1$

   $= 2x^{1+1}$

   $= 2x^2$

   Rewrite the expression, substituting $2x^2$ for $2x \bullet x$.

   $2x \bullet x + 2x \bullet 18 + (-1) \bullet x + (-1) \bullet 18$

   $= 2x^2 + 2x \bullet 18 + (-1) \bullet x + (-1) \bullet 18$

3. Simplify any remaining products.

   The coefficient of a term can be multiplied by a number: $ax \bullet b = abx$.

   $2x^2 + 2x \bullet 18 + (-1) \bullet x + (-1) \bullet 18$

   $= 2x^2 + 36x - x - 18$

4. Combine any like terms.

   $2x^2 + 36x - x - 18$

   $= 2x^2 + 35x - 18$

   The result of $(2x - 1)(x + 18)$ is $2x^2 + 35x - 18$.

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Instruction

Example 2

Find the product of $(x^3 + 9x)(-x^2 + 11)$.

1. Distribute the first polynomial over the second.

   Ensure that any negatives are included in the products where appropriate.

   $(x^3 + 9x)(-x^2 + 11)$

   $= x^3 \bullet (-x^2) + x^3 \bullet 11 + 9x \bullet (-x^2) + 9x \bullet 11$

2. Use properties of exponents to simplify like exponential expressions.

   To multiply terms that have the same base (in this case, x), keep this base and add the exponents: $x^m \bullet x^n = x^{(m+n)}$.

   $= x^3 \bullet (-x^2) + x^3 \bullet 11 + 9x \bullet (-x^2) + 9x \bullet 11$

   $= -x^{3+2} + x^3 \bullet 11 - 9x^{1+2} + 9x \bullet 11$

   $= -x^5 + x^3 \bullet 11 - 9x^3 + 9x \bullet 11$

3. Simplify any remaining products.

   The coefficient of a term can be multiplied by a number: $ax \bullet b = abx$.

   $-x^5 + 11 \bullet x^3 - 9x^3 + 9x \bullet 11$

   $= -x^5 + 11x^3 - 9x^3 + 99x$

4. Combine any like terms.

   $-x^5 + 11x^3 - 9x^3 + 99x$

   $= -x^5 + 2x^3 + 99x$

   The result of $(x^3 + 9x)(-x^2 + 11)$ is $-x^5 + 2x^3 + 99x$.

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Instruction

Example 3

Find the product of $(3x + 4)(x^2 + 6x + 10)$.

1. Distribute the first polynomial over the second.

   Multiply each term in the first polynomial by each term in the second polynomial.

   $(3x + 4)(x^2 + 6x + 10)$

   $= 3x \bullet x^2 + 3x \bullet 6x + 3x \bullet 10 + 4 \bullet x^2 + 4 \bullet 6x + 4 \bullet 10$

2. Use properties of exponents to simplify any expressions.

   $3x \bullet x^2 + 3x \bullet 6x + 3x \bullet 10 + 4 \bullet x^2 + 4 \bullet 6x + 4 \bullet 10$

   $= 3x^3 + 18x^2 + 3x \bullet 10 + 4 \bullet x^2 + 4 \bullet 6x + 4 \bullet 10$

3. Simplify any remaining products.

   $3x^3 + 18x^2 + 3x \bullet 10 + 4 \bullet x^2 + 4 \bullet 6x + 4 \bullet 10$

   $= 3x^3 + 18x^2 + 30x + 4x^2 + 24x + 40$

4. Combine any like terms.

   Only terms with the same variable raised to the same power can be combined.

   The sum can first be rewritten with the exponents in descending order.

   $3x^3 + 18x^2 + 30x + 4x^2 + 24x + 40$

   $= 3x^3 + 18x^2 + 4x^2 + 30x + 24x + 40$

   $= 3x^3 + 22x^2 + 54x + 40$

   The result of $(3x + 4)(x^2 + 6x + 10)$ is $3x^3 + 22x^2 + 54x + 40$.

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Instruction

Example 4

Find the product of $(x + y + 1)(x^2 + 4y - 5)$.

1. Distribute the first polynomial over the second.

   Multiply each term in the first polynomial by each term in the second polynomial.

   $(x + y + 1)(x^2 + 4y - 5)$

   $= x \bullet x^2 + x \bullet 4y + x \bullet (-5) + y \bullet x^2 + y \bullet 4y + y \bullet (-5) + 1 \bullet x^2 + 1 \bullet 4y + 1 \bullet (-5)$

2. Use properties of exponents to simplify any expressions.

   $x \bullet x^2 + x \bullet 4y + x \bullet (-5) + y \bullet x^2 + y \bullet 4y + y \bullet (-5) + 1 \bullet x^2 + 1 \bullet 4y + 1 \bullet (-5)$

   $= x^3 + x \bullet 4y + x \bullet (-5) + y \bullet x^2 + 4y^2 + y \bullet (-5) + 1 \bullet x^2 + 1 \bullet 4y + 1 \bullet (-5)$

3. Simplify any remaining products.

   $x^3 + x \bullet 4y + x \bullet (-5) + y \bullet x^2 + 4y^2 + y \bullet (-5) + 1 \bullet x^2 + 1 \bullet 4y + 1 \bullet (-5)$

   $= x^3 + 4xy - 5x + x^2y + 4y^2 - 5y + x^2 + 4y - 5$

4. Combine any like terms.

   Only terms with the same variable raised to the same power can be combined.

   The sum can first be rewritten with the exponents in descending order.

   When two variables are in a term, such as $x^n y^m$, both n and m, the powers of the two variables, must be the same to combine the terms.

   $x^3 + 4xy - 5x + x^2y + 4y^2 - 5y + x^2 + 4y - 5$

   $= x^3 + x^2 + x^2y - 5x + 4xy + 4y^2 - 5y + 4y - 5$

   $= x^3 + x^2 + x^2y - 5x + 4xy + 4y^2 - y - 5$

   The result of $(x + y + 1)(x^2 + 4y - 5)$ is $x^3 + x^2 + x^2y - 5x + 4xy + 4y^2 - y - 5$.

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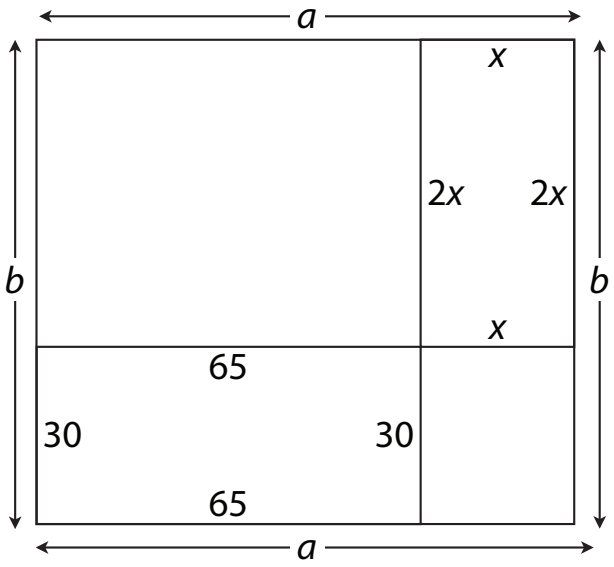
Problem-Based Task 1.2.2: Architectural Area

SMP	
1 ✓	2
3	4 ✓
5	6
7	8

An architect is creating a template, or reusable pattern, of the design of a bathroom. One part of the bathroom has a standard size in order to fit a standard bathtub, and one part of the bathroom can vary based on what the customer wants. The architect's template is shown, and all units are in inches. The formula for finding the area of a rectangle is $A = lw$, or in this case, $A = ab$. Find an expression to determine the total area of the bathroom for any value of x.



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Problem-Based Task 1.2.2: Architectural Area

Coaching

a. What is an expression for a in terms of x?

b. What is an expression for b in terms of x?

c. Using the formula for area and the expressions for a and b, write a formula for the area of the bathroom in terms of x.

d. What is the simplified expression for the area of the bathroom in terms of x?

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Instruction

Problem-Based Task 1.2.2: Architectural Area

Coaching Sample Responses

a. What is an expression for a in terms of x?

The length of a contains two labeled lengths: x and 65.

The sum of these two lengths, in inches, equals a: $(x + 65)$.

b. What is an expression for b in terms of x?

The length of b contains two labeled lengths: $2x$ and 30.

The sum of these two lengths, in inches, equals b: $(2x + 30)$.

c. Using the formula for area and the expressions for a and b, write a formula for the area of the bathroom in terms of x.

Replace a and b in the formula with the expressions found in parts a and b.

$A = ab = (x + 65)(2x + 30)$

d. What is the simplified expression for the area of the bathroom in terms of x?

Find the product of the two polynomials.

Apply the Distributive Property, multiplying each term in the first polynomial by each term in the second polynomial.

$(x + 65)(2x + 30) = 2x^2 + 30x + 130x + 1950$

Simplify the expression by combining like terms.

$2x^2 + 30x + 130x + 1950$

$= 2x^2 + 160x + 1950$

The simplified expression for the area of the bathroom written in terms of x is $(2x^2 + 160x + 1950)$ inches^2.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.2.2: Multiplying Polynomials

Find each product.

1. $(x + 10)(x - 7)$

2. $(3x + 5)(x^3 + 4x)$

3. $(2x + 1)(x^4 - 6x + 3)$

4. $(x^5 - 2)(x^2 + 2x + 4)$

5. $(2x^2 + x - 6)(10x + 4)$

6. $(-x^3 - x^2 + 2)(x^3 + 3x^2 + 2)$

The area of a rectangle is found using the formula $A = lw$, where l is the length of the rectangle and w is the width. Multiply each pair of factors and express the area of each rectangle as a single polynomial in terms of x.

7. $l = x + 14$; $w = 3x + 1$

8. $l = x^2 - 8$; $w = -x + 12$

9. $l = x^2 - 4$; $w = 5x + 10$

10. $l = 4x^2 + 8$; $w = 2x^2 - 3$

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Progress Assessment

Circle the letter of the best answer.

1. What is the result of $(x^2 - 6) + (3x^2 + 4)$?
   a. $4x - 10$
   b. $4x^2 + 10$
   c. $x^2 - 2$
   d. $4x^2 - 2$

2. What is the result of $(5x + 2) - (x^3 + x^2 - 9)$?
   a. $4x^3 + x^2 + 11$
   b. $-x^3 - x^2 + 5x + 11$
   c. $x^3 + x^2 + 5x - 7$
   d. $-x^2 + 5x - 11$

3. What is the result of $(8x^2 + 7x - 2) - (x^2 - 3x + 1)$?
   a. $7x^2 + 10x - 3$
   b. $7x^2 + 4x - 3$
   c. $8x^2 + 4x - 1$
   d. $8x^2 + 10x - 2$

4. What is the result of $(x - 12)(3x + 4)$?
   a. $3x^2 - 48$
   b. $x^2 - 32x - 48$
   c. $3x^2 - 36x + 48$
   d. $3x^2 - 32x - 48$

5. What is the result of $(-x^3 + x^2 + 2)(5x + 2)$?
   a. $-5x^4 - 2x^3$
   b. $-5x^4 + 5x^3 + 10x$
   c. $-5x^4 + 3x^3 + 2x^2 + 10x + 4$
   d. $-x^3 + 5x^2 + 10x + 4$

6. What is the result of $(9x^3 + 4x^2 + 3) + (-x^3 + 2x^2 - 16)$?
   a. $8x^3 + 6x^2 - 13$
   b. $10x^3 + 6x^2 + 19$
   c. $16x - 13$
   d. $10x^3 + 2x^2 + 19$

7. What is the result of $(-5x + 8) + (x^2 - 10x)$?
   a. $x^2 - 15x + 8$
   b. $-6x - 2$
   c. $x^2 + 5x + 8$
   d. $-x^2 - 15x$

continued

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Assessment

8. What is the result of $(x^3 - 10x^2) - (3x^3 + 10x^2 - 9)$?

   a. $4x^3 - 9$
   b. $-2x^3 - 20x^2 + 9$
   c. $-4x^3 + 9$
   d. $-24x + 9$

9. What is the result of $(x^3 - 2x^2)(x + 2)$?

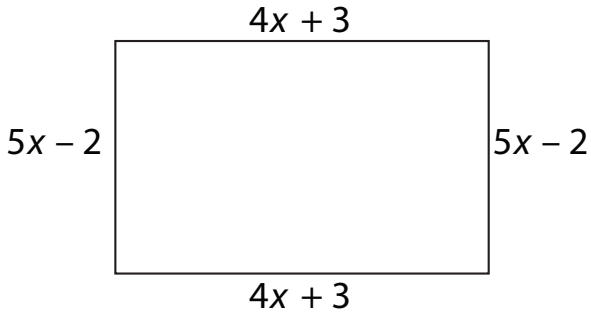
   a. $x^4 + 2x^3$
   b. $x^4 - 2x^2 + 2x^3$
   c. $x^4 - 4x^2$
   d. $x^4 - 4x^3 - 4x^2$

10. What is the result of $(-4x^2 + 1)(x^2 - 3x + 6)$?

    a. $4x^4 + 12x^3 + 25x^2 - 3x + 6$
    b. $-4x^4 + 12x^3 - 23x^2 - 3x + 6$
    c. $-4x^2 - 3x + 6$
    d. $x^2 - 3x + 6$

Use the given information and the diagram that follows to answer the questions.

11. The perimeter of a rectangle is the sum of the side lengths: perimeter = $2l + 2w$, where l is the rectangle's length and w is the rectangle's width. The area of a rectangle is the product of the side lengths: $A = lw$.



    a. What is the perimeter of the rectangle in simplest form?

    b. What is the area of the rectangle in simplest form?

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. What is the real part of the complex number $34 - 18i$?

   a. 18
   b. –18
   c. i
   d. 34

2. Which value is equal to i^{46}?

   a. i
   b. –1
   c. 1
   d. $-i$

3. What is the result of $(18 - 10i) + (44 + 16i)$?

   a. $34 - 54i$
   b. $6 + 62i$
   c. $62 - 6i$
   d. $62 + 6i$

4. What is the result of $(-1 + 5i)(6 + 2i)$?

   a. $-16 + 28i$
   b. $4 + 28i$
   c. $-5 + 7i$
   d. 16

5. What is the result of $(-2 + 4i)(6 - 2i)$?

   a. 4
   b. $4 + 2i$
   c. $-20 + 28i$
   d. $-4 + 28i$

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Lesson 3: Operating with Complex Numbers

Instruction

Common Core State Standards

N–CN.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

N–CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

SMP	
1 ✓	2 ✓
3	4 ✓
5	6 ✓
7 ✓	8

Essential Questions

1. How are complex numbers and real numbers related?
2. What types of numbers can be the sum or difference of complex numbers?
3. What type of number is the product of a complex number and its conjugate?

WORDS TO KNOW

complex conjugate	the complex number that when multiplied by another complex number produces a value that is wholly real; the complex conjugate of $a + bi$ is $a - bi$
complex number	a number in the form $a + bi$, where a and b are real numbers, and i is the imaginary unit
complex number system	all numbers of the form $a + bi$, where a and b are real numbers, including complex numbers (neither a nor b equal 0), real numbers ($b = 0$), and imaginary numbers ($a = 0$)
imaginary number	any number of the form bi, where b is a non-zero real number and $i = \sqrt{-1}$
imaginary unit, *i*	the letter i, used to represent the non-real value, $i = \sqrt{-1}$
real numbers	the set of all rational and irrational numbers
wholly imaginary	a complex number that has a real part equal to 0; written in the form $a + bi$, where a and b are real numbers, i is the imaginary unit, $a = 0$, and $b \neq 0$: $0 + bi$
wholly real	a complex number that has an imaginary part equal to 0; written in the form $a + bi$, where a and b are real numbers, i is the imaginary unit, $b = 0$, and $a \neq 0$: $a + 0i$

SRB p. U1-35

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Instruction

Recommended Resources

- Khan Academy. "Adding and Subtracting Complex Numbers."

  http://www.walch.com/rr/00087

  This interactive site provides practice with adding and subtracting complex numbers. The site also includes a video tutorial breaking down sums and differences if additional support is needed.

- Math Worksheets Go! "Multiplying Complex Numbers."

  http://www.walch.com/rr/00088

  Teachers can download free PDF worksheets (with answer keys) for students to practice multiplying complex numbers, including complex conjugates.

- The Physics Classroom. "Parallel Circuits."

  http://www.walch.com/rr/00089

  This site provides essential background knowledge on series and parallel electrical circuits, which are cited in several examples in this lesson.

- Purplemath.com. "Complex Numbers: Introduction."

  http://www.walch.com/rr/00090

  This site offers a supplemental explanation of complex numbers, including defining complex numbers and introducing operations with complex numbers.

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Lesson 1.3.1: Defining Complex Numbers, *i*, and i^2

Warm-Up 1.3.1

Martin has been given the task of preparing for the Drama Club's year-end party. The club's faculty advisor has given Martin a budget and a plan to follow.

1. Martin plans to buy a package of 20 cookies, and knows that only 7 club members will eat them. If he were to give each of the 7 people the same number of cookies, how many cookies would be left over?

2. Martin plans to buy 18 slices of cheese pizza. If 6 members each ate the same number of slices, how many slices would they each get, and how many slices would be left over?

3. Martin was given a budget of $80 to buy T-shirts for the club. He wants to purchase as many shirts as possible. Each shirt costs $9. How many T-shirts can he buy, and how much money will he have left over after buying the shirts?

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Instruction

Lesson 1.3.1: Defining Complex Numbers, *i*, and i^2

Common Core State Standard

N–CN.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

Warm-Up 1.3.1 Debrief

1. Martin plans to buy a package of 20 cookies, and knows that only 7 club members will eat them. If he were to give each of the 7 people the same number of cookies, how many cookies would be left over?

   $20 \div 7 = 2$ remainder 6

   Each person would get 2 cookies, and there would be 6 cookies left over.

2. Martin plans to buy 18 slices of cheese pizza. If 6 members each ate the same number of slices, how many slices would they each get, and how many slices would be left over?

   $18 \div 6 = 3$ remainder 0

   Each person would get 3 slices, and there would be no slices left over.

3. Martin was given a budget of $80 to buy T-shirts for the club. He wants to purchase as many shirts as possible. Each shirt costs $9. How many T-shirts can he buy, and how much money will he have left over after buying the shirts?

   $80 \div 9 = 8$ remainder 8

   Martin can buy 8 T-shirts, and he will have $8 left over.

Connection to the Lesson

- In this lesson, students will simplify powers of the imaginary unit i.
- To simplify i^n, where $n > 4$, students will need to find the remainder when dividing an exponent by 4.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- simplifying expressions using properties of exponents (8.EE.1)
- finding quotients that include remainders (4.NBT.6)
- understanding the real number system (no standard)

Introduction

We know that the square root of –1, or $\sqrt{-1}$, is not a real number because there is no number that when squared will result in –1. French mathematician René Descartes suggested the imaginary unit i be defined so that $i^2 = -1$. The imaginary unit enables us to solve problems that we would not otherwise be able to solve. Problems involving electricity often use the imaginary unit.

Key Concepts

- All rational and irrational numbers are real numbers.
- The **imaginary unit *i*** is used to represent the non-real value, $\sqrt{-1}$.
- An **imaginary number** is any number of the form bi, where b is a non-zero real number and $i = \sqrt{-1}$.
- Real numbers and imaginary numbers can be combined to create the **complex number system**.
- A **complex number** contains two parts: a real part and an imaginary part.
- All complex numbers are of the form $a + bi$, where a and b are real numbers and i is the imaginary unit.
- In the general form of a complex number, a is the real part and bi is the imaginary part.
- If $a = 0$, the complex number $a + bi$ is **wholly imaginary** and contains no real part: $0 + bi = bi$.
- If $b = 0$, the complex number $a + bi$ is **wholly real** and contains no imaginary part: $a + (0)i = a$.
- Expressions containing imaginary numbers can be simplified.
- To simplify i^n for $n > 4$, divide n by 4, and use the properties of exponents to rewrite the exponent.
- The exponent can be rewritten using the quotient: $n \div 4 = m$ remainder r, or $\frac{n}{4} = m + \frac{r}{4}$, where r is the remainder when dividing n by 4, and m is a whole number.

SRB p. U1-37

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Instruction

- The equation $\frac{n}{4} = m + \frac{r}{4}$ can be rewritten as $n = 4m + r$, and r is equal to 0, 1, 2, or 3. Use the Product of Powers Property to rewrite i^n.

  $i^n = i^{4m+r} = i^{4m} \bullet i^r$

- $i^4 = 1$, so i raised to a power that is a multiple of 4 will also equal 1: $i^{4m} = (i^4)^m = (1)^m = 1$.

- The expression i^r will determine the value of i^n.

- Use i^0, i^1, i^2, and i^3 to find i^n.

  If $r = 0$, then:

  $i^n = i^{4m+r}$

  $i^n = i^{4m} \bullet i^r$

  $i^n = 1 \bullet i^0$

  $i^n = 1 \bullet 1 = 1$

  If $r = 1$, then:

  $i^n = i^{4m+r}$

  $i^n = i^{4m} \bullet i^r$

  $i^n = 1 \bullet i^1$

  $i^n = i$, or $\sqrt{-1}$

  If $r = 2$, then:

  $i^n = i^{4m+r}$

  $i^n = i^{4m} \bullet i^r$

  $i^n = 1 \bullet i^2$

  $i^n = 1 \bullet -1 = -1$

  If $r = 3$, then:

  $i^n = i^{4m+r}$

  $i^n = i^{4m} \bullet i^r$

  $i^n = 1 \bullet i^3$

  $i^n = 1 \bullet -i = -i$, or $-\sqrt{-1}$

- Only the value of the remainder when n is divided by 4 is needed to simplify i^n.

  $i^0 = 1$, $i^1 = i$, $i^2 = -1$, and $i^3 = -i$

- Properties of exponents, along with replacing i with its equivalent value of $\sqrt{-1}$, can be used to simplify the expression i^n. Start with $n = 0$.

  $i^0 = 1$

  $i^1 = i = \sqrt{-1}$

  $i^2 = \left(\sqrt{-1}\right)^2 = -1$

- When simplifying i^3, use the property $i^3 = i^2 \bullet i^1$, and the values of i^2 and i.

  $i^3 = i^2 \bullet i = (-1) \bullet \sqrt{-1} = -\sqrt{-1}$, or $-i$

- When simplifying i^4, use the property $i^4 = i^2 \bullet i^2$, and the value of i^2.

  $i^4 = i^2 \bullet i^2 = (-1) \bullet (-1) = 1$

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Instruction

Common Errors/Misconceptions

- incorrectly identifying the real and imaginary parts of a complex number
- assuming that a real number isn't part of the complex number system
- incorrectly dividing a power of i by 4
- incorrectly simplifying the expressions i^2, i^3, and i^4, including stating $i^2 = 1$

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Instruction

Guided Practice 1.3.1

Example 1

Identify the real and imaginary parts of the complex number $8+\frac{1}{3}i$.

1. Identify the real part of the complex number.

   Identify the part that is not a multiple of i.

   8 is not a multiple of i.

   The real part of $8+\frac{1}{3}i$ is 8.

2. Identify the imaginary part of the complex number.

   Identify the part that is a multiple of i.

   $\frac{1}{3}$ is a multiple of i.

   The imaginary part of $8+\frac{1}{3}i$ is the term $\frac{1}{3}i$.

Example 2

Rewrite the complex number $2i$ using a radical.

1. Replace i with $\sqrt{-1}$.

   $2i = 2 \bullet \sqrt{-1}$

2. Place the squared value of any whole multiples of i under the radical sign.

   $2 \bullet \sqrt{-1} = \sqrt{2^2 \bullet (-1)}$

3. Simplify the radical expression.

   $\sqrt{2^2 \bullet (-1)} = \sqrt{4 \bullet (-1)} = \sqrt{-4}$

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Instruction

Example 3

Rewrite the radical $\sqrt{-32}$ using the imaginary unit i.

1. Rewrite the radicand as the product of –1 and a positive value.

   $\sqrt{-32} = \sqrt{(-1) \bullet 32}$

2. Rewrite the radical $\sqrt{-1}$ as i.

   $\sqrt{(-1) \bullet 32} = \sqrt{-1} \bullet \sqrt{32} = i \bullet \sqrt{32}$

3. If possible, rewrite the real number radicand as the product of two factors, where one factor is the largest perfect square factor of the radicand.

   32 = 16 • 2, and 16 is the factor of 32 that is the largest perfect square.

   $i \bullet \sqrt{32} = i \bullet \sqrt{16 \bullet 2}$

4. Rewrite the radicand as the product of two roots, and simplify the square root of the perfect square. Then, write the result as the coefficient of i.

   $i \bullet \sqrt{16 \bullet 2} = i \bullet \sqrt{4^2 \bullet 2} = 4i\sqrt{2}$

Example 4

Simplify i^{57}.

1. Find the remainder of the power of i when divided by 4.

   14 • 4 = 56; therefore, 57 ÷ 4 = 14 remainder 1.

   The remainder is 1.

2. Use the remainder to simplify the power of i.

   $i^{57} = (i^4)^{14} \bullet i^1 = 1 \bullet i^1 = i$

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Instruction

Example 5

Impedance, Z, is the measure of a circuit's opposition to the flow of current. Complex numbers are used to represent the impedance of a circuit. The resistance, R, is the real part of the impedance, and the reactance, X, is the coefficient of the imaginary unit i. So, impedance is $R + Xi$, where R and X are both measured in ohms. A certain circuit has a resistance of 18 ohms and a reactance of 2 ohms. Use a complex number to represent the circuit's impedance.

1. Use the resistance to write the real part of the complex number.

   The resistance is 18 ohms; therefore, the real part of the number is 18.

2. Use the reactance to write the imaginary part of the complex number.

   The reactance is the coefficient of i.

   The reactance is 2 ohms; therefore, the coefficient of i is 2.

   The imaginary part of the number is $2i$.

3. The complex representation of impedance is the sum of the real and imaginary parts.

   The circuit's impedance in ohms is $18 + 2i$.

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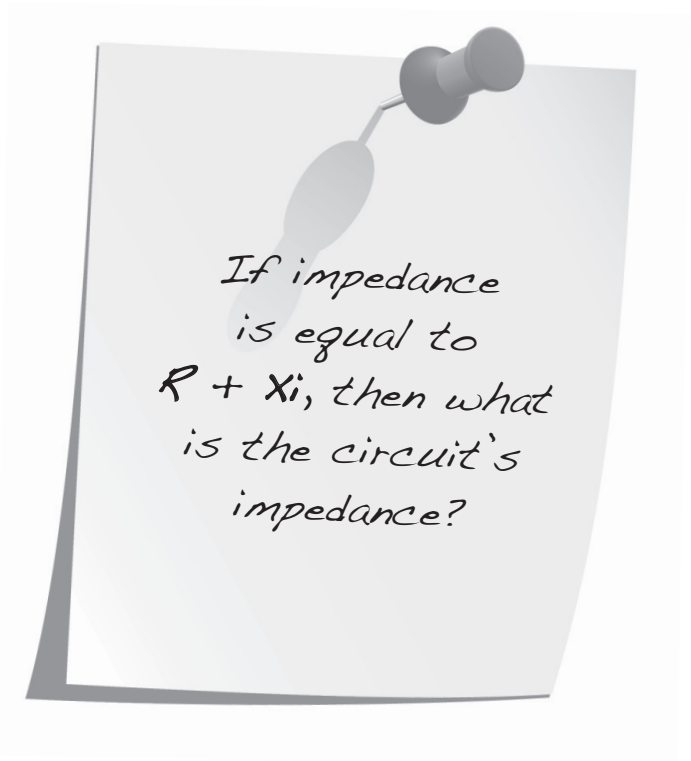
Problem-Based Task 1.3.1: Representing Impedance

SMP	
1	2
3	4 ✓
5	6 ✓
7	8

Impedance, *Z*, is the measure of a circuit's opposition to the flow of current. Complex numbers are used to represent the impedance of a circuit. A certain circuit has a resistance of 21 ohms and a reactance of 1.25 ohms. If impedance is equal to $R + Xi$, then what is the circuit's impedance?



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Problem-Based Task 1.3.1: Representing Impedance

Coaching

a. Which quantity will be the real part of the complex number?

b. Which quantity will be the multiple of i in the complex number?

c. How is the multiple of i used to write the imaginary part of the complex number?

d. How are the real and imaginary parts used to write a complex number?

e. What is the circuit's impedance?

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Instruction

Problem-Based Task 1.3.1: Representing Impedance

Coaching Sample Responses

a. Which quantity will be the real part of the complex number?

The resistance, 21, is the real part of the complex number.

b. Which quantity will be the coefficient of i in the complex number?

The reactance, 1.25, is the coefficient of i in the complex number.

c. How is the coefficient of i used to write the imaginary part of the complex number?

The product of the coefficient of i and i is the imaginary part: $1.25 \cdot i = 1.25i$.

d. How are the real and imaginary parts used to write a complex number?

The sum of the real and imaginary parts is a complex number.

$21 + 1.25i$

e. What is the circuit's impedance?

The circuit's impedance is $21 + 1.25i$.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.3.1: Defining Complex Numbers, *i*, and i^2

Identify the real and imaginary parts of each complex number.

1. $64 - 7i$

2. $39i$

Simplify the radical and use the imaginary unit *i*.

3. $\sqrt{-162}$

4. $\sqrt{-49}$

Rewrite each imaginary number using a radical instead of the imaginary unit *i*.

5. $4i$

6. $5i\sqrt{5}$

Simplify each imaginary number using the properties of exponents.

7. i^{102}

8. i^{15}

Write a complex number to represent the impedance of each circuit. The resistance, *R*, is the real part, and the reactance, *X*, is the coefficient of the imaginary unit *i*.

9. $R = 34$ ohms; $X = 3$ ohms

10. $R = 13$ ohms; $X = 2.4$ ohms

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Lesson 1.3.2: Adding and Subtracting Complex Numbers

Warm-Up 1.3.2

Angela's class is putting on a performance. Tickets to the show are $8 each, and snacks are sold during the performance for $1 each.

1. Let x equal the number of tickets sold. Write an expression to show the total money earned from ticket sales.

2. Angela estimates that the number of snacks sold will be approximately equal to half the number of tickets sold. Use x, the number of tickets sold, to write an expression to estimate the total money earned from snacks.

3. Use the expressions for money earned from tickets sold and money earned from snacks sold to write an expression that can be used to estimate the total amount of money earned.

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Instruction

Lesson 1.3.2: Adding and Subtracting Complex Numbers

Common Core State Standard

N–CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Warm-Up 1.3.2 Debrief

1. Let x equal the number of tickets sold. Write an expression to show the total money earned from ticket sales.

   Each ticket sells for \$8. The total money earned from ticket sales is the number of tickets sold, x, times the price of each ticket, \$8.

   $8 \bullet x = 8x$

2. Angela estimates that the number of snacks sold will be approximately equal to half the number of tickets sold. Use x, the number of tickets sold, to write an expression to estimate the total money earned from snacks.

   If x is the number of tickets sold, and only half of ticket holders buy snacks, then $\frac{1}{2}x$ or $0.5x$ represents the number of people who buy snacks.

   The total money earned from snacks is found by multiplying the number of snacks sold, $0.5x$, by the cost of each snack, \$1.

   $0.5x \bullet 1 = 0.5x$

3. Use the expressions for money earned from tickets sold and money earned from snacks sold to write an expression that can be used to estimate the total amount of money earned.

   The total money earned is the sum of the money earned from tickets and money earned from snacks: $8x + 0.5x$.

   Add the coefficients of the variable x to simplify the expression.

   $8x + 0.5x = (8 + 0.5)x = 8.5x$

Connection to the Lesson

- In this lesson, students will expand their understanding of finding the sum or difference of algebraic terms to finding the sum or difference of two imaginary numbers by adding the coefficients of i.
- Students will also find sums and differences of complex numbers with both real and imaginary parts.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- combining like terms in polynomials (8.EE.7b)
- using the Commutative Property of Addition to reorder the terms in sums and differences (6.EE.3)
- simplifying powers of i (N–CN.1)

Introduction

Finding sums and differences of complex numbers is similar to finding sums and differences of expressions involving numeric and algebraic quantities. The real part of a complex number is similar to the numeric quantities of an algebraic expression, and the imaginary part of a complex number is similar to the algebraic quantities of an algebraic expression. Before finding sums or differences, each complex number should be in the form $a + bi$. If i is raised to a power of n greater than 4, use the remainder of $n \div 4$ to simplify i^n.

Key Concepts

- To add or subtract two complex numbers, first add or subtract the real parts of the complex number, then add or subtract the coefficients of i.
- The resulting sum or difference of the real parts and the imaginary parts is the solution.
- In the following expressions, a, b, c, and d are real numbers.
- When finding the sum of two complex numbers, combine like terms.

  $(a + bi) + (c + di) = a + c + bi + di = (a + c) + (b + d)i$

- $a + c$ is the real part of the sum, and $(b + d)i$ is the imaginary part of the sum.
- When finding a difference, rewrite the difference as the sum of the opposite of the complex number being subtracted, then combine like terms.

  $(a + bi) - (c + di) = (a + bi) + (-c - di) = a + bi - c - di = (a - c) + (b - d)i$

- $a - c$ is the real part of the difference, and $(b - d)i$ is the imaginary part of the difference.
- The sum or difference of two complex numbers can be wholly real (having only real parts), wholly imaginary (having only imaginary parts), or complex (having both real and imaginary parts).

SRB p. U1-43

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Instruction

Common Errors/Misconceptions

- incorrectly subtracting two complex numbers by not rewriting the difference as the addition of an opposite
- adding or subtracting two unlike powers of i, such as $2i^2 + 3i$

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Guided Practice 1.3.2

Example 1

Is $(6 + 5i) + (8 - 3i)$ wholly real or wholly imaginary, or does it have both a real and an imaginary part?

1. Find the sum of the real parts.

   The real parts are 6 and 8.

   $6 + 8 = 14$

2. Find the sum of the imaginary parts by summing the coefficients of i.

   The imaginary parts are $5i$ and $-3i$.

   $5i + (-3i) = [5 + (-3)]i = 2i$

3. Write the solution as the sum of the real and imaginary parts.

   $14 + 2i$

4. Use the form of the sum to determine if it is wholly real or wholly imaginary, or if it has both a real and an imaginary part.

   A wholly real number has no imaginary part, a wholly imaginary number has no real part, and a complex number includes both parts.

   $14 + 2i$ has a real part, 14, and an imaginary part, $2i$.

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Instruction

Example 2

Is $(5 + 6i) – (5 – 3i)$ wholly real or wholly imaginary, or does it have both a real and an imaginary part?

1. Rewrite the subtraction as the addition of the opposite.

   $(5 + 6i) – (5 – 3i)$

   $= (5 + 6i) + (–5 + 3i)$

   $= 5 + 6i – 5 + 3i$

2. Find the sum or difference of the real parts.

   $5 – 5 = 0$

3. Find the sum or difference of the imaginary parts.

   $6i + 3i = 9i$

4. Find the sum of the real and imaginary parts.

   $0 + 9i = 9i$

5. Use the form of the sum to determine if it is wholly real or wholly imaginary, or if it has both a real and an imaginary part.

   $9i$ has only an imaginary part, $9i$, so the difference is wholly imaginary.

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Instruction

Example 3

Is $(12 + 4i) + (-18 - 4i)$ wholly real or wholly imaginary, or does it have both a real and an imaginary part?

1. Find the sum or difference of the real parts.

   $12 + (-18) = -6$

2. Find the sum or difference of the imaginary parts.

   $4i + (-4i) = 0i$

3. Find the sum of the real and imaginary parts.

   $-6 + 0i = -6$

4. Use the form of the sum to determine if it is wholly real or wholly imaginary, or if it has both a real and an imaginary part.

   -6 has only a real part, -6, so the sum is wholly real.

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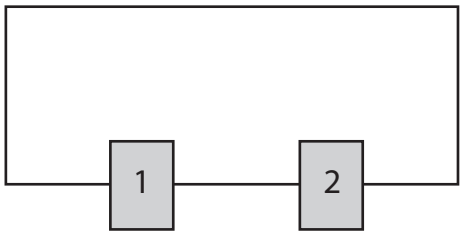
Instruction

Example 4

A circuit in series is a circuit in which the power flows in only one direction and goes through each part of the circuit. A flashlight with two batteries is a series circuit, because the power goes through the batteries to the lightbulb. Impedance, Z, is the measure of a circuit's opposition to the flow of current. The impedance of an element in the circuit can be represented using the complex number $R + Xi$, where R is the element's resistance and X is the element's reactance. If two elements are used in a circuit in series, the total impedance is the sum of the impedance of each element. The following diagram of a circuit contains two elements, 1 and 2, in series.



The total impedance of the circuit is the sum of the impedance of elements 1 and 2. Element 1 has a resistance of 25 ohms and a reactance of 1 ohm. Element 2 has a resistance of 20 ohms and a reactance of 1.5 ohms. What is the total impedance of the circuit?

1. Write the impedance of each element as a complex number.

   Impedance = $R + Xi$

   Element 1: $25 + i$

   Element 2: $20 + 1.5i$

2. The two elements are in series in a circuit. The total impedance of the circuit is the sum of the impedance of each element. Write the sum.

   Total impedance: $(25 + i) + (20 + 1.5i)$

3. Evaluate the sum.

   $(25 + i) + (20 + 1.5i)$

   $= 25 + 20 + i + 1.5i$

   $= 45 + 2.5i$

   The total impedance is $45 + 2.5i$.

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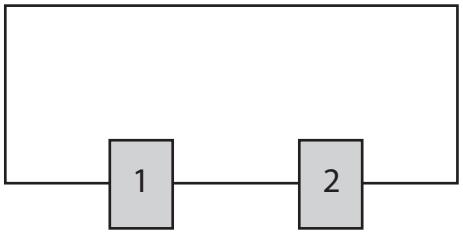
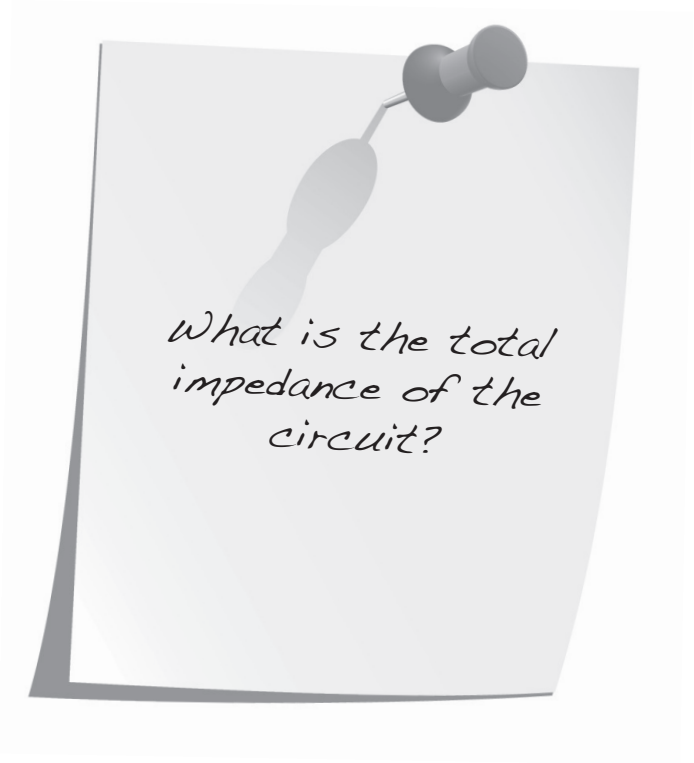
Problem-Based Task 1.3.2: Elements in Series in a Circuit

SMP	
1 ✓	2 ✓
3	4 ✓
5	6
7	8

The impedance of a circuit can be represented using the complex number $R + Xi$, where R is the element's resistance and X is the element's reactance. Both R and X are measured in ohms. If two elements are in a circuit in series, the total impedance is the sum of the impedance of each element. The following diagram of a circuit contains two elements, 1 and 2, in series.



Element 1 has a resistance of 30.5 ohms and a reactance of 2.8 ohms. Element 2 has a resistance of 19 ohms and a reactance of 3 ohms. What is the total impedance of the circuit?



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Problem-Based Task 1.3.2: Elements in Series in a Circuit

Coaching

a. How can the resistance and reactance be used to write a complex number to represent the impedance of each element?

b. What is the impedance of each element?

c. What operation is used to find the total impedance if the elements are in series?

d. Describe how to find the sum or difference of two complex numbers by using their real and imaginary parts.

e. What is the total impedance of the circuit?

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Instruction

Problem-Based Task 1.3.2: Elements in Series in a Circuit

Coaching Sample Responses

a. How can the resistance and reactance be used to write a complex number to represent the impedance of each element?

The impedance of each element can be written in the form $R + Xi$, where R is the resistance and X is the reactance.

b. What is the impedance of each element?

Element 1: $30.5 + 2.8i$

Element 2: $19 + 3i$

c. What operation is used to find the total impedance if the elements are in series?

Addition is used to find total impedance. Therefore, find the sum of the impedance of each element.

d. Describe how to find the sum or difference of two complex numbers by using their real and imaginary parts.

To find the sum of two complex numbers, find the sum of the real parts and the sum of the imaginary parts.

To find the difference of two complex numbers, rewrite the difference as the sum of the opposite of the complex number being subtracted, then find the sum of the real parts and the sum of the imaginary parts.

e. What is the total impedance of the circuit?

$(30.5 + 2.8i) + (19 + 3i)$

$= (30.5 + 19) + (2.8 + 3)i$

$= 49.5 + 5.8i$

The total impedance of the circuit is $49.5 + 5.8i$.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.3.2: Adding and Subtracting Complex Numbers

Find each sum or difference. Identify whether each sum or difference is wholly real or wholly imaginary, or if it has both a real and an imaginary part.

1. $(8 - 4i) + (11 + 4i)$

2. $(15 + 3i) + (14 + 8i)$

3. $(20 - 10i) - (13 - i)$

4. $(7 + 14i) - (7 - 14i)$

5. $(1 + i) + (22 + 12i)$

6. $(-5 + 2i) - (5 + 2i)$

7. $(-43 - 13i) - (-31 + i)$

8. $(9 + 6i) + (-9 + 6i)$

Use the following information to solve problems 9 and 10.

> The impedance of an element can be written in the form $R + Xi$, where R is the resistance and X is the reactance. Both R and X are measured in ohms. For two elements in series in a circuit, the total impedance is the sum of each element's impedance. Find the total impedance of two given elements if the elements are in series in a circuit.

9. Element 1: $R = 10.5$ ohms, $X = 2.1$ ohms

   Element 2: $R = 12$ ohms, $X = 1.7$ ohms

10. Element 1: $R = 33$ ohms, $X = 4$ ohms

    Element 2: $R = 38$ ohms, $X = 3.6$ ohms

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Lesson 1.3.3: Multiplying Complex Numbers

Warm-Up 1.3.3

A frame shop cuts custom mats to fit the size of any picture. A customer wants several mats cut for different sizes of pictures. He wants each mat cut to be 6 inches longer than the length of the picture and 5 inches wider than the width of the picture. Let x be the length of a picture, and y be the width. The cost of the mat is based on the perimeter and area of the mat before the hole for the picture is cut out. The perimeter of a rectangle is found using $2l + 2w$, where l is the rectangle's length and w is the rectangle's width. The area of a rectangle is found using lw.

1. Using x to represent the length of a picture, write an expression to show the length of a mat.

2. Using y to represent the width of a picture, write an expression to show the width of a mat.

3. What is the perimeter of any mat for a picture with length x and width y?

4. What is the area of any mat for a picture with length x and width y?

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Lesson 1.3.3: Multiplying Complex Numbers

Common Core State Standard

N–CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Warm-Up 1.3.3 Debrief

1. Using x to represent the length of a picture, write an expression to show the length of a mat.

   The length of a mat is 6 inches longer than the length of the picture.

   Expression: $x + 6$

2. Using y to represent the width of a picture, write an expression to show the width of a mat.

   The width of a mat is 5 inches longer than the width of the picture.

   Expression: $y + 5$

3. What is the perimeter of any mat for a picture with length x and width y?

   Use the formula for perimeter, $2l + 2w$, and replace l with the expression for mat length and w with the expression for mat width. Then, distribute and combine like terms.

   $2l + 2w = 2(x + 6) + 2(y + 5) = 2x + 12 + 2y + 10 = 2x + 2y + 22$

4. What is the area of any mat for a picture with length x and width y?

   Use the formula for area, lw, and replace l with the expression for mat length and w with the expression for mat width.

   $lw = (x + 6)(y + 5)$

   Apply the Distributive Property and multiply each term in the first binomial by each term in the second binomial.

   $(x + 6)(y + 5) = xy + 5x + 6y + 30$

Connection to the Lesson

- Students will extend their understanding of multiplying binomials to find the products of two complex numbers in the form $a + bi$.
- Students will simplify expressions by combining like terms.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- finding the product of two binomials (A–APR.1)
- simplifying powers of i (N–CN.1)
- adding two fractions with different denominators (5.NF.1)

Introduction

The product of two complex numbers is found using the same method that is used to multiply two binomials. As when multiplying binomials, both terms in the first complex number need to be multiplied by both terms in the second complex number. The product of the two binomials $x + y$ and $x – y$ is the difference of squares: $x^2 – y^2$. If y is an imaginary number, this difference of squares will be a real number because $i \bullet i = –1$: $(a + bi)(a – bi) = a^2 – (bi)^2 = a^2 – b^2(–1) = a^2 + b^2$.

Key Concepts

- Simplify any powers of i before evaluating products of complex numbers.
- In the following expressions, a, b, c, and d are real numbers.
- To multiply two complex numbers, apply the Distributive Property. Then, use the definition of i^2 to substitute –1 for i^2. Finally, combine like terms.

  $(a + bi) \bullet (c + di)$

  $= a \bullet c + a \bullet di + bi \bullet c + bi \bullet di$

  $= ac + (ad)i + (bc)i + (bd)i^2$

  $= ac + (ad + bc)i + bd(–1)$

  $= ac + (ad + bc)i – bd$

  $= (ac – bd) + (ad + bc)i$

- $ac – bd$ is the real part of the product, and $ad + bc$ is the coefficient of the imaginary unit i in the imaginary part of the product.
- A **complex conjugate** is a complex number that when multiplied by another complex number produces a value that is wholly real.
- The product of a complex number and its conjugate is a real number.
- The complex conjugate of $a + bi$ is $a – bi$, and the complex conjugate of $a – bi$ is $a + bi$.

SRB p. U1-49

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- The product of a complex number and its conjugate is the difference of squares, $a^2 - (bi)^2$, which can be simplified. The result is a wholly real number.

  $a^2 - (bi)^2 = a^2 - b^2i^2 = a^2 - b^2 \bullet (-1) = a^2 + b^2$

Common Errors/Misconceptions

- incorrectly finding the product of two complex numbers
- incorrectly identifying the complex conjugate of $a + bi$ as a value such as $-a + bi$, $a + bi$, or $-a - bi$

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Guided Practice 1.3.3

Example 1

Find the result of $i \cdot 5i$.

1. Multiply the two terms.

   $i \cdot 5i = 5i^2$

2. Simplify any powers of i.

   $i^2 = -1$

   Substitute -1 in for i^2.

   $5i^2 = 5(-1) = -5$

   The product of $i \cdot 5i$ is -5.

Example 2

Find the result of $(7 + 2i)(4 + 3i)$.

1. Apply the Distributive Property and multiply both terms in the first polynomial by both terms in the second polynomial.

   $(7 + 2i)(4 + 3i)$

   $= 7 \cdot 4 + 7 \cdot 3i + 2i \cdot 4 + 2i \cdot 3i$

2. Evaluate or simplify each term.

   $7 \cdot 4 + 7 \cdot 3i + 2i \cdot 4 + 2i \cdot 3i$

   $= 28 + 21i + 8i + 6i^2$

   $= 28 + 21i + 8i + 6(-1)$

   $= 28 + 21i + 8i - 6$

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3. Combine any real parts and any imaginary parts.

$28 + 21i + 8i - 6$

$= 28 - 6 + 21i + 8i$

$= 22 + 29i$

Example 3

Find the complex conjugate of $5 - i$. Use multiplication to verify your answer.

1. Find the complex conjugate.

   The complex conjugate of a number of the form $a - bi$ is $a + bi$; therefore, the complex conjugate of $5 - i$ is $5 + i$.

2. Multiply both terms in the first polynomial by both terms in the second polynomial, and then simplify the result.

$(5 - i)(5 + i)$

$= 25 + 5i - 5i - i^2$

$= 25 - i^2$

$= 25 - (-1) = 26$

3. Verify that the product of the complex number and its conjugate contains no imaginary units, i.

   The product of $(5 - i)$ and $(5 + i)$ is 26, which is a real number. Therefore, $(5 + i)$ is the complex conjugate of $(5 - i)$.

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Example 4

A parallel circuit has multiple pathways through which current can flow. The following diagram of a circuit contains two elements, 1 and 2, in parallel.

The impedance Z of an element can be represented using the complex number $R + Xi$, where R is the element's resistance and X is the element's reactance. Both are measured in ohms. If two elements are in a circuit in parallel, the total impedance is the sum of the reciprocals of each impedance. If the impedance of element 1 is Z_1 and the impedance of element 2 is Z_2, the total impedance of the two elements in parallel is $\frac{1}{Z_1}+\frac{1}{Z_2}$.

Element 1 has a resistance of 10 ohms and a reactance of 3 ohms. Element 2 has a resistance of 15 ohms and a reactance of 2 ohms. What is the total impedance of the circuit? Leave your result as a fraction.

1. Write each impedance as a complex number.

   Impedance = $R + Xi$

   Element 1: $10 + 3i$

   Element 2: $15 + 2i$

2. Find the reciprocal of each impedance.

   Element 1: $\frac{1}{10+3i}$

   Element 2: $\frac{1}{15+2i}$

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Instruction

3. Find a common denominator of the two fractions.

   Find the product of the two denominators. This will be the common denominator.

   $(10 + 3i)(15 + 2i)$

   $= 150 + 20i + 45i + 6i^2$

   $= 150 + 65i - 6 = 144 + 65i$

4. Find an equivalent fraction for each element with a common denominator.

   Multiply the numerator and denominator of the fraction for element 1 by the denominator of element 2. Use the product from the previous step to simplify $(10 + 3i)(15 + 2i)$.

   $$\frac{1}{10+3i} \bullet \frac{15+2i}{15+2i} = \frac{15+2i}{144+65i}$$

   Multiply the numerator and denominator of the fraction for element 2 by the denominator of element 1. Use the product from the previous step to simplify $(10 + 3i)(15 + 2i)$.

   $$\frac{1}{15+2i} \bullet \frac{10+3i}{10+3i} = \frac{10+3i}{144+65i}$$

5. Find the total impedance by summing the reciprocals of the impedance of each element.

   $$\frac{1}{10+3i} + \frac{1}{15+2i}$$
   $$= \frac{15+2i}{144+65i} + \frac{10+3i}{144+65i}$$
   $$= \frac{15+10+2i+3i}{144+65i}$$
   $$= \frac{25+5i}{144+65i}$$

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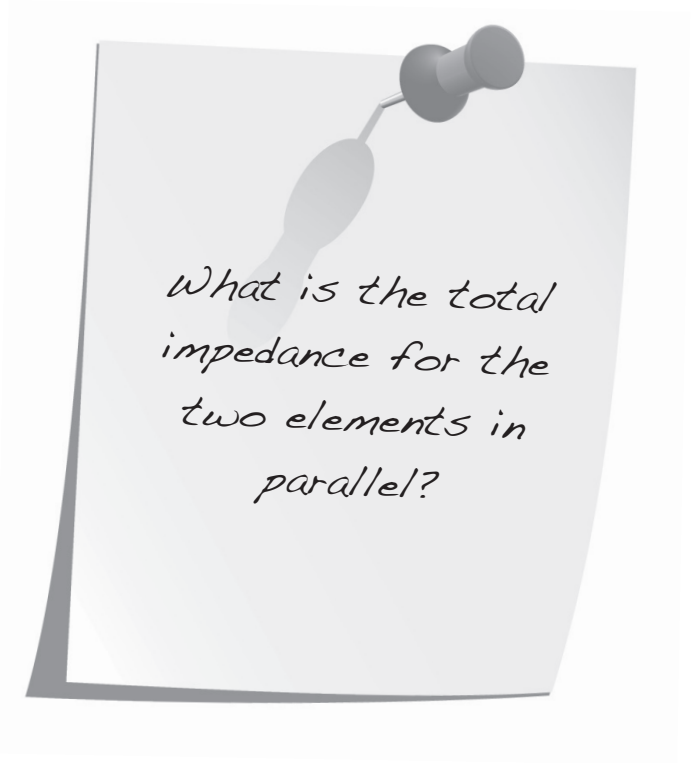
Problem-Based Task 1.3.3: Elements in Parallel in a Circuit

SMP	
1 ✓	2
3	4
5	6
7 ✓	8

The impedance Z of an element can be represented using the complex number $R + Xi$, where R is the element's resistance and X is the element's reactance. Both R and X are measured in ohms. The following diagram of a circuit contains two elements, 1 and 2, in parallel.

If the impedance of element 1 is $Z_1 = 15 + i$, and the impedance of element 2 is $Z_2 = 10 + 2i$, the total impedance of the two elements in parallel is $\frac{1}{Z_1} + \frac{1}{Z_2}$. What is the total impedance for the two elements in parallel? Leave your response as a fraction.



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Problem-Based Task 1.3.3: Elements in Parallel in a Circuit

Coaching

a. What is the reciprocal of each impedance?

b. How can a common denominator be calculated?

c. Find the common denominator.

d. Use the factor needed to calculate the common denominator to write an equivalent reciprocal for each impedance.

e. What is the sum of these reciprocals, written as a fraction?

f. What is the total impedance for the two elements in parallel, expressed as a fraction?

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Problem-Based Task 1.3.3: Elements in Parallel in a Circuit

Coaching Sample Responses

a. What is the reciprocal of each impedance?

$$\frac{1}{Z_1}=\frac{1}{15+i} \qquad \frac{1}{Z_2}=\frac{1}{10+2i}$$

b. How can a common denominator be calculated?

The product of the two existing denominators can be used as the common denominator.

c. Find the common denominator.

Apply the Distributive Property and substitute the definition of i^2 to simplify the product.

$(15 + i)(10 + 2i)$

$= 150 + 30i + 10i + 2i^2$

$= 150 + 30i + 10i + 2(–1)$

$= 148 + 40i$

d. Use the factor needed to calculate the common denominator to write an equivalent reciprocal for each impedance.

$$\frac{1}{Z_1}=\frac{1}{15+i}=\frac{1}{15+i}\bullet\frac{10+2i}{10+2i}=\frac{10+2i}{148+40i}$$

$$\frac{1}{Z_2}=\frac{1}{10+2i}=\frac{1}{10+2i}\bullet\frac{15+i}{15+i}=\frac{15+i}{148+40i}$$

e. What is the sum of these reciprocals, written as a fraction?

$$\frac{10+2i}{148+40i}+\frac{15+i}{148+40i}=\frac{25+3i}{148+40i}$$

f. What is the total impedance for the two elements in parallel, expressed as a fraction?

The total impedance for the two elements in parallel is $\frac{25+3i}{148+40i}$.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 1.3.3: Multiplying Complex Numbers

Find each product.

1. $(8 + 2i)(3 + i)$

2. $(5 - i)(6 + 4i)$

3. $(-7 + 5i)(12 + 9i)$

4. $(1 + i)(-15 + 2i)$

5. $(20 - 13i)(-4 + i)$

Find the complex conjugate of each number. Find the product of the complex number and its conjugate to verify your answer.

6. $34 + 14i$

7. $30 - 6i$

8. $-1 + i$

Use the following information to solve problems 9 and 10.

> The impedance Z of an element can be represented using the complex number $R + Xi$, where R is the element's resistance and X is the element's reactance. For two elements in a circuit in parallel, where the impedance of element 1 is Z_1 and the impedance of element 2 is Z_2, the total impedance of the two elements is $\frac{1}{Z_1} + \frac{1}{Z_2}$. Calculate the total impedance for each pair of elements. Leave your response as a fraction.

9. Element 1: $11 + i$
   Element 2: $14 + 2i$

10. Element 1: $30 + 4i$
    Element 2: $29 + 3i$

SWB p. 73; SRB Practice 1.3.3 p. U1-54

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Progress Assessment

Circle the letter of the best answer.

1. What is the imaginary part of the complex number $-53 + 29i$?

   a. -53
   b. $29i$
   c. $-29i$
   d. 53

2. Which value is equal to i^{51}?

   a. -1
   b. 1
   c. $-i$
   d. i

3. What is the result of $(25 + 16i) - (4 + 2i)$?

   a. $21 + 18i$
   b. $21 - 14i$
   c. $21 + 14i$
   d. $18 - 21i$

4. What is the result of $(10 + i)(5 - 8i)$?

   a. $50 - 8i$
   b. $15 - 7i$
   c. $42 - 85i$
   d. $58 - 75i$

5. What is the result of $3i(1 + 4i)$?

   a. $-12 + 3i$
   b. $1 + 7i$
   c. $1 + 12i$
   d. $12 + 3i$

6. What is the real part of the complex number $\frac{5}{9} - 3i$?

   a. -3
   b. $3i$
   c. $\frac{5}{9}$
   d. $-3i$

continued

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Assessment

7. Which value is equal to i^{29}?

   a. -1
   b. 1
   c. $-i$
   d. i

8. What is the result of $\left(24+i^{20}\right)+\left(-10+18i\right)$?

   a. $14 + 19i$
   b. $15 + 18i$
   c. $34 + 19i$
   d. $35 + 18i$

9. What is the result of $(15 + 8i)(15 - 8i)$?

   a. 289
   b. 161
   c. $225 - 64i$
   d. $161 + 240i$

10. What is the result of $(1 - 2i)(3 - 4i)$?

    a. $11 - 10i$
    b. $3 + 8i$
    c. $3 - 6i$
    d. $-5 - 10i$

Use what you have learned about operating with complex numbers to complete the problem that follows.

11. A circuit in parallel consists of two elements. The impedance of element 1 is $Z_1 = 21 + 2i$ and the impedance of element 2 is $Z_2 = 11 + 3i$. The sum of $\frac{1}{Z_1}+\frac{1}{Z_2}$ is the total impedance of the two elements in a circuit in parallel. What is the total impedance? Leave your answer as a fraction.

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Unit 1 Assessment

Circle the letter of the best answer.

1. Which sum is irrational?

   a. $\frac{2}{5}+\sqrt[3]{27}$

   b. $4^{6}+49^{\frac{1}{2}}$

   c. $\sqrt[3]{1000}+2.1$

   d. $12^{\frac{2}{3}}+35$

2. Let $\sqrt[5]{x^{2}}=3$. What is the approximate value of x?

   a. 1.55

   b. 7.50

   c. 15.59

   d. 243.00

3. What is the result of $(-6x + 7) + (19x - 1)$?

   a. $25x + 8$

   b. $13x + 6$

   c. $19x$

   d. $25x + 6$

4. What is the result of $\left(x^{5}+x\right)-\left(4x+12\right)$?

   a. $-2x - 12$

   b. $x^{5}-3x-12$

   c. $x^{5}+5x+12$

   d. $x^{5}-3x+12$

5. What is the result of $(-x^2 + 7)(2x^2 + 3x)$?

   a. $-x^{4}+11x^{2}+21x$

   b. $2x^{4}+3x^{3}+14x^{2}+21x$

   c. $-2x^{4}-3x^{3}+14x^{2}+21x$

   d. $-2x^2 + 11x + 21$

6. What is the real part of the complex number $-31 + 15i$?

   a. 15

   b. –31

   c. i

   d. $15i$

continued

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7. Which value is equal to i^{29}?

   a. i
   b. $-i$
   c. 1
   d. -1

8. What is the result of $(18 - i) + (29 + 4i)$?

   a. 47
   b. $47 + 3i$
   c. $-11 - 5i$
   d. $-11 + 3i$

9. What is the value of $125^{\frac{2}{3}}$?

   a. 5
   b. $\sqrt{125}$
   c. 25
   d. 15,625

10. What is the result of $(7 + 3i) - (2 - 4i)$?

    a. $9 - i$
    b. $9 + 7i$
    c. $5 - i$
    d. $5 + 7i$

11. What is the result of $(3 + 5i)(6 - 4i)$?

    a. $38 + 18i$
    b. $-2 + 18i$
    c. $18 + 18i$
    d. $-2 - 20i$

12. What is the result of $(-8 + 2i)(1 - 3i)$?

    a. $-2 + 26i$
    b. $-7 - i$
    c. $-14 + 26i$
    d. $-8 - 6i$

continued

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Assessment

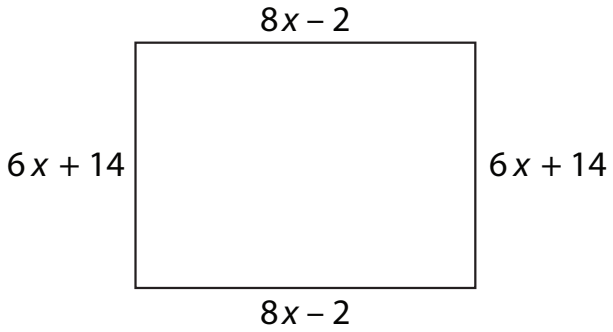
Use your knowledge of polynomials, rational numbers, and complex numbers to answer the following questions.

13. A town conducts a census of its population biannually, or every 2 years. The equation $y = 31{,}000(1.04)^{\frac{t}{2}}$ can be used to estimate the town's population, where t is the number of years after the year 2000.

    a. What was the town's approximate population in 2006?

    b. What will be the town's estimated population in 2025?

14. The perimeter of a rectangle is the sum of the lengths of its sides. The area of a rectangle is the product of the rectangle's length and width. If l is the length of a rectangle and w is the width, then its perimeter is equal to $2l + 2w$ and its area is equal to lw. Use the rectangle shown to answer the questions that follow.



    a. What is the rectangle's perimeter, expressed as a polynomial in terms of x?

    b. What is the rectangle's area, expressed as a polynomial in terms of x?

15. The impedance of element 1 is $Z_1 = 5 + 2i$ and the impedance of element 2 is $Z_2 = 10 + i$. The elements are in a circuit in parallel, and the sum of $\frac{1}{Z_1} + \frac{1}{Z_2}$ is the total impedance of the circuit.

    a. What is a common denominator for finding the total impedance?

    b. What is the total impedance? Leave your answer as a fraction.

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Answer Key

Lesson 1: Working with the Number System

Pre-Assessment, p. U1-1

1. b
2. a
3. c
4. c
5. d

Warm-Up 1.1.1, p. U1-4

1. Because both towns begin with the same population and Tinyville has a higher growth rate, it will have a greater population in 5 years.
2. $p_t = 10{,}000 \cdot (1.018)^t$
3. The approximate population of Tinyville in 10 years will be 12,801 people.

Practice 1.1.1: Defining, Rewriting, and Evaluating Rational Exponents, p. U1-15

1. $\sqrt[4]{5}$
2. $\sqrt[9]{g^2}$ or $\left(\sqrt[9]{g}\right)^2$
3. $-\left(\sqrt[7]{10^3}\right)$
4. $20^{\frac{3}{2}}$
5. $r^{\frac{s}{6}}$
6. 55.902
7. 3.344
8. 1,024 bacteria
9. $10,266.02
10. $2,441.64

Warm-Up 1.1.2, p. U1-17

1. approximately $208.03
2. $225.00

Practice 1.1.2: Rational and Irrational Numbers and Their Properties, p. U1-30

1. $\frac{1}{g^{\frac{4}{9}}}$
2. $8^{\frac{41}{70}}$
3. $19^{\frac{2}{3}}$
4. rational
5. irrational
6. rational
7. $x = 216$
8. $d = 76.4$
9. $p = 1.06$
10. $r = 51.84$; $y = 5(51.84)^t$

Progress Assessment, p. U1-32

1. d
2. d
3. c
4. a
5. c
6. a
7. c
8. b
9. b
10. a
11. a. approximately 55,167 people
    b. approximately 57,668 people

Lesson 2: Operating with Polynomials

Pre-Assessment, p. U1-34

1. b
2. d
3. d
4. a
5. b

Warm-Up 1.2.1, p. U1-36

1. 37 feet
2. $(23 + x)$ feet
3. $(x + 7)$ feet

Practice 1.2.1: Adding and Subtracting Polynomials, p. U1-46

1. $7x^3 - 3$
2. $x^4 + x^3 + 8x + 2$
3. $-4x^2 + 22x + 20$
4. $5x^5 - 4x^4 - 3x^2 - 2x$
5. $x^2 - 12x - 9$
6. $7x^4 - 2x^3 + 9$
7. $(4x + 64)$ cm
8. $(8x + 16)$ cm
9. $(2x^2 + 16x + 2)$ cm
10. $(x^2 + 6x + 9)$ cm

Warm-Up 1.2.2, p. U1-48

1. 96 ft^2
2. 108 ft^2
3. x^3 ft^2

Practice 1.2.2: Multiplying Polynomials, p. U1-60

1. $x^2 + 3x - 70$
2. $3x^4 + 5x^3 + 12x^2 + 20x$
3. $2x^5 + x^4 - 12x^2 + 3$
4. $x^7 + 2x^6 + 4x^5 - 2x^2 - 4x - 8$
5. $20x^3 + 18x^2 - 56x - 24$
6. $-x^6 - 4x^5 - 3x^4 + 4x^2 + 4$
7. $(3x^2 + 43x + 14)$ units^2
8. $(-x^3 + 12x^2 + 8x - 96)$ units^2
9. $(5x^3 + 10x^2 - 20x - 40)$ units^2
10. $(8x^4 + 4x^2 - 24)$ units^2

Progress Assessment, p. U1-61

1. d
2. b
3. a
4. d
5. c
6. a
7. a
8. b
9. c
10. b
11. a. $(18x + 2)$ feet
    b. $(20x^2 + 7x - 6)$ ft^2

Lesson 3: Operating with Complex Numbers

Pre-Assessment, p. U1-63

1. d
2. b
3. d
4. a
5. d

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Warm-Up 1.3.1, p. U1-66

1. Each person would get 2 cookies, and there would be 6 cookies left over.
2. Each person would get 3 slices, and there would be no slices left over.
3. Martin can buy 8 T-shirts, and he will have $8 left over.

Practice 1.3.1: Defining Complex Numbers, i, and i^2, p. U1-77

1. real: 64; imaginary: $-7i$
2. real: 0 or none; imaginary: $39i$
3. $9i\sqrt{2}$
4. $7i$
5. $\sqrt{-16}$
6. $\sqrt{-125}$
7. -1
8. $-i$
9. $34 + 3i$
10. $13 + 2.4i$

Warm-Up 1.3.2, p. U1-78

1. $8x$
2. $0.5x$
3. $8.5x$

Practice 1.3.2: Adding and Subtracting Complex Numbers, p. U1-89

1. 19; wholly real
2. $29 + 11i$; both real and imaginary parts
3. $7 - 9i$; both real and imaginary parts
4. $28i$; wholly imaginary
5. $23 + 13i$; both real and imaginary parts
6. -10; wholly real
7. $-12 - 14i$; both real and imaginary parts
8. $12i$; wholly imaginary
9. $22.5 + 3.8i$
10. $71 + 7.6i$

Warm-Up 1.3.3, p. U1-90

1. Expression for the length of a mat: $x + 6$
2. Expression for the width of a mat: $y + 5$
3. Perimeter of any mat: $2x + 2y + 22$
4. Area of any mat: $xy + 5x + 6y + 30$

Practice 1.3.3: Multiplying Complex Numbers, p. U1-101

1. $22 + 14i$
2. $34 + 14i$
3. $-129 - 3i$
4. $-17 - 13i$
5. $-67 + 72i$
6. $34 - 14i$; 1,352
7. $30 + 6i$; 936
8. $-1 - i$; 2
9. $\frac{25+3i}{152+36i}$
10. $\frac{59+7i}{858+206i}$

Progress Assessment, p. U1-102

1. b
2. c
3. c
4. d
5. a
6. c
7. d
8. b
9. a
10. d
11. The total impedance is $\frac{32+5i}{225+85i}$.

Unit Assessment p. U1-104

1. d
2. c
3. b
4. b
5. c
6. b
7. a
8. b
9. c
10. d
11. a
12. a
13. a. approximately 34,871 people
    b. approximately 50,615 people
14. a. $(28x + 24)$ units
    b. $(48x^2 + 100x - 28)$ units^2
15. a. $48 + 25i$
    b. $\frac{15+3i}{48+25i}$

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Student Book Answer Key

Lesson 1: Working with the Number System

Practice 1.1.1: Defining, Rewriting, and Evaluating Rational Exponents, p. U1-9

1. $\sqrt[2]{(-14)^4}$ or $(-14)^2$
2. $\sqrt[12]{8^w}$
3. $9^{\frac{1}{7}}$
4. $j^{\frac{k}{4}}$
5. $(-12)^{\frac{6}{8}}$ or $(-12)^{\frac{3}{4}}$
6. 1.732
7. –243
8. 1,966,405 people
9. $1,131.75
10. 22,563 subscribers

Practice 1.1.2: Rational and Irrational Numbers and Their Properties, p. U1-18

1. $n^{\frac{9}{22}}$
2. $15^{\frac{36}{5}}$
3. $20^{\frac{17}{6}}$
4. rational
5. irrational
6. irrational
7. $w = 343$
8. $m = -5.4$
9. $n = 4$
10. $s = 1.05$; $y = 600(1.05)^w$

Lesson 2: Operating with Polynomials

Practice 1.2.1: Adding and Subtracting Polynomials, p. U1-27

1. 22
2. $-7x^3 - x^2 - 6$
3. $-x^3 + x^2 + 2x - 14$
4. $4x^6 + x^3 - x^2$
5. $-x^3 + 6x^2 + x - 6$
6. $8x^3 + 2x^2 - 7$
7. $(10x - 14)$ cm
8. $(11x + 24)$ cm
9. $(23x + 23)$ cm
10. $(4x^2 + 24x - 8)$ cm

Practice 1.2.2: Multiplying Polynomials, p. U1-34

1. $x^2 + 11x + 24$
2. $x^5 - 9x^3 + 3x^2 - 27$
3. $2x^3 + 21x^2 + 4x - 60$
4. $3x^6 + 24x^5 - 15x^4 - x^2 - 8x + 5$
5. $x^5 + 2x^4 - 2x^3 - x^2 + 2x - 6$
6. $12x^4 - x^3 + 15x^2 + 4x$
7. $(2x^2 - 23x + 60)$ units^2
8. $(-x^5 - x^4 + 2x^2 + 2x)$ units^2
9. $(5x^3 + 2x^2 + 5x + 2)$ units^2
10. $(24x^2 - 45x + 21)$ units^2

Lesson 3: Operating with Complex Numbers

Practice 1.3.1: Defining Complex Numbers, *i*, and i^2, p. U1-42

1. real: –51; imaginary: $0i$ or none
2. real: 48; imaginary: $32i$
3. $3i\sqrt{3}$
4. $12i$
5. $-\sqrt{-100}$
6. $\sqrt{-115}$
7. $-i$
8. i
9. $50 + 6i$
10. $29 + 3.1i$

Practice 1.3.2: Adding and Subtracting Complex Numbers, p. U1-48

1. 8; wholly real
2. $16 - 6i$; both real and imaginary parts
3. $-15 + i$; both real and imaginary parts
4. $-6i$; wholly imaginary
5. $-13 + 2i$; both real and imaginary parts
6. $36 - 27i$; both real and imaginary parts
7. $7i$; wholly imaginary
8. $-5 - i$; both real and imaginary parts
9. $56 + 2i$
10. $38.5 + 4.3i$

Practice 1.3.3: Multiplying Complex Numbers, p. U1-54

1. $4 + 22i$
2. $-26 - 2i$
3. $52 + 4i$
4. $-91 - 77i$
5. $-1 - 123i$
6. $42 - i$
7. $-24 - 10i$
8. $-13 + 4i$
9. $\frac{57+4i}{808+114i}$
10. $\frac{40+7i}{387+139i}$

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Station Activities Set 1: Operations with Complex Numbers

Instruction

Goal: To give students practice in adding, subtracting, multiplying, and dividing complex numbers; to help students recognize the relationship between complex numbers in radical form and numbers in $a + bi$ form

Common Core State Standards

N–CN.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

N–CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

N–CN.3 (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Student Activities Overview and Answer Key

Station 1

Students will race the other members of their group to complete addition and subtraction problems involving complex numbers. Students will check one another's work.

Answers

1. $3 + 8i$
2. $13 - 4i$
3. $22 + 5i$
4. $26 + 3i$
5. 5
6. $3a + 4gi$
7. $-26 + 9i$
8. 4
9. $7 - 7i$
10. $-4 + 17i$

Station 2

Students will work with their group members to multiply complex numbers.

Answers

1. $-13 + 11i$
2. $-2 + 34i$
3. $1 + 8i$
4. $\frac{1}{2} + \frac{81}{4}i$
5. $-8 - 12i$
6. $13 - i$
7. $47 - 28i$
8. $50 + 10i$
9. 85

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Instruction

Station 3

Students will work in groups to identify the conjugate c of the denominator and simplify division problems.

Answers

1. $c = 5 + 6i$

   $\frac{3}{61} + \frac{28}{61}i$
2. $c = 2 - i$

   $\frac{11}{5} + \frac{2}{5}i$
3. $c = 3 + 2i$

   $\frac{11}{13} + \frac{16}{13}i$
4. $c = 3 + 2i$

   $\frac{5}{13} + \frac{12}{13}i$
5. $c = 7 - 3i$

   1
6. $c = 2 - i$

   $\frac{13}{5} - \frac{14}{5}i$
7. $c = 5 - 3i$

   $\frac{12}{17} - \frac{14}{17}i$
8. $c = 4 - 2i$

   $\frac{1}{2} - \frac{1}{2}i$

Station 4

Students will work in groups to simplify expressions involving complex numbers, which are sometimes in radical form and sometimes in $a + bi$ form.

Answers

1. $8 + \frac{1}{2}$
2. $3 + 4i$
3. $-2 + 3i$
4. $-20 - 20i$
5. $\frac{2}{3} - \frac{2}{3}i$
6. $\frac{-8}{13} + \frac{12}{13}i$
7. 13
8. $\frac{-254}{221} + \frac{667}{221}i$

Materials List/Setup

No materials needed

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. How do we represent the square root of –1?
2. What is a complex number?
3. How do you find the conjugate of a complex number?
4. What is the product of a complex number $a + bi$ and its conjugate?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. i
2. A complex number is one that can be expressed as $a + bi$, where a and b are real numbers.
3. If a complex number is expressed as $a + bi$, its conjugate is $a – bi$.
4. $a^2 + b^2$

Possible Misunderstandings/Mistakes

- Incorrectly multiplying polynomials
- Incorrectly finding the conjugate of a complex number
- Making simple arithmetical errors in adding, subtracting, multiplying, and dividing
- Not understanding the relationship between imaginary numbers in radical form and in $a + bi$ form
- Not recognizing that $i^2 = –1$ and $i^4 = 1$

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Station 1

Race your group members to complete the addition and subtraction problems. Show all your work. When you have all finished, check one another's work.

1. $(1 + 3i) + (2 + 5i)$

2. $(3 + 7i) + (10 - 11i)$

3. $(18 + 3i) + (4 + 2i)$

4. $(16 + 2i) + (10 + i)$

5. $(4i - 7) + (12 - 4i)$

6. $(a + gi) + (2a + 3gi)$

7. $(7i - 8) - (18 - 2i)$

8. $(3i + 2) - (3i - 2)$

9. $(10 - 5i) - (3 + 2i)$

10. $(2 + 10i) - (6 - 7i)$

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Station 2

Simplify each expression.

1. $(1 + 3i)(2 + 5i)$

2. $(3 + 7i)(4 + 2i)$

3. $(-1 + 2i)(3 - 2i)$

4. $\left(\frac{1}{4} + 2i\right)(10 + i)$

5. $(2i - 3)4i$

6. $(3 - i)(4 + i)$

7. $(8 + 3i)(4 - 5i)$

8. $(10 - 2i^3)(4 + 1)$

9. $(9 + 2i)(9 - 2i)$

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Station 3

Work with your group to identify the conjugate c of the denominator and then simplify each division problem. Show all your work.

1. $\frac{3+2i}{5-6i}$

2. $\frac{4+3i}{2+i}$

3. $\frac{5+2i}{3-2i}$

4. $\frac{3+2i}{3-2i}$

5. $\frac{7+3i}{7+3i}$

6. $\frac{8-3i}{2+i}$

7. $\frac{6-2i}{5+3i}$

8. $\frac{3-i}{4+2i}$

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Station 4

Work with your group to simplify each expression. State your answer in the form $a + bi$. Show all your work.

1. $8 + \sqrt{-\frac{1}{4}}$

2. $\sqrt{-16} + 3$

3. $\sqrt{-9} - 2$

4. $\sqrt{-25}\left(\sqrt{-16} - 4\right)$

5. $\frac{4}{3 + \sqrt{-9}}$

6. $\left(2\sqrt{-4}\right)\left(\frac{1}{3 - \sqrt{-4}}\right)$

7. $\left(3 + \sqrt{-4}\right)\left(3 - \sqrt{-4}\right)$

8. $\frac{1}{4 + \sqrt{-1}} + \frac{2 + 2\sqrt{-36}}{3 - \sqrt{-4}}$

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Instruction

Goal: To provide opportunities for students to develop concepts and skills related to adding, subtracting, and multiplying polynomials

Common Core State Standard

A–APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Student Activities Overview and Answer Key

Station 1

Students will be given 20 blue algebra tiles, 20 red algebra tiles, 20 green algebra tiles, and 20 yellow algebra tiles. Students work together to model polynomials with algebra tiles. Then they add the polynomials using the algebra tiles.

Answers

1. $8x^2 + xy + 5y^2$
2. Answers will vary. Possible answer: We combined same-color algebra tiles.
3. Answers will vary. Possible answer: We used the fact that the sum of a negative xy tile and a positive xy tile is equal to 0 in order to find pairs of the same colored algebra tiles that canceled each other out.
4. Additive Inverse Property
5. $12y^2 - 12xy - 5x^2 - 4$
6. Answers will vary.
7. Answers will vary. Possible answer: We used the fact that the sum of a negative tile and a positive tile that both represent the same variable is equal to 0.
8. $5a^3 - 3a^2b^2 + 10b^3$
9. $10x^2 - y^2 - 15xy + 4$
10. $16c^3 - 8a^3 + 3ac^2 - 7$

Station 2

Students will be given 20 blue algebra tiles, 20 red algebra tiles, 20 green algebra tiles, and 20 yellow algebra tiles. Students will work together to model polynomials with algebra tiles. Then they will subtract polynomials using the algebra tiles.

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Answers

1. $5x^2 + 5xy + 4y^2$
2. Answers will vary.
3. $3x^2$, $2xy$, $2y^2$
4. $-5x^2 - 5xy - 4y^2$
5. No, because the sign of the terms in the second polynomial changes to the opposite sign.
6. $7x^2 + 5xy + 11y^2$
7. Answers will vary. Possible answer: We matched like terms and then performed subtraction.
8. Answers will vary. Possible answer: We added negative terms because subtracting a negative number is the same as adding the opposite of that number.
9. $-2a^4 - 4a^2b^2 + 6b^3 + 6$
10. $6c^2 - 6bc - 18$

Station 3

Students will be given a number cube. Students will use the number cube to populate coefficients of polynomials. Then they will multiply polynomials using the Distributive Property.

Answers

1. Answers will vary. Possible answer: $2x$ and $(3x + y - 2)$
2. Distributive Property
3. Answers will vary. Possible answer: $2x(3x + y - 2) = 6x^2 + 2xy - 4x$
4. Answers will vary. Possible answer: $-3x^2$ and $(-4x + 7xy - 8)$
5. Answers will vary. Possible answer: $-3x^2(-4x + 7xy - 8) = 12x^3 - 21x^3y + 24x^2$
6. The sign of each term changed to the opposite sign because we multiplied by a negative value.
7. $(x + 3)$ and $(x - 4)$
8. Distributive Property
9. $(x + 3)(x - 4)$

   $x^2 - 4x + 3x - 12$

   $x^2 - x - 12$
10. We combined the like terms $-4x$ and $3x$.

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Instruction

Station 4

Students will be given six index cards with the following polynomials written on them:

$x - 1$; $6x^2 - 3x + 1$; $3x^2 - 2x + 5$; $3 + x$; $2x^2 + 3x - 1$; $-6x^2 + 5x - 8$

Students will also find three index cards (operation cards) that each have a different operation symbol written on them: addition (+), subtraction (–), or multiplication (•).

Students will work together to create problems given the solutions. Students will explain their strategies for choosing the cards and the operations. Then, students will create their own addition and subtraction problems by picking polynomial cards.

Answers

1. $(3x^2 - 2x + 5) - (2x^2 + 3x - 1)$; strategy responses may vary. Sample response: Since the result is a trinomial with a positive leading coefficient, look at the leading coefficients of the trinomials to determine what combination and operation yield a coefficient of 1 on the x^2 term.
2. $(x - 1)(3 + x)$; strategy responses may vary. Sample response: Since the leading coefficient of the trinomial is 1, look at the leading coefficients of the trinomials to determine if any combination yields 1 for the x^2 term. We know from problem 1 that there is a combination of trinomials that yields an x^2 term of 1, but it doesn't match the result in the linear term. We tried multiplying the binomials because the multiplication of two binomials can also yield a trinomial.
3. $(6x^2 - 3x + 1) + (-6x^2 + 5x - 8)$; strategy responses may vary. Sample response: Since the result is a binomial, determine if adding the two binomials yields the desired result. Since it does not, determine if any of the trinomials have an x^2 term that would add or subtract to 0.
4. $(x - 1) + (3 + x)$; strategy responses may vary. Sample response: Since the result is a binomial, determine if adding the two binomials yields the desired result.
5. $(-6x^2 + 5x - 8) + (3x^2 - 2x + 5)$; strategy responses may vary. Sample response: Since the result is a trinomial with a negative leading coefficient, look at the leading coefficients of the trinomials to determine what combination and operation yield a coefficient of –3 on the x^2 term.
6. Answers will vary.
7. Answers will vary.

Materials List/Setup

Station 1 20 blue algebra tiles; 20 red algebra tiles; 20 green algebra tiles; 20 yellow algebra tiles

Station 2 20 blue algebra tiles; 20 red algebra tiles; 20 green algebra tiles; 20 yellow algebra tiles

Station 3 number cube

Station 4 six index cards with the following polynomials written on them:
$x - 1$; $6x^2 - 3x + 1$; $3x^2 - 2x + 5$; $3 + x$; $2x^2 + 3 - 1$; $-6x^2 + 5x - 8$
three index cards that each have a different operation symbol written on them:
addition (+), subtraction (–), or multiplication (•)

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. How do you add polynomials?
2. How do you subtract polynomials?
3. What happens to the exponents of the variables when you add or subtract polynomials?
4. How do you multiply polynomials?
5. How do you deal with the exponents of the variables when multiplying polynomials?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. Combine like terms.
2. Add the opposite of the second polynomial to the first polynomial.
3. Exponents of like terms remain the same when adding or subtracting polynomials.
4. Multiply each term in the first polynomial by each term in the second polynomial using the Distributive Property.
5. The exponents of like variables are added.

Possible Misunderstandings/Mistakes

- Incorrectly adding exponents when adding polynomials
- Incorrectly subtracting exponents when subtracting polynomials
- Not using the product rule on exponents when multiplying polynomials
- Adding or subtracting unlike terms

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Station 1

At this station, you will find 20 blue algebra tiles, 20 red algebra tiles, 20 green algebra tiles, and 20 yellow algebra tiles. Work as a group to model each polynomial by placing the tiles next to the polynomials. Then find the sum. Write your answer in the space provided below each problem.

- Use the blue algebra tiles to model the x^2 term.
- Use the red algebra tiles to represent the xy term.
- Use the green algebra tiles to represent the y^2 term.
- Use the yellow algebra tiles to represent the constant.

1. Given: $\begin{array}{r} 3x^2 + 2xy + 2y^2 \\ +\ 5x^2 - xy + 3y^2 \\ \hline \end{array}$. Model the polynomial and find the sum.

2. How did you use the algebra tiles to model the problem?

3. How did you model the $-xy$ term?

4. What property did you use on the xy terms?

5. Model the following problem using the algebra tiles. Show your work, and write your answer in the space provided.

   $(4y^2 - 12xy + 5x^2) + (-10x^2 + 8y^2 - 4)$

continued

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6. How did you use the algebra tiles to model problem 5?

7. How did you deal with negative terms during addition?

Work together to add each polynomial. Show your work, and write your answer in the space below each problem.

8. Given:
$$\begin{array}{rr} & 2a^3 + a^2b^2 + 3b^3 \\ + & 3a^3 - 4a^2b^2 + 7b^3 \\ \hline \end{array}$$

9. $-10xy - 3 + 2x^2 - 5y^2 + 4y^2 + 8x^2 - 5xy + 7$

10. $8c^3 + 3ac^2 + 4a^3 + 8c^3 - 12a^3 - 7$

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Station 2

At this station, you will find 20 blue algebra tiles, 20 red algebra tiles, 20 green algebra tiles, and 20 yellow algebra tiles. Work as a group to model each polynomial by placing the tiles next to the polynomials. Then find the difference. Write your answer in the space provided below each problem.

- Use the blue algebra tiles to model the x^2 term.
- Use the red algebra tiles to represent the xy term.
- Use the green algebra tiles to represent the y^2 term.
- Use the yellow algebra tiles to represent the constant.

1. Given: $\begin{array}{r} 8x^2 + 7xy + 6y^2 \\ \underline{-\,(3x^2 + 2xy + 2y^2)} \end{array}$. Model the polynomial and find the difference.

2. How did you use the algebra tiles to model the problem?

3. To what terms in the bottom polynomial does the subtraction sign apply?

4. Find the difference: $\begin{array}{r} 3x^2 + 2xy + 2y^2 \\ \underline{-\,(8x^2 + 7xy + 6y^2)} \end{array}$. Write your answer in the space provided.

continued

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5. Is your answer from problem 1 the same as your answer from problem 4? Why or why not?

6. Model the following subtraction problem using the algebra tiles, then solve. Show your work, and write your answer in the space provided.

$$\begin{array}{r} 2x^2 + 5y^2 + 9xy \\ -\ (4xy - 5x^2 - 6y^2) \\ \hline \end{array}$$

7. How did you arrange the algebra tiles to model problem 6?

8. How did you deal with negative terms during subtraction?

For problems 9 and 10, work together to subtract each polynomial. Show your work, and write your answer in the space provided.

9. $$\begin{array}{r} a^4 - a^2b^2 + 4b^3 + 8 \\ -\ (3a^4 + 3a^2b^2 - 2b^3 + 2) \\ \hline \end{array}$$

10. Subtract $8c^2 + 2bc + 10$ from $-4bc + 14c^2 - 8$.

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Station 3

At this station, you will find a number cube. As a group, roll the number cube. Write the result in the box.

Given: $\square x(3x + y - 2)$

1. Identify the two polynomials you've created.

2. What property can you use to multiply these polynomials?

3. Multiply the polynomials. Show your work.

As a group, roll the number cube. Write the result in the box.

Given: $-\square x^2(-4x + 7xy - 8)$

4. Identify the two polynomials you've created.

5. Multiply the polynomials. Show your work.

6. What happened to the signs of each term of the polynomial in the parentheses? Explain your answer.

continued

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Use the given information to complete problems 7–10.

Given: $(x + 3)(x - 4)$

7. Identify the two polynomials given.

8. What method can you use to multiply these polynomials?

9. Multiply the polynomials. Show your work.

10. What extra steps did you take when multiplying $(x + 3)(x - 4)$ versus $-\square x^2(-4x + 7xy - 8)$?

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Station 4

At this station, you will find six index cards with the following polynomials written on them:

$x - 1$; $6x^2 - 3x + 1$; $3x^2 - 2x + 5$; $3 + x$; $2x^2 + 3x - 1$; $-6x^2 + 5x - 8$

You will also find three operation cards, each with an addition, subtraction, or multiplication symbol written on them: +, –, •.

Work as a group to find the two polynomials and corresponding operation that yield the results that follow by using the cards to set up a problem.

1. $x^2 - 5x + 6$

   Problem:

   What strategies did you use to determine the problem?

2. $x^2 + 2x - 3$

   Problem:

   What strategies did you use to determine the problem?

continued

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3. $2x - 7$

   Problem:

   What strategies did you use to determine the problem?

4. $2x + 2$

   Problem:

   What strategies did you use to determine the problem?

5. $-3x^2 + 3x - 3$

   Problem:

   What strategies did you use to determine the problem?

continued

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Place the polynomial cards in a pile and shuffle them.

6. Pick the top two cards from the polynomial pile and add the two expressions. Write the problem and the solution in the space provided.

7. Pick the top two cards from the polynomial pile and subtract one expression from the other. Write the problem and the solution in the space provided.

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APPENDIX

Problem-Based Task Implementation Guides

Introduction

This addendum contains a collection of Implementation Guides (IGs) designed to support the use of the Problem-Based Tasks found throughout this unit.

Each Implementation Guide includes an overview and suggestions for introducing and facilitating the task. These contain specific recommendations for selected Standards for Mathematical Practice (SMPs) to highlight, guidance on addressing common errors and misconceptions associated with the task, and ideas for monitoring and coaching students as they work through the task. Finally, there is guidance for debriefing the task, with special attention to the SMPs and relevant Key Concepts from the instruction, as well as information on suitable extensions and alternate strategies.

Throughout the Implementation Guides, suggestions are provided in a question/prompt format to support your efforts to elicit student thinking and facilitate classroom discourse.

We developed these Implementation Guides for our Problem-Based Tasks at the request of math educators, and with advice and feedback from mathematics supervisors and teachers. Please let us know if they are helpful. Visit us at www.walch.com, follow us on Twitter (@WalchEd), or e-mail suggestions to customerservice@walch.com.

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Implementation Guide

Problem-Based Task 1.1.1: Population Growth

Common Core State Standards

N–RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.*

N–RN.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Task Overview

Focus

If a census is taken only once every 10 years, how can you use that data to estimate what the population will be after a specific number of years that isn't a multiple of 10? How do you use the growth rate over a 10-year period to estimate the growth rate over a 1-year period? In this task, students will use a town's current census and estimated 10-year growth rate to estimate the population before the next census is taken.

This activity will provide practice with:

- simplifying fractions
- applying the order of operations
- applying the exponential growth formula
- raising a base to a non-integer, rational exponent
- interpreting the solution in terms of the context of the problem

Introduction

This task should be used to explore or apply the use of rational exponents. Students should already know how to apply the order of operations, and it will be beneficial if they have been exposed to the exponential growth formula and how to raise a number to a rational exponent, although this task may be used to introduce and explore these topics. Graphing may be done using graphing technology.

Begin by reading the problem and clarifying the meaning of the following terms:

census	a count of the population; usually taken every 10 years
population growth rate	the percentage by which a population grows within a given time period

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Implementation Guide

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 5**: Use appropriate tools strategically.

  Students might struggle with using the calculator effectively and/or efficiently. Students often struggle with raising a quantity to a power other than 2 or 3 since calculators lack specific buttons for rational exponents. Encourage students to think about how they raise a quantity to the fourth power and to follow the same procedure. If students attempt to type in the entire expression all at once, they might omit parentheses. This will result in an incorrect answer. Encourage students to think about the order of operations and why parentheses are necessary.

- **SMP 6**: Attend to precision.

  Students might want to write down the final number exactly as it appears in their calculator as their solution. Ask students to think about what level of precision makes sense for the context of the problem. Ask students if you can have a fraction of a person. Also, students might want to round before finishing their calculations. Encourage students to use the "ans" feature on their calculators so the answer will be as precise as possible until the last step, when rounding should take place. Make sure that students use precise mathematical language when discussing the problem, such as "y sub-zero" for y_0.

- **SMP 7**: Look for and make use of structure.

  The population growth model is a model students have seen before. Ask students to think about what the various parts of the formula represent structurally, particularly the base and the exponent. (**Answer:** The base represents growth because it's greater than 1. The 1 indicates 100%, while the 0.35 indicates the 35% percent increase over the 10-year period. The exponent represents t divided by 10 years because the growth occurred over 10 years.)

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- not following the order of operations

  Make sure that students apply the exponent to the base before multiplying by the current population.

- miscopying or misreading the formula

  Make sure students understand what each variable in the formula represents, and that they carefully substitute each value into the correct position in the formula. Verify that students are writing the exponent, $\frac{t}{10}$, as an exponent and not as a factor.

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Implementation Guide

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.
- If students don't know how to start, have them reread the problem and write what they know.

  Ask students for the general population growth formula for this situation.

  (**Answer:** $y = y_0 \bullet (1+r)^{\frac{t}{10}}$)

  Ask students to verbally identify what each of the variables represents. (**Answer:** "The variable y is the population quantity we are seeking, y_0 is the current population, r is the estimated growth rate over 10 years, and t is the time in years.")
- If students aren't sure which values replace which variables, ask them to reread the problem, paying particular attention to where the problem gives the definition of the variables.

  Ask, "Which variables can you replace with the given quantities, and what are these quantities?" (**Answer:** y_0, r, and t are all given. $y_0 = 42{,}000$, $r = 0.35$, and $t = 8$.)

  Ask students to substitute these values into the equation without simplifying.

  (**Answer:** $y = 42{,}000 \bullet (1+0.35)^{\frac{8}{10}}$)
- Ask students to explain how they think the base shows exponential growth. (**Answer:** The base of the exponent is greater than 1, which indicates growth.)

  Ask students if taking 35% of a number yields a larger or a smaller number. (**Answer:** a smaller number)

  Ask students if the population is shrinking or growing. (**Answer:** It is growing.)

  Ask students how they might account for growth rather than decline by adding one number to the base. (**Answer:** Add 1 to the growth rate.)

  Ask students what adding 1 to the growth rate signifies. (**Answer:** It signifies 100% + 35%.)
- Ask students to simplify the expression inside the parentheses, and then to simplify the exponent, and write the result. (**Answer:** $y = 42{,}000 \bullet (1.35)^{\frac{4}{5}}$)
- Have students make predictions about the approximate value of the population, and ask them to explain their predictions. (**Sample answer:** "The base is being raised to an exponent that is less than 1, so once the exponent is applied, the result will be less than the original base. However, the result will still be greater than 1. This means I'll be multiplying the current population by a number that's a little greater than 1. As a result, the population will increase, but only by a little bit.")

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Ask students to think about how raising a base to an exponent equal to, less than, or greater than 1 affects the base. (**Answer:** "Raising a base to 1 results in a quantity that is equal to the base, raising a base to an exponent less than 1 results in a quantity that is less than the original base, and raising a base to an exponent greater than 1 results in a quantity that is larger than the original base.")

Ask students whether the exponent in this problem is equal to, less than, or greater than 1, and to explain how they know. (**Answer:** "We are raising the base to an exponent less than 1 because the numerator in the exponent is smaller than the denominator.")

- Students might not be clear about what a rational exponent represents. Ask students to rewrite the base and the exponent using a different structure, after first discussing what it means to raise a base to an exponent of $\frac{4}{5}$. (**Answer:** "It means to take the fifth root of the base and then raise that result to fourth power: $(1.35)^{\frac{4}{5}} = \left(\sqrt[5]{1.35}\right)^4$, or to raise the base to the fourth power and take the fifth root of the result: $(1.35)^{\frac{4}{5}} = \sqrt[5]{1.35^4}$. Both of these methods yield the correct result.")
- Ask students to simplify the full expression. (**Answer:** $y = 42{,}000 \cdot (1.27136) \approx 53{,}397$)
- After students have found the answer, ask them to think about the context of the problem and the level of precision that makes sense for the context. (**Answer:** "This answer is approximate because the value 1.27136 is rounded off." *Note*: Depending on how students round, they may have a decimal in their result. Because you cannot have a fraction of a person, it makes sense to round to the nearest whole number. Additionally, because this is an estimated population for a future date, it would also make sense to round to the nearest 10 or 100.)
- Ask if anyone solved the problem graphically and have a volunteer explain this method. (**Sample answer:** "I used my calculator to graph the equation and then I used the calculator to find the *y*-value when $x = 8$.")
- Ask these students if they were satisfied with the level of precision with which they were able to determine their answer. (**Answer:** Answers will vary. Have students justify their thinking.)

  Have students consider the level of precision for either method of solving. Ask them if the level of precision matters and, if so, to give examples. (**Sample answer:** "The level of precision probably doesn't matter because the prediction is an estimate only, but the estimate should not include a portion of a person.")

  Have students comment on the accuracy of the level of precision for the estimate. (**Answer:** "The more precise you try to be with your estimate, the more likely you are to be inaccurate. It's better to round to the nearest 100.")

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Debriefing the Task

Students may have elected to solve the problem by graphing. Ask these students to show their work and explain their methodology. If the solutions found by graphing differ from the solutions found using algebraic methods, have the students who graphed reason why that is, and ask them how they could bring the numbers into closer agreement with results that were found algebraically. (**Answer:** Inaccuracy is involved when reading numbers from a graph. If students read their answer from the graph without using the "calculate" feature of the calculator, the answer can only be as precise as the least precise tick mark. If students are not aware of how to calculate a precise value on a graph, show them how to use the "calculate" feature on their calculators.)

Connecting to Key Concepts

Make explicit connections to key concepts:

- An exponential expression contains a base and an exponent. A base is the quantity that is being raised to a power, and the power is determined by the exponent.

  In this task, the base is 1.35, the exponent is $\frac{4}{5}$, and the power is the value of $(1.35)^{\frac{4}{5}}$.
- Roots can be expressed using rational exponents instead of radical symbols. For example, $\sqrt[n]{a} = a^{\frac{1}{n}}$ and $\sqrt[n]{x^m} = x^{\frac{m}{n}}$.

  In this task, $\sqrt[5]{1.35^4} = (1.35)^{\frac{4}{5}}$.

Extending the Task

- To extend the task, ask students to find the estimated yearly growth rate and create a table of values showing the population for each year from the current year to 8 years from today. Ask students to demonstrate and verify that the yearly growth rate (1.03) holds for each year.
- Another option is to have students graph the function and then analyze the graph.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Sample answer:** "On my graphing calculator, I used the '^' button to raise the base to a rational exponent, and I put the rational exponent in parentheses. I also used parentheses around the base to make sure the order of operations was followed.")

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- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Sample answer:** "I considered the context of the problem when deciding the level of precision needed for my results. I rounded to the nearest whole number because you can't have a fraction of a person. When I rounded, I decided that rounding to the nearest 10 or even 100 makes sense because we are predicting into the future and our model won't be 100% accurate.")
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** "I used the structure of the exponential growth equation to make predictions about my answer and to check my solution.")

Alternate Strategies or Solutions

- Students may use a graph to solve the problem.
- Students may analyze the data in a table on their graphing calculators. They can enter the function for Y1, then set up the table from $x = 1$ to $x = 10$ to look at the population each year for the first 10 years. The population for year 8, the year asked for in this task, appears on the table.

Technology

Students can use graphing technology.

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Problem-Based Task 1.1.2: Estimating Depreciation

Common Core State Standards

N–RN.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents.

N–RN.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Task Overview

Focus

Assuming that a car depreciates in value at a constant rate each year, how can you determine that rate if you know how much you paid for the car when it was new, how much it's worth now, and how much time has passed since you bought it new? Do you have to wait for the start of the next year so that you can use integer exponents to calculate the value? In this task, students will use what they know about exponents and rates of growth and decay to solve for a decay rate using a non-integer, rational exponent.

This activity will provide practice with:

- substituting for variables
- determining the independent and dependent variables
- isolating a variable that has a rational exponent
- raising a number to a rational exponent
- applying the properties of exponents and extending them to rational exponents
- applying the order of operations
- analyzing a solution for reasonableness
- interpreting a solution in terms of a context

Introduction

This task should be used to explore or apply the skill of developing an exponential model for a real-life situation. Students should already be familiar with exponential models from previous lessons.

Begin by reading the problem and clarifying the meaning of *depreciate*:

depreciate to decrease in value

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 5:** Use appropriate tools strategically.

  Students will most likely be using a scientific or graphing calculator that has the ability to store the latest calculation for use in future calculations. Be prepared to show students how to efficiently use this feature, especially if calculating the value of the car for a specific number of years. Students will often write intermediate calculations on a piece of paper and retype them into the calculator, even though this is unnecessary. Also, students often struggle with raising a quantity to a rational power on a calculator since there are no buttons for this. Encourage students to think about how they raise a quantity to a power of 4 and to follow the same procedure. Stress the importance of placing the rational exponent in parentheses.

- **SMP 6:** Attend to precision.

  Students might want to write down the final number exactly as it appears on their calculator as their solution. Ask students to think about what level of precision makes sense for the context of the problem. (**Answer:** "Because the problem involves estimating future values for a car, it would make sense to round to the nearest 100 or 1,000.") Also, students might want to round before finishing their calculations. Encourage students to store intermediate results on their calculators to maintain precision as much as possible until the last step, when rounding should occur.

- **SMP 7:** Look for and make use of structure.

  Students might have difficulty seeing that the rational exponent 2.5 can be converted into a fraction. Ask students to think about how they can change the form of the exponent so that it is easier to simplify the variable with that exponent (**Answer:** "I can rewrite the exponent as a fraction.")

  Ask students how to use the structure of the fraction to find the power to which you can raise both sides of the equation in order to isolate the variable. (**Answer:** "You find the multiplicative inverse, or the reciprocal, of the fraction by 'flipping' the fraction.")

  Ask students about the structure of the equation and what type of model this equation follows. (**Answer:** "This is an exponential decay model.")

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Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- forgetting to isolate the variable with the exponent before solving for the variable

  When solving $16{,}905 = 22{,}000 \cdot d^{2.5}$ for d, the first step is to isolate $d^{2.5}$ by dividing both sides by 22,000.

- attempting to eliminate the rational exponent by treating it as a factor rather than an exponent

  Students may try to eliminate the exponent by dividing by it or multiplying by the reciprocal.

  Point out that to eliminate the rational exponent and isolate the variable d, you must write the exponent 2.5 as a fraction, $\frac{5}{2}$, and then raise both sides of the equation to this power.

- converting the decimal to a fraction incorrectly

  Explain that 2.5 can be read "two and a half." Another way to write "two and a half" is $2\frac{1}{2}$. Written as an improper fraction, this becomes $\frac{5}{2}$.

- making division errors when isolating the variable

  Make sure students isolate the variable by dividing 16,905 by 22,000 and not the other way around.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.
- If students have difficulty starting the task:
  - Ask them to identify which variable needs to be determined in the equation $c = 22{,}000 \cdot d^t$. (**Answer:** "We are trying to find the value of the variable d.")
  - Ask students how they know this information. (**Answer:** "The equation has three variables: c, d, and t. The only value not given in the problem statement is d.")
- If students have difficulty determining which numbers to substitute for which variables, guide them to think about the structure of the equation and what we are solving for. Ask:
  - "What type of model does this equation follow?" (**Answer:** an exponential model)

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- "What is the general form of an exponential equation?" (**Answer:** $y = a \cdot b^x$)
- "What does each variable in the general form represent in terms of this context?" (**Answer:** "The variable y represents the car's value after x years, a represents the initial value, and b represents the rate of growth or decay.")
- "From the general form, which variable are we trying to determine?" (**Answer:** "We are trying to determine b, or the rate of growth/decay.")
- Ask students whether this situation reflects exponential growth or decay, and how they know. (**Answer:** "This situation represents exponential decay because the value of the car is decreasing.")
- From this information, ask students to substitute in the known values for the corresponding variables in the given equation, $c = 22{,}000 \cdot d^t$. (**Answer:** "Comparing the given equation $c = 22{,}000 \cdot d^t$ to the general form $y = a \cdot b^x$, c corresponds to y, 22,000 corresponds to a, d corresponds to b, and t corresponds to x. I can substitute known values for c and t: $c = 16{,}905$ (the current value of the car) and $t = 2.5$ years. The resulting equation is $16{,}905 = 22{,}000 \cdot d^{2.5}$.")

- If students are uncertain about how to work with a variable that has an exponent:
  - Encourage students to isolate the variable as much as possible by using inverse operations. (**Answer:** Divide both sides of the equation by 22,000 to get $0.768 \approx d^{2.5}$.)
  - Ask, "How would you handle solving for a variable if the exponent was 2?" (**Answer:** "I would take the square root of both sides.") "What if the exponent was 3?" (**Answer:** "I would take the cube root of both sides.")
  - "What's another way to write the square root and the cube root?" (**Answer:** "By using the exponents $\frac{1}{2}$ and $\frac{1}{3}$.")
  - "What is the relationship between an exponent of 2 on a squared variable and an exponent of $\frac{1}{2}$ when taking the square root?" (**Answer:** "The numbers 2 and $\frac{1}{2}$ are multiplicative inverses.")
  - "How can you use this structure here to solve for the variable?" (**Answer:** "I can rewrite 2.5 as a fraction and then find its multiplicative inverse.")
  - "What is 2.5 written as a fraction?" (**Answer:** $\frac{5}{2}$)
  - "What is the multiplicative inverse of $\frac{5}{2}$?" (**Answer:** $\frac{2}{5}$)

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- "How can you use the multiplicative inverse to solve for the variable?" (**Answer:** "I can raise both sides of the equation to that power; when raising a power to a power, multiply the exponents. When I multiply multiplicative inverses, the result is 1. This means that the exponent on the variable is now 1.")
- Ask students to show how they raise both sides of the equation to the power of $\frac{2}{5}$ symbolically. (**Answer:** $(0.768)^{\frac{2}{5}} \approx \left(d^{\frac{5}{2}}\right)^{\frac{2}{5}}$)

- Once students have solved the resulting equation for *d*, ask:
  - "What does the answer to this equation mean?" (**Answer:** "This is the approximated depreciation rate of Yasmina's car each year. Because the value for *d* is 0.90 or, as a percentage, 90%, this means that each year, her car retains 90% of its previous value.")
  - "How can this information be used to create a model for the value of Yasmina's car for the value of her car at any time?" (**Answer:** "Substitute the value of *d* into the equation given in the task: $c = 22{,}000 \bullet (0.90)^t$.")
- For students who complete the Coaching questions:
  - Once students have calculated the value of Yasmina's car after 6 years, ask them to write the solution in a sentence to interpret its meaning. (**Answer:** "After 6 years, the car will be worth approximately $11,692, or about half of its original value.")
  - Ask, "Do you think that it's reasonable for the car to lose nearly half of its original value after 6 years?" (**Sample answer:** "I was surprised to find out how quickly cars can depreciate.")

Debriefing the Task

Compare students' strategies for solving the problem. Encourage students to explain their thinking as they were solving the problem, particularly the connections they were able to recognize between this model and previous exponential decay models they have seen. Ask students to draw similarities between solving this equation and solving other exponential decay equations. (**Answer:** "This problem and other exponential decay problems I've seen had the same general structure, $y = a \bullet b^x$, where $b < 0$ because it represented decay.")

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Connecting to Key Concepts

Make explicit connections to key concepts:

- The properties of exponents apply even when the result is an irrational number.

  When raising a power to a power, you multiply exponents.

- To solve an equation that has a rational exponent, raise both sides to the multiplicative inverse (reciprocal) of the exponent.

  In this task, the exponent on the variable is $\frac{5}{2}$, so raise both sides of the equation to the power of $\frac{2}{5}$.

Extending the Task

- To extend the task, have students use the model that they developed to calculate the estimated value of Yasmina's car after some particular number of years. Then, have students graph the model and analyze the graph of the function. Ask students to describe what they notice about the graph. (**Sample answer:** "The graph shows that the value of the car depreciates less each year as the years go by, so that the car never reaches a value of $0.")

- Another way to extend the task is to have students look up various cars' prices when new, and then find the cars' values depending on the condition of the car after a given number of years. Students could determine the depreciation rate of a car that they are familiar with, and see if the depreciation rate is similar to that of Yasmina's car. The following website can be used to find current values, though researching an older car's original price will require additional investigation.

  Kelley Blue Book. "Get Your Car Value."

  http://www.walch.com/rr/07002

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** "On my calculator, I used the memory and 'ans' features to help calculate the value of the car for a given value of *t*, and I used the '^' button to raise numbers to rational exponents.")

- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** "I used precise language when describing what I did, and I thought about the best way to round the calculated values in this situation.")

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- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** "I recognized that this situation represents an exponential decay model, and that I was being asked to find the decay rate. I reasoned that since this problem models exponential decay, the base must be a number less than 1. Furthermore, I reasoned that the number being multiplied, 0.90^t, must be the starting value. When I substituted known values into the equation and realized that the exponent 2.5 was rational, I remembered that equations in this form can be solved by raising both sides of the equation to the power that is the multiplicative inverse of the given exponent.")

Alternate Strategies or Solutions

- When asked to evaluate the model for $t = 6$, students may choose to graph the function using technology and use the graph to calculate the value of the car after 6 years.
- Students may also use the table feature of a graphing calculator to find this value.

Technology

Students can use scientific calculators and/or graphing technology to help solve this problem.

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Problem-Based Task 1.2.1: Cabin Perimeter

Common Core State Standard

A–APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Task Overview

Focus

How are expressions used to represent unknown quantities? How are these expressions combined to express relationships? In this task, students will write an expression modeling the perimeter of a cabin that has both known and unknown dimensions.

This activity will provide practice with:

- representing unknown quantities with variables
- writing expressions from figures
- substituting expressions
- adding polynomials

Introduction

This task should be used to explore or apply the skill of representing unknown quantities using polynomial expressions and adding polynomial expressions.

Begin by reading the problem and clarifying the meaning of the terms *expression* and *perimeter*, along with the meaning of *blueprint*:

blueprint a detailed design plan or technical drawing that shows how something will be made

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Some students will be overwhelmed by all of the variables on the diagram. They might have trouble determining how to approach the task. Ask students to think about what they already know about perimeter. Then, ask them to describe how to find the perimeter of the cabin, even if it's not in terms of x.

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- **SMP 7:** Look for and make use of structure.

  Students might see immediately that the length of each side labeled a can be rewritten using x and 6, and the length of each side labeled b can be rewritten using x and 4. If students can't see this structure, encourage them to label side lengths not already labeled in the blueprint.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  Students might see that there are two pairs of congruent sides. Encourage students to look for different ways of writing the expression for the perimeter; for example, $(x + 6) + (x + 4) + (x + 6) + (x + 4) = P$ or $2(x + 6) + 2(x + 4) = P$.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- confusing the formulas for perimeter and area

  Remind students that the perimeter is found by *adding* the side lengths, and the area is found by *multiplying* the length and the width.

- writing the expression in terms of a and/or b

  Explain to students that the expression is to be written in terms of x, which means that a and b must be rewritten using x, and that a and b cannot be used in the expression.

- distributing incorrectly or not at all

  Remind students that to distribute over addition, you must multiply each term in parentheses by the factor.

- substituting incorrectly

  Remind students to use parentheses when substituting an expression for a variable.

- adding terms that are not like terms

  Explain to students that only like terms can be added (or subtracted), and that like terms have the same variable(s) raised to the same power.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students are unsure of how to start the problem, ask them what they know about *perimeter*. (**Answer:** "Perimeter is the distance around a figure.")

  Ask students to describe how to calculate the perimeter of a figure. (**Answer:** "The perimeter is found by adding the lengths of all the sides.")

  Ask students if they can identify the perimeter of the cabin even if it isn't in terms of x. (**Answer:** $P = 2a + 2b$)

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- If students struggle with how to proceed, ask them to look at the sides labeled with a total length of a. Focus on a single side as needed. Ask students to identify how to use the information given to write the length of each side labeled a in terms of x. (**Answer:** "Each side labeled a is made up of lengths x and 6; therefore, a written in terms of x is $x + 6$.")
- If students can see how to rewrite a as $x + 6$, they will probably be able to move forward with rewriting b as $x + 4$. If not, ask students to look at the sides labeled with a total length of b. Focus on a single side as needed. Encourage students to turn the paper sideways if they are having trouble seeing what other lengths comprise b. (**Answer:** "Each side labeled b is made up of lengths x and 4; therefore, b written in terms of x is $x + 4$.")
- If students believe they are done after writing the expressions for a and b in terms of x, ask them to reread the problem and restate the purpose of the task. (**Answer:** "The purpose is to write an expression in terms of x for the total perimeter.")
- When writing the initial expression to determine the total perimeter of the cabin, students might forget to use parentheses when substituting values for the length and width into the given formula, $P = 2a + 2b$. Ask the following:
  - "What do you need to add in order to find the perimeter of the cabin?" (**Answer:** "I need to add the lengths of the four sides of the cabin.")
  - "What is the expression in terms of x for the length, a?" (**Answer:** $x + 6$)
  - "What is the expression in terms of x for the width, b?" (**Answer:** $x + 4$)
- If students end up with an expression that looks like $2x + 6 + 2x + 4$, they most likely did not use parentheses when substituting values into the formula $P = 2a + 2b$. Ask students to go back to the original equation, $P = 2a + 2b$, and perform the substitutions again. If students forgot to use parentheses, give them this sample problem: "Is $2(x + 1)$ equal to $2x + 1$? Why or why not?" (**Answer:** "No. The first expression results in $2x + 2$, whereas the second expression remains $2x + 1$. They are not equal.")

  At this point, students might see their error. If not, ask them to visualize walking their way around the cabin, adding each side as they go along. Then, ask students to compare that result to their original calculation. (**Answer:** Students will arrive at something similar to $(x + 6) + (x + 4) + (x + 6) + (x + 4) = P$.)
- If students leave their answer as $2(x + 6) + 2(x + 4)$, ask them what simplified answers look like in general. (**Answer:** "Simplified answers generally don't have parentheses, and like terms have been combined.") Ask students to look at their answers again, and rewrite them in simplest terms. (**Answer:** $4x + 20$)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

Compare students' strategies and explanations for creating the algebraic expression. Focus on the use of precise mathematical language, including the identification of the variables, as well as the process for simplifying the expression.

Connecting to Key Concepts

Make explicit connections to key concepts:

- A monomial is an expression with one term, consisting of a number, a variable, or the product of a number and variable(s).

  The lengths of the cabin sides are initially expressed as a or b; each is a monomial.

- A polynomial is a monomial or a sum of monomials that contains variables, numeric quantities, or both. The variables of polynomials are raised to integer powers greater than 0.

  The task asks for the side lengths to be expressed in terms of x, which requires a polynomial expression.

- Each part of a polynomial is called a term. A term is a number, a variable, or the product of a number and variable(s).

  When a and b are rewritten in terms of x, the resulting expressions ($x + 6$ and $x + 4$) are both polynomials with two terms.

- Like terms are terms that contain the same variables raised to the same power. Numeric quantities are like terms.

  In the expressions for a and b, the like terms are x and the constants (6 and 4).

- To add polynomials, combine like terms.

  For the side lengths, first combine the terms with x, and then combine the terms with the numbers.

- The result of adding two polynomials is also a polynomial, so the system of polynomials is closed under addition.

  The initial expression used to describe the perimeter, $P = 2a + 2b$, is a polynomial; its simplified form, $4x + 20$, is also a polynomial.

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Extending the Task

- To extend the task, replace x in the diagram with a more complicated expression, such as $3x + 1$, and have students find the new perimeter using this measurement. (**Answer:** $12x + 24$)
- Ask students to return to the square on the inside of the cabin and start increasing each side length incrementally by 1. Have students identify and discuss patterns. Ask students to think about and discuss the connection between the amount of materials needed for the cabin based on the given blueprint, and the amount of materials needed when 1 foot is added to each dimension x.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** "Initially, I didn't know how to start solving the problem, but I used what I knew about perimeter to break the problem down into smaller parts by looking at the length of each individual side.")
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** "I looked at the side length a and noticed that it was made up of sections with lengths of x and 6. Similarly, side b was made up of sections with lengths of x and 4. With this information, I was able to build expressions for the side lengths in terms of x.")
- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** "Instead of adding the same expression twice, I noticed I could multiply the expression by 2.")

Alternate Strategies or Solutions

- Students might choose to label all parts of the figure that aren't labeled before starting the problem.
- Students might not set up the problem using distribution, but might use addition instead: $(x + 6) + (x + 4) + (x + 6) + (x + 4) = P$. Encourage students to make use of the repeated structure to rewrite using distribution: $2(x + 6) + 2(x + 4) = P$.

Technology

No technology is needed for this task.

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Implementation Guide

Problem-Based Task 1.2.2: Architectural Area

Common Core State Standard

A–APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Task Overview

Focus

How do we multiply polynomial expressions by other polynomial expressions? Why would we need to multiply polynomial expressions? In this task, students will calculate the total area of a bathroom given the constraint that one section of the bathroom must have specific dimensions in order to accommodate a standard-size bathtub. However, the dimensions of the remainder of the bathroom can vary.

This activity will provide practice with:

- representing unknown quantities with variables
- writing expressions from figures
- substituting expressions
- multiplying and adding polynomials

Introduction

This task should be used to explore or apply the skill of representing unknown quantities using polynomial expressions and adding polynomial expressions.

Begin by reading the problem and clarifying any questions about the architect's template. Review the meaning of the terms *area*, *expression*, and *polynomial*.

area	the amount of space inside the boundary of a two-dimensional figure
expression	a combination of variables, quantities, and mathematical operations; 4, $8x$, and $b + 10^2$ are all expressions
polynomial	a monomial or the sum of monomials

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Some students will be overwhelmed by all of the variables on the diagram. They might have trouble determining how to approach the task. Ask students to think about what they already know about area. Ask students if they can find the area of any of the smaller figures shown on the architect's template. Then, ask students how these figures make up the larger figure.

- **SMP 4:** Model with mathematics.

  The situation presented in this task can be modeled algebraically. Encourage students to think about how to represent the total lengths and widths shown on the bathroom template (*a* and *b*, respectively) using polynomial expressions. Then, ask students to substitute those expressions into the formula for area. The result serves as the model for the total area of the bathroom.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- confusing the formulas for perimeter and area

  Remind students that the area is found by *multiplying* the length and the width, and the perimeter is found by *adding* the side lengths.

- distributing or multiplying the polynomials incorrectly

  Remind students that each term in the first set of the parentheses must be multiplied by the each term in the second set of parentheses.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students are unsure of how to start the problem, suggest that they break the task into smaller parts. Ask the following:
  - "What is the overall shape of the bathroom?" (**Answer:** "The bathroom represented on the template is a rectangle.")
  - "How is the area of a rectangle calculated?" (**Answer:** "The area of a rectangle is calculated by multiplying the length by the width.")

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- "What smaller lengths shown on the template make up the total length, a?" (**Answer:** "The total length, a, is the sum of the smaller lengths, 65 and x.")
- "What smaller widths shown on the template make up the total width, b?" (**Answer:** "The total width, b, is the sum of the smaller widths, 30 and $2x$.")
- "How can you use the given measurements for the smaller lengths and widths to write expressions for a and b in terms of x?" (**Answer:** "The length, a, can be written as $x + 65$; the width, b, can be written as $2x + 30$.")

- If students believe they are done after writing the expressions for a and b in terms of x, ask them to reread the problem and restate the purpose of the task. (**Answer:** "The purpose is to write an expression in terms of x for the total area of the bathroom.")
- When writing the initial expression to determine the total area of the bathroom, students might forget to use parentheses when substituting values for the length and width into the given formula, $A = ab$. Ask the following:
  - "What do you need to multiply in order to find the total area of the bathroom?" (**Answer:** "I need to multiply the total length of the bathroom, a, by the total width of the bathroom, b.")
  - "What is the expression in terms of x for the length, a?" (**Answer:** $x + 65$)
  - "What is the expression in terms of x for the width, b?" (**Answer:** $2x + 30$)
- Ask students to think about how to set up and solve an equation to multiply the total length by the total width. (**Answer:** "I can enclose each polynomial expression in parentheses, and then distribute.")
- Students who understand the concept of multiplying the polynomial expression for the length by the polynomial expression for the width still might carry out the distribution process incorrectly. Encourage students to rewrite the multiplication step-by-step, and to show the distribution of each term in the first polynomial to the terms in the second polynomial. (**Answer:** $(x + 65)(2x + 30) = (x \bullet 2x) + (x \bullet 30) + (65 \bullet 2x) + (65 \bullet 30) = 2x^2 + 30x + 130x + 1950 = 2x^2 + 160x + 1950$)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Compare students' strategies and explanations for creating the algebraic expression. Focus on the use of precise mathematical language, including the identification of the variables, as well as the process for simplifying the expression.

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Connecting to Key Concepts

Make explicit connections to key concepts:

- To multiply two polynomials, multiply each term in the first polynomial by each term in the second.

  In this task, the total length and total width are written as polynomials, each of which has two terms. The terms in the first polynomial are multiplied by the terms in the second polynomial to calculate the area of the bathroom.

- The Distributive Property can be used to simplify the product of two or more polynomials. For example, if each polynomial has two terms, with real numbers a, b, c, and d, then $(a + b)(c + d) = (a + b)c + (a + b)d = ac + bc + ad + bd$.

  In this task, two polynomials that each have two terms are multiplied using the Distributive Property: $(x + 65)(2x + 30) = (x \bullet 2x) + (x \bullet 30) + (65 \bullet 2x) + (65 \bullet 30)$.

- To find the product of a variable with a coefficient and a number, multiply the coefficient by the number. If a and b are real numbers, then $ax \bullet b = abx$.

  In this task, the polynomials are comprised of variables and numeric terms that are multiplied.

- After multiplying all terms, simplify the expression by combining like terms.

  The linear terms in the expression can be combined: $2x^2 + 30x + 130x + 1950 = 2x^2 + 160x + 1950$.

Extending the Task

- To extend the task, replace x and $2x$ in the template with more complicated polynomials, such as $(x + 1)$ and $(2x – 1)$. Have students write and solve an expression to determine the total area of the new bathroom for any values of x. (**Answer:** $2x^2 + 161x + 1914$)

- Another option could be to graph the area model, $2x^2 + 160x + 1950$, and use the graph to find the area for various values of x.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** "Initially, I didn't know how to start solving the problem, but I used what I knew about area to break the problem down into smaller parts by looking at the length of each side by itself.")

- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** "I used the numeric value given for a portion of the total length, a, and added the variable portion to write an expression for the length. I repeated this process to write an expression for the total width, b. I used the formula for area, $A = lw$, to multiply the two expressions and find the model for the total area of the bathroom.")

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Alternate Strategies or Solutions

Although the solution method presented in the Coaching Sample Responses is based on writing and then multiplying polynomials that represent the total length and total width, students may also choose to find the areas of the smaller rectangles and then add the results. This method will yield the same expression for the area: $2x^2 + 160x + 1950$. For instance:

Dimensions	Area in square inches
$65 \cdot 2x$	$130x$
$x \cdot 2x$	$2x^2$
$65 \cdot 30$	1950
$x \cdot 30$	$30x$

The sum of the areas is $130x + 2x^2 + 1950 + 30x$. When like terms are combined, the new expression is $160x + 2x^2 + 1950$ or, rewritten in descending order, $2x^2 + 160x + 1950$.

Technology

Students may wish to use a calculator when multiplying terms. Students may use graphing calculators to carry out the second extension.

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Implementation Guide

Problem-Based Task 1.3.1: Representing Impedance

Common Core State Standard

N–CN.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

Task Overview

Focus

What are imaginary numbers and how are they used in real world applications? Students will explore the concept of impedance and examine its connection to the complex number system.

This activity will provide practice with:

- substituting
- representing complex numbers
- examining the concept of impedance

Introduction

This task should be used to explore or to apply the concept of representing complex numbers.

Begin by reading the problem and clarifying the meaning of an electrical circuit, as well as the definition of impedance. Review the following terms:

circuit	a path that electricity follows or flows through
current	the flow of electricity
element	part of a circuit
impedance	the measure of a circuit's opposition to the flow of current
ohms	a unit for measuring electrical resistance

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 4:** Model with mathematics.

  Students might have difficulty understanding the concept of impedance, that it contains real and imaginary components. Students might have difficulty teasing out the information in the problem statement. Encourage students to write the formula given, Impedance = $R + Xi$. Ask them what information they have been given and what they information they seek.

- **SMP 6:** Attend to precision.

  Students often want to jump right into calculations. Here students will most likely need to read and reread the problem statement to discern what information contains the real part and which information contains the imaginary component of the impedance. Encourage students to pay close attention to the details given in the problem and match them with the formula given for the problem. Ask students to think about what identifies a part of a complex number as the imaginary part.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- switching the real and imaginary numbers so that the impedance is written as $1.25 + 21i$ instead of $21 + 1.25i$

  Suggest to students that they read and identify the variables in the formula and then reread the problem statement and associate the numbers given with the variables in the formula.

- writing the imaginary part of the number first instead of the real part as in $1.25i + 21$

  Remind students that when writing complex numbers, always start with the real part first.

- combining 21 and 1.25 to arrive at 22.25 or $22.25i$ as the final impedance

  Explain to students that impedance is a complex number comprised of a real and a complex part. Remind students that real parts and imaginary parts cannot be combined.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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- Students might be very confused about what is going on in the problem statement. Direct students to the formula and ask them if they can identify the variables based on what they read in the problem statement. (**Answer:** Impedance = $R + Xi$; R is the resistance associated with an element in a circuit and the X is the reactance associated with the current of that same element in a circuit.)
- Ask students to comment about whether or not i is a variable and ask them to explain how they know. (**Answer:** The lowercase i is not a variable but a constant. It is equal to $\sqrt{-1}$ and is always equal to $\sqrt{-1}$; therefore, it is not a variable.)
- If students are still stuck as to where the numbers belong in expressing and modeling the impedance, ask them, "What is impedance comprised of?" (**Answer:** resistance and reactance)
- Ask students to identify the resistance associated with the element (**Answer:** 21 volts)
- Ask students to identify the reactance associated with the element. (**Answer:** 1.25 milliamperes) Finally, ask students to substitute these numbers into the formula. (**Answer:** Impedance = $21 + 1.25i$)
- Some students might leave off the i in the current portion of the impedance. Ask students to reflect on what i means. (**Answer:** $i = \sqrt{-1}$)
- If students try to add 21 and 1.25, ask students whether combining real and imaginary parts can be done, and have students explain why or why not. (**Answer:** No, real and imaginary parts cannot be combined. The imaginary part is multiplied by $\sqrt{-1}$ and that cannot be added to a real number. Only imaginary parts can be added to imaginary parts and real parts to real parts.)

Debriefing the Task

Compare methods used by the groups and discuss advantages and disadvantages of the varying strategies.

Connecting to Key Concepts

Make explicit connections to key concepts:

- All rational and irrational numbers are real numbers.

  The imaginary unit i is used to represent the non-real value, $\sqrt{-1}$.
- An imaginary number is any number of the form bi, where b is a real number, $i = \sqrt{-1}$, and $b \neq 0$.

  In the task, the current portion of the impedance is the imaginary number $1.25i$.

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- Real numbers and imaginary numbers can be combined to create a complex number system.

  A complex number contains two parts: a real part and an imaginary part.

- All complex numbers are of the form $a + bi$, where a and b are real numbers and i is the imaginary unit.

  The impedance results in a complex number $21 + 1.25i$, where 21 is the real part and $1.25i$ is the imaginary part.

Extending the Task

To extend the task, use different values for resistance and reactance in the problem set-up. For example, ask students to model the impedance of an element with a resistance of 30 volts and a reactance of 1.75 milliamperes. Ask them to research the concept of rotation as it applies to the imaginary dimension. Point students to resources such as the following site:

- Better Explained, Kalid Azad. "A Visual, Intuitive Guide to Imaginary Numbers."

  http://www.walch.com/rr/07008

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 4, ASK**: "How did you use mathematics to model this particular scenario?" (**Answer:** I used the resistance and the reactance given in the problem statement and substituted this information into the given formula for impedance to model the total impedance of the element of the circuit.)
- **For SMP 6, ASK**: "How did you make sure you attended to precision?" (**Answer:** I made sure to use the imaginary part in the complex number as an imaginary part instead of using it incorrectly as a real part. I kept the real and imaginary components separate to create the complex impedance.)

Technology

Students can complete the extension of this task using a computer or tablet and the Internet.

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Problem-Based Task 1.3.2: Elements in Series in a Circuit

Common Core State Standard

N–CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex nu

Task Overview

Focus

Why would complex numbers need to be added ? What real-world applications contain combining complex numbers? Students will examine an electrical circuit to explore how impedance is calculated when elements are in a series.

This activity will provide practice with:

- substituting values into a formula
- representing complex numbers
- adding complex numbers

Introduction

This task should be used to explore or apply representing and adding complex numbers.

Begin by reading the problem and clarifying the meaning of an electrical circuit. Review the following terms:

circuit	a path that electricity follows or flows through
current	the flow of electricity
element	part of a circuit
impedance	the measure of a circuit's opposition to the flow of current
ohms	a unit for measuring electrical resistance
series	the current flows through one element and then another without branching

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students might be confused by the term "series" and not know how to handle the problem. Encourage students to reread the problem. Ask students if the problem explains impedance of elements in series.

  Students will need to add complex numbers. Encourage students to recall what they know about combining like terms, and ask them to think of similarities between like terms and complex numbers.

- **SMP 2:** Reason abstractly and quantitatively.

  Students might have difficulty associating the given resistance and reactance with the given impedance formula. Encourage students to create an expression for the total impedance of the circuit using variables. Then ask students to list the values of the variables that have been given. From there, students can make the substitutions.

- **SMP 4:** Model with mathematics.

  Once students have sorted out the expression for the total impedance, they might need guidance with determining which components are real and which are imaginary.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- adding real parts to imaginary parts

  Remind students that real parts and imaginary parts cannot be combined.

- not simplifying answers

  Remind students to combine real parts to real parts, and imaginary parts to imaginary parts.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.
- Before they begin the task, encourage students to reread the problem, and ask them what they are asked to find. (**Answer:** What is the total impedance of the circuit?)

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- Ask students how the total impedance is calculated. (**Answer:** The total impedance is calculated by summing the elements in the series.)
- Ask students how many elements there are in the series. (**Answer:** There are 2 elements in series.)
- Ask students what the formula is for impedance. (**Answer:** *Impedance* = $R + Xi$)
- Ask students what each variable represents in the formula. (**Answer:** R is the resistance of the element, and X is the reactance of the element. Lowercase i is the imaginary number $\sqrt{-1}$.)
- As students work to find the total impedance, ask them how they can write an expression to model the total impedance of the circuit. (**Answer:** *Impedance* = $R_1 + X_1i + R_2 + X_2i$.)
- Ask students to identify the values for each of the variables (**Answer:** $R_1 = 30.5$ ohms, $I_1 = 2.8$ ohms, $V_2 = 19$ ohms, and $I_1 = 3$ ohms.)
- If students try to combine all the terms into one term, ask the process of adding complex numbers. (**Answer:** Add the real parts to real parts, and the imaginary parts to imaginary parts; Impedance = $(30.5 + 19) + (2.8i + 3i) = 49.5 + 5.8i$.)

Debriefing the Task

Compare students' strategies and explanations for creating the algebraic expression. Focus on the use of precise mathematical language, including the identification of the variables, as well as the process for simplifying the expression.

Connecting to Key Concepts

Make explicit connections to key concepts:

- To add or subtract two complex numbers, first add or subtract the real parts of the complex number, then add or subtract the coefficients of i.

  In this task, the real parts are 30.5 and 19; therefore, 30.5 + 19 = 49.5. The coefficients of i are 2.8 and 3; therefore, the result is $5.8i$.

- The resulting sum or difference of the real parts and the imaginary parts is the solution.

  Combining the real and imaginary parts gives the total impedance of $30.5 + 49.5i$.

Extending the Task

To extend the task, have students learn to graph complex numbers on the complex plane, and use Argand diagrams to add complex numbers graphically. This tutorial from Khan Academy provides practice with adding and subtracting complex numbers graphically.

- Khan Academy. "Graphically Add and Subtract Complex Numbers."

  http://www.walch.com/rr/07009

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you demonstrate perseverance?" (**Answer:** I read and reread the problem until I had an understanding of what I was being asked to find. I also wrote down the information I was given, including the formula for impedance.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I reasoned abstractly by translating the definition of a circuit with elements in series into an expression with variables.) "Which of your strategies represent quantitative reasoning?" (**Answer:** Substituting the quantities into the expression, performing the calculations, and finalizing the model.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I created an expression using the concept of elements in series in a circuit, used the operation of addition of complex numbers, and simplified the expression to give the final model.)

Alternate Strategies or Solutions

Students could model the addition of the complex part by formally using the distributive property $(2.8 + 3)i = 5.8i$.

Technology

Students can use a computer or tablet for the extension of the task.

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Problem-Based Task 1.3.3: Elements in Parallel in a Circuit

Common Core State Standard

N–CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Task Overview

Focus

When would you need to multiply two complex numbers? What practical applications involve multiplying complex numbers? Students will examine the impedance of a circuit with elements of the circuit in parallel that require multiplying complex numbers in order to find the total impedance of the circuit.

This activity will provide practice with:

- substituting
- calculating the common denominator with complex numbers
- adding fractions with of complex numbers

Introduction

This task should be used to explore or to apply adding and multiplying complex numbers.

Begin by reading the problem and clarifying the meaning of the following terms:

circuit	a path that electricity follows or flows through
current	the flow of electricity
element	part of a circuit
impedance	a measure of how much resistance a circuit has to the current
parallel circuit	the current branches out and flows through two or more paths at onc

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students might have trouble getting stared because the topic of impedance with respect to elements in parallel is unfamiliar to them. Encourage students to reread the problem and to review what information they have been given. If necessary, walk students through substituting the impedance of the first element into the formula for the total impedance.

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Encourage students to think about what they already know about adding fractions with unlike denominators.

Students might have trouble with working with numbers that aren't concrete, that are difficult to visualize, and aren't tangible. Encourage students to work through the math that they know, adding and multiplying. They might be surprised to find that they've calculated the correct answer in the end.

- **SMP 7:** Look for and make use of structure.

  Students might immediately see the connection between adding complex number fractions and real number fractions. Encourage students to see the similarities – the structure is the same in that students must find a common denominator first before adding.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- taking the reciprocal of each part of the complex number instead of the reciprocal of the entire complex number

  Show students the difference between taking the reciprocal of 2 and 4 and then adding them versus taking the reciprocal of $2 + 4$, so add $\frac{1}{2}+\frac{1}{4}$ and compare to $i = \frac{1}{2+4}$.
- adding numerators and unlike denominators or adding numerators and like denominators

  When adding fractions, first, find the common denominator, then ad numerators. The common denominator stays the same.
- adding real parts to imaginary parts

  Remind students that real parts and imaginary parts cannot be combined.
- leaving i^2 in the answer

  Remind students that i^2 simplifies to –1.
- forgetting to add powers of *i* when multiplying *i* by *i*.

  Remind students that multiplying two *i*s together is like multiplying two *x*s together in that the powers add.
- distributing incorrectly

  Remind students that each term in the first set of the parentheses must be multiplied by each term in the second set of parentheses.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.
- Students might have trouble getting started and might not understand what the formula for total impedance is saying.

  Ask students what the "1" above the Z_1 and Z_2 means. (It means to take the reciprocal of the given impedances.)

  If students can't answer that question, ask them about the relationship between 2 and $\frac{1}{2}$, 3 and $\frac{1}{3}$, 4 and $\frac{1}{4}$, etc. (**Answer:** They are reciprocals of each other.)
  Now return to the previous question about placing a 1 above the given impedances and ask, "What is happening there?" (**Answer:** We are adding the reciprocal of the impedances together.)
- If students say they don't know how to do add them or how to go on, ask students to write down what they know so far. (**Answer:** Impedance = $\frac{1}{Z_1} + \frac{1}{Z_2} = \frac{1}{15+i} + \frac{1}{10+2i}$)

  Students might say that they still don't know how to handle this situation. Refer students to adding real fractions with unlike denominators, like $\frac{1}{2} + \frac{1}{3}$. Ask students how they would handle this situation. (**Answer:** Find a common denominator.)

  Ask students how they would find a common denominator. (**Answer:** Find the least common multiple or the smallest number they both divide into evenly.)

  Ask students what multiple always works as a common multiple that can be used as a common denominator and why? (**Answer:** Multiplying the denominators together because each is a factor of the product.)
- Ask students how they might treat this addition problem like a real number fraction problem. (**Answer:** Find the common denominator.)

  Ask students for the common denominator (**Answer:** $(15 + i)(10 + 2i) = 150 + 30i + 10i + 2i^2 = 148 + 40i$)

  If students distribute incorrectly ask them about the structure of their multiplication. Ask, "How did you set up your multiplication problem?" (**Answer:** I used two sets of parentheses to prepare for distribution.)

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If students did not distribute properly, ask students how distribution works. (**Answer:** Multiply each term in the first set of parentheses by each term in the second set of parentheses.)

If students forgot to add the powers of i when multiplying i by i, ask students how to multiply i by i. (**Answer:** Add the powers, so $i \bullet i = i^2$.)

- Ask students about the conceptual meaning of multiplying i by i. (**Answer:** $\frac{1}{2}+\frac{1}{4}$. When you square a square root, you get what's under the square root.)

  If students add real parts to imaginary parts, ask them what i equals. ($i = \frac{1}{2+4}$)

  If students try to add real to imaginary parts also ask students if combining real and imaginary parts can be done and why or why not. (**Answer:** No, real and imaginary parts cannot be combined. The imaginary part is multiplied by $\frac{1}{2}$ and that cannot be added to a real number. Only imaginary parts can be added to imaginary parts and real parts to real parts.)

- Students might be stuck after they find the common denominator. Ask students to return to the real number fraction example of $\frac{1}{2}+\frac{1}{3}$. After students find the common denominator here, what do they do next before they can add the fractions together? (**Answer:** Multiply each fraction top and bottom by the factor that will create the common denominator. In this case the first fraction $\frac{1}{2}$ would be multiplied by $\frac{3}{3}$ and the second fraction $\frac{1}{3}$ would be multiplied by $\frac{2}{2}$. This would give both fractions a denominator of 6.)
  - Ask students why this can be done. (**Answer:** Multiplying by $\frac{3}{3}$ or by $\frac{2}{2}$ is the same as multiplying by 1 and multiplying by is the multiplicative identity. When you multiply by the multiplicative identity you get back what you started with. In other words anything multiplied by is anything. So you are just rewriting the fraction in a different way, you aren't changing its value.)

- Ask students what needs to be done next with the complex fraction after having found the common denominator. (**Answer:** Multiply the first fraction, $\frac{1}{15+i}$, top and bottom, by $10 + 2i$, and multiply the second fraction, $\frac{1}{10+2i}$, top and bottom, by $15 + i$; $\frac{1}{3}$; $\frac{1}{4}$.)

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- Ask students to think about how they would handle the structure of the real number fraction at this point. (**Answer:** Add the numerators and keep the denominator)
- Ask students to add the complex fraction, $\frac{1}{Z_1}+\frac{1}{Z_2}$. If students add the denominators, return to a more concrete example. Ask students how much of a pizza would have been eaten if 1 person ate $\frac{3}{8}$ and another person ate $\frac{2}{8}$ or $\frac{1}{4}$. (**Answer:** The total amount of pizza eaten is $\frac{5}{8}$.)

  Ask students how they handled the addition of those fractions with respect to adding the numerators and denominators and why. (**Answer:** Added the numerators and kept the denominators the same because het denominator represents the whole and we are adding the numerators as parts of the whole.)

  Ask students to reflect on how they should handle adding the numerators and denominators of the complex fractions. (**Answer:** Add the numerators but keep the denominator the same.)
- Students might believe that they are finished, but they are not. Ask students to review the formula for total impedance and reflect on why their answer is not yet final. (**Answer:** The formula for impedance gives the total as the reciprocal of the total. We need to take the reciprocal of the previous answer to arrive at the total impedance. $\frac{1}{15+i}+\frac{1}{10+2i}$.)

  If students have a difficult time seeing why you can take the reciprocal of both sides, have students set up and solve the proportion for Z_t. (**Answer:** $\frac{1}{2}+\frac{1}{3}$)

Debriefing the Task

Compare methods used by the groups and discuss advantages and disadvantages of the varying strategies.

Connecting to Key Concepts

In the following equations, let *a, b, c,* and *d* be real numbers.

- Find the product of the first terms, outside terms, inside terms, and last terms. (*Note*: The imaginary unit *i* follows the product of real numbers.)

  $(a + bi) \bullet (c + di) = ac$ (product of the first terms) $+ adi$ (product of the outside terms) $+$ bci (product of the inside terms) $+ bidi$ (product of the last terms) $= ac + adi + bci + bdi\ 2 = ac + bd(-1) + adi + bci = (ac - bd) + (ad + bc)i$
- $ac - bd$ is the real part of the product, and $ad + bc$ is the multiple of the imaginary unit i in the imaginary part of the product.

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Extending the Task

To extend the task, have students rationalize the denominator of the total impedance. For example, have students find the complex conjugate of $25 + 3i$. (**Answer:** The complex conjugate is $25 - 3i$.)

Also, ask students to multiply the numerator and the denominator by the complex conjugate and then simplify. (**Answer:** $\frac{1}{2}+\frac{1}{3}$)

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK**: "How did you demonstrate perseverance?" (**Answer:** I had a difficult time getting started, but I thought about what the what over the impedances meant by connecting that to the reciprocals of real numbers and I kept going with multiplying and adding even though the numbers were hard to visualize.)
- **For SMP 7, ASK**: "How did you make use of structure when solving the problem?" (**Answer:** I transferred what I knew about adding fractions with unlike denominators in the real number system to adding fractions with unlike denominators in the complex number system.)

Technology

Students can use calculators to multiply the numbers in the binomials.

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Common Core State Standards

Mathematics II

Integrated Pathway

Teacher Resource

Unit 2

WALCH
INTEGRATED MATH

This program was developed and reviewed by experienced math educators who have both academic and professional backgrounds in mathematics. This ensures: freedom from mathematical errors, grade level appropriateness, freedom from bias, and freedom from unnecessary language complexity.

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Standards Correlations

Each lesson in this Integrated Pathway: Mathematics II program was written specifically to address the Common Core State Standards. Each lesson lists the standards covered in all the sub-lessons, and each sub-lesson lists the standards addressed in that particular section. In this section, you'll find a comprehensive list mapping the sub-lessons to the CCSS.

Guide to Common Core State Standards Annotation

As you use this program, you will come across a symbol included with the Common Core standards for some of the lessons and activities. These symbols are explained below.

Symbol: ⋆

Denotes: Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (⋆).

From http://www.walch.com/CCSS/00003

Symbol: (+)

Denotes: College and Career Readiness Standards

Advanced mathematics standards that are required in higher-level courses such as advanced statistics may also be included in lower-level courses. These additional standards are denoted by (+). According to the Common Core State Standards Initiative, "the evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6–8."

From http://www.walch.com/CCSS/00004

Connections to Future Courses

This section provides a map between topics introduced in each unit of this course and subsequent courses where each topic is revisited and built upon.

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CCSS INTEGRATED PATHWAY: MATHEMATICS II STANDARDS CORRELATIONS

Unit 2: Quadratic Functions and Modeling				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Analyzing Quadratic Functions**			
	2.1.1	Graphing Quadratic Functions	F–IF.7a★	U2-4
	2.1.2	Interpreting Various Forms of Quadratic Functions	F–IF.7a★ F–IF.8a	U2-27
Lesson 2	**Interpreting Quadratic Functions**			
	2.2.1	Interpreting Key Features of Quadratic Functions	F–IF.4★	U2-55
	2.2.2	Identifying the Domain of a Quadratic Function	F–IF.5★	U2-77
	2.2.3	Identifying the Average Rate of Change	F–IF.6★	U2-91
Lesson 3	**Building Functions**			
	2.3.1	Building Functions from Context	F–BF.1a★	U2-113
	2.3.2	Operating on Functions	F–BF.1b★	U2-133
Lesson 4	**Graphing Other Functions**			
	2.4.1	Square Root and Cube Root Functions	F–IF.7b★	U2-155
	2.4.2	Absolute Value and Step Functions	F–IF.7b★	U2-190
	2.4.3	Piecewise Functions	F–IF.7b★	U2-219
Lesson 5	**Analyzing Functions**			
	2.5.1	Analyzing Exponential Functions	F–IF.8b	U2-254
	2.5.2	Comparing Properties of Functions Given in Different Forms	F–IF.9 F–LE.3★	U2-268
Lesson 6	**Transforming Functions**			
	2.6.1	Replacing $f(x)$ with $f(x) + k$ and $f(x + k)$	F–BF.3	U2-295
	2.6.2	Replacing $f(x)$ with $k \cdot f(x)$ and $f(k \cdot x)$	F–BF.3	U2-315
Lesson 7	**Finding Inverse Functions**			
	2.7.1	Finding Inverse Functions	F–BF.4a	U2-346

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TOPICS FROM FUTURE COURSES: CCSS INTEGRATED PATHWAY MATHEMATICS II

Math II	Topic introduced	Course/unit where addressed	How addressed
Unit 2	Quadratic functions for analyzing, solving, and transforming square root, cube root, and absolute value functions	Math III Units 3, 4A, and 4B	extended to rational, logarithmic, periodic, and polynomial functions
	Inverse functions	Math III Unit 4A	finding inverses for other function families
		Math IV	composing functions and verifying inverses

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Conceptual Activities

Use these interactive open education and/or Desmos resources to build conceptual understanding of mathematical ideas. (*Note*: Activity links will be monitored and repaired or replaced as necessary.)

- Desmos. "Card Sort: Parabolas."

  http://www.walch.com/ca/01022

  Find the shape of a parabola by using its form to reveal its characteristics. The activity begins with a review of both the characteristics and forms of a parabola, then moves on to determine characteristics of the graph of a parabola given in standard form, vertex form, or intercept form.

- Desmos. "Domain and Range Introduction."

  http://www.walch.com/ca/01049

  In this activity, students practice finding the domain and range of piecewise functions. Students begin with an informal exploration of domain and range using a graph, and build up to representing the domain and range of piecewise functions using inequalities.

- Desmos. "Free-Range Functions."

  http://www.walch.com/ca/01028

  This activity challenges students to strengthen their ideas about the range of quadratic functions.

- Desmos. "Polygraph: Absolute Value."

  http://www.walch.com/ca/01050

  This activity is designed to spark vocabulary-rich conversations about transformations of the absolute value parent function. Key vocabulary terms that may appear in student questions include *translation*, *shift*, *slide*, *dilation*, *stretch*, *horizontal*, *vertical*, and *reflect*.

- Desmos. "Polygraph: Piecewise Functions."

  http://www.walch.com/ca/01052

  This activity is designed to spark vocabulary-rich conversations about piecewise functions. Key vocabulary terms that may appear in student questions include *piecewise*, *continuous*, and *interval*.

- Desmos. "Polygraph: Square Root Functions."

  http://www.walch.com/ca/01032

  This activity is designed to spark vocabulary-rich conversations about transforming square root functions. Key vocabulary terms that may appear in student questions include *translation*, *reflection*, *intercept*, and *quadrant*.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. What is the vertex of the parabola with the equation $f(x) = 4(x - 1)^2 + 9$?

   a. (–1, –9)
   b. (–1, 9)
   c. (1, –9)
   d. (1, 9)

2. What are the x-intercepts of the parabola with the equation $g(x) = (x - 2)(x + 5)$?

   a. –2 and 5
   b. –5 and 2
   c. 1 and 3
   d. –10 and 3

3. What is the vertex of the parabola with the equation $f(x) = 3x^2 - 12x + 4$?

   a. (–4, –3)
   b. (–6, –2)
   c. (2, –8)
   d. (–2, 40)

4. Which of the following functions does not have a maximum?

   a. $f(x) = 2(x - 4)(x + 6)$
   b. $f(x) = –5(x + 8)^2 – 3$
   c. $f(x) = –4x^2 – 16x + 5$
   d. $f(x) = –5x^2 – 20x + 3$

5. A beginner pilot is participating in a flight simulation. In the simulation, the plane is in distress and losing altitude. The plane's altitude can be modeled by the equation $a(t) = 0.5(t - 11)^2 + 1$, where $a(t)$ measures the altitude in feet and t is the number of seconds that have passed since the trouble was noticed. Will the plane crash? How do you know?

   a. No, the plane will not crash. The minimum height of the plane is 1 foot.
   b. No, the plane will not crash. The value of a is positive.
   c. Yes, the plane will crash. The minimum height is 11 feet.
   d. Yes, the plane will crash. The value of a is less than 1.

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Lesson 1: Analyzing Quadratic Functions

Instruction

Common Core State Standards

SMP	
1 ✓	2 ✓
3 ✓	4 ✓
5	6
7	8

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

F–IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

Essential Questions

1. How is a quadratic equation similar to a linear equation? How is it different?
2. How is a quadratic equation similar to an exponential equation? How is it different?
3. In what situations is it appropriate to use a quadratic model?

WORDS TO KNOW

axis of symmetry of a parabola — the line through the vertex of a parabola about which the parabola is symmetric. For a quadratic function in the form $f(x) = ax^2 + bx + c$, the equation of the axis of symmetry is $x = \frac{-b}{2a}$.

extrema — the minima or maxima of a function

factored form of a quadratic function — the intercept form of a quadratic equation, written as $f(x) = a(x - p)(x - q)$, where p and q are the x-intercepts of the function; also known as the *intercept form of a quadratic function*

intercept — the point where a line or a curve intersects the x- or y-axis

intercept form of a quadratic function — the factored form of a quadratic equation, written as $f(x) = a(x - p)(x - q)$, where p and q are the x-intercepts of the function; also known as the *factored form of a quadratic function*

SRB p. U2-1

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Instruction

maximum	the greatest value of y in the graph of a function
minimum	the least value of y in the graph of a function
parabola	the U-shaped graph of a quadratic function
quadratic function	a function that can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. The graph of a quadratic function is a parabola.
standard form of a quadratic function	a quadratic function written as $f(x) = ax^2 + bx + c$, where a is the coefficient of the quadratic term, b is the coefficient of the linear term, and c is the constant term
vertex form of a quadratic function	a quadratic function written as $f(x) = a(x - h)^2 + k$, where the vertex of the parabola is the point (h, k); the form of a quadratic equation where the vertex can be read directly from the equation
vertex of a parabola	the point on a parabola that is closest to the directrix and lies on the axis of symmetry; its y-coordinate is either a maximum or a minimum
***x*-intercept**	the x-coordinate of the point where a line or a curve intersects the x-axis
***y*-intercept**	the y-coordinate of the point where a line or a curve intersects the y-axis

Recommended Resources

- IXL Learning. "Solve an Equation Using the Zero Product Property."

  http://www.walch.com/rr/00115

  This website gives a series of problems and scores them immediately, with feedback provided for incorrect answers. Users solve quadratic equations by setting factors equal to 0. This activity is meant as a review.

- PhET Interactive Simulations. "Equation Grapher 2.02."

  http://www.walch.com/rr/00116

  This website allows users to compare the graphs of various self-created equations.

- West Texas A&M University Virtual Math Lab. "Graphs of Quadratic Functions."

  http://www.walch.com/rr/00117

  This tutorial offers a review and examples for writing and graphing quadratic functions in different forms, as well as practice problems with worked solutions for reference.

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Lesson 2.1.1: Graphing Quadratic Functions

Warm-Up 2.1.1

The table represents the amount of money a car owner spends on repairs as a function of the number of oil changes the owner gets each year.

Yearly number of oil changes	Amount spent on car repairs ($)
0	500
1	420
2	340
3	260
4	180
5	100

1. Write a linear model of the amount the owner will spend on car repairs as a function of the number of oil changes in 1 year.

2. What is the *y*-intercept? What does it represent?

3. What is the rate of change? What does it mean for the owner?

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Instruction

Lesson 2.1.1: Graphing Quadratic Functions

Common Core State Standard

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

Warm-Up 2.1.1 Debrief

1. Write a linear model of the amount the owner will spend on car repairs as a function of the number of oil changes in 1 year.

   $y = -80x + 500$, where x is the number of oil changes in 1 year and y is the total amount spent on car repairs in 1 year.

2. What is the y-intercept? What does it represent?

   The y-intercept is 500. This is the amount spent on car repairs if the owner does not get any oil changes over the course of a year.

3. What is the rate of change? What does it mean for the owner?

   The rate of change is the slope of the function, –80. For the owner, it means that for every oil change performed, he will save $80 on car repairs.

Connection to the Lesson

- Students will extend their knowledge of interpreting linear functions to identify and interpret the key features of a quadratic function.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- graphing functions by creating a table of values (A–CED.2★)
- graphing using technology (F–IF.7a★)
- identifying key features of linear functions (F–IF.4★)

Introduction

You may recall that a line is the graph of a linear function and that all linear functions can be written in the form $f(x) = mx + b$, where m is the slope and b is the y-intercept. The solutions to a linear function are the infinite set of points on the line. In this lesson, you will learn about a second type of function known as a quadratic function.

Key Concepts

- A **quadratic function** is a function that can be written in the form $f(x) = ax^2 + bx + c$, where x is the variable, a, b, and c are real numbers, and $a \neq 0$. This form is also known as the **standard form of a quadratic function**, where a is the coefficient of the quadratic term, b is the coefficient of the linear term, and c is the constant term.
- Quadratic functions can be graphed on a coordinate plane.
- One method of graphing a quadratic function is to create a table of at least five x-values and calculate the corresponding y-values.
- Once graphed, all quadratic functions will have a U-shape called a **parabola**.
- Distinguishing characteristics can be used to describe, draw, and compare quadratic functions. These characteristics include the y-intercept, x-intercepts, the maximum or minimum of the function, and the axis of symmetry.
- The **intercept** of a graph is the point where a line or a curve intersects the x- or y-axis.
- The ***x*-intercept** is the x-coordinate of the point where a line or a curve intersects the x-axis. It is written as $(x, 0)$.
- The x-intercepts of a quadratic function occur when the parabola intersects the x-axis at $(x, 0)$.

SRB p. U2-4

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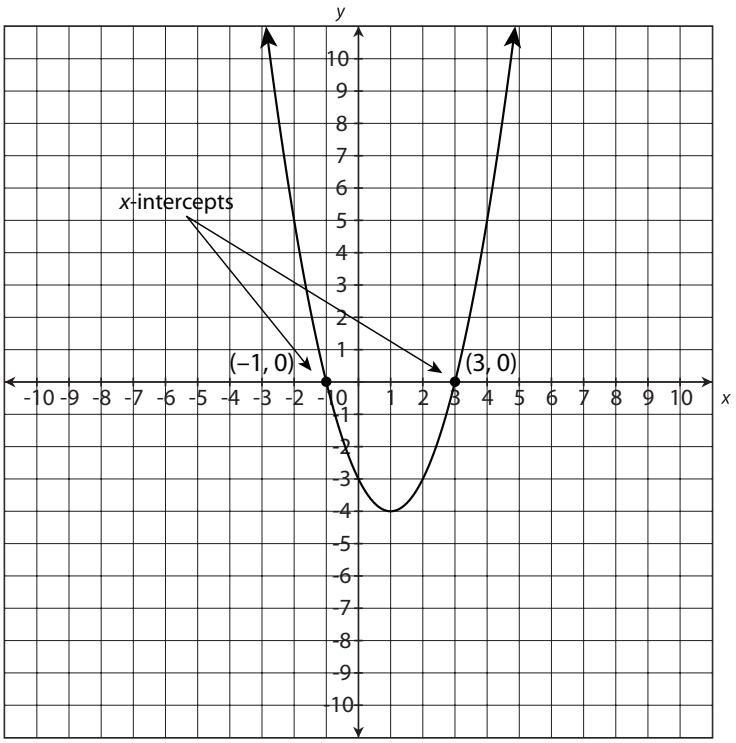
Instruction

- The following graph of a quadratic function, $f(x) = x^2 - 2x - 3$, shows the location of the parabola's x-intercepts.



- Note that the x-intercepts of this function are (–1, 0) and (3, 0).
- The equation of the x-axis is $y = 0$; therefore, the x-intercepts can also be found in a table by identifying which values of x have a corresponding y-value that is 0.
- The following table of values corresponds to the function $f(x) = x^2 - 2x - 3$. Notice that the same x-intercepts noted in the graph can be found where the table shows y is equal to 0.

x	y
–2	5
–1	0
0	–3
1	–4
2	–3
3	0

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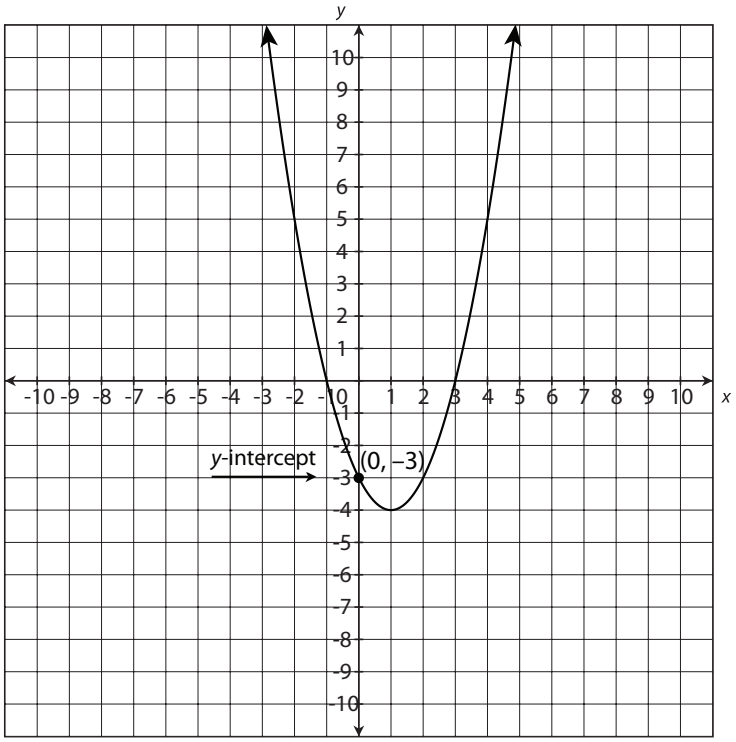
Instruction

- The ***y*-intercept** of a quadratic function is the y-coordinate of the point where a line or a curve intersects the y-axis. It is written as $(0, y)$.
- The y-intercept of a quadratic is the c value of the quadratic function when written in standard form.
- The following graph of a quadratic function, $f(x) = x^2 - 2x - 3$, shows the location of the parabola's y-intercept.



- Note that the y-intercept of this function is $(0, -3)$. The c-value of the function is -3.
- The **axis of symmetry of a parabola** is the line through the vertex of a parabola about which the parabola is symmetric.
- For a quadratic function in the form $f(x) = ax^2 + bx + c$, the equation of the axis of symmetry is $x = \frac{-b}{2a}$.

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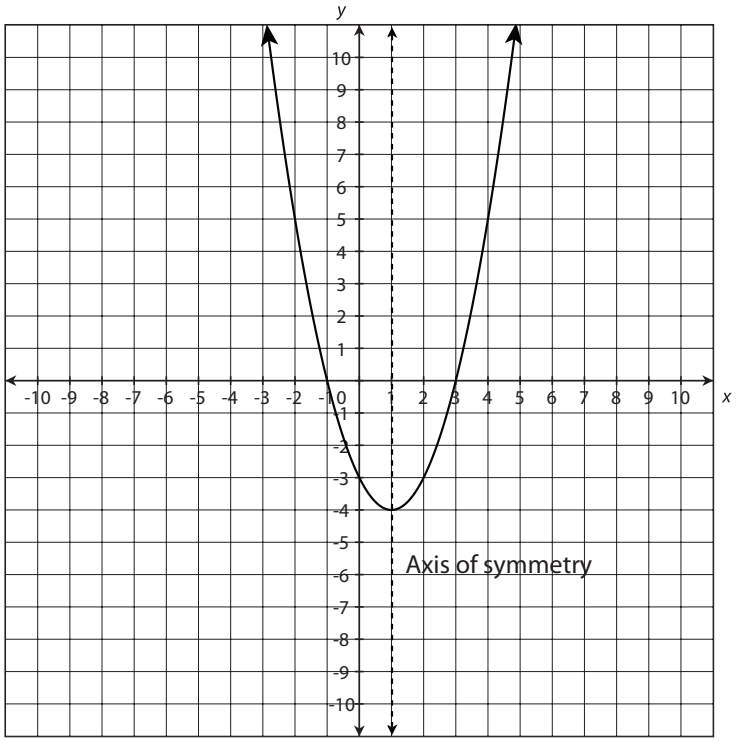
Instruction

- The equation of the axis of symmetry for the function $f(x) = x^2 - 2x - 3$ is $x = 1$ because the vertical line through 1 is the line that cuts the parabola in half.



- The **vertex of a parabola** is the point on a parabola that is the maximum or minimum of the function.
- The **maximum** is the greatest value of y in the graph of a function, and the **minimum** is the least value of y.
- The **extrema** of a graph are the minima or maxima of a function. In other words, an extremum is the function value that achieves either a minimum or maximum.
- The vertex of a parabola lies on the axis of symmetry.
- The vertex is often written as (h, k).
- The formula $x = \frac{-b}{2a}$ is also used to find the x-coordinate of the vertex.
- To find the y-coordinate, substitute the value of x into the original function, $(h,k) = \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

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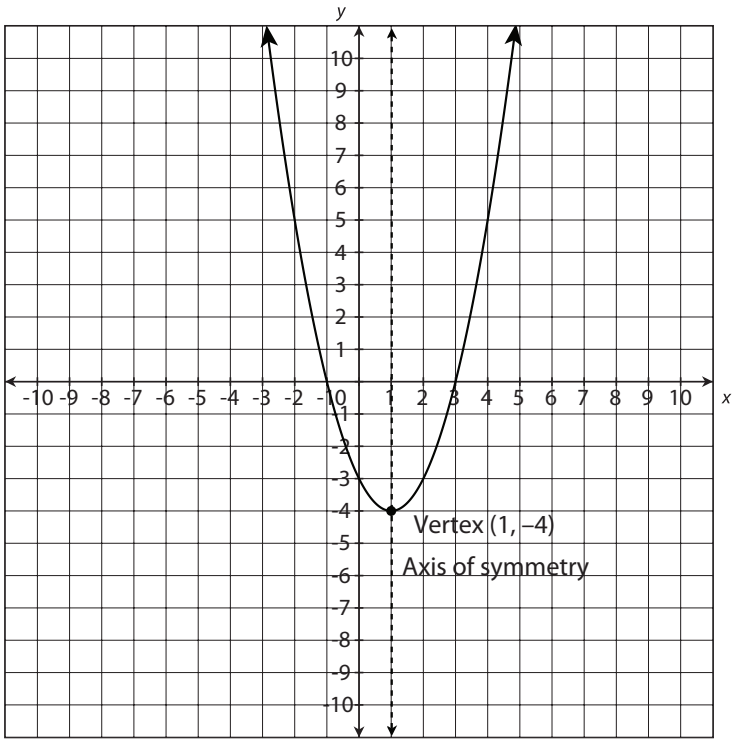
Instruction

- The graph that follows shows the relationship between the vertex and the axis of symmetry of a parabola.



- Notice that the vertex of the function $f(x) = x^2 - 2x - 3$ is (1, –4).
- If you know the x-intercepts of the graph, or any two points on the graph with the same y-value, the x-coordinate of the vertex is the point halfway between the values of the x-coordinates.
- For x-intercepts $(r, 0)$ and $(s, 0)$, the x-coordinate of the vertex is $\frac{r+s}{2}$.
- From the equation of a function in standard form, you can determine if the function has a maximum or a minimum based on the sign of the coefficient of the quadratic term, a.
- If $a > 0$, then the parabola opens up and therefore has a minimum value. If $a < 0$, the parabola opens down and therefore has a maximum value.
- The value of a in the function $f(x) = x^2 - 2x - 3$ is 1; therefore, the vertex is a minimum.

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Instruction

- To graph a function using a graphing calculator, follow these general steps for your calculator model.

On a TI-83/84:

Step 1: Press the [Y=] button.

Step 2: Type the function into Y1, or any available equation. Use the [X, T, θ, n] button for the variable x. Use the [x^2] button for a square.

Step 3: Press [WINDOW]. Enter values for Xmin, Xmax, Ymin, and Ymax. The Xscl and Yscl are arbitrary. Leave Xres = 1.

Step 4: Press [GRAPH].

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Type the function next to $f1(x)$, or any available equation, and press [enter]. Use the [X] button for the variable x. Use the [x^2] button for a square.

Step 4: To change the viewing window, press [menu]. Select 4: Window/ Zoom and select A: Zoom – Fit.

Common Errors/Misconceptions

- incorrectly replacing x with 0 instead of y when determining the x-intercept (and vice versa)
- using the incorrect sign when calculating the x-coordinate of the vertex

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Instruction

Guided Practice 2.1.1

Example 1

Given the function $f(x) = x^2$, identify the key features of the graph: the extremum, vertex, and y-intercept. Then sketch the graph.

1. Determine the extremum of the graph.

   The extreme value is a minimum when $a > 0$. It is a maximum when $a < 0$.

   Because $a = 1$, the graph opens upward and the quadratic has a minimum.

2. Determine the vertex of the graph.

   The minimum value occurs at the vertex.

   The vertex is of the form $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

   Use the original function $f(x) = x^2$ to find the values of a and b in order to find the x-coordinate of the vertex.

$x = \frac{-b}{2a}$	Formula to find the x-coordinate of the vertex of a parabola
$x = \frac{-(0)}{2(1)}$	Substitute 1 for a and 0 for b.
$x = 0$	Simplify.

   The x-coordinate of the vertex is 0.

   Substitute 0 into the original equation to find the y-coordinate.

$f(x) = x^2$	Original equation
$f(0) = (0)^2$	Substitute 0 for x.
$f(0) = 0$	Simplify.

   The y-coordinate of the vertex is 0.

   The vertex is located at (0, 0).

SWB p. 9; SRB p. U2-10

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3. Determine the y-intercept of the graph.

   The y-intercept occurs when $x = 0$.

   The y-intercept of the function $f(x) = x^2$ is the same as the vertex, (0, 0).

   When the equation is written in standard form, the y-intercept is c.

4. Graph the function.

   Create a table of values and axis of symmetry to identify points on the graph.

   The axis of symmetry goes through the vertex, so the axis of symmetry is $x = 0$.

   For each point to the left of the axis of symmetry, there is another point the same distance on the right side of the axis and vice versa.

   Choose at least two values of x that are to the right and left of 0.

   Let's start with $x = 2$.

$f(x) = x^2$	Original equation
$f(2) = (2)^2$	Substitute 2 for x.
$f(2) = 4$	Simplify.

   An additional point is (2, 4).

   (2, 4) is 2 units to the right of the vertex. The point (–2, 4) is 2 units to the left of the vertex, so (–2, 4) is also on the graph.

   To find another set of points on the graph, let's evaluate the original equation for $x = 3$.

$f(x) = x^2$	Original equation
$f(3) = (3)^2$	Substitute 3 for x.
$f(3) = 9$	Simplify.

   An additional point is (3, 9).

   (3, 9) is 3 units to the right of the vertex. The point (–3, 9) is 3 units to the left of the vertex, so (–3, 9) is also on the graph.

(*continued*)

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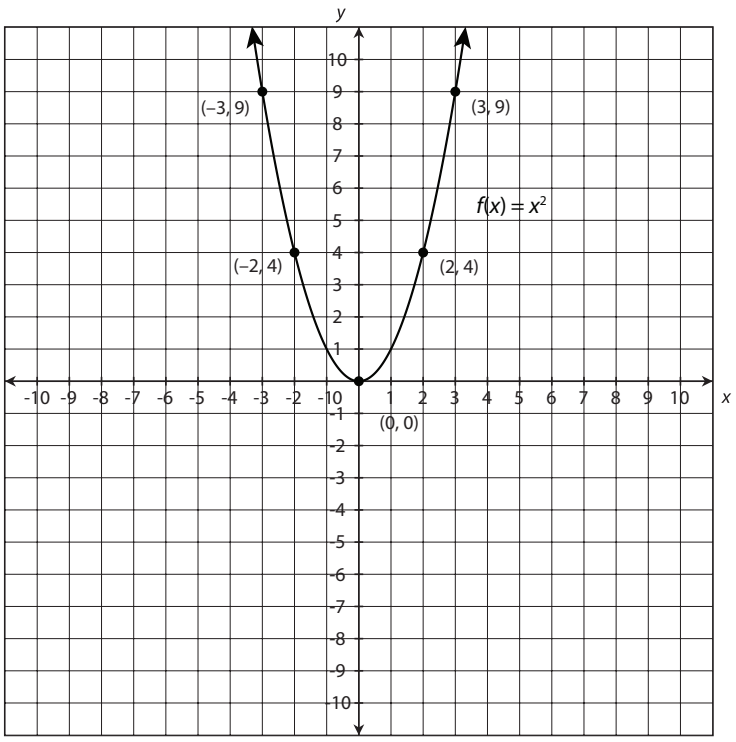
Instruction

Plot the points and join them with a smooth curve.



Example 2

Given the function $f(x) = -2x^2 + 16x - 30$, identify the key features of the graph: the extremum, vertex, and y-intercept. Then sketch the graph.

1. Determine the extremum of the graph.

   The extreme value is a minimum when $a > 0$. It is a maximum when $a < 0$.

   Because $a = -2$, the graph opens downward and the quadratic has a maximum.

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Instruction

2. Determine the vertex of the graph.

The maximum value occurs at the vertex.

The vertex is of the form $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

Use the original equation $f(x) = -2x^2 + 16x - 30$ to find the values of a and b in order to find the x-coordinate of the vertex.

$x = \frac{-b}{2a}$	Formula to find the x-coordinate of the vertex of a parabola
$x = \frac{-(16)}{2(-2)}$	Substitute –2 for a and 16 for b.
$x = 4$	Simplify.

The x-coordinate of the vertex is 4.

Substitute 4 into the original equation to find the y-coordinate.

$f(x) = -2x^2 + 16x - 30$	Original equation
$f(4) = -2(4)^2 + 16(4) - 30$	Substitute 4 for x.
$f(4) = 2$	Simplify.

The y-coordinate of the vertex is 2.

The vertex is located at (4, 2).

3. Determine the y-intercept of the graph.

The y-intercept occurs when $x = 0$.

Substitute 0 for x in the original equation.

$f(x) = -2x^2 + 16x - 30$	Original equation
$f(0) = -2(0)^2 + 16(0) - 30$	Substitute 0 for x.
$f(0) = -30$	Simplify.

The y-intercept is (0, –30).

When the quadratic equation is written in standard form, the y-intercept is c.

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Instruction

4. Graph the function.

Use symmetry to identify additional points on the graph.

The axis of symmetry goes through the vertex, so the axis of symmetry is $x = 4$.

For each point to the left of the axis of symmetry, there is another point the same distance on the right side of the axis and vice versa.

The point (0, –30) is on the graph, and 0 is 4 units to the left of the axis of symmetry.

The point that is 4 units to the right of the axis is 8, so the point (8, –30) is also on the graph.

Determine two additional points on the graph.

Choose an x-value to the left or right of the vertex and find the corresponding y-value.

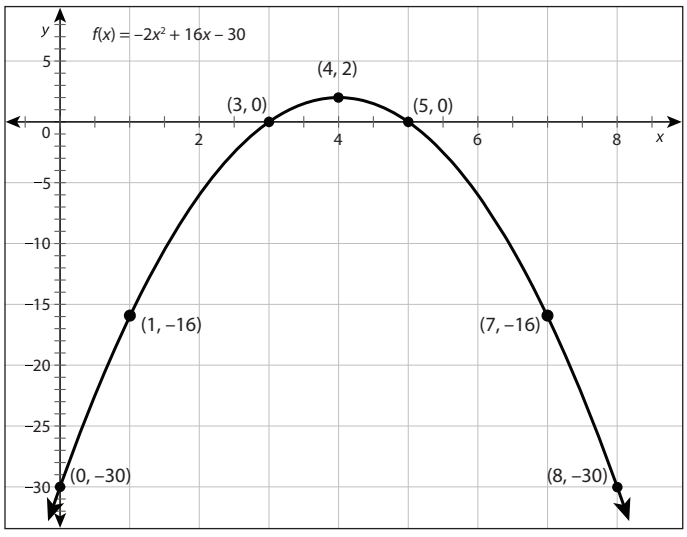
$f(x) = -2x^2 + 16x - 30$	Original equation
$f(1) = -2(1)^2 + 16(1) - 30$	Substitute 1 for x.
$f(1) = -16$	Simplify.

An additional point is (1, –16).

(1, –16) is 3 units to the left of the axis of symmetry.

The point that is 3 units to the right of the axis is 7, so the point (7, –16) is also on the graph.

Plot the points and join them with a smooth curve.



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Instruction

Example 3

Given the function $f(x) = x^2 + 6x + 9$, identify the key features of its graph: the extremum, vertex, and y-intercept. Then sketch the graph.

1. Determine the extremum of the graph.

   The extreme value is a minimum when $a > 0$. It is a maximum when $a < 0$.

   Because $a = 1$, the graph opens upward and the quadratic has a minimum.

2. Determine the vertex of the graph.

   The minimum value occurs at the vertex.

   The vertex is of the form $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

   Use the original function $f(x) = x^2 + 6x + 9$ to find the values of a and b in order to find the x-coordinate of the vertex.

$x = \frac{-b}{2a}$	Formula to find the x-coordinate of the vertex of a parabola
$x = \frac{-(6)}{2(1)}$	Substitute 1 for a and 6 for b.
$x = -3$	Simplify.

   The x-coordinate of the vertex is –3.

   Substitute –3 into the original equation to find the y-coordinate.

$f(x) = x^2 + 6x + 9$	Original equation
$f(-3) = (-3)^2 + 6(-3) + 9$	Substitute –3 for x.
$f(-3) = 0$	Simplify.

   The y-coordinate of the vertex is 0.

   The vertex is located at (–3, 0).

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Instruction

3. Determine the y-intercept of the graph.

The y-intercept occurs when $x = 0$.

Substitute 0 for x in the original equation.

$f(x) = x^2 + 6x + 9$	Original equation
$f(0) = (0)^2 + 6(0) + 9$	Substitute 0 for x.
$f(0) = 9$	Simplify.

The y-intercept is (0, 9).

4. Graph the function.

Use symmetry to identify an additional point on the graph.

The axis of symmetry goes through the vertex, so the axis of symmetry is $x = -3$.

For each point to the left of the axis of symmetry, there is another point the same distance on the right side of the axis and vice versa.

The point (0, 9) is on the graph, and 0 is 3 units to the right of the axis of symmetry.

The point that is 3 units to the left of the axis is –6, so the point (–6, 9) is also on the graph.

Determine two additional points on the graph.

Choose an x-value to the left or right of the vertex and find the corresponding y-value.

$f(x) = x^2 + 6x + 9$	Original equation
$f(-1) = (-1)^2 + 6(-1) + 9$	Substitute –1 for x.
$f(-1) = 4$	Simplify.

An additional point is (–1, 4).

(–1, 4) is 2 units to right of the axis of symmetry.

The point that is 2 units to the left of the axis is –5, so the point (–5, 4) is also on the graph.

(*continued*)

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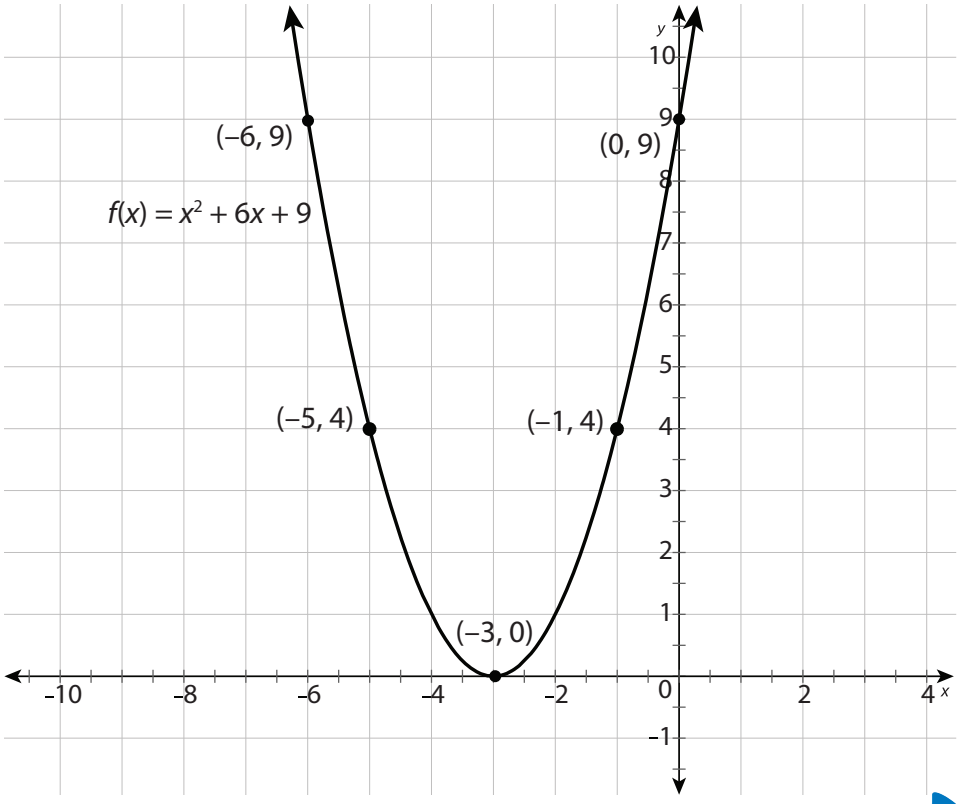
Instruction

Plot the points and join them with a smooth curve.



Example 4

Given the function $f(x) = -2x^2 - 12x - 10$, identify the key features of its graph: the extremum, vertex, and y-intercept. Then sketch the graph.

1. Determine the extremum of the graph.

   The extreme value is either a minimum, when $a > 0$, or a maximum, when $a < 0$.

   Because $a = -2$, the graph opens down and the quadratic has a maximum.

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Instruction

2. Determine the vertex of the graph.

The vertex is of the form $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

Use the original function $f(x) = -2x^2 - 12x - 10$ to find the values of a and b in order to find the x-coordinate of the vertex.

$x = \frac{-b}{2a}$	Formula to find the x-coordinate of the vertex of a parabola
$x = \frac{-(-12)}{2(-2)}$	Substitute –2 for a and –12 for b.
$x = -3$	Simplify.

The x-coordinate of the vertex is –3.

Substitute –3 into the original equation to find the y-coordinate.

$f(x) = -2x^2 - 12x - 10$	Original equation
$f(-3) = -2(-3)^2 - 12(-3) - 10$	Substitute –3 for x.
$f(-3) = 8$	Simplify.

The y-coordinate of the vertex is 8.

The vertex is (–3, 8).

3. Determine the y-intercept of the graph.

The y-intercept occurs when $x = 0$.

Substitute 0 for x in the original equation.

$f(x) = -2x^2 - 12x - 10$	Original equation
$f(0) = -2(0)^2 - 12(0) - 10$	Substitute 0 for x.
$f(0) = -10$	Simplify.

The y-intercept is (0, –10).

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Instruction

4. Graph the function.

Use symmetry to identify another point on the graph.

Because 0 is 3 units to the right of the axis of symmetry, the point 3 units to the left of the axis will have the same value, so (–6, –10) is also on the graph.

Determine two additional points on the graph.

Choose an x-value to the left or right of the vertex and find the corresponding y-value.

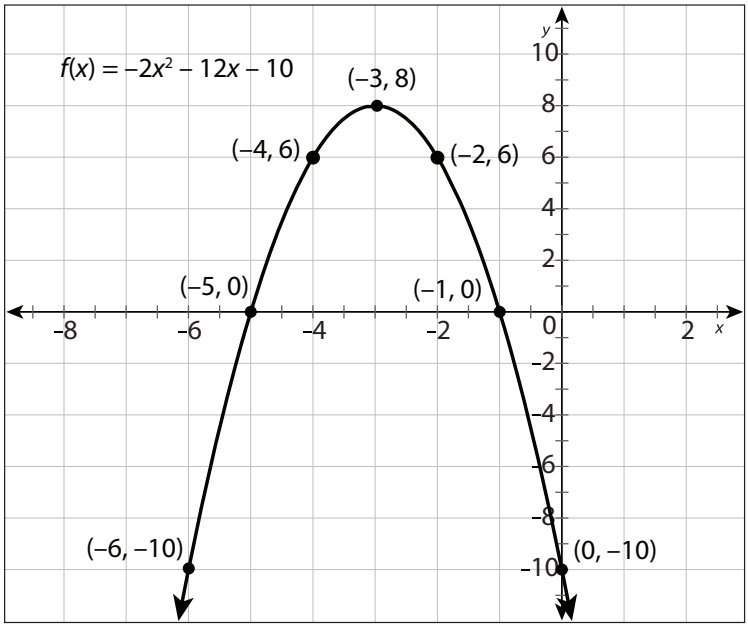
$f(x) = -2x^2 - 12x - 10$	Original equation
$f(0) = -2(-2)^2 - 12(-2) - 10$	Substitute –2 for x.
$f(-2) = 6$	Simplify.

An additional point is (–2, 6).

(–2, 6) is 1 unit to right of the axis of symmetry.

The point that is 1 unit to the left of the axis is –4, so the point (–4, 6) is also on the graph.

Plot the points and join them with a smooth curve.



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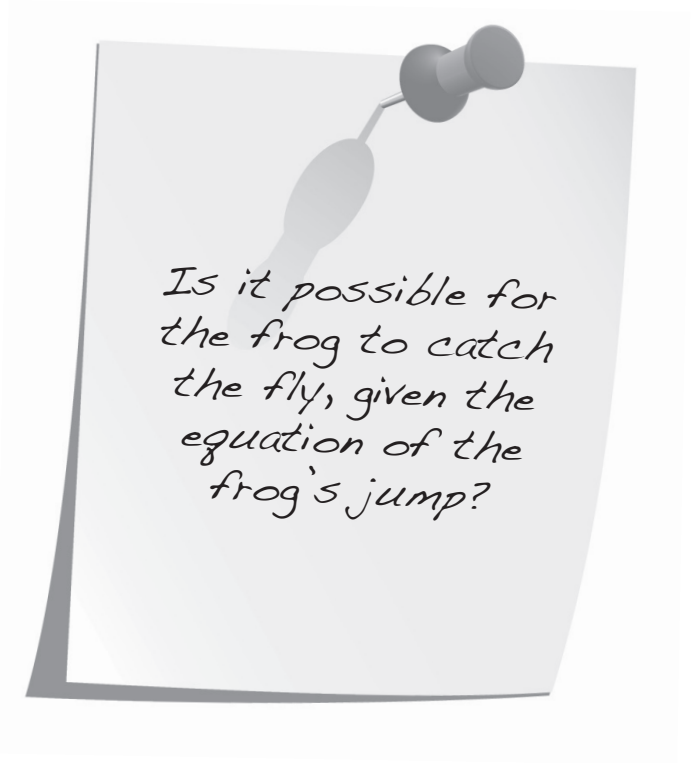
Problem-Based Task 2.1.1: How High Can a Frog Jump?

SMP	
1 ✓	2 ✓
3	4 ✓
5	6
7	8

A frog is about to hop from the bank of a creek. The path of the jump can be modeled by the equation $h(x) = -x^2 + 4x + 1$, where $h(x)$ is the frog's height above the water and x is the number of seconds since the frog jumped. A fly is cruising at a height of 5 feet above the water. Is it possible for the frog to catch the fly, given the equation of the frog's jump?



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Problem-Based Task 2.1.1: How High Can a Frog Jump?

Coaching

a. Can the frog catch the fly without jumping? How do you know?

b. Is the extremum a maximum or a minimum? Why?

c. What is the vertex of the equation?

d. If the frog jumps to catch the fly, is it possible for the frog to catch the fly?

e. Sketch the graph of the paths of the frog and fly.

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Instruction

Problem-Based Task 2.1.1: How High Can a Frog Jump?

Coaching Sample Responses

a. Can the frog catch the fly without jumping? How do you know?

No, the frog cannot catch the fly without jumping. The frog's initial height is when $t = 0$, represented by the y-intercept.

$h(x) = -x^2 + 4x + 1$

$h(0) = -(0)^2 + 4(0) + 1$

$h(0) = 1$

The frog's initial height above the water is 1 foot when it's sitting on the bank of the creek, so the frog cannot catch a fly that is 5 feet above the water.

b. Is the extremum a maximum or a minimum? Why?

The extremum is a maximum. The value of $a = -1$ and $-1 < 0$, so the parabola opens down.

c. What is the vertex of the equation?

The vertex of the equation can be found by completing the square or by using $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

Use the original equation $h(x) = -x^2 + 4x + 1$, where $a = -1$ and $b = 4$, to find the x-coordinate of the vertex.

$$x = \frac{-b}{2a}$$

$$x = \frac{-(4)}{2(-1)}$$

$$x = 2$$

The x-coordinate of the vertex is 2.

To find the y-coordinate of the vertex, substitute 2 into the original equation.

$h(x) = -(2)^2 + 4(2) + 1$

$h(x) = 5$

The vertex of the equation is (2, 5).

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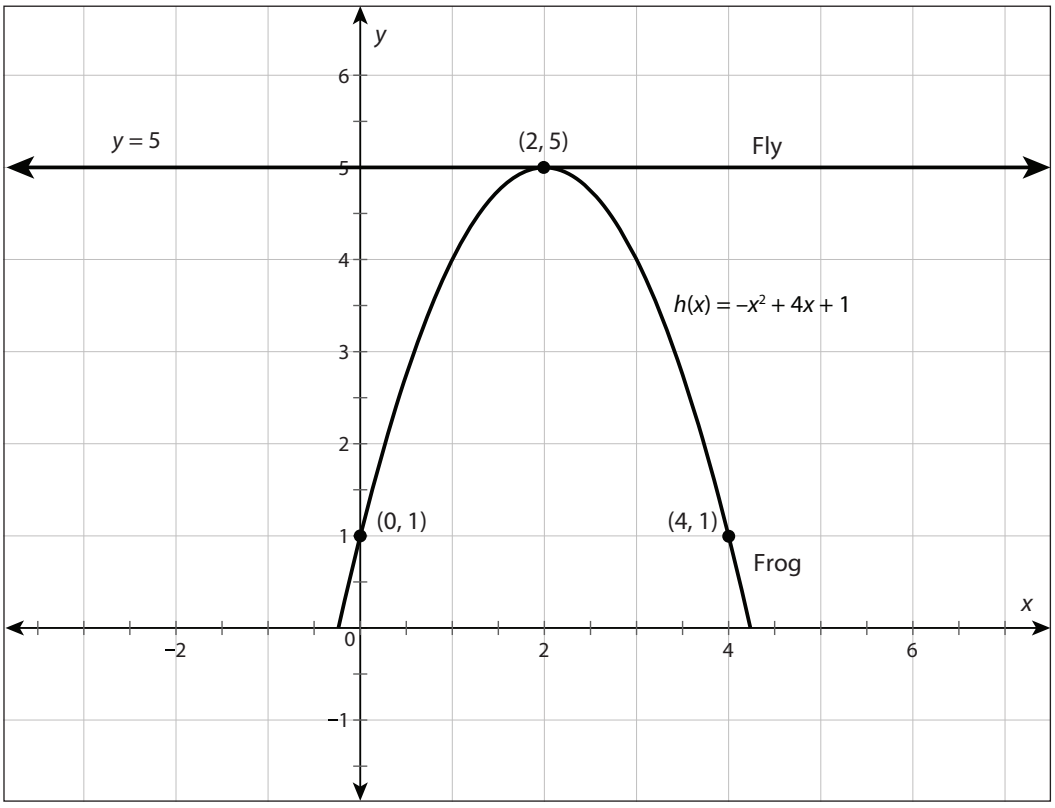
d. If the frog jumps to catch the fly, is it possible for the frog to catch the fly?

Yes. The maximum height of the frog is the y-coordinate of the vertex, 5. It is possible for the frog to catch the fly because the frog can jump as high or higher than the fly.

e. Sketch the graph of the paths of the frog and fly.

Plot the points of the parabola and connect them with a smooth curve. The fly's path is a horizontal line at $y = 5$.



Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2.1.1: Graphing Quadratic Functions

For each function that follows, identify the *y*-intercept, vertex, and extremum. Then, sketch the graph of the function.

1. $f(x) = x^2 + 6x + 5$

2. $f(x) = x^2 + 2x - 15$

3. $f(x) = -x^2 + 10x - 9$

4. $f(x) = -x^2 - 4x$

5. $f(x) = x^2 - 4x - 12$

6. $f(x) = \frac{1}{2}x^2 + 2x$

7. $f(x) = -x^2 - 4x - 3$

For each problem that follows, determine whether the function has a minimum or maximum, identify the maximum or minimum, and identify the *y*-intercept.

8. Suppose the distance between a boomerang and the person who threw it follows a quadratic relationship in terms of the time *t* since it was thrown. The equation that models this situation is given by $d(t) = -\frac{1}{2}t^2 + 6t$.

9. The path of a snowboarder performing stunts is given by the equation $f(x) = -16t^2 + 24t + 16$, where *t* is time in seconds and $f(x)$ is the height of the snowboarder.

10. The flight of a boulder launched from a catapult follows the quadratic equation $H(x) = -x^2 + 6x + 16$, where $H(x)$ represents the height of the boulder in feet and *x* is the horizontal distance in feet the boulder travels after launch.

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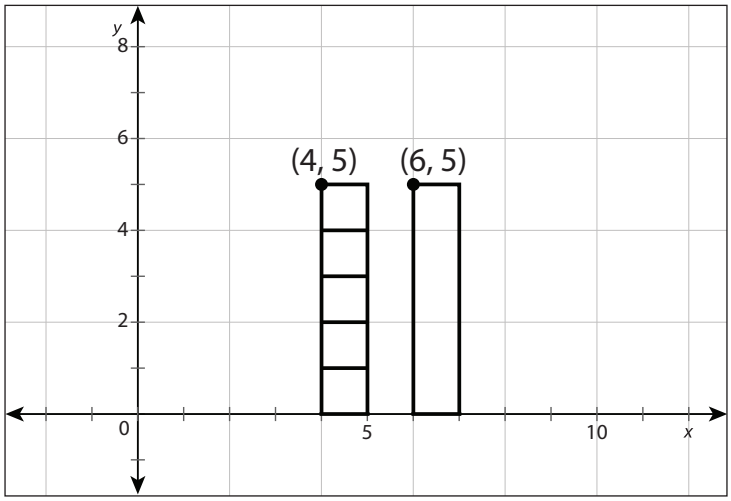
Lesson 2.1.2: Interpreting Various Forms of Quadratic Functions

Warm-Up 2.1.2

Joe is trying to use a hose to spray water over a stack of moving boxes and into a bird bath. The graph represents a stack of boxes on the left that the water must clear and a birdbath on the right that the water must fill. The water will follow a parabolic path.



1. One possible path the water could travel is given by $y=-\frac{1}{5}x^2+\frac{8}{5}x+\frac{9}{5}$, where y represents the height in feet and x represents the horizontal distance traveled in feet. What is the vertex of this parabola?

2. Determine the second x-intercept if one x-intercept of the path of the water is –1.

3. What is the maximum value of the quadratic function?

4. Sketch the graph of the path of the water.

5. Based on the graph, will the water clear the boxes? If it clears the boxes, will the water fill the birdbath?

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Lesson 2.1.2: Interpreting Various Forms of Quadratic Functions

Common Core State Standards

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

F–IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

Warm-Up 2.1.2 Debrief

1. One possible path the water could travel is given by $y=-\frac{1}{5}x^2+\frac{8}{5}x+\frac{9}{5}$, where y represents the height in feet and x represents the horizontal distance traveled in feet. What is the vertex of this parabola?

   The vertex is of the form $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

   Use the original equation, $y=-\frac{1}{5}x^2+\frac{8}{5}x+\frac{9}{5}$, to find the values of a and b in order to find the x-coordinate of the vertex.

   $x=\frac{-b}{2a}$ Formula to find the x-coordinate of the vertex of a parabola

   $x=\frac{-\left(\frac{8}{5}\right)}{2\left(-\frac{1}{5}\right)}$ Substitute $-\frac{1}{5}$ for a and $\frac{8}{5}$ for b.

   $x = 4$ Simplify.

   The x-coordinate of the vertex is 4.

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Substitute 4 into the original equation to find the *y*-coordinate of the vertex.

$y = -\frac{1}{5}x^2 + \frac{8}{5}x + \frac{9}{5}$ Original equation

$y = -\frac{1}{5}(4)^2 + \frac{8}{5}(4) + \frac{9}{5}$ Substitute 4 for *x*.

$y = 5$ Simplify.

The *y*-coordinate of the vertex is 5.

The vertex is located at (4, 5).

2. Determine the second *x*-intercept if one *x*-intercept of the path of the water is –1.

   Use symmetry to identify the second *x*-intercept.

   The axis of symmetry goes through the vertex, so the axis of symmetry is $x = 4$.

   For each point to the left of the axis of symmetry, there is another point the same distance on the right side of the axis.

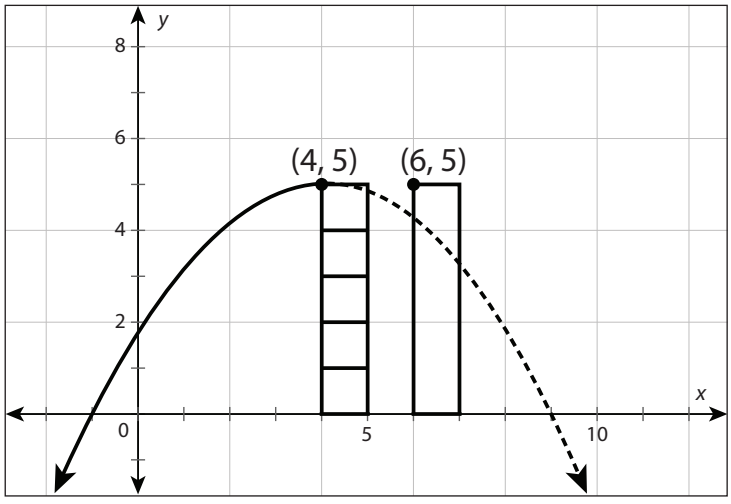
   The point (–1, 0) is on the graph; $x = -1$ is 5 units to the left of the axis of symmetry.

   The point that is 5 units to the right of the axis is 9, so the point (9, 0) is also on the graph. The other *x*-intercept is 9.

3. What is the maximum value of the quadratic function?

   The maximum value is the *y*-coordinate of the vertex, 5.

4. Sketch the graph of the path of the water.



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5. Based on the graph, will the water clear the boxes? If it clears the boxes, will the water fill the birdbath?

   No, the water will not clear the boxes. If the boxes were not in the way, the water would still not fill the birdbath because the birdbath is too tall.

Connection to the Lesson

- Students will extend their knowledge of identifying key features of a quadratic function in standard form in order to identify key features of a quadratic function written in alternative forms.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- applying the order of operations (5.OA.1)
- identifying key features of a quadratic function in standard form (F–IF.4★)

Introduction

Quadratic equations can be written in several forms, including standard form, vertex form, and factored form. While each form is equivalent, certain forms easily reveal different features of the graph of the quadratic function. In this lesson, you will learn to use the various forms of quadratic functions to show the key features of the graph and determine how these key features relate to the characteristics of a real-world situation.

Key Concepts

Standard Form

- Recall that the standard form, or general form, of a quadratic function is written as $f(x) = ax^2 + bx + c$, where a is the coefficient of the quadratic term, b is the coefficient of the linear term, and c is the constant term.
- When a function is written in standard form, the y-intercept is the value of c.
- The vertex of the function can be found by first determining the value of x, $x = \frac{-b}{2a}$, and then finding the corresponding y-value, $y = f\left(\frac{-b}{2a}\right)$.
- The vertex is often written as $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.
- If $a > 0$, the function has a minimum and the graph opens up.
- If $a < 0$, the function has a maximum and the graph opens down.

Vertex Form

- The **vertex form of a quadratic function** is written as $f(x) = a(x - h)^2 + k$.
- In vertex form, the maximum or minimum of the function is identified using the vertex of the parabola, the point (h, k).

SRB p. U2-21

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Instruction

- If $a > 0$, the function has a minimum. The minimum is the value of k, the y-coordinate of the vertex.
- If $a < 0$, the function has a maximum. The maximum is the value of k, the y-coordinate of the vertex.
- Because the axis of symmetry goes through the vertex, the axis of symmetry can be identified from vertex form as $x = h$.
- The graph of a quadratic function is symmetric about the axis of symmetry.

Factored Form

- The **factored form**, or **intercept form**, of a quadratic function is written as $f(x) = a(x - p)(x - q)$.
- Recall that the x-intercepts of a function are the x-values when $y = 0$.
- In factored form, the x-intercepts of the function are identified as p and q.
- Recall that the y-intercept of a function is the point at which the function intersects the y-axis.
- To determine the y-intercept, substitute 0 for x and simplify.
- The axis of symmetry can be identified from the factored form since it passes through the midpoint between the x-intercepts. Therefore, the axis of symmetry is $x = \frac{p+q}{2}$.
- To determine the vertex of the parabola, calculate the y-value that corresponds to the x-value of the axis of symmetry.
- If $a > 0$, the function has a minimum and the graph opens up.
- If $a < 0$, the function has a maximum and the graph opens down.

Common Errors/Misconceptions

- confusing the attributes of different forms
- incorrectly identifying x-intercepts of the factored form
- incorrectly identifying the vertex as a maximum or minimum

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Guided Practice 2.1.2

Example 1

Suppose that the flight of a launched bottle rocket can be modeled by the function $f(x) = -(x - 1)(x - 6)$, where $f(x)$ measures the height above the ground in meters and x represents the horizontal distance in meters from the launching spot at the point (1, 0). How far does the bottle rocket travel in the horizontal direction from launch to landing? What is the maximum height the bottle rocket reaches? How far has the bottle rocket traveled horizontally when it reaches its maximum height? Graph the function.

1. Identify the x-intercepts of the function.

   In the function, $f(x)$ represents the height of the bottle rocket. At launch and landing, the height of the bottle rocket is 0.

   The function $f(x) = -(x - 1)(x - 6)$ is of the form $f(x) = a(x - p)(x - q)$, where p and q are the x-intercepts.

   The x-intercepts of the function are 1 and 6.

   Find the distance between the two points to determine how far the bottle rocket traveled in the horizontal direction.

   $6 - 1 = 5$

   The bottle rocket traveled 5 meters in the horizontal direction from launch to landing.

2. Determine the maximum height of the bottle rocket.

   The maximum height occurs at the vertex.

   Find the x-coordinate of the vertex using the formula $x = \frac{p+q}{2}$.

$x = \frac{p+q}{2}$	Formula to determine the x-coordinate of the vertex of a parabola
$x = \frac{6+1}{2}$	Substitute 6 for p and 1 for q.
$x = 3.5$	Simplify.

   The x-coordinate of the vertex is 3.5.

(*continued*)

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Use this value to determine the vertex of the function.

$f(x) = -(x - 1)(x - 6)$	Original function
$f(3.5) = -[(3.5) - 1][(3.5) - 6]$	Substitute 3.5 for x.
$f(3.5) = -(2.5)(-2.5)$	Simplify.
$f(3.5) = 6.25$	Multiply.

The y-coordinate of the vertex is 6.25.

The maximum height reached by the bottle rocket is 6.25 meters.

3. Determine the horizontal distance from the launch point to the maximum height of the bottle rocket.

   We know that the bottle rocket is launched from the point (1, 0) and reaches a maximum height at (3.5, 6.25). Subtract the x-value of the two points to find the distance traveled horizontally.

   $3.5 - 1 = 2.5$

   Another method is to take the total distance traveled horizontally from launch to landing and divide by 2 to find the same answer. This is because the maximum value occurs halfway between the x-intercepts of the function.

   $$\frac{5}{2} = 2.5$$

   The bottle rocket travels 2.5 meters horizontally when it reaches its maximum.

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4. Graph the function.

Use a graphing calculator or complete a table of values. Use the *x*-intercepts and vertex as three of the known points. Choose *x*-values on either side of the vertex for two additional *x*-values.

x	*y*
1	0
2	
3.5	6.25
5	
6	0

To determine the *y*-coordinates of the additional points, substitute each *x*-value into the original function and evaluate.

$f(x) = -(x - 1)(x - 6)$	Original function
$f(2) = -[(2) - 1][(2) - 6]$	Substitute 2 for *x*.
$f(2) = -(1)(-4)$	Simplify.
$f(2) = 4$	Multiply.

$f(x) = -(x - 1)(x - 6)$	Original function
$f(5) = -[(5) - 1][(5) - 6]$	Substitute 5 for *x*.
$f(5) = -(4)(-1)$	Simplify.
$f(5) = 4$	Multiply.

Fill in the missing table values.

x	*y*
1	0
2	**4**
3.5	6.25
5	**4**
6	0

(*continued*)

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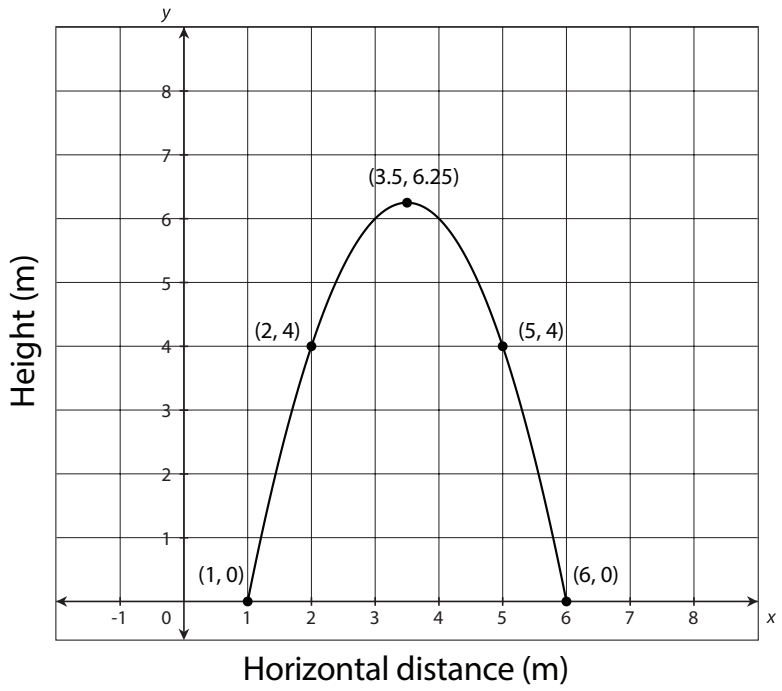
Instruction

Plot the points on a coordinate plane and connect them using a smooth curve.

Since the function models the flight of a bottle rocket, it is important to only show the portion of the graph where both horizontal distance and height are positive.



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Example 2

Reducing the cost of an item can result in a greater number of sales. The revenue function that predicts the revenue in dollars, $R(x)$, for each \$1 change in price, x, for a particular item is $R(x) = -100(x - 7)^2 + 28{,}900$. What is the maximum value of the function? What does the maximum value mean in the context of the problem? What price increase maximizes the revenue and what does it mean in the context of the problem? Graph the function.

1. Determine the maximum value of the function.

   The function $R(x) = -100(x - 7)^2 + 28{,}900$ is written in vertex form, $f(x) = a(x - h)^2 + k$, where (h, k) is the vertex.

   The vertex of the function is (7, 28,900); therefore, the maximum value is 28,900.

2. Determine what the maximum value means in the context of the problem.

   The maximum value of 28,900 means that the maximum revenue resulting from increasing the price by x dollars is \$28,900.

3. Determine the price increase that will maximize the revenue and what it means in the context of the problem.

   The maximum value occurs at the vertex, (7, 28,900).

   This means an increase in price of \$7 will result in the maximum revenue.

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4. Graph the function.

Use a graphing calculator or complete a table of coordinates. Use the vertex as one known point. Choose x-values on either side of the vertex to have four additional x-values.

x	y
0	
5	
7	28,900
9	
14	

To determine the y-coordinates of the additional points, substitute each x-value into the original function and solve.

$R(x) = -100(x - 7)^2 + 28{,}900$ Original function

$R(0) = -100[(0) - 7]^2 + 28{,}900$ Substitute 0 for x.

$R(0) = 24{,}000$ Simplify.

$R(x) = -100(x - 7)^2 + 28{,}900$ Original function

$R(5) = -100[(5) - 7]^2 + 28{,}900$ Substitute 5 for x.

$R(5) = 28{,}500$ Simplify.

$R(x) = -100(x - 7)^2 + 28{,}900$ Original function

$R(9) = -100[(9) - 7]^2 + 28{,}900$ Substitute 9 for x.

$R(9) = 28{,}500$ Simplify.

$R(x) = -100(x - 7)^2 + 28{,}900$ Original function

$R(14) = -100[(14) - 7]^2 + 28{,}900$ Substitute 14 for x.

$R(14) = 24{,}000$ Simplify.

(*continued*)

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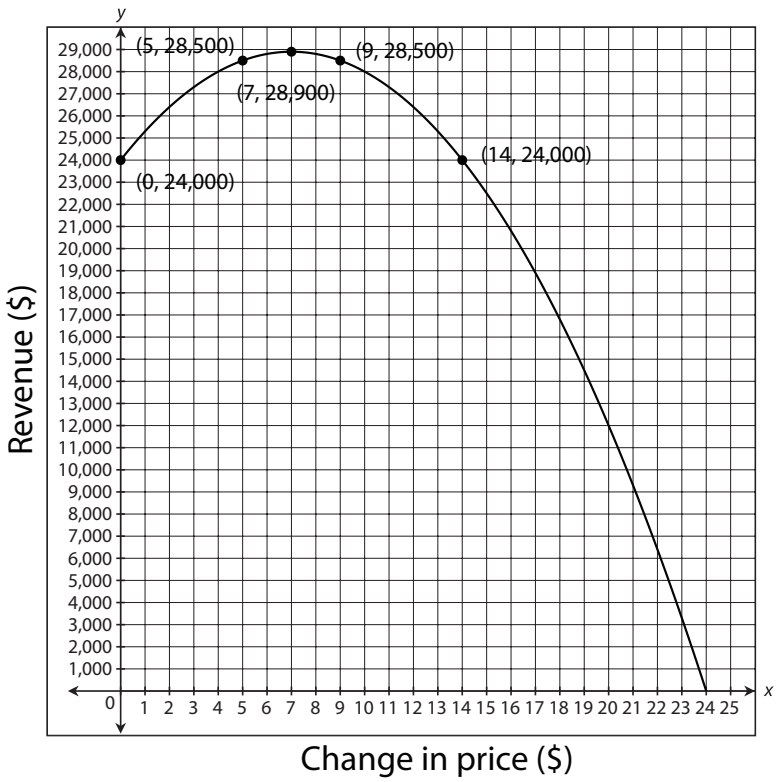
Fill in the missing table values.

x	y
0	**24,000**
5	**28,500**
7	28,900
9	**28,500**
14	**24,000**

Notice that the points (0, 24,000) and (14, 24,000) are the same horizontal distance from the vertex on either side. The same is true for (5, 28,500) and (9, 28,500).

Plot the points on a coordinate plane and connect using a smooth curve.



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Instruction

Example 3

A football is kicked and follows a path given by $f(x) = -0.03x^2 + 1.8x$, where $f(x)$ represents the height of the ball in feet and x represents the horizontal distance in feet. What is the maximum height the ball reaches? What horizontal distance maximizes the height? Graph the function.

1. Determine the maximum height of the ball.

   The function $f(x) = -0.03x^2 + 1.8x$ is written in standard form, $f(x) = ax^2 + bx + c$, where $a = -0.03$, $b = 1.8$, and $c = 0$.

   The maximum occurs at the vertex, $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

   Determine the x-coordinate of the vertex.

$x = \frac{-b}{2a}$	Formula to determine the x-coordinate of the vertex of a parabola
$x = \frac{-(1.8)}{2(-0.03)}$	Substitute values for a and b.
$x = 30$	Simplify.

   Determine the y-coordinate of the vertex.

$f(x) = -0.03x^2 + 1.8x$	Original function
$f(30) = -0.03(30)^2 + 1.8(30)$	Substitute 30 for x.
$f(30) = 27$	Simplify.

   The vertex is (30, 27), so the maximum value is 27 feet.

   The maximum height the ball reaches is 27 feet.

2. Determine the horizontal distance of the ball when it reaches its maximum height.

   This horizontal distance is determined by the x-coordinate of the vertex.

   The vertex is (30, 27).

   The ball will have traveled 30 feet in the horizontal direction when it reaches its maximum height.

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Instruction

3. Graph the function.

Use a graphing calculator or complete a table of coordinates. Use the vertex as one known point. Choose x-values on either side of the vertex to have four additional x-values.

x	y
5	
20	
30	27
40	
55	

To determine the y-coordinates of the additional points, substitute each x-value into the original function and solve.

$f(x) = -0.03x^2 + 1.8x$	Original function
$f(5) = -0.03(5)^2 + 1.8(5)$	Substitute 5 for x.
$f(5) = 8.25$	Simplify.

$f(x) = -0.03x^2 + 1.8x$	Original function
$f(20) = -0.03(20)^2 + 1.8(20)$	Substitute 20 for x.
$f(20) = 24$	Simplify.

$f(x) = -0.03x^2 + 1.8x$	Original function
$f(40) = -0.03(40)^2 + 1.8(40)$	Substitute 40 for x.
$f(40) = 24$	Simplify.

$f(x) = -0.03x^2 + 1.8x$	Original function
$f(55) = -0.03(55)^2 + 1.8(55)$	Substitute 55 for x.
$f(55) = 8.25$	Simplify.

(*continued*)

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Instruction

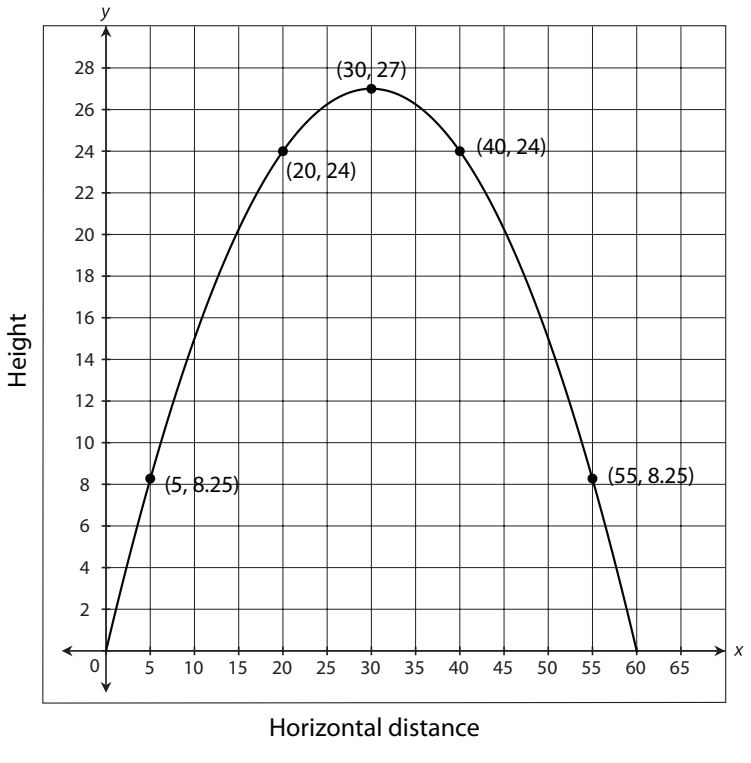
Fill in the missing table values.

x	y
5	**8.25**
20	**24**
30	27
40	**24**
55	**8.25**

Notice that the points (5, 8.25) and (55, 8.25) are the same horizontal distance from the vertex on either side. The same is true for (20, 24) and (40, 24).

Plot the points on a coordinate plane and connect them using a smooth curve.

Since the function models the path of a kicked football, it is important to only show the portion of the graph where both height and horizontal distance are positive.



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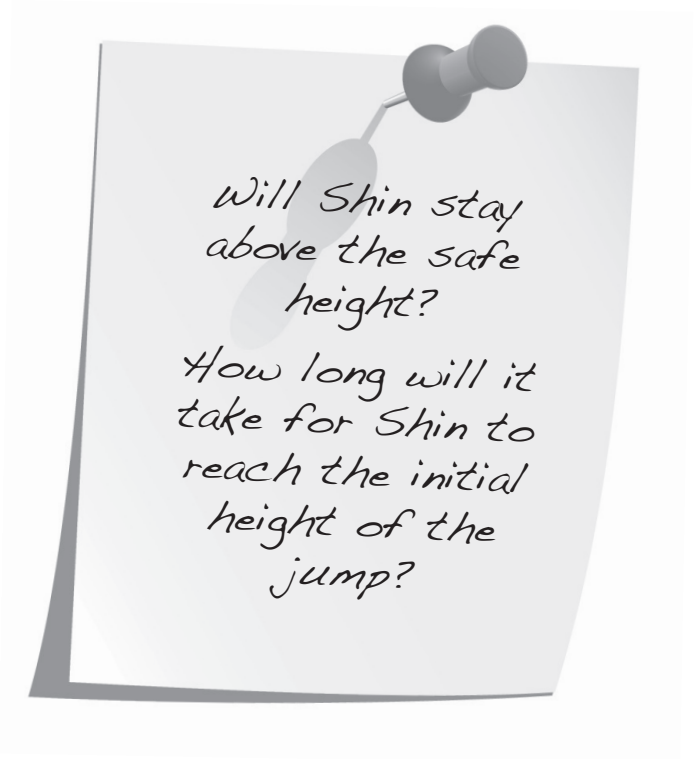
Problem-Based Task 2.1.2: Is the Glider Safe?

SMP	
1 ✓	2
3 ✓	4 ✓
5	6
7	8

Shin is a beginner hang glider. He's practicing jumping from a certain height, dipping initially, and then rising. Shin should dip to a height no lower than 6 feet above the ground, which is considered a safe height, before changing direction and beginning to rise. The position of Shin's hang glider is given by $y = (x - 4)(x - 6)$, with x representing the time in seconds since Shin starts the initial jump and y representing the distance in feet from the safe height. Will Shin stay above the safe height? How long will it take for Shin to reach the initial height of the jump?



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Problem-Based Task 2.1.2: Is the Glider Safe?

Coaching

a. What is the minimum height Shin reaches?

b. Does a negative answer make sense in the context of the problem? Explain.

c. Find the time at which the minimum height occurs.

d. What are the *x*-intercepts and what do they represent?

e. Will Shin stay above the safe height? Explain.

f. How long will it take for Shin to reach his initial height?

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Instruction

Problem-Based Task 2.1.2: Is the Glider Safe?

Coaching Sample Responses

a. What is the minimum height Shin reaches?

The minimum occurs at the vertex. Write the equation in standard form.

$y = (x - 4)(x - 6)$	Original equation
$y = x^2 - 6x - 4x + 24$	Distribute.
$y = x^2 - 10x + 24$	Simplify.

The vertex is of the form $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

Use the equation $y = x^2 - 10x + 24$ to find the values of a and b in order to find the x-coordinate of the vertex.

$x = \frac{-b}{2a}$	Formula to determine the x-coordinate of the vertex of a parabola
$x = \frac{-(-10)}{2(1)}$	Substitute 1 for a and –10 for b.
$x = 5$	Simplify.

The x-coordinate of the vertex is 5.

Substitute 5 into the original equation to find the y-coordinate.

$y = x^2 - 10x + 24$	Original equation
$y = (5)^2 - 10(5) + 24$	Substitute 5 for x.
$y = -1$	Simplify.

The y-coordinate of the vertex is –1.

The vertex is located at (5, –1). The minimum height that Shin reaches is –1 foot.

b. Does a negative answer make sense in the context of the problem? Explain.

Yes. The value of –1 means 1 foot below the safe height.

c. Find the time at which the minimum height occurs.

The x-coordinate of the vertex is 5.

Shin reaches the minimum 5 seconds after the jump.

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d. What are the x-intercepts and what do they represent?

The x-intercepts can be identified from the original equation written in factored form.

$y = (x - 4)(x - 6)$

The x-intercepts are 4 and 6.

The x-intercepts represent times when the glider goes from a safe height to an unsafe height or vice versa.

The glider is at the safe height for the first 4 seconds, and then again after 6 seconds.

e. Does Shin stay above the safe height? Explain.

No. Shin is above the safe height for the first 4 seconds after the jump, and again after 6 seconds.

He dips below the safe height between 4 seconds and 6 seconds.

For 2 seconds, the glider is below the safe height.

f. How long will it take for Shin to reach his initial height?

The initial height occurs when $x = 0$.

The axis of symmetry passes through the vertex, $(5, -1)$.

The axis of symmetry is $x = 5$.

Zero is 5 units to the left of the axis of symmetry.

The y-value is the same for the point 5 units to the right of the axis of symmetry.

At $x = 10$, the height will be the same as the initial height.

Substitute 0 and 10 for x to check.

$y = (x - 4)(x - 6)$	$y = (x - 4)(x - 6)$
$y = [(0) - 4][(0) - 6]$	$y = [(10) - 4][(10) - 6]$
$y = 24$	$y = 24$

Shin will reach his initial height of 24 feet above the safe height (or 30 feet above the ground) after 10 seconds.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2.1.2: Interpreting Various Forms of Quadratic Functions

Use the given functions to complete all parts of problems 1–3.

1. $f(x) = x^2 - 6x + 8$
   a. Identify the y-intercept.
   b. Identify the vertex.
   c. Identify whether the function has a maximum or minimum.

2. $f(x) = -0.5(x + 2)(x - 4)$
   a. Identify the x-intercepts.
   b. Identify the y-intercept.
   c. Identify the axis of symmetry.
   d. Identify the vertex.

3. $f(x) = -16(x - 1)^2 + 10$
   a. Identify the vertex.
   b. Identify whether the function has a maximum or minimum.

Use the given information in each scenario that follows to complete the remaining problems.

4. A bird is descending toward a lake to catch a fish. The bird's flight can be modeled by the equation $h(t) = t^2 - 14t + 40$, where $h(t)$ is the bird's height above the water in feet and t is the time in seconds since you saw the bird. Graph the function. What is the vertex? What does the minimum value mean in the context of the problem?

5. A military pilot fires a test missile whose path can be modeled by the equation $f(x) = -(x - 40)(x + 2)$, where $f(x)$ is the height of the missile in miles and x is the number of seconds since the missile was fired. Graph this function. What are the x-intercepts and what do they mean in the context of the problem? After how many seconds is the height of the missile the same as the initial height?

continued

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6. The path of a snowboarder performing stunts is given by the equation $f(x) = -16(x - 2)(x + 1)$, where x is time in seconds and $f(x)$ is the height of the snowboarder above the ground. Graph the function. What are the x-intercepts? Explain the meaning of the x-intercepts in the context of the problem. How long does the stunt last?

7. The flight of a paper airplane follows the quadratic equation $H(x) = -(x - 3)^2 + 25$, where $H(x)$ represents the height of the paper airplane in feet, and x is the horizontal distance in feet the airplane travels after it is thrown. Graph the function. What is the vertex? Explain the meaning of the vertex in the context of the problem.

8. The height of a golfer's ball is given by the equation $y = -16x^2 + 32x$, where y represents the height in feet and x represents the time in seconds. Graph the function. What is the vertex and what does it mean in the context of the problem?

9. The revenue, $R(x)$, generated by an increase in price of x dollars for an item is represented by the equation $R(x) = -5(x - 15)(x + 5)$. Graph the function. What are the x-intercepts and what do they represent in the context of the problem? What value of x maximizes the revenue?

10. Reducing the cost of an item can result in a greater number of sales. The revenue function that predicts the revenue in dollars, $R(x)$, for each \$1 decrease in price, x, is $R(x) = -(x - 7)^2 + 289$. Graph the function. What is the vertex and what does it represent in the context of the problem?

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Progress Assessment

Circle the letter of the best answer.

1. What characteristic of a parabola can be identified when the standard form of a quadratic is given?
   a. extremum
   b. vertex
   c. x-intercept(s)
   d. y-intercept

2. What characteristic of a parabola can be identified when the factored form of a quadratic is given?
   a. extremum
   b. vertex
   c. x-intercept(s)
   d. y-intercept

3. What are the x-intercepts of the function $f(x) = x^2 + 6x + 8$?
   a. 6 and 8
   b. –8 and –6
   c. 2 and 4
   d. –4 and –2

4. What is the vertex of the function $f(x) = -5x^2 + 10x + 15$, and what does this vertex represent?
   a. The vertex is (1, 20); it represents the maximum of the function.
   b. The vertex is (1, 20); it represents the minimum of the function.
   c. The vertex is (0, 15); it represents the maximum of the function.
   d. The vertex is (0, 15); it represents the minimum of the function.

5. The line of symmetry for a parabola does not always contain which point?
   a. extremum
   b. midpoint of the x-intercepts
   c. vertex
   d. y-intercept

6. What characteristic of a quadratic function is found in its vertex form?
   a. a
   b. vertex
   c. x-intercept(s)
   d. y-intercept

continued

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7. If the value of a is negative, then how many x-intercepts will the graph have?

   a. 0

   b. 1

   c. 2

   d. It cannot be determined from the information.

8. The path of a circus performer shot from a cannon can be described by the equation $h(x) = -2x^2 + 8x + 10$, where $h(x)$ is the height of the performer and x is the number of seconds that have passed since the cannon was fired. Which of the following points shows the extremum for the function?

   a. (–1, 0)

   b. (0, 10)

   c. (2, 18)

   d. (5, 0)

9. The path of a water balloon can be described by the equation $h(x) = -2x^2 + 8x + 10$, where $h(x)$ is the height of the water balloon in feet and x is the number of seconds that have passed since it was launched. Which of the following points does not make sense within the context of the problem?

   a. (–1, 0)

   b. (0, 10)

   c. (2, 18)

   d. (5, 0)

10. A missile is launched along the path determined by the equation $f(x) = -2x^2 + 72x$, where $f(x)$ is the height of the missile in feet x seconds after it has been launched. A plane is flying at a height of 650 feet. Is the plane in danger? Why or why not?

    a. Yes. The value of a is negative.

    b. Yes. The missile reaches a height of greater than 650 feet.

    c. No. The missile does not reach a height of 650 feet.

    d. No. The y-intercept is the origin.

Use what you have learned about forms of quadratic functions to answer the following questions.

11. The function $f(t) = -16t^2 + 40t + 3$ models the height $f(t)$ in feet of a golf ball t seconds after being hit.

    a. What is the height of the golf ball after 2 seconds?

    b. After how many seconds does the golf ball reach its maximum height?

    c. What is the maximum height of the golf ball?

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Which function is even?

   a. $f(x) = 5x^2 + 3$

   b. $f(x) = 4x^3 - 2x$

   c. $f(x) = 4x^3 + 5x^2 - 2x + 2$

   d. $f(x) = 6x^5 + 7x^2$

2. For the function $f(x) = -(x - 2)^2 + 0.25$, what is the value of x that changes the function from increasing to decreasing?

   a. 0.25

   b. 2

   c. –2

   d. –0.25

3. The function $h(t) = -16(t - 1.5)(t + 1.5)$ represents the height of a nut t seconds after falling from a tree. What is a reasonable domain for this function?

   a. –1.5 to 1.5 seconds

   b. 0 to 0.75 seconds

   c. 0 to 1.5 seconds

   d. 0 to 3 seconds

4. What is the average rate of change of the function $f(x) = 0.25x^2 - 3x + 0.45$ from $x = 0$ to $x = 0.25$?

   a. –2.9

   b. –0.7

   c. –0.3

   d. –2.4

5. What is the average rate of change of the function $g(x) = \frac{5}{6}x^2 - 7$ from $x = -7$ to $x = -6.5$?

   a. –6.5

   b. –7.0

   c. –0.5

   d. –11.25

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Lesson 2: Interpreting Quadratic Functions

Instruction

Common Core State Standards

SMP	
1 ✓	2 ✓
3	4 ✓
5	6 ✓
7	8

F–IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F–IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F–IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

Essential Questions

1. What information can be gathered by analyzing the key features of a quadratic function?
2. What properties must be true for a function to be identified as odd, even, or neither?
3. What are the applications of the domain of a quadratic function?
4. Why is it important to find the average rate of change when calculating the slope of a quadratic function?

WORDS TO KNOW

average rate of change the ratio of the difference of output values to the difference of the corresponding input values: $\frac{f(b)-f(a)}{b-a}$; a measure of how a quantity changes over some interval

concave down a graph of a curve that is bent downward, such as a quadratic function with a maximum value

concave up a graph of a curve that is bent upward, such as a quadratic function with a minimum value

SRB p. U2-35

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Instruction

concavity	the arch of a curve, either upward or downward. For a quadratic function with positive concavity, the vertex is the minimum or lowest point of the curve. For a quadratic function with negative concavity, the vertex is the maximum or highest point of the curve.
decreasing	the interval of a function for which the output values are becoming smaller as the input values are becoming larger
domain	the set of all input values for which a relation or function is defined; the set of x-values that are valid for a relation or function
end behavior	the behavior of the graph as x approaches positive infinity and as x approaches negative infinity
even function	a function that, when evaluated for $-x$, results in a function that is the same as the original function; $f(-x) = f(x)$
extrema	the minima or maxima of a function
increasing	the interval of a function for which the output values are becoming larger as the input values are becoming larger
key features of a quadratic function	characteristics used to describe, draw, and compare quadratic functions. These characteristics are: the x-intercept(s); the y-intercept; values of x for which the function is increasing and decreasing and for which the graph of the function is positive and negative; the function's minimum or maximum; the axis of symmetry; and the end behavior of the graph of the function
neither	describes a function that, when evaluated for $-x$, does not result in the opposite of the original function (odd) or the original function (even)
odd function	a function that, when evaluated for $-x$, results in a function that is the opposite of the original function; $f(-x) = -f(x)$
slope	the measure of the rate of change of one variable with respect to another variable; slope = $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$; the slope in the equation $y = mx + b$ is m

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Recommended Resources

- ChiliMath. "Finding the Domain and Range of a Function."

  http://www.walch.com/rr/00112

  This website provides a summary of finding the domain and range for various types of functions as well as practice problems. Illustrated examples walk through finding the domain and range for different situations.

- JamesRahn.com. "Rate of Change."

  http://www.walch.com/rr/00113

  This website provides a summary, practice, and an answer key for problems related to average rate of change. The intended audience is precalculus and calculus students; however, the summary is written so students of all levels feel comfortable exploring the concept.

- MathIsFun.com. "Even and Odd Functions."

  http://www.walch.com/rr/00114

  This website provides a summary and practice problems for even functions, odd functions, and functions that are neither odd nor even. The site also illustrates the differences in behavior for each type of function.

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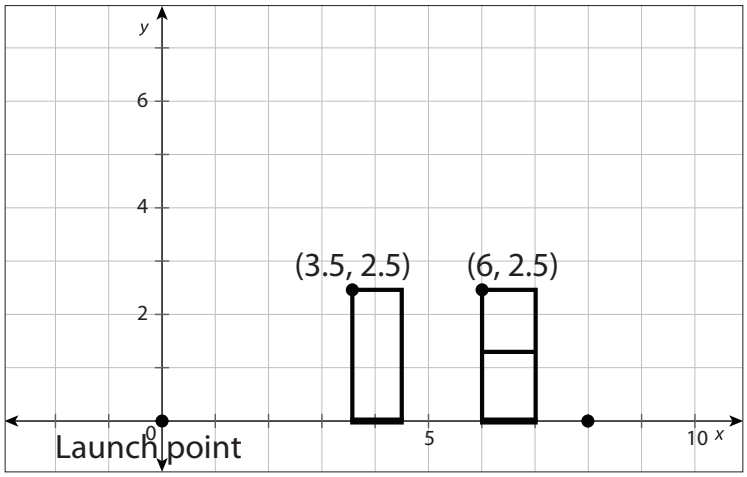
Lesson 2.2.1: Interpreting Key Features of Quadratic Functions

Warm-Up 2.2.1

The object of a popular video game is to launch a boulder to knock over boxes, buildings, and other items. The graph shows an obstacle on the left that the boulder must clear in order to knock over the stack of boxes on the right. The boulder will follow a parabolic path and will launch from (0, 0) and end at (8, 0).



1. What are the *x*-intercepts for the parabola formed by the path of the boulder?

2. What is the axis of symmetry for the parabola formed by the path of the boulder? How do you know?

3. One possible path for the boulder is $y = -\frac{3}{8}x^2 + 3x$. What is the vertex of the parabola created by this equation?

4. Will the boulder clear the obstacle? How do you know?

5. Will the boulder knock down the boxes? How do you know?

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Instruction

Lesson 2.2.1: Interpreting Key Features of Quadratic Functions

Common Core State Standard

F–IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

Warm-Up 2.2.1 Debrief

1. What are the *x*-intercepts for the parabola formed by the path of the boulder?

   The *x*-intercepts occur where the graph intersects the *x*-axis when $y = 0$. The *x*-intercepts are 0 and 8.

2. What is the axis of symmetry for the parabola formed by the path of the boulder? How do you know?

   The axis of symmetry is a vertical line through the middle of the parabola. You can use the midpoint between the two *x*-intercepts to find the axis of symmetry. The midpoint between 0 and 8 is 4, so the axis of symmetry is $x = 4$.

3. One possible path for the boulder is $y = -\frac{3}{8}x^2 + 3x$. What is the vertex of the parabola created by this equation?

   The axis of symmetry goes through the vertex. The *x*-coordinate of the vertex is 4. To find the *y*-coordinate, substitute 4 into the equation $y = -\frac{3}{8}x^2 + 3x$.

   $y = -\frac{3}{8}x^2 + 3x$ Original equation

   $y = -\frac{3}{8}(4)^2 + 3(4)$ Substitute 4 for *x*.

   $y = -\frac{3}{8}(16) + 12$ Simplify.

   $y = 6$

   The vertex is (4, 6).

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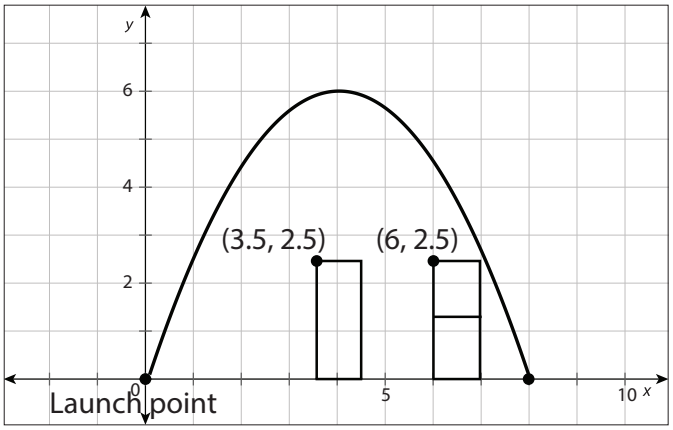
4. Will the boulder clear the obstacle? How do you know?

   Yes, it will. Sketch the boulder's path using (0, 0), (4, 6), and (8, 0), and you can see that the path is above the obstacle.



5. Will the boulder knock down the boxes? How do you know?

   No, it will not. The top right corner of the boxes is at the point (7, 2.5). If you substitute 7 for x in the equation of the parabola, you find that the parabola contains the point (7, 2.625), which is slightly above the boxes. The boulder will not come in contact with the boxes, though it is close.

Connection to the Lesson

- Students will identify key features of quadratic functions such as the x-intercepts and vertex from the equation of the function.
- Students will use key features to graph the quadratic function.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- knowing the standard form of quadratic functions (F–IF.8a)
- using graphing technology to model quadratic functions (F–IF.4★)

Introduction

The tourism industry thrives on being able to provide travelers with an amazing travel experience. At the same time, tourism professionals need to know exactly how successful they are in their businesses. Tour planners might use quadratic models to determine when profits are increasing or decreasing, when they maximized, and how profits change month to month by looking at the key features of the quadratic functions. In this lesson, you will review the definitions of key features of a quadratic function and how to use graphs, tables, and verbal descriptions to identify and apply the key features.

Key Concepts

- The **key features of a quadratic function** are characteristics used to describe, draw, and compare quadratic functions. These characteristics are: the x-intercept(s); the y-intercept; values of x for which the function is increasing and decreasing and for which the graph of the function is positive and negative; the function's minimum or maximum; the axis of symmetry; and the end behavior of the graph of the function.
- Recall each of the forms of quadratic functions.

  Standard Form

  - The standard form, or general form, of a quadratic function is written as $f(x) = ax^2 + bx + c$, where a is the coefficient of the quadratic term, b is the coefficient of the linear term, and c is the constant term.
  - The y-intercept is the value of c.
  - The vertex of the function can be found by first determining the x-coordinate, and then substituting this in the function to find the corresponding y-coordinate.

  Vertex Form

  - The vertex form of a quadratic function is written as $f(x) = a(x - h)^2 + k$.
  - The vertex of the parabola is (h, k).
  - The axis of symmetry is identified from vertex form as $x = h$.

SRB p. U2-38

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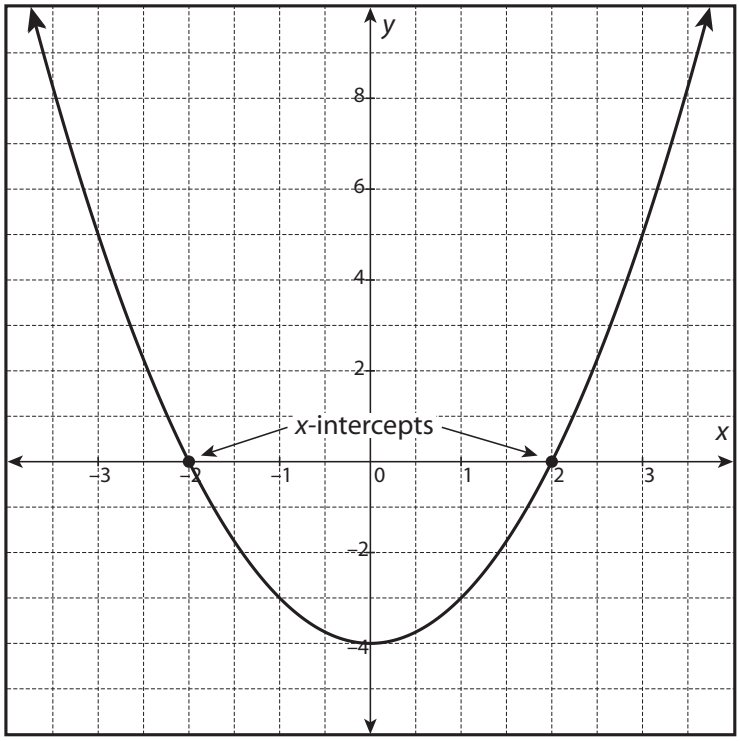
Instruction

Factored Form

- The factored form, or intercept form, of a quadratic function is written as $f(x) = a(x - p)(x - q)$.
- The x-intercepts of the function are p and q.

- The x-intercepts of a quadratic function are where the parabola intersects the x-axis. In the graph that follows, the x-intercepts are 2 and –2.



- All points on the x-axis have a y-coordinate of 0; therefore, the x-intercepts can also be found by identifying the value(s) of x when $y = 0$.
- The following table of values corresponds to the parabola shown. Notice that the x-intercepts, 2 and –2, can be found where the table shows that y is equal to 0.

x	*y*
–4	12
–2	0
0	–4
2	0
4	12

- The x-intercepts are the x-coordinates of the points where the curve intersects the x-axis. The ordered pair that corresponds to an x-intercept is of the form $(x, 0)$. The x-intercepts are also the solutions of the related quadratic equation, $ax^2 + bx + c = 0$.

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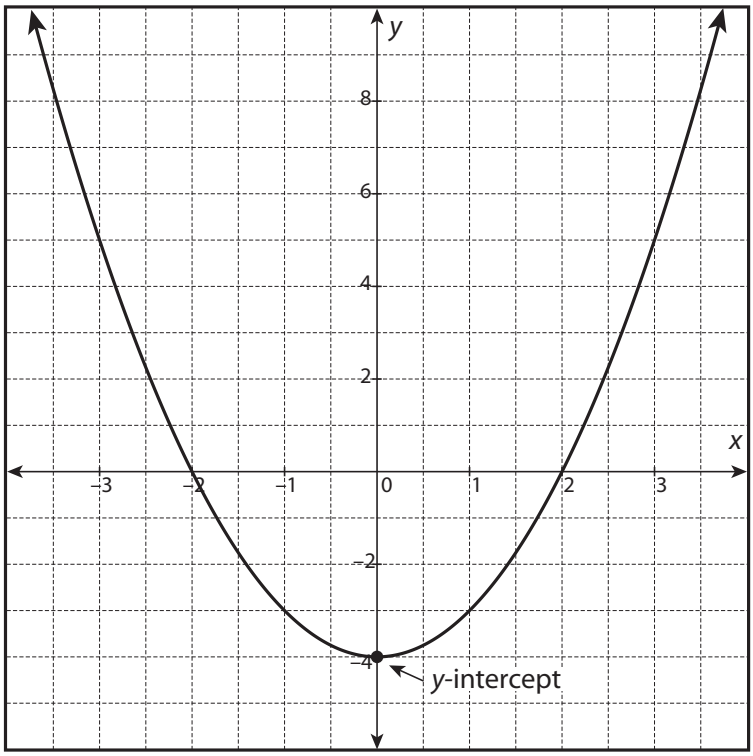
Instruction

- The *y*-intercept of a quadratic function is where the parabola intersects the *y*-axis. In the following graph, the *y*-intercept is –4.



- All points on the *y*-axis have an *x*-coordinate of 0; therefore, the *y*-intercept can also be found by identifying the value of *y* when $x = 0$.
- The following table of values corresponds to the parabola shown. Notice that the *y*-intercept, –4, can be found where the table shows that *x* is equal to 0.

x	*y*
–4	12
–2	0
0	–4
2	0
4	12

- The *y*-intercept is the *y*-coordinate of the point where the curve intersects the *y*-axis. The ordered pair that corresponds to a *y*-intercept is of the form (0, *y*).

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Instruction

- Recall that the vertex is the maximum or minimum of the function.
- The vertex is also the point where a parabola changes from increasing to decreasing or from decreasing to increasing.
- **Increasing** refers to the interval of a function for which the output values are becoming larger as the input values are becoming larger.
- **Decreasing** refers to the interval of a function for which the output values are becoming smaller as the input values are becoming larger.
- Recall that parabolas are symmetric to a vertical line that passes through the vertex, called the axis of symmetry.
- The reflection of every point on a parabola across the axis of symmetry is another point on the parabola. A point and its reflection are equidistant from the axis of symmetry.
- Read the graph from left to right to determine when the function is increasing or decreasing. Trace the path of the graph with a pencil tip going from left to right. If your pencil tip goes down as you move toward increasing values of x, then the function is decreasing. If your pencil tip goes up as you move toward increasing values of x, then the function is increasing.
- If the graph of a quadratic function has a minimum value, it will start by decreasing toward the vertex; after the vertex, it will increase.
- If the graph of a quadratic function has a maximum value, it will start by increasing toward the vertex; after the vertex, it will decrease.
- The vertex is called an extremum. Extrema are the maxima or minima of a function.
- The **concavity** of a parabola refers to the arch of the curve, either upward or downward.
- A quadratic function that has a minimum value has positive concavity and is **concave up** because the graph of the function is bent upward.
- A quadratic function that has a maximum value has negative concavity and is **concave down** because the graph of the function is bent downward.
- The graphs that follow demonstrate examples of parabolas as they decrease and then increase, and vice versa. Trace the path of each parabola from left to right with your pencil to see the difference.

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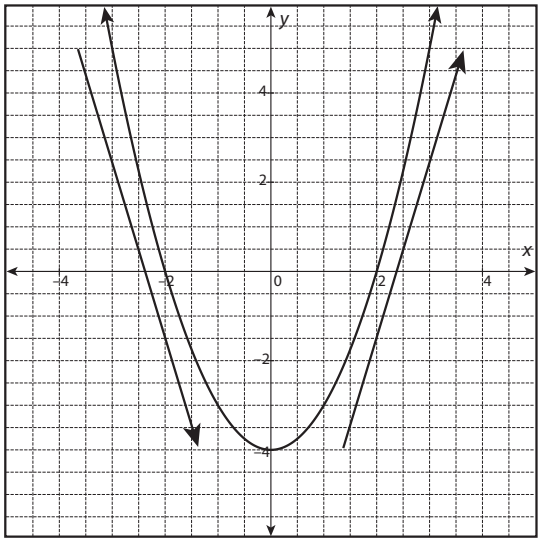
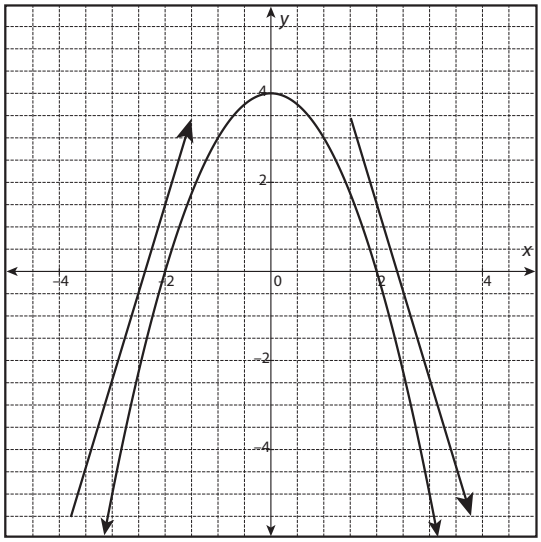
Instruction

Decreasing then Increasing	Increasing then Decreasing
Vertex: (0, –4); minimum	Vertex: (0, 4); maximum
$x < 0$: decreasing	$x < 0$: increasing
$x > 0$: increasing	$x > 0$: decreasing
Direction: concave up	Direction: concave down
	

- **End behavior** is the behavior of the graph as x approaches positive or negative infinity.
- If the highest exponent of a function is even, and the coefficient of the same term is positive, then the function is approaching positive infinity as x approaches both positive and negative infinity.
- If the highest exponent of a function is even, but the coefficient of the same term is negative, then the function is approaching negative infinity as x approaches both positive and negative infinity.

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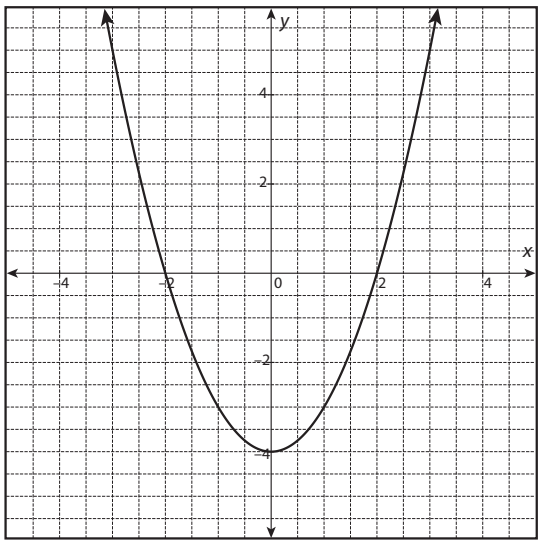
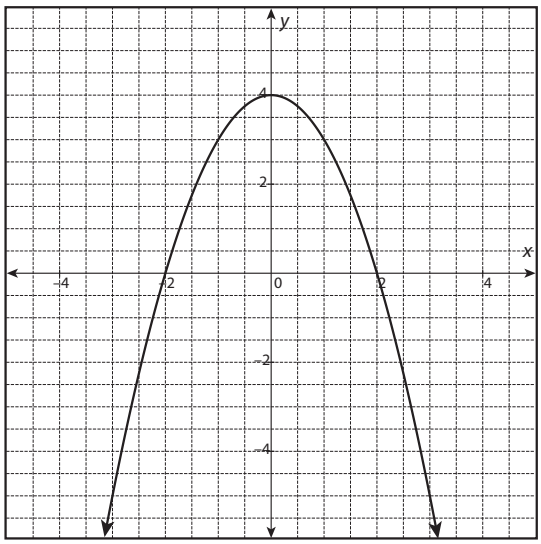
Instruction

Even and Positive	Even and Negative
$f(x) = x^2 - 4$	$f(x) = -x^2 + 4$
Highest exponent: 2	Highest exponent: 2
Coefficient of x^2: positive	Coefficient of x^2: negative
As x approaches positive infinity, $f(x)$ approaches positive infinity.	As x approaches positive infinity, $f(x)$ approaches negative infinity.
As x approaches negative infinity, $f(x)$ approaches positive infinity.	As x approaches negative infinity, $f(x)$ approaches negative infinity.
	

- Functions can be defined as odd or even based on the output when evaluating the function for $-x$.
- For an **odd function**, $f(-x) = -f(x)$. That is, if you evaluate a function for $-x$, the resulting function is the opposite of the original function.
- For an **even function**, $f(-x) = f(x)$. That is, if you evaluate a function for $-x$, the resulting function is the same as the original function.
- If evaluating the function for $-x$ does not result in the opposite of the original function or the original function, then the function is **neither** odd nor even.
- All quadratic functions have an even power, but they are not all even functions. For example, the function $f(x) = x^2 - 2x$ is neither even nor odd because $f(-x) \neq x^2 - 2x$ and $f(-x) \neq -(x^2 - 2x)$.

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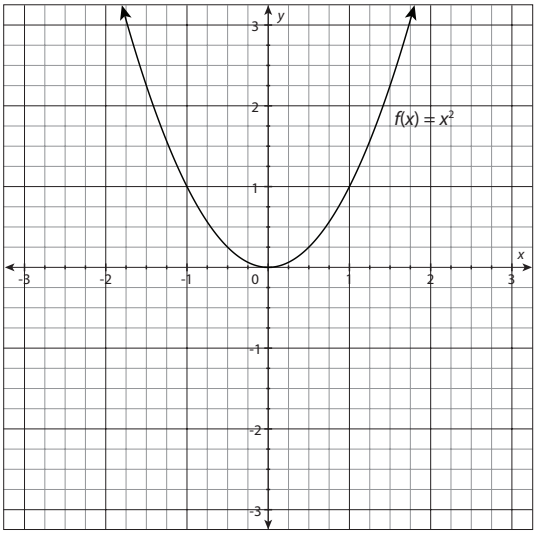
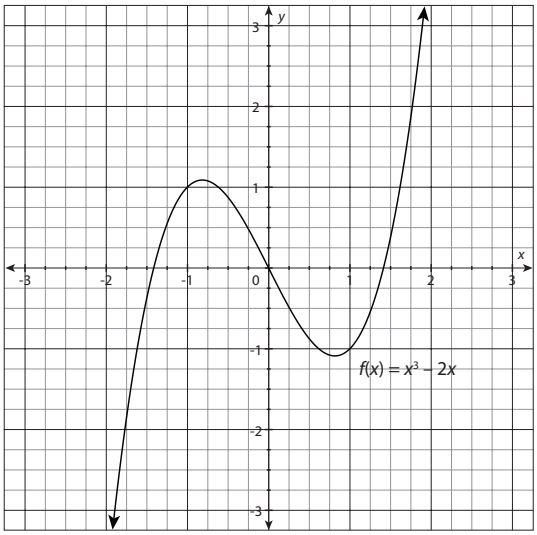
Instruction

- The graphs of even functions are symmetric with respect to the *y*-axis. If the graph is folded along the *y*-axis, the two sides match up perfectly.
- The graphs of odd functions are symmetric with respect to the origin. The graphs of odd functions look the same right-side up as they do upside down.

Example of an even function	Example of an odd function
	

Common Errors/Misconceptions

- incorrectly identifying when a function is increasing or decreasing
- making sign errors when determining if a function is odd, even, or neither

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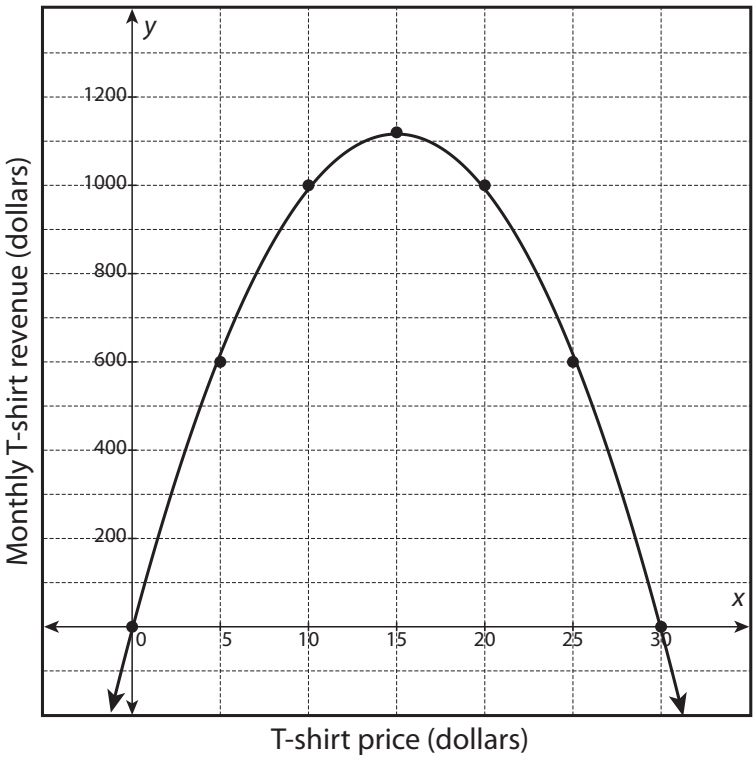
Instruction

Guided Practice 2.2.1

Example 1

A local store's monthly revenue from T-shirt sales, $f(x)$, as a function of price, x, is modeled by the function $f(x) = -5x^2 + 150x - 7$. Use the equation and graph to answer the following questions: At what prices is the revenue increasing? Decreasing? What is the maximum revenue? What prices yield no revenue? Is the function even, odd, or neither?



1. Determine when the function is increasing and decreasing.

   Use your pencil to determine for which x-values the function is increasing and decreasing.

   Moving from left to right, trace your pencil along the function.

   The function increases until it reaches the vertex, then decreases.

   The vertex of this function has an x-value of 15.

   The revenue is increasing when the price per shirt is less than \$15, or when $x < 15$.

   The revenue is decreasing when the price per shirt is more than \$15, or when $x > 15$.

SWB p. 29; SRB p. U2-45

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Instruction

2. Determine the maximum revenue.

Use the vertex of the function to determine the maximum revenue.

The T-shirt price that maximizes revenue is \$15, or $x = 15$.

The maximum revenue is the corresponding y-value.

Since it is difficult to estimate accurately from this graph, substitute the value of x into the function and solve.

$f(x) = -5x^2 + 150x - 7$	Original function
$f(15) = -5(15)^2 + 150(15) - 7$	Substitute 15 for x.
$f(15) = 1118$	Simplify.

The maximum revenue is \$1,118.

3. Determine the prices that yield no revenue.

Identify the x-intercepts.

The x-intercepts are 0 and 30, so the store has no revenue when the shirts cost \$0 and when the shirts cost \$30 or more.

4. Determine if the function is even, odd, or neither.

Evaluate the function for $-x$.

$f(x) = -5x^2 + 150x - 7$	Original function
$f(-x) = -5(-x)^2 + 150(-x) - 7$	Substitute $-x$ for x.
$f(-x) = -5x^2 - 150x - 7$	Simplify.

Since $f(-x)$ is neither the original function nor the opposite of the original function, the function is not even or odd; it is neither.

5. Use the graph of the function to verify that the function is neither odd nor even.

Since the function is not symmetric over the y-axis or the origin, the function is neither even nor odd.

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Instruction

Example 2

A quadratic function has a minimum value of –5 and x-intercepts of –8 and 4. What is the value of x that minimizes the function? For what values of x is the function increasing? Decreasing?

1. Determine the x-value that minimizes the function.

   The graphs of quadratic functions are symmetric about the axis of symmetry, the line that divides the parabola in half and passes through the vertex.

   The x-value that minimizes the function is the midpoint of the two x-intercepts.

   Find the midpoint of the x-intercepts by taking the average of the two values.

   $$x = \frac{-8+4}{2} = -2$$

   The value of x that minimizes the function is –2.

2. Determine when the function is increasing and decreasing.

   Use the vertex to determine for which x-values the function is increasing and for which x-values it is decreasing.

   The minimum value is –5 and the vertex of the function is (–2, –5).

   From left to right, the function decreases as it approaches the minimum and then increases.

   The function is decreasing when $x < -2$ and increasing when $x > -2$.

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Instruction

Example 3

The table shows the predicted temperatures for a summer day in Woodland, California. At what times is the temperature increasing? Decreasing?

Time	Temperature (°F)
8 A.M.	52
10 A.M.	64
12 P.M.	72
2 P.M.	78
4 P.M.	81
6 P.M.	76

1. Use the table to determine approximate intervals of increasing and decreasing temperatures.

   Examine what is happening to the temperatures as the day progresses from morning to evening.

   The values are increasing when they get progressively larger in the table, and they are decreasing when they get progressively smaller in the table.

   At 4 P.M. the temperature is at its highest, 81°. Before this time, the temperatures are increasing, and after this time, the temperatures are decreasing.

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2. Use graphing technology to verify the information that is assumed from the table.

On a TI-83/84:

Step 1: Press [STAT].

Step 2: Press [ENTER] to select Edit.

Step 3: Enter *x*-values into L1. Enter times based on a 24-hour clock for times after 12 P.M. For example, 1 P.M. should be entered as hour 13.

Step 4: Enter *y*-values into L2.

Step 5: Press [2ND][Y=].

Step 6: Press [ENTER] twice to turn on the Stat Plot.

Step 7: Press [ZOOM][9] to select ZoomStat and show the scatter plot.

Step 8: Press [STAT].

Step 9: Arrow to the right to select Calc.

Step 10: Press [5] to select QuadReg.

Step 11: Enter [L1][,][L2], Y_1. To enter Y_1, press [VARS] and arrow over to the right to "Y-VARS." Select 1: Function. Select 1: Y_1.

Step 12: Press [ENTER] to see the graph of the data and the quadratic equation.

On a TI-Nspire:

Step 1: Press the [home] key and select the Lists & Spreadsheet icon.

Step 2: Name Column A "time" and Column B "temperature."

Step 3: Enter *x*-values under Column A. Enter times based on a 24-hour clock for times after 12 P.M. For example, 1 P.M. should be entered as hour 13.

Step 4: Enter *y*-values under Column B.

Step 5: Select Menu, then 3: Data, and then 6: Quick Graph.

Step 6: Press [enter].

Step 7: Move the cursor to the *x*-axis and choose "time."

(*continued*)

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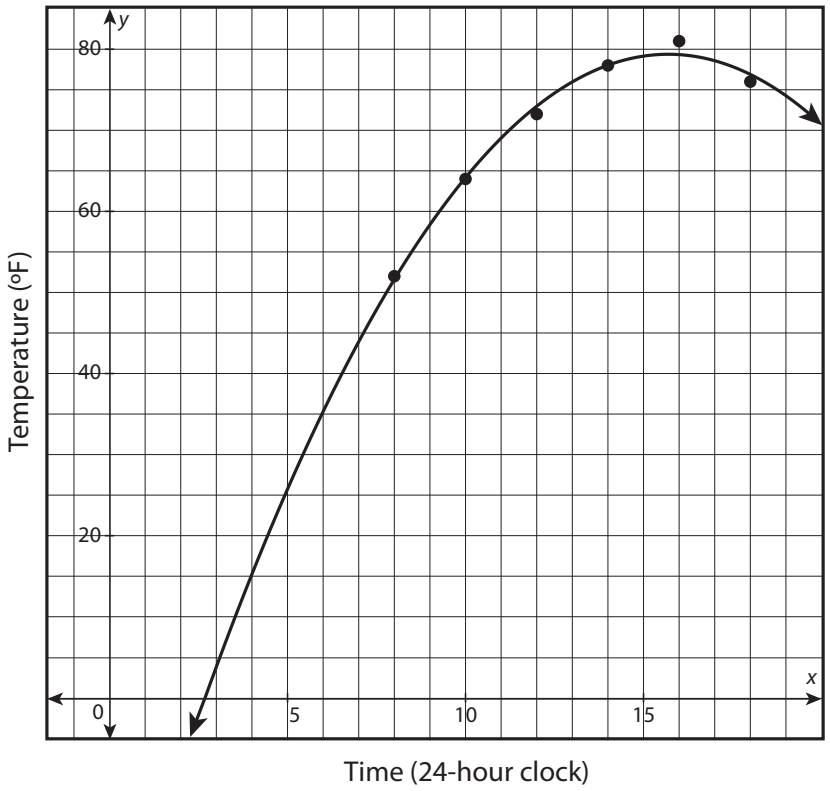
Step 8: Move the mouse to the *y*-axis and choose "temperature."

Step 9: Select Menu, then 4: Analyze, then 6: Regression, and then 4: Show Quadratic.

Step 10: Move the cursor over the equation and press the center key in the navigation pad to drag the equation for viewing, if necessary.



3. State your conclusion.

The highest temperature in the table occurs at the maximum *y*-value on the curve, or in this case, the time at which the temperature goes from increasing to decreasing.

The highest temperature is 81°, and this occurs at 4 P.M.

The maximum temperature appears to happen at hour 16, according to the quadratic model, or at around 4 P.M.

The high temperature in the graphed quadratic model is approximately 79°, which is slightly less than 81°, the predicted temperature for that hour.

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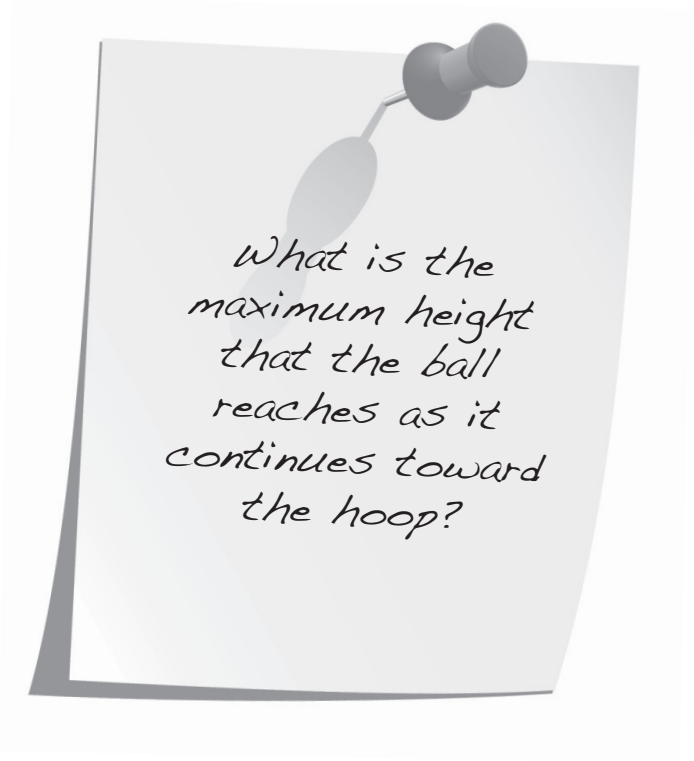
Problem-Based Task 2.2.1: One-on-One Basketball

SMP	
1 ✓	2
3	4 ✓
5	6 ✓
7	8

You and a friend are playing one-on-one basketball at the park. You aim at the hoop and release the ball, which follows a parabolic path. The table represents the ball's horizontal distance from you and the ball's height as it travels toward the center of the hoop, which is represented by the point (14, 10). Use a quadratic model to determine for what horizontal distances the height of the ball is increasing and decreasing. What is the maximum height that the ball reaches as it continues toward the hoop?

Distance from shooter (feet)	Height of basketball (feet)
4	10
6	12
12	12
14	10



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Problem-Based Task 2.2.1: One-on-One Basketball

Coaching

a. Is there enough information in the table to estimate where the ball is increasing and decreasing, and at what distance it reaches its maximum height?

b. What's the best method to develop a quadratic model given a table of values?

c. How do you determine the x-values for which the function is increasing and decreasing?

d. How do you determine the maximum value of the parabola?

e. What is the maximum height that the ball reaches as it continues toward the hoop?

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Instruction

Problem-Based Task 2.2.1: One-on-One Basketball

Coaching Sample Responses

a. Is there enough information in the table to estimate where the ball is increasing and decreasing, and at what distance it reaches its maximum height?

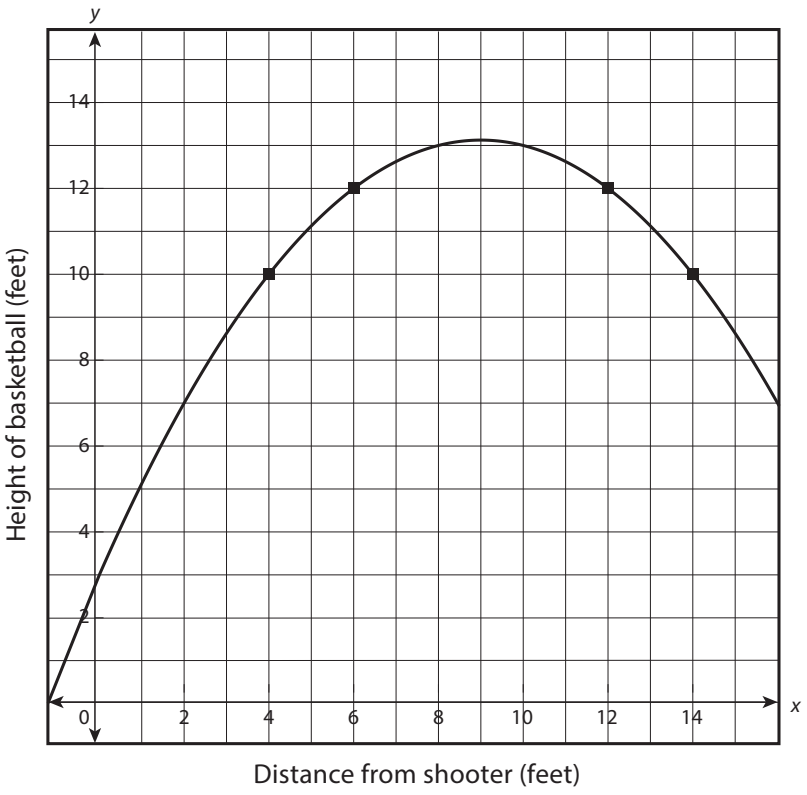
The heights, or output values, are the symmetric coordinates about some vertical line, the axis of symmetry.

It is difficult to estimate where the function is increasing and decreasing since this depends on the vertex.

b. What's the best method to develop a quadratic model given a table of values?

A graphing calculator is the best method because it will fill in all of the blanks not included in the table. The table only gives a snapshot of the basketball's travels along the way.

Entering the table of values into a graphing calculator yields the quadratic model $f(x) = -0.125x^2 + 2.25x + 3$. The quadratic model and plotted points from the table are shown.



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Instruction

c. How do you determine the *x*-values for which the function is increasing and decreasing?

This function has a maximum and therefore is increasing first, then decreasing.

The function will increase until reaching the vertex, which appears have an *x*-coordinate of 9.

This can be checked by using the fact that the axis of symmetry is halfway between symmetric points, and the vertex lies on the axis of symmetry.

If we use the symmetric points (6, 12) and (12, 12), the *x*-coordinates are 6 and 12. Calculate the midpoint between 6 and 12:

$$x = \frac{6+12}{2} = \frac{18}{2} = 9$$

The basketball's height is increasing when its horizontal distance from the shooter is less than 9 feet and decreasing when its horizontal distance from the shooter is greater than 9 feet.

d. How do you determine the maximum value of the parabola?

The maximum of the parabola occurs at the vertex.

We've determined that the parabola's axis of symmetry is $x = 9$, so we know that the *x*-coordinate of the vertex is 9.

From the graph, we can estimate the maximum to be about 13.1 feet.

However, substituting the *x*-value into the quadratic model will yield a more accurate estimate.

$f(x) = -0.125(9)^2 + 2.25(9) + 3 = 13.125$

e. What is the maximum height that the ball reaches as it continues toward the hoop?

The maximum height the ball reaches is about 13.1 feet.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2.2.1: Interpreting Key Features of Quadratic Functions

For each of the given quadratic functions, use graphing technology to answer the following questions: What are the x-values for which the function is increasing? Decreasing? What is the maximum or minimum value of the function? What are the intercepts? Is the function even, odd, or neither?

1. $f(x) = 3x^2 - 2x - 5$

2. $g(x) = -3x^2 + 10x + 1$

3. $y = 5x^2 + 10x + 11$

4. $h(x) = 2x^2 - 4x - 11$

Given the following descriptions of quadratic functions, answer the questions: What is the value of x that minimizes or maximizes the function? For what values of x is the function increasing? Decreasing?

5. A function has a minimum value of –20 and x-intercepts of –1.72 and 0.38.

6. A function has a maximum value of 12.375 and x-intercepts of 0.41 and 1.84.

7. A function has a minimum value of –8.675 and x-intercepts of 1.23 and –0.48.

8. A function has a minimum value of –8.167 and x-intercepts of 1.3 and –1.

continued

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Use the tables and scenarios that follow to complete the remaining problems.

9. You and a friend are playing softball. You throw the ball toward your friend's mitt so that the ball follows a parabolic path. The table represents the ball's horizontal distance from you and the ball's height as it travels from a starting position of (0, 4). Use a quadratic model to determine for what distances the height of the ball is increasing and decreasing.

Distance from you (feet)	Height of softball (feet)
0	4
20	10
30	10
40	8

10. The table shows the predicted temperatures for an autumn day in Annapolis, Maryland. Use a quadratic model to determine the maximum temperature that Annapolis reaches on this day.

Time	Temperature (°F)
11 A.M.	59
2 P.M.	63
5 P.M.	63
8 P.M.	58
11 P.M.	56

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Lesson 2.2.2: Identifying the Domain of a Quadratic Function

Warm-Up 2.2.2

Gina works in a video game store over her summer break. She earns $8 per hour plus commission on the video games and gaming consoles she sells. This month, her store offered an incentive of an extra day off to the employee who sells the most copies of a certain new game. She makes $3 in commission for each game sold.

1. Write a linear model to represent Gina's take-home pay as a function of the number of games she sells in one 8-hour workday.

2. What is a reasonable domain for this function?

3. What does the domain represent?

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Instruction

Lesson 2.2.2: Identifying the Domain of a Quadratic Function

Common Core State Standard

F–IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

Warm-Up 2.2.2 Debrief

1. Write a linear model to represent Gina's take-home pay as a function of the number of games she sells in one 8-hour workday.

   $y = 3x + 64$, where 3 is the rate of change of Gina's take-home pay, x represents the number of games she sells, and 64 is the total hourly pay she receives for working 8 hours.

2. What is a reasonable domain for this function?

   A reasonable domain is all whole numbers, since selling a negative number of games is impossible, as is selling fractional parts of games.

3. What does the domain represent?

   The domain represents the possible number of games that Gina could sell in one 8-hour workday.

Connection to the Lesson

- Students will extend their knowledge of identifying and interpreting the domain of a linear function to the domain of a quadratic function.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- using graphing technology to interpret quadratic functions (F–IF.4★)
- understanding the difference between domain and range (F–IF.1)

Introduction

The **domain** of a function is all input values that satisfy the function without restriction. Often, the domain is all real numbers; however, with application problems, there may be important restrictions. For example, negative or fractional input values may not make sense in the context of the problem. In this lesson, you will practice identifying the domain of a quadratic function given an equation, a graph, or a real-world context.

Key Concepts

- When a quadratic function does not have a specified interval and is not an application of a real-world situation, its domain is all real numbers.
- In other words, the input values exist from negative infinity to infinity: $-\infty < x < \infty$.

Common Errors/Misconceptions

- identifying domain with output values rather than with input values

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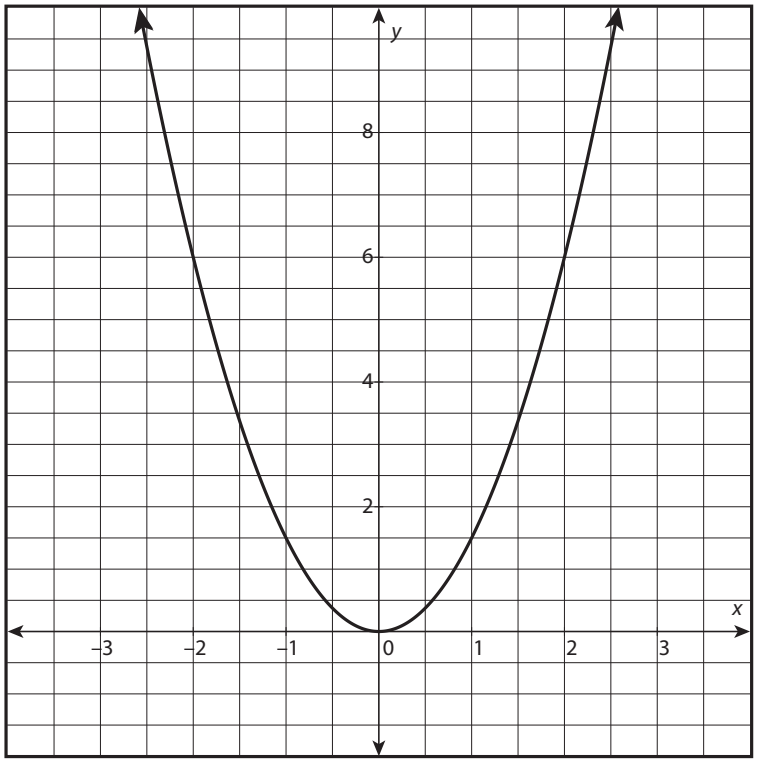
Guided Practice 2.2.2

Example 1

Describe the domain of the quadratic function $g(x) = 1.5x^2$.

1. Sketch a graph of the function.



2. Describe what will happen if the function continues.

   Looking at the function, you can see that the function will continue to increase upward and the function will continue to grow wider.

   Growing wider without end means that the domain of this function is all real numbers, or $-\infty < x < \infty$.

SWB p. 39; SRB p. U2-54

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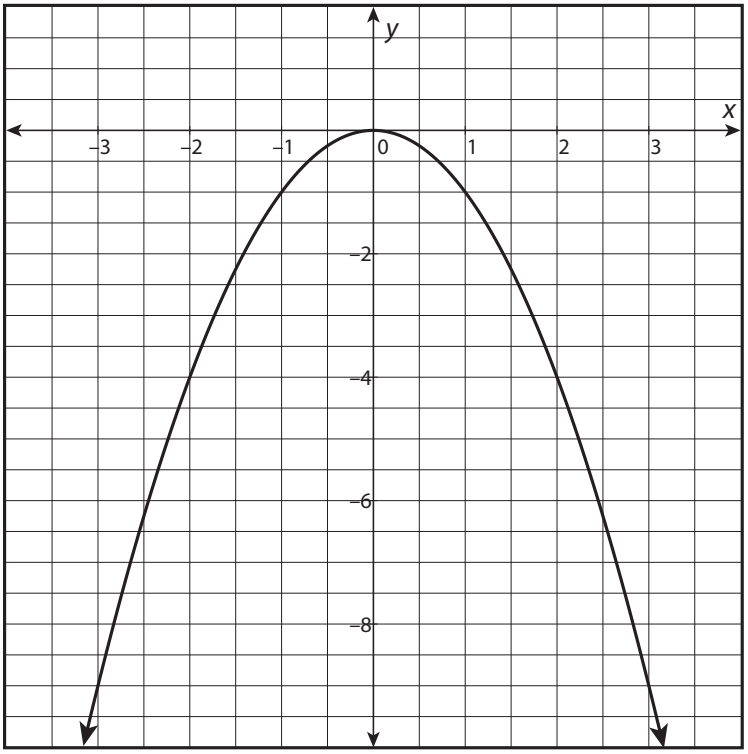
Instruction

Example 2

Describe the domain of the following function.



1. Describe what happens to the width of the parabola as x approaches positive and negative infinity.

   The function continues to grow wider and wider as x approaches both positive and negative infinity.

2. Determine the domain of the function.

   Growing wider without end means that the domain of this function is all real numbers, or $-\infty < x < \infty$.

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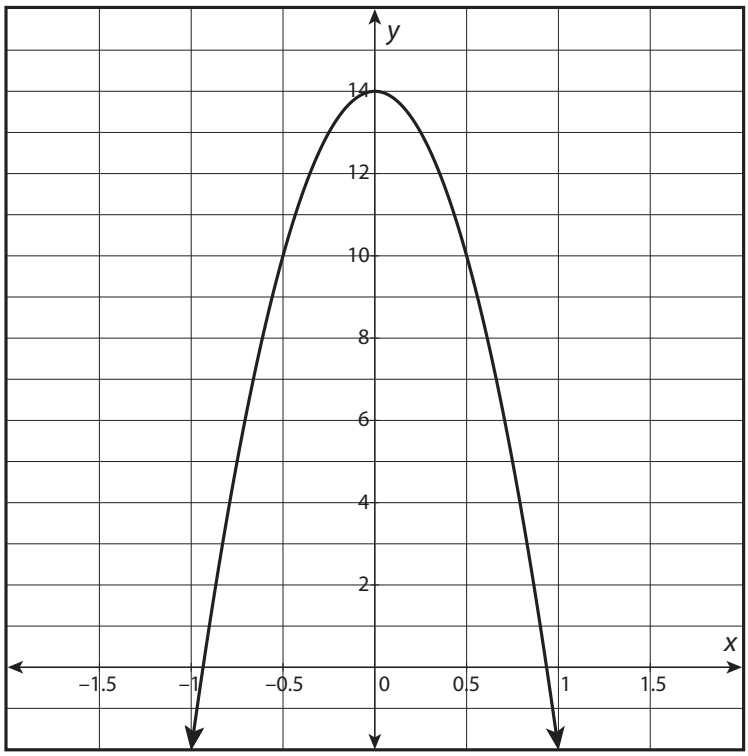
Instruction

Example 3

Amit is a diver on the swim team. Today he's practicing by jumping off a 14-foot platform into the pool. Amit's height in feet above the water is modeled by $f(x) = -16x^2 + 14$, where x is the time in seconds after he leaves the platform. About how long will it take Amit to reach the water? Describe the domain of this function.

1. Sketch a graph of the situation.



2. Identify the x-intercepts of the function.

   Using graphing technology, the x-intercepts occur when $x \approx -0.94$ and $x \approx 0.94$.

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Instruction

3. Determine a reasonable domain for this context.

   The x-values in this problem represent the time in seconds after Amit leaves the platform. In this context, negative values of x do not make sense.

   The positive x-intercept represents the moment Amit makes contact with the water. It will take him 0.94 second to reach the water.

   A reasonable domain of this context is 0 to 0.94 seconds, or $0 < x < 0.94$.

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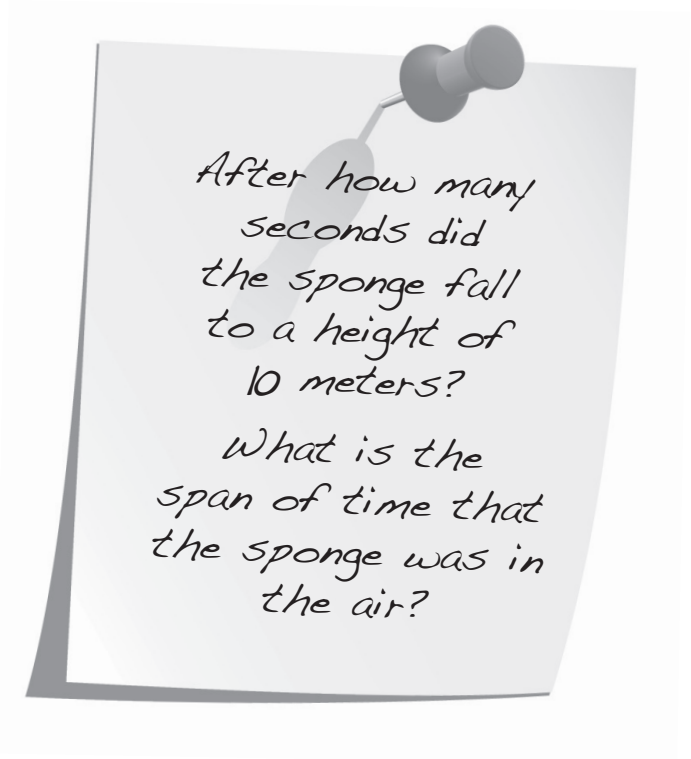
Problem-Based Task 2.2.2: Window Washers

SMP	
1 ✓	2 ✓
3	4
5	6 ✓
7	8

A window washer tossed a wet sponge from a height of 10 meters above ground level to his coworker above him. The sponge reached its maximum height of 11.25 meters exactly 0.5 second later, but the coworker did not catch the sponge and it fell to the ground. After how many seconds did the sponge fall to a height of 10 meters? What is the span of time that the sponge was in the air?



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Problem-Based Task 2.2.2: Window Washers

Coaching

a. What are the known values?

b. What are the unknown values?

c. What type of model is needed in order to find the unknowns?

d. What is the vertex form of a quadratic function?

e. How can the known information be substituted into the vertex form of a quadratic function?

f. How can you find the value of a, and what is the quadratic model that represents this situation?

g. After how many seconds did the sponge fall to a height of 10 meters?

h. What is a reasonable domain of the function? How long was the sponge in the air?

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Instruction

Problem-Based Task 2.2.2: Window Washers

Coaching Sample Responses

a. What are the known values?

We know the maximum height of the sponge, 11.25 meters.

The sponge reached the maximum height after 0.5 second.

These two values represent the vertex, which can be written as the point (0.5, 11.25).

We also know the starting point of the sponge, (0, 10).

b. What are the unknown values?

We do not know how long it took for the sponge to fall to a height of 10 meters, and we do not know the span of time that the sponge was in the air.

c. What type of model is needed in order to find the unknowns?

A quadratic model will help determine the unknown values.

d. What is the vertex form of a quadratic function?

The vertex form of a quadratic function is $f(x) = a(x - h)^2 + k$, where (h, k) is the vertex, x represents the time in seconds, and $f(x)$ is the height of the sponge above the ground.

e. How can the known information be substituted into the vertex form of a quadratic function?

Substitute the vertex, (0.5, 11.25), and the starting point, (0, 10), into the vertex form of a quadratic function.

$f(x) = a(x - h)^2 + k$

$(10) = a[(0) - (0.5)]^2 + (11.25)$

$10 = 0.25a + 11.25$

f. How can you find the value of a, and what is the quadratic model that represents this situation?

Solve the equation for a. Then, substitute the values of a, h, and k into $f(x) = a(x - h)^2 + k$.

$10 = 0.25a + 11.25$

$-1.25 = 0.25a$

$a = -5$

Recall that $h = 0.5$ and $k = 11.25$. Therefore, the quadratic model is $f(x) = -5(x - 0.5)^2 + 11.25$.

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g. After how many seconds did the sponge fall to a height of 10 meters?

Use symmetry to determine how many seconds it took after the sponge was thrown for it to fall to a height of 10 meters.

The starting point, (0, 10), is 0.5 unit to the left of the axis of symmetry, $x = 0.5$. Therefore, the point that is 0.5 unit to the right of the axis of symmetry is (1, 10). Check by substituting (1, 10) into the model $f(x) = -5(x - 0.5)^2 + 11.25$. The result, $10 = 10$, is a true statement.

The sponge fell to a height of 10 meters 1 second after being thrown.

h. What is a reasonable domain of the function? How long was the sponge in the air?

A reasonable domain is the interval of time that the sponge was in the air. This interval begins at $x = 0$, which represents when the sponge was thrown, and ends at the greatest x-intercept, which represents when the sponge hit the ground. Use graphing technology or algebra to find the greatest x-intercept. The sponge hit the ground at $x = 2$, or 2 seconds after being tossed. Therefore, the sponge was in the air for 2 seconds. This span of time can be represented as an inequality: $0 < x < 2$.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2.2.2: Identifying the Domain of a Quadratic Function

Use graphing technology to determine the domain of each quadratic function.

1. $y = -x^2 + 7x + 1$

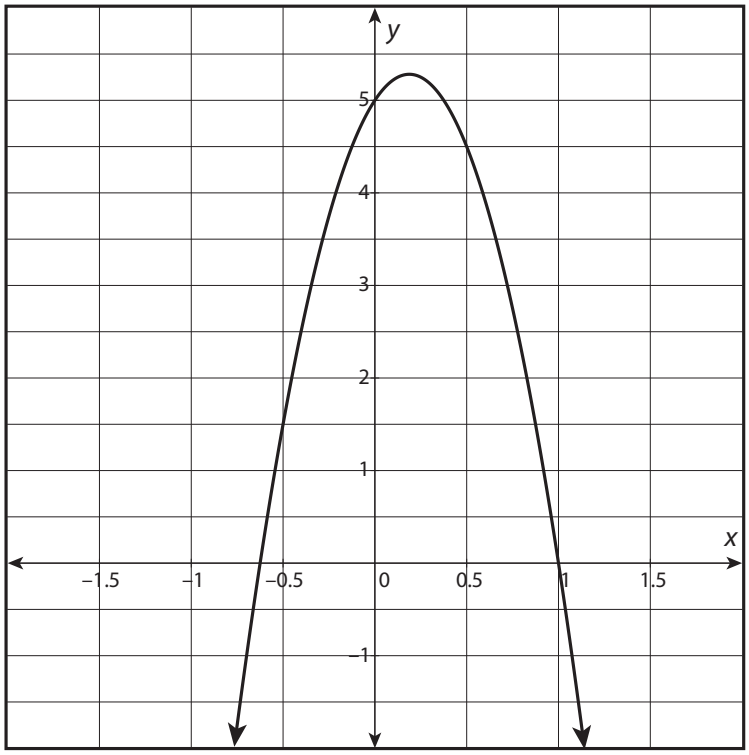
2. $y = -\frac{3}{5}x^2 + 21x - 3$

3. $f(x) = 4x^2 + 5x - 12$

4. $g(x) = x^2 + 12x - 8$

Describe the domain of each of the following functions in words and as an inequality.

5.



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SWB p. 43; SRB Practice 2.2.2 p. U2-58

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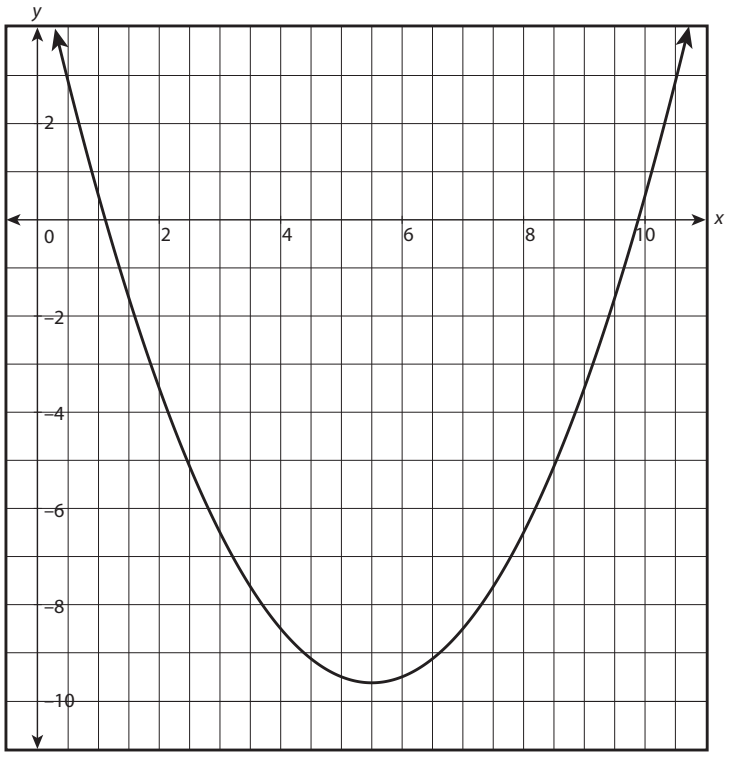

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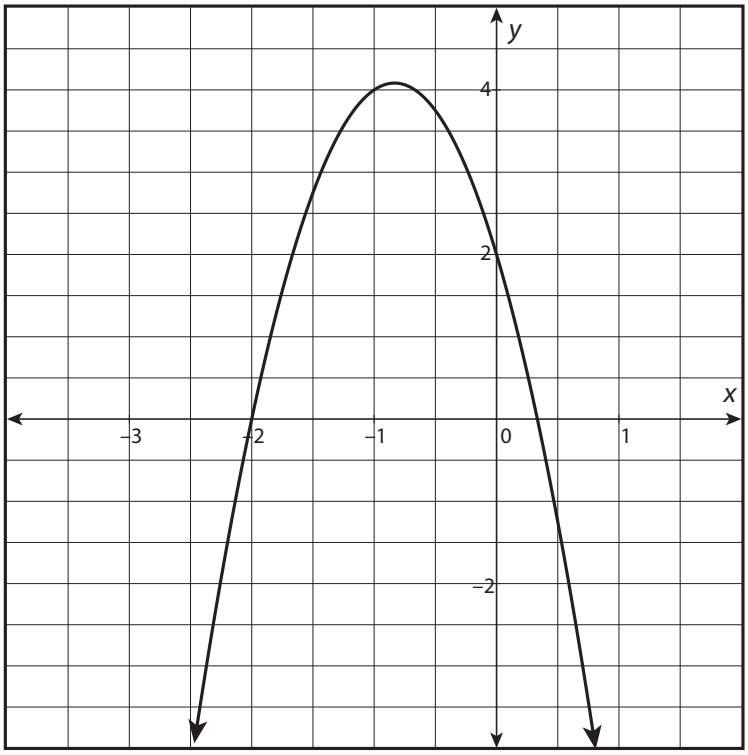

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Use the given information to solve the following problems.

8. A soccer ball is kicked from the ground and travels a parabolic path. The path can be modeled by the function $h(t) = -5t^2 + 19.5t$, where $h(t)$ is the height of the soccer ball in meters above the ground t seconds after being kicked. Assuming the ball lands on level ground, about how long is the ball in the air?

9. A golf ball is shot from the ground using a practice cannon and travels a parabolic path. The path of the ball can be modeled by the function $h(t) = -16t^2 + 150t$, where $h(t)$ is the height of the golf ball in meters above the ground t seconds after being shot. Assuming the ball lands on level ground, about how long does it take the golf ball to hit the ground?

10. The senior class is putting on a talent show to raise money for their senior trip. In the past, the profit from the talent show could be modeled by the function $P(x) = -16x^2 + 600x - 4000$, where x represents the ticket price in dollars. What is a reasonable domain for this function? For what domain value will the profits be maximized?

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Lesson 2.2.3: Identifying the Average Rate of Change

Warm-Up 2.2.3

Data on a certain car shows that its gas mileage can be modeled by a linear function. The car yields 45 miles per gallon (mpg) when driven at 40 miles per hour (mph) and 25 mpg when driven at 80 mph.

1. What is the rate of change of this car's gas mileage?

2. Create the linear model of the car's gas mileage as a function of its speed.

3. What does the rate of change in this model tell you about the car's gas mileage? Explain.

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Instruction

Lesson 2.2.3: Identifying the Average Rate of Change

Common Core State Standard

F–IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

Warm-Up 2.2.3 Debrief

1. What is the rate of change of this car's gas mileage?

   The rate of change of a linear function is the slope, or the ratio of the difference in output values to the difference in input values.

   The problem gives two pairs of values that can be substituted into the slope formula: (40, 45) and (80, 25).

   $m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope formula

   $m = \frac{25 - 45}{80 - 40}$ Substitute (40, 45) for (x_1, y_1) and (80, 25) for (x_2, y_2).

   $m = \frac{-20}{40}$ Simplify.

   $m = -\frac{1}{2}$

   The rate of change of this car's gas mileage is $-\frac{1}{2}$.

2. Create the linear model of the car's gas mileage as a function of its speed.

   Since the rate of change is the slope, we have the value for m in the slope-intercept form of a linear equation, $y = mx + b$.

   The problem gives two points, either of which can be substituted into the slope-intercept form to determine the value of b, the y-intercept.

   $y = mx + b$ Slope-intercept form

   $y = -\frac{1}{2}x + b$ Substitute $-\frac{1}{2}$ for m.

   $45 = -\frac{1}{2}(40) + b$ Substitute 40 for x and 45 for y.

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$45 = -20 + b$ Simplify.

$65 = b$ Solve for b.

The linear model for this car's gas mileage as a function of its speed is $G(x) = -\frac{1}{2}x + 65$, where $G(x)$ is the gas mileage at a speed of x.

3. What does the rate of change in this model tell you about the car's gas mileage? Explain.

   The rate of change for this car's gas mileage means the car's mileage will be reduced by 1 mile per gallon for every 2 miles per hour that the speed of the car increases. This is true because a linear function's slope, and therefore its rate of change, is constant.

Connection to the Lesson

- Students will extend their knowledge of finding a rate of change for linear functions to finding the average rate of change for quadratic functions.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- using graphing technology to model quadratic functions (F–IF.4★)
- evaluating quadratic functions for specific values of x (F–IF.2)
- finding the slope or rate of change of linear functions (8.F.4)

Introduction

The average running speed of a human being is 5–8 mph. Imagine you set out on an 8-mile run that takes you 1 hour. You run often, so you run consistently and at the same speed for the entire hour. The rate of change of your position for your runs is always 8 mph and can be modeled linearly, because the rate of change is constant. However, a friend of yours, averaging the same distance in an hour, hasn't built up the endurance needed to run 8 miles consistently. Sometimes your friend runs 9 mph, sometimes he stops to rest, sometimes he walks, and then he resumes running at 8 mph. He might do this several times in the hour, sometimes running faster than you, sometimes slower, and sometimes not running at all. The rate of change of your friend's speed is not constant and cannot be modeled linearly. The rate of change for your friend's speed, the ratio of the change in the output of a function to the change in the input for specific intervals, is inconsistent. However, he may still have an *average* rate of change of 8 mph. In this lesson, you will practice calculating the average rate of change of quadratic functions over various intervals.

Key Concepts

- The **average rate of change** of a function is the rate of change between any two points on the graph of a function; it is a measure of how a quantity changes over a given interval.
- The average rate of change can be found by calculating the ratio of the difference of output values to the difference of the corresponding input values, $\frac{f(b)-f(a)}{b-a}$, from $x = a$ to $x = b$. This formula is often referred to as the average rate of change formula.
- Recall that the **slope** of a linear function is found using the formula $m=\frac{y_2-y_1}{x_2-x_1}$.
- Although the formula for calculating the average rate of change looks different from the formula used to find the slope of a linear function, they are essentially the same. Both formulas express the ratio of the difference of the output values to the difference of the input values. That is, both formulas are used to find the rate of change between two given points.

SRB p. U2-61

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Instruction

- The rate of change of a linear function is always constant, whereas the average rate of change of a quadratic function is not constant.
- Choosing different pairs of points will result in different rates of change.

Common Errors/Misconceptions

- assuming the rate of change is the same for every interval of a quadratic function

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Guided Practice 2.2.3

Example 1

Calculate the average rate of change for the function $f(x) = x^2 + 6x + 9$ between $x = 1$ and $x = 3$.

1. Evaluate the function for $x = 3$.

$f(x) = x^2 + 6x + 9$	Original function
$f(3) = (3)^2 + 6(3) + 9$	Substitute 3 for x.
$f(3) = 36$	Simplify.

2. Evaluate the function for $x = 1$.

$f(x) = x^2 + 6x + 9$	Original function
$f(1) = (1)^2 + 6(1) + 9$	Substitute 1 for x.
$f(1) = 16$	Simplify.

3. Use the average rate of change formula to determine the average rate of change between $x = 1$ and $x = 3$.

Average rate of change $= \frac{f(b) - f(a)}{b - a}$	Average rate of change formula
Average rate of change $= \frac{f(3) - f(1)}{3 - 1}$	Substitute 1 for a and 3 for b.
Average rate of change $= \frac{36 - 16}{2}$	Substitute the values for $f(3)$ and $f(1)$.
Average rate of change = 10	Simplify.

The average rate of change of $f(x) = x^2 + 6x + 9$ between $x = 1$ and $x = 3$ is 10.

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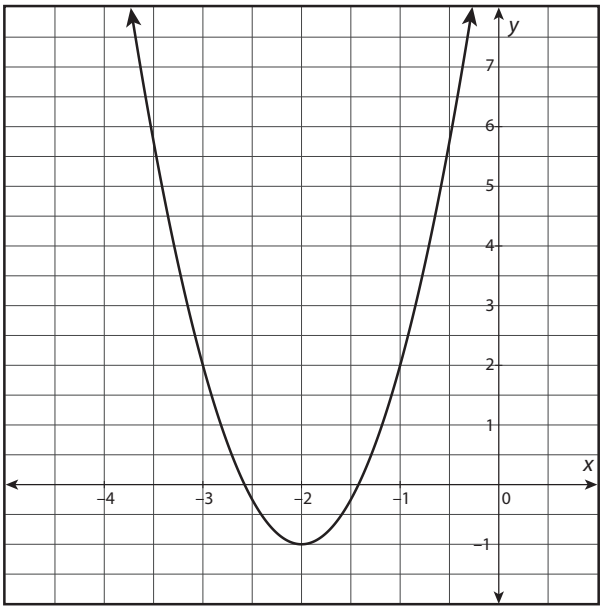
Instruction

Example 2

Use the graph of the function to calculate the average rate of change between $x = -3$ and $x = -2$.



1. Use the graph to identify $f(-2)$.

   According to the graph, $f(-2) = -1$.

2. Use the graph to identify $f(-3)$.

   According to the graph, $f(-3) = 2$.

3. Use the average rate of change formula to calculate the average rate of change between $x = -3$ and $x = -2$.

Average rate of change $= \dfrac{f(b)-f(a)}{b-a}$	Average rate of change formula
Average rate of change $= \dfrac{f(-2)-f(-3)}{(-2)-(-3)}$	Substitute -3 for a and -2 for b.
Average rate of change $= \dfrac{-1-2}{1}$	Substitute the values for $f(-3)$ and $f(-2)$.
Average rate of change $= -3$	Simplify.

The average rate of change of the function between $x = -3$ and $x = -2$ is -3.

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Instruction

Example 3

For the function $g(x) = (x - 3)^2 - 2$, is the average rate of change greater between $x = -1$ and $x = 0$ or between $x = 1$ and $x = 2$?

1. Calculate the average rate of change between $x = -1$ and $x = 0$.

   Evaluate the function at $x = -1$ and $x = 0$.

For $x = -1$:	**For $x = 0$:**
$g(x) = (x - 3)^2 - 2$	$g(x) = (x - 3)^2 - 2$
$g(-1) = [(-1) - 3]^2 - 2$	$g(0) = [(0) - 3]^2 - 2$
$g(-1) = 14$	$g(0) = 7$

Average rate of change $= \frac{g(b) - g(a)}{b - a}$	Average rate of change formula
Average rate of change $= \frac{g(0) - g(-1)}{0 - (-1)}$	Substitute -1 for a and 0 for b.
Average rate of change $= \frac{7 - 14}{1}$	Substitute the values for $g(-1)$ and $g(0)$.
Average rate of change $= -7$	Simplify.

The average rate of change between $x = -1$ and $x = 0$ is -7.

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Instruction

2. Calculate the average rate of change between $x = 1$ and $x = 2$.

Evaluate the function at $x = 1$ and $x = 2$.

For $x = 1$:

$g(x) = (x - 3)^2 - 2$

$g(1) = [(1) - 3]^2 - 2$

$g(1) = 2$

For $x = 2$:

$g(x) = (x - 3)^2 - 2$

$g(2) = [(2) - 3]^2 - 2$

$g(2) = -1$

Average rate of change $= \frac{g(b) - g(a)}{b - a}$	Average rate of change formula
Average rate of change $= \frac{g(2) - g(1)}{2 - 1}$	Substitute 1 for a and 2 for b.
Average rate of change $= \frac{-1 - 2}{1}$	Substitute the values for $g(1)$ and $g(2)$.
Average rate of change $= -3$	Simplify.

The average rate of change between $x = 1$ and $x = 2$ is -3.

3. Compare the averages.

Since $-3 > -7$, the average rate of change of $g(x) = (x - 3)^2 - 2$ is greater between $x = 1$ and $x = 2$ than it is between $x = -1$ and $x = 0$.

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Instruction

Example 4

Find the average rate of change between $x = -0.75$ and $x = -0.25$ for the following function.

x	$g(x)$
–1	0
–0.75	3.44
–0.5	6.25
–0.25	8.44
0	10
0.25	10.94

1. Identify the output values when $x = -0.75$ and $x = -0.25$.

   Refer to the table.

   When $x = -0.75$, $g(x) = 3.44$.

   When $x = -0.25$, $g(x) = 8.44$.

2. Calculate the average rate of change between $x = -0.75$ and $x = -0.25$ by applying the average rate of change formula.

   Average rate of change $= \dfrac{g(b) - g(a)}{b - a}$ — Average rate of change formula

   Average rate of change $= \dfrac{g(-0.25) - g(-0.75)}{-0.25 - (-0.75)}$ — Substitute –0.75 for a and –0.25 for b.

   Average rate of change $= \dfrac{8.44 - 3.44}{-0.25 - (-0.75)}$ — Substitute the values for $g(-0.75)$ and $g(-0.25)$.

   Average rate of change $= \dfrac{5}{0.5}$ — Simplify.

   Average rate of change = 10

   The average rate of change of the function between $x = -0.75$ and $x = -0.25$ is 10.

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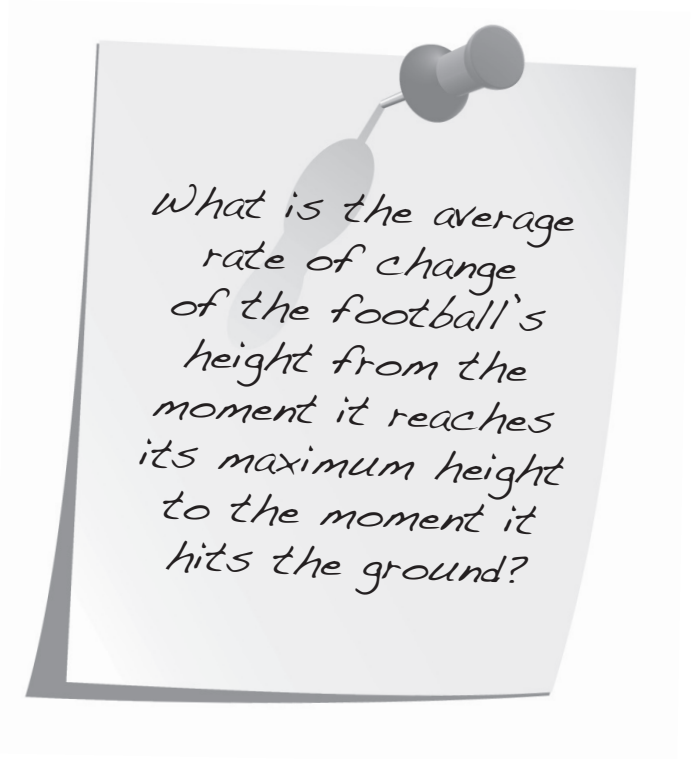
Problem-Based Task 2.2.3: Is the Maximum High Enough?

SMP	
1 ✓	2 ✓
3	4
5	6 ✓
7	8

It is Super Bowl season and teams that have made the play-offs have specialists evaluating every aspect of their field game. One particular team received news that their recently injured kicker's field goal kick is modeled by the function $h(x) = -16(x - 1)^2 + 16$, where $h(x)$ is the height of the ball in feet x seconds after it is kicked. If the football needs to clear a 10-foot goalpost, will the ball make it over if this particular team member kicks it? What is the average rate of change of the football's height from the moment it reaches its maximum height to the moment it hits the ground?



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Problem-Based Task 2.2.3: Is the Maximum High Enough?

Coaching

a. What are the unknown values?

b. What do you need to know in order to determine whether the football will clear the goalpost?

c. What do you need to know in order to find the average rate of change of the football's height from its maximum height to the ground?

d. If the football needs to clear a 10-foot goalpost, will the ball make it over if this particular team member kicks it?

e. What is the average rate of change of the football's height from the moment it reaches its maximum height to the moment it hits the ground?

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Instruction

Problem-Based Task 2.2.3: Is the Maximum High Enough?

Coaching Sample Responses

a. What are the unknown values?

The maximum height of the football and the average rate of change from the maximum height to the ground are unknown values.

b. What do you need to know in order to determine whether the football will clear the goal?

You need to know the *y*-coordinate of the vertex to determine the maximum height that the football reaches.

If the value is greater than 10 feet, then the ball will clear the goalpost.

c. What do you need to know in order to find the average rate of change of the football's height from its maximum height to the ground?

You need to know the *x*-coordinate of the vertex, $x = 1$, and the *x*-intercept that indicates the moment the ball hits the ground. Use symmetry to determine that the ball will hit the ground when $x = 2$.

d. If the football needs to clear a 10-foot goalpost, will the ball make it over if this particular team member kicks it?

The function is written in vertex form, so the vertex is (1, 16). The maximum height that this player can kick the ball is 16 feet, so the ball will clear the goalpost.

e. What is the average rate of change of the football's height from the moment it reaches its maximum height to the moment it hits the ground?

The ball hits the ground when $x = 2$, or 2 seconds after big kicked.

The maximum height of the ball occurs at the vertex, when $x = 1$, or 1 second after being kicked.

$h(2) = -16[(2) - 1]^2 + 16 = 0$

$h(1) = -16[(1) - 1]^2 + 16 = 16$

Average rate of change $= \frac{h(2) - h(1)}{2 - 1} = \frac{0 - 16}{2 - 1} = \frac{-16}{1} = -16$

The average rate of change of the football from its maximum height to the moment it hits the ground is –16 ft/s.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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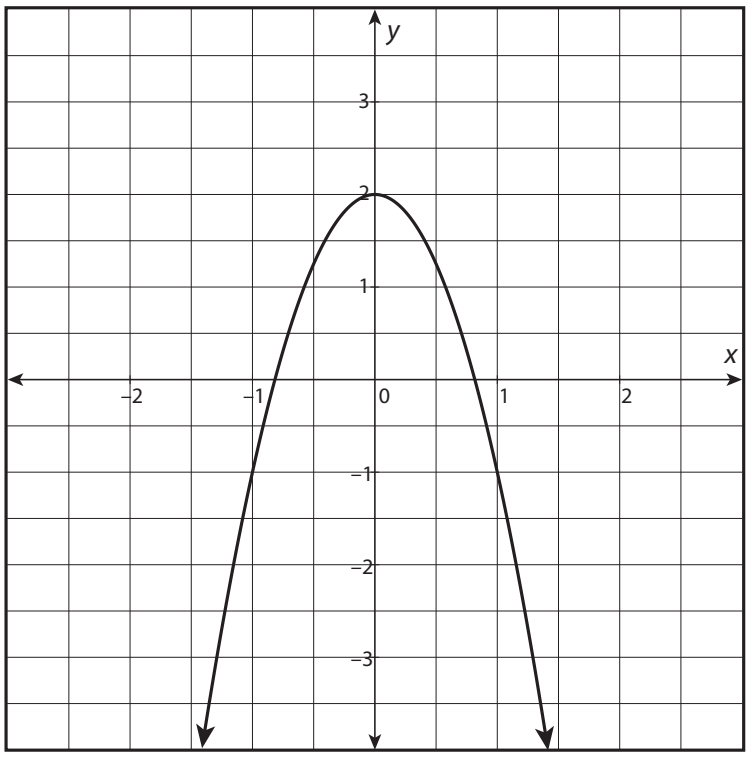
Practice 2.2.3: Identifying the Average Rate of Change

For problems 1–6, calculate the average rate of change of each function between $x = -1$ and $x = 1$.

1. $f(x) = 2(x + 1)^2 - 3$

2. $g(x) = 4 - 3(x - 1)^2$

3. $h(x) = x^2 - 4x + 6$

4. 

5.

x	*y*
–2	–1
–1.5	–1.75
–1	–4
–0.5	–7.75
0	–13
0.5	–19.75
1	–28
1.5	–37.75

continued

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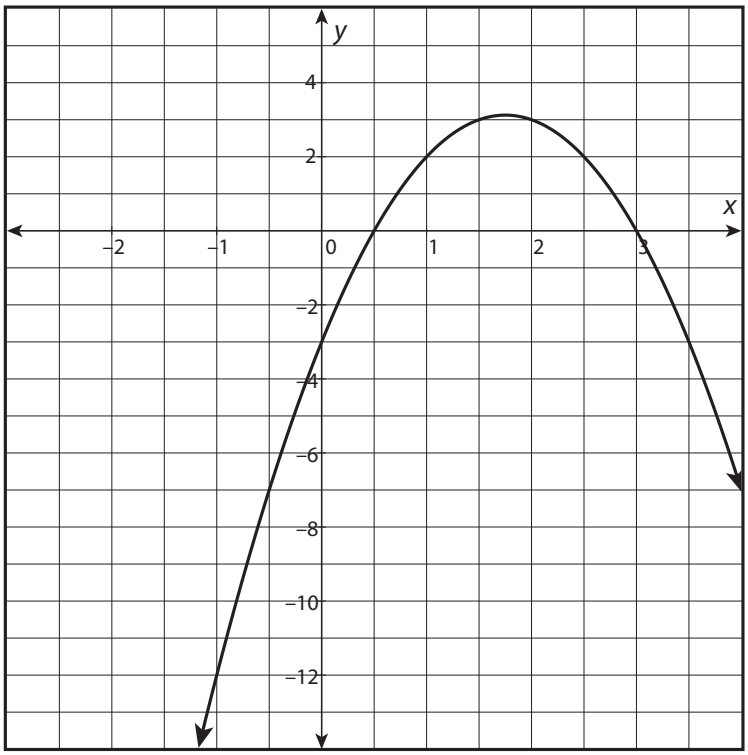
6.



For problems 7–9, determine whether the average rate of change is greater between $x = -2$ and $x = 0$ or between $x = 0$ and $x = 2$.

7. $y = \frac{1}{2}(x+2)^2 - 3$

8. $a(x) = -x^2 + 8x + 3$

9. $f(x) = 5x^2 - 6x + 4$

Read the scenario and use the information in it to answer the question.

10. A drop of rain falls from a height of 1,400 feet above the ground. The function $h(t) = -16t^2 + 1400$ is used to model the raindrop's height, $h(t)$, in feet t seconds after it starts to fall. What is the raindrop's average rate of change between 2 seconds and 3 seconds after it falls?

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Progress Assessment

Circle the letter of the best answer.

1. The table shows the predicted temperatures for a summer day in Salt Lake City, Utah. At what times is the temperature increasing?

Time	Temperature (°F)
6 A.M.	63
9 A.M.	76
12 P.M.	86
3 P.M.	89
6 P.M.	86
9 P.M.	76

   a. from 6 A.M. until 6 P.M.
   b. from 6 A.M. until 3 P.M.
   c. from 6 A.M. until 5:30 P.M.
   d. from 6 A.M. until 6:30 P.M.

2. Which function is neither even nor odd?

   a. $f(x) = 8x^2 + 4$
   b. $h(x) = x^5 + x^3$
   c. $g(x) = -\frac{3}{4}(x-5)^2 + 3$
   d. $y = -2x^3 + x$

3. A quadratic function has a maximum value of 22 and x-intercepts of –6 and 8. What is the value of x that maximizes the function?

   a. 8
   b. 2
   c. 1
   d. 11

4. A tourist jumps off a 16-foot cliff to swim with pink river dolphins in Bolivia. The tourist's height in feet above the water is modeled by $h(t) = -16t^2 + 16$, where t is the time in seconds after the tourist jumps from the cliff. About how long will it take the tourist to reach the water?

   a. 2 seconds
   b. 0.5 second
   c. 1.5 seconds
   d. 1 second

continued

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5. A local company manufactures custom truck parts for big rigs. The company's average profit can be modeled by $a(p) = -3(p - 20)(p)$, where $a(p)$ is the average profit in dollars per p parts produced. What is a reasonable domain for this function?

   a. 0 to 10 parts
   b. 0 to 14 parts
   c. 0 to 18 parts
   d. 0 to 20 parts

6. What is the average rate of change of the function $f(x) = 5x^2 + 20x$ between $x = 0$ and $x = 0.5$?

   a. 22.5
   b. 11.25
   c. –22.5
   d. –11.25

7. The following table shows the values given by a function. Use the table to determine the average rate of change of the function between $x = -2.3$ and $x = -2.2$.

x	*y*
–2.5	–168.8
–2.4	–161.3
–2.3	–153.9
–2.2	–146.5
–2.1	–139.2

   a. 7.4
   b. 74
   c. 0.1
   d. 10

continued

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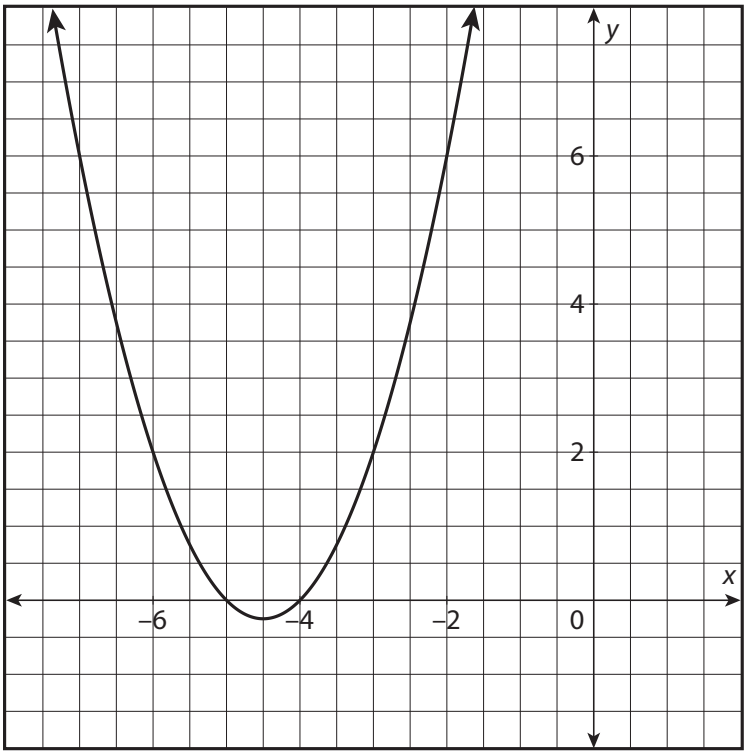
8. Use the graphed function to determine the average rate of change between $x = -5$ and $x = -3$.



a. 0
b. 1
c. 2
d. 3

9. The population of a large town is modeled by $P(t) = -0.86(t - 4.9)^2 + 589$, where $P(t)$ is the population in thousands t years since 1982. What is a reasonable domain for this function?

a. 0 to 4 years
b. 4 to 31 years
c. 0 to 31 years
d. 0 to 4.9 years

10. Find the average rate of change of $g(x) = -3x^2 + 12x - 2$ between $x = 0.1$ and $x = 0.2$.

a. 1.11
b. 0.1
c. 11.1
d. 0.11

continued

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Assessment

Read the scenario and use the information in it to answer the questions that follow.

11. Typically, cars achieve their best mileage when traveling at modest speeds. The gas mileage $G(s)$ for a certain new car can be modeled by $G(s)=-\frac{1}{32}s^2+3s-39$ when the speed s of the car is between 15 and 70 miles per hour, inclusive.

    a. What is the average rate of change in gas mileage when the car's speed increases from 65 mph to 70 mph?

    b. At what values of s is the gas mileage decreasing?

    c. What is the car's best gas mileage, and at what speed is it reached?

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. What is the area of a rectangle with one side of length x and the other 3 less than twice x?

   a. $2x - 3$
   b. $2(x - 3)$
   c. $x(2x - 3)$
   d. $2x^2 - 3$

2. The height of a triangle is twice the base. If the area is 16 square inches, what is the length of the base?

   a. $2\sqrt{2}$
   b. 4
   c. 2
   d. $4\sqrt{2}$

3. Given $f(x) = 4x^2 + 6x - 2$ and $g(x) = 5x^2 - 2x + 7$, find $(f - g)(x)$.

   a. $-x^2 + 8x + 5$
   b. $-x^2 + 4x + 5$
   c. $-x^2 + 8x - 9$
   d. $-x^2 + 4x - 9$

4. Given $f(x) = 2x + 9$ and $g(x) = 4x - 3$, find $(f \bullet g)(x)$.

   a. $8x^2 + 7x - 27$
   b. $8x^2 + 30x + 27$
   c. $6x^2 + 7x - 27$
   d. $8x^2 + 30x - 27$

5. Given $f(x) = (x - 7)(x + 4)$ and $g(x) = x + 4$, find $\left(\frac{f}{g}\right)(x)$.

   a. $x - 7$
   b. $x - 6$
   c. $-2x - 7$
   d. $x + 7$

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Lesson 3: Building Functions

Instruction

Common Core State Standards

SMP
1 ✓ 2
3 4
5 6 ✓
7 ✓ 8 ✓

F–BF.1 Write a function that describes a relationship between two quantities.★

a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*

Essential Questions

1. What is the connection between the vertex of a parabola and maximum/minimum values?
2. What is the connection between the x^2 coefficient and a maximum or minimum?
3. What is concavity?
4. How can mathematical expressions be used to describe real-world problems?

WORDS TO KNOW

concavity	the arch of a curve, either upward or downward. For a quadratic function with positive concavity, the vertex is the minimum or lowest point of the curve. For a quadratic function with negative concavity, the vertex is the maximum or highest point of the curve.
curve	the graphical representation of the solution set for $y = f(x)$; in the special case of a linear equation, the curve will be a straight line
function	a relation in which each element in the domain is mapped onto exactly one element in the range; that is, for every value of x, there is exactly one value of y
leading coefficient	the coefficient of the term with the highest power
vertex of a parabola	the point at which the curve changes direction; its y-coordinate is either a maximum or a minimum

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Instruction

Recommended Resources

- Khan Academy. "Quadratic Equations."

  http://www.walch.com/rr/00118

  This tutorial offers lessons and videos showing how to factor quadratic equations.

- WolframAlpha. "Mathematical Functions."

  http://www.walch.com/rr/00120

  Users can choose from several different types of functions and input values. The site will show how the values look as an equation, provide results of the equation, and graph it.

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Lesson 2.3.1: Building Functions from Context

Warm-Up 2.3.1

The height of a falling object (that is, its distance from the surface of a planet) can be approximated by the quadratic equation $h(t) = a - \frac{1}{2}gt^2$, where a is the starting height, g is the gravitational constant of the planet, and t is the free-fall time in seconds. Each planet has a different gravitational constant based on its mass. Use the information in the following problems to determine how long it will take an object to fall from a given height to the surface of a particular planet. Round your answers to the nearest hundredth.

1. The gravitational constant of Earth is approximately 32 ft/s^2. If an object is dropped from 1,200 feet above Earth, how many seconds will pass before the object hits the ground?

2. The gravitational constant of Mars is approximately 12.5 ft/s^2. If an object is dropped from 1,200 feet above Mars, how many seconds will pass before the object hits the surface of Mars?

3. The gravitational constant of Jupiter is approximately 84.5 ft/s^2. If an object is dropped from 1,200 feet above Jupiter, how many seconds will pass before the object hits the surface of Jupiter?

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Instruction

Lesson 2.3.1: Building Functions from Context

Common Core State Standard

F–BF.1 Write a function that describes a relationship between two quantities.★

a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

Warm-Up 2.3.1 Debrief

1. The gravitational constant of Earth is approximately 32 ft/s^2. If an object is dropped from 1,200 feet above Earth, how many seconds will pass before the object hits the ground?

   Using the general equation given, build an equation specific to an object falling from 1,200 feet above Earth.

   The general equation is $h(t) = a - \frac{1}{2}gt^2$.

   Substitute 1,200 for a and 32 for g.

   This yields the equation $h(t) = 1200 - \frac{1}{2}(32)t^2$, where t is in seconds.

   When the object hits the ground, its height will be 0. Therefore, let $h(t) = 0$ and solve for t.

   $h(t) = 1200 - \frac{1}{2}(32)t^2$

   $0 = 1200 - \frac{1}{2}(32)t^2$

   $0 = 1200 - 16t^2$

   $16t^2 = 1200$

   $t^2 = 75$

   $t = \pm\sqrt{75}$

   $t \approx 8.66$

   It will take approximately 8.66 seconds for the object to hit the ground.

2. The gravitational constant of Mars is approximately 12.5 ft/s^2. If an object is dropped from 1,200 feet above Mars, how many seconds will pass before the object hits the surface of Mars?

   Using the general equation given, build an equation specific to an object falling from 1,200 feet above Mars.

   Substitute 1,200 for a and 12.5 for g.

   $h(t) = 1200 - \frac{1}{2}(12.5)t^2$

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Instruction

When the object hits the surface of Mars, its height will be 0. Set $h(t) = 0$ and solve for t.

$$h(t) = 1200 - \frac{1}{2}(12.5)t^2$$

$$0 = 1200 - \frac{1}{2}(12.5)t^2$$

$$0 = 1200 - 6.25t^2$$

$$6.25t^2 = 1200$$

$$t^2 = 192$$

$$t = \pm\sqrt{192}$$

$$t \approx 13.86$$

It will take approximately 13.86 seconds for the object to hit the surface of Mars.

3. The gravitational constant of Jupiter is approximately 84.5 ft/s^2. If an object is dropped from 1,200 feet above Jupiter, how many seconds will pass before the object hits the surface of Jupiter?

   Using the general equation given, build an equation specific to an object falling from 1,200 feet above Jupiter.

   Substitute 1,200 for a and 84.5 for g.

   $$h(t) = 1200 - \frac{1}{2}(84.5)t^2$$

   When the object hits the surface of Jupiter, its height will be 0. Set $h(t) = 0$ and solve for t.

   $$h(t) = 1200 - \frac{1}{2}(84.5)t^2$$

   $$0 = 1200 - \frac{1}{2}(84.5)t^2$$

   $$0 = 1200 - 42.25t^2$$

   $$42.25t^2 = 1200$$

   $$t^2 \approx 28.40$$

   $$t \approx \pm\sqrt{28.40}$$

   $$t \approx 5.33$$

   It will take approximately 5.33 seconds for the object to hit the surface of Jupiter.

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Instruction

Connection to the Lesson

- Students will be asked to use existing forms of quadratic equations, such as vertex form, to build equations describing specific conditions.
- Students will solve equations for specific values.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- multiplying linear expressions (7.EE.1)
- factoring quadratic equations (F–IF.8a)
- finding the value of *a* in the vertex form of a quadratic equation given the vertex and a point on the parabola (F–IF.8a)
- finding the *x*- and *y*-coordinates of the vertex of a parabola (F–IF.8a)

Introduction

A **function** is a relation in which each element in the domain is mapped onto exactly one element in the range; that is, for every value of *x*, there is exactly one value of *y*. When writing functions based on real-world information, first we must carefully analyze the words describing the situation. For example, if we say an unknown length is 2 more than twice another length, we must be able to translate the given information into an algebraic expression. In this case there are two lengths, with one length dependent on the other. If we let *x* represent the second length, then the first length is 2 more (+) than twice (2 ×) *x*, or $2 + 2x$.

In this lesson, the problems will involve different contexts that quadratic equations describe. Notice that every problem will either require you to identify the two distinct linear expressions whose product will build the quadratic, or to identify the vertex of the parabola along with one other point on the parabola (which will allow you to build the quadratic in the vertex form of a quadratic equation).

Key Concepts

- The product of two linear factors is a quadratic expression.
- The vertex of a parabola is the point at which the curve changes direction. Its *y*-coordinate is either a maximum or minimum for the function.
- The vertex of a parabola given in standard form, $f(x) = ax^2 + bx + c$, can be found using $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.
- The vertex of a parabola given in vertex form, $f(x) = a(x - h)^2 + k$, is (h, k).
- The **curve** of a parabola is the graphical representation of the solution set for the equation of the parabola.
- A quadratic equation can be built using the vertex and any other point on the parabola.

SRB p. U2-71

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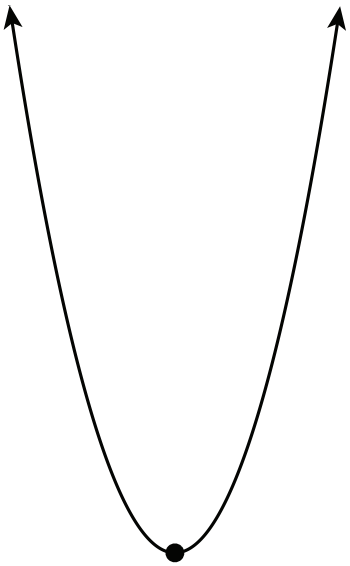
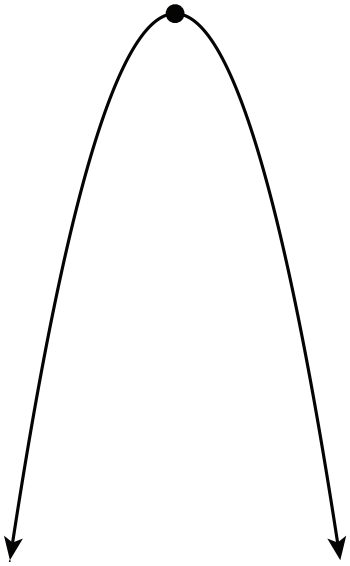
Instruction

- **Concavity** refers to the direction the parabola faces, upward or downward.
- The vertex of an upward-facing (or concave up) parabola is at the minimum point of the parabola.
- Conversely, the vertex of a downward-facing (or concave down) parabola is at the maximum point of the parabola.

Concave Up	Concave Down
	
The vertex is a minimum.	The vertex is a maximum.

- The **leading coefficient** of an expression or equation is the coefficient of the term with the highest power.
- In a quadratic equation, the leading coefficient is the number that is being multiplied by x^2.
- The leading coefficient of the equation of a parabola determines its concavity.
- If the leading coefficient is positive, the parabola is concave up and the graph has a minimum.
- If the leading coefficient is negative, the parabola is concave down and the graph has a maximum.

Common Errors/Misconceptions

- thinking that when given a fixed perimeter, the corresponding area of the figure is constant
- thinking that a positive leading coefficient yields a maximum value instead of a minimum value
- mistakenly believing that adding linear expressions will build quadratics; they must be multiplied

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Instruction

Guided Practice 2.3.1

Example 1

A farmer is building a rectangular pen using 100 feet of electric fencing and the side of a barn. On two of the parallel sides of the pen, the farmer will install 4-foot wide gates. These also require electric fencing. The farmer wants to maximize the area of the pen. How long should he make each side of the fence in order to create the maximum area?

1. Write the expressions that describe the length of each side of the pen.

   Starting with one of the sides that is perpendicular to the barn, we can say the length is an unknown amount, x, plus the 4-foot gate, or $x + 4$.

   The length of the second side of the fence that is perpendicular to the barn will be the same length as the first.

   The side that is parallel to the barn is whatever amount of fence is left over after creating the two perpendicular sides.

   The total amount of fencing is 100 feet, and there are two sides of length $(x + 4)$, so the length of the side that is parallel to the barn is $100 - 2(x + 4)$.

   Simplifying the expression, we get $100 - 2x - 8$, or $92 - 2x$.

2. Build the equation that describes the area of the pen.

   Remember that area equals length times width, or $A = l \bullet w$.

   Let $l = x + 4$ and $w = 92 - 2x$.

$A(x) = (x + 4)(92 - 2x)$	Substitute values for length and width.
$A(x) = 92x - 2x^2 + 368 - 8x$	Multiply.
$A(x) = -2x^2 + 84x + 368$	Reorder and simplify.

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3. To find the maximum area, use the vertex.

   Note that the leading coefficient of the equation is negative. This means the graph of the equation will be a downward-facing parabola. Therefore, the vertex of the parabola will describe the greatest amount of area.

   $A(x) = -2x^2 + 84x + 368$

   Given that the general form of a quadratic function is $y = ax^2 + bx + c$, we can determine that $a = -2$, $b = 84$, and $c = 368$.

   Find the x-coordinate of the vertex by substituting in a and b values from the quadratic function into the expression $\frac{-b}{2a}$:

   $$\frac{-b}{2a} = \frac{-(84)}{2(-2)} = \frac{-84}{-4} = 21$$

   The x-coordinate of the vertex is 21.

   Find the y-coordinate of the vertex by substituting the x-coordinate from the vertex into the quadratic function.

$A(x) = -2x^2 + 84x + 368$	Quadratic function
$A(21) = -2(21)^2 + 84(21) + 368$	Substitute 21 for x.
$A(21) = -882 + 1764 + 368$	Simplify, then solve.
$A(21) = 1250$	

   The maximum area of the pen is 1,250 ft^2.

4. Finally, use the x-value from the vertex to find the lengths of each side of the pen.

   $x = 21$

   Each side that is perpendicular to the barn is equal to $x + 4$.

   $x + 4 = (21) + 4 = 25$ feet Substitute 21 for x.

   The side that is parallel to the barn is equal to $92 - 2x$.

   $92 - 2x = 92 - 2(21) = 50$ feet Substitute 21 for x.

   The sides of the pen should be 25 feet long and 50 feet long.

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Instruction

Example 2

An amusement park has commissioned the design of a steel roller coaster with a drop section that is modeled by a parabola. Part of the roller coaster's track will go through an underground tunnel. In this section, the roller coaster will dip 12 feet below ground level. The roller coaster will dip below ground level at a horizontal distance of 24 feet from the peak just before the drop and reemerge to ground level at a horizontal distance of 36 feet from the peak just before the drop. Find an equation of the parabola that describes the drop and the height of the roller coaster at the peak.

1. Determine the coordinates of the points on the parabola.

   Let the x-axis represent ground level and the y-intercept represent the peak of this section of the roller coaster.

   Since the horizontal distances are given as 24 feet and 36 feet, it can be determined that the roller coaster dips below ground level at (24, 0) and reemerges at (36, 0).

2. Establish the linear factors and the equation of the parabola.

   From the x-intercepts, the linear factors of the equation are $(x - 24)$ and $(x - 36)$. Therefore, the equation of the parabola is $y = a(x - 24)(x - 36)$.

3. Find the vertex of the parabola.

   Expand the factored form of the equation of the parabola into the general form, $y = ax^2 + bx + c$.

$y = a(x - 24)(x - 36)$	Equation of the parabola
$y = a(x^2 - 36x - 24x + 864)$	Multiply the binomial factors.
$y = a(x^2 - 60x + 864)$	Simplify.
$y = ax^2 - 60ax + 864a$	Distribute a.

   The x-coordinate for the vertex of a quadratic equation is $\frac{-b}{2a}$.
   Substitute the values from the quadratic function to determine the x-coordinate of the vertex.

(*continued*)

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$$\frac{-b}{2a}=\frac{-(-60a)}{2(a)}=\frac{30a}{a}=30$$

The x-coordinate is 30.

Combine this information with the given minimum of the drop, 12 feet below ground level, to find the vertex of the equation: (30, –12).

4. Find the value of a.

Substitute the point (30, –12) into the factored form of the quadratic equation and solve for a.

$y = a(x - 24)(x - 36)$	Factored form of the quadratic equation
$-12 = a(30 - 24)(30 - 36)$	Substitute 30 and –12 for x and y.
$-12 = a(6)(-6)$	Simplify, then solve for a.
$-12 = -36a$	

$$a=\frac{1}{3}$$

After solving for a, the equation of the parabola is $y=\frac{1}{3}(x-24)(x-36)$.

5. Find the height of the peak of this section of the roller coaster.

The height of the peak is the roller coaster's vertical distance above ground level. Since the peak occurs at the y-intercept, where the x-coordinate equals 0, set $x = 0$ in the quadratic equation and solve for y.

$$y=\frac{1}{3}(0-24)(0-36)$$

$$y=\frac{1}{3}(-24)(-36)$$

$$y=\frac{1}{3}(864)$$

$$y=288$$

The height of the roller coaster's peak just before the drop is 288 feet.

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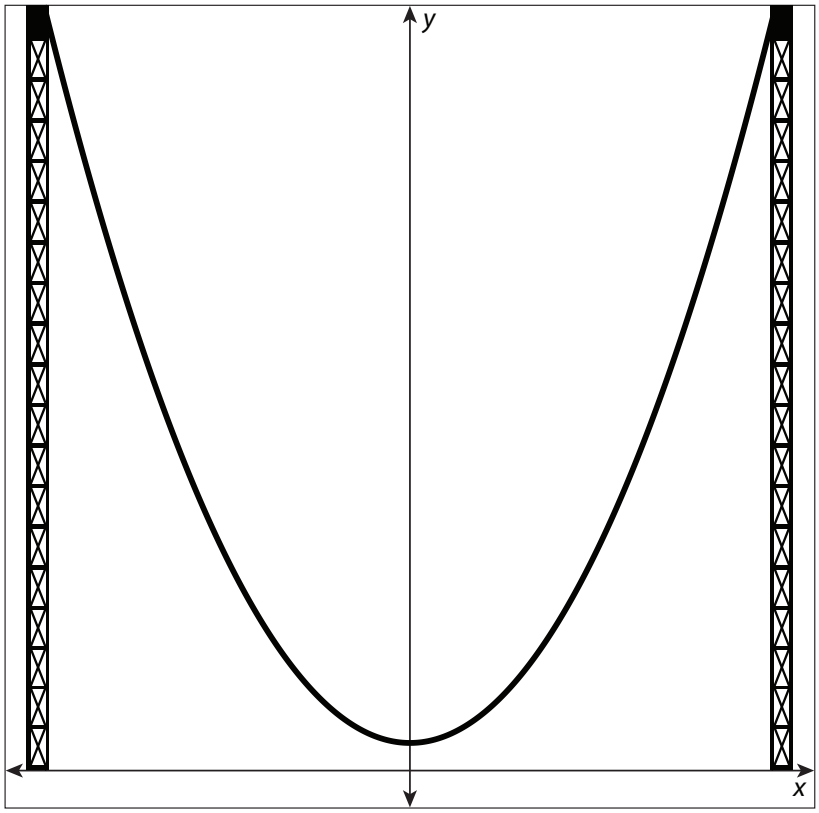
Example 3

Each side of a suspension bridge has a cable that is secured at either end of the span by two supporting towers. The cable is attached to the tops of the two towers. In the section between the two towers, the cable forms a parabolic curve. At its lowest point, the cable is 15 feet above the surface of the bridge. The towers are 400 feet apart, and the vertical distance from the surface of the bridge to the top of each tower is 415 feet. What is a quadratic equation that describes the curve of the cable between the towers?



1. Determine the vertex of the parabola.

   Since the cable is attached to the tops of the towers, this means the cable starts 415 feet up from the surface of the bridge. The cable dips down to a height of 15 feet above the surface of the bridge. If we allow the *y*-axis to represent the midpoint between the towers, and the *x*-axis to represent the surface of the bridge, then we can say the vertex of the parabola is (0, 15).

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2. Find the equation of the parabola.

Substitute the vertex (0, 15) into the vertex form of a quadratic equation, $y = a(x - h)^2 + k$, so that $h = 0$ and $k = 15$.

$y = a(x - h)^2 + k$	Vertex form of a quadratic equation
$y = a(x - 0)^2 + 15$	Substitute 0 for h and 15 for k.
$y = ax^2 + 15$	Simplify.

The equation of the parabola formed by the bridge cable is $y = ax^2 + 15$.

3. Find another point on the parabola.

To solve for a, we need any other point on the parabola to substitute into the equation for x and y. In this case, we have been given the height of the supporting towers (415 feet) and their distance apart (400 feet). Since we let the axis of symmetry of the parabola be the y-axis, we can determine where the top of each tower lies on the parabola.

The y-coordinate for the top of each tower is 415, the height.

To find the x-coordinate, divide the horizontal distance between the towers, 400 feet, by 2.

$$\frac{400}{2} = 200$$

Since the y-axis is the axis of symmetry of the parabola, and each y-coordinate is positive, we can determine that one tower is in Quadrant I, (x, y). The other tower is in Quadrant II, $(-x, y)$. Substitute the x- and y-values into these forms to determine where the tops of the towers are located.

Tower in Quadrant I: $(x, y) = (200, 415)$

Tower in Quadrant II: $(-x, y) = (-200, 415)$

The tops of the towers lie at (–200, 415) and (200, 415). Therefore, we know the points (–200, 415) and (200, 415) lie on the parabola.

(*continued*)

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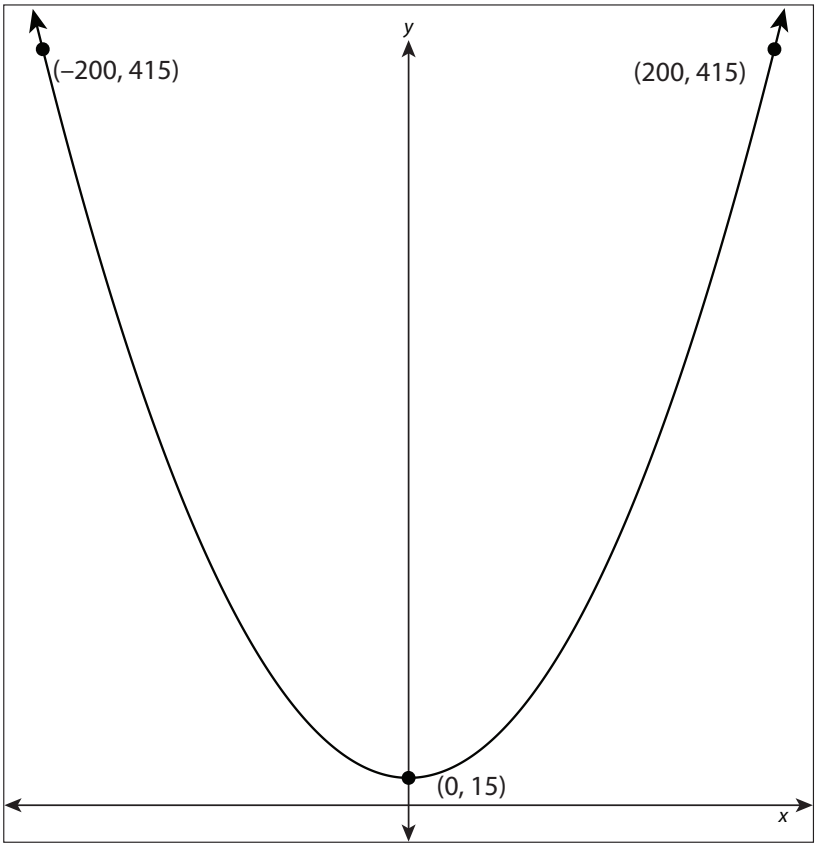

4. Solve for a.

   Take one of the known points that lies on the parabola and substitute the x- and y-values into the equation of the parabola. Either point will do. Let's use (200, 415).

$y = ax^2 + 15$	Equation of the parabola
$(415) = a(200)^2 + 15$	Substitute 200 for x and 415 for y.
$415 = a(40{,}000) + 15$	Simplify.
$400 = a(40{,}000)$	Subtract 15 from both sides.
$\frac{400}{40{,}000} = a$	Divide both sides by 40,000.
$a = \frac{1}{100}$	Simplify.

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Instruction

5. Complete the equation of the parabola that describes the curve of the cable.

Substitute $\frac{1}{100}$ for a in the equation of the parabola.

$y = ax^2 + 15$, where $a = \frac{1}{100}$

$y = \frac{1}{100}x^2 + 15$

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Problem-Based Task 2.3.1: To Drill or Not to Drill?

SMP	
1	2
3	4
5	6 ✓
7 ✓	8 ✓

Cal owns and operates a small oil field in Texas. The field has 75 oil wells, and each well produces 945 barrels of oil per day. There is enough land in the oil field for Cal to drill more wells, but every additional well will cause oil production to drop by 3 barrels per day for each well. Cal believes that it would be profitable to build more wells in the field, but he is not sure how many to build. Building too few wells won't result in the most possible profit. Building too many wells will not be cost effective and will actually cut his profits. Help Cal create a model that allows him to predict the effect that more wells will have on oil production. Use this model to determine the maximum number of wells Cal should have on his land.

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Problem-Based Task 2.3.1: To Drill or Not to Drill?

Coaching

a. Write an expression to describe adding an unknown number of wells to Cal's existing wells.

b. Write an expression to describe the effect of drilling more wells on the number of barrels of oil each well produces per day.

c. What is the function that describes oil production in relation to the number of wells?

d. How many barrels of oil does the field produce with 75 wells?

e. What is the vertex of the parabola that describes this situation?

f. Does this vertex represent a maximum or a minimum? Explain.

g. Use the vertex to determine the number of wells that will produce the maximum amount of oil.

h. Use the vertex to determine the maximum number of barrels the oil field can produce.

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Instruction

Problem-Based Task 2.3.1: To Drill or Not to Drill?

Coaching Sample Responses

a. Write an expression to describe adding an unknown number of wells to Cal's existing wells.

There are 75 wells already. Let x be the number of oil wells Cal wants to add.

The expression $75 + x$ represents the number of existing wells plus the unknown number of new wells.

b. Write an expression to describe the effect of drilling more wells on the number of barrels of oil each well produces per day.

Currently, each well produces 945 barrels of oil per day.

For each well added, 3 fewer barrels of oil will be produced by each well.

Subtract 3 times the unknown number of new wells (x) from the current output per individual well.

$945 - 3x$

c. What is the function that describes oil production in relation to the number of wells?

The number of barrels produced each day is the product of the number of wells and the number of barrels of oil that can be produced by each well.

$f(x) = (75 + x)(945 - 3x)$

$f(x) = 70{,}875 + 720x - 3x^2$

Rearranging the terms in order of descending powers yields the following function.

$f(x) = -3x^2 + 720x + 70{,}875$

d. How many barrels of oil does the field produce with 75 wells?

Since there are 75 wells already, this means that there are 0 wells added. Evaluate the oil production function at $f(0)$.

$f(x) = -3x^2 + 720x + 70{,}875$

$f(0) = -3(0)^2 + 720(0) + 70{,}875$

$f(0) = 70{,}875$

The oil field produces 70,875 barrels of oil when it has 75 wells.

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e. What is the vertex of the parabola that describes this situation?

To find the coordinates of the vertex, identify a, b, and c in the equation of the function.

$f(x) = -3x^2 + 720x + 70{,}875$

Thus, $a = -3$, $b = 720$, and $c = 70{,}875$.

The x-coordinate for the vertex of a parabola is $\frac{-b}{2a}$. Substitute values for a and b into this expression to find x.

$$x = \frac{-b}{2a} = \frac{-(720)}{2(-3)} = \frac{-720}{-6} = 120$$

The y-coordinate of the vertex of the parabola can be determined by substituting 120 for x in the function to find $f(120)$.

$f(x) = -3x^2 + 720x + 70{,}875$

$f(120) = -3(120)^2 + 720(120) + 70{,}875$

$f(120) = 114{,}075$

The vertex of the parabola is (120, 114,075).

f. Does this vertex represent a maximum or a minimum? Explain.

The coefficient for the x^2 term is negative, so the graph of $f(x)$ will be a downward-facing parabola. Therefore, the vertex of the parabola, (120, 114,075), will be the maximum point on the curve.

g. Use the vertex to determine the number of wells that will produce the maximum amount of oil.

Interpret the data: x is the number of wells added. The x-value of the vertex is 120. Therefore, 120 is the maximum number of added wells. There are already 75 wells in the field. This means the maximum number of wells is 75 + 120 = 195 wells.

h. Use the vertex to determine the maximum number of barrels the oil field can produce.

Interpret the data: y represents the barrels of oil produced each day, so 114,075 is the maximum number of barrels the oil field can produce each day.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2.3.1: Building Functions from Context

Use your knowledge of quadratic functions to complete each problem that follows.

1. Expand the linear factors of $f(x)$, where $f(x) = (x - 10)(x + 2)$.

2. Let $g(x) = 3x^2 + 4x - 36$. What is $g(2)$?

3. The product of two consecutive odd integers is 2,915. Build a function that can be used to solve for the integers. What are the two integers?

Use the following scenario to complete problems 4 and 5.

> A suspension bridge has two supporting towers with a cable secured at either end of the span and then draped between the towers. In the section between the two supporting towers, the cable forms a parabolic curve. At the lowest point of the curve, the cable is 20 feet from the surface of the bridge. The supporting towers are 500 feet apart, and rise 420 feet from the surface of the bridge to the top of the tower. Use the height of the left tower as the *y*-intercept.

4. What are the vertex and the *y*-intercept?

5. What is the equation in vertex form of the suspension cable?

continued

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Use the following scenario to complete problems 6 and 7.

A museum curator needs to frame a rectangular painting. The painting is 24 inches by 16 inches.

6. If the frame is to be of width x, what is a function for the area of the painting, including the frame?

7. What would be the area of just the frame if the value of x was determined to be 2.5 inches?

Use the following scenario to complete problems 8 and 9.

An amusement park is building a wooden roller coaster with a drop section modeled by a parabola. The roller coaster will travel through an underground tunnel. In this section, the roller coaster will dip 9 feet below ground level. The roller coaster will dip below ground level a horizontal distance of 32 feet from the peak just before the drop and reemerge to ground level a horizontal distance of 80 feet from the peak just before the drop.

8. What is the equation of the parabola that models the drop from the peak to the reemergence at ground level?

9. What is the height of the peak just before this drop?

Use the given information to write a function for problem 10.

10. A family just bought a puppy and wants to fence in part of the yard. The fenced-in area is a rectangular section of the yard that is separate from all other structures on the property. The family has purchased 70 feet of fencing. What function models the area of the pen?

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Lesson 2.3.2: Operating on Functions

Warm-Up 2.3.2

The student council is selling lanyards with the school's name on them. Each lanyard costs $1.50 to purchase from the manufacturer. The student council is selling each lanyard for $5.00.

1. Write a function $f(x)$ that models the cost of buying the lanyards before reselling them.

2. Write a function $g(x)$ that models the income from selling the lanyards.

3. What is the profit from selling one lanyard?

4. Write a function $h(x)$ that models the profit of the sale of the lanyards in terms of $f(x)$ and $g(x)$, and then simplify the function.

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Lesson 2.3.2: Operating on Functions

Common Core State Standard

F–BF.1 Write a function that describes a relationship between two quantities.★

b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*

Warm-Up 2.3.2 Debrief

1. Write a function $f(x)$ that models the cost of buying the lanyards before reselling them.

   Each lanyard costs $1.50 to purchase from the manufacturer.

   Let x represent the number of lanyards purchased.

   $f(x) = 1.5x$

2. Write a function $g(x)$ that models the income from selling the lanyards.

   Each lanyard is sold for $5.00.

   Let x represent the number of lanyards sold.

   $g(x) = 5x$

3. What is the profit from selling one lanyard?

   To find the profit, subtract the amount paid to the lanyard manufacturer (1.50) from the price charged by the student council (5.00).

   $5.00 - 1.50 = 3.50$

   The student council makes $3.50 in profit for selling one lanyard.

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Instruction

4. Write a function $h(x)$ that models the profit of the sale of the lanyards in terms of $f(x)$ and $g(x)$, and then simplify the function.

   Since the cost to purchase a lanyard must be subtracted from the price for which each lanyard is sold, subtract $f(x)$ from $g(x)$ to find $h(x)$.

$h(x) = (g - f)(x)$	Set up the equation.
$h(x) = g(x) - f(x)$	Simplify.
$h(x) = (5x) - (1.5x)$	Substitute values for $g(x)$ and $f(x)$.
$h(x) = 3.5x$	Simplify.

   The simplified function is $h(x) = 3.5x$.

Connection to the Lesson

- Students will combine functions by using the four basic operations.
- Students will extend their experience to building new functions that are quadratic.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- adding, subtracting, multiplying, and dividing functions (F–BF.1b★)

Introduction

As is true with linear and exponential functions, we can perform operations on quadratic functions. Such operations include addition, subtraction, multiplication, and division. This lesson will focus on adding, subtracting, multiplying, and dividing functions to create new functions. The lesson will also explore the effects of dividing a quadratic by one of its linear factors.

Key Concepts

Operations with Functions

- Functions can be added, subtracted, multiplied, and divided.
- For two functions $f(x)$ and $g(x)$:
  - The addition of the functions is represented as $(f+g)(x)=f(x)+g(x)$.
  - The subtraction of the functions is represented as $(f-g)(x)=f(x)-g(x)$.
  - The multiplication of the functions is represented as $(f \bullet g)(x)=f(x) \bullet g(x)$.
  - The division of the functions is represented as $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$.
- Adding and subtracting linear expressions from a quadratic will yield a quadratic.
- Multiplying and dividing a quadratic by anything other than a constant will not yield a quadratic.

Restricted Domains

- When considering the division of a quadratic by a linear factor, it is possible to create a linear expression with a restricted domain. For example:
  - For $f(x) = (x + 2)(x + 3)$ and $g(x) = x + 3$, $\left(\frac{f}{g}\right)(x)$ can be found like this:

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{(x+2)(x+3)}{x+3}=\frac{(x+2)\cancel{(x+3)}}{\cancel{x+3}}=x+2$$

  - In simpler terms, $\left(\frac{f}{g}\right)(x)=x+2$.

SRB p. U2-84

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- Remember that the denominator of a fraction cannot equal 0. Before the function was simplified, the denominator was $x + 3$. Set the denominator equal to 0 and solve for x to find the restricted value(s) in the domain: $x + 3 = 0$, so $x \neq -3$.
- Given the similar function $h(x) = x + 2$, the domain and range are both all real numbers. However, since $f(x)$ is divided by $g(x)$, the domain of $\left(\frac{f}{g}\right)(x) = x + 2$ from the preceding example is all real numbers *except* for $x = -3$ and the range is all real numbers *except* for $y = -1$.
- This is because the restricted value of the domain (–3) is substituted into the simplified form of $\left(\frac{f}{g}\right)(x) = x + 2$ to find the corresponding value of y:
  - $\left(\frac{f}{g}\right)(-3) = (-3) + 2$
  - $\left(\frac{f}{g}\right)(-3) = -1$
  - Therefore, since $x \neq -3$, then $y \neq -1$.

Common Errors/Misconceptions

- forgetting to restrict the domain when dividing functions
- not realizing that functions must be of the same variable for like terms to be combined
- having difficulty moving from the formal notation to a workable problem where functions can be used with operations

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Guided Practice 2.3.2

Example 1

Let $f(x) = x^2 - 3x + 4$ and $g(x) = x^2 + 6x - 3$. Build a new function, $h(x)$, for which $h(x) = (f + g)(x)$.

1. Expand the new function, $h(x)$, into a form where substitution can be used.

   $h(x) = (f + g)(x) = f(x) + g(x)$

   The new function is expanded as $h(x) = f(x) + g(x)$.

2. Add the functions.

$f(x) = x^2 - 3x + 4$ and $g(x) = x^2 + 6x - 3$	Given functions from problem statement
$h(x) = f(x) + g(x)$	Expanded form
$h(x) = (x^2 - 3x + 4) + (x^2 + 6x - 3)$	Substitute values for $f(x)$ and $g(x)$.
$h(x) = 2x^2 + 3x + 1$	Combine like terms.

   The new function is $h(x) = 2x^2 + 3x + 1$.

Example 2

Let $f(x) = 3x + 4$ and $g(x) = 5x - 2$. Build a new function, $h(x)$, for which $h(x) = (f \bullet g)(x)$.

1. Expand the new function, $h(x)$, into a form where substitution can be used.

   $h(x) = (f \bullet g)(x) = f(x) \bullet g(x)$

   The new function is expanded as $h(x) = f(x) \bullet g(x)$.

2. Multiply the functions.

$f(x) = 3x + 4$ and $g(x) = 5x - 2$	Given functions from problem statement
$h(x) = f(x) \bullet g(x)$	Expanded form
$h(x) = (3x + 4)(5x - 2)$	Substitute values for $f(x)$ and $g(x)$.
$h(x) = 15x^2 - 6x + 20x - 8$	Multiply.
$h(x) = 15x^2 + 14x - 8$	Combine like terms.

   The new function is $h(x) = 15x^2 + 14x - 8$.

SWB p. 71; SRB p. U2-86

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Instruction

Example 3

For $f(x) = (3x - 2)(x + 5)$ and $g(x) = x + 5$, find $\left(\frac{f}{g}\right)(x)$. What type of function is the quotient of $\left(\frac{f}{g}\right)(x)$? Are there restrictions on the domain and range of $\left(\frac{f}{g}\right)(x)$?

1. Since the functions are being divided, write the functions $f(x)$ and $g(x)$ as a fraction.

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{(3x - 2)(x + 5)}{x + 5}$$

2. Simplify the rational expression and define the type of equation of the simplified form.

Divide away the monomial $(x + 5)$ from the numerator and denominator of the fraction:

$$\left(\frac{f}{g}\right)(x) = \frac{(3x-2)\cancel{(x+5)}}{\cancel{x+5}} = 3x - 2$$

The function $\left(\frac{f}{g}\right)(x)$ is a linear function, and can be written as $y = 3x - 2$.

3. Look at the original fraction to see if there are restricted values on the domain.

In this case, $x \neq -5$ because $(-5) + 5 = 0$ and division by 0 is undefined. Substitute $x = -5$ into the simplified equation to determine the restricted value(s) of y.

$y = 3x - 2 = 3(-5) - 2 = -17$

Since $x \neq -5$, then $y \neq -17$.

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Instruction

Example 4

Zane is a textiles designer. His latest project is to design a rectangular area rug for a hotel lobby. The dimensions of the lobby are such that one pair of walls is twice the length of the other pair of walls. The rug must lay centered in the lobby, with each edge of the rug exactly 3 feet from the wall. What is the function in terms of x that describes the area of the lobby? What is the function in terms of x that describes the area of the rug? What is the function that describes the area of the lobby left uncovered by the rug?

1. Define a function that describes the area of the lobby.

   Using x for the unknown quantity, the dimensions of the lobby are x feet wide by $2x$ feet long.

   Since *area* = *length* • *width*, the area of the lobby can be described by the function $f(x)$.

   $f(x) = l \cdot w$

   $f(x) = (2x)(x)$

   $f(x) = 2x^2$

2. Define a function that describes the area of the rug.

   The dimensions of the rug are 3 feet less than the length of the walls on either side, so the shorter side is $x - 2(3) = x - 6$ and the longer side is $2x - 2(3) = 2x - 6$.

   Therefore, the area of the rug can be described by substituting values for the length (the longer side) and width (the shorter side) into the function $g(x)$.

   $g(x) = l \cdot w$

   $g(x) = (2x - 6)(x - 6)$

   $g(x) = 2x^2 - 18x + 36$

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Instruction

3. Define a function that represents the area of the lobby left uncovered by the rug.

   The area of the lobby left uncovered by the rug can be described by subtracting the area of the rug, $g(x)$, from the area of the lobby, $f(x)$.

   $(f - g)(x) = f(x) - g(x)$

   $(f - g)(x) = (2x^2) - (2x^2 - 18x + 36)$

   $(f - g)(x) = 18x - 36$

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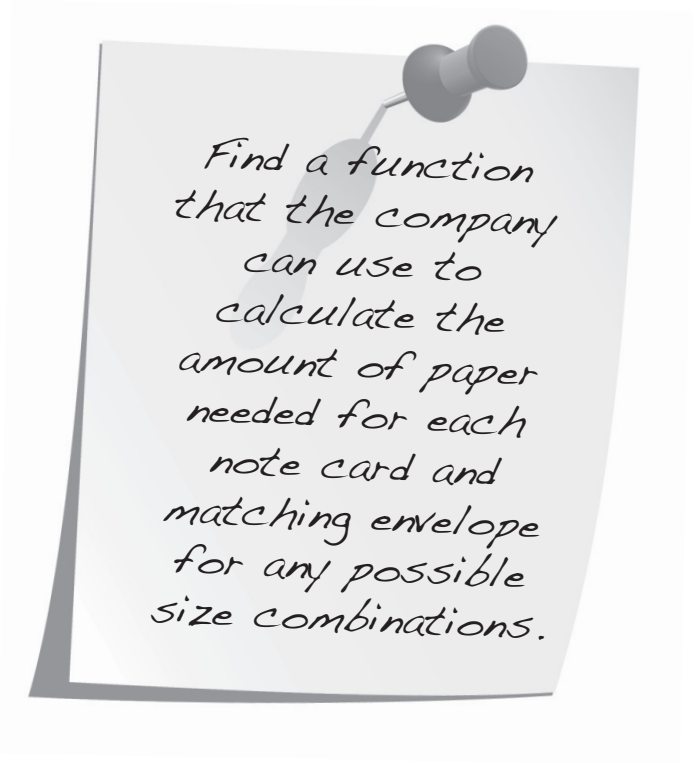
Problem-Based Task 2.3.2: Pushing Envelopes

SMP	
1 ✓	2
3	4
5	6 ✓
7	8

A stationery company makes flat note cards and matching envelopes in various sizes. Regardless of the size of the note card, the ratio of the height to the width is always 3 : 5. Every envelope is made to be precisely 2 mm wider and 3 mm taller than its matching note card. Finally, the amount of paper required to construct each envelope is equal to 3 times the area of the front of the envelope. Find a function that the company can use to calculate the amount of paper needed for each note card and matching envelope for any possible size combinations.



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Problem-Based Task 2.3.2: Pushing Envelopes

Coaching

a. What are the equations for the height and width of any size note card?

b. What is the equation that describes the area of any size note card and therefore the amount of paper needed?

c. Use the equations for the height and width of any size note card to write similar equations for the height and width of the corresponding envelope.

d. Using the equations created up to this point, what is the equation that describes the amount of paper needed for an envelope for a given note card?

e. Build an equation that describes the total amount of paper needed for each note card and envelope pair.

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Instruction

Problem-Based Task 2.3.2: Pushing Envelopes

Coaching Sample Responses

a. What are the equations for the height and width of any size note card?

Each note card the company makes has a height-to-width ratio of 3 : 5. Therefore, the height of each note card can be represented as $3x$, where x is an arbitrary length. Similarly, the width of each note card can be represented as $5x$.

$h(x) = 3x$

$w(x) = 5x$

b. What is the equation that describes the area of any size note card and therefore the amount of paper needed?

The area of each note card can be determined by multiplying the height by the width.

$(h \bullet w)(x) = 3x \bullet 5x = 15x^2$

c. Use the equations for the height and width of any size note card to write similar equations for the height and width of the corresponding envelope.

The height of each envelope is 3 mm greater than the height of each note card. Therefore, the height of each envelope is $h(x) + 3 = 3x + 3$.

The width of each envelope is 2 mm greater than the width of each note card. Therefore, the width of each envelope is $w(x) + 2 = 5x + 2$.

d. Using the equations created up to this point, what is the equation that describes the amount of paper needed for an envelope for a given note card?

The amount of paper required to construct an envelope is equal to 3 times the area of the front of the envelope. The area of the front of each envelope can be determined by multiplying the height and width.

$(h(x) + 3)(w(x) + 2) = (3x + 3)(5x + 2) = 15x^2 + 21x + 6$

Multiply this amount by 3 to determine the amount of paper needed for each envelope.

$3(15x^2 + 21x + 6) = 45x^2 + 63x + 18$

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e. Build an equation that describes the total amount of paper needed for each note card and envelope pair.

The amount of paper needed for one note card is $(h \bullet w)(x)$.

The amount of paper needed for one envelope is $3[(h(x) + 3)(w(x) + 2)]$.

Add the two functions to find the total amount of paper needed for one note card and envelope pair. Let $j(x)$ represent the total amount of paper for the note card and envelope combined.

$j(x) = (h \bullet w)(x) + 3[(h(x) + 3)(w(x) + 2)]$

$j(x) = (3x)(5x) + 3[(3x + 3)(5x + 2)]$

$j(x) = 15x^2 + 3(15x^2 + 21x + 6)$

$j(x) = 15x^2 + 45x^2 + 63x + 18$

$j(x) = 60x^2 + 63x + 18$

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2.3.2: Operating on Functions

Use your knowledge of operations on functions to complete the problems that follow.

1. Let $f(x) = x^2 + 7x + 12$ and $g(x) = 5x + 23$. Build a new function, $h(x)$, for which $h(x) = (f + g)(x)$.

2. Let $s(x) = x^2 + 12x + 13$ and $t(x) = 19x + 3$. Build a new function, $u(x)$, for which $u(x) = (s - t)(x)$.

3. Let $j(x) = 2x + 1$ and $k(x) = 11x - 3$. Build a new function, $m(x)$, for which $m(x) = (j \bullet k)(x)$.

4. Let $f(x) = (x + 3)(x + 4)$ and $g(x) = x + 3$. Build a new function, $h(x)$, for which $h(x) = \left(\frac{f}{g}\right)(x)$. State any restrictions on the domain and range.

5. Let $g(x) = -3x^2 - 11x + 15$ and $h(x) = -x^2 + 3$. Build a new function, $k(x)$, for which $k(x) = (h - g)(x)$.

continued

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6. Sinopa is designing a rectangular area rug for a room. The dimensions of the room are such that one pair of walls is 3 feet longer than the other pair of walls. The rug must lay centered in the room with each edge 2 feet from each wall. Let x represent the width of the room. What is the function in terms of x that describes the area of the room? What is the function in terms of x that describes the area of the rug? What is the function that describes the area of the room left uncovered by the rug?

7. A city parks department increased the lengths of both sides of a park by the same amount. As a result, the park is 120 feet by 200 feet. Build the functions that represent the original side lengths, and write the area function of the original park using an operation of the functions you built.

8. The altitude of a triangular banner is 4 feet longer than twice its base. Define the functions that describe the altitude and base of the triangle, and then use these functions to build an area function of the banner.

9. The surface area of a cylinder is found by adding the top, bottom, and side surface areas. The top and bottom of a cylinder are circles with an area of πr^2, where r is the radius of the circle. The side surface of a cylinder is given by the expression $2\pi rh$, where r is the radius of the circles on the top and bottom of the cylinder, and h is the height of the cylinder. Define a function $f(r)$ that describes the area of the top and bottom of the cylinder. Define a function $g(r)$ that describes the surface area of the side of the cylinder. Using $f(r)$ and $g(r)$, define a function that describes the surface area of a cylinder.

10. If the cylinder described in the previous problem were cut into thirds that are parallel to the top and bottom, resulting in 3 cylinders of equal height, what would be the new function for the total surface area of the cylinders?

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Progress Assessment

Circle the letter of the best answer.

1. A rancher is building a rectangular pen using 150 feet of fencing. The pen will be enclosed on 3 sides, with the fourth side shared by an already existing pen. What is a function that describes the area of the new pen?

   a. $f(x) = -2x^2 - 150x$
   b. $f(x) = 2x^2 - 300x$
   c. $f(x) = 2x - 300$
   d. $f(x) = -2x^2 + 150x$

2. Given the vertex of a parabola of (4, –8) that passes through the point (2, 20), what is the vertex form of the quadratic equation?

   a. $y = -7(x + 4)^2 - 8$
   b. $y = 7(x - 4)^2 + 8$
   c. $y = -7(x + 4)^2 + 8$
   d. $y = 7(x - 4)^2 - 8$

3. The area of a memorial park was determined by the height of a flagpole in the center of the memorial. The park is rectangular, and the length of the north side of the park is 9 times the height of the flagpole. The east side of the park is 11 times the height of the flagpole. If the height of the flagpole is h feet, what is the area of the park?

   a. $(h + 9)(h + 11)$ ft^2
   b. $(9h + 11h)$ ft^2
   c. $99h^2$ ft^2
   d. $(h^2 - 99)$ ft^2

4. The dimensions of a desktop are such that the length is 5 inches shorter than twice its width. What expression describes the area of the desktop in terms of its width, w?

   a. $w(2w - 5)$ in^2
   b. $2w(w - 5)$ in^2
   c. $w(5w - 2)$ in^2
   d. $(w + 2)(w - 5)$ in^2

5. A suspension bridge has two supporting towers and cables secured at either end of the span. The section of the cables between the two supporting towers forms a parabolic curve. At the lowest point, the cables are 12 feet from the surface of the bridge. The supporting towers are 600 feet apart and 412 feet tall. What is a quadratic equation that describes the curve of the cables between the towers? Use the vertex as the y-intercept.

   a. $f(x) = \frac{1}{225}x^2 + 12$
   b. $f(x) = \frac{1}{600}(x - 12)^2$
   c. $f(x) = \frac{1}{300}x^2 + 12$
   d. $f(x) = \frac{1}{412}(x - 12)^2 + 300$

continued

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Assessment

6. Let $f(x) = 3x^2 + x$ and $g(x) = -7x + 3$. Which function represents a new function, $h(x)$, for which $h(x) = (f + g)(x)$?

   a. $h(x) = -21x^3 + 2x^2 + 3x$
   b. $h(x) = 3x^2 - 6x + 3$
   c. $h(x) = -4x^2 + x + 3$
   d. $h(x) = 3x^2 + 8x + 3$

7. Let $s(x) = 5x^2 + 12x + 2$ and $t(x) = x^2 + 10x - 4$. Which function represents a new function, $u(x)$, for which $u(x) = (s - t)(x)$?

   a. $u(x) = 4x^2 + 2x + 6$
   b. $u(x) = 6x^2 + 22x - 2$
   c. $u(x) = 4x^2 - 2x + 6$
   d. $u(x) = 6x^2 - 22x + 2$

8. The surface area of a cylinder is defined as $B(r) = 2\pi rh + 2\pi r^2$, where r is the radius of the cylinder and h is the height. Which function defines the total surface area of a cylinder with a height that is 3 less than twice its radius?

   a. $C(r) = 2\pi r^2 + 2\pi r$
   b. $C(r) = 4\pi r^2 - 4\pi r$
   c. $C(r) = 6\pi r^2 - 6\pi r$
   d. $C(r) = 8\pi r^2 + 8\pi r$

9. Let $f(x) = (x - 4)(x - 10)$ and $g(x) = x - 10$. What value can the expression $\left(\frac{f}{g}\right)(x)$ never equal?

   a. 5
   b. 6
   c. 10
   d. –10

10. Let $f(x) = -x - 2$ and $g(x) = 7x + 3$. Which function represents a new function, $h(x)$, for which $h(x) = (f \bullet g)(x)$?

    a. $h(x) = 7x^2 - 17x - 6$
    b. $h(x) = -7x^2 - 11x - 6$
    c. $h(x) = 6x + 1$
    d. $h(x) = -7x^2 - 17x - 6$

Read the scenario and follow the directions to complete problem 11.

11. Custom Frame Company designs picture frames with mats. A mat is a border around the picture. The mat must be cut out in the middle so that the border is always 4 inches around the picture. One picture that is being framed is 2 inches longer than it is wide. Let x represent the length of the picture.

    a. Build a function that describes the area of the picture.
    b. Build a function that describes the area of the mat before the mat is cut.
    c. Define a function that describes the area of the mat after it has been cut.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. What is the domain of $y=2\sqrt{(x-8)}$?

   a. $x \geq 0$ c. $x \geq 8$

   b. $x \leq -8$ d. $x \leq 0$

2. Which function has a point of inflection at (–2, 5)?

   a. $y=2\sqrt[3]{x}+5$ c. $y=-2\sqrt[3]{(x-5)}$

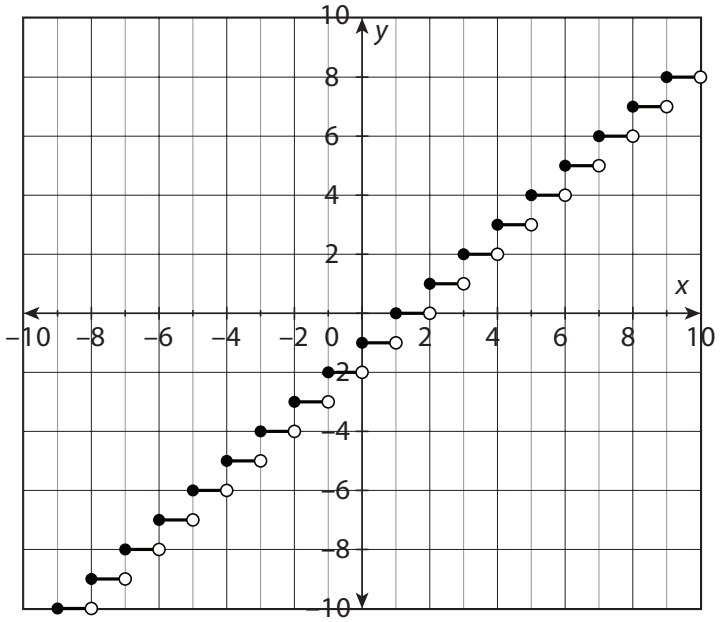
   b. $y=\sqrt[3]{(x-2)}-5$ d. $y=5\sqrt[3]{(x+2)}+5$

3. What are the coordinates of the minimum point of the graph of $y=|x+3|-1$?

   a. (–3, –1) c. (0, –1)

   b. (3, –1) d. (–1, 3)

4. Which function is shown in the graph?



   a. $y=\lceil x \rceil-1$ c. $y=\lfloor x \rfloor+1$

   b. $y=-\lceil x \rceil-1$ d. $y=\lfloor x \rfloor-1$

continued

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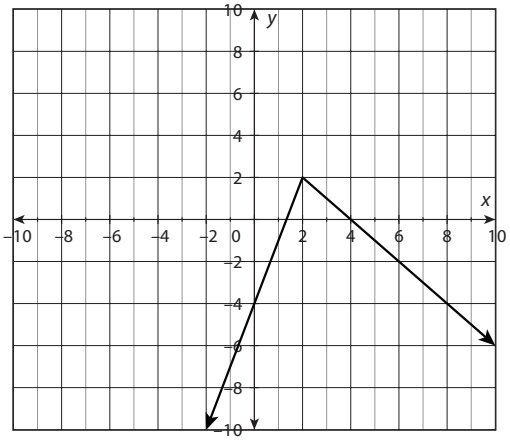
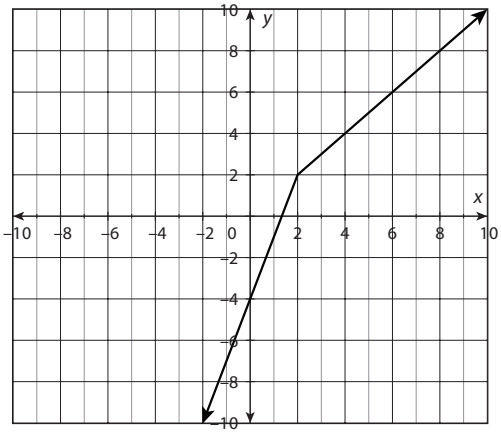
Assessment

5. Which graph shows the function $y=\begin{cases}3x-4, \text{ if } x\le 2\\ -x, \text{ if } x>2\end{cases}$?

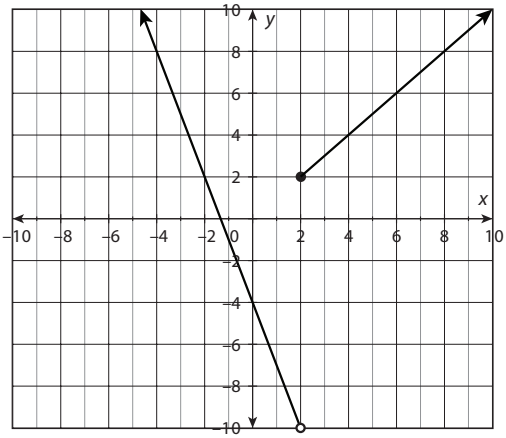
a.



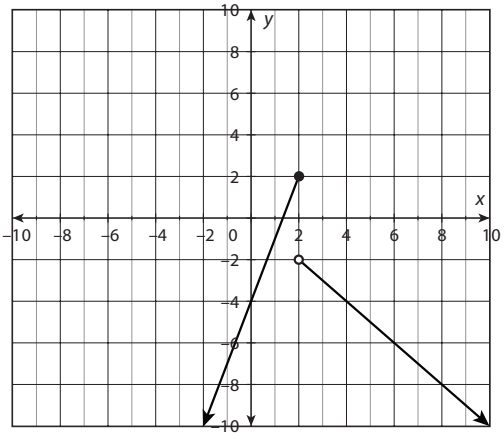
c.



b.



d.



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Lesson 4: Graphing Other Functions

Instruction

Common Core State Standard

SMP	
1	2 ✓
3	4 ✓
5 ✓	6 ✓
7	8

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

Essential Questions

1. How can the algebraic representation of a square root or cube root function be used to create a graph of the function?
2. What is an important feature of an absolute value function, and why is it important?
3. What are examples of functions that have a domain of all real numbers but a restricted range?

WORDS TO KNOW

absolute value	a number's distance from 0 on a number line; the positive value of a quantity
absolute value function	a function with a variable inside absolute value symbols
ceiling function	also known as the least integer function; a function represented as $y = \lceil x \rceil$. For any input x, the output is the smallest integer greater than or equal to x; for example, $\lceil -3 \rceil = -3$, $\lceil 2.1 \rceil = 3$, and $\lceil -2.1 \rceil = -2$.
cube root	For any real numbers a and b, if $a^3 = b$, then a is a cube root of b. The cube root of b is written using a radical: $\sqrt[3]{b}$.
cube root function	a function that contains the cube root of a variable or a variable expression. The general form is $y = a\sqrt[3]{(x-h)} + k$, where a, h, and k are real numbers.
decreasing function	a function such that as the independent values increase, the dependent values decrease
domain	the set of all input values for which a relation or function is defined; the set of x-values that are valid for a relation or function

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Instruction

extrema	the minima or maxima of a function
floor function	also known as the greatest integer function; a function represented as $y=\lfloor x \rfloor$. For any input x, the output is the largest integer less than or equal to x; for example, $\lfloor -3 \rfloor = -3$, $\lfloor 2.1 \rfloor = 2$, and $\lfloor -2.1 \rfloor = -3$.
greatest integer function	also known as the floor function; a function represented as $y=\lfloor x \rfloor$. For any input x, the output is the largest integer less than or equal to x; for example, $\lfloor -3 \rfloor = -3$, $\lfloor 2.1 \rfloor = 2$, and $\lfloor -2.1 \rfloor = -3$.
increasing function	a function such that as the independent values increase, the dependent values also increase
least integer function	also known as the ceiling function; a function represented as $y=\lceil x \rceil$. For any input x, the output is the smallest integer greater than or equal to x; for example, $\lceil -3 \rceil = -3$, $\lceil 2.1 \rceil = 3$, and $\lceil -2.1 \rceil = -2$.
piecewise function	a function that is defined by two or more expressions on separate portions of the domain
point of inflection	the point on a curve where the concavity changes from concave up to concave down or vice versa
radical function	a function with the independent variable under a radical sign. The general form is $y=a\sqrt[n]{(x-h)}+k$, where n is a positive integer root and a, h, and k are real numbers.
range	the set of all outputs of a relation or function; the set of y-values for which a function is defined
restricted domain	a subset of a function's defined domain
restricted range	a subset of a function's defined range
square root	For any real numbers a and b, if $a^2 = b$, then a is a square root of b. The square root of b is written using a radical: $\sqrt{b}$.
square root function	a function that contains a square root of a variable or a variable expression
step function	a function that is a series of disconnected constant functions, and whose graph resembles a staircase

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Instruction

Recommended Resources

- AlgebraLAB. "Graphing Absolute Value Functions."

  http://www.walch.com/rr/00148

  This site introduces how to graph absolute value functions. It includes information about how adding or subtracting constants changes the graph, and provides an opportunity to describe how the graphs of different absolute value functions relate to the graph of $y = |x|$. Examples are provided along with solutions.

- Interactive Mathematics. "Split Functions (Piecewise-Defined Functions)."

  http://www.walch.com/rr/00149

  This website provides examples of piecewise functions, including step functions. Algebraic representations, graphs, and practice problems are all provided.

- MathIsFun.com. "Floor and Ceiling Functions."

  http://www.walch.com/rr/00150

  This site provides examples of step functions, including floor functions, ceiling functions, and fractional part functions. Interactive practice problems offer encouragement and coaching when the wrong answer is given.

- Southern Illinois University. "Math 150—Topic 2: Piecewise-Defined Functions."

  http://www.walch.com/rr/00151

  These review pages establish a connection between absolute value functions and piecewise functions. Sample exercises and solutions to graph absolute value and piecewise functions are included.

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Lesson 2.4.1: Square Root and Cube Root Functions

Warm-Up 2.4.1

Michael has 24 feet of wooden fencing. He is using all 24 feet of fencing to create one rectangular garden. The perimeter of a rectangle is the sum of its four sides, and the area of a rectangle is its length times its width, or *area* = *length* • *width*. For Michael's garden, if x = the length of the rectangle, then the width = $12 - x$.

1. Write a function that shows all the possible areas of rectangular gardens Michael could make with the 24 feet of fencing.

2. If x is the rectangle's length, what is the domain of x?

3. Create a graph of the function on the determined domain.

4. Which rectangle length and width will result in the largest garden area?

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Instruction

Lesson 2.4.1: Square Root and Cube Root Functions

Common Core State Standard

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

Warm-Up 2.4.1 Debrief

1. Write a function that shows all the possible areas of rectangular gardens Michael could make with the 24 feet of fencing.

   For a rectangle, *area* = *length* • *width*. If x = length, then width = $12 - x$. The area, y, of a garden with length x and width $12 - x$ is $y = (x)(12 - x)$. Simplify the equation by finding the product.

   $y = (x)(12 - x)$

   $y = 12x - x^2$

2. If x is the rectangle's length, what is the domain of x?

   The domain of x is all possible values of x. Both the length and width of the rectangle must be greater than 0.

   length > 0

   $x > 0$

   width > 0

   $12 - x > 0$

   $12 > x$

   x is greater than 0 but less than 12. The domain of x is $0 < x < 12$.

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Instruction

3. Create a graph of the function on the determined domain.

   The highest power of the function $y = 12x - x^2$ is 2, so the graph of the function will be a parabola. Evaluate the function at x-values in the domain to find a few points on the graph. Evaluate the function at the minimum and maximum x-values (0 and 12), as well as a few points in between.

   - When $x = 0$:

     $y = 12(0) - (0^2) = 0$

     Since $x > 0$, the point (0, 0) is not on the graph, but can be drawn as an open circle to represent a bound on the graph.

   - When $x = 5$:

     $y = 12(5) - (5^2) = 60 - 25 = 35$

     Point on the graph: (5, 35)

   - When $x = 6$:

     $y = 12(6) - (6^2) = 72 - 36 = 36$

     Point on the graph: (6, 36)

   - When $x = 7$:

     $y = 12(7) - (7^2) = 84 - 49 = 35$

     Point on the graph: (7, 35)

   - When $x = 12$:

     $y = 12(12) - (12^2) = 144 - 144 = 0$

     Since $x < 12$, the point (12, 0) is not on the graph, but can be drawn as an open circle to represent a bound on the graph.

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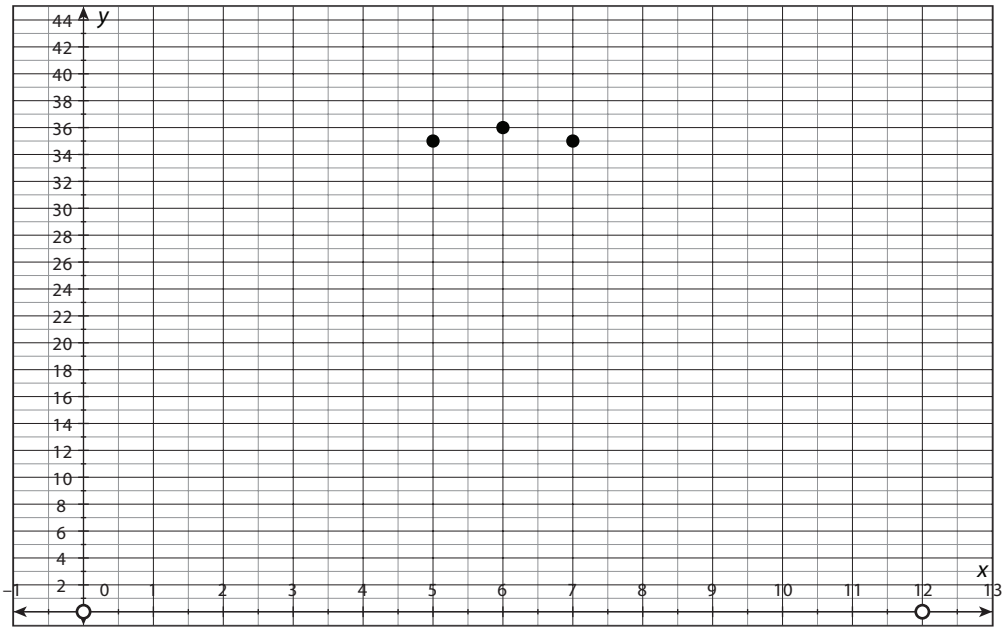
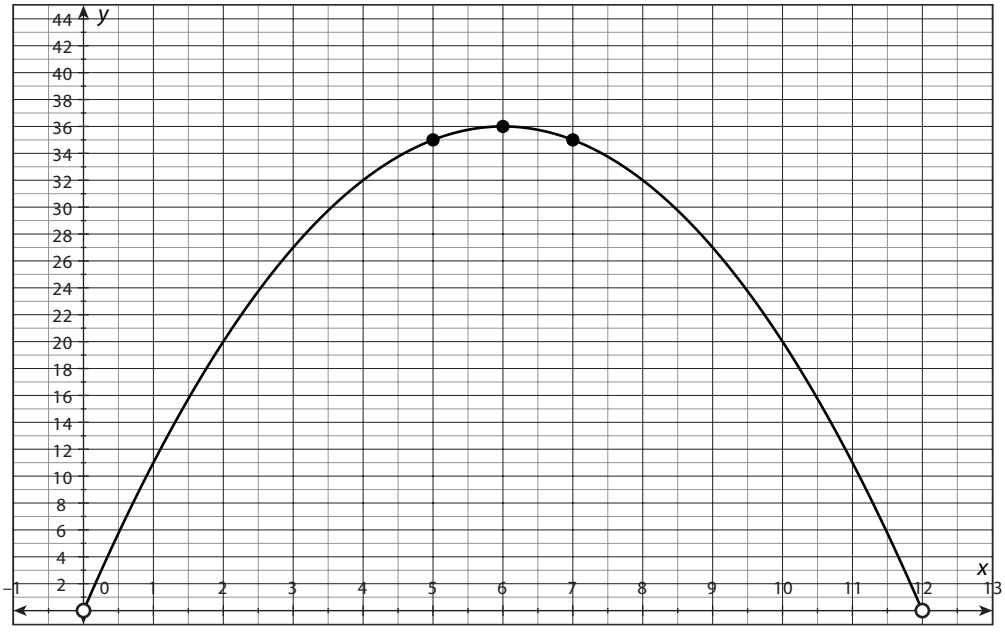
Instruction

Graph a few points on the graph to determine the general shape of the graph.



Draw a parabola using the graphed points.



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4. Which rectangle length and width will result in the largest garden area?

   The maximum area, 36 feet^2, occurs at $x = 6$. If the length = x and the width = $12 - x$, then the length and width that produce the greatest area are:

   length = x = 6 feet

   width = $12 - x = 12 - 6 = 6$ feet

   A rectangle with a length and width of 6 feet will give the largest garden area for a rectangle with a perimeter of 24 feet.

Connection to the Lesson

- In this lesson, students will graph square root functions and cube root functions.
- Students will need to determine the domain and range of a square root function before graphing.
- Students will need to find ordered pairs for a given function.
- Students will need to use ordered pairs and other key points, such as minima and maxima, to draw graphs.
- Students will analyze a graph to solve a problem.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- finding the domain and range of functions (F–IF.1)
- evaluating functions for given values (F–IF.2)
- finding ordered pairs by evaluating functions (8.F.1)
- creating graphs using ordered pairs (5.G.1)
- evaluating squares and cubes of real numbers, both with and without a calculator (8.EE.2)

Introduction

As with all functions, square root and cube root functions can be expressed algebraically and graphically. A function's equation can be used to determine values on the graph of the function. When evaluating square root functions, we must keep in mind restrictions on the domain that arise because of the fact that the square root of a negative value is not a real number. Identifying key points on the function can also help translate between algebraic and graphic representations.

Key Concepts

- The domain of a function is the set of all input values for which the function is defined. It is the set of x-values that are valid for the function.
- A **restricted domain** of a function is a subset of a function's defined domain. The application of a function to a real-world scenario often requires that the domain be restricted, or limited, to values that make sense for the scenario. For example, for a function such as $y = 2x$, the function is defined for all real values of x. However, if x is the number of tickets sold and y is the total money earned, the restricted domain of $y = 2x$ is $x \geq 0$, because the number of tickets sold cannot be negative.
- The **range** of a function is the set of all outputs of the function; in other words, the range is the set of y-values for which a function is defined.
- A **restricted range** of a function is a subset of a function's defined range. If a function's domain has been restricted to match a real-world scenario, the range will also be restricted, based on the restricted domain.
- The extrema of a function are the maxima or minima of the function and are key points on the graph.

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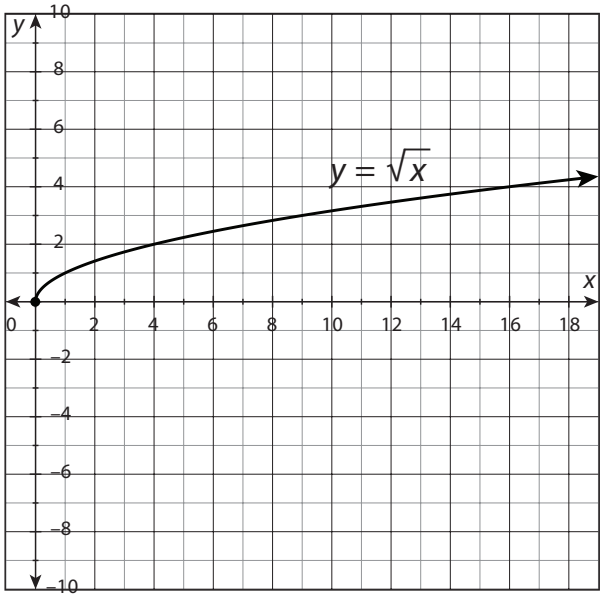
Instruction

- If the independent variable increases and the dependent variable also increases, a function is said to be an **increasing function**.
- If the independent variable increases and the dependent variable decreases, a function is said to be a **decreasing function**.
- A function with the independent variable under a radical sign is called a **radical function**.
- The general form of a radical function is $y = a\sqrt[n]{(x-h)} + k$, where n is a positive integer root and a, h, and k are real numbers.
- The domain of a radical function with the variable x is all values of x for which the radical is defined.
- The range of a radical function with the variable x is all outputs that correspond to the domain of the radical function.

Square Root Functions

- For any real numbers a and b, if $a^2 = b$, then a is a **square root** of b. The square root of b is represented using a radical symbol: $\sqrt{b}$. The square of the square root of b, or $\left(\sqrt{b}\right)^2$, is b.
- The square root of a negative value is not a real number; the square root of a negative number is only defined in the set of complex numbers.
- A **square root function** is a function that contains a square root of a variable or a variable expression. The general shape of a square root function is similar to the graph of $y = \sqrt{x}$:



- The general form of a square root function is $y = a\sqrt{(x-h)} + k$, where a, h, and k are real numbers.

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Instruction

- A square root function is always increasing if a is greater than 0 and is always decreasing if a is less than 0.
- The domain of a radical function with a root of 2 and the variable x is all values of x that result in a positive value under the radical symbol.
- For example, to find the domain of the square root function $y=\sqrt{x+7}$, solve $x + 7 \geq 0$ for x.

  $x + 7 \geq 0$

  $x \geq -7$

  The domain of $y=\sqrt{x+7}$ is $x \geq -7$.
- The range of a radical function with a root of 2 and a variable of x is the outputs that correspond to the domain of x.
- For example, to find the range of the square root function $y=\sqrt{x+7}$, use the domain of $x \geq -7$. The minimum value of the domain determines either the minimum or maximum of the range. Replace x with the minimum domain value to find the corresponding minimum or maximum of the range.

  $y=\sqrt{(-7)+7}$

  $y=\sqrt{0}$

  $y = 0$
- To determine if the remaining outputs are greater than or less than the calculated value, evaluate the function on another value in the domain.
- Since $x \geq -7$, 2 is another value in the domain of $y=\sqrt{x+7}$.

  $y=\sqrt{2+7}$

  $y=\sqrt{9}$

  $y = 3$
- The chosen value in the range of $y=\sqrt{x+7}$, 3, is greater than the calculated value at the minimum of the domain. Therefore, the y-values are increasing as x increases, and 0 is the minimum of the range. The range of $y=\sqrt{x+7}$ is $y \geq 0$.
- The graph of a square root function has an extreme value at the maximum or minimum of its domain and range.
- For example, the square root function $y=\sqrt{x+7}$ has a domain of $x \geq -7$ and a range of $y \geq 0$. The minimum of the domain is –7, and the minimum of the range is 0. The minimum of the function is the point (–7, 0).

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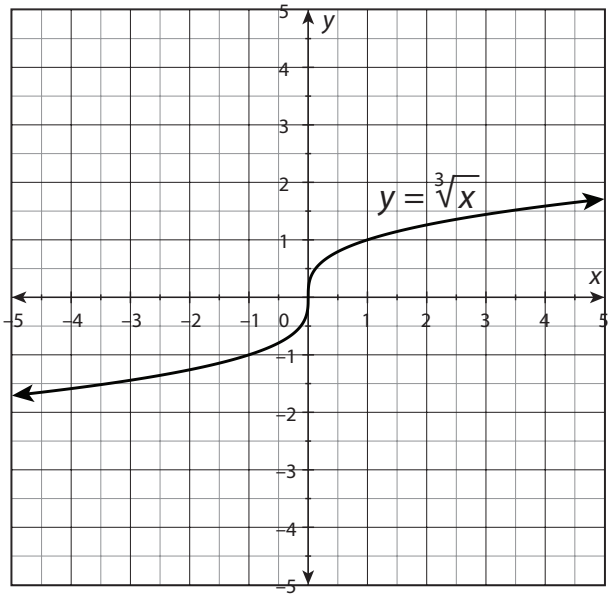
Instruction

- To graph a square root function, identify the domain and range of the function, plot the minimum or maximum of the function, and plot a few additional points to identify the shape of the function.

Cube Root Functions

- For any real numbers a and b, if $a^3 = b$, then a is a **cube root** of b. The cube root of b is written using a radical symbol: $\sqrt[3]{b}$. The cube of the cube root of b, or $\left(\sqrt[3]{b}\right)^3$, is b.
- The domain and range of a cube root function are all real numbers.
- A **cube root function** contains the cube root of a variable or a variable expression. The general shape of a cube root function is similar to the graph of $y = \sqrt[3]{x}$:



- The general form of a cube root function is $y = a\sqrt[3]{(x-h)} + k$, where a, h, and k are real numbers.
- A cube root function is always increasing if a is greater than 0 and is always decreasing if a is less than 0.
- The **point of inflection** is the point on a curve where the concavity changes from concave up to concave down or vice versa. The point of inflection for a cube root function of the form $y = a\sqrt[3]{(x-h)} + k$ is determined by the x-value where the expression under the cube root symbol equals 0.

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Instruction

- For example, let $y=\sqrt[3]{(x-1)}+3$. The expression under the cube root symbol is $(x - 1)$. Find the value of x that makes the expression equal 0 by solving $x - 1 = 0$ for x.

  $x - 1 = 0$

  $x = 1$

- Evaluate the function at the determined value of x to find the corresponding y-value.

  $y=\sqrt[3]{(x-1)}+3$

  $y=\sqrt[3]{[(1)-1]}+3$

  $y=\sqrt[3]{0}+3$

  $y = 3$

- The point (1, 3) is a point of inflection on the graph of the function $y=\sqrt[3]{(x-1)}+3$.
- To graph a cube root function, choose four x-coordinates, two on either side of the point of inflection, and substitute them into the function to find the corresponding y-coordinates. This will give you a total of five points to graph. Then, plot the five points and connect them with a smooth curve.
- Functions can also be graphed using a graphing calculator. The scale of the displayed graph can be changed to adjust the viewed portion of the graph. A square root can be written as a rational power of $\frac{1}{2}$, and a cube root can be written as a rational power of $\frac{1}{3}$. On a calculator, you can either use the square root or cube root symbols, or you can enter the exponents as fractions in parentheses: (1/2) or (1/3).

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Instruction

On a TI-83/84:

Step 1: Press [Y=]. If anything is already entered, press [CLEAR].

Step 2: Enter the expression, using x as the variable. Enter the expression in Y_1. A square root can be written as a rational exponent; for example, $y=\sqrt{x+7}$ is the same as $y=(x+7)^{\frac{1}{2}}$.

Step 3: Press [GRAPH].

To change the scale:

Step 4: Press [WINDOW].

Step 5: Set the Xmin, Xmax, Xscl, Ymin, Ymax, and Yscl.

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Enter the expression at the bottom of the application, using x as the variable. A root can be written as a rational power; for example, $y=\sqrt{x+7}$ is the same as $y=(x+7)^{\frac{1}{2}}$.

Step 4: Press [enter] after writing the expression in order to plot it on the graph.

To change the scale:

Step 5: Press [menu].

Step 6: Select 4: Window/Zoom.

Step 7: Select 1: Window Settings.

Step 8: Tab between the fields to edit the values. Set the XMin, XMax, YMin, and YMax. Leave the XScale and YScale on Auto.

Common Errors/Misconceptions

- plotting too few points on a graph to understand the behavior of the function on the graphed domain and range
- incorrectly calculating square roots and cube roots when identifying points on a graph
- forgetting the relationship between an algebraic function and key points on the graph

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Instruction

Guided Practice 2.4.1

Example 1

The function $y = 0.176\sqrt{x+30}$ has a restricted domain of $-30 \le x \le 0$. Determine the restricted range of the function, then use a graph to estimate the value of y when $x = -10$.

1. Determine the range of the function.

   The range of the function is the values of y that correspond with the restricted domain of x. A square root function is either always increasing or always decreasing, so the minimum and maximum values of the range will occur at the minimum and maximum values of the domain. Evaluate the function at the minimum and maximum domain values to find the minimum and maximum range values.

   Let $x = -30$.

$y = 0.176\sqrt{x+30}$	Equation of the square root function
$y = 0.176\sqrt{(-30)+30}$	Substitute –30 for x.
$y = 0.176\sqrt{0}$	Simplify.
$y = 0.176(0)$	
$y = 0$	

   One of the extreme values of the range is $y = 0$.

   Let $x = 0$.

$y = 0.176\sqrt{x+30}$	Equation of the square root function
$y = 0.176\sqrt{(0)+30}$	Substitute 0 for x.
$y = 0.176\sqrt{30}$	Simplify.
$y \approx 0.176(5.477)$	Use a calculator to find $\sqrt{30}$.
$y \approx 0.964$	Simplify.

   The other extreme value of the range is at about $y = 0.964$.

   Therefore, the restricted range of y is $0 \le y \le 0.964$.

SWB p. 81; SRB p. U2-101

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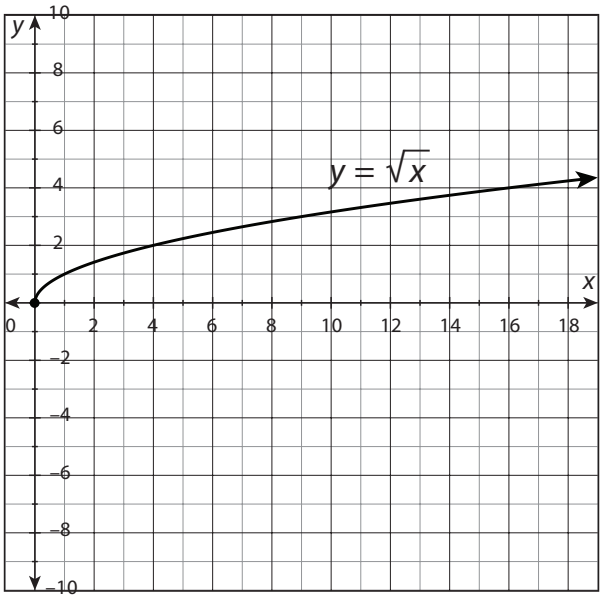
Instruction

2. Find at least three points on the graph of the function, including the end points.

Two key points on the graph of the function with a restricted domain are at the minimum and maximum. For this increasing function with a restricted domain and range, the minimum of the function occurs at the point where the x- and y-values are at their minimum values: (–30, 0). The maximum of the function occurs at the point where the x- and y-values are at their maximum values: (0, 0.964).

Evaluate the function at a value of x that is closer to –30 than 0, since the shape of the square root graph $y = \sqrt{x}$ looks like this:



Let $x = -26$.

$y = 0.176\sqrt{x+30}$ Equation of the square root function

$y = 0.176\sqrt{(-26)+30}$ Substitute –26 for x.

$y = 0.176\sqrt{4}$ Simplify.

$y = 0.176(2)$

$y = 0.352$

A third point on the function is (–26, 0.352).

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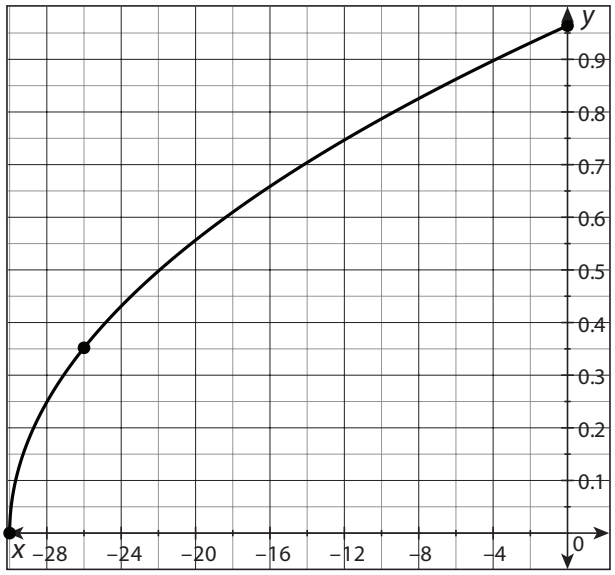
3. Plot the three points and sketch the graph.



You could alternately use a calculator to create the graph.

On a TI-83/84:

Step 1: Press [Y=]. If anything is already entered, press [CLEAR].

Step 2: Enter the expression using x as the variable, in Y_1. Write the root as a rational power: [(0.176)][×][(][(X+30)][^][(1÷2)][)].

Step 3: Press [GRAPH].

To change the scale:

Step 4: Press [WINDOW].

Step 5: Set the Xmin to –30, Xmax to 0, Xscl to 2, Ymin to 0, Ymax to 1, and Yscl to 0.1.

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Enter the expression at the bottom of the application, using x as the variable. Write the root as a rational power: [(0.176)][×][(][(X+30)][^][(1÷2)][)].

Step 4: Press [enter] to plot the expression on the graph.

(*continued*)

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To change the scale:

Step 5: Press [menu].

Step 6: Select 4: Window.

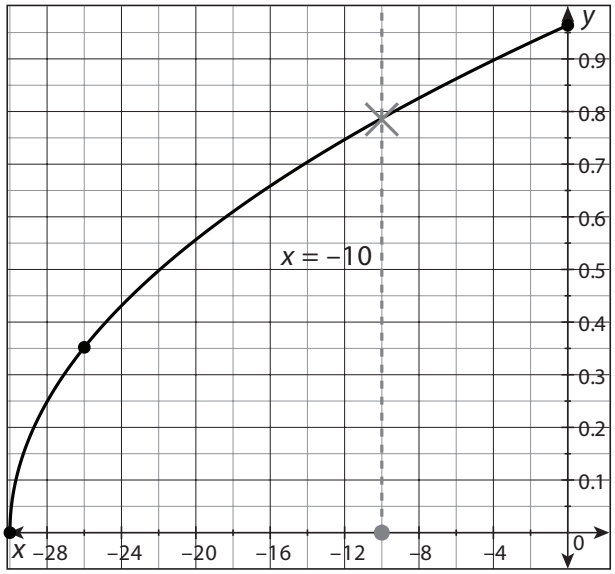
Step 7: Select 1: Window Settings.

Step 8: Tab between the fields to edit the values. To see the intersection of the two graphs, set the XMin to –30, XMax to 0, YMin to 0, and YMax to 1. Leave the XScale and YScale on Auto.

4. Use the graph to estimate the function's output at the given input.

The problem asks to estimate the value of y when $x = -10$.

Find $x = -10$ on the graph.



(*continued*)

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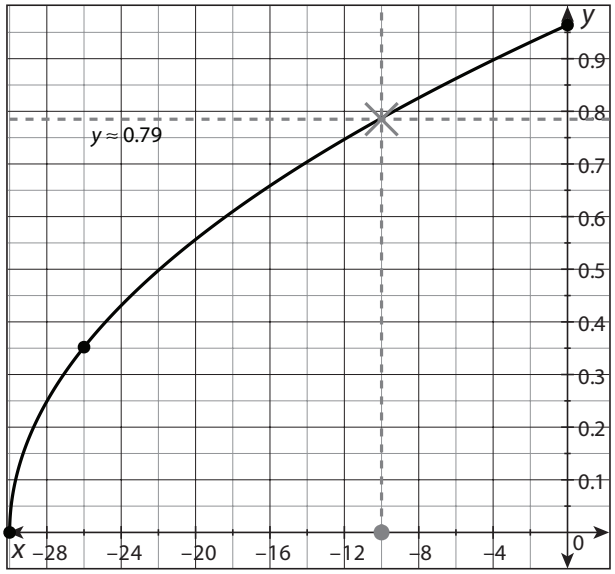
Instruction

Approximate the y-value.



When $x = -10$, y is approximately 0.79.

Check this answer by substituting –10 for x in the original equation and solving for y.

$y = 0.176\sqrt{x+30}$	Original equation
$y = 0.176\sqrt{(-10)+30}$	Substitute –10 for x.
$y = 0.176\sqrt{20}$	Simplify.
$y \approx 0.787$	

The estimation from the graph is accurate.

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Instruction

Example 2

Compare the domain, range, graph, and points of inflection on the graph of $y=-4\sqrt[3]{x}+2$ to the graph of $y=\sqrt[3]{x}$. How are these differences reflected in the algebraic equations?

1. Determine the domain and range of $y=\sqrt[3]{x}$.

   The function $y=\sqrt[3]{x}$ is defined for all real values of x, and the coordinating outputs are all real values of y. The domain and range of $y=\sqrt[3]{x}$ are both all real numbers.

2. Find at least five points on the graph of $y=\sqrt[3]{x}$, including the point of inflection.

   The point of inflection for a function of the form $y=a\sqrt[3]{(x-h)}+k$ is determined by the x-value where the expression under the cube root symbol equals 0. In this case, the expression under the cube root symbol is x, so the point of inflection occurs when $x = 0$. Evaluate the function at $x = 0$ to find the y-coordinate of the point.

$y=\sqrt[3]{x}$	Equation of the cube root function
$y=\sqrt[3]{0}=0$	Substitute 0 for x and solve.

   The point of inflection is at (0, 0).

   Next, evaluate the function for four more x-values, two on either side of 0. Evaluate the function at $x = 1$, $x = –1$, $x = 8$, and $x = –8$ because these numbers are perfect cubes, which simplifies the evaluation process.

$y=\sqrt[3]{x}$	Equation of the cube root function
$y=\sqrt[3]{(1)}=1$	Substitute 1 for x and simplify.

   There is a point at (1, 1).

$y=\sqrt[3]{x}$	Equation of the cube root function
$y=\sqrt[3]{(-1)}=-1$	Substitute –1 for x and simplify.

   There is a point at (–1, –1).

(*continued*)

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Instruction

$y = \sqrt[3]{x}$ Equation of the cube root function

$y = \sqrt[3]{(8)} = 2$ Substitute 8 for x and simplify.

There is a point at (8, 2).

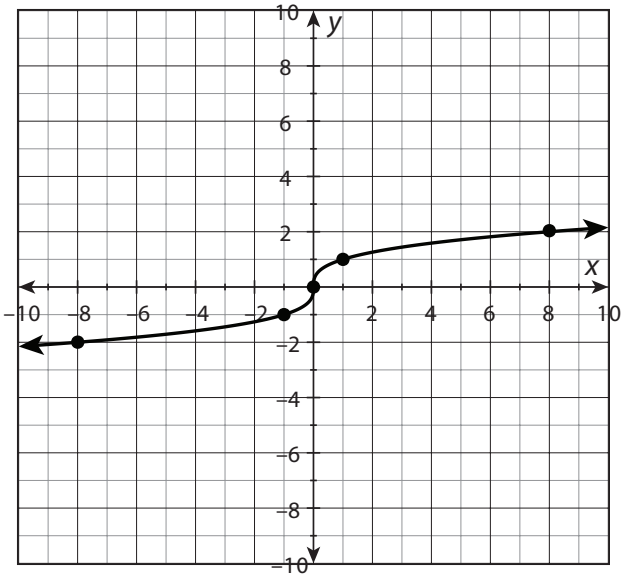
$y = \sqrt[3]{x}$ Equation of the cube root function

$y = \sqrt[3]{(-8)} = -2$ Substitute –8 for x and simplify.

There is a point at (–8, –2).

3. Plot the points and graph $y = \sqrt[3]{x}$.

Plot the five known points and connect them with a smooth curve.



A calculator can also be used to create the graph.

On a TI-83/84:

Step 1: Press [Y=]. If anything is already entered, press [CLEAR].

Step 2: Enter the expression, using x as the variable, in Y_1. Write the root as the rational power, $x^{\frac{1}{3}}$: [(][X][^][(][1][÷][3][)][)].

Step 3: Press [GRAPH].

(*continued*)

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Instruction

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Enter the expression at the bottom of the application, using x as the variable. Write the root as the rational power, $x^{\frac{1}{3}}$:

[(][X][^][(][1][÷][3][)][)].

Step 4: Press [enter] to plot the expression on the graph.

4. Determine the domain of $y=-4\sqrt[3]{x}+2$.

   A cube root function is defined on any real value of x, so the domain is all real numbers.

5. Determine the range of $y=-4\sqrt[3]{x}+2$.

   A cube root function with a domain of all real numbers also has a range of all real numbers.

6. Find at least five points on the function $y=-4\sqrt[3]{x}+2$, including the point of inflection.

   The point of inflection for a function of the form $y=a\sqrt[3]{(x-h)}+k$ is determined by the x-value where the expression under the cube root symbol equals 0. In this case, the expression under the cube root symbol is x, so the point of inflection occurs when $x = 0$. Evaluate the function at $x = 0$ to find the y-coordinate of the point.

$y=-4\sqrt[3]{x}+2$	Equation of the cube root function
$y=-4\sqrt[3]{(0)}+2$	Substitute 0 for x.
$y = -4(0) + 2$	Simplify.
$y = 2$	

   The point of inflection is at (0, 2).

(*continued*)

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Instruction

Next, evaluate the function for four more x-values, two on either side of 0. Evaluate the function at $x = 1$, $x = -1$, $x = 8$, and $x = -8$ because these numbers are perfect cubes, which simplifies the evaluation process.

$y = -4\sqrt[3]{x} + 2$	Equation of the cube root function
$y = -4\sqrt[3]{(1)} + 2$	Substitute 1 for x.
$y = -4(1) + 2$	Simplify.
$y = -2$	

There is a point at (1, –2).

$y = -4\sqrt[3]{x} + 2$	Equation of the cube root function
$y = -4\sqrt[3]{(-1)} + 2$	Substitute –1 for x.
$y = -4(-1) + 2$	Simplify.
$y = 6$	

There is a point at (–1, 6).

$y = -4\sqrt[3]{x} + 2$	Equation of the cube root function
$y = -4\sqrt[3]{(8)} + 2$	Substitute 8 for x.
$y = -4(2) + 2$	Simplify.
$y = -6$	

There is a point at (8, –6).

$y = -4\sqrt[3]{x} + 2$	Equation of the cube root function
$y = -4\sqrt[3]{(-8)} + 2$	Substitute –8 for x.
$y = -4(-2) + 2$	Simplify.
$y = 10$	

There is a point at (–8, 10).

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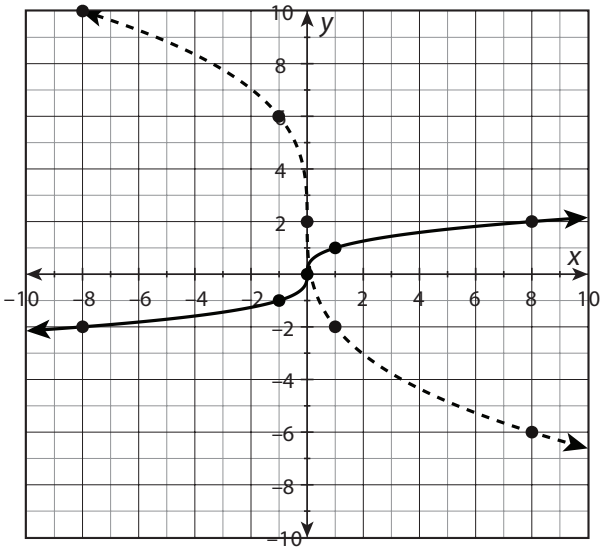
Instruction

7. Plot the points and graph $y=-4\sqrt[3]{x}+2$ on the same coordinate plane as the graph of $y=\sqrt[3]{x}$.

   Plot the five points and connect them with a smooth curve.



   The functions can also be graphed on a calculator. If the function $y=\sqrt[3]{x}$ is already graphed, add the graph of $y=-4\sqrt[3]{x}+2$. On a TI-83 or 84, enter the second function in Y_2. On a TI-Nspire, enter the second function at the bottom of the screen.

8. Compare the domains and ranges of the functions $y=\sqrt[3]{x}$ and $y=-4\sqrt[3]{x}+2$.

   The domain and range of both functions are the same: all real numbers.

9. Compare the points of inflection of $y=\sqrt[3]{x}$ and $y=-4\sqrt[3]{x}+2$, and note how the equations are related to the differences between these points.

   The *x*-values of the points of inflection, where the expression under the cube root equals 0, is the same for both functions. However, the point of inflection of the graph of $y=-4\sqrt[3]{x}+2$ is shifted up 2 units due to the "+ 2" outside the cube root.

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Instruction

10. Compare the general shapes of $y=\sqrt[3]{x}$ and $y=-4\sqrt[3]{x}+2$, and note how the equations are related to the differences.

The function $y=\sqrt[3]{x}$ is always increasing, whereas the function $y=-4\sqrt[3]{x}+2$ is always decreasing because of the negative factor outside the cube root symbol, which reflects the graph over the y-axis. The graph of $y=-4\sqrt[3]{x}+2$ is stretched vertically, due to the factor of –4. The graph of the function $y=-4\sqrt[3]{x}+2$ is shifted up 2 units, due to the addend of + 2 outside the cube root.

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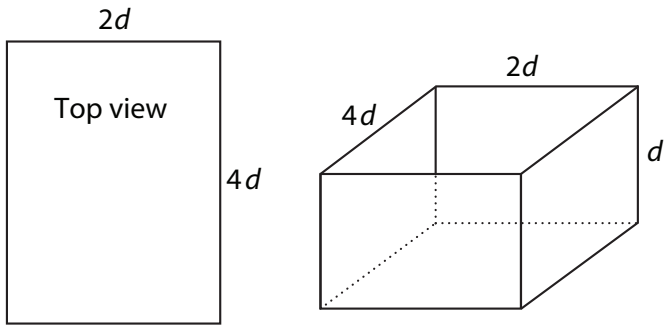
Instruction

Example 3

The Stephens family is building a rectangular inground pool. The depth, length, and width will all be related, as shown in the diagrams.



The depth is d, the length is $4d$, and the width is $2d$. All dimensions are measured in feet. The volume of the pool is the product of the three dimensions: $4d \bullet 2d \bullet d = 8d^3$. The depth, d, can be represented for a pool with any volume using the function $d = \sqrt[3]{\frac{1}{8}V} = \frac{1}{2}\sqrt[3]{V}$. The Stephens family would like a pool with a volume of at least 1,000 ft^3 and no more than 11,000 ft^3. Create a graph to show the Stephens family all the possible values of d for the desired volumes.

1. Determine the domain of the function.

   The domain is the set of all possible input values. The input values for the function $d = \frac{1}{2}\sqrt[3]{V}$ are all possible volumes of the pool. The cube root is defined for all values of V, but the domain is restricted by the application of the function. The minimum volume is 1,000 ft^3, and the maximum volume is 11,000 ft^3. The restricted domain of the function is $1000 \leq V \leq 11{,}000$.

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2. Determine the range of the function.

The range is the set of all possible outputs, based on the inputs. The general form of a cube root function is $y = a\sqrt[3]{(x-h)} + k$. If $a > 0$, then the function is increasing. The minimum value of the range will be at the minimum value of the domain, and the maximum value of the range will be at the maximum value of the domain. The function $d = \frac{1}{2}\sqrt[3]{V}$ has an a value of $\frac{1}{2}$, so the function is increasing. Evaluate the function at the minimum value of the domain to find the minimum value of the range.

$d = \frac{1}{2}\sqrt[3]{V}$ Equation of the cube root function

$d = \frac{1}{2}\sqrt[3]{1000}$ Substitute 1,000 for V.

$d = \frac{1}{2}(10)$ Simplify.

$d = 5$

The minimum of the range is 5 feet.

Evaluate the function at the maximum value of the domain to find the maximum value of the range.

$d = \frac{1}{2}\sqrt[3]{V}$ Equation of the cube root function

$d = \frac{1}{2}\sqrt[3]{11{,}000}$ Substitute 11,000 for V.

Use a calculator to approximate the cube root of 11,000.

$d \approx \frac{1}{2}(22.2)$ Simplify using the approximated value of $\sqrt[3]{11{,}000}$.

$d \approx 11.1$

The maximum value of the range is approximately 11.1 feet.

The restricted range of the function is approximately $5 \leq d \leq 11.1$.

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Instruction

3. Find several points on the graph of the function.

Two points on the graph of the function are at the minimum and maximum. For this increasing function with a restricted domain and range, the minimum of the function is the point at the minimum input and output: (1,000, 5). The maximum of the function is the point at the maximum input and output: (11,000, 11.1).

A cube root function has a point of inflection where the quantity under the cube root symbol equals 0. For the function $d = \frac{1}{2}\sqrt[3]{V}$, when the quantity under the cube root equals 0 ($V = 0$), it is less than the minimum value of the restricted domain, 1,000. The point of inflection is not on the portion of the function related to the problem statement.

Find a third point whose x-coordinate is in the domain of the function. The value 1,728 is a perfect cube (12 • 12 • 12) and is in the domain of V. Evaluate the function at $V = 1728$.

$d = \frac{1}{2}\sqrt[3]{V}$ Equation of the cube root function

$d = \frac{1}{2}\sqrt[3]{1728}$ Substitute 1,728 for V.

$d = \frac{1}{2}(12)$ Simplify.

$d = 6$

The point (1,728, 6) is a third point on the graph.

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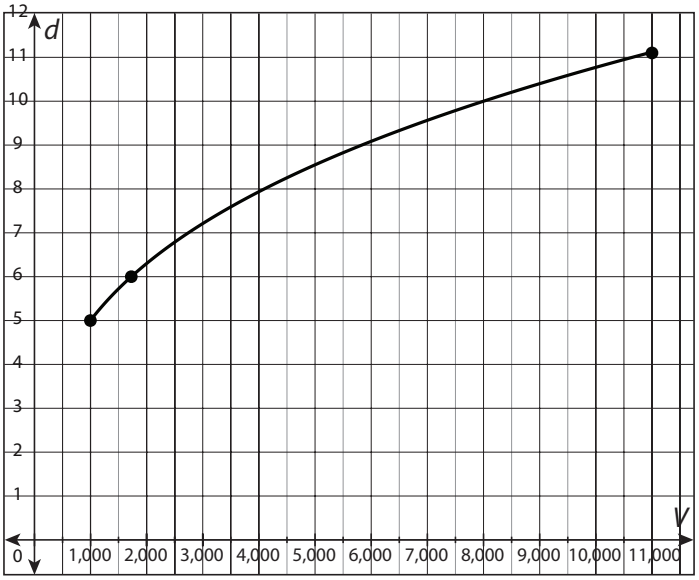
Instruction

4. Plot the points and sketch the graph.



A calculator can also be used to create a graph of a cube root function.

On a TI-83/84:

Step 1: Press [Y=]. If anything is already entered, press [CLEAR].

Step 2: Enter the function, using x as the variable, in Y_1. Write the root as the rational power, $\left(\frac{1}{2}\right)x^{\frac{1}{3}}$: [(][1][÷][2][)][×][(][X][^][(][1][÷][3][)][)].

Step 3: Press [GRAPH].

To change the scale:

Step 4: Press [WINDOW].

Step 5: Set the Xmin to 1,000, Xmax to 11,000, Xscl to 1,000, Ymin to 0, Ymax to 12, and Yscl to 1.

(*continued*)

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Instruction

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Enter the function at the bottom of the application, using x as the variable. Write the root as the rational power, $\frac{1}{2}x^{\frac{1}{3}}$: [(][1][÷][2][)][×][(][X][^][(][1][÷][3][)][)].

Step 4: Press [enter] to plot the function on the graph.

To change the scale:

Step 5: Press [menu].

Step 6: Select 4: Window.

Step 7: Select 1: Window Settings.

Step 8: Tab between the fields to edit the values. To see the intersection of the two graphs, set the XMin to 1,000, XMax to 11,000, YMin to 0, and YMax to 12. Leave the XScale and YScale on Auto.

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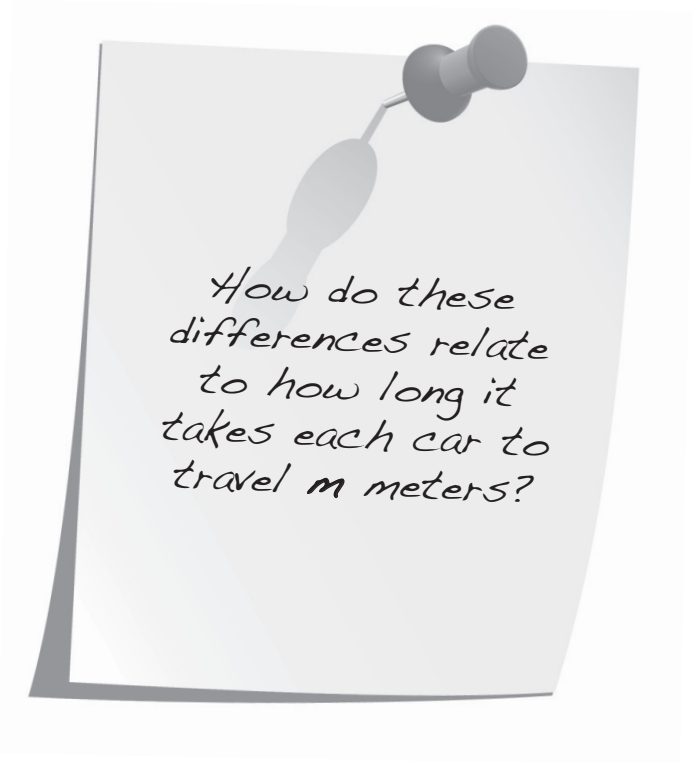
Problem-Based Task 2.4.1: Auto Acceleration

SMP	
1	2 ✓
3	4 ✓
5 ✓	6
7	8

Car manufacturers test how quickly a car is capable of accelerating by recording how long it takes the car to accelerate from 0 miles per hour to 60 miles per hour. Sofasta Motors' newest car, the Accelerator, reached 60 miles per hour in 8.1 seconds, which is an acceleration of 3.3 meters per second^2. Vroomy Vehicles' car, the VV1, reached 60 miles per hour in 9.1 seconds, which is an acceleration of 2.9 meters per second^2. The time t_a, in seconds, it takes the Accelerator to travel m meters can be represented using the function $t_a = 0.778\sqrt{m}$. The time t_v, in seconds, it takes the VV1 to travel m meters can be represented using the function $t_v = 0.830\sqrt{m}$. How do the differences in the graphs of the functions relate to the differences in the algebraic representations? How do these differences relate to how long it takes each car to travel m meters?



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Problem-Based Task 2.4.1: Auto Acceleration

Coaching

a. What is a reasonable domain for the function $t_a = 0.778\sqrt{m}$?

b. What is the corresponding range?

c. What are three points on the graph of $t_a = 0.778\sqrt{m}$?

d. Use the domain, range, and three points to sketch the graph of $t_a = 0.778\sqrt{m}$.

e. What is a reasonable domain for the function $t_v = 0.830\sqrt{m}$?

f. What is the corresponding range?

g. What are three points on the graph of $t_v = 0.830\sqrt{m}$?

h. Use the domain, range, and three points to create the graph of $t_v = 0.830\sqrt{m}$ on the same coordinate plane as the graph of t_a.

i. How are the two graphs of the functions similar?

j. How are the two graphs of the functions different?

k. How are the algebraic representations of each function related to the similarities and differences between the two graphs?

l. What do these differences show about the time it takes the two cars to travel the same distance?

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Instruction

Problem-Based Task 2.4.1: Auto Acceleration

Coaching Sample Responses

a. What is a reasonable domain for the function $t_a = 0.778\sqrt{m}$?

The square root of a negative number is not a real number, so the function is only defined for $m \geq 0$.

The domain for this function is $m \geq 0$.

b. What is the corresponding range?

Because the domain is $m \geq 0$, the range can be found by substituting $m = 0$ into the function.

$t_a = 0.778\sqrt{m}$

$t_a = 0.778\sqrt{0}$

$t_a = 0.778(0)$

$t_a = 0$

The corresponding range is $t_a \geq 0$.

c. What are three points on the graph of $t_a = 0.778\sqrt{m}$?

The minimum point, at the beginning of the domain and range, is one key point on the graph.

Minimum: (0, 0)

Evaluate the function at two values of m that are greater than 0. Let's use $m = 4$ and $m = 10$.

Let $m = 4$.	**Let $m = 10$.**
$t_a = 0.778\sqrt{4}$	$t_a = 0.778\sqrt{10}$
$t_a = 0.778(2)$	$t_a \approx 0.778(3.162)$
$t_a = 1.556$	$t_a \approx 2.460$

Two more points on the graph of the function are (4, 1.556) and about (10, 2.460).

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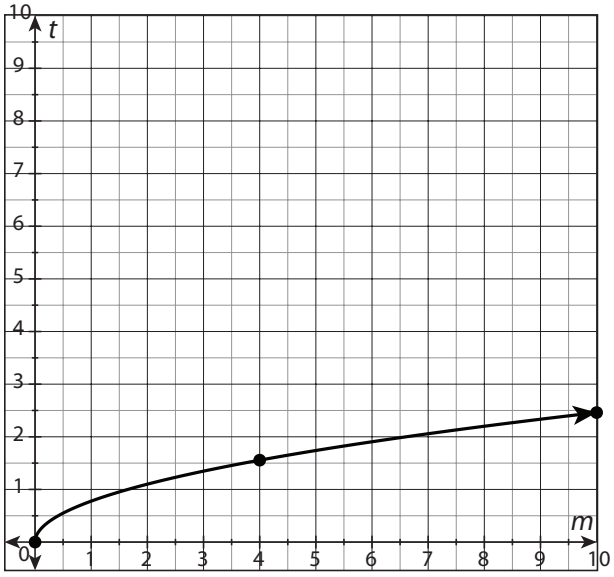
d. Use the domain, range, and three points to sketch the graph of $t_a = 0.778\sqrt{m}$.



e. What is a reasonable domain for the function $t_v = 0.830\sqrt{m}$?

The square root of a negative number is not a real number, so the function is only defined for $m \geq 0$. Use the same domain restriction as the function t_a: $m \geq 0$.

f. What is the corresponding range?

Because the domain is $m \geq 0$, the range can be found by substituting $m = 0$ into the function.

$t_v = 0.830\sqrt{m}$

$t_v = 0.830\sqrt{0}$

$t_v = 0.830(0)$

$t_v = 0$

The corresponding range is $t_v \geq 0$.

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g. What are three points on the graph of $t_v = 0.830\sqrt{m}$?

The minimum point, at the beginning of the domain and range, is one key point on the graph.

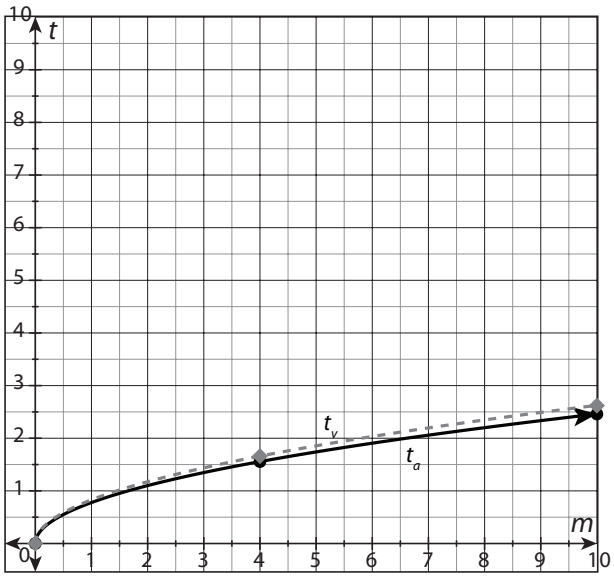
Minimum: (0, 0)

Evaluate the function at two values of m that are greater than 0. Let's use $m = 4$ and $m = 10$.

Let $m = 4$.	**Let $m = 10$.**
$t_v = 0.830\sqrt{4}$	$t_v = 0.830\sqrt{10}$
$t_v = 0.830(2)$	$t_v \approx 0.830(3.162)$
$t_v = 1.66$	$t_v \approx 2.624$

Two more points on the graph of the function are (4, 1.66) and about (10, 2.624).

h. Use the domain, range, and three points to create the graph of $t_v = 0.830\sqrt{m}$ on the same coordinate plane as the graph of t_a.



i. How are the two graphs of the functions similar?

Both graphs have the same general shape and same minimum at (0, 0).

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j. How are the two graphs of the functions different?

Except for the point (0, 0), the graph of t_v is always above the graph of t_a, and the distance between the two graphs is increasing as x gets larger.

k. How are the algebraic representations of each function related to the similarities and differences between the two graphs?

The function t_v has a greater number being multiplied by the square root, and this graph is always above the graph of the function t_a. Neither function has any addends, either under the square root symbol or outside it, which is why both graphs have a minimum at (0, 0). Both graphs are increasing, because the number that is multiplied by the square root is positive.

l. What do these differences show about the time it takes the two cars to travel the same distance?

The outputs of both functions are time to travel a given distance. The car that has the greater t-values takes longer to travel a given distance. If the two cars accelerate following the functions t_a and t_v, and start accelerating at the same time, the Accelerator will always be behind the VV1.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2.4.1: Square Root and Cube Root Functions

For problems 1–4, create a graph of each function. Note the domain, range, and key points for each function.

1. $y=\sqrt{x+1}$

2. $y=2\sqrt[3]{x}$

3. $y=-\sqrt[3]{x-4}+3$

4. $y=3\sqrt{x}-10$

For problems 5 and 6, create a graph showing each pair of functions. Describe any similarities and differences between the graphs, including domain, range, maximum or minimum points, and points of inflection. Describe how any similarities and differences are shown in the algebraic functions.

5. $f(x)=\sqrt{x}$ and $g(x)=-\sqrt{x+8}$

6. $f(x)=\sqrt[3]{x}$ and $g(x)=5\sqrt[3]{x}+9$

continued

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Use the following information to complete problems 7 and 8.

The radius of a sphere, r, with a given volume, V, is $r = \sqrt[3]{\left(\frac{3}{4\pi}\right)} \bullet \sqrt[3]{V}$.

7. Create a graph to show the radii for spheres with different volumes. Only include a reasonable domain and range in the graph.

8. The volume of an NBA basketball is between 433.5 and 450.3 in^3. Use the graph of $r = \sqrt[3]{\left(\frac{3}{4\pi}\right)} \bullet \sqrt[3]{V}$ to estimate the range of the radii of NBA basketballs.

Read the following scenario and use the information in it to complete problems 9 and 10.

Two soccer players, Aurora and Isla, each kick a ball from the same goal line. The balls each decelerate as they travel down the soccer field. The time in seconds, t, it takes Aurora's ball to travel m meters is $t = 1.061\sqrt{m}$. The time it takes Isla's ball to travel m meters is $t = 0.849\sqrt{m}$.

9. Graph the two functions on the same coordinate plane. Describe any similarities and differences between the graphs.

10. Using the graphs from problem 9, compare how long it will take Aurora's ball and Isla's ball to travel the 90-meter length of a soccer field.

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Lesson 2.4.2: Absolute Value and Step Functions

Warm-Up 2.4.2

Emery works at a clothing store. He earns extra money based on how much clothing he sells. He earns 15% on the sales he makes, in addition to the flat salary of $400 he makes each week. He can determine how much money he earns in a week, m, by using the following equation: $m = 0.15s + 400$, where s represents his weekly sales, in dollars.

1. If Emery sells $1,400 worth of clothing in a week, how much money will he earn that week?

2. Create a graph that shows Emery's weekly earnings for any sales amount.

3. Use the graph to estimate how much Emery needs to sell in order to earn $750 in a week.

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Instruction

Lesson 2.4.2: Absolute Value and Step Functions

Common Core State Standard

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

Warm-Up 2.4.2 Debrief

1. If Emery sells $1,400 worth of clothing in a week, how much money will he earn that week?

   $1,400 is the amount of Emery's weekly sales. To determine how much he earns for the week, replace s with 1,400 in the equation $m = 0.15s + 400$.

$m = 0.15s + 400$	Original equation
$m = 0.15(1400) + 400$	Substitute 1,400 for s.
$m = 210 + 400$	Simplify.
$m = 610$	

   If Emery sells $1,400 worth of clothing in a week, he will earn $610.

2. Create a graph that shows Emery's weekly earnings for any sales amount.

   The highest power of the variable in the equation $m = 0.15s + 400$ is 1, so the equation is linear. Two points are needed to graph a linear equation. One point, determined by the work in the previous problem, is (1,400, 610). Find a second point on the line at $s = 0$.

$m = 0.15s + 400$	Original equation
$m = 0.15(0) + 400$	Substitute 0 for s.
$m = 0 + 400$	Simplify.
$m = 400$	

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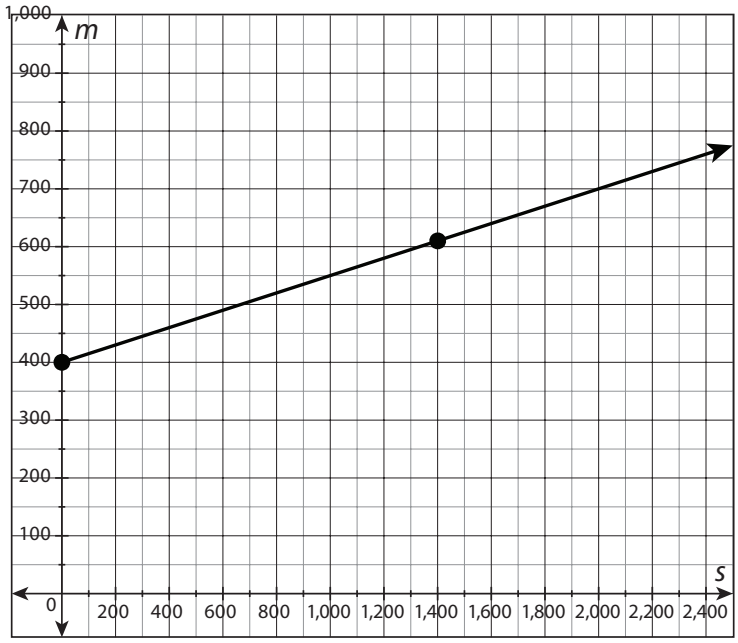
Plot the two points, then draw a line through the points.



3. Use the graph to estimate how much Emery needs to sell in order to earn $750 in a week.

   Find $750 on the *y*-axis. Find the point on the function where $m = 750$, and estimate the value of *s*. To earn $750, Emery needs to sell approximately $2,300 worth of clothes.

Connection to the Lesson

- Students will determine restricted domains and ranges to fit problem statements and use graphs to solve application problems.
- Students will graph absolute value functions and step functions using algebraic representations.
- Students will evaluate functions at different inputs to find points on the graph.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- graphing a linear function (A–CED.2★)
- finding the absolute value of a quantity (6.NS.7c)
- finding domains and ranges of functions (F–IF.1)
- differentiating between inputs and outputs (F–IF.2)
- determining restricted domains and ranges for application problems (F–IF.4★)

Introduction

A function defined for the domain of all real numbers can still have a graph that is disconnected or that suddenly changes direction. Both absolute value functions and step functions can be evaluated at any real input, but the ranges of the graphs are restricted by the functions. The graph of an absolute value function includes a sharp corner, called the vertex, and the graph of a step function looks like a series of disconnected horizontal segments.

Key Concepts

Absolute Value Functions

- The **absolute value** of a number is its distance from 0 on a number line. An absolute value is the positive value of a quantity. For example, the absolute value of 2, written as $|2|$, is 2, and $|-2|=2$.
- An **absolute value function** is a function with a variable inside absolute value symbols.
- The general form of an absolute value function is $y=a|x-h|+k$.
- The domain of an absolute value function is all real numbers.

SRB p. U2-119

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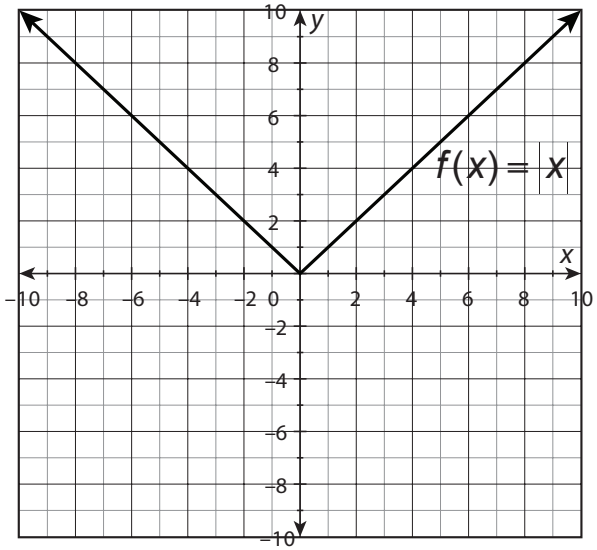
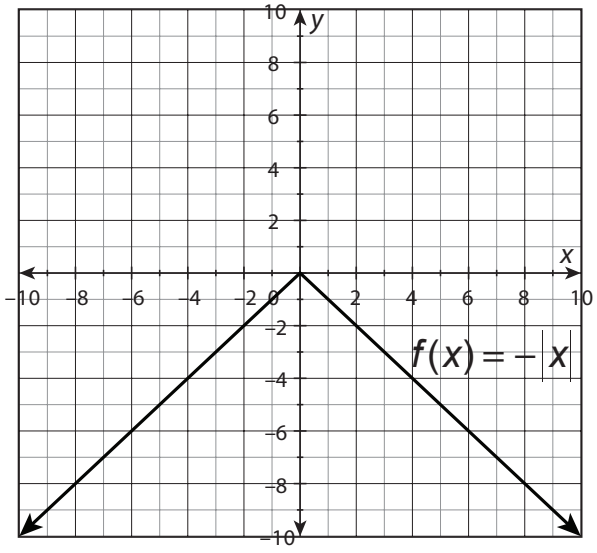
Instruction

- If a in the general form $y = a|x - h| + k$ is positive, the function has the following general shape.



- If a in the general form $y = a|x - h| + k$ is negative, the function has the following general shape.



- For an absolute value function of the form $y = a|x - h| + k$, there is one extreme value for the range at $y = k$. This occurs at the vertex of the absolute value graph. If a in the general form $y = a|x - h| + k$ is positive, the extreme value is at the minimum of the range, and there is no maximum of the range. If a in the general form $y = a|x - h| + k$ is negative, the extreme value is at the maximum of the range, and there is no minimum of the range.
- For example, let $y = |x + 4| - 5$. The minimum range value of this absolute value function is $y = -5$, and there is no maximum of the range. The range of the function is $y \geq -5$.
- The coordinates of the vertex of the graph of an absolute value function are the values of h and k when the function is in standard form, $y = a|x - h| + k$.

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- The graph of an absolute value function consists of two rays that meet at the vertex.
- To graph an absolute value function, first find the vertex. Next, plot one point to the left and one point to the right of the vertex. Finally, draw two rays, starting at the vertex and extending through each plotted point.
- In applications, one or both of these rays may instead be line segments, based on the restricted domain and range of the application.
- The graph of an absolute value function can be viewed as the graph of two linear functions, each on a restricted domain, meeting at the vertex of the absolute value function.
- A graphing calculator can be used to graph an absolute value function.

On a TI-83/84:

Step 1: Press [Y=]. If anything is already entered, press [CLEAR].

Step 2: Press [MATH]. Arrow to "NUM," then select 1: abs(.

Step 3: Enter the expression contained within the absolute value bars, using *x* as the variable. Enter the remaining part of the expression by moving the cursor to the left and/or right of the absolute value bars.

Step 4: Press [GRAPH].

To change the scale:

Step 5: Press [WINDOW].

Step 6: Set the Xmin, Xmax, Xscl, Ymin, Ymax, and Yscl.

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Enter the expression at the bottom of the application, using *x* as the variable. Type "abs(x)" to enter an absolute value; for example, typing "abs(x+4)–5" is equivalent to the expression $|x+4|-5$.

Step 4: Press [enter] to plot the expression on the graph.

To change the scale:

Step 5: Press [menu].

Step 6: Select 4: Window.

Step 7: Select 1: Window Settings.

Step 8: Tab between the fields to edit the values. Set the XMin, XMax, YMin, and YMax. Leave the XScale and YScale on Auto.

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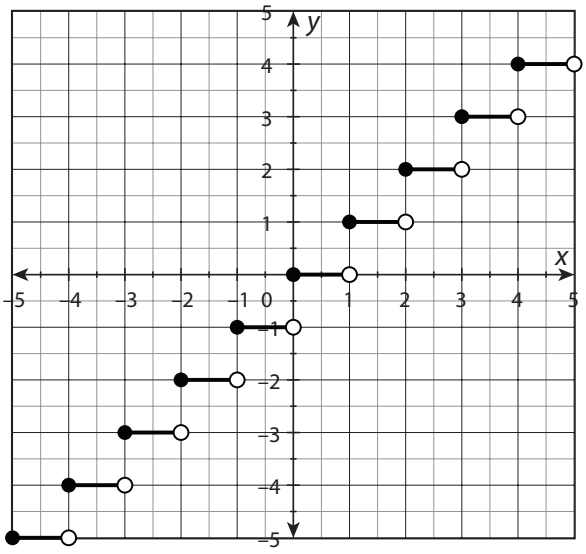
Step Functions

- A **step function** is a function that is a series of disconnected constant functions. Each *x*-value is assigned only one *y*-value on the function. The graph of a step function resembles a staircase.
- One example of a step function is the **greatest integer function**, or **floor function**, represented using $\lfloor\ \rfloor$: $y=\lfloor x \rfloor$. For any input *x*, the output is the largest integer less than or equal to *x*. For example, $\lfloor -3 \rfloor = -3$, $\lfloor 2.1 \rfloor = 2$, and $\lfloor -2.1 \rfloor = -3$.
- Points that are approached by the function, but not included, can be represented by open circles.
- The graph of $y=\lfloor x \rfloor$, which can be used to create variations of the greatest integer function, is:



- The domain of the greatest integer function is all real numbers, since the function can be evaluated at any input.
- The outputs of the greatest integer function are integers. Multiplying the greatest integer function by a factor, or adding a constant, will change the type of outputs of the function.
- The range of the function depends on whether the greatest integer function has an addend or is being multiplied by a constant.
- The jumps from one step to the next in the graph occur at *x*-values where the expression in the greatest integer symbols is an integer.
- The output of the greatest integer function is constant on each step.
- The steps of a greatest integer function are either always increasing or always decreasing.
- A graphing calculator can be used to graph the greatest integer function.

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Instruction

On a TI-83/84:

Step 1: Press [Y=]. If anything is already entered, press [CLEAR].

Step 2: Press [MATH]. Arrow to "NUM," then select 5: int(.

Step 3: Enter the expression contained within the greatest integer function, using x as the variable. Enter the remaining part of the expression by moving the cursor to the left and/or right of the int() expression. For example, "int(x+3)" is equivalent to the expression $\lfloor x+3 \rfloor$.

Step 4: Press [GRAPH].

To change the scale:

Step 5: Press [WINDOW].

Step 6: Set the Xmin, Xmax, Xscl, Ymin, Ymax, and Yscl.

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Enter the expression at the bottom of the application, using x as the variable. Press the catalog key (to the right of the number 9). Select floor() to enter a greatest integer expression; for example, typing "floor(x+3)" is equivalent to the expression $\lfloor x+3 \rfloor$.

Step 4: Press [enter] to plot the expression on the graph.

To change the scale:

Step 5: Press [menu].

Step 6: Select 4: Window.

Step 7: Select 1: Window Settings.

Step 8: Tab between the fields to edit the values. Set the XMin, XMax, YMin, and YMax. Leave the XScale and YScale on Auto.

- Another example of a step function is the **least integer function**, or **ceiling function**. It is represented using $\lceil\ \rceil$: $y=\lceil x \rceil$. For any input x, the output is the smallest integer greater than or equal to x. For example, $\lceil -3 \rceil = -3$, $\lceil 2.1 \rceil = 3$, and $\lceil -2.1 \rceil = -2$.

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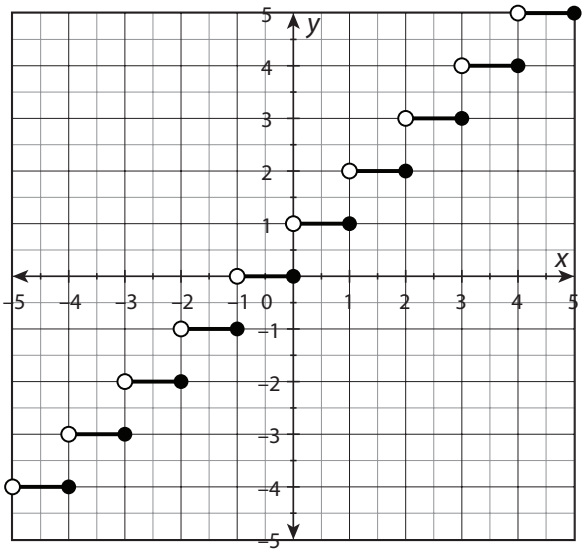
Instruction

- The graph of $y = \lceil x \rceil$, which can be used to create variations of the least integer function, is shown as follows.



- The domain of the least integer function is all real numbers, since it can be evaluated at any input.
- The outputs of the least integer function are integers. Multiplying the least integer function by a factor, or adding a constant, will change the type of outputs of the function.
- The range of the function depends on whether the least integer function has an addend or is being multiplied by a constant.
- The jumps from one step to the next in the graph occur at *x*-values where the expression in the least integer symbols is an integer.
- The output of the least integer function is constant on each step.
- The steps of a least integer function are either always increasing or always decreasing.
- A graphing calculator can be used to graph the least integer function.

On a TI-83/84:

Step 1: Press [Y=]. If anything is already entered, press [CLEAR].

Step 2: Press [MATH]. Arrow to "NUM," then select 5: int(.

(*continued*)

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Instruction

Step 3: Enter the expression contained within the least integer function, using x as the variable. Enter a negative sign before the entire expression in the least integer brackets. Enter a negative sign before the int(function. Enter the remaining part of the expression by moving the cursor to the left and/or right of the int() expression. For example, "–int(–(x+3))" is equivalent to the expression $\lceil x+3 \rceil$.

Step 4: Press [GRAPH].

To change the scale:

Step 5: Press [WINDOW].

Step 6: Set the Xmin, Xmax, Xscl, Ymin, Ymax, and Yscl.

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Enter the expression at the bottom of the application, using x as the variable. Press the catalog key (to the right of the number 9). Select ceiling() to enter a least integer expression; for example, typing "ceiling(x+3)" is equivalent to the expression $\lceil x+3 \rceil$.

Step 4: Press [enter] to plot the expression on the graph.

To change the scale:

Step 5: Press [menu].

Step 6: Select 4: Window.

Step 7: Select 1: Window Settings.

Step 8: Tab between the fields to edit the values. Set the XMin, XMax, YMin, and YMax. Leave the XScale and YScale on Auto.

- The graph of a step function, such as the greatest integer function or least integer function, can be viewed as the graph of many constant functions, each on a restricted domain.

Common Errors/Misconceptions

- incorrectly identifying integers greater than or less than negative values
- entering items into the calculator with incorrectly placed parentheses
- connecting steps on a step function instead of showing the jump between output values
- drawing a curved graph of an absolute value function
- incorrectly determining the extreme value of an absolute value function, potentially by inverting x- and y-values

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Guided Practice 2.4.2

Example 1

A software program can show a user how far he or she is from a location on a map. Zadie is walking in a straight line down her street. For any time t, in seconds, her distance d from her home, in feet, can be represented by the function $d = 4|t - 20|$. Create a graph to show Zadie's distance from her house. Which point on the graph shows when Zadie has reached her house?

1. Determine a domain for the problem statement.

   Since the input, t, is time, t can be greater than or equal to 0. The domain is $t \geq 0$.

2. Determine the range for the given domain.

   For an absolute value function of the form $y = a|x - h| + k$, there is one extreme value for the range at $y = k$. If a in the general form $y = a|x - h| + k$ is positive, the extreme value is the minimum of the range, and there is no maximum of the range. If a in the general form $y = a|x - h| + k$ is negative, the extreme value is the maximum of the range, and there is no minimum of the range.

   For the function $d = 4|t - 20|$, $a > 0$, and $k = 0$. The minimum of the range is $d = 0$. The range of the function is $d \geq 0$.

3. Find the vertex of the graph of the absolute value function.

   The coordinates of the vertex of the graph of an absolute value function are the values of h and k when the function is in standard form, $y = a|x - h| + k$.

   Therefore, the coordinates of the vertex are (20, 0).

4. Determine whether the vertex is a minimum or a maximum.

   If a in the general form $y = a|x - h| + k$ is positive, the vertex is a minimum of the function. If a in the general form $y = a|x - h| + k$ is negative, the vertex is a maximum of the function. For the function $d = 4|t - 20|$, $a > 0$.

   Thus, the vertex, (20, 0), is a minimum of the function.

SWB p. 91; SRB p. U2-126

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5. Find at least two additional points on the graph. One should have an input less than the t-coordinate of the vertex, and the other should have an input greater than the t-coordinate of the vertex.

   The input value of the vertex is $t = 20$. The domain of the function is restricted to $t \geq 0$, since t represents time. Evaluate the function at $t = 0$ to find the point at the minimum of the domain. This point has an input less than the input of the vertex.

   $d = 4|t - 20|$ Original function

   $d = 4|0 - 20|$ Substitute 0 for t.

   $d = 4|-20|$ Simplify.

   $d = 4(20)$

   $d = 80$

   The point at the minimum of the domain is (0, 80).

   Evaluate the function at a value of t greater than 20.

   Let $t = 30$.

   $d = 4|t - 20|$ Original function

   $d = 4|30 - 20|$ Substitute 30 for t.

   $d = 4|10|$ Simplify.

   $d = 4(10)$

   $d = 40$

   A point with an input greater than 20 is (30, 40).

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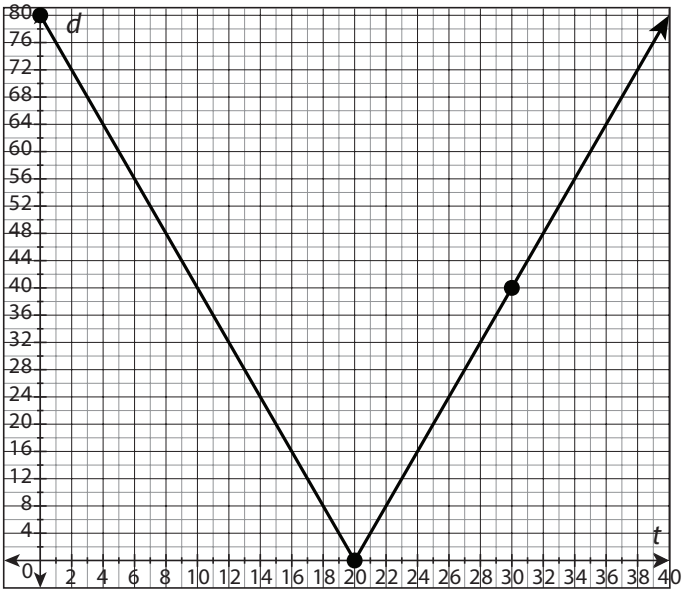
6. Use the three points to create the graph of the function on the determined domain.

   Plot (0, 80), (20, 0), and (30, 40) on the domain $t \geq 0$. An absolute value graph will have two rays with endpoints at the vertex. Each ray will go through the points less than and greater than the vertex.



   A calculator can also be used to graph an absolute value function.

   On a TI-83/84:

   Step 1: Press [Y=]. If anything is already entered, press [CLEAR].

   Step 2: Press [4] then press [MATH]. Arrow to "NUM," then select 1: abs(.

   Step 3: Enter the expression contained within the absolute value bars, using x as the variable, and then close the parentheses: press [X][–][2][0][)].

   Step 4: Press [GRAPH].

   To change the scale:

   Step 5: Press [WINDOW].

   Step 6: Set the Xmin to 0, Xmax to 40, Xscl to 4, Ymin to 0, Ymax to 81, and Yscl to 8.

(*continued*)

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Instruction

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Enter the expression at the bottom of the application, using x as the variable. Type [4][×][A][B][S][(][X][–][2][0][)].

Step 4: Press [enter] to plot the expression on the graph.

To change the scale:

Step 5: Press [menu].

Step 6: Select 4: Window.

Step 7: Select 1: Window Settings.

Step 8: Tab between the fields to edit the values. Set the XMin to 0, XMax to 40, YMin to 0, and YMax to 81. Leave the XScale and YScale on Auto.

7. Interpret the graph to find when Zadie will arrive at her house.

The y-value of the graph is Zadie's distance from her house. When $y = 0$, Zadie is at her house. Zadie will arrive at her house after 20 seconds.

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Instruction

Example 2

Cynthia is starting a house painting business. She charges $20 per hour, and any partial hour worked is rounded up to the next hour. This fee schedule can be modeled by the ceiling function $f(x) = 20\lceil x \rceil$, where x represents the number of hours worked and $f(x)$ represents her fee. Find the domain and range of the function, then graph it.

1. Determine the domain for the problem situation.

   For any job that she does, Cynthia will work more than 0 hours. There is no indication of an upper bound. The domain is $x > 0$.

2. Determine the range for the given domain.

   Because the factor of 20 is positive, and the coefficient of x is also positive, this step function will always be increasing. The lower boundary of the range will occur at the lower boundary of the domain, when $x = 0$. Evaluate the function at $x = 0$.

$f(x) = 20\lceil x \rceil$	Original function
$f(0) = 20\lceil (0) \rceil$	Substitute 0 for x.
$f(0) = 20(0)$	Evaluate $\lceil 0 \rceil$.
$f(0) = 0$	Simplify.

   When $x = 0$, $y = 0$. Because 0 is not included in the domain ($x > 0$), this means that 0 is not included in the range. Since there is no maximum domain value, there is no maximum range value.

   The range is $f(x) > 0$.

3. Use a calculator to graph the function.

   Follow the directions specific to your calculator model.

   On a TI-83/84:

   Step 1: Press [Y=]. If anything is already entered, press [CLEAR].

   Step 2: The "int(" function in the calculator is for the greatest integer function, but we want to graph a least integer function. To do this, enter a negative sign before the entire expression, then enter a negative sign before the expression inside the greatest integer brackets. Press [–][2][0], then press [MATH]. Arrow to "NUM," then select 5: int(. Press [–][x][)].

   Step 3: Press [GRAPH].

(*continued*)

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Instruction

On a TI-Nspire:

Step 1: Press the [home] key.

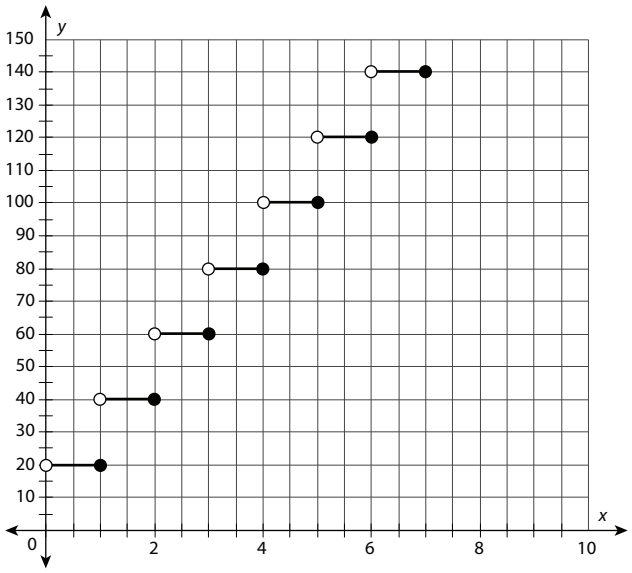
Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Enter the expression at the bottom of the application, using x as the variable. Press [2][0], then press the catalog key (to the right of the number 9). Select ceiling() to enter a least integer expression. Press [enter].

Step 4: Type [x], then press [enter].

Step 5: Adjust the window if necessary, then press [enter] to see the graph.

The resulting graph should resemble the following:



Example 3

Compare the domain, range, graph, and vertices for the functions $y=-3|x+1|$ and $y=|x|$. How are these differences seen in the algebraic equations?

1. Determine the domain and range of $y=|x|$.

   The absolute value function is defined for all real values of x, so the domain is all real numbers. The absolute value of a number is its distance from 0, which is always positive. The range of values is $y \geq 0$.

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2. Find at least three points on the graph of $y = |x|$, including the vertex.

The x-coordinate of the vertex is where the expression inside the absolute value symbols is equal to 0. The expression inside the absolute value symbols is x, so the x-coordinate of the vertex is 0. Evaluate the function at $x = 0$ to find the y-coordinate of the vertex.

$y = \|x\|$	Original function
$y = \|0\|$	Substitute 0 for x.
$y = 0$	Simplify.

The vertex of the graph of this absolute value function is (0, 0).

Evaluate the function at an input less than 0 and an input greater than 0.

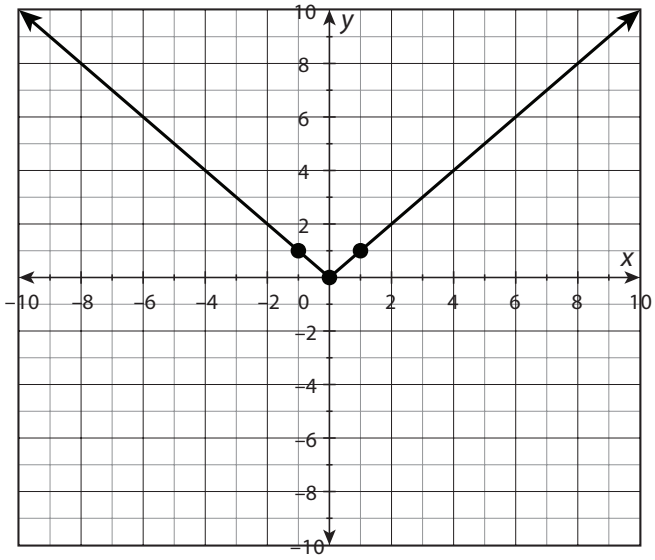
If $x = -1$, $y = |-1| = 1$. (–1, 1) is a point on the function.

If $x = 1$, $y = |1| = 1$. (1, 1) is a point on the function.

Because the y-values of both points are greater than the y-value of the vertex, the vertex is a minimum point of the function.

3. Plot the points and graph $y = |x|$.

Draw two rays. The endpoint of each ray is the vertex. One ray goes through the point with the x-value to the left of the vertex, and the other ray goes through the point to the right of the vertex.



(*continued*)

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Instruction

A calculator can also be used to create a graph.

On a TI-83/84:

Step 1: Press [Y=]. If anything is already entered, press [CLEAR].

Step 2: Press [MATH]. Arrow to "NUM," then select 1: abs(.

Step 3: Enter the expression contained within the absolute value bars, using x as the variable: abs[(][X][)].

Step 4: Press [GRAPH].

To change the scale:

Step 5: Press [WINDOW].

Step 6: Set the Xmin to –10, Xmax to 10, Xscl to 2, Ymin to –10, Ymax to 10, and Yscl to 2.

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Enter the expression at the bottom of the application, using x as the variable. Type [A][B][S][(][X][)].

Step 4: Press [enter] to plot the expression on the graph.

To change the scale:

Step 5: Press [menu].

Step 6: Select 4: Window.

Step 7: Select 1: Window Settings.

Step 8: Tab between the fields to edit the values. Set the XMin to –10, XMax to 10, YMin to –10, and YMax to 10. Leave the XScale and YScale on Auto.

4. Determine the domain of $y = -3|x+1|$.

The absolute value function is defined for all real values of x, so the domain is all real numbers.

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Instruction

5. Determine the range of $y = -3|x+1|$.

   For an absolute value function of the form $y = a|x-h|+k$, there is one extreme value for the range at $y = k$. If a in the general form $y = a|x-h|+k$ is positive, the extreme value is the minimum of the range, and there is no maximum of the range. If a in the general form $y = a|x-h|+k$ is negative, the extreme value is the maximum of the range, and there is no minimum of the range.

   For the function $y = -3|x+1|$, $a < 0$, and $k = 0$. This is a maximum value, and the range of the function is $y \leq 0$.

6. Find at least 3 points on the graph of $y = -3|x+1|$, including the vertex.

   The x-coordinate of the vertex is where the expression inside the absolute value symbols is equal to 0. The expression inside the absolute value symbols is $x + 1$. Find the value of x that makes the expression equal to 0.

$x + 1 = 0$	Set the expression equal to 0.
$x = -1$	Solve.

   The y-coordinate of the vertex is the extreme value of the range. The maximum of the range is $y = 0$.

   The coordinates of the vertex are (–1, 0).

   Evaluate the function at an input less than –1 and an input greater than –1.

(*continued*)

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Let $x = -2$.

$y = -3|x+1|$ Original function

$y = -3|(-2)+1|$ Substitute –2 for x.

$y = -3|-1|$ Simplify.

$y = -3(1)$

$y = -3$

(–2, –3) is a point on the graph.

Let $x = 0$.

$y = -3|x+1|$ Original function

$y = -3|(0)+1|$ Substitute 0 for x.

$y = -3|1|$ Simplify.

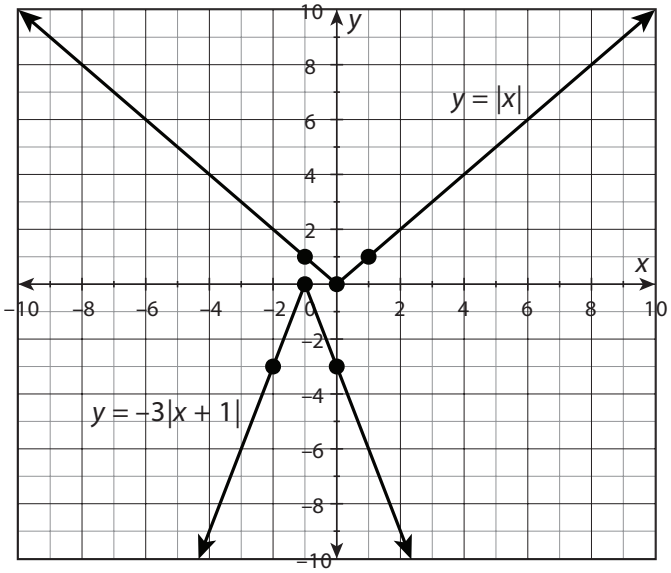
$y = -3(1)$

$y = -3$

(0, –3) is a point on the graph.

7. Plot the points and graph $y = -3|x+1|$ on the same coordinate plane as the graph of $y = |x|$.

Plot the three points of the function $y = -3|x+1|$ and draw two rays that begin at the vertex (–1, 0) and pass through the other two points.



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Instruction

8. Compare the domains and ranges of the functions $y=-3|x+1|$ and $y=|x|$. Note how the equation is related to any differences between the domains and ranges.

   The domains of both functions are the same: all real numbers. The range of $y=-3|x+1|$ is $y \le 0$, where the range of $y=|x|$ is $y \ge 0$. This is because the coefficient of the absolute value expression is negative in the function $y=-3|x+1|$.

9. Compare the vertices of $y=-3|x+1|$ and $y=|x|$, and note how the equation is related to the differences between these points.

   In the graph of $y=-3|x+1|$, the vertex is a maximum, and on the graph of $y=|x|$, the vertex is a minimum. This is because the absolute value in $y=-3|x+1|$ is multiplied by a negative number. The vertex on the graph of $y=-3|x+1|$ is shifted 1 unit to the left. This is because the expression in the absolute value symbols contains an addend of + 1.

10. Compare the general shapes of $y=-3|x+1|$ and $y=|x|$, and note how the equation is related to the differences.

    Both functions are in a v-shape. The graph of the function $y=-3|x+1|$ is narrower than the graph of the function $y=|x|$, due to the factor of 3 outside the absolute value symbols. The graph of $y=-3|x+1|$ also opens downward, due to the negative factor outside the absolute value symbols.

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Problem-Based Task 2.4.2: Fancy Wheels

SMP	
1	2 ✓
3	4 ✓
5 ✓	6
7	8

Fancy Wheels Car Service rents out two types of chauffeured vehicles: luxury limos and sedans. The company's owner is a big supporter of the local high school, so she offers special pricing to students for prom night.

The cost of renting a luxury limo is modeled by the function $f(x)=\lfloor 0.2x \rfloor+10$, and the cost of renting a sedan can be modeled by the function $g(x)=\lfloor 0.5x \rfloor+2$. For both functions, x represents the number of miles driven, and $f(x)$ and $g(x)$ represent the total cost in dollars. Assume that no student has to travel more than 25 miles in all. How do the differences in the graphs of the functions relate to the differences in the algebraic representations? What do these differences show about the cost of renting each type of car?

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Problem-Based Task 2.4.2: Fancy Wheels

Coaching

a. What is the domain for the function $f(x)=\lfloor 0.2x \rfloor+10$ in the context of the problem?

b. What is the corresponding range in the context of the problem?

c. Use a calculator to graph $f(x)=\lfloor 0.2x \rfloor+10$, and then sketch the graph.

d. What is the domain for the function $g(x)=\lfloor 0.5x \rfloor+2$?

e. What is the corresponding range?

f. Graph $g(x)=\lfloor 0.5x \rfloor+2$ on the calculator on the same coordinate plane as $f(x)=\lfloor 0.2x \rfloor+10$. Sketch the graph on the same coordinate plane you used for part c.

g. How are the graphs of the functions similar?

h. How are the graphs of the functions different?

i. How are the algebraic representations of each function related to the similarities and differences between the two graphs?

j. What do these differences show about the cost of renting the two different types of car?

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Instruction

Problem-Based Task 2.4.2: Fancy Wheels

Coaching Sample Responses

a. What is the domain for the function $f(x)=\lfloor 0.2x \rfloor+10$ in the context of the problem?

The domain is the set of x-values that are valid for the function in the context of the problem. Because x represents the number of miles driven, it cannot be negative. The problem states that no student will travel more than 25 miles, so 25 is the upper bound for x. The restricted domain is $0 \le x \le 25$.

b. What is the corresponding range in the context of the problem?

The extreme values of the range will be at the extreme values of the domain. Find the smallest and largest values of the range by evaluating the function for the smallest and largest values of the domain.

For $x = 0$:

$f(x)=\lfloor 0.2x \rfloor+10$

$f(0)=\lfloor 0.2(0) \rfloor+10$

$f(0)=\lfloor 0 \rfloor+10$

$f(0) = 10$

For $x = 25$:

$f(x)=\lfloor 0.2x \rfloor+10$

$f(25)=\lfloor 0.2(25) \rfloor+10$

$f(25)=\lfloor 5 \rfloor+10$

$f(25) = 15$

The minimum value of the range is 10, and the maximum value of the range is 15. Because the function is a greatest integer function, and there is no number multiplied by the greatest integer and no addend inside the greatest integer function symbols, the output values will be integers.

The restricted range is $10 \le f(x) \le 15$, inclusive.

c. Use a calculator to graph $f(x)=\lfloor 0.2x \rfloor+10$, and then sketch the graph.

On a TI-83/84:

Step 1: Press [Y=]. If anything is already entered, press [CLEAR].

Step 2: Press [MATH]. Arrow to "NUM," then select 5: int(.

Step 3: Enter the expression contained within the greatest integer function, using x as the variable. Enter the remaining part of the expression by moving the cursor to the left and/or right of the int() expression. Type [(][0][.][2][×][X][)][+][1][0].

Step 4: Press [GRAPH].

To change the scale:

Step 5: Press [WINDOW].

Step 6: Set the Xmin to 0, Xmax to 26, Xscl to 2, Ymin to 0, Ymax to 26, and Yscl to 2.

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Instruction

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Enter the expression at the bottom of the application, using x as the variable. Press the catalog key (to the right of the number 9). Select floor() to enter a greatest integer expression. Type [F][L][O][O][R][(][0][.][2][×][X][)][+][1][0].

Step 4: Press [enter] to plot the expression on the graph.

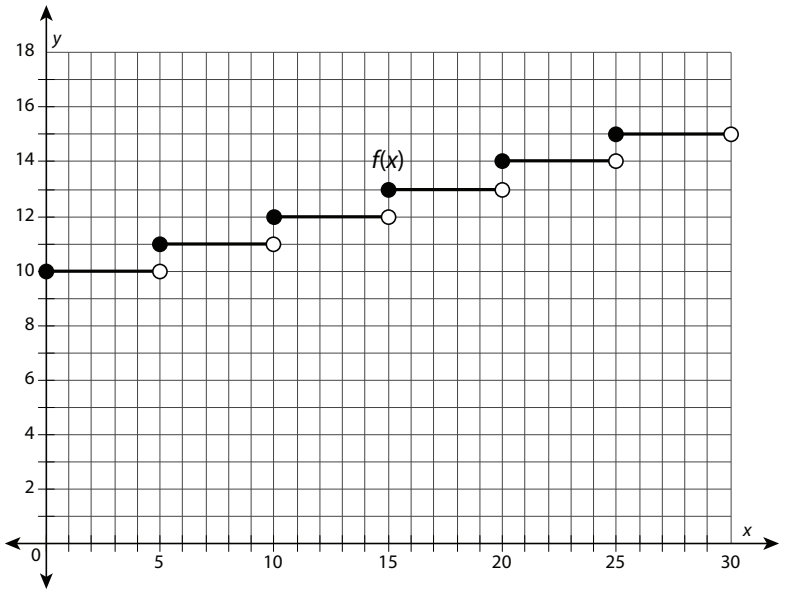
To change the scale:

Step 5: Press [menu].

Step 6: Select 4: Window.

Step 7: Select 1: Window Settings.

Step 8: Tab between the fields to edit the values. Set the XMin to 0, XMax to 26, YMin to 0, and YMax to 18. Leave the XScale and YScale on Auto.



d. What is the domain for the function $g(x)=\lfloor 0.5x \rfloor+2$?

As was the case with the domain for the first function, the domain for $g(x)$ is the set of x-values that are valid for the function in the context of the problem. Because x represents the number of miles driven, it cannot be negative. The problem states that no student will travel more than 25 miles, so 25 is the upper bound for x. The domain is $0 \le x \le 25$.

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e. What is the corresponding range?

The extreme values of the range will be at the extreme values of the domain. Find the smallest and largest values of the range by evaluating the function for the smallest and largest values of the domain.

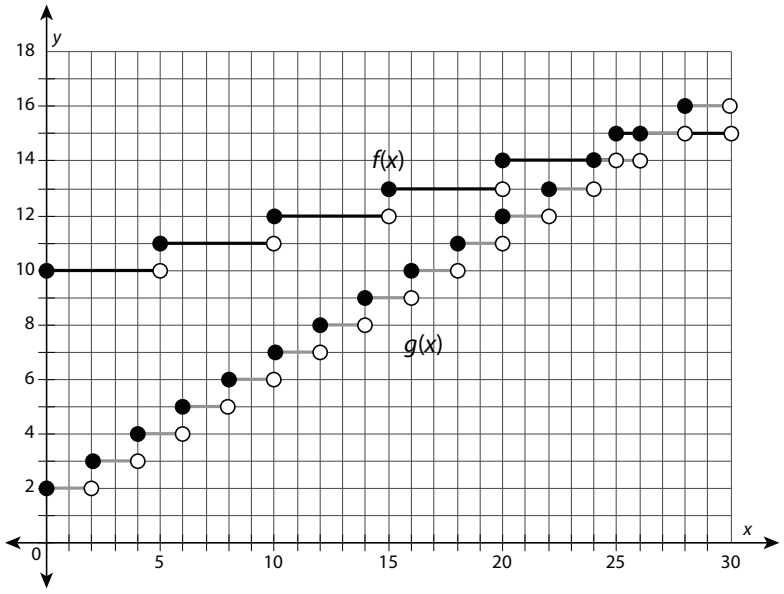
For $x = 0$:	**For $x = 25$:**
$g(x)=\lfloor 0.5x \rfloor+2$	$g(x)=\lfloor 0.5x \rfloor+2$
$g(0)=\lfloor 0.5(0) \rfloor+2$	$g(25)=\lfloor 0.5(25) \rfloor+2$
$g(0)=\lfloor 0 \rfloor+2$	$g(25)=\lfloor 12.5 \rfloor+2$
$g(0) = 0 + 2$	$g(25) = 12 + 2$
$g(0) = 2$	$g(25) = 14$

The minimum value of the range is 2, and the maximum value of the range is 14. Because the function is a greatest integer function, and there is no number multiplied by the greatest integer and no addend inside the greatest integer function symbols, the output values will be integers.

The range is $2 \le g(x) \le 14$.

f. Graph $g(x)=\lfloor 0.5x \rfloor+2$ on the calculator on the same coordinate plane as $f(x)=\lfloor 0.2x \rfloor+10$. Sketch the graph on the same coordinate plane you used for part c.

On the TI-83/84, enter the expression in Y_2. On the TI-Nspire, enter the expression in the bottom panel and press [enter].



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g. How are the graphs of the functions similar?

Both graphs are step functions, made up of many disconnected constant functions. The steps of both graphs are increasing.

h. How are the graphs of the functions different?

The graph for the chauffeured sedan, $g(x)$, is increasing at a faster rate than the graph for the chauffeured limo, $f(x)$. Up until the point where $x = 24$, the values of $f(x)$ are greater than the values of $g(x)$.

i. How are the algebraic representations of each function related to the similarities and differences between the two graphs?

The coefficient of the variable inside the greatest integer expression determines how quickly each function will increase. The coefficient of x for the function $f(x)$ is 0.2, and the coefficient of x for the function $g(x)$ is 0.5. Similar to the slope of linear functions, the graph of the function with the larger coefficient will rise more quickly.

The addends outside the greatest integer expression determine the y-intercepts of the graphs.

j. What do these differences show about the cost of renting the two different types of car?

The starting cost for renting the sedan is \$2, and the starting cost of renting the limo is \$10. The cost of renting the limo increases by \$1 for every 5 miles, or portion of 5 miles, driven. The cost of renting the sedan increases by \$1 for every 2 miles, or portion of 2 miles, driven. For the first 24 miles, it is more expensive to rent the limo. For anything between 24 miles and 25 miles, the price is the same. For more than 25 miles, it would be more expensive to rent the sedan, but since none of the students travel more than 25 miles, this is not a consideration.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2.4.2: Absolute Value and Step Functions

For problems 1–4, create a graph of each function. Note the domain, range, and any vertices. Note that $\lfloor\ \rfloor$ represents the greatest integer function, and $\lceil\ \rceil$ represents the least integer function.

1. $y=|x-6|$

2. $y=-\lceil x \rceil+1$

3. $y=10|x|+7$

4. $y=\lfloor 2.5x \rfloor-12$

For problems 5–8, create a graph showing each pair of functions. Describe any similarities and differences between the graphs, including domain, range, and any vertices. Describe how any similarities and differences are shown in the algebraic functions.

5. $f(x)=|x|$ and $g(x)=-4|x-9|$

6. $f(x)=|x|$ and $g(x)=|x+8|-15$

continued

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7. $f(x)=\lfloor x \rfloor$ and $g(x)=\lfloor 3x+2 \rfloor$

8. $f(x)=\lceil x \rceil$ and $g(x)=2\lceil x \rceil-10$

Read the following scenario and use the information in it to complete problems 9 and 10.

> A package delivery service charges a flat rate of \$1.50 plus \$2.50 per pound that the package weighs. Partial pounds are rounded up to the next full pound. This situation can be modeled by the function $f(x)=2.50\lceil x \rceil+1.50$, where x represents the number of pounds and $f(x)$ represents the total cost in dollars.

9. Identify the domain and range of the function. Create a graph to show the pricing for this delivery service.

10. A rival company is planning to open soon, and the pricing schedule for this new business will be modeled by the function $g(x)=\lceil 0.5x \rceil+10$, where x represents the number of pounds and $g(x)$ represents the total cost in dollars. Identify the domain and range of the function. Graph this function on the same coordinate plane as the previous function, and compare the two graphs. Which business offers a better deal?

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Lesson 2.4.3: Piecewise Functions

Warm-Up 2.4.3

After a hockey player hits a hockey puck, the puck travels at a constant rate of 45 meters per second. The distance in meters, d, traveled by the puck after s seconds can be found using the function $d = 45s$.

1. How long will it take for the puck to travel half the length of a 60-meter hockey rink?

2. Create a graph to show the distance traveled by a puck traveling for any number of seconds. Include a reasonable domain and range in the graph.

3. A second player hits a hockey puck at a constant rate of 50 meters per second, and the distance d_2 can be determined after any time s using the function $d_2 = 50s$. Graph this function on the same graph as the first function.

4. How are the two graphs similar? How are they different? Relate any similarities and differences to the algebraic representations of the functions.

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Instruction

Lesson 2.4.3: Piecewise Functions

Common Core State Standard

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

Warm-Up 2.4.3 Debrief

1. How long will it take for the puck to travel half the length of a 60-meter hockey rink?

   Half the length of a 60-meter rink is 60 • 0.5 = 30 meters. To find the time it takes to travel 30 meters, solve the function $d = 45s$ for $d = 30$.

$d = 45s$	Original function
$30 = 45s$	Substitute 30 for d.
$0.67 = s$	Divide both sides by 45.

   The puck will travel 30 meters in 0.67 second.

2. Create a graph to show the distance traveled by a puck traveling for any number of seconds. Include a reasonable domain and range in the graph.

   In the function $d = 45s$, s is the time in seconds. The minimum value of s is 0. A reasonable maximum time is the time it takes the puck to travel the 60-meter length of the hockey rink. Solve $d = 45s$ for $d = 60$.

$d = 45s$	Original function
$60 = 45s$	Substitute 60 for d.
$1.33 \approx s$	Divide both sides by 45.

   The maximum time it takes for the puck to travel the length of the 60-meter hockey rink is approximately 1.33 seconds.

   Therefore, a reasonable restricted domain is $0 \leq s \leq 1.33$.

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Instruction

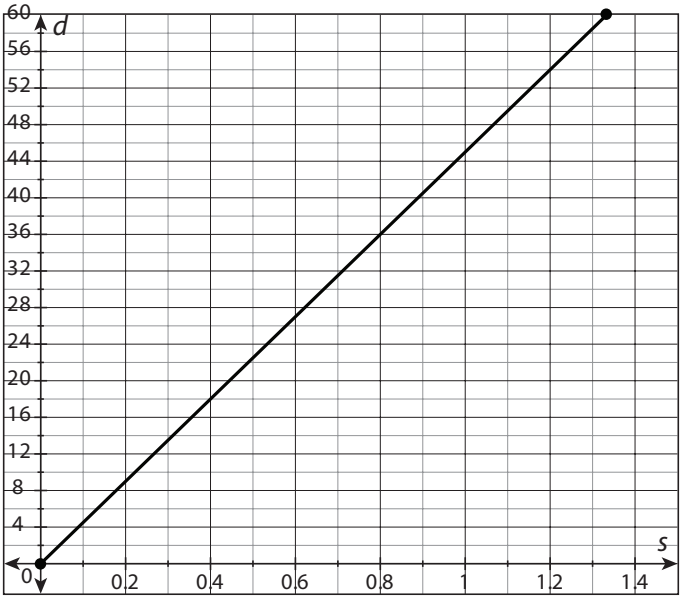
The minimum of the range is at the minimum of the domain, since $d = 45s$ is an increasing function. Evaluate $d = 45s$ at $s = 0$.

$d = 45s$	Original function
$d = 45(0)$	Substitute 0 for s.
$d = 0$	Simplify.

The maximum of the range is at the maximum of the domain. The maximum of the domain was found by solving the function for $d = 60$, so the maximum of the range is 60.

A reasonable restricted range is $0 \leq d \leq 60$.

Use the maximum and minimum points, (0, 0) and (1.33, 60), to draw the graph of the function.



3. A second player hits a hockey puck at a constant rate of 50 meters per second, and the distance d_2 can be determined after any time s using the function $d_2 = 50s$. Graph this function on the same graph as the first function.

   Find two points on the function on the same domain as $d = 45s$. Evaluate the function $d_2 = 50s$ at $s = 0$ and $s = 1$.

For $s = 0$:	**For $s = 1$:**
$d_2 = 50s$	$d_2 = 50s$
$d_2 = 50(0)$	$d_2 = 50(1)$
$d_2 = 0$	$d_2 = 50$

Two points on the line are (0, 0) and (1, 50). Plot the two points and draw a line through them.

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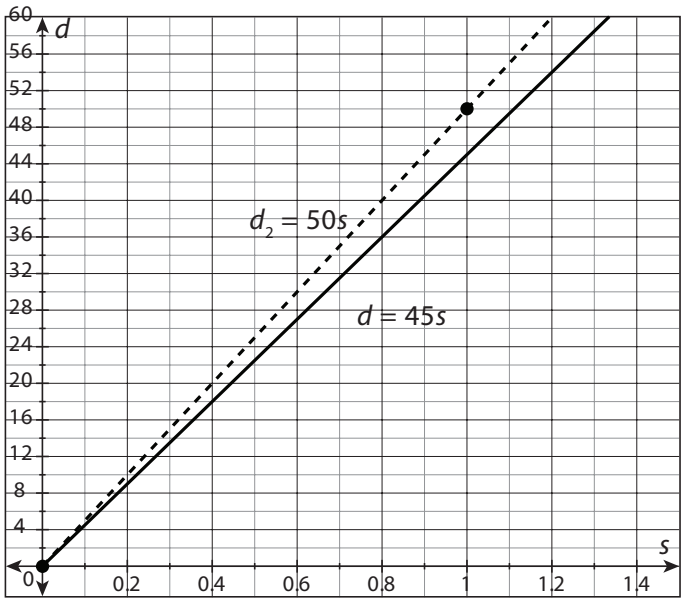
Instruction



4. How are the two graphs similar? How are they different? Relate any similarities and differences to the algebraic representations of the functions.

   Both graphs are linear; we can tell this by looking at the equations of the functions, both of which have 1 as the highest power of the variable, s. The graph of the second hockey puck, $d_2 = 50s$, is steeper than the graph of $d = 45s$. In the algebraic representation, the slope of d_2, 50, is greater than the slope of d, 45.

Connection to the Lesson

- In this lesson, students will translate between algebraic and graphical representations.
- Students will relate differences and similarities in graphs to differences and similarities in equations.
- Students will identify reasonable restricted domains for problem scenarios.
- Students will plot linear functions on restricted domains.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- plotting polynomial functions given an algebraic equation (8.F.1)
- determining the domain and range of a function given an algebraic equation (F–IF.1)
- determining restricted domains and ranges for problem situations (F–IF.4★)

Introduction

Some functions look as though they are made up of two or more different functions. This is because some functions are defined differently on different portions of their domains. Recall that a function cannot have two outputs for the same input, so the domains do not overlap.

Key Concepts

- A **piecewise function** is a function that is defined by two or more expressions on separate portions of the domain.
- A piecewise function is written using a large brace. For example, the absolute value function $y = |x|$ can be written as the piecewise function $y = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$. Note that both portions of the function cannot include $x = 0$ in the domain, even though the output is the same for both expressions.
- To graph a piecewise function, graph each portion of the function on its restricted domain.
- Graphs of piecewise functions may or may not be connected.
- One piece of a piecewise function can be for a single input value, an interval of values, or for infinitely many values.
- If the function is represented by an interval, be sure to mark each endpoint clearly. If the endpoint is defined for the function, use a closed circle. If the endpoint is undefined for the function, use an open circle.

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Instruction

- A calculator can also be used to graph a piecewise function.

On a TI-83/84:

Step 1: Put the calculator in Dot mode to show the graph as disconnected. To do this, press [MODE], then arrow down to "CONNECTED." Arrow to the right, select "DOT," and then press [ENTER].

Step 2: Press [Y=]. If anything is already entered, press [CLEAR].

Step 3: Enter the first expression of the piecewise function in parentheses.

Step 4: Enter the domain of the first expression in parentheses in the same line as the expression. To find < and > options, press [2ND] [MATH].

Step 5: Enter the addition symbol, +.

Step 6: Enter the second expression of the piecewise function in parentheses.

Step 7: Enter the domain of the second expression in parentheses.

Step 8: Press [GRAPH].

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Press the key to display math expressions (the key directly to the right of the 9 key) and select the icon that shows a piecewise function; the 7th icon has two pieces, and the 8th icon has three pieces.

Step 4: Use the [tab] key to enter the expressions and domain restrictions. Press [enter] to plot the expressions on the graph.

Common Errors/Misconceptions

- defining one point on a domain with two different expressions, even if the expressions are equivalent at that point
- connecting graphs of different components of a piecewise function that should be disconnected

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Instruction

Guided Practice 2.4.3

Example 1

A cell-phone plan charges customers a monthly fee, which includes 500 minutes of talk time. After 500 talk minutes, the customer is charged \$0.10 a minute. The total monthly charges, y, for any number of talk minutes, x, can be represented using the piecewise function $y = \begin{cases} 50, \text{ if } 0 \le x \le 500 \\ 50 + 0.10(x - 500), \text{ if } x > 500 \end{cases}$. Create a graph to show the monthly charges for any number of talk minutes.

1. Determine the domain and range of the first piece of the function.

   The domain of the first piece of the function, $y = 50$, is $0 \le x \le 500$, which is given in the function. The function $y = 50$ is a constant function, and the range is $y = 50$.

2. Find at least two points on the graph of the first piece of the function.

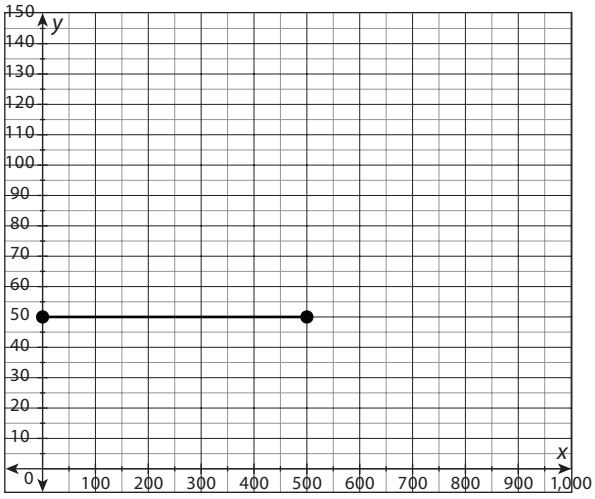
   The first piece of the function is the constant function $y = 50$, so y is equal to 50 for any x-value on the domain of this piece. Find the points at the beginning and end of the domain of the function.

   For $x = 0$ (the beginning of the domain of the function), $y = 50$.

   For $x = 500$ (the end of the domain of the function), $y = 50$.

   Therefore, two points on the graph of $y = 50$ are (0, 50) and (500, 50).

3. Use the two points to graph the first piece of the function.



SWB p. 101; SRB p. U2-141

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4. Determine the domain and range for the second piece of the function.

The domain of the second piece of the function, $y = 50 + 0.10(x - 500)$, is $x > 500$, which is given in the function. This linear function has a positive slope, so the function will always be increasing. The minimum of the function will be at the smallest value on the domain. Evaluate the function at $x = 500$. This value isn't included in this piece of the function, so it can be represented by an open circle on the graph.

$y = 50 + 0.10(x - 500)$	Original function
$y = 50 + 0.10[(500) - 500]$	Substitute 500 for x.
$y = 50$	Simplify.

The range is all y-values greater than 50.

Domain: $x > 500$

Range: $y > 50$

5. Find at least two points on the graph of the second piece of the function.

The second piece of the function is the increasing linear function $y = 50 + 0.10(x - 500)$. Find the points at the beginning of the domain and another point on the function.

For $x = 500$ (the beginning of the domain of the function), $y = 50$.

To determine another point on the function, choose an x-value greater than 500. Let's use $x = 600$.

$y = 50 + 0.10(x - 500)$	Original function
$y = 50 + 0.10[(600) - 500]$	Substitute 600 for x.
$y = 60$	Simplify.

Therefore, two points are (500, 50) and (600, 60).

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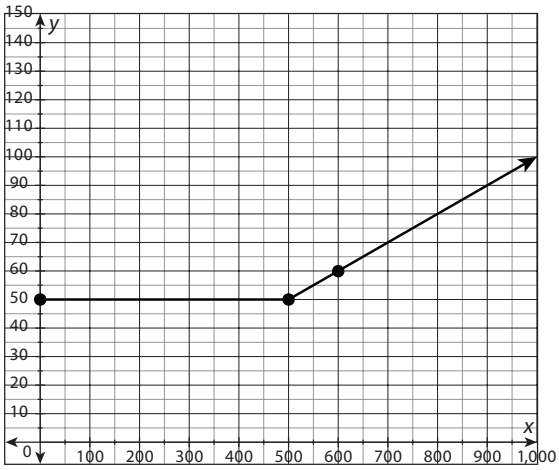
6. Use the two points to graph the second piece of the function.

   Note that the point (500, 50) is an open circle for the second piece of the function, but since this point is included in the graph for the first piece of the function, the open circle is filled in when both pieces are graphed.



A calculator can also be used to graph a piecewise function.

On a TI-83/84:

Step 1: Put the calculator in Dot mode to show the graph as disconnected. To do this, press [MODE], then arrow down to "CONNECTED." Arrow to the right, select "DOT," and then press [ENTER].

Step 2: Press [Y=]. If anything is already entered, press [CLEAR].

Step 3: Enter the first expression of the piecewise function in parentheses: (50).

Step 4: Enter the domain of the first expression in parentheses in the same line as the expression. To find < and > options, press [2ND][MATH]: (50)(x<500).

Step 5: Enter the addition symbol, +: (50)(x<500)+.

Step 6: Enter the second expression of the piecewise function in parentheses: (50)(x<500)+(50+0.1(x–500))(x>500).

Step 7: Enter the domain of the second expression in parentheses.

Step 8: Press [GRAPH].

(*continued*)

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On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Press the key to display math expressions (the key directly to the right of the 9 key) and select the icon that shows a piecewise function; the seventh icon has two pieces.

Step 4: Use the [tab] key to enter the expressions 50 and $50 + 0.10(x - 500)$ and domain restrictions of $x < 500$ and $x > 500$ for each expression. Press [enter].

Example 2

One hockey player passes a puck to a second player. The second player hits the puck instantly. The total meters, m, traveled by the hockey puck after s seconds can be represented using the function $m = \begin{cases} 48s, \text{ if } 0 \le s \le 1 \\ 40s + 8, \text{ if } 1 < s < 2 \end{cases}$. Create a graph to show the distance traveled by the puck after s seconds.

1. Determine the domain and range for the first piece of the function.

   The domain of the first piece of the function, $m = 48s$, is $0 \le s \le 1$, which is given in the function. The function $m = 48s$ is a linear function, since the greatest power of the variable is 1. The function is increasing because the slope, 48, is positive. The minimum of the range will be at the minimum of the domain, and the maximum of the range will be at the maximum of the domain. Evaluate the function at the minimum and maximum values of the domain to find the minimum and maximum values of the range.

For $s = 0$:	**For $s = 1$:**
$m = 48s$	$m = 48s$
$m = 48(0)$	$m = 48(1)$
$m = 0$	$m = 48$

   The range of the first piece of the function is $0 \le m \le 48$.

2. Find at least two points on the graph of the first piece of the function.

   The minimum and maximum values of the first piece of the function are each a point on the graph. The two points are (0, 0) and (1, 48).

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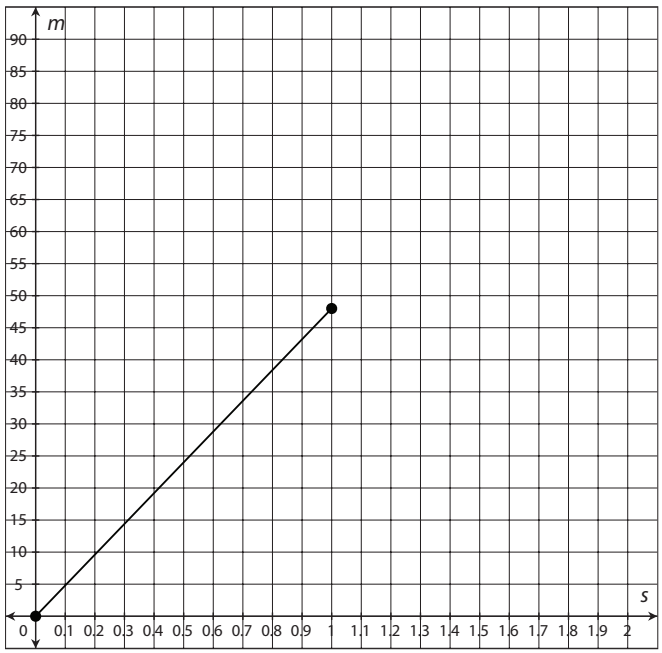
3. Use the two points to graph the first piece of the function.

   The graph will be a line segment with the minimum and maximum as endpoints.



4. Determine the domain and range for the second piece of the function.

   The domain of the second piece of the function, $m = 40s + 8$, is defined by the function on the domain $1 < s < 2$. It is also a linear increasing function, so the minimum and maximum of the range will be at the minimum and maximum of the domain. Evaluate the function at the minimum and maximum values of the domain. Keep in mind that these endpoints are not included on the graph of the function.

For $s = 1$:	**For $s = 2$:**
$m = 40s + 8$	$m = 40s + 8$
$m = 40(1) + 8$	$m = 40(2) + 8$
$m = 40 + 8$	$m = 80 + 8$
$m = 48$	$m = 88$

   The range of the function is $48 < m < 88$.

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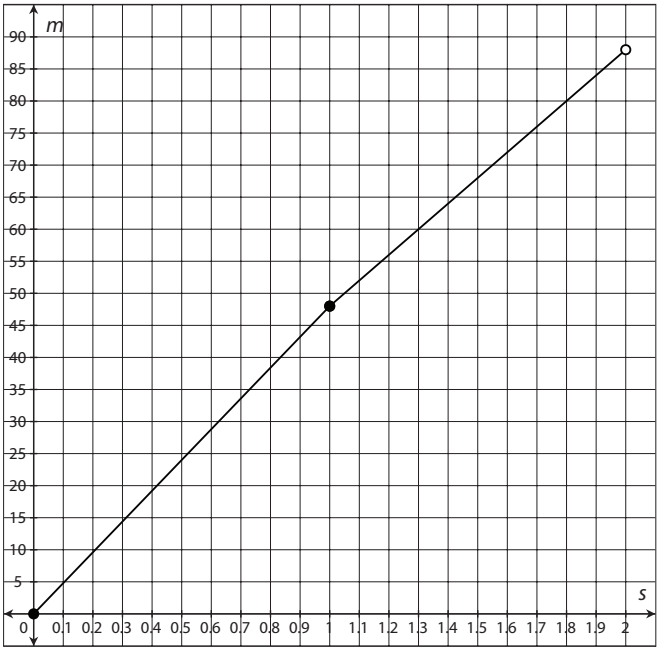
5. Find at least two points on the graph of the second piece of the function.

   The points at the minimum and maximum of the function can be used to graph the function. The points are not included, and should be shown as open circles on the graph.

   Based on the values determined in the previous step, the points are (1, 48) and (2, 88).

6. Use the two points to graph the second piece of the function.

   Note that the open circle at the beginning of the second piece of the graph is "filled in" by the endpoint from the first piece of the graph.



A calculator can also be used to graph a piecewise function.

On a TI-83/84:

Step 1: Put the calculator in Dot mode to show the graph as disconnected. To do this, press [MODE], then arrow down to "CONNECTED." Arrow to the right, select "DOT," and then press [ENTER].

Step 2: Press [Y=]. If anything is already entered, press [CLEAR].

Step 3: Enter the first expression of the piecewise function in parentheses: (48x).

(*continued*)

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Step 4: Enter the domain of the first expression in parentheses in the same line as the expression. To find < and > options, press [2ND][MATH]: (48x)(x<1).

Step 5: Enter the addition symbol, +: (48x)(x<1)+.

Step 6: Enter the second expression of the piecewise function in parentheses: (48x)(x<1)+(40x+8)(x>1).

Step 7: Enter the domain of the second expression in parentheses.

Step 8: Press [GRAPH].

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Press the key to display math expressions (the key directly to the right of the 9 key) and select the icon that shows a piecewise function; the seventh icon has two pieces.

Step 4: Use the [tab] key to enter the expressions $48x$ and $40x + 8$ and domain restrictions of $x < 1$ and $x > 1$ for each expression. Press [enter] to plot the expressions.

Example 3

Create a graph of the function $y = \begin{cases} 2, \text{ if } 0 \le x < 4 \\ x^2 - 14, \text{ if } x \ge 4 \end{cases}$.

1. Determine the domain and range for the first piece of the function.

   The domain of the first piece of the function, $y = 2$, is $0 \le x < 4$, which is given in the function. The function $y = 2$ is a constant function, since the output, or y-value, will be 2 for any input of x. The range of the first piece is $y = 2$.

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2. Find at least two points on the graph of the first piece of the function.

   Find the points at the minimum and maximum domain values.

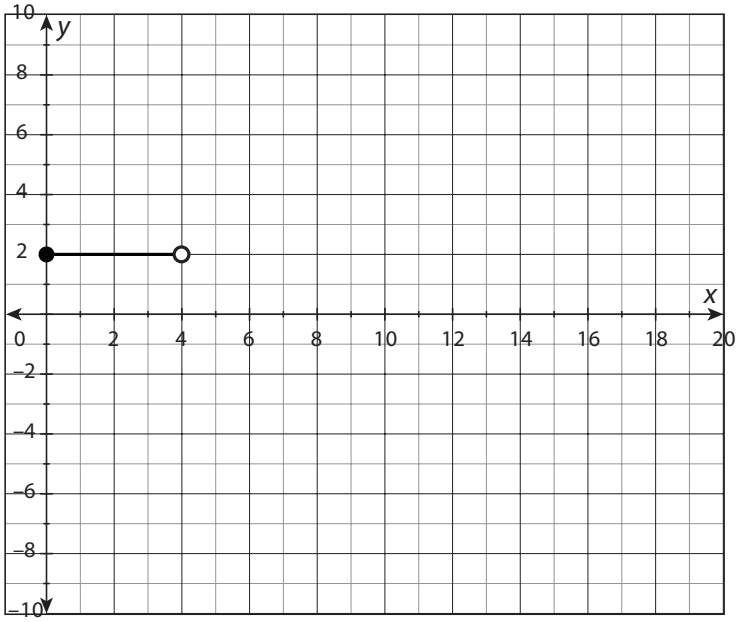
   For $x = 0$, $y = 2$.

   For $x = 4$, $y = 2$.

   Thus, (0, 2) and (4, 2) are two points easily found.

3. Use the two points to graph the first piece of the function.

   The graph is a horizontal line segment with the minimum and maximum x-values as endpoints. The endpoint (4, 2) is marked by an open circle because it is not defined for this piece of the function.



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4. Determine the domain and range for the second piece of the function.

The second piece of the function, $y = x^2 - 14$, is defined by the function on the domain $x \geq 4$. It is a quadratic function, and its vertex is at the point $(0, -14)$. Because the coefficient of x^2 is positive, the vertex is a minimum for the parabola, and the parabola is increasing for $x > 0$. Since this quadratic function is only defined on the domain $x \geq 4$, the given piece of the function is always increasing. The minimum range value will be at the minimum value of the domain. Evaluate the function at the minimum value of the domain to find the minimum of the range.

Let $x = 4$.

$y = x^2 - 14$	Original function
$y = (4^2) - 14$	Substitute 4 for x.
$y = 16 - 14$	Simplify.
$y = 2$	

Therefore, the minimum of the range is $y \geq 2$.

5. Find at least two points on the graph of the second piece of the function.

The point at the minimum can be used to graph the function. Find a second point on the domain $x \geq 4$.

Let $x = 5$.

$y = x^2 - 14$	Original function
$y = 5^2 - 14$	Substitute 5 for x.
$y = 25 - 14$	Simplify.
$y = 11$	

Two points on the function are (4, 2) and (5, 11).

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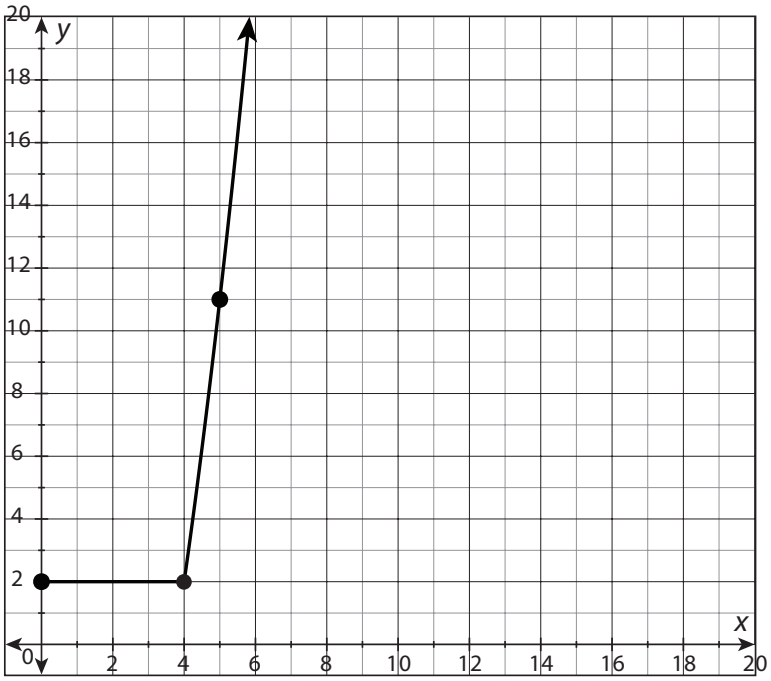
6. Use the two points to graph the second piece of the function.

   Keep in mind that the graph of a quadratic function is not linear, but curved. Also, the open circle at the end of the first piece is "filled in" by the endpoint of the second piece.



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On a TI-83/84:

Step 1: Put the calculator in Dot mode to show the graph as disconnected. To do this, press [MODE], then arrow down to "CONNECTED." Arrow to the right, select "DOT," and then press [ENTER].

Step 2: Press [Y=]. If anything is already entered, press [CLEAR].

Step 3: Enter the first expression of the piecewise function in parentheses: (2).

Step 4: Enter the domain of the first expression in parentheses in the same line as the expression. To find < and > options, press [2ND][MATH]: (2)(x<4).

Step 5: Enter an additional symbol, +: (2)(x<4)+.

(*continued*)

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Step 6: Enter the second expression of the piecewise function in parentheses: (2)(x<4)+((x^2)–14)(x>4).

Step 7: Enter the domain of the second expression in parentheses.

Step 8: Press [GRAPH].

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Press the key to display math expressions (the key directly to the right of the 9 key) and select the icon that shows a piecewise function; the seventh icon has two pieces.

Step 4: Use the [tab] key to enter the expressions 2 and $(x^2) - 14$ and domain restrictions of $x < 4$ and $x > 4$ for each expression. Press [enter] to plot the expressions on the graph.

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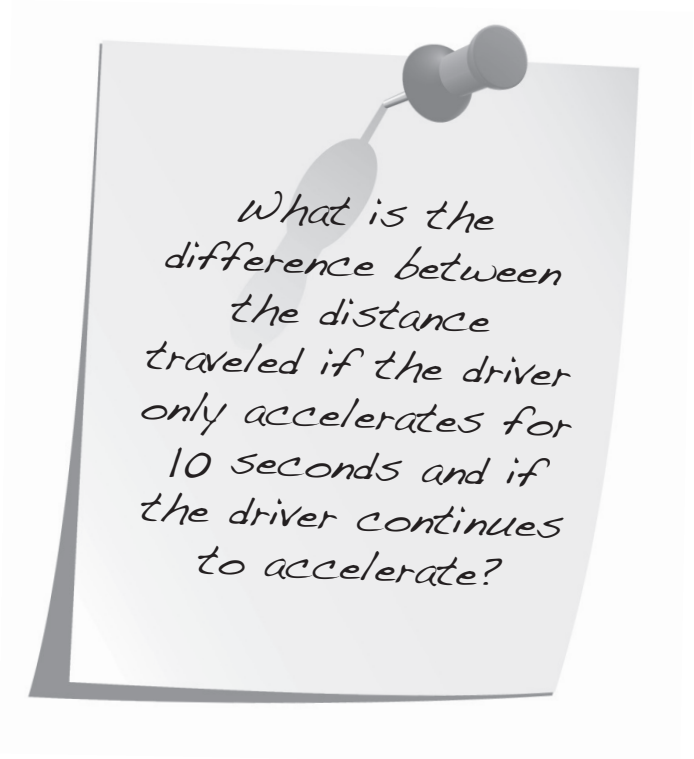
Problem-Based Task 2.4.3: Stop and Go

SMP	
1	2 ✓
3	4 ✓
5	6 ✓
7	8

A driver is stopped at a red light. When the light turns green, the driver will accelerate, then drive at a steady speed. The total distance traveled by the car, d, in meters, at any time t, in seconds, can be represented using the piecewise function $d_1 = \begin{cases} 1.34t^2;\ 0 \le t \le 10 \\ 26.8t - 134;\ t > 10 \end{cases}$. If the driver continues to accelerate, rather than driving at a steady rate after 10 seconds, the function $d_2 = 1.34t^2$ represents the driver's distance traveled. What is the difference between the distance traveled if the driver only accelerates for 10 seconds and if the driver continues to accelerate? Use graphs and algebraic representations to explain your answer.



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Problem-Based Task 2.4.3: Stop and Go

Coaching

a. What is a reasonable domain to use for both d_1 and d_2?

b. Create a graph of the piecewise function, d_1.

c. How are the two pieces of function d_1 different?

d. Create a graph of the quadratic function, d_2.

e. Are there any similarities between the graphs of d_1 and d_2?

f. How are these similarities seen in the algebraic equations?

g. Describe the differences between d_1 and d_2, including descriptions of both the graphs and the algebraic equations.

h. What do these differences and similarities tell you about how far a car would travel following each function?

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Problem-Based Task 2.4.3: Stop and Go

Coaching Sample Responses

a. What is a reasonable domain to use for both d_1 and d_2?

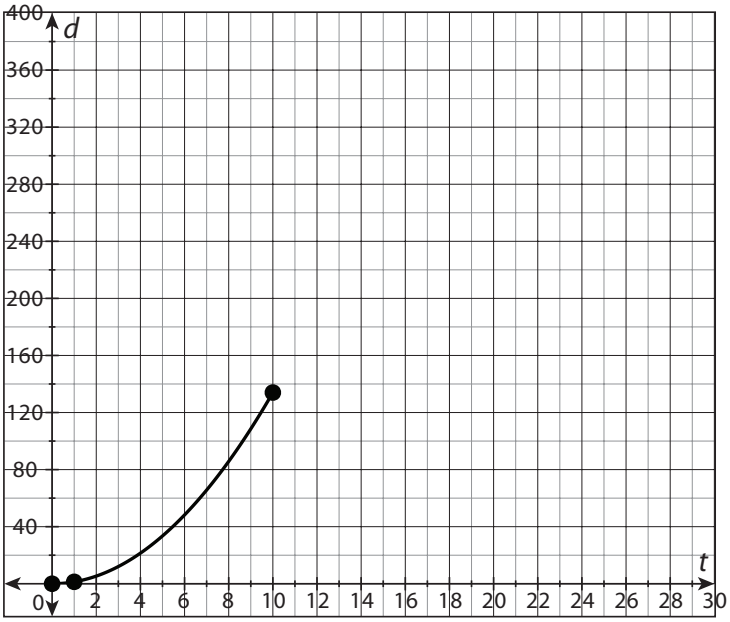
The minimum value of the piecewise function is 0. Use a maximum value that is larger than 10 so that both pieces of the piecewise function can be observed. Let's use $t = 30$ as the maximum of the domain. Therefore, a reasonable domain is $0 \leq t \leq 30$.

b. Create a graph of the piecewise function, d_1.

The first part of the function is quadratic. It may help to find three points on this piece of the function to understand its shape. Evaluate the function at three values of t on the domain of the first piece, including the start and end values of the domain.

For $t = 0$:	**For $t = 1$:**	**For $t = 10$:**
$d_1 = 1.34(0^2)$	$d_1 = 1.34(1^2)$	$d_1 = 1.34(10^2)$
$d_1 = 1.34(0)$	$d_1 = 1.34(1)$	$d_1 = 1.34(100)$
$d_1 = 0$	$d_1 = 1.34$	$d_1 = 134$

(0, 0), (1, 1.34), and (10, 134) are all points on the first piece of the function d_1.



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The second piece of the function is linear, so only two points on the line are needed. Find the value of the function at the minimum of the domain, $t = 10$, and the determined maximum, $t = 30$.

For $t = 10$:

$d_1 = 26.8(10) - 134$

$d_1 = 268 - 134$

$d_1 = 134$

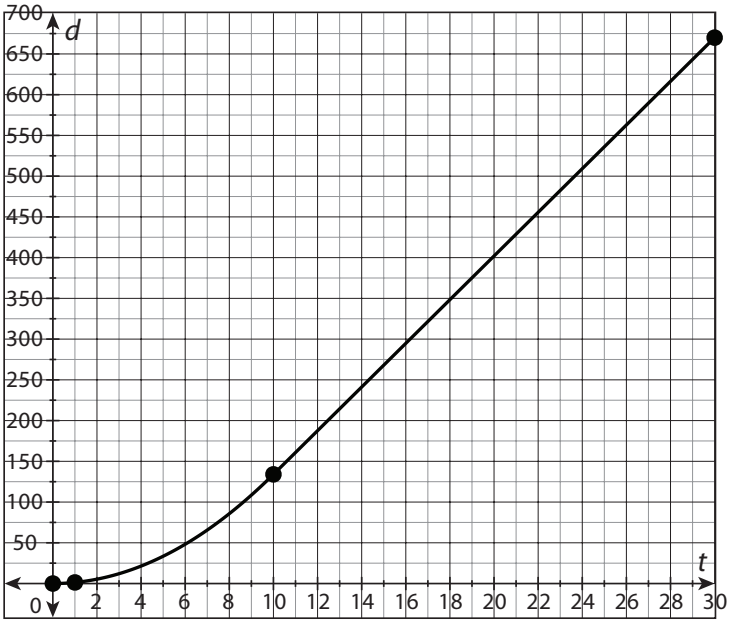
For $t = 30$:

$d_1 = 26.8(30) - 134$

$d_1 = 804 - 134$

$d_1 = 670$

The point (10, 134) is not included on this piece of the function, but it is shown with a closed circle on the following graph because it is a point on the first piece of the function. The point (30, 670) is on the second piece of the function.



c. How are the two pieces of function d_1 different?

The first piece of the function, the quadratic piece, is curved. The second piece of the function, the linear piece, is a line. The distance traveled is increasing at a constant rate for the second piece of the function. The distance is increasing at a variable rate for the first piece of the function.

d. Create a graph of the quadratic function, d_2.

The graph of d_1 for $0 \le d \le 10$ is the same as the graph of d_2, since they are defined by the same algebraic equation. The first three determined points on d_1 are also on d_2. Find a fourth point on d_2, for $t > 10$, to see the shape of the rest of d_2.

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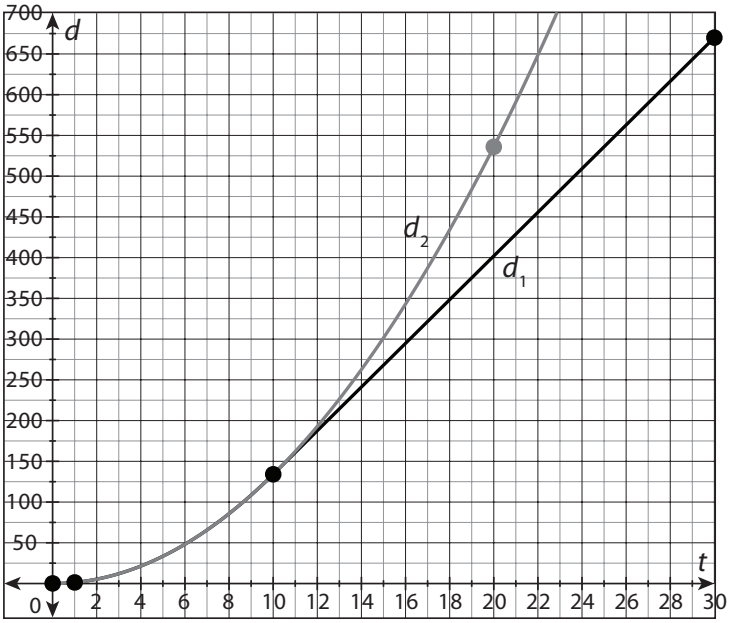
Let $t = 20$.

$d_2 = 1.34(20^2)$

$d_2 = 1.34(400)$

$d_2 = 536$

The point (20, 536) is a fourth point on the graph of d_2.



e. Are there any similarities between the graphs of d_1 and d_2?

Yes. The graphs are identical for $0 \le t \le 10$.

f. How are these similarities seen in the algebraic equations?

The algebraic equations for $0 \le t \le 10$ are the same for both functions.

g. Describe the differences between d_1 and d_2, including descriptions of both the graphs and the algebraic equations.

The graphs of the functions for $t > 0$ are both increasing, but the graph of d_2 is increasing faster than the graph of d_1. The graph of d_1 is increasing at a constant rate for $t > 10$, because the function is linear, whereas the graph of d_2 is increasing at a changing rate because the function is quadratic.

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h. What do these differences and similarities tell you about how far a car would travel following each function?

The output, d, is the total distance traveled. For $x > 10$, d_2 is always greater than d_1. This means a car following the function d_2 will travel farther than a car following the function d_1, if both cars began traveling at the same time. The car following the function d_2 will also continue to increase its speed, whereas the car following the function d_1 would drive at the same speed for $t > 10$ seconds.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2.4.3: Piecewise Functions

For problems 1–6, create a graph for each of the following piecewise functions.

1. $y=\begin{cases}7x, \text{ if } x<3\\ x^2, \text{ if } x\geq 3\end{cases}$

2. $y=\begin{cases}-6, \text{ if } 1\leq x\leq 8\\ x-10, \text{ if } x>8\end{cases}$

3. $y=\begin{cases}-3x+1, \text{ if } x<2\\ 4x-13, \text{ if } x\geq 2\end{cases}$

4. $y=\begin{cases}-x^2+1, \text{ if } -1\leq x<1\\ 10, \text{ if } x\geq 1\end{cases}$

5. $y=\begin{cases}-x+12, \text{ if } x<15\\ x-18, \text{ if } x\geq 15\end{cases}$

6. $y=\begin{cases}2x^2, \text{ if } -2\leq x\leq 2\\ -x^2+12, \text{ if } x>2\end{cases}$

continued

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Read the following scenario and use the information in it to complete problems 7 and 8.

Saar is walking to school. He looks at his watch, realizes he is running late, and starts jogging. Saar's distance to school in miles, d, at any time in minutes t, can be represented using the function $d = \begin{cases} -0.1t + 1.5, \text{ if } 0 \le t < 8 \\ -0.2t + 2.3, \text{ if } t \ge 8 \end{cases}$.

7. Create a graph to show Saar's distance from school at any time t.

8. How long does it take Saar to reach school? How can you tell from the graph?

Read the following scenario and use the information in it to complete problems 9 and 10.

Saar compares this walk to school to his walk to school yesterday, when he walked at a steady pace. His distance from school yesterday could be represented by the function $d_2 = -0.12t + 1.5$.

9. Graph the function for yesterday's walk on the same coordinate plane as the graph of today's walk.

10. Describe any differences in the domain, range, and slope of the two graphs. On which day did Saar reach school in less time?

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Progress Assessment

Circle the letter of the best answer.

1. What is the range of $y=-\sqrt{(x+2)}-1$?

   a. $y \le 0$
   b. $y \le -1$
   c. $y \ge 0$
   d. $y \ge -1$

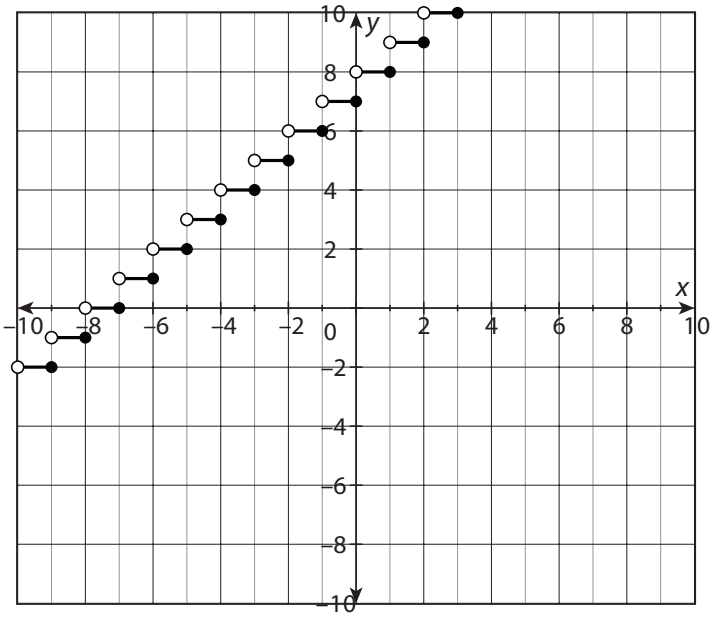
2. Which function has a point of inflection at (0, –1)?

   a. $y=-\sqrt{(x+1)}$
   b. $y=\sqrt[3]{x}-1$
   c. $y=\sqrt[3]{(x+1)}$
   d. $y=\sqrt[3]{x}+1$

3. What are the coordinates of the minimum point of the graph of $y=2|x-1|$?

   a. (0, 0)
   b. (–1, 0)
   c. (0, 1)
   d. (1, 0)

4. Which function is shown in the graph?



   a. $y=\lceil x+7 \rceil$
   b. $y=\lceil x \rceil-7$
   c. $y=\lfloor x-7 \rfloor$
   d. $y=x+7$

continued

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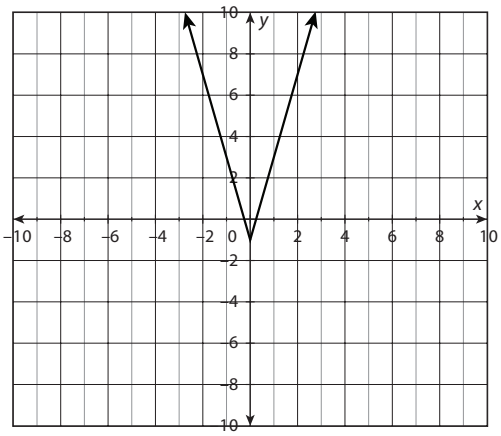
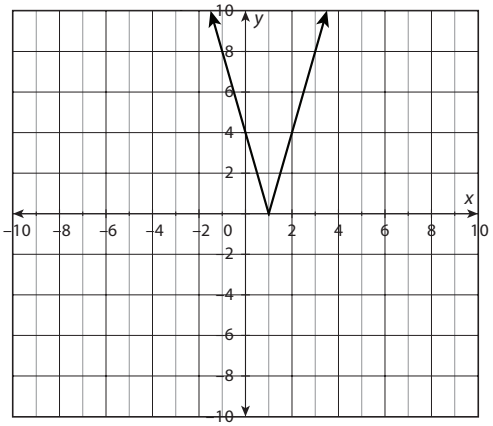
Assessment

5. Which graph shows the function $y = 4|x-1|$?

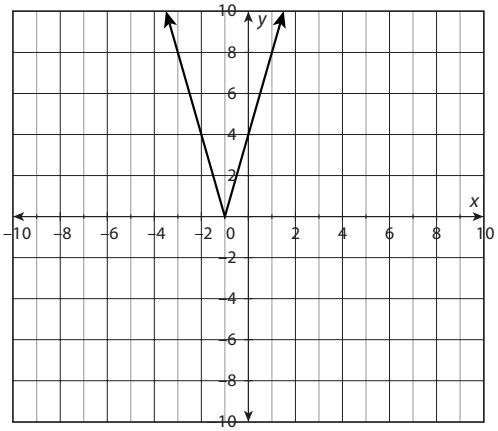
a.



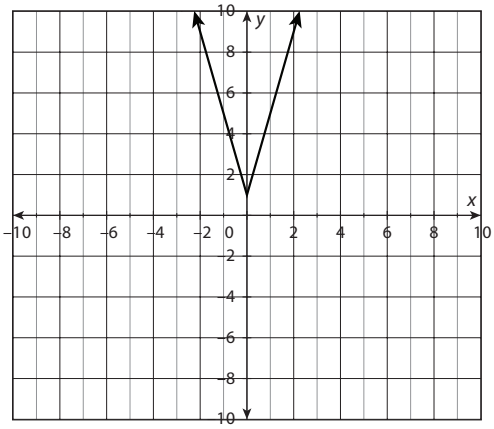
c.



b.



d.



6. What is the domain of $y = \sqrt{(x+12)} + 3$?

a. $x \leq 0$

b. $x \leq 3$

c. $x \leq 12$

d. $x \geq -12$

7. Which function has a point of inflection at (–3, 8)?

a. $y = 3\sqrt[3]{(x+3)} + 8$

b. $y = 3\sqrt[3]{x} + 8$

c. $y = \sqrt[3]{(x-3)} + 8$

d. $y = -3\sqrt[3]{x} + 8$

continued

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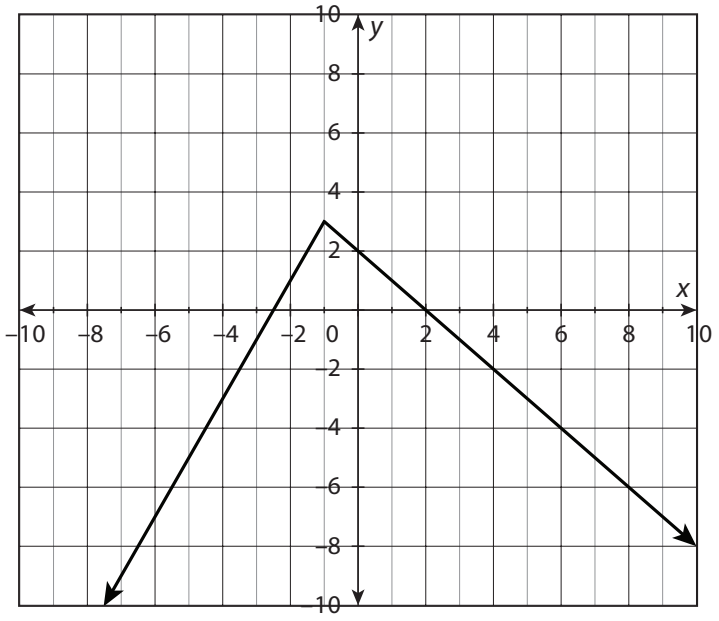
8. What are the coordinates of the maximum point of the graph of $y=-|x+4|+10$?

   a. (–4, 10)
   b. (4, 10)
   c. (0, –10)
   d. (–10, 4)

9. Which function is shown in the graph?



   a. $y=\begin{cases}2x-5, \text{ if } x<-1\\ x+2, \text{ if } x>-1\end{cases}$
   b. $y=\begin{cases}x, \text{ if } x<-1\\ -2x+2, \text{ if } x>-1\end{cases}$
   c. $y=\begin{cases}2x+5, \text{ if } x<-1\\ -x+2, \text{ if } x>-1\end{cases}$
   d. $y=\begin{cases}x-5, \text{ if } x<-1\\ -x+2, \text{ if } x>-1\end{cases}$

continued

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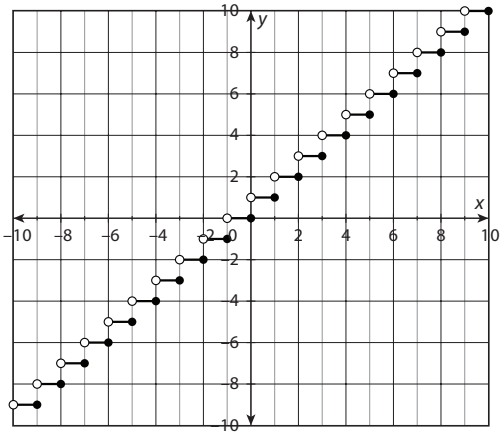
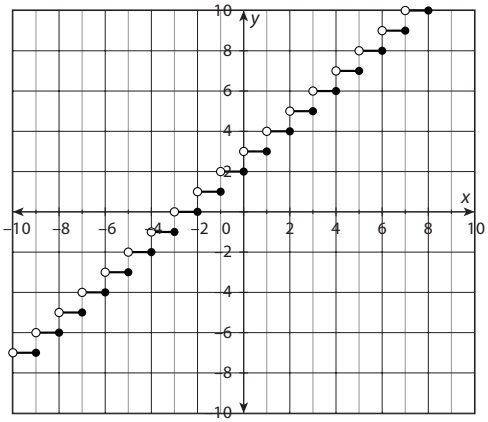
10. Which graph shows the function $y = \lceil 2x \rceil$?

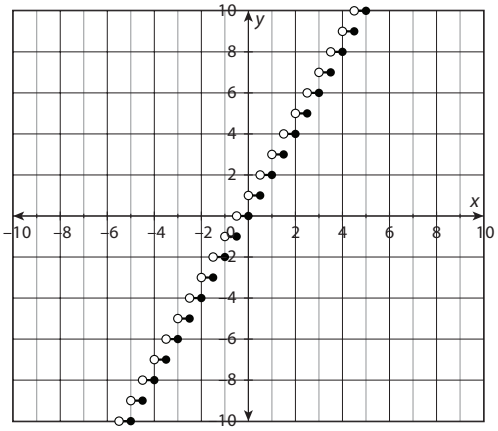
a.



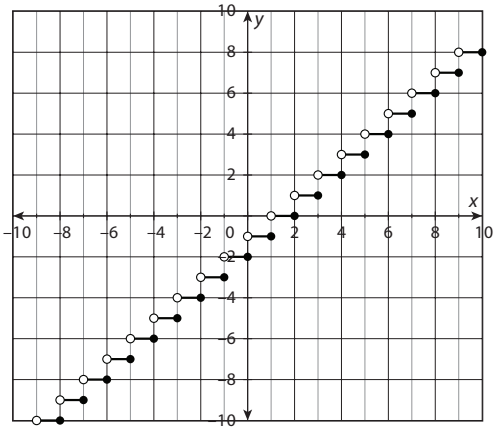
c.



b.



d.



Use what you have learned about functions to solve the problem that follows.

11. Orion is walking his dog, Duke. Orion and Duke start their walk moving pretty quickly, but then change to a slower pace after Duke gets tired. Their total distance walked, d, in miles, at any time t, in minutes, can be represented by the following function.

$$d = \begin{cases} 0.2t, \text{ if } 0 < t < 15 \\ 0.15t + 0.75, \text{ if } 15 \leq t < 30 \end{cases}$$

a. What is the domain of t?

b. Identify any extreme values of d.

c. Create a graph to show the total distance walked at any time t.

d. Approximately how far did Orion and Duke walk after 25 minutes?

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. If $f(x) = 18(1.34)^x$, what is the percent rate of change of y for each unit of x?
   a. 1.34%
   b. 3.4%
   c. 34%
   d. 134%

2. If $f(x) = 40(0.75)^{5x}$, what is the percent rate of change of y for each unit of x?
   a. 0.24%
   b. 24%
   c. 0.75%
   d. 75%

3. Which of the following quadratic functions has a vertex with the greatest y-coordinate?
   a. $A(x) = -2(x + 7)^2 + 3$
   b. $B(x) = 2x^2 + 4x + 4$
   c. $C(x) = (x - 2)(x + 4)$
   d. It cannot be determined from this information.

continued

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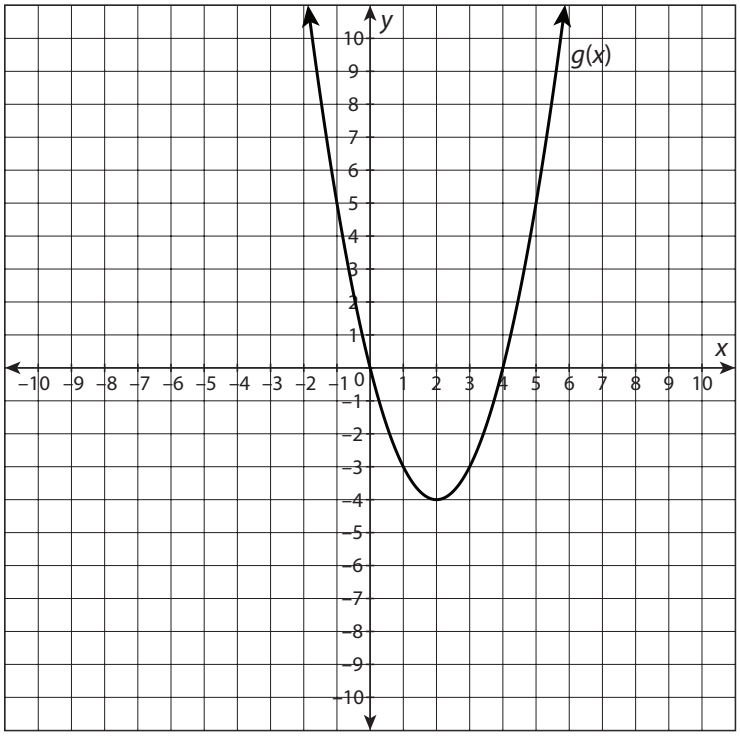
4. Which of the following statements is true about the quadratic functions $f(x)$ and $g(x)$?

x	$f(x)$
–3	8
–2	4
0	2
2	8
3	14



a. The function $f(x)$ has a greater maximum than the function $g(x)$.

b. The function $g(x)$ has a greater maximum than the function $f(x)$.

c. The function $f(x)$ has a greater minimum than the function $g(x)$.

d. The function $g(x)$ has a greater minimum than the function $f(x)$.

continued

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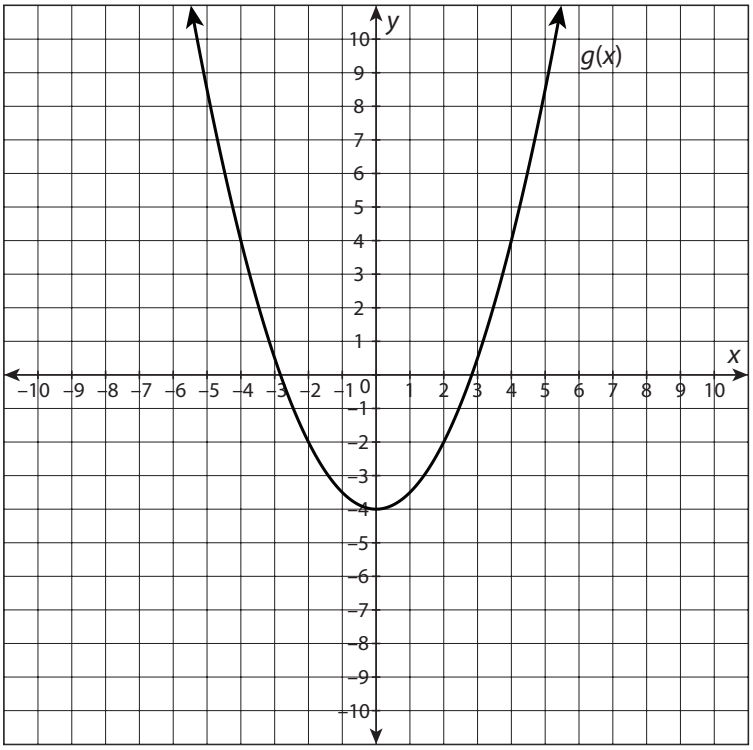
5. Which of the following statements is true about the functions $f(x)$ and $g(x)$?

$f(x) = 2^x - 5$



a. The y-intercept of the function $f(x)$ is less than the y-intercept of the function $g(x)$.

b. The y-intercept of the function $f(x)$ is greater than the y-intercept of the function $g(x)$.

c. The y-intercept of the function $f(x)$ is equal to the y-intercept of the function $g(x)$.

d. The y-intercepts cannot be determined.

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Lesson 5: Analyzing Functions

Instruction

Common Core State Standards

SMP	
1	2 ✓
3 ✓	4 ✓
5	6 ✓
7	8

F–IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, *and classify them as representing exponential growth or decay.*

F–IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

F–LE.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.★

Essential Questions

1. How is the rate of change of an exponential function related to the exponential expression?
2. How is the percent rate of change of an exponential function different from the rate of change of a linear function?
3. How is a quadratic equation similar to a linear equation? How is it different?
4. How is a quadratic equation similar to an exponential equation? How is it different?

WORDS TO KNOW

decay factor	$1 - r$ in the exponential decay model $f(t) = a(1 - r)^t$, or b in the exponential function $f(x) = ab^x + k$ if $0 < b < 1$; the multiple by which a quantity decreases over time
decay rate	r in the exponential decay model $f(t) = a(1 - r)^t$
exponential decay	an exponential equation with a base, b, that is between 0 and 1 exclusive (that is, $0 < b < 1$); an example is the formula $y = a(1 - r)^t$, where a is the initial value, $(1 - r)$ is the base (with $0 < r < 1$), t is the variable exponent, and y is the final value

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Instruction

exponential decay model	an exponential function, $f(t) = a(1 - r)^t$, where $f(t)$ is the final output value, a is the initial value, r is the percent decrease per time period (expressed as a decimal), and t is the variable exponent, usually representing time
exponential function	a function whose independent variable is in the exponent; the general form of its equation is $f(x) = ab^x + k$, where a is the initial value, b is the base, x is the input value, k is the vertical shift, and $f(x)$ is the output
exponential growth	an exponential equation with a base, b, greater than 1 ($b > 1$); an example is the formula $f(t) = a(1 + r)^t$, where a is the initial value, $(1 + r)$ is the base (with $r > 0$), t is the variable exponent, and $f(t)$ is the final value
exponential growth model	an exponential function, $f(t) = a(1 + r)^t$, where $f(t)$ is the final output value, a is the initial value, r is the percent increase per time period (expressed as a decimal), and t is the variable exponent, usually representing time
first difference	in a set of data, the change in the y-value when the x-value is increased by 1
growth factor	$1 + r$ in the exponential growth model $f(t) = a(1 + r)^t$, or b in the exponential function $f(x) = ab^x + k$ if $b > 1$; the factor by which a quantity increases over time
growth rate	the rate of increase in size per unit of time; r in the exponential growth model $f(t) = a(1 + r)^t$
interval	the continuous set of all real numbers between two given numbers. The two numbers on the ends of the interval are the endpoints. The endpoints may or may not be included in the interval depending on whether the interval is open, closed, or half-open/half-closed.
linear function	a first-degree equation that can be written in the form $f(x) = mx + b$, in which m is the slope of the line and b is the y-intercept. The graph of a linear function is a straight line.
quadratic function	a function that can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. The graph of a quadratic function is a parabola.
second difference	in a set of data, the change in successive first differences

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Recommended Resources

- Oswego City School District Regents Exam Prep Center. "Exponential Growth and Decay."

  http://www.walch.com/rr/00152

  This test preparation site defines and compares exponential growth and decay. Provided examples include algebraic equations, tables of values, and graphical representations. The site also links to calculator instructions for graphing exponential functions.

- Oswego City School District Regents Exam Prep Center. "The Fable of the Chess Board and the Grains of Wheat."

  http://www.walch.com/rr/00153

  This site relates the legend of how the inventor of chess tricked a king into promising a reward that turned out to be greater than the king bargained for. Users can explore the concept of exponential growth demonstrated by the legend by completing a chart, writing a function, and then graphing the function. Answers are provided.

- Virtual Nerd. "How Do You Determine If a Graph Represents a Linear, Exponential, or Quadratic Function?"

  http://www.walch.com/rr/00154

  This video tutorial covers how to determine whether a graph represents a linear, quadratic, or exponential function.

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Lesson 2.5.1: Analyzing Exponential Functions

Warm-Up 2.5.1

If a cube has a side length of s, then the volume of the cube, V, is represented by the equation $V = s^3$. Claire is moving, and is packing her belongings into cube-shaped boxes. A moving company sells cube-shaped boxes with three different side lengths: 15 inches, 18 inches, and 24 inches. The price, p, of each box can be determined using the equation $p = 0.0005 \cdot V$, where p is the price in dollars and V is the box's volume.

1. What is the volume of a cube-shaped box with 18-inch sides?

2. What is the price of a cube-shaped box with 18-inch sides?

3. Claire decides to buy one 15-inch box and two 24-inch boxes. What is the total volume of the three boxes?

4. What is the price of one 15-inch box and two 24-inch boxes?

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Lesson 2.5.1: Analyzing Exponential Functions

Common Core State Standard

F–IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, *and classify them as representing exponential growth or decay.*

Warm-Up 2.5.1 Debrief

1. What is the volume of a cube-shaped box with 18-inch sides?

   Use the equation $V = s^3$ to find the volume of a cube with $s = 18$.

$V = s^3$	Given equation
$V = (18)^3$	Substitute 18 for s.
$V = 5832$	Simplify.

   The volume of an 18-inch cube-shaped box is 5,832 in^3.

2. What is the price of a cube-shaped box with 18-inch sides?

   Use the volume of an 18-inch box, 5,832 in^3, to find the price of the box.

   The price of a box can be determined using the given equation, $p = 0.0005 \cdot V$.

   Substitute 5,832 for V to find the price of a box with a volume of 5,832 in^3.

$p = 0.0005 \cdot V$	Given equation
$p = 0.0005 \cdot 5832$	Substitute 5,832 for V.
$p = 2.916$	Simplify.

   The price of an 18-inch cube-shaped box, rounded to the nearest whole cent, is $2.92.

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Instruction

3. Claire decides to buy one 15-inch box and two 24-inch boxes. What is the total volume of the three boxes?

   To find the total volume, find the volume of each box, and then add the volumes of the boxes.

   Calculate the volume of a 15-inch box.

$V = s^3$	Given equation
$V = (15)^3$	Substitute 15 for s.
$V = 3375$	Simplify.

   The volume of a 15-inch cube-shaped box is 3,375 in^3.

   Calculate the volume of a 24-inch box.

$V = s^3$	Given equation
$V = (24)^3$	Substitute 24 for s.
$V = 13{,}824$	Simplify.

   The volume of a 24-inch cube-shaped box is 13,824 in^3.

   The volume of a 15-inch box is 3,375 in^3. The volume of a 24-inch box is 13,824 in^3. Claire is buying one 15-inch box and two 24-inch boxes. The total volume of the three boxes is $3375 + 2(13{,}824) = 31{,}023$ in^3.

4. What is the price of one 15-inch box and two 24-inch boxes?

   The price of the three boxes can be found using the total volume of the three boxes.

   The volume of the three boxes is 31,023 in^3. The price is $p = 0.0005 \cdot V$. Substitute 31,023 for V and solve.

$p = 0.0005 \cdot V$	Given equation
$p = 0.0005 \cdot 31{,}023$	Substitute 31,023 for V.
$p = 15.5115$	Simplify.

   The price of one 15-inch box and two 24-inch boxes, rounded to the nearest whole cent, is $15.51.

Connection to the Lesson

- Students will evaluate expressions involving both products and exponents as they analyze exponential functions.

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Prerequisite Skills

This lesson requires the use of the following skills:

- identifying the base and power of an exponent (6.EE.1)
- simplifying exponential expressions with integer exponents (8.EE.1)
- applying the order of operations to evaluate expressions (5.OA.1)

Introduction

If the relationship between two quantities can be represented using a linear equation, such as $y = 2x$, then the rate of change of the function is constant—that is, the rate of change does not vary. For example, if you are paid a flat rate of $10 each time you rake your neighbor's yard, and never spend that money, the amount of money you have from raking increases by $10 each time you rake. The $10 you add is a constant rate of change.

Some quantities change at a rate that is not constant. Take the example of Burmese pythons from Southeast Asia being introduced into the Florida Everglades. There are no natural predators for Burmese pythons in Florida, so the python population has grown more and more each year. The rate of change in the number of pythons from year to year is not constant, because each year there are more pythons that lay more eggs, which become more pythons, and so on. (Though park officials have tried to remove the pythons to protect native species in the Everglades, the officials cannot keep up with the pythons' increasing rate of growth.)

Exponential functions, such as $f(x) = 2^x$, show change that is not constant. Rather, the rate itself varies each time the quantity is measured.

Key Concepts

- An **exponential function** is a function whose independent variable is in the exponent; the general form of its equation is $f(x) = ab^x + k$, where a is the initial value, b is the base, x is the input value, k is the vertical shift, and $f(x)$ is the output.
- An exponential function can be used to model exponential growth (increase) or decay (decrease) of quantities over time.
- If $b > 1$, then there is **exponential growth**, and b is the **growth factor**, or the factor by which a quantity increases over time. The growth factor is the expression $1 + r$ in the exponential growth model $f(t) = a(1 + r)^t$, or b in the exponential function $f(x) = ab^x + k$ if $b > 1$.
- The general form of an exponential function modeling growth is $f(t) = a(1 + r)^t$.

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Instruction

- This form is known as the **exponential growth model**, where $f(t)$ is the final value, a is the initial value, r is the **growth rate** or percent increase per time period (expressed as a decimal), and t is the variable exponent, usually representing time.
- A function that represents exponential growth is an increasing function, in which the dependent variable increases as the independent variable increases.
- If $0 < b < 1$, then there is **exponential decay**, and b is the **decay factor**, or factor by which a quantity decreases over time. The decay factor is the expression $1 - r$ in the exponential decay model $f(t) = a(1 - r)^t$, or b in the exponential function $f(x) = ab^x + k$ if $0 < b < 1$.
- The general form of an exponential function modeling decay is $f(t) = a(1 - r)^t$.
- This is known as the **exponential decay model**, where $f(t)$ is the final value, a is the initial value, r is the **decay rate** or percent decrease per time period (expressed as a decimal), and t is the variable exponent, usually representing time.
- A function that represents exponential decay is a decreasing function, in which the dependent variable decreases as the independent variable increases.
- Understanding exponential growth and decay allows us to interpret exponential functions that represent many real-world situations.
- A helpful property of exponents is the Power of a Power Property.

Power of a Power Property

If a is a real number and m and n are rational numbers, then $(a^m)^n = a^{m \cdot n}$.

Examples:

$$\left(3^5\right)^2 = 3^{5 \cdot 2} = 3^{10} = 59{,}049$$

$$\left(1.02^{\frac{1}{4}}\right)^4 = 1.02^{\left(\frac{1}{4} \cdot 4\right)} = 1.02^1 = 1.02$$

Common Errors/Misconceptions

- interpreting the rate r as a constant linear rate
- incorrectly applying the order of operations to evaluate the expression ab^t
- identifying $b > 1$ as exponential decay and $b < 1$ as exponential growth

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Instruction

Guided Practice 2.5.1

Example 1

A school tracks the total number of students enrolled each year. The school uses the change in the total number of students to estimate how many students have been enrolled in the school each year since 2000. If t is the number of years after 2000, the total number of students, $f(t)$, can be estimated using the function $f(t) = 250(0.98)^t$. How is the total number of students changing each year?

1. Identify the yearly rate of change in the function.

   In the function $f(t) = 250(0.98)^t$, $b = 0.98$, and $0 < b < 1$. Therefore, the function represents exponential decay, which can be modeled by the form $f(t) = a(1 - r)^t$. To find the yearly rate of change, r, set b equal to $1 - r$ and solve for r.

   $0.98 = 1 - r$

   $0.98 - 1 = -r$

   $-0.02 = -r$

   $0.02 = r$

   The rate of change is 0.02. In this decay model, this indicates a decrease of 2% per year.

2. Describe how the rate of change relates to the change of the dependent quantity.

   The rate of change explains how the dependent quantity is growing or decaying over the given unit of time. The dependent quantity is the number of students, and the unit of time is "each year." The function $f(t) = 250(0.98)^t$ shows exponential decay, and the rate of change is 0.02, or 2%. The independent variable t is the time in years. The total number of students is decreasing by 2% each year.

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Instruction

Example 2

A bank offers a savings account with interest that is compounded monthly. In other words, the interest earned is added to the account every month instead of once a year. Dillon opened a savings account with \$500. If t is the time in years the account has been open, the balance in his account, $f(t)$, is $f(t) = 500(1.004)^{12t}$. What is the estimated yearly exponential growth rate? Describe how this rate relates to the yearly change of the balance in Dillon's account.

1. Use the Power of a Power Property to simplify the exponential expression.

   The expression $(1.004)^{12t}$ can be simplified using the Power of a Power Property to remove the factor of the exponent.

$f(t) = 500(1.004)^{12t}$	Original function
$f(t) = 500\left(1.004^{12}\right)^{t}$	Apply the Power of a Power Property.
$f(t) = 500(1.049)^{t}$	Simplify.

2. Identify the yearly rate of change in the function.

   In the function $f(t) = 500(1.049)^{t}$, $b = 1.049$ and $b > 1$. Therefore, the function represents exponential growth, which can be modeled by the form $f(t) = a(1 + r)^{t}$. To find the yearly rate of change, r, set b equal to $1 + r$ and solve for r.

   $1.049 = 1 + r$

   $1.049 - 1 = r$

   $0.049 = r$

   The rate of change is 0.049, or an increase of 4.9% per year.

3. Describe how the rate of change relates to the change of the dependent quantity.

   The rate of change explains how the dependent quantity is growing or decaying over the given unit of time. The yearly rate of change in the function $f(t) = 500(1.004)^{12t}$ is 4.9%. The independent variable t is the time in years. The total balance of the account, in dollars, is increasing by approximately 4.9% each year.

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Instruction

Example 3

A number of bacteria, $f(t)$, at any time t, in hours, can be estimated using the function $f(t) = 3000(1.24)^t$. What was the initial size of the bacteria colony? Is the bacteria population exponentially decaying or growing?

1. Identify the value of the dependent quantity at $t = 0$.

   In an exponential function of the form $f(t) = a(1 + r)^t$, a is the value of the dependent quantity, $f(t)$, when $t = 0$. In the function $f(t) = 3000(1.24)^t$, the factor a is 3,000. Recall that any number raised to a power of 0 is equal to 1. Therefore, the initial size of the bacteria colony was 3,000 bacteria.

2. Identify the hourly rate of change in the function.

   In the function $f(t) = 3000(1.24)^t$, $b = 1.24$ and $b > 0$. Therefore, there is exponential growth, which can be modeled by the form $f(t) = a(1 + r)^t$. To find the ~~yearly~~ rate of change, r, set b equal to $1 + r$ and solve for r.

   $1.24 = 1 + r$

   $1.24 - 1 = r$

   $0.24 = r$

   The rate of change is 0.24, or an increase of 24% per hour.

3. Describe how the rate of change relates to the change of the dependent quantity.

   The rate of change explains how the dependent quantity is growing or decaying over the given unit of time. The hourly rate of change in the function $f(t) = 3000(1.24)^t$ is 24%. The independent variable t is the time in hours. The total number of bacteria in the colony is increasing by approximately 24% every hour, so the bacteria population is exponentially growing.

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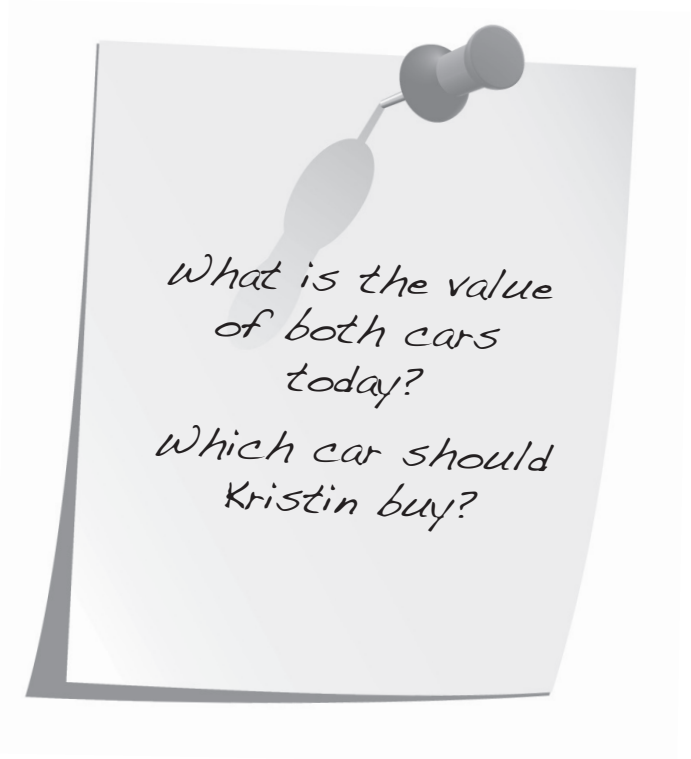
Problem-Based Task 2.5.1: Which Car Is a Better Buy?

SMP	
1	2 ✓
3 ✓	4
5	6 ✓
7	8

Kristin is buying a new car, and is trying to decide between two particular models. Both cars are the same price today. Kristin knows that the value of each car will change over time. According to her research, the first car, the Roadvana, will have a value of v_1 dollars after t years, modeled by the equation $v_1 = 22{,}000(0.90)^{2t}$. The second car, the Savannah, will have a value of v_2 dollars after t years, modeled by the equation $v_2 = 22{,}000(0.86)^{t}$. Kristin wants to buy the car that will have a higher value in the future. What is the value of both cars today? Which car should Kristin buy?



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Problem-Based Task 2.5.1: Which Car Is a Better Buy?

Coaching

a. Which quantity of each equation represents the value of the car at $t = 0$, or today?

b. Rewrite the equation for the value of the Roadvana with a simplified exponential expression.

c. What is the yearly rate of change in the value of the Roadvana?

d. What is the yearly rate of change in the value of the Savannah?

e. Does each car's value grow or decay over time?

f. Which car will have a higher value in the future?

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Instruction

Problem-Based Task 2.5.1: Which Car Is a Better Buy?

Coaching Sample Responses

a. Which quantity of each equation represents the value of the car at $t = 0$, or today?

In an exponential function of the form $f(t) = a(1 + r)^t$ or $f(t) = a(1 - r)^t$, a is the value of the dependent quantity, $f(t)$, when $t = 0$.

Identify a in each equation to find the value of each car today.

The value of the Roadvana is represented by $v_1 = 22{,}000(0.90)^{2t}$. In this equation, the factor 22,000 represents a. Therefore, the initial value of the Roadvana is $22,000.

The value of the Savannah is represented by $v_2 = 22{,}000(0.86)^t$. The factor 22,000 represents a. Therefore, the initial value of the Savannah is also $22,000.

b. Rewrite the equation for the value of the Roadvana with a simplified exponential expression.

Use the Power of a Power Property to simplify the exponential expression $(0.90)^{2t}$.

$$0.90^{2t} = \left(0.90^2\right)^t = 0.81^t$$

Replace the exponential expression $(0.90)^{2t}$ with the simplified expression in the equation for v_1.

$v_1 = 22{,}000(0.90)^{2t}$

$v_1 = 22{,}000(0.81)^t$

c. What is the yearly rate of change in the value of the Roadvana?

In the exponential function $v_1 = 22{,}000(0.81)^t$, $b = 0.81$ and $0 < b < 1$. Therefore, this is an exponential decay model. To find the rate of change, set b equal to $1 - r$ and solve for r.

$0.81 = 1 - r$

$0.81 - 1 = -r$

$-0.19 = -r$

$0.19 = r$

The yearly rate of change for the Roadvana is 0.19, or a decrease of 19% per year.

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d. What is the yearly rate of change in the value of the Savannah?

The function $v_2 = 22{,}000(0.86)^t$ is also an example of exponential decay, because $b = 0.86$ and $0 < b < 1$. To find the rate of change, r, set b equal to $1 - r$ and solve for r.

$0.86 = 1 - r$

$0.86 - 1 = -r$

$-0.14 = -r$

$0.14 = r$

The yearly rate of change for the Savannah is 0.14, or a decrease of 14% per year.

e. Does each car's value grow or decay over time?

The yearly rate of change is a decrease for both cars. The values of both cars decay over time.

f. Which car will have a higher value in the future?

The car with the higher future value is the car with a lower yearly rate of decrease. The Savannah has a lower decrease rate of 14%; therefore, it will have a higher value in the future.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2.5.1: Analyzing Exponential Functions

Find the percent rate of change of $f(t)$ for each unit of t. State whether the function shows exponential growth or decay.

1. $f(t) = 110(0.95)^t$

2. $f(t) = 1.08(1.07)^t$

3. $f(t) = 30(0.90)^{4t}$

4. $f(t) = 63(0.87)^{11t}$

5. $f(t) = 500(1.15)^{2t}$

Use the information to complete problems 6–8.

> The deer population, p, in a forest preserve t years after 2005 can be estimated using the function $p(t) = 440(0.92)^t$.

6. What was the size of the deer population in 2005?

continued

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7. What is the yearly rate of change of the population?

8. The wolf population may be related to the deer population. The wolf population, w, can be estimated t years after 2005 using the function $w(t) = 84(0.98)^{2t}$. Which population is changing faster? Explain your answer.

Use the information to complete problems 9 and 10.

Neal opens a savings account that earns interest monthly. He can estimate the total dollars in his account, $d(t)$, t years after opening the account by using $d(t) = 4000(1.0008)^{12t}$.

9. How much money did Neal initially put into the account?

10. What is the yearly rate of change of the account? Is it growing or decaying?

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Lesson 2.5.2: Comparing Properties of Functions Given in Different Forms

Warm-Up 2.5.2

Two seagulls dive into the ocean. The given functions represent the height of each seagull above the surface of the ocean as a function of the seagull's horizontal distance from a certain buoy. For each pair of functions, determine which bird descends deeper into the ocean.

1. $f(x) = 3(x - 2)^2 - 5$ or $g(x) = \{(-8, 0), (-6, -4), (-4, 0)\}$

2. $f(x) = 3x^2 - 12x + 7$ or $g(x) = \frac{1}{2}(x+2)^2 - 6$

3. $f(x) = 2x^2 - 8x + 11$ or the function outlined in the following table:

x	−3	−1	1	3	5
$g(x)$	11	6	3	2	3

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Instruction

Lesson 2.5.2: Comparing Properties of Functions Given in Different Forms

Common Core State Standards

F–IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

F–LE.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.★

Warm-Up 2.5.2 Debrief

1. $f(x) = 3(x-2)^2 - 5$ or $g(x) = \{(-8, 0), (-6, -4), (-4, 0)\}$

   The function $f(x) = 3(x-2)^2 - 5$ is written in vertex form and has a vertex of $(2, -5)$, which is lower than the minimum value of $(-6, -4)$ in the function for $g(x)$.

   The first bird descends deeper into the ocean.

2. $f(x) = 3x^2 - 12x + 7$ or $g(x) = \frac{1}{2}(x+2)^2 - 6$

   The function $g(x) = \frac{1}{2}(x+2)^2 - 6$ is written in vertex form and has a vertex of $(-2, -6)$. Using the formula $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$, we find that the vertex of the function $f(x) = 3x^2 - 12x + 7$ is $(2, -5)$.

   The vertex of the function $g(x) = \frac{1}{2}(x+2)^2 - 6$ is lower than that of the function $f(x) = 3x^2 - 12x + 7$. Therefore, the second bird descends deeper into the ocean.

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3. $f(x) = 2x^2 - 8x + 11$ or the function outlined in the following table:

x	–3	–1	1	3	5
$g(x)$	11	6	3	2	3

To find the vertex of $f(x) = 2x^2 - 8x + 11$, use the formula $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$. The function represented by the table of values has a vertex of (3, 2), which is lower than the vertex (2, 3) of the function $f(x) = 2x^2 - 8x + 11$.

The second bird descends closer to the water, but neither bird reaches the surface of the water.

Connection to the Lesson

- Students will extend their knowledge of quadratics to making comparisons between quadratics presented in different forms.
- Students will extend their knowledge of quadratics to making comparisons among quadratic functions, linear functions, and exponential functions presented in different forms.

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Prerequisite Skills

This lesson requires the use of the following skills:

- finding the vertex of a parabola (F–IF.7a★)
- finding the *x*-intercepts of a parabola (F–IF.7a★)
- identifying and graphing linear, quadratic, and exponential functions (A–REI.10, A–CED.2★, F–IF.7e)
- writing an equation for a simple exponential function (A–CED.1★)

Introduction

Mathematical relationships can be presented through different forms: tables, equations, graphs, and words. This section explores how to compare linear, quadratic, and exponential functions presented in all four forms.

Key Concepts

Linear Functions

- A **linear function** is a first-degree equation that can be written in the form $f(x) = mx + b$, in which m is the slope and b is the y-intercept. The graph of a linear function is a straight line.
- Linear functions have a constant slope.
- As the value of x increases by 1, $f(x)$ will increase by a constant value.
- The rate of change of a linear function will always remain constant; however, the function will either increase, decrease, or remain constant.
- In a set of data, the change in y when x increases by 1 is called a **first difference**. For a linear function, the first difference is a constant.

Quadratic Functions

- A quadratic function is a function that can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$.
- The graph of a quadratic function is a U-shaped curve called a parabola.
- The value of a determines whether the function has a maximum or a minimum.
- If a is negative, the parabola opens down and the quadratic function has a maximum.
- If a is positive, the parabola opens up and the quadratic function has a minimum.

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- There is neither a constant rate of change nor a constant factor of the variable in a quadratic function.
- In a quadratic model, the first difference is not a constant, but the *change* in the first differences is a constant.
- The change in successive first differences is called a **second difference**.
- Compare the *y*-values of the vertices of quadratic functions to determine the greatest maximum or least minimum values when comparing two or more quadratic functions.

Exponential Functions

- Exponential functions are written in the form $f(x) = ab^x$.
- The variable of an exponential function is part of the exponent.
- As the value of x increases, the value of $f(x)$ will increase or decrease by a constant factor.
- Exponential functions either increase or decrease.
- For exponential functions, the rate of change between *y*-values increases or decreases by a constant factor.
- An **interval** is the continuous set of all real numbers between two given numbers.
- The rate of change of an exponential function varies depending on the interval observed.
- Graphs of exponential functions of the form $f(x) = ab^x$, where b is greater than 1, will increase faster than graphs of linear functions of the form $f(x) = mx + b$, where m is positive.
- A quantity that increases exponentially will eventually exceed a quantity that increases linearly or quadratically.

Common Errors/Misconceptions

- misidentifying the appropriate type of function (linear, quadratic, or exponential)
- thinking that an equation with a higher *y*-intercept has a higher maximum value
- incorrectly determining the rate of change
- assuming that the rate of change of a function is linear by only referencing one interval

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Guided Practice 2.5.2

Example 1

Which function has a greater y-intercept, $f(x) = 8x - 2$ or $g(x) = 2(x - 3)(x + 1)$?

1. Make a general observation.

   $f(x) = 8x - 2$ is a linear function of the form $f(x) = mx + b$.

   The constant -2 is the y-intercept of the function.

   $g(x) = 2(x - 3)(x + 1)$ is a quadratic function in factored form, $g(x) = a(x - p)(x - q)$.

2. Determine the y-intercept of each function.

   The y-intercept of the function $f(x)$ is given in the function as -2.

   To determine the y-intercept of the function $g(x)$, substitute 0 for x and solve for $g(x)$.

$g(x) = 2(x - 3)(x + 1)$	Original function
$g(0) = 2(0 - 3)(0 + 1)$	Substitute 0 for x.
$g(0) = 2(-3)(1)$	Simplify.
$g(0) = -6$	

   The y-intercept of the function $g(x) = 2(x - 3)(x + 1)$ is -6.

3. Compare the y-intercept of each function.

   The y-intercept of $f(x) = 8x - 2$ is -2. The y-intercept of the function $g(x) = = 2(x - 3)(x + 1)$ is -6. The function $f(x) = 8x - 2$ has a greater y-intercept.

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Example 2

Three students are shooting wads of paper with a rubber band, aiming for a trash can in the front of the room. The height of each student's paper wad in feet is given as a function of the time in seconds. Which student's paper wad flies the highest?

- The path of Alejandro's paper wad is modeled by the equation $f(x) = -x^2 + 2x + 7$.
- Melissa's paper wad is estimated to reach the heights shown in the table.

x	0	2	3	4
y	3	6	7	6

- After 3 seconds, Connor's paper wad achieves a maximum height of 6.5 feet above the floor.

1. Determine if each situation represents a quadratic function.

   Alejandro's path is represented by a quadratic function written in standard form.

   In general, a projectile follows a parabolic path, gaining height until reaching a maximum, then descending. We can assume that Melissa's and Connor's paper wads will follow a parabolic path.

2. Verify that the extremum for each function is the maximum value of each function.

   The value of a in the equation for Alejandro's paper wad is a negative value. Therefore, the function has a maximum value.

   The y-values in the table for Melissa's paper wad increase then decrease, so the function has a maximum value.

   The maximum height of the path of Connor's paper wad is given.

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3. Determine the vertex of each function.

Find the vertex of Alejandro's function using $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

$x = \frac{-b}{2a}$ Formula to find the x-coordinate of the vertex

$x = \frac{-(2)}{2(-1)}$ Substitute –1 for a and 2 for b.

$x = 1$ Simplify.

Use the given function to find $f\left(\frac{-b}{2a}\right) = f(1)$.

$f(x) = -x^2 + 2x + 7$ Original equation

$f(1) = -(1)^2 + 2(1) + 7$ Substitute 1 for x.

$f(1) = -1 + 2 + 7$ Simplify.

$f(1) = 8$

The vertex of Alejandro's function is at (1, 8).

Find the vertex for Melissa's paper wad using symmetry. Her paper wad's function is symmetric about the vertex at (3, 7).

The vertex for Connor's paper wad is given as (3, 6.5).

4. Use the vertices to determine whose paper wad goes the highest.

Compare the y-values of the vertices.

$8 > 7 > 6.5$

Alejandro's paper wad flies the highest, at 8 feet.

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Instruction

Example 3

Which of the following quadratic functions has a vertex with a greater y-value: $f(x) = 2x^2 - 12x + 25$, or $g(x)$ as presented in the table?

x	–4	–3	–2	0	2
$g(x)$	7	8	7	–1	–17

1. Verify for each function whether the vertex is a minimum or maximum.

   For $f(x)$, a is positive, which means that the vertex is a minimum.

   For $g(x)$, as x increases, the y-values first increase and then decrease. This indicates the vertex is a maximum.

2. Find the vertex for each function.

   For $f(x)$, the vertex is of the form $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

   Use the original function, $f(x) = 2x^2 - 12x + 25$, to find the values of a and b in order to find the x-value of the vertex.

   $x = \frac{-b}{2a}$ Formula to find the x-coordinate of the vertex of a quadratic

   $x = \frac{-(-12)}{2(2)}$ Substitute 2 for a and –12 for b.

   $x = 3$ Simplify.

   The x-coordinate of the vertex is 3.

   Substitute 3 into the original function to find the y-coordinate.

   $f(x) = 2x^2 - 12x + 25$ Original function

   $f(3) = 2(3)^2 - 12(3) + 25$ Substitute 3 for x.

   $f(3) = 7$ Simplify.

   The y-coordinate of the vertex is 7.

(*continued*)

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The vertex of $f(x)$ is (3, 7).

For $g(x)$, the points are symmetric about $x = -3$. Therefore, the line of symmetry contains (–3, 8), which must be the vertex.

3. Determine which vertex has a greater y-value.

   Compare the y-values of each function.

   $7 < 8$

   The function $g(x)$ has a vertex with a greater y-value.

Example 4

You are considering investing \$5,000 in one of two mutual funds. The first fund will pay \$500 each year. The second fund is predicted to have end-of-year balances determined by the function $A(x) = 5000 \bullet (1 + 0.075)^x$. Which fund should you choose if you want to withdraw your money after 5 years? Which fund should you choose if you want to invest the money for 10 years?

1. Determine whether the function for the first fund is linear, quadratic, or exponential.

   The first fund has a constant rate of change of \$500 each year; therefore, the function for the first fund is linear.

2. Determine whether the function for the second fund is linear, quadratic, or exponential.

   From the form of the function $A(x)$, we can see that it is exponential.

3. Determine which fund you should choose if you want to withdraw your money after 5 years.

   At the end of the fifth year, the first fund would contain \$5000 + \$500(5) = \$7500.

   The second fund would contain $\$5000 \bullet (1.075)^5 \approx \$7178.15$ after 5 years.

   If you want to withdraw your money after 5 years, you should choose the first fund.

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Instruction

4. Determine which fund you should choose if you want to withdraw your money after 10 years.

   At the end of the fifth year, the first fund would contain $\$5000 + \$500(10) = \$10{,}000$.

   The second fund would contain $\$5000 \bullet (1.075)^{10} \approx \$10{,}305.16$ after 10 years.

   If you want to withdraw your money after 10 years, you should choose the second fund.

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Instruction

Example 5

Suppose that you have been offered a position at a prestigious company. You may choose how your salary is paid. Option 1 is described by the quadratic equation $S(x) = 2500x^2 + 2500x + 60{,}000$, where x is the number of years you are with the company and $S(x)$ is the yearly salary in dollars. Option 2 has a starting yearly salary of \$35,000, but you will get a 25% raise each year. Make a table of values for each salary and graph both functions on a coordinate plane. If you plan to work for this company for 5 years, which option should you choose? If you plan to work for this company until you retire at age 70, which option should you choose?

1. Write an equation for Option 2.

   Since you have been given the equation for Option 1 already, start by determining the equation for Option 2. Option 2 is an exponential function because it increases by the same factor (125%) each year. We can write the function by multiplying the initial salary by 1.25 raised to the number of years.

   $T(x) = 35{,}000(1.25)^x$

2. Make a table of values for each salary.

   Salaries listed in the table that follows are in thousands of dollars, rounded to the nearest tenth. Year 0 represents the starting salary.

Year	0	1	2	3	4	5	6	7	8	9	10	11
Option 1	60	65	75	90	110	135	165	200	240	285	335	390
Option 2	35	43.8	54.7	68.4	85.4	106.8	133.5	166.9	208.6	260.8	326.0	407.5

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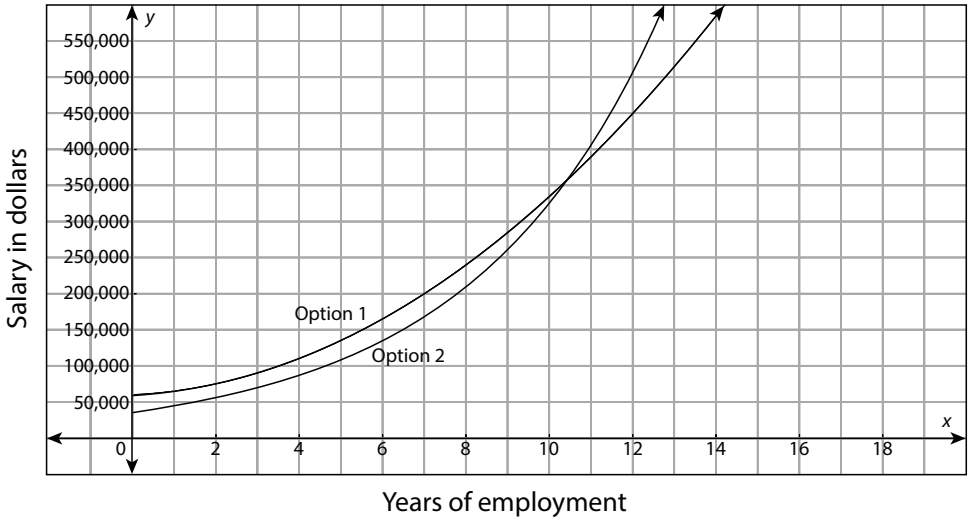
Instruction

3. Graph both functions on the same coordinate plane.

   Plot the points from the table and connect them with a smooth curve. Label each function clearly.



4. Determine which option you should choose if you plan to work for this company for only 5 years.

   The table of values shows us that Option 1 yields a salary of $135,000 in the fifth year, compared to $106,800 for Option 2.

   If you plan to work for this company for only 5 years, you should select Option 1.

5. Determine which option you should choose if you plan to work for this company until retirement at age 70.

   We can see from the graph that although Option 2 pays less for the first 10 years, from year 11 on it is the better choice.

   Exponential functions eventually exceed quadratic functions.

   If you plan to work for this company until retirement, you should select Option 2.

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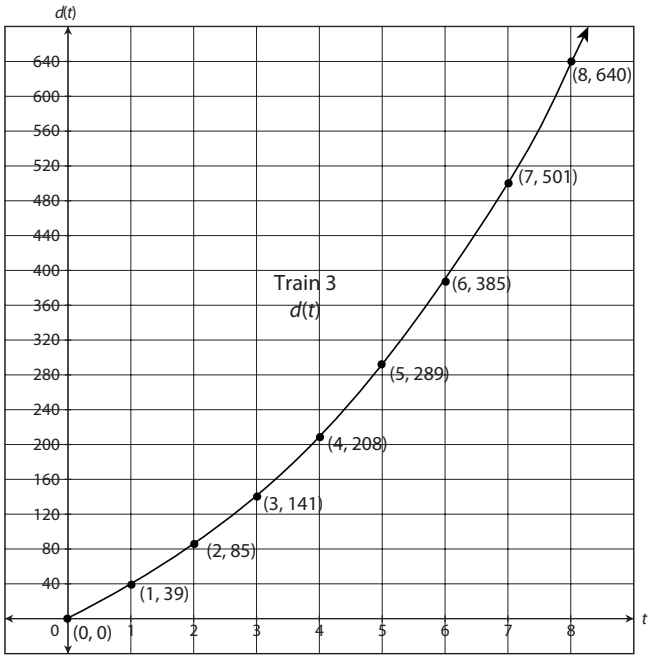
Problem-Based Task 2.5.2: On to Washington!

SMP	
1	2 ✓
3 ✓	4 ✓
5	6
7	8

An emergency meeting has been called on Capitol Hill in Washington, D.C. As a newly elected representative for the state of Georgia, you need to find a way to travel to Washington. No flights are available at a time that fits your schedule, but there are three trains that you could take. The distance between Atlanta and Washington, D.C., is 639 miles. Train 1 is a nonstop train that leaves at 10 A.M. and travels at 66 mph. Train 2 also leaves at 10 A.M., but has a few stops that slow it down. The total distance traveled each hour by Train 2 is given in the table, where t is in hours and $d(t)$ is in miles. Train 3 leaves at noon. The distance traveled for Train 3 can be modeled by the following graph, where t is the time in hours beginning at noon, and d is the distance traveled in miles. Is it possible to determine which train will arrive in Washington first? Which train should you select if you want to minimize time spent on the train? Explain.

t	0	1	2	3	4	5	6	7	8	9	10
$d(t)$	0	48	100	156	216	280	348	420	496	576	660



Is it possible to determine which train will arrive in Washington first? Which train should you select if you want to minimize time spent on the train?

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Problem-Based Task 2.5.2: On to Washington!

Coaching

a. Could a table of values help to compare all three trains? If so, create the table.

b. What kind of model describes the movement of each train? Explain.

c. At what time will Train 1 arrive?

d. At what time will Train 2 arrive?

e. At what time will Train 3 arrive?

f. Is it possible to determine which train will arrive in Washington, D.C., first?

g. Which train should you select if you need to arrive in Washington, D.C., as early as possible? Explain.

h. Which train should you select if you want to minimize the amount of time spent on the train? Explain.

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Instruction

Problem-Based Task 2.5.2: On to Washington!

Coaching Sample Responses

a. Could a table of values help to compare all three trains? If so, create the table.

Yes, a table of values could help to compare the distances traveled in relation to time for all three trains. Use the graph to approximate the distances for Train 3.

Time	10 A.M.	11 A.M.	12 P.M.	1 P.M.	2 P.M.	3 P.M.	4 P.M.	5 P.M.	6 P.M.	7 P.M.	8 P.M.
Train 1	0	66	132	198	264	330	396	462	528	594	660
Train 2	0	48	100	156	216	280	348	420	496	576	660
Train 3	0	0	0	39	85	141	208	289	385	501	640

b. What kind of model describes the movement of each train? Explain.

The first train is linear, because it travels at a constant rate of 66 mph. For the second train, the increase in distance is not constant, but the second differences are constant. Therefore, the second train follows a quadratic model.

The distance traveled by the third train increases as the hours pass, but not by a constant multiple. Neither the first nor the second differences are constant. However, the ratio of the differences is nearly constant, so the third train appears to follow an exponential model.

c. At what time will Train 1 arrive?

Train 1 travels at a constant speed of 66 mph.

To travel 639 miles, it takes $639 \div 66 = 9.68$ hours, or about 9 hours and 41 minutes.

If the train leaves Atlanta at 10 A.M., it will arrive in Washington, D.C., at about 7:41 P.M.

d. At what time will Train 2 arrive?

According to the table, the train will go 576 miles in 9 hours and 660 miles in 10 hours. The distance from Atlanta to Washington is 639 miles, which is between these two points, about three quarters of the way from 9 hours to 10 hours. The train will arrive between 7 P.M. and 8 P.M., shortly after 7:45 P.M.

e. At what time will Train 3 arrive?

The distance traveled by Train 3 is represented by the following graph. To find out when this train will have traveled 639 miles, graph the horizontal line $d(t) = 639$ and observe where the functions intersect. They intersect when x is approximately 8. Because Train 3 leaves at 12 P.M., it will arrive 8 hours later, at approximately 8 P.M.

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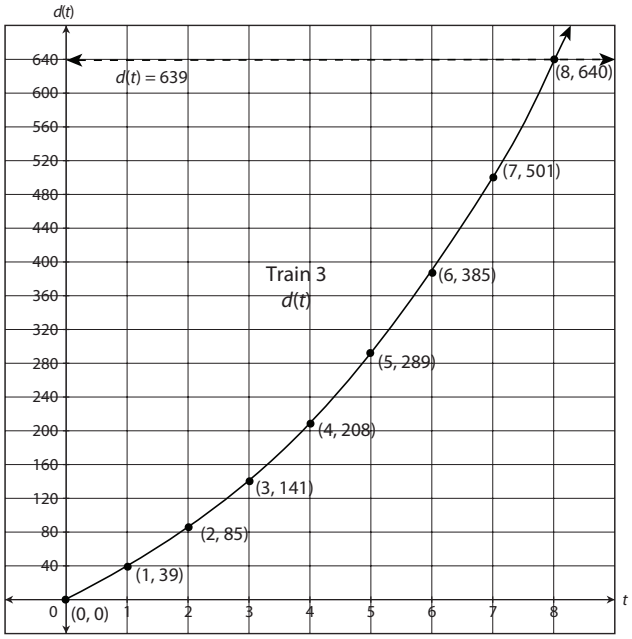

f. Is it possible to determine which train will arrive in Washington, D.C., first?

Train 1 would arrive in Washington, D.C., after 9 hours 41 minutes, or at 7:41 P.M.

Train 2 would arrive in Washington, D.C., shortly after 7:45 P.M.

Using the graph, Train 3 would arrive in Washington, D.C., after approximately 8 hours, or at about 8:00 P.M.

Train 1 will arrive in Washington, D.C., first.

g. Which train should you select if you need to arrive in Washington, D.C., as early as possible? Explain.

Because Train 1 will arrive in Washington, D.C., first, Train 1 is the one you should select if you need to arrive as early as possible.

h. Which train should you select if you want to minimize the amount of time spent on the train? Explain.

Train 3 leaves Atlanta 2 hours later than Trains 1 and 2, but arrives only 18 minutes later than Train 1. You should select Train 3 to minimize the amount of time spent on the train.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2.5.2: Comparing Properties of Functions Given in Different Forms

Use the following information to solve problems 1–3.

Natalie is considering which method of travel—car, train, or plane—would be best to travel the flight distance of 747 miles from Atlanta to New York City. Use this distance for each problem. One rule of thumb is to estimate traveling by car at 60 mph. The train can be modeled by the equation $T(x) = -2.4x^2 + 90.8x + 1.59$, where x represents the number of hours and $T(x)$ represents the number of miles traveled. The table represents the time and distance traveled during the plane trip.

Hours	0	0.5	1	1.5	2	2.5
Miles	0	149	300	455	612	747

1. If the car and the train leave Atlanta at the same time, which one arrives in New York City first? Is this mode of travel faster for the entire trip?

2. Estimate the vertex for the train. Is this vertex reasonable within the context of the problem? Why or why not?

3. If the car and the train both leave Atlanta at 7 A.M. and the plane leaves Atlanta at 4:30 P.M., determine which would arrive in New York City first.

Use the following information to solve problems 4–7.

Three students are saving money. One student, Elsie, puts $800 into an account that earns 5.5% each year. Frank puts $840 into an account that earns 4% per year. Gina puts $750 into an account that earns 6% per year. The amount of money in Elsie's account can be modeled with the function $E(t) = 800(1.055)^t$, where t is the number of years. The functions that model Frank's and Gina's accounts are $F(t) = 840(1.04)^t$ and $G(t) = 750(1.06)^t$.

4. Who has the most money in his or her account after 2 years?

5. Who has the most money in his or her account after 5 years?

continued

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6. How many years does it take for the amount in Gina's account to catch up with the amount in Frank's?

7. If all three students are trying to save $1,200, who will reach his or her goal first?

Use the given information to answer questions 8–10.

8. Which of the following parabolas has the vertex with the smallest y-value: a parabola with two x-intercepts with $a > 0$, or a parabola with two x-intercepts with $a < 0$?

9. Which function achieves a lower maximum value: a parabola with no x-intercepts and $a < 0$, or a parabola with two x-intercepts and $a < 0$?

10. You've been offered jobs at two different companies, and each company offers a different form of payment. At one company, each day's take-home pay will be the sum of that day's wages plus the previous day's wages, starting with $1 on the first day and continuing as shown in the table. The other company will pay you $0.01 on the first day, then double that amount every subsequent day, and is modeled by the equation $y = 0.01(2)^{(x-1)}$. On what day does the exponential function surpass the quadratic model?

Day	1	2	3	4	5
Take-home pay ($)	1	3	6	10	15

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Progress Assessment

Circle the letter of the best answer.

1. If $f(x) = 1.05(0.40)^x$, what is the percent rate of change of y for each unit of x?
   a. 0.4%
   b. 4%
   c. 5%
   d. 60%

2. If $f(x) = 11(1.16)^x$, what is the value of y at $x = 0$?
   a. 1.16
   b. 11
   c. 12.76
   d. 16

3. If $f(x) = 77(1.08)^{8x}$, what is the percent rate of change of y for each unit of x?
   a. 1.08%
   b. 77%
   c. 80%
   d. 85%

4. If $f(x) = 9(0.64)^x$, which statement is true?
   a. The value of y decays exponentially.
   b. The value of y grows exponentially.
   c. The value of y grows at a linear rate.
   d. The value of y is 9 times the value of x.

5. Which function shows exponential decay?
   a. $f(x) = 1.1(1.1)^x$
   b. $f(x) = 1.1(1.01)^x$
   c. $f(x) = 1.1(0.99)^x$
   d. $f(x) = 1.1(1.99)^x$

6. Given two quadratic functions, what characteristics do you look at to see which achieves a higher value?
   a. a-values and vertices
   b. a-values and y-intercepts
   c. a-values and x-intercepts
   d. x- and y-intercepts

continued

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7. Which of the following quadratic functions has a vertex with a lower y-value?

   a. $f(x) = 3(x + 1)(x - 2)$

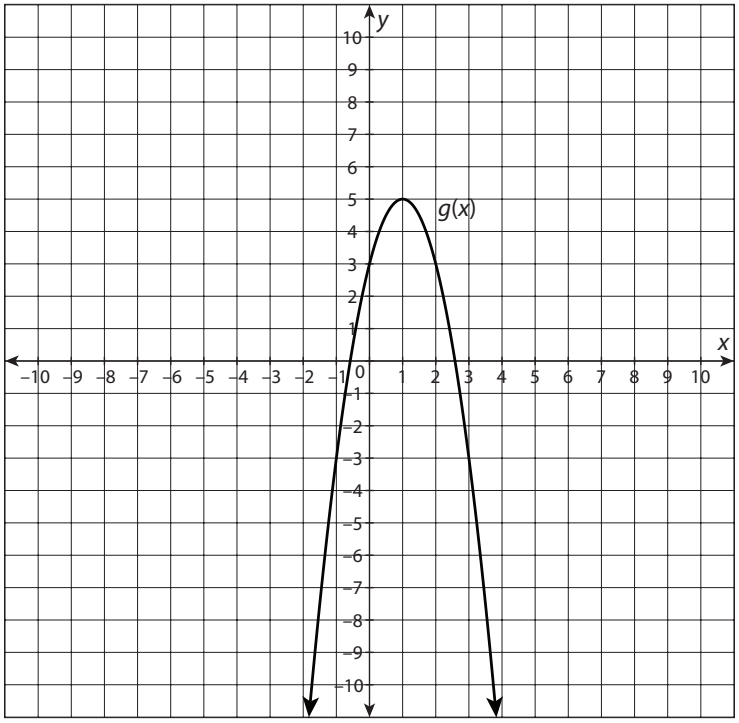
   b. $g(x) = 3x^2 + 2x + 1$

   c. $h(x) = 3(x + 1)^2 + 2$

   d. It cannot be determined from this information.

8. Which of the following statements is true about the functions $f(x)$ and $g(x)$?

x	$f(x)$
–1	–1
0	2
1	3
2	2
3	–1



   a. The function $f(x)$ has a greater maximum than the function $g(x)$.

   b. The function $g(x)$ has a greater maximum than the function $f(x)$.

   c. The function $f(x)$ has a greater minimum than the function $g(x)$.

   d. The function $g(x)$ has a greater minimum than the function $f(x)$.

continued

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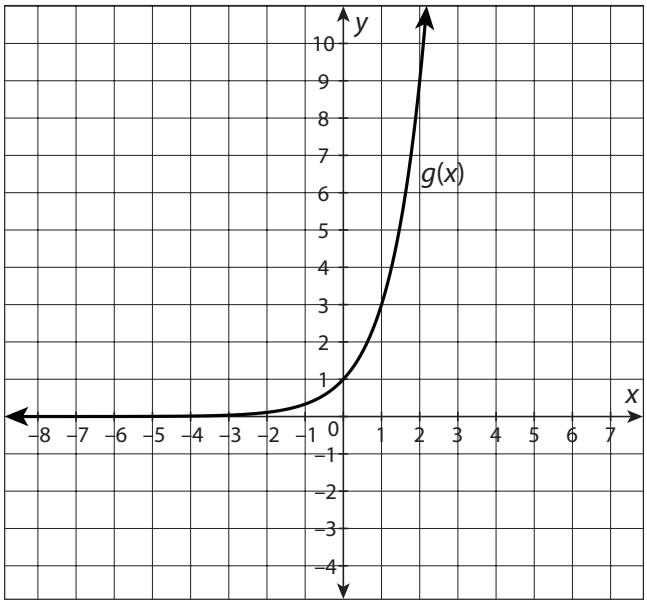
9. Which of the following statements is true about the functions $f(x)$ and $g(x)$?

$f(x) = -x^2 + 2x + 2$



a. The y-intercept of the function $f(x)$ is less than the y-intercept of the function $g(x)$.

b. The y-intercept of the function $f(x)$ is greater than the y-intercept of the function $g(x)$.

c. The y-intercept of the function $f(x)$ is equal to the y-intercept of the function $g(x)$.

d. The y-intercepts cannot be determined.

continued

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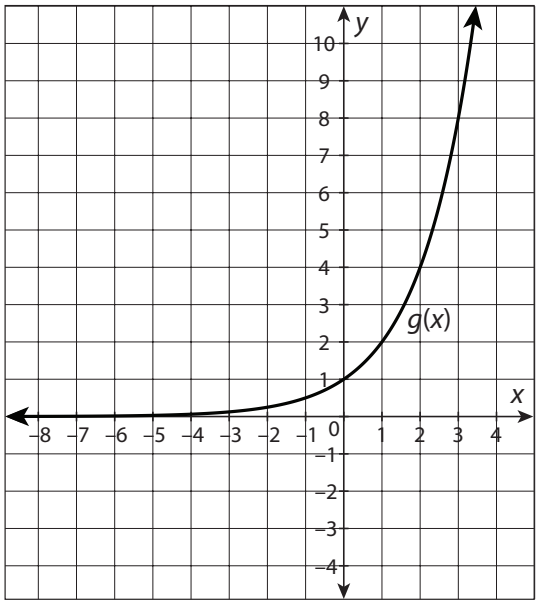
10. Which of the following statements is true about the functions $f(x)$, $g(x)$, and $h(x)$?

$f(x) = x^2 + 7x + 10$



x	$h(x)$
–3	–13
–1	–7
0	–4
1	–1
3	5

a. The values of $f(x)$ will eventually exceed both $g(x)$ and $h(x)$.

b. The values of $g(x)$ will eventually exceed both $f(x)$ and $h(x)$.

c. The values of $h(x)$ will eventually exceed both $f(x)$ and $g(x)$.

d. The values of these functions cannot be compared.

Use what you have learned about exponential functions to answer the following questions.

11. Harry deposits money into a savings account. The total amount of money in the account, $m(t)$, t years after opening the account, can be estimated using the function $m(t) = 3200(1.03)^t$.

a. How much money did Harry initially deposit into the account?

b. What is the percent rate of change of the money in the account each year?

c. Is the total amount of money in the account growing or decaying? Explain.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Given $f(x) = x^2 + 2x + 1$, what is the shift in the vertex of the graph of $f(x + 4)$?
   a. The vertex shifts 4 units to the left.
   b. The vertex shifts 4 units up.
   c. The vertex shifts 4 units to the right.
   d. The vertex shifts 4 units down.

2. The vertex of $f(x)$ is $(2, -3)$. What is the vertex of $f(x) + 3$?
   a. $(5, -3)$
   b. $(-1, -3)$
   c. $(2, -6)$
   d. $(2, 0)$

3. Given $f(x) = x^2 + 1$, what is the value of $5 \cdot f(x)$ when $x = 3$?
   a. 15
   b. 50
   c. 16
   d. 150

4. If the vertex of $f(x)$ is $(4, 7)$, what is the vertex of $f(x - 2) + 3$?
   a. $(6, 10)$
   b. $(2, 10)$
   c. $(7, 5)$
   d. $(7, 9)$

continued

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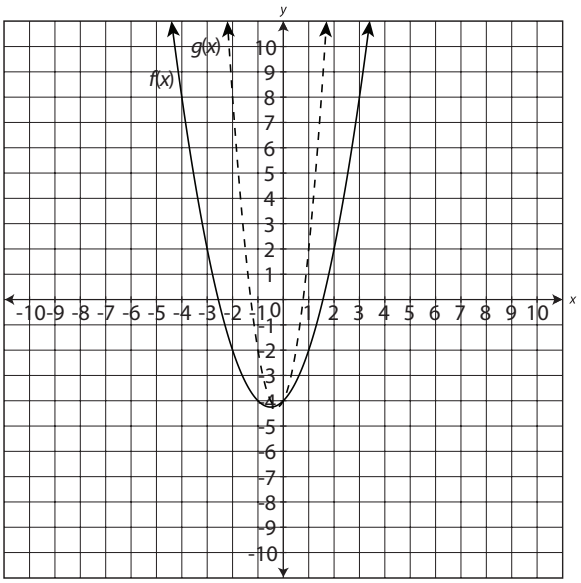
5. Consider the graphs of the functions $f(x)$ and $g(x)$. The equation for $f(x)$ is $f(x) = x^2 + x - 4$. What could be the equation for $g(x)$?



a. $g(x) = 2x^2 + 2x - 8$

b. $g(x)=\frac{1}{2}x^2+\frac{1}{2}x-2$

c. $g(x) = 4x^2 + 2x - 4$

d. $g(x)=\frac{1}{4}x^2+\frac{1}{2}x-4$

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Lesson 6: Transforming Functions

Instruction

Common Core State Standard

F–BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

SMP	
1 ✓	2 ✓
3	4 ✓
5	6
7 ✓	8 ✓

Essential Questions

1. How is a function $f(x)$ affected when multiplied by a constant k?
2. How is a function $f(x)$ affected when x is multiplied by a constant k?
3. What are the effects of k on the graph of $f(x + k)$ compared to $f(x)$?
4. What are the effects of k on the graph of $f(x) + k$ compared to $f(x)$?

WORDS TO KNOW

horizontal compression	squeezing of the graph of a function or geometric figure toward the y-axis
horizontal stretch	stretching of the graph of a function or geometric figure away from the y-axis
transformation	a change in the position, shape, or size of the graph of a function that occurs when a constant is either added to or multiplied by the function or the function's independent variable. Transformations include translations, reflections, dilations, compressions, and rotations.
translation	a type of transformation in which a constant is added to the independent or dependent variable of a function so that the shape, size, and orientation of the graph of the function remain the same, but the position of the graph moves horizontally and/or vertically
vertical compression	squeezing of the graph of a function or geometric figure toward the x-axis
vertical stretch	stretching of the graph of a function or geometric figure away from the x-axis

SRB p. U2-175

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Instruction

Recommended Resources

- MathIsFun.com. "Function Transformations."

  http://www.walch.com/rr/00121

  This site offers a simple, illustrated guide to how changing values in a function transforms the related graph. At the bottom of the page, users will find interactive review questions.

- Seeing Math. "Quadratic Transformer Applet."

  http://www.walch.com/rr/00123

  This applet allows users to flip, skew, and otherwise transform a graph, and then view the equation of their graph in polynomial, vertex, or *x*-intercept form. The "Transform Function" button lets users create a new function by transforming an existing function. The applet requires Java to run.

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Lesson 2.6.1: Replacing $f(x)$ with $f(x) + k$ and $f(x + k)$

Warm-Up 2.6.1

As a fund-raiser for the senior prom, the student council has decided to charge a small fee for parking in the school parking lot. Students can purchase yearly parking passes for $5 each. The school spent a total of $10 to buy a year's supply of the tags the students hang in their windows to verify that they paid for the parking spot.

1. Build a function that models the profit the school will make selling parking passes.

2. Graph the function.

3. What would be the effect on the graph if the school were able to find a company that only charges $8 for the tags?

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Instruction

Lesson 2.6.1: Replacing $f(x)$ with $f(x) + k$ and $f(x + k)$

Common Core State Standard

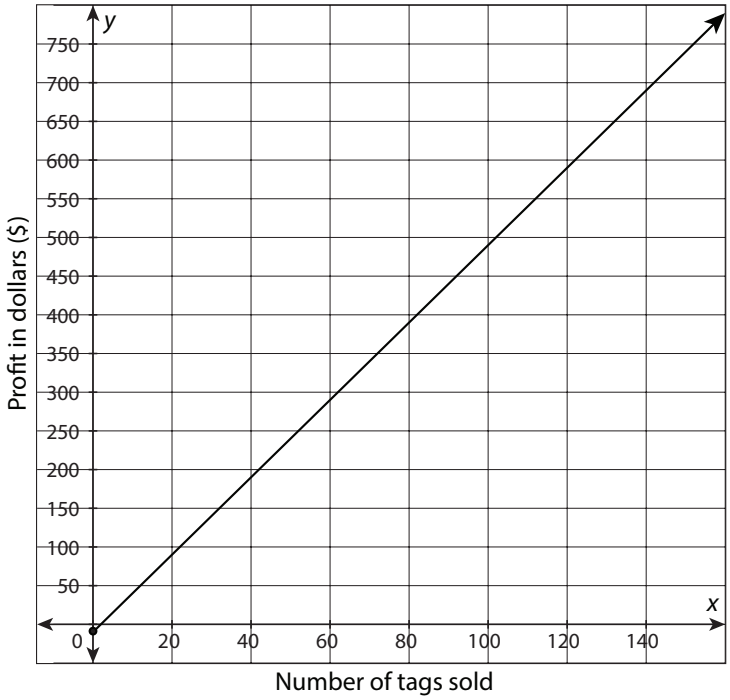
F–BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Warm-Up 2.6.1 Debrief

1. Build a function that models the profit the school will make selling parking passes.

   Each parking pass sells for $5, but the school spent $10 up front to purchase the window tags. The function that models this situation is $f(x) = 5x - 10$.

2. Graph the function.



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3. What would be the effect on the graph if the school were able to find a company that only charges $8 for the tags?

   The profit would increase by $2. This would affect the *y*-intercept by changing it from –10 to –8. The graph would move up 2 units.

Connection to the Lesson

- Students will extend their study of the effects of transformations on linear and exponential functions to quadratic functions.
- Students will graph functions and analyze those graphs.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- graphing quadratic functions (A–REI.10)
- evaluating quadratic functions (F–IF.2)

Introduction

You can change a function's position or shape by adding a constant to that function or to the independent variable in the function, or by multiplying the function or independent variable by a constant. This is called a **transformation**. When adding a constant, you can transform a function in two distinct ways. The first is a transformation on the independent variable of the function; that is, given a function $f(x)$, we add some constant k to x: $f(x)$ becomes $f(x + k)$. The second is a transformation on the dependent variable; given a function $f(x)$, we add some constant k to $f(x)$: $f(x)$ becomes $f(x) + k$.

In this lesson, we consider the transformation of a quadratic function by a constant k, either when k is added to the independent variable, x, or when k is added to the dependent variable, $f(x)$. Given $f(x)$ and a constant k, we will observe the transformations $f(x) + k$ and $f(x + k)$, and examine how transformations affect the vertex of the graph of a quadratic function.

Key Concepts

- To determine the effects of the constant on a graph, compare the vertex of the original function to the vertex of the transformed function.
- Neither $f(x + k)$ nor $f(x) + k$ will change the shape, size, or orientation of the function so long as k is a constant.
- Transformations that do not change the shape, size, or orientation of the function but do move it horizontally and/or vertically are called **translations**.
- Translations are performed by adding a constant to the independent or dependent variable.

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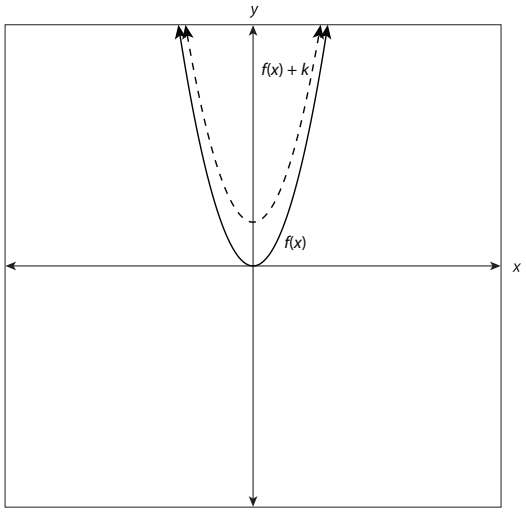
Instruction

Vertical Translations—Adding a Constant to the Dependent Variable, $f(x) + k$

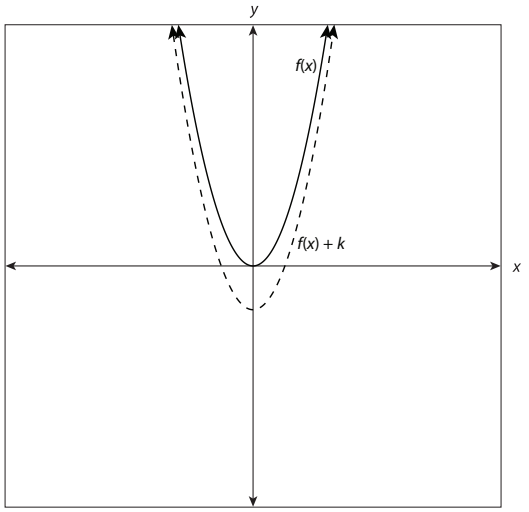
- $f(x) + k$ moves the graph of the function k units up or down depending on whether k is greater than or less than 0.
- If k is positive in $f(x) + k$, the graph of the function will be moved up.
- If k is negative in $f(x) + k$, the graph of the function will be moved down.

Vertical translations: $f(x) + k$

When k is positive, $k > 0$, the graph moves up:



When k is negative, $k < 0$, the graph moves down:



Horizontal Translations—Adding a Constant to the Independent Variable, $f(x + k)$

- $f(x + k)$ moves the graph of the function k units to the right or left depending on whether k is greater than or less than 0.
- If k is positive in $f(x + k)$, the function will be moved to the left.
- If k is negative in $f(x + k)$, the function will be moved to the right.

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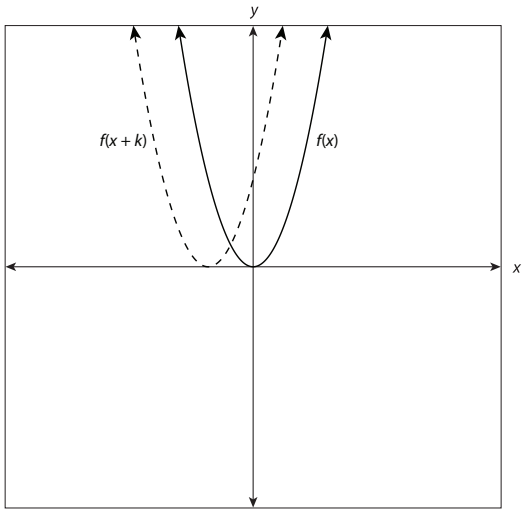
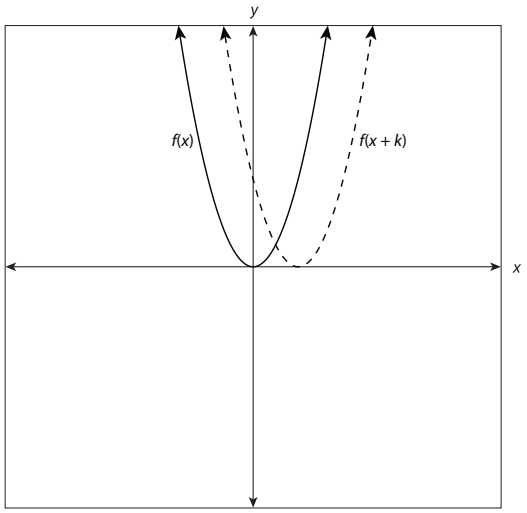
Instruction

Horizontal translations: $f(x + k)$

When k is positive, $k > 0$, the graph moves left:



When k is negative, $k < 0$, the graph moves right:



Common Errors/Misconceptions

- incorrectly moving the graph in the direction opposite that indicated by k, especially in horizontal shifts; for example, moving the graph left when it should be moved right
- incorrectly moving the graph left and right versus up and down (and vice versa) when operating with $f(x + k)$ and $f(x) + k$

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Instruction

Guided Practice 2.6.1

Example 1

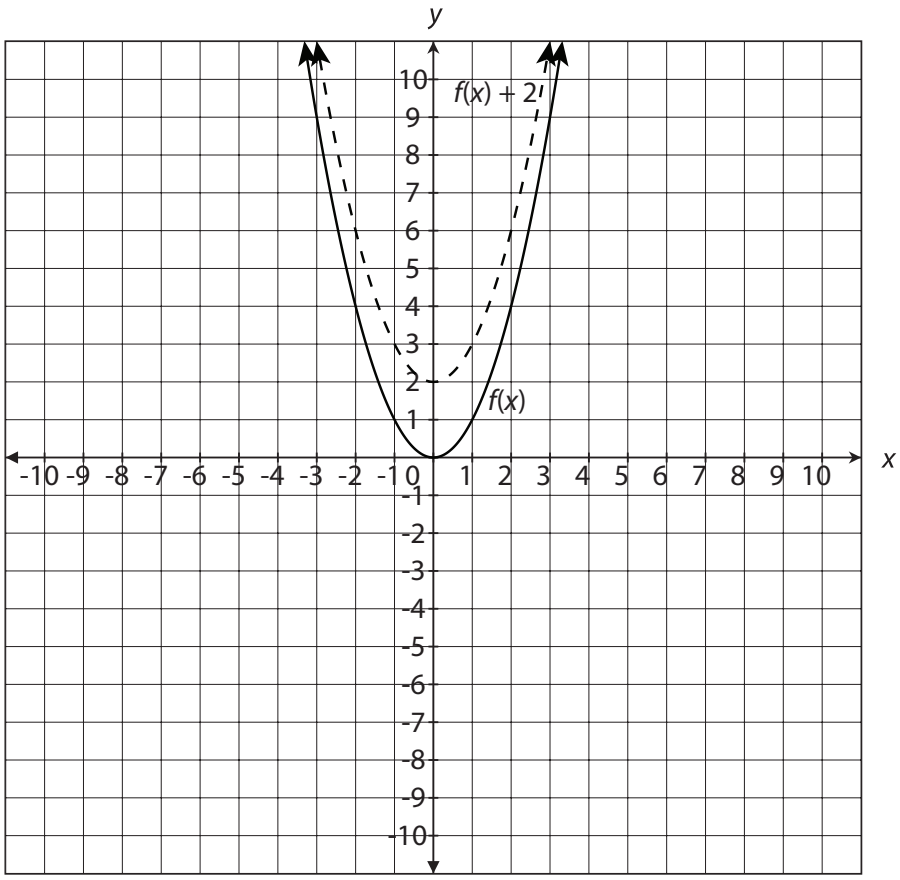
Consider the function $f(x) = x^2$ and the constant $k = 2$. What is $f(x) + k$? How are the graphs of $f(x)$ and $f(x) + k$ different?

1. Substitute the value of k into the function.

   If $f(x) = x^2$ and $k = 2$, then $f(x) + k = x^2 + 2$.

2. Use a table of values to graph the functions on the same coordinate plane.

x	$f(x)$	$f(x) + 2$
–2	4	6
–1	1	3
0	0	2
1	1	3
2	4	6



SWB p. 131; SRB p. U2-179

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Instruction

3. Compare the graphs of the functions.

   Notice the shape and horizontal position of the two graphs are the same. The only difference between the two graphs is that the value of $f(x) + 2$ is 2 more than $f(x)$ for all values of x. In other words, the transformed graph is 2 units up from the original graph.

Example 2

Consider the function $f(x) = x^2$ and the constant $k = -3$. What is $f(x) + k$? How are the graphs of $f(x)$ and $f(x) + k$ different?

1. Substitute the value of k into the function.

   If $f(x) = x^2$ and $k = -3$, then $f(x) + k = x^2 - 3$.

2. Use a table of values to graph the two functions on the same coordinate plane.

x	$f(x)$	$f(x) - 3$
–4	16	13
–2	4	1
0	0	–3
2	4	1
4	16	13

(*continued*)

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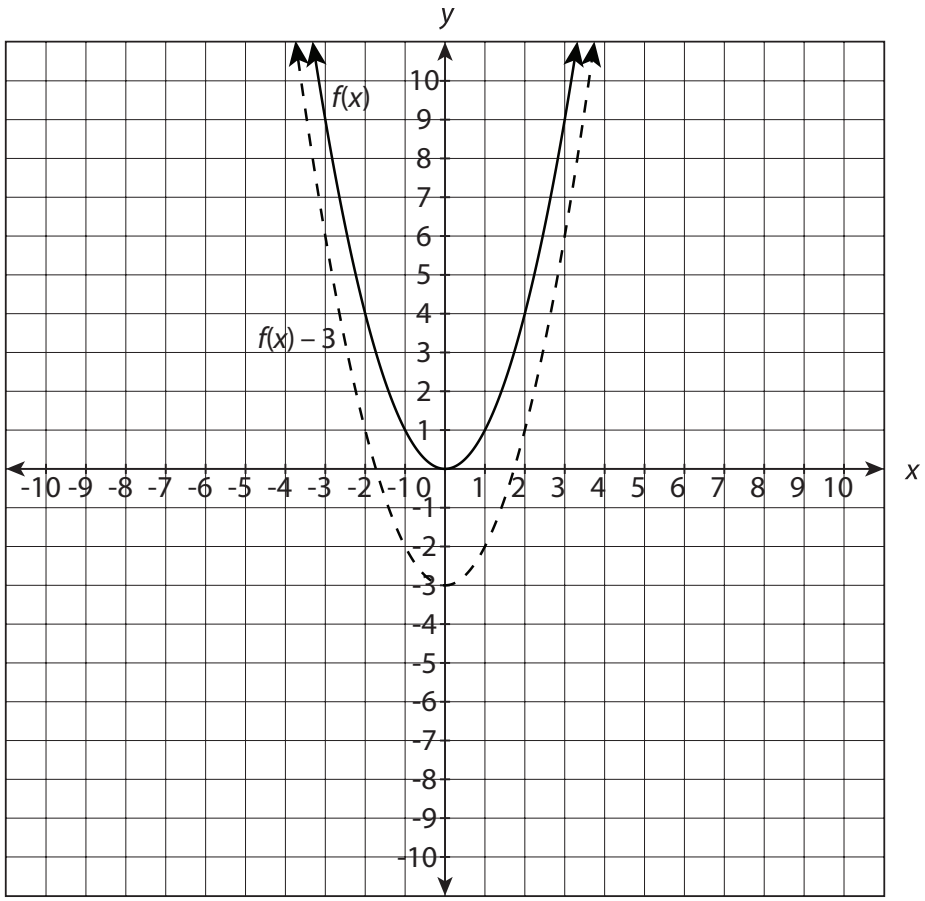
Instruction



3. Compare the graphs of the functions.

   Notice both the shape and horizontal position of the two graphs are the same. The only difference between the two graphs is that the value of $f(x) - 3$ is 3 less than $f(x)$ for all values of x.

   The easiest point to analyze on the graphs is the vertex. For $f(x)$, the vertex occurs at (0, 0). For the graph of $f(x) - 3$, the vertex occurs at (0, –3). The transformed graph has moved down 3 units.

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Instruction

Example 3

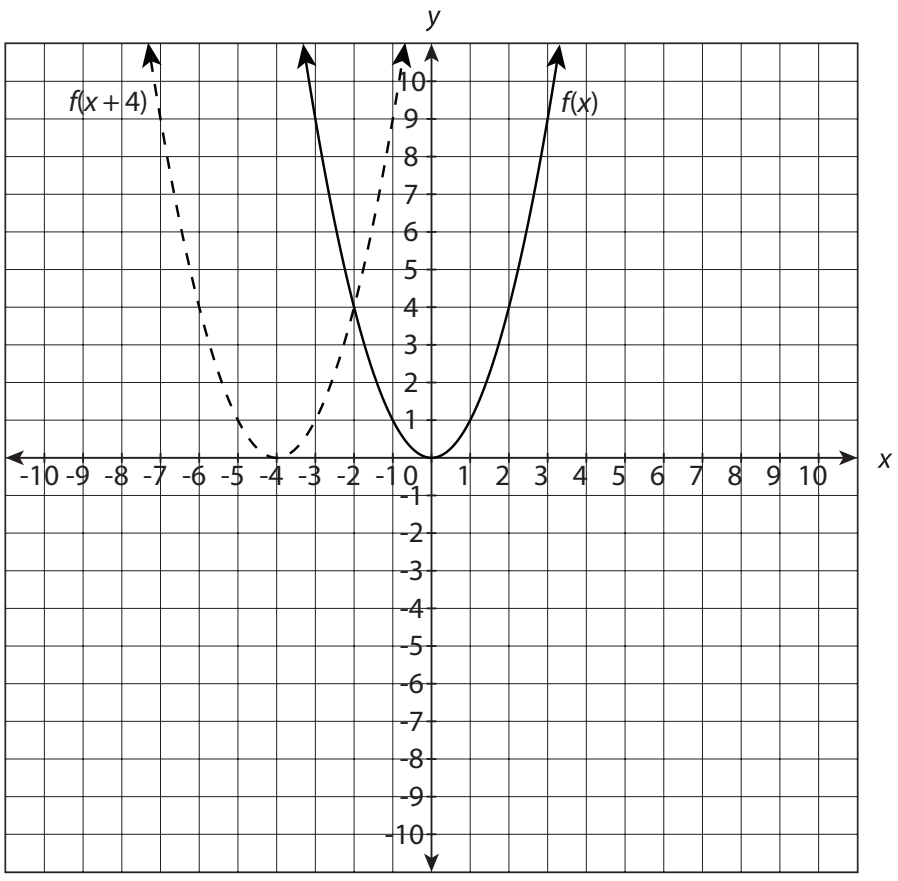
Consider the function $f(x) = x^2$, its graph, and the constant $k = 4$. What is $f(x + k)$? How are the graphs of $f(x)$ and $f(x + k)$ different?

1. Substitute the value of k into the function.

   If $f(x) = x^2$ and $k = 4$, then $f(x + k) = f(x + 4) = (x + 4)^2$.

2. Use a table of values to graph the two functions on the same coordinate plane.

x	$f(x)$	$f(x+4)$
–6	36	4
–4	16	0
–2	4	4
0	0	16
2	4	36
4	16	64



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Instruction

3. Compare the graphs of the functions.

   Notice the shape and vertical position of the two graphs are the same. The only difference between the two graphs is that every point on the curve of $f(x)$ has been shifted 4 units to the left in the graph of $f(x + 4)$.

Example 4

Consider the function $f(x) = x^2$ and the constant $k = -1$. What is $f(x + k)$? How are the graphs of $f(x)$ and $f(x + k)$ different?

1. Substitute the value of k into the function.

   If $f(x) = x^2$ and $k = -1$, then $f(x + k) = f(x - 1) = (x - 1)^2$.

2. Using vertex or standard form, graph the two functions on the same coordinate plane.

   Recall from Example 2 that the vertex of the parent function, $f(x) = x^2$, is (0, 0).

   The vertex of the transformed function, $f(x - 1) = (x - 1)^2$, is (1, 0). This can be verified by using $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

   To use this formula, expand the function so that it's in the form of a quadratic, then find a, b, and c.

   $f(x - 1) = (x - 1)^2 = x^2 - 2x + 1$

   $a = 1$, $b = -2$, and $c = 1$

   The x-coordinate of the vertex is given by $x = \frac{-b}{2a} = \frac{-(-2)}{2(1)} = 1$.

   Substitute the x-value of the vertex into the function.

   $f(x - 1) = (x - 1)^2$

   $f(1) = [(1) - 1]^2$

   $f(1) = 0$

   Therefore, the vertex is (1, 0).

(*continued*)

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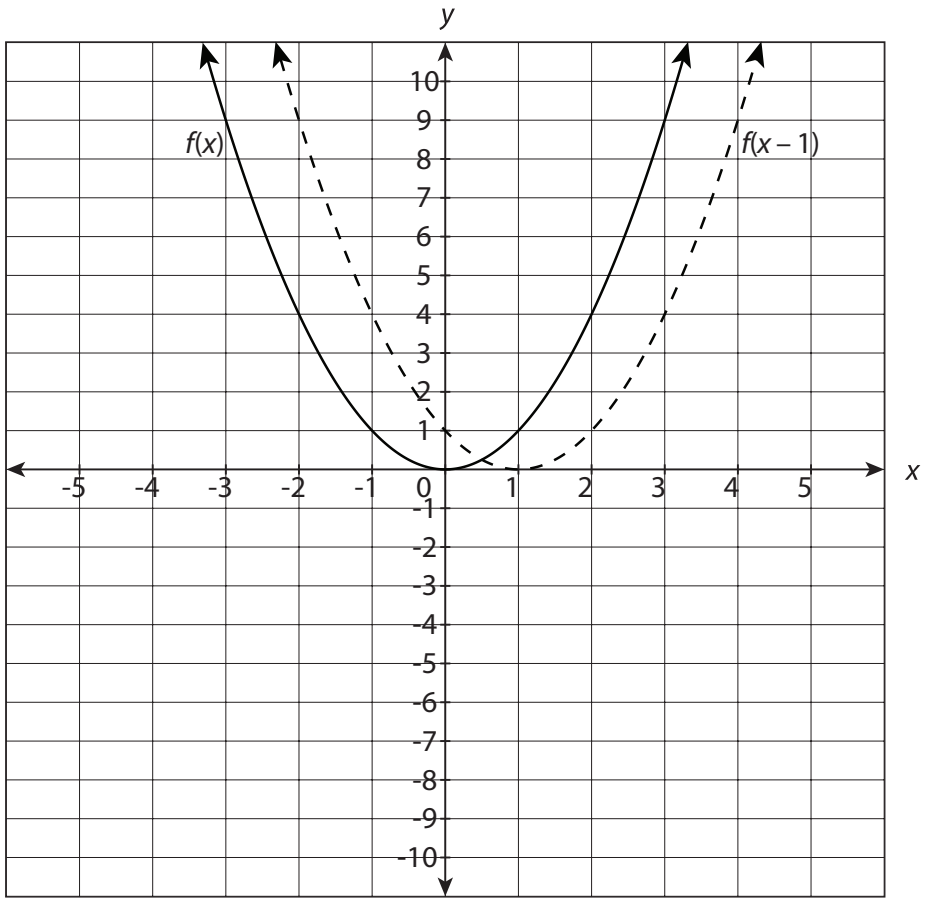
Instruction

Graph the functions.



3. Compare the graphs of the functions.

   Notice both the shape and vertical position of the two graphs are the same. The only difference between the two graphs is that every point on the curve of $f(x)$ has been shifted 1 unit to the right in the graph of $f(x - 1)$.

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Instruction

Example 5

The revenue function for a model helicopter company is modeled by the curve $f(x) = -5x^2 + 400x$, where x is the number of helicopters built per month and $f(x)$ is the revenue. The owner wants to include rent in the revenue equation to determine the company's profit per month. The company pays \$2,250 per month to rent its warehouse. In terms of $f(x)$, what equation now describes the company's profit per month? Compare the vertices of the original function and the transformed function.

1. Build the new function.

   The rent is being subtracted from the entire function. Therefore, the function follows the format $f(x) + k$, where k is negative.

   The company makes $f(x) = -5x^2 + 400x$ dollars per month. The company also pays \$2,250 dollars per month in rent. Therefore, the new function that describes the company's profit, $P(x)$, is as follows:

   $P(x) = f(x) - \text{rent} = f(x) - 2250 = -5x^2 + 400x - 2250$

2. Determine the vertex of $f(x)$.

   The vertex of $f(x)$ has the x-coordinate $\frac{-b}{2a} = \frac{-(400)}{2(-5)} = \frac{-400}{-10} = 40$.

   Evaluate $f(x)$ at the x-coordinate of the vertex to find the y-coordinate.

   $f(x) = -5x^2 + 400x$

   $f(40) = -5(40)^2 + 400(40)$

   $f(40) = -8000 + 16{,}000$

   $f(40) = 8000$

   The vertex of $f(x)$ is (40, 8000).

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Instruction

3. Determine the vertex of $P(x)$.

   The vertex of $P(x)$ has the same x-coordinate as $f(x)$ since the values of a and b are the same. The x-coordinate of the vertex of $P(x)$ is 40.

   Evaluate $P(x)$ at the x-coordinate of the vertex to find the y-coordinate.

   $P(x) = -5x^2 + 400x - 2250$

   $P(40) = -5(40)^2 + 400(40) - 2250$

   $P(40) = 8000 - 2250$

   $P(40) = 5750$

   The vertex of $P(x)$ is (40, 5750).

4. Compare the vertices.

   The vertex of $P(x)$ is 2,250 units lower than the vertex of $f(x)$. The model was shifted down 2,250 units.

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Problem-Based Task 2.6.1: The Catch

SMP	
1	2 ✓
3	4 ✓
5	6
7 ✓	8

On the last play of a football game, the offense is on the opposing team's 35-yard line. The offense is losing by 4 points, but can win by making a touchdown. The quarterback backs away 5 yards from behind the line of scrimmage and throws the ball to his receiver, who makes the catch at the goal line for the touchdown and the win. The quarterback's release point is 6 feet above the ground, the same height at which the receiver caught the ball. Also, the ball was thrown such that its maximum height was 15 feet above the ground.

Given this information, build and graph the equation of the football's path, with the x-axis representing the distance from the line of scrimmage and the y-axis as the height of the football above the ground, with distances measured in yards.

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Problem-Based Task 2.6.1: The Catch

Coaching

a. What is the total horizontal distance the ball was thrown?

b. What is the height difference between the release of the ball and its maximum height?

c. What are the x-intercepts of the equation describing the path of the ball, disregarding the point of release and allowing the parabola to begin at the origin, (0, 0)?

d. What value of a gives the parabola the correct arch?

e. Using the value of a and the vertex from part d, write the equation of the path of the ball in vertex form.

f. Translate the equation to account for the line of scrimmage.

g. Translate the equation to account for the height of the point of release and catch.

h. Graph the equation.

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Problem-Based Task 2.6.1: The Catch

Coaching Sample Responses

a. What is the total horizontal distance the ball was thrown?

The ball was thrown from 5 yards behind the 35-yard line to the goal line. The ball traveled 40 yards.

b. What is the height difference between the release of the ball and its maximum height?

The ball reached a height of 15 feet, or 5 yards, and was thrown from a height of 6 feet, or 2 yards. Therefore, the height difference between the release of the ball and the maximum height is 3 yards.

c. What are the *x*-intercepts of the equation describing the path of the ball, disregarding the point of release and allowing the parabola to begin at the origin, (0, 0)?

If the path of the ball started at the origin (0, 0), then $x = 0$ is one of the *x*-intercepts. If the ball was caught 40 yards later, then it was caught at the coordinates (40, 0), making $x = 40$ the other *x*-intercept of the equation.

d. What value of *a* gives the parabola the correct arch?

The vertex of the path of the ball must happen at midway between the *x*-intercepts $\left(\frac{40-0}{2}=20\right)$, and from part b we know the height difference was 3 yards. Therefore, the vertex of the equation is (20, 3).

Using vertex form, one of the known coordinates (0, 0), and the vertex, we can establish the value of *a* for the path of the ball.

$$y=a(x-20)^2+3$$
$$0=a(0-20)^2+3$$
$$-3=a(400)$$
$$-\frac{3}{400}=a$$

e. Using the value of *a* and the vertex from part d, write the equation of the path of the ball in vertex form.

$$f(x)=-\frac{3}{400}(x-20)^2+3$$

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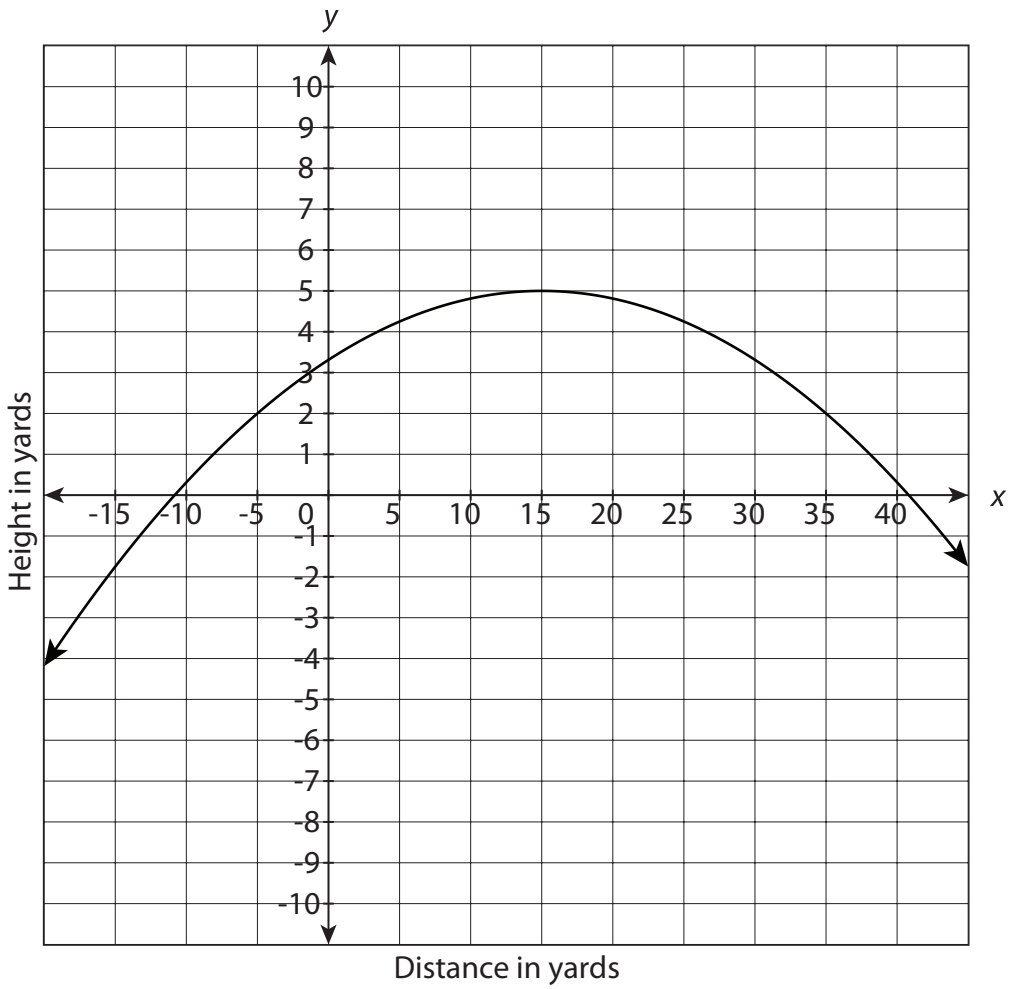
f. Translate the equation to account for the release point.

$$f(x+5)=-\frac{3}{400}(x+5-20)^2+3=-\frac{3}{400}(x-15)^2+3$$

g. Translate the equation to account for the height of the point of release and catch.

$$f(x+5)+2=\left[-\frac{3}{400}(x-15)^2+3\right]+2=-\frac{3}{400}(x-15)^2+5$$

h. Graph the function.



Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2.6.1: Replacing $f(x)$ with $f(x) + k$ and $f(x + k)$

For problems 1–3, let $f(x) = x^2$. Write a function that translates $f(x)$ as described.

1. 2 units to the left

2. 3 units up

3. 5 units to the right and 2 units down

For problems 4–6, let $f(x) = x^2$. Graph $g(x)$ by translating the graph of $f(x)$. State the vertex of the translated function.

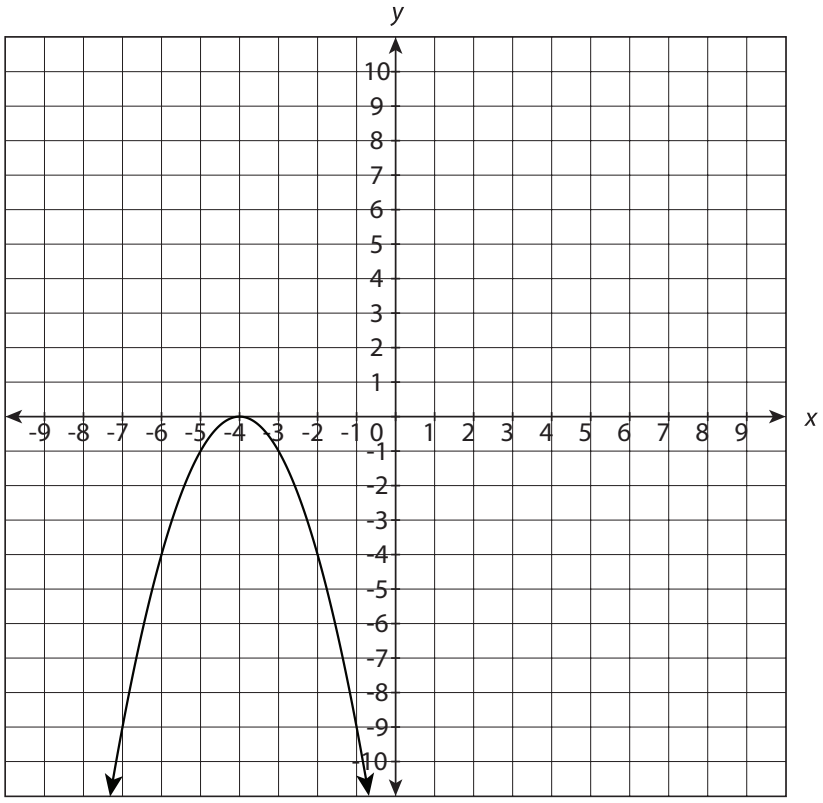
4. $g(x) = (x - 2)^2$

5. $g(x) = x^2 - 4$

6. $g(x) = (x - 5)^2 - 2$

Use what you know about translations of functions to solve each problem.

7. The following graph is a translation of $f(x) = -x^2$. Write an equation for the graph and state the value of k that was used to transform the function.



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8. A mother and her daughter went golfing. The mother hit first. Her ball followed the path modeled by the equation $f(x) = -0.0009x^2 + 0.2088x$ in the direction of the hole, and landed 18 yards short of the hole. The daughter teed off 18 yards closer to the hole because she is a beginner. She realized that if she could hit the ball on the same trajectory as her mother, her ball would land right by the hole. What is the equation that describes the path that the daughter's ball should follow?

9. A basketball is thrown from a height of 4 feet so that its path is modeled by the function $f(x) = -0.03x^2 + 1.3x + 4$. If the exact same shot is taken from a balcony that is 12 feet above where the original shooter was standing, how far away will the ball hit the ground? What is the equation that models this shot?

10. Simon has a toy that launches hollow plastic balls. The launched balls always follow a path modeled by the function $f(x) = -\frac{1}{8}(x-8)^2 + 8$ when the launcher is at the "origin." If the launcher is lifted up 2 feet and moved forward 5 feet, will a launched ball land in a basket that is on a 4-foot high stool 20 feet from the origin? What is the function that models this new launcher position?

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Lesson 2.6.2: Replacing $f(x)$ with $k \bullet f(x)$ and $f(k \bullet x)$

Warm-Up 2.6.2

An architectural firm is designing a one-story, square office building that is required to have a 4-foot space in every exterior wall for an exit. The area of the office building is given by the equation $f(x) = (x + 4)(x + 4) = (x + 4)^2$.

1. Given $f(x) = (x + 4)^2$, what would be the area of the building if $x = 20$ feet?

2. If the firm used the same blueprint with the area found in problem 1 to design a building with 3 identical floors, what would be the total area of the new building's office space?

3. If the firm were to double the length of each wall in the original design, what would be the new area of the one-story building?

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Lesson 2.6.2: Replacing $f(x)$ with $k \bullet f(x)$ and $f(k \bullet x)$

Common Core State Standard

F–BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Warm-Up 2.6.2 Debrief

1. Given $f(x) = (x + 4)^2$, what would be the area of the building if $x = 20$ feet?

   Evaluate $f(x) = (x + 4)^2$ for $x = 20$.

   $f(20) = (20 + 4)^2$

   $f(20) = (24)^2$

   $f(20) = 576$

   The area of the building would be 576 ft^2.

2. If the firm used the same blueprint with the area found in problem 1 to design a building with 3 identical floors, what would be the total area of the new building's office space?

   If the area is 576 ft^2 when the office building is only 1 floor, then the area of 3 identical floors would be $3 \bullet 576 = 1728$ ft^2. Written in function notation, $3 \bullet f(20) = 3 \bullet 576 = 1728$ ft^2.

   The area of the new building would be 1,728 ft^2.

3. If the firm were to double the length of each wall in the original design, what would be the new area of the one-story building?

   Build the new function.

   The original function is $f(x) = (x + 4)^2$.

   If the firm doubled the length of each office wall, then x would be twice as long, or $2x$.

   Therefore, the function that gives the area for the newly designed office building is $f(2x) = (2x + 4)^2$.

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Evaluate $f(2x) = (2x + 4)^2$ for $x = 20$.

$f(2 \bullet 20) = (2 \bullet 20 + 4)^2$

$f(40) = (40 + 4)^2$

$f(40) = (44)^2$

$f(40) = (44)^2 = 1936$

The new area of the one-story building would be 1,936 ft^2.

Connection to the Lesson

- Students will extend their understanding of multiplication of numbers to the effects of multiplication in functions.
- Students will explore the effects on a graph when $f(x)$ is replaced with $k \bullet f(x)$.
- Students will explore the effects on a graph when $f(x)$ is replaced with $f(k \bullet x)$.
- Students will learn how these function transformations differ when applied to quadratics.

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Prerequisite Skills

This lesson requires the use of the following skills:

- graphing quadratic functions (A–REI.10)
- finding intercepts and vertices of quadratic functions (F–IF.7a★)

Introduction

Functions can be transformed in two distinct ways: by transforming the independent variable of the function (such as by multiplying the independent variable by k), and by transforming the function as a whole (such as by multiplying the dependent variable by k). Previously, we saw how adding some constant k to the variable of a function or to the entire function affected the graph of the function.

In this lesson, we will see how multiplying by a constant affects the graph of a quadratic function. Given $f(x)$ and a constant k, we will observe the transformations $f(k \cdot x)$ and $k \cdot f(x)$.

Key Concepts

Graphing and Points of Interest

- In the graph of a function, there are key points of interest that define the graph and represent the characteristics of the function.
- When a function is transformed, the key points of the graph define the transformation.
- Key points that can be used to graph the translation of a quadratic function are the vertex and the x- and y-intercepts.

Multiplying the Dependent Variable by a Constant, k: $k \cdot f(x)$

- In general, multiplying a function by a constant will stretch or shrink (compress) the graph of $f(x)$ vertically.
- If $k > 1$, the graph of $f(x)$ will stretch vertically by a factor of k (so the parabola will appear narrower).
- A **vertical stretch** pulls the parabola and stretches it away from the x-axis.
- If $0 < k < 1$, the graph of $f(x)$ will shrink or compress vertically by a factor of k (so the parabola will appear wider).
- A **vertical compression** squeezes the parabola towards the x-axis.

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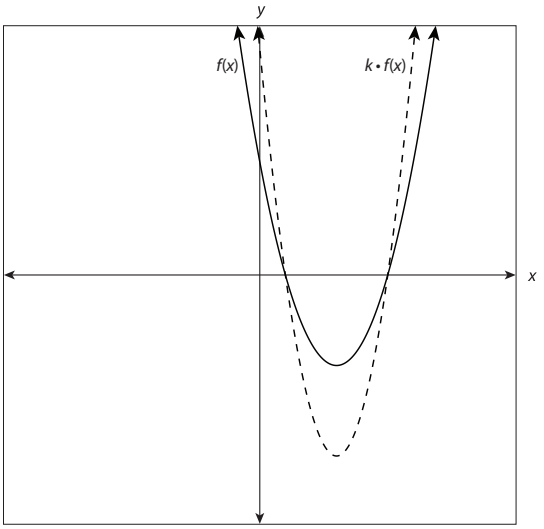
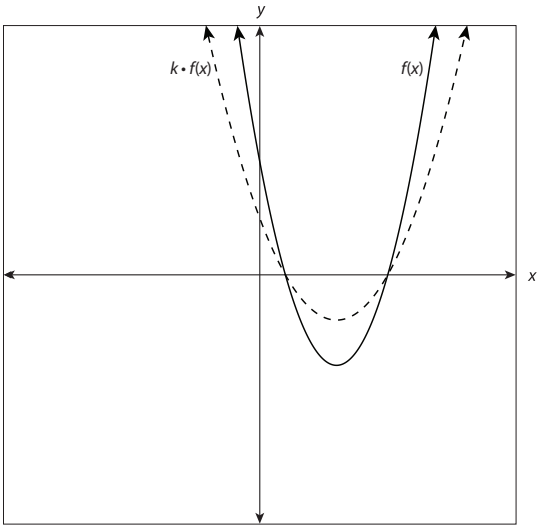
- If $k < 0$, the parabola will be first stretched or compressed and then reflected over the x-axis.
- When the graph of a function is stretched, compressed, or reflected over the x-axis, the x-intercepts (zeros) remain the same, as does the x-coordinate of the vertex (the axis of symmetry).
- While $k \bullet f(x) = f(k \bullet x)$ can be true, generally $k \bullet f(x) \neq f(k \bullet x)$.

Vertical stretches: when $k > 1$ in $k \bullet f(x)$



- The graph is stretched vertically by a factor of k.
- The x-coordinate of the vertex remains the same.
- The y-coordinate of the vertex changes.
- The x-intercepts remain the same.

Vertical compressions: when $0 < k < 1$ in $k \bullet f(x)$



- The graph is compressed vertically by a factor of k.
- The x-coordinate of the vertex remains the same.
- The y-coordinate of the vertex changes.
- The x-intercepts remain the same.

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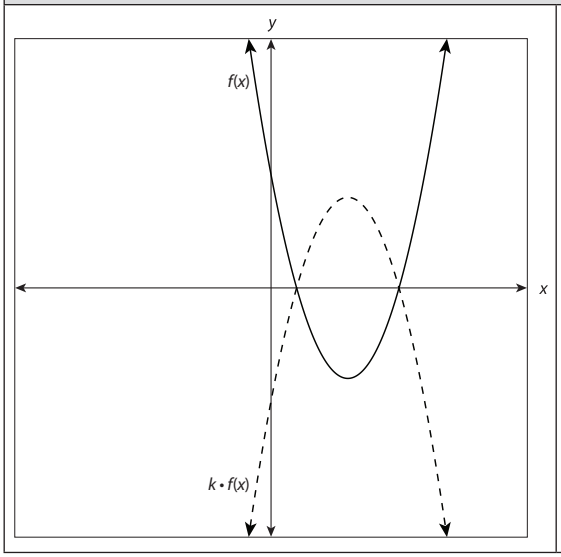
Reflections over the *x*-axis: when $k < 0$ in $k \cdot f(x)$



- The parabola is reflected over the *x*-axis.
- The *x*-coordinate of the vertex remains the same.
- The *y*-coordinate of the vertex changes.
- The *x*-intercepts remain the same.
- When $k < 0$, first perform the vertical stretch or compression, and then reflect the function over the *x*-axis.

Multiplying the Independent Variable by a Constant, *k*: $f(k \cdot x)$

- In general, multiplying the independent variable in a function by a constant will stretch or shrink the graph of the function horizontally.
- If $k > 1$, the graph of $f(x)$ will shrink or compress horizontally by a factor of $\frac{1}{k}$ (so the parabola will appear narrower).
- A **horizontal compression** squeezes the parabola toward the *y*-axis.
- If $0 < k < 1$, the graph of $f(x)$ will stretch horizontally by a factor of $\frac{1}{k}$ (so the parabola will appear wider).
- A **horizontal stretch** pulls the parabola and stretches it away from the *y*-axis.
- If $k < 0$, the graph is first horizontally stretched or compressed and then reflected over the *y*-axis.
- The *y*-intercept remains the same, as does the *y*-coordinate of the vertex.
- When a constant *k* is multiplied by the variable *x* of a function $f(x)$, the interval of the intercepts of the function is increased or decreased depending on the value of *k*.

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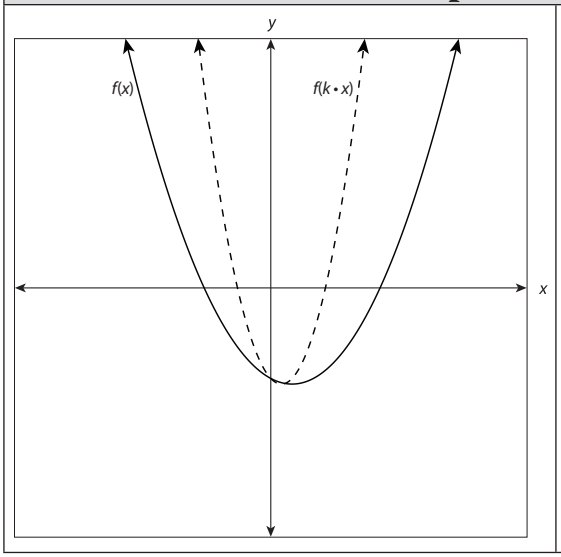
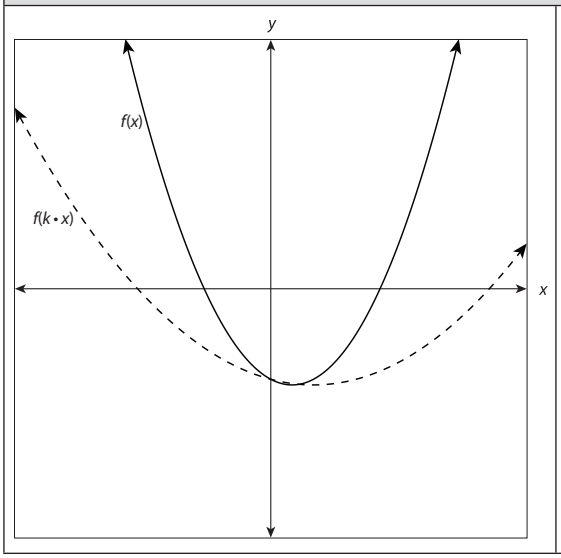
Horizontal compressions: when $k > 1$ in $f(k \cdot x)$



- The graph is compressed horizontally by a factor of $\frac{1}{k}$.
- The y-coordinate of the vertex remains the same.
- The x-coordinate of the vertex changes.
- The y-intercept remains the same.

Horizontal stretches: when $0 < k < 1$ in $f(k \cdot x)$



- The graph is stretched horizontally by a factor of $\frac{1}{k}$.
- The y-coordinate of the vertex remains the same.
- The x-coordinate of the vertex changes.

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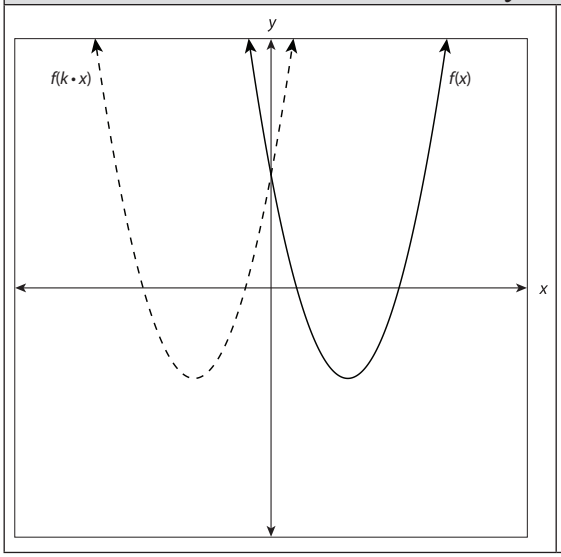
Instruction

Reflections over the *y*-axis: when $k < 0$ in $f(k \cdot x)$



- The parabola is reflected over the *y*-axis.
- The *y*-coordinate of the vertex remains the same.
- The *x*-coordinate of the vertex changes.
- The *y*-intercept remains the same.
- When $k < 0$, first perform the horizontal compression or stretch, and then reflect the function over the *y*-axis.

Common Errors/Misconceptions

- thinking that multiplying the dependent variable by a constant *k* yields the same equation as multiplying the independent variable by the same constant *k* in all cases (i.e., that $k \cdot f(x)$ is always equal to $f(k \cdot x)$, but this is not always true)
- forgetting to substitute all values of *x* with $k \cdot x$ when working with the transformation $f(k \cdot x)$
- forgetting to square the constant *k* when substituting $f(k \cdot x)$ into $ax^2 + bx + c$
- confusing horizontal with vertical transformations and vice versa
- confusing stretches and compressions

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Guided Practice 2.6.2

Example 1

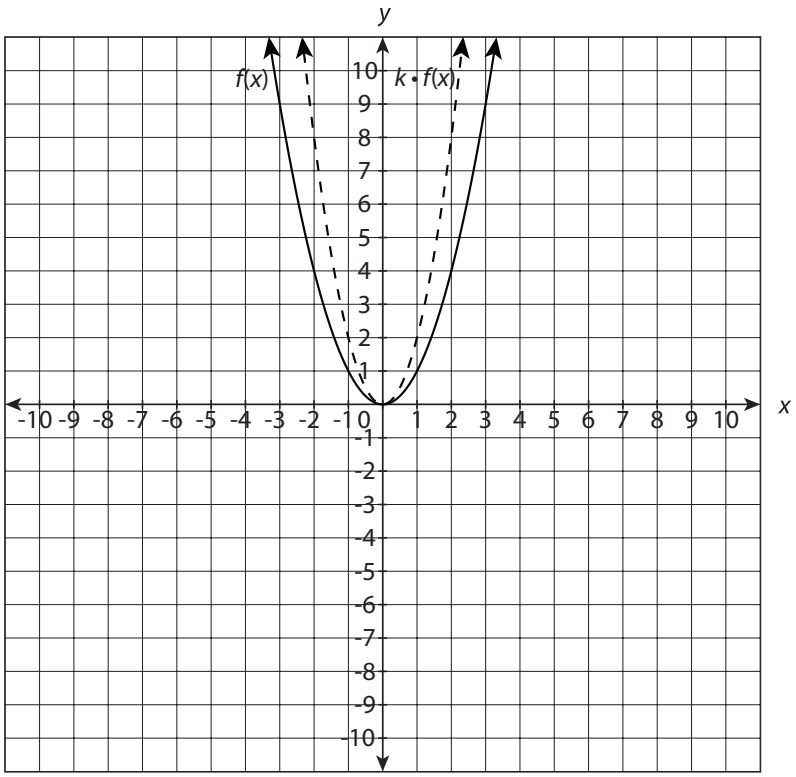
Consider the function $f(x) = x^2$, its graph, and the constant $k = 2$. What is $k \bullet f(x)$? How are the graphs of $f(x)$ and $k \bullet f(x)$ different? How are they the same?

1. Substitute the value of k into the function.

   If $f(x) = x^2$ and $k = 2$, then $k \bullet f(x) = 2 \bullet f(x) = 2x^2$.

2. Use a table of values to graph the functions.

x	$f(x)$	$k \bullet f(x)$
–2	4	8
–1	1	2
0	0	0
1	1	2
2	4	8



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Instruction

3. Compare the graphs.

   Notice the position of the vertex has not changed in the transformation of $f(x)$. Therefore, both equations have same x- and y-intercepts. However, notice the inner graph, $2x^2$, is narrower than the graph of x^2 because the y-coordinate of $2 \bullet f(x)$ is twice as much as the y-coordinate of $f(x)$. Since $k > 1$, and is being multiplied by the entire function as a whole, the graph of $f(x)$ will stretch vertically by a factor of 2. The parabola appears narrower.

Example 2

Consider the function $f(x) = x^2 - 81$, its graph, and the constant $k = 3$. What is $f(k \bullet x)$? How do the vertices and intercepts of $f(x)$ and $f(k \bullet x)$ compare?

1. Substitute the value of k into the function.

   If $f(x) = x^2 - 81$ and $k = 3$, then $f(k \bullet x) = f(3x) = (3x)^2 - 81 = 9x^2 - 81$.

2. Using a graphing calculator, graph the two functions on the same coordinate plane.

   From step 1, the equations are as follows: $f(x) = x^2 - 81$ and $f(k \bullet x) = 9x^2 - 81$.

   On a TI-83/84:

   Step 1: Press [Y=].

   Step 2: Type in the equation for $f(x)$ at Y_1: [X,T, θ, n][x^2][–][81]. Press [ENTER].

   Step 3: Type in the equation for $f(k \bullet x)$ at Y_2: [9][X,T, θ, n][x^2][–][81].

   Step 4: Press [GRAPH].

   Step 5: If necessary, set the viewing window by pressing [WINDOW]. A list of suggested parameters follows. Use the arrow keys to navigate between fields. Xmin = –10, Xmax = 10, Xscale = 1, Ymin = –90, Ymax = 50, Yscale = 10, Xres = 1.

   Step 6: Press [GRAPH].

(*continued*)

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On a TI-Nspire:

Step 1: From the home screen, choose a graphing window.

Step 2: Type in the equation for $f(x)$ at $f1(x)$: [x][x^2][–][81]. Press [enter].

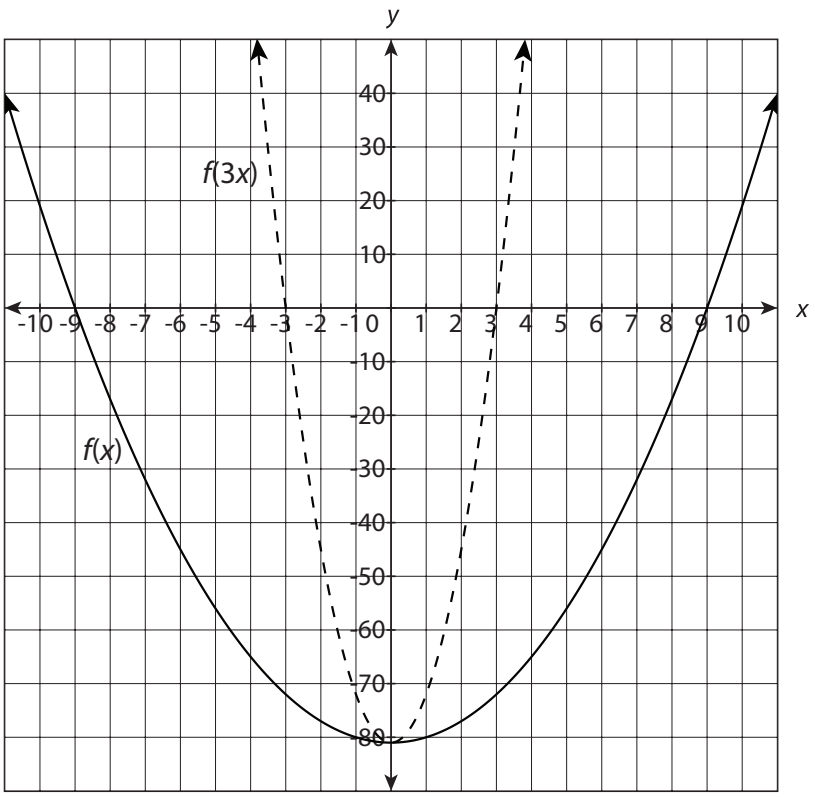
Step 3: Use the navigation pad to move the cursor over to the double arrows in the lower left corner and press the "click" key.

Step 4: Type in the equation for $f(k \cdot x)$ at $f2(x)$: [9][x][x^2][–][81]. Press [enter].

Step 5: Set the viewing window by pressing [menu] and selecting 4: Window/Zoom and then 1: Window Settings.

Step 6: Fill in the fields as follows using the [tab] key to navigate between fields: Xmin = –10, Xmax = 10, Xscale = 1, Ymin = –90, Ymax = 50, Yscale = 10, Xres = 1.

Step 7: Press [tab] to navigate to "OK" and press [enter].



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Instruction

3. Compare the graphs.

   Notice the position of the vertex has not changed in the transformation of $f(x)$, so the y-intercept remains the same. However, notice that the inner graph, $f(3x) = 9x^2 - 81$, is narrower than $f(x)$. The x-intercepts change. Specifically, the interval between the x-intercepts becomes smaller. Since $k > 1$, and is being multiplied by the independent variable of the function, x, the graph of $f(x)$ will shrink horizontally by a factor of $\frac{1}{3}$, so the parabola appears narrower.

Example 3

Consider the function $f(x) = x^2 - 6x + 8$, its graph, and the constant $k = -1$. What is $k \bullet f(x)$? How do the graphs of $f(x)$ and $k \bullet f(x)$ compare?

1. Substitute the value of k into the function.

   If $f(x) = x^2 - 6x + 8$ and $k = -1$, then $k \bullet f(x) = -f(x) = -(x^2 - 6x + 8) = -x^2 + 6x - 8$.

2. Using a graphing calculator, graph the two functions on the same coordinate plane.

   From step 1, the equations are as follows: $f(x) = x^2 - 6x + 8$ and $k \bullet f(x) = -x^2 + 6x - 8$. Refer to Example 1 for steps for graphing. Use a viewing window of –10 to 10 on the x-axis and from –10 to 10 on the y-axis. Each axis has a scale of 1.

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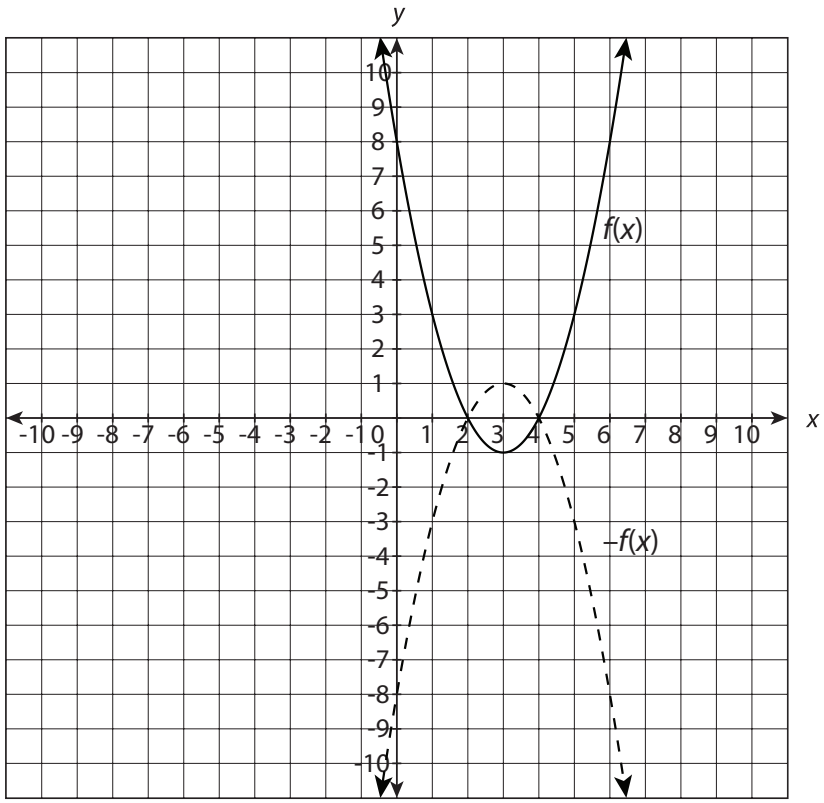

3. Compare the graphs.

   Notice that $f(x)$ is reflected over the x-axis for $k \cdot f(x)$ because k is negative. Also notice that the x-intercepts of the two parabolas are the same. This is because k is being multiplied by the entire function. Therefore, the x-intercepts stay the same.

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Example 4

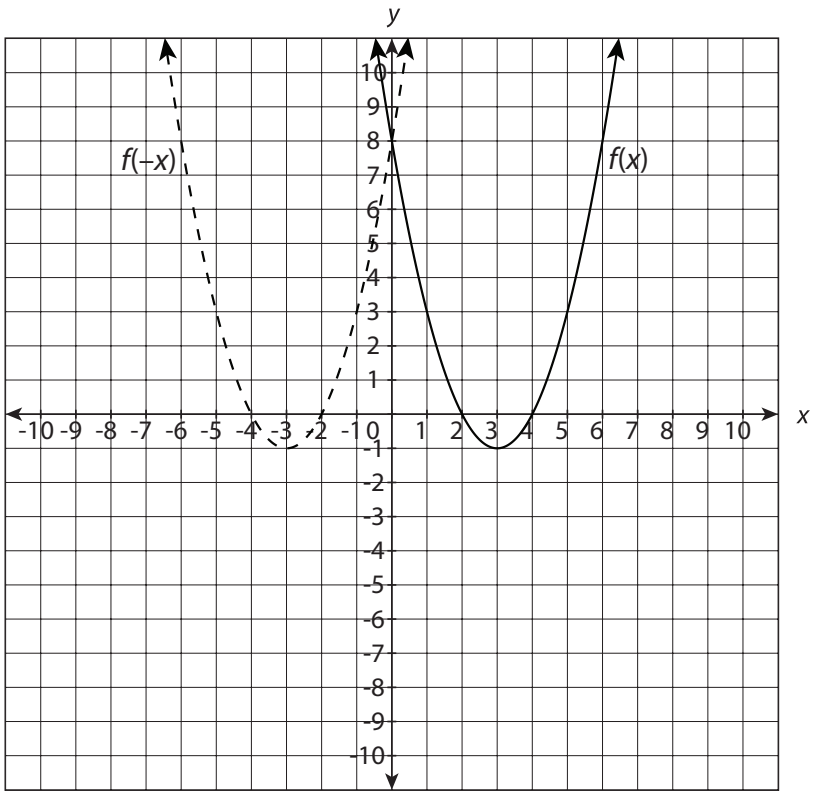
Consider the function $f(x) = x^2 - 6x + 8$, its graph, and the constant $k = -1$. What is $f(k \bullet x)$? How do the graphs of $f(x)$ and $f(k \bullet x)$ compare?

1. Substitute the value of k into the function.

   If $f(x) = x^2 - 6x + 8$ and $k = -1$, then $f(k \bullet x) = f(-x) = (-x)^2 - 6(-x) + 8 = x^2 + 6x + 8$.

2. Using a graphing calculator, graph the two functions on the same coordinate plane.

   From step 1, the equations are as follows: $f(x) = x^2 - 6x + 8$ and $k \bullet f(x) = x^2 + 6x + 8$. Refer to Example 1 for steps for graphing. Use a viewing window of –10 to 10 on the x-axis and –10 to 10 on the y-axis. Each axis has a scale of 1.



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3. Compare the graphs.

   Notice that $f(x)$ is reflected over the y-axis for $f(k \cdot x)$ because k is negative. Also notice that the x-intercepts of the equations have changed, but the y-intercept remains the same.

Example 5

The dimensions of a rectangular garden edged with wood are such that the longer sides are 3 times the length of the shorter sides. Keeping the same ratio of side lengths, which would result in having a larger garden area: making the existing sides 5 times longer, or building 4 more gardens that are equal in size to the existing garden?

1. Write an equation that models the area of the existing garden.

   Area is length times width, and the length of the garden is 3 times longer than its width. Therefore, the area can be modeled by the equation $a(x) = x \cdot 3x = 3x^2$, where x is the width (the garden's shorter side).

2. Use the equation to determine the area of the garden when the sides of the existing garden are made 5 times longer.

   The garden's width would now be $5x$, and the area of the garden would be $a(5x) = 3(5x)^2 = 3(25x^2) = 75x^2$.

3. Use the equation from step 1 to determine the total garden area when 4 more gardens equal in size to the existing garden are built, for a total of 5 gardens.

   If the area of the existing garden is $a(x) = 3x^2$, then the area of 5 gardens that are the same size is $5a(x) = 5(3x^2) = 15x^2$.

4. Compare the results of the two scenarios.

   The area of the garden in the first scenario is $75x^2$ square units.

   The area of the 5 gardens in the second scenario is $15x^2$ square units.

   The area of the garden in the first scenario is 5 times the area of the gardens in the second scenario; therefore, making the sides of the existing garden 5 times longer would result in a larger area.

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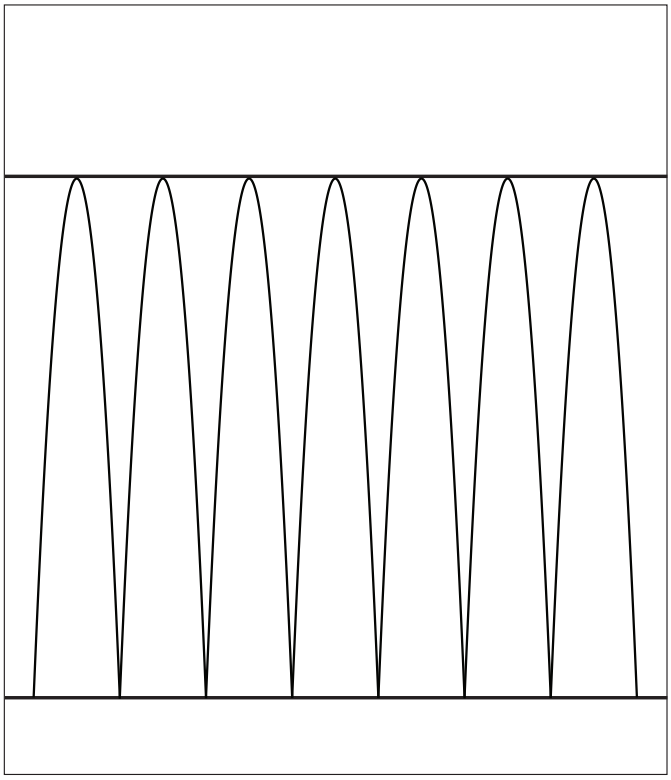
Problem-Based Task 2.6.2: Fewer Parabolas, Please

SMP	
1 ✓	2
3	4 ✓
5	6
7	8 ✓

A city hired a civil engineering firm to draw up plans for a 420-foot bridge that would use 7 downward-facing parabolic arches to support the span. The resulting plans called for the arches to be 75 feet high, with a distance of 60 feet between the bases of each arch, as shown in the following diagram.

After seeing the drawings, city councilors asked the civil engineering firm to make a second set of drawings using only 5 evenly spaced parabolic curves but covering the same 420-foot span. What are the equations of the 5 parabolas in the new plan? Let the *y*-axis represent the axis of symmetry for the left-most parabola, and let the *x*-axis represent the bottom of the arches.



What are the equations of the 5 parabolas in the new plan?

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Problem-Based Task 2.6.2: Fewer Parabolas, Please

Coaching

a. Identify the vertex and x-intercepts for the first of the original 7 parabolas.

b. Build the equation for the left-most parabola in vertex form using the coordinates of the vertex.

c. Solve for a using a point on the parabola.

d. Write the equation that describes the first of the 7 parabolas.

e. Determine whether keeping the same height and span of the bridge while reducing the number of arches is an $k \cdot f(x)$ problem or an $f(k \cdot x)$ problem.

f. What is the value of k?

g. What is the transformed equation for the left-most parabola?

h. What is the distance between the bases of each arch if the span must be 420 feet?

i. What type of transformation problem must be used to create the remaining four arches?

j. Use the information from parts h and i to create the equations for the four remaining arches.

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Instruction

Problem-Based Task 2.6.2: Fewer Parabolas, Please

Coaching Sample Responses

a. Identify the vertex and *x*-intercepts for the first of the original 7 parabolas.

Let the *y*-axis be the line of symmetry for the parabola. The *x*-value of the vertex is 0. The *y*-value of the vertex is the maximum value of the arch. The arches are to be 75 feet tall, so the vertex is (0, 75).

The *x*-intercepts are the points on the graph where $y = 0$. The arches are 60 feet apart at their bases. Divide by 2 to determine the distance each base is from the axis of symmetry.

$60 \div 2 = 30$

Since the *y*-axis is the axis of symmetry, the *x*-intercepts are –30 and 30.

b. Build the equation for the left-most parabola in vertex form using the coordinates of the vertex.

Substitute the known information to build the vertex form of the equation, then simplify.

$f(x) = a(x - h)^2 + k$

$f(x) = a(x - 0)^2 + 75$

$f(x) = ax^2 + 75$

c. Solve for *a* using a point on the parabola.

Substitute one of the *x*-intercepts into $f(x) = ax^2 + 75$ to solve for *a*.

$f(x) = ax^2 + 75$

$0 = a(30)^2 + 75$

$-75 = a \bullet 900$

$$-\frac{75}{900} = a = -\frac{1}{12}$$

d. Write the equation that describes the first of the 7 parabolas.

$$f(x) = -\frac{1}{12}x^2 + 75$$

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Instruction

e. Determine whether keeping the same height and span of the bridge while reducing the number of arches is an $k \bullet f(x)$ problem or an $f(k \bullet x)$ problem.

The problem is asking for a horizontal stretch, so that each arch has the same height as in the previous plan, but a greater width. This is an $f(k \bullet x)$ problem.

f. What is the value of k?

Starting with the current function, consider the ratio of the number of arches in the old and new plans. We now want 5 parabolas instead of the 7 we had before, so our ratio is 5 : 7.

Remember that a value of k between 0 and 1 will increase the interval between the x-intercepts, so we want our value to be $k = \frac{5}{7}$.

g. What is the transformed equation for the left-most parabola?

Substitute the value of k into $f(k \bullet x)$.

$$f(k \bullet x) = f\left(\frac{5}{7}x\right)$$

$$f\left(\frac{5}{7}x\right) = -\frac{1}{12}\left(\frac{5}{7}x\right)^2 + 75$$

$$f\left(\frac{5}{7}x\right) = -\frac{1}{12}\left(\frac{25}{49}\right)x^2 + 75$$

$$f\left(\frac{5}{7}x\right) = -\frac{25}{588}x^2 + 75$$

h. What is the distance between the bases of each arch if the span must be 420 feet?

The problem states the span of the 5 parabolas must stay at 420 feet. There are 5 parabolas. Therefore, divide 420 feet by 5.

$$\frac{420}{5} = 84$$

The distance between each pair of bases for each arch is 84 feet.

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Instruction

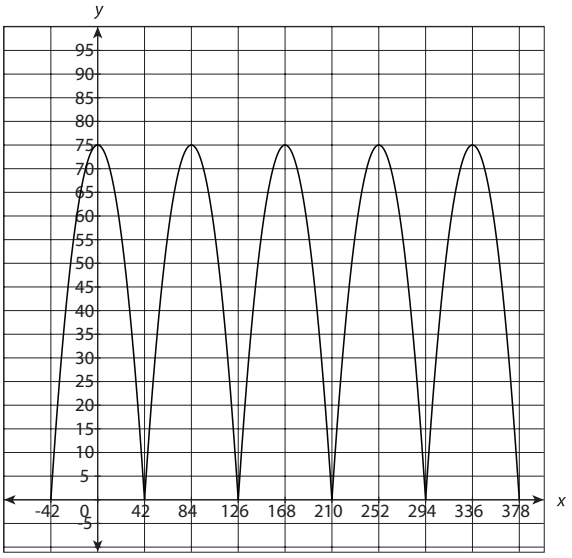
i. What type of transformation problem must be used to create the remaining four arches?

The first arch needs to be translated to the right 4 times. This is an $f(x + k)$ problem.

j. Use the information from parts h and i to create the equations for the four remaining arches.

For each of the remaining 4 parabolas, translate the function to the right by 84 units each, the distance between the arches.

- Equation 2: $f(x)=-\frac{25}{588}(x-84)^2+75$
- Equation 3 (translating Equation 2 by 84 units to the right): $f(x)=-\frac{25}{588}(x-168)^2+75$
- Equation 4 (translating Equation 3): $f(x)=-\frac{25}{588}(x-252)^2+75$
- Equation 5 (translating Equation 4): $f(x)=-\frac{25}{588}(x-336)^2+75$



Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2.6.2: Replacing $f(x)$ with $k \bullet f(x)$ and $f(k \bullet x)$

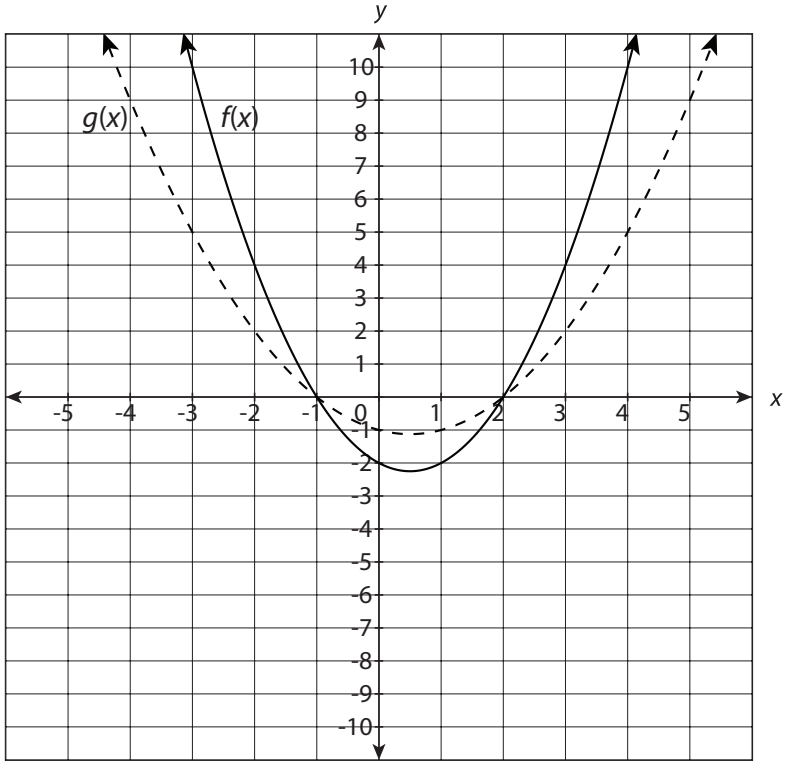
Use what you have learned about transformations of functions to solve problems 1 and 2.

1. For the function $f(x) = x^2 + x - 6$, find $2 \bullet f(x)$, and describe the changes that occur to the graph of $f(x)$ as a result of multiplying the function by 2. Check your answers by comparing the two functions on your graphing calculator.

2. For the function $f(x) = x^2 + x$, find $f(3x)$, and describe the changes that occur to the graph of $f(x)$ as a result of multiplying the variable x by 3. Check your answers by comparing the two functions on your graphing calculator.

Use the graphs and the given information to complete problems 3 and 4.

3. Consider the graphs of the functions $f(x)$ and $g(x)$. The equation for $f(x)$ is $f(x) = x^2 - x - 2$. What could be the equation for $g(x)$?



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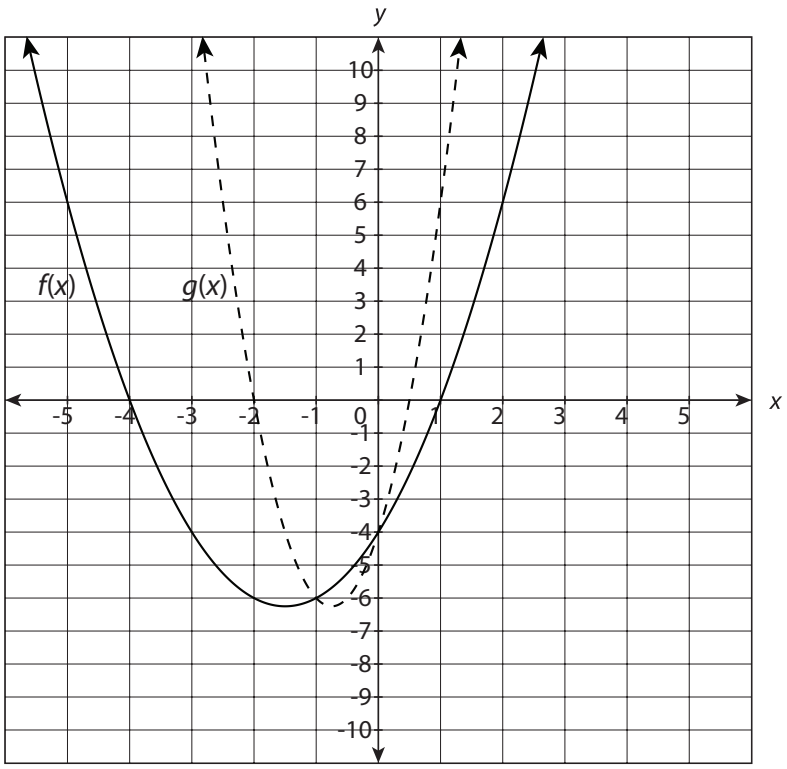
4. Consider the graphs of the functions $f(x)$ and $g(x)$. The equation for $f(x)$ is $f(x) = x^2 + 3x - 4$. What could be the equation for $g(x)$?



Complete each of the following tasks for the functions in problems 5–7.

- Graph $f(x)$ and $g(x)$ on your graphing calculator.
- Determine the scale factor and the transformation(s): horizontal stretch, horizontal compression, vertical stretch, vertical compression, reflection over the x-axis, or reflection over the y-axis.
- Describe the similarities and differences of the graphs.

5. $f(x) = x^2 - x - 2$; $g(x) = -2f(x)$

6. $f(x) = x^2 - x - 2$; $g(x) = f(-2x)$

7. $f(x) = x^2 - 1$; $g(x) = -\frac{1}{2} \cdot f(x)$

continued

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Read each scenario and use the given information to solve problems 8–10.

8. A farmer has a rectangular goat pen such that one side is 2 times as long as the other side. He would like to have more space for his goats, and he is deciding between two options. He could either double the lengths of the sides of the existing pen, or he could build a second pen of the same size as the first. Which option would give him the most area for his goats? Explain your answer in terms of $k \bullet f(x)$ and $f(k \bullet x)$.

9. A company that produces skateboards knows the equation that models profit per month is $f(x) = 3x^2 + 300x$, where x is the price charged per skateboard. If the company plans to expand with the hopes of doubling its profits, should the new model for the company's profit be $f(2x)$, $f\left(\frac{1}{2}x\right)$, $2 \bullet f(x)$, or $\frac{1}{2} \bullet f(x)$? Explain.

10. Jada and Jayla are twins on the same softball team. They can each hit the ball so that it follows a path modeled by the equation $f(x) = -0.01x^2 + 0.98x + 2$. Jada says that the ball would go farther if it followed the path $g(x) = f(2x)$. Jayla says the ball would go farther if it followed the path $g(x) = 2 \bullet f(x)$. Who is correct? Which equation for $g(x)$ would allow the ball to achieve the same height as the ball in the original equation?

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Progress Assessment

Circle the letter of the best answer.

1. The vertex of $f(x)$ is $(3, -4)$. What is the vertex of $f(x) + k$?

   a. $(3k, 4k)$
   b. $(3, -4 + k)$
   c. $(3k, -4k)$
   d. $(3, 4 + k)$

2. Given the function $f(x) = x^2$, what is the vertex of $f(x) - 4$?

   a. $(0, 4)$
   b. $(0, -4)$
   c. $(-4, 0)$
   d. $(4, 0)$

3. If $f(x) = x^2$, what is the equation of $g(x)$ if $f(x)$ is translated to the left 3 units and down 2 units?

   a. $g(x) = (x + 3)^2 - 2$
   b. $g(x) = (x - 3)^2 - 2$
   c. $g(x) = (x + 2)^2 - 3$
   d. $g(x) = (x - 2)^2 + 3$

4. Given $f(x) = x^2$ and $g(x) = (x - 5)^2$, what is the shift of the vertex for $g(x)$ from $f(x)$?

   a. to the right 5 units
   b. to the left 5 units
   c. up 5 units
   d. down 5 units

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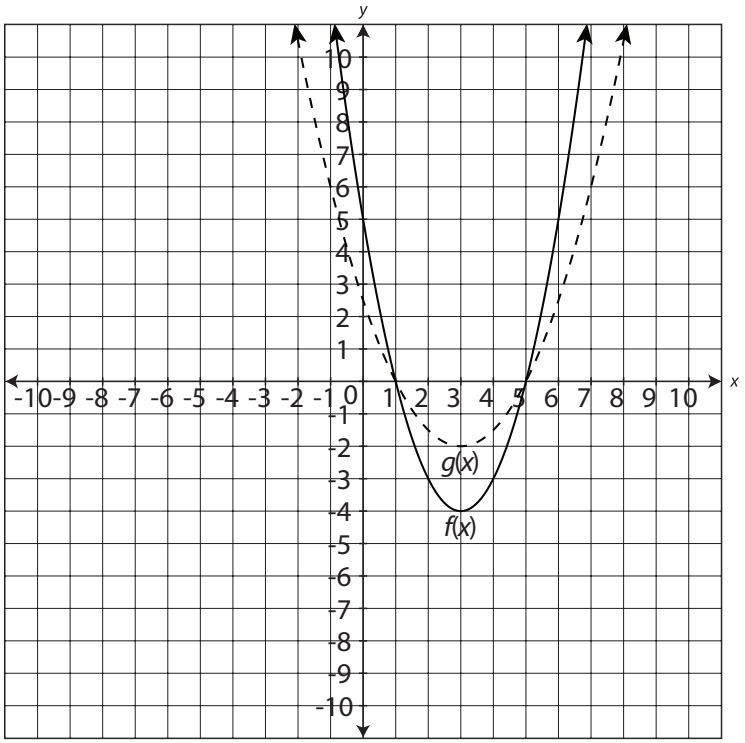
5. Consider the graphs of $f(x)$ and $g(x) = \frac{1}{2} \cdot f(x)$. What is the transformation, including the scale factor?



a. The function is compressed vertically by a factor of 2.

b. The function is compressed horizontally by a factor of 2.

c. The function is stretched vertically by a factor of $\frac{1}{2}$.

d. The function is compressed vertically by a factor of $\frac{1}{2}$.

6. Given $f(x) = x^2 - 3x + 108$, what is the transformation of $f(3x)$?

a. The function is stretched vertically by a factor of 3.

b. The function is compressed vertically by a factor of $\frac{1}{3}$.

c. The function is stretched horizontally by a factor of 3.

d. The function is compressed horizontally by a factor of $\frac{1}{3}$.

continued

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7. All of the following conditions are true of horizontal compressions *except*:

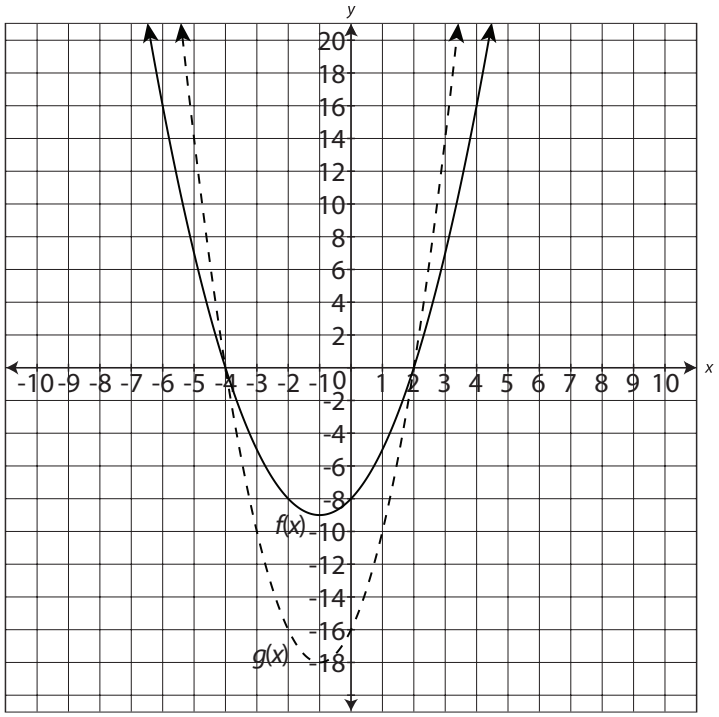
   a. The *y*-coordinate of the vertex remains the same.

   b. The *x*-intercepts remain the same.

   c. The scale factor is equal to $\frac{1}{k}$.

   d. The *x*-coordinate of the vertex changes.

8. Consider the graphs of the functions $f(x)$ and $g(x)$. The equation for $f(x)$ is $f(x) = x^2 + 2x - 8$. What could be an equation for $g(x)$?



   a. $g(x) = 4x^2 + 4x - 8$

   b. $g(x) = 2x^2 + 4x - 16$

   c. $g(x) = 0.5x^2 + x - 4$

   d. $g(x) = 0.25x^2 + x - 8$

continued

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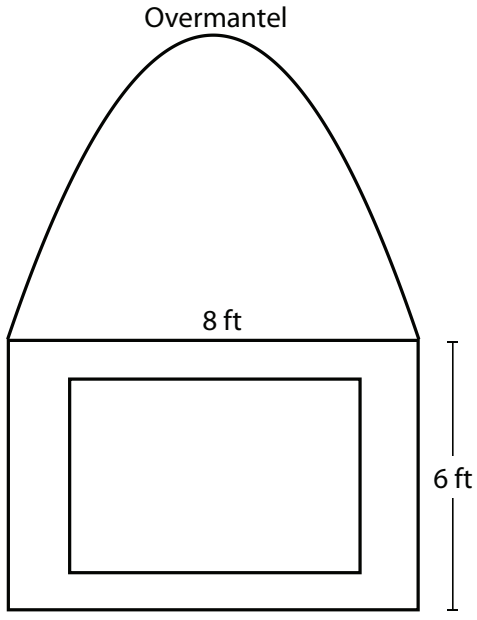
9. An architectural firm is designing a park for a small city. The center of the park will have a large monument consisting of a parabolic arch. The blueprint calls for the space in between the bases of the arch to be 80 feet. The planned height of the monument is 400 feet. City officials reviewed the blueprint and told the architects that the arch could be no taller than 380 feet because of planes taking off from a nearby airport. If $f(x)$ describes the original plans for the arch, what describes the equation of the monument with the same bases but only 380 feet tall?

   a. $f(0.95x)$
   b. $f(x - 0.95)$
   c. $0.95 \cdot f(x)$
   d. $1.05 \cdot f(x)$

10. A couple remodeling their home bought an antique concrete overmantel to go on the top of their new fireplace. The overmantel is in the shape of a parabola. The room is 20 feet wide, easily accommodating the overmantel, which is 8 feet wide, base to base. The fireplace is centered in the room, and the top of the fireplace is 6 feet off the ground. If the parabolic overmantel is described by $f(x)$ when it sits on the floor at the edge of the left wall, what equation describes the overmantel once it's placed over the fireplace?



   a. $f(x - 8) + 6$
   b. $f(x - 6) + 6$
   c. $f(x + 6) - 6$
   d. $f(x + 8) + 6$

continued

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Assessment

Use the information given in the problem to answer the questions that follow.

11. Aiden is dribbling a basketball down the court toward two defenders. He decides to stop 20 feet out from the basket for a jump shot. He knows the shot must clear the tall jumping defenders at no less than 11 feet, and go far enough to make the hoop. The ball passes over the defenders, but falls 1 foot short of the center of the hoop. Given that $f(x)$ describes the path of the basketball, answer the following questions.

    a. Which type of transformation would have enabled the ball to enter the hoop when thrown from the same distance, 20 feet from the hoop?

    b. What is the value of k for this transformation?

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Pre-Assessment

Circle the letter of the best answer.

1. Which function is the inverse of $f(x) = x - 15$?

   a. $f^{-1}(x) = -x - 15$
   b. $f^{-1}(x) = x - 15$
   c. $f^{-1}(x) = -x + 15$
   d. $f^{-1}(x) = x + 15$

2. If $f^{-1}(x) = \frac{x}{6} + 8$, what is $f(x)$?

   a. $f(x) = 6x - 48$
   b. $f(x) = 6x - 8$
   c. $f(x) = 6x + 48$
   d. $f(x) = \frac{x}{6} + 48$

3. Which function is the inverse of $f(x) = x^2 - 5$ if the domain is $x \geq 0$?

   a. $f^{-1}(x) = \sqrt{x} + 5$
   b. $f^{-1}(x) = \sqrt{x + 5}$
   c. $f^{-1}(x) = -\sqrt{x - 5}$
   d. $f^{-1}(x) = x^2 + 5$

4. Which function is not one-to-one?

   a. $f(x) = 10x$
   b. $f(x) = x + 10$
   c. $f(x) = \frac{x}{10}$
   d. $f(x) = 10x^2$

5. If $f^{-1}(x) = 3\sqrt{x}$, what is $f(x)$?

   a. $f(x) = 3x^2$ if $x \geq 0$
   b. $f(x) = 9x^2$ if $x \geq 0$
   c. $f(x) = \frac{1}{9}x^2$ if $x \geq 0$
   d. $f(x) = \frac{1}{3}x^2$ if $x \geq 0$

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Lesson 7: Finding Inverse Functions

Instruction

Common Core State Standard

F–BF.4 Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example, $f(x) = 2x^3$ or $f(x) = (x+1)/(x–1)$ for $x \neq 1$.*

SMP	
1 ✓	2 ✓
3	4
5	6
7 ✓	8

Essential Questions

1. What is the relationship between a function and its inverse?
2. What is the relationship between the independent and dependent quantities in a function and its inverse?
3. How can the inverse of a function that is not one-to-one be found?

WORDS TO KNOW

function	a relation in which each element in the domain is mapped onto exactly one element in the range; that is, for every value of x, there is exactly one value of y
function notation	a way to name a function using $f(x)$ as the dependent variable instead of y
inverse function	the function that results from switching the x- and y-variables in a given function; the inverse of $f(x)$ is written as $f^{-1}(x)$
inverse operation	the operation that reverses the effect of another operation. Addition and subtraction are inverse operations, and multiplication and division are inverse operations.
linear function	a first-degree equation that can be written in the form $f(x) = mx + b$, in which m is the slope of the line and b is the y-intercept. The graph of a linear function is a straight line.
one-to-one	a relationship in which each x-value is mapped onto exactly one y-value, and each y-value is mapped onto exactly one x-value

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Instruction

quadratic function — a function that can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. The graph of a quadratic function is a parabola.

Recommended Resources

- Khan Academy. "Introduction to Function Inverses."

  http://www.walch.com/rr/00155

  This video tutorial shows how to algebraically determine the inverse of a linear or quadratic function.

- Math Warehouse. "Inverse of a Function."

  http://www.walch.com/rr/00156

  This site explores how to determine the inverse of the function algebraically. Interactive practice problems show how to find the inverses of linear and simple quadratic functions.

- Purplemath.com. "Finding the Inverse of a Function."

  http://www.walch.com/rr/00157

  This site models how to determine the inverse of a function algebraically, with worked examples and explanations. Situations where the original function isn't one-to-one, such as $y = x^2$, are also explained.

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Lesson 2.7.1: Finding Inverse Functions

Warm-Up 2.7.1

Ari received $100 as a birthday gift and has decided to save this money as well as add $20 of his own money each month. The total money Ari has saved can be estimated using the equation $y = 20x + 100$, where x is the number of months and y is the total amount of money saved.

1. How much money will Ari have saved after 6 months?

2. How long will it take Ari to save $400?

3. Suppose Ari decided to save $50 each month. The amount of money in his savings account after x months can now be estimated using the equation $y = 50x + 100$. How long will it take Ari to save $400 with his new savings plan?

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Instruction

Lesson 2.7.1: Finding Inverse Functions

Common Core State Standard

F–BF.4 Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example, $f(x) = 2x^3$ or $f(x) = (x+1)/(x–1)$ for $x \neq 1$.*

Warm-Up 2.7.1 Debrief

1. How much money will Ari have saved after 6 months?

   Replace x with 6 in the equation $y = 20x + 100$ to determine how much money Ari has saved after 6 months.

$y = 20x + 100$	Original equation
$y = 20(6) + 100$	Substitute 6 for x.
$y = 120 + 100$	Simplify.
$y = 220$	

   Ari will have saved $220 after 6 months.

2. How long will it take Ari to save $400?

   Replace y with 400 in the equation $y = 20x + 100$ to determine how long it will take Ari to save $400.

$y = 20x + 100$	Original equation
$(400) = 20x + 100$	Substitute 400 for y.
$300 = 20x$	Simplify.
$15 = x$	

   It will take Ari 15 months to save $400.

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Instruction

3. Suppose Ari decided to save $50 each month. The amount of money in his savings account after x months can now be estimated using the equation $y = 50x + 100$. How long will it take Ari to save $400 with his new savings plan?

   Replace y with 400 in the new equation, $y = 50x + 100$, and solve for x.

$y = 50x + 100$	New equation
$(400) = 50x + 100$	Substitute 400 for y.
$300 = 50x$	Simplify.
$6 = x$	

   It will take Ari 6 months to save $400 with the new savings plan.

Connection to the Lesson

- Students will use inverse operations to find inverses of linear and quadratic functions.
- Students will apply inverse operations to solve for a variable.
- Students will identify the dependent and independent quantities in a function.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- understanding the definition of a function (F–IF.1)
- identifying independent and dependent variables (6.EE.9)
- determining the domain and range of linear and quadratic functions (F–IF.1)
- applying inverse operations to isolate a variable, including taking square roots (7.EE.4a)
- using function notation (F–IF.2)
- simplifying expressions involving square roots (5.OA.1)

Introduction

An inverse function can be thought of as any function that undoes something that has already been done. In our everyday lives, examples of actions that are done and then undone could include turning off the faucet after it was turned on, taking off your jacket after putting it on, or even solving a puzzle and then taking it apart again. The inverse of an algebraic function can be used to solve problems for which the dependent quantity is known and the independent quantity is unknown.

Key Concepts

- Recall that a function is a relation in which each element in the domain is mapped onto exactly one element in the range; that is, for every value of x, there is exactly one value of y.
- **Function notation** is the use of $f(x)$, which means "function of x," instead of y as the dependent variable in the equation of a function. For example, $f(x) = 2x + 1$ and $y = 2x + 1$ are equivalent functions.
- The inverse of a function, called an **inverse function**, is the function that results from switching the x- and y-variables in a given function.
- The inverse of $f(x)$ is written as $f^{-1}(x)$.
- The inverse function identifies all of the possible outputs of the function and the corresponding inputs. The outputs of a function are the independent values of the inverse of the function, and the inputs of the function are the dependent values of the inverse of the function.
- The domain of a function's inverse is the range of the original function.
- The range of a function's inverse is the domain of the original function.

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Instruction

- Consider a function with the following inputs and corresponding outputs. Notice that inputs and outputs of the function and inverse function are reversed.

Function	
Input	**Output**
–3	–17
–1	–11
1	–6
3	–3

Inverse function	
Input	**Output**
–17	–3
–11	–1
–6	1
–3	3

- To algebraically find the inverse of a function with x as the independent variable and y as the dependent variable, switch x and y, then solve for y in terms of x by using **inverse operations**, or operations that reverse the effect of other operations.
- The following table outlines the inverses of some common mathematical operations.

Original operation	Inverse operation
Addition	Subtraction
Subtraction	Addition
Multiplication	Division
Division	Multiplication
Square	Taking the square root
Square root	Squaring the number

- In order to find the inverse of various functions, recall the following about linear and quadratic functions.
- A linear function is a first-degree equation that can be written in the form $f(x) = mx + b$, where m is the slope of the line and b is the y-intercept. The graph of a linear function is a straight line.
  - The domain of a linear function is all real numbers.
  - The range of a linear function is all real numbers.
- A quadratic function is in the form $y = a(x – h)^2 + k$, where a, h, and k are real numbers.
  - The highest power of a quadratic function is 2.
  - The domain of a quadratic function is all real numbers.
  - The range of a quadratic function of the form $y = a(x – h)^2 + k$ has an extreme value at $y = k$. If $a > 0$, then the range is $y \geq k$. If $a < 0$, then the range is $y \leq k$.

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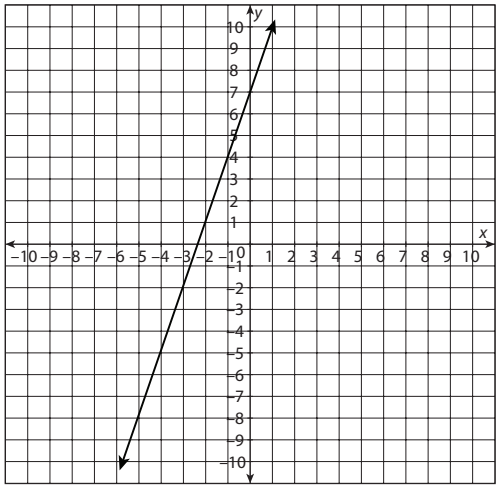
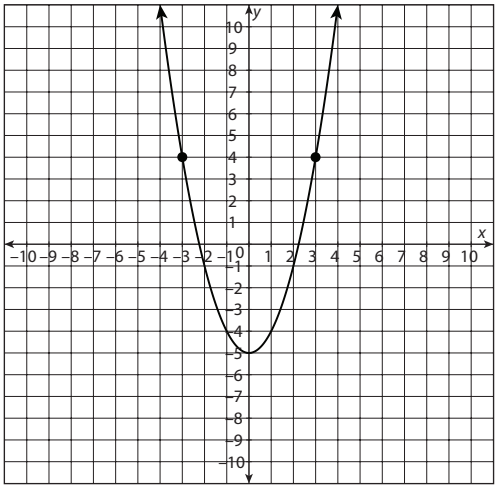
Instruction

- If a function is **one-to-one**, meaning that each x-value is mapped onto exactly one y-value and each y-value is mapped onto exactly one x-value, then the inverse of the function is also a function.
- A linear function is one-to-one because each y-value is associated with a unique x-value.
- A quadratic function is a function that can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$.
- A quadratic function is not one-to-one. Look at the following diagrams. Notice that in the parabola on the right, each y-value is not associated with a unique x-value, as shown by the two points.

One-to-one



Not one-to-one



- The domain of a function that isn't one-to-one can be restricted to find an inverse function.
- A quadratic function can be restricted to a domain of $x \geq h$ or $x \leq h$ in order for it to be one-to-one.
- If a quadratic function of the form $f(x) = a(x - h)^2 + k$ is restricted to a domain of $x \geq h$ or $x \leq h$, the range of the restricted function is equal to the range of the original function $f(x)$.
- The restricted domain of a function is the range of the function's inverse.

Common Errors/Misconceptions

- misinterpreting the superscript in inverse function notation, $f^{-1}(x)$, as a power of $f(x)$
- incorrectly performing inverse operations to find the inverse of a function

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Instruction

Guided Practice 2.7.1

Example 1

Lana is driving home from her friend's house. She is driving at a steady speed, and her distance from her home, $f(x)$, in miles, can be represented by the function $f(x) = -40x + 15$, where x is her driving time in hours. Find the inverse function $f^{-1}(x)$ to show when, in hours, Lana will be x miles from home.

1. Determine if the function is one-to-one.

   The function $f(x) = -40x + 15$ is linear because the highest power of x is 1. A linear function is one-to-one, so $f(x)$ is one-to-one.

2. Rewrite the function $f(x)$ in the form "$y =$."

   If $f(x) = -40x + 15$, then $y = -40x + 15$.

3. Switch x and y in the original equation of the function.

$y = -40x + 15$	Original equation
$x = -40y + 15$	Switch x and y.

4. Solve the new equation for y in terms of x by using inverse operations.

$x = -40y + 15$	New equation
$x + 40y = 15$	Add $40y$ to both sides.
$40y = -x + 15$	Subtract x from both sides.
$y = \frac{-x+15}{40}$	Divide both sides by 40.
$y = -\frac{1}{40}x + \frac{3}{8}$	Simplify.

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5. Replace y with $f^{-1}(x)$ to show that the equation is the inverse of $f(x)$.

$$f^{-1}(x)=-\frac{1}{40}x+\frac{3}{8}$$

The inverse of $f(x) = -40x + 15$ is $f^{-1}(x)=-\frac{1}{40}x+\frac{3}{8}$.

The inverse function $f^{-1}(x)=-\frac{1}{40}x+\frac{3}{8}$ shows when, in hours, Lana will be x miles from home.

Example 2

A high school is selling tickets to a school play. The school is using the money earned from selling tickets to pay for the play. Any extra money after expenses will be profit. The profit, in dollars, can be represented using the equation $p(x) = 12x – 600$, where x is the number of tickets sold. Find the inverse of $p(x)$ to show the number of tickets that need to be sold in order to earn a given profit.

1. Determine if the function is one-to-one.

   The function $p(x) = 12x – 600$ is linear because the highest power of x is 1. A linear function is one-to-one, so $p(x)$ is one-to-one.

2. Rewrite the function $p(x)$ in the form "$y =$."

   If $p(x) = 12x – 600$, then $y = 12x – 600$.

3. Switch x and y in the original equation of the function.

$y = 12x – 600$	Original equation
$x = 12y – 600$	Switch x and y.

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4. Solve the new equation for y by using inverse operations.

$x = 12y - 600$	New equation
$x - 12y = -600$	Subtract $12y$ from both sides.
$-12y = -x - 600$	Subtract x from both sides.
$y = \frac{x + 600}{12}$	Divide both sides by -12.
$y = \frac{1}{12}x + 50$	Simplify.

5. Replace y with $p^{-1}(x)$ to show that the equation is the inverse of $p(x)$.

$$p^{-1}(x) = \frac{1}{12}x + 50$$

The inverse of $p(x) = 12x - 600$ is $p^{-1}(x) = \frac{1}{12}x + 50$.

The inverse function $p^{-1}(x) = \frac{1}{12}x + 50$ shows the number of tickets that need to be sold, $p^{-1}(x)$, in order to earn a profit of x dollars.

Example 3

Find the inverse function of $f(x) = 4x^2$. Use a restricted domain so the inverse is a function.

1. Determine if the function is one-to-one.

   The function $f(x) = 4x^2$ is quadratic because the highest power of x is 2. A quadratic function is not one-to-one, so $f(x)$ is not one-to-one.

2. Determine a restricted domain for $f(x)$ on which the function is one-to-one.

   A quadratic function of the form $f(x) = a(x - h)^2 + k$ can be restricted to a domain of $x \geq h$ or $x \leq h$ in order for it to be one-to-one.

   For the function $f(x) = 4x^2$, $h = 0$ and $k = 0$, so the vertex is at the origin.

   The function $f(x)$ can be restricted to $x \geq 0$ in order for it to be one-to-one.

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3. Rewrite the function $f(x)$ in the form "$y =$."

   If $f(x) = 4x^2$, then $y = 4x^2$.

4. Switch x and y in the original equation of the function.

$y = 4x^2$	Original equation
$x = 4y^2$	Switch x and y.

5. Solve the new equation for y by using inverse operations.

$x = 4y^2$	New equation
$\frac{x}{4} = y^2$	Divide both sides by 4.
$\pm\sqrt{\frac{x}{4}} = y$	Take the square root of both sides.
$\frac{\sqrt{x}}{2} = y$	Simplify. Because the domain of the original function was restricted to $x \geq 0$, we only need to consider values of y such that $y \geq 0$, so the negative root can be disregarded.
$y = \frac{\sqrt{x}}{2}$	

6. Replace y with $f^{-1}(x)$ to show that the equation is the inverse of $f(x)$.

$$f^{-1}(x) = \frac{\sqrt{x}}{2}$$

The inverse of $f(x) = 4x^2$ on the restricted domain of $x \geq 0$ is $f^{-1}(x) = \frac{\sqrt{x}}{2}$.

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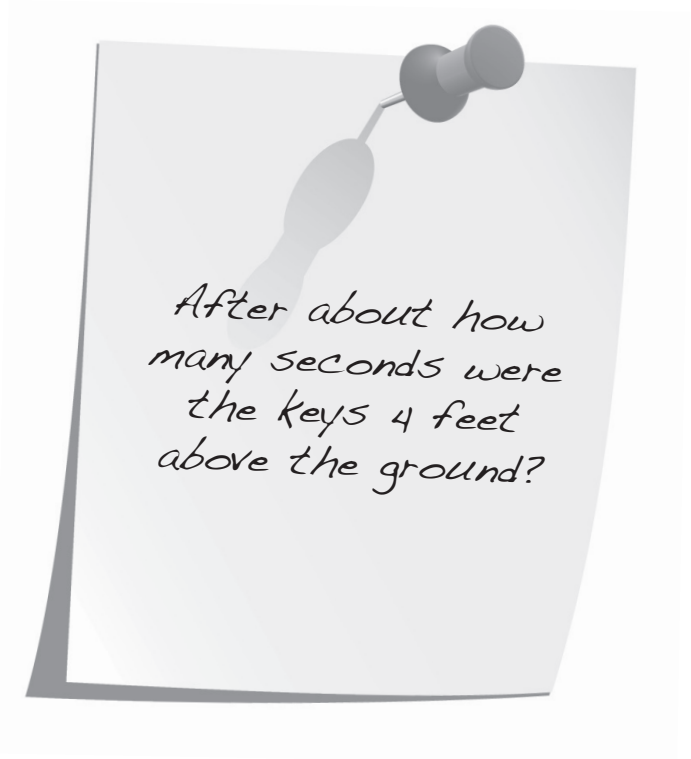
Problem-Based Task 2.7.1: Falling Keys

SMP	
1 ✓	2 ✓
3	4
5	6
7 ✓	8

Piper drops her keys from a third-story window to a friend standing on the sidewalk. The distance between the keys and the ground, in feet, can be represented by the function $f(x) = -16x^2 + 32$, where x is the time in seconds. Piper would like to be able to determine the time, in seconds, at which the keys were any given distance from the ground. What is the function that uses the height, in feet from the ground, as the independent variable? After about how many seconds were the keys 4 feet above the ground?



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Problem-Based Task 2.7.1: Falling Keys

Coaching

a. What type of function is represented by the equation $y = -16x^2 + 32$?

b. Generally, what is the domain of this type of equation?

c. Why does the domain need to be restricted?

d. What is the restricted domain?

e. What is the range of $f(x)$ on the restricted domain?

f. What is the algebraic equation of the function's inverse?

g. Identify the independent and dependent quantities of the function's inverse.

h. What are the domain and range of the function's inverse?

i. After about how many seconds were the keys 4 feet above the ground?

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Instruction

Problem-Based Task 2.7.1: Falling Keys

Coaching Sample Responses

a. What type of function is represented by the equation $y = -16x^2 + 32$?

The equation $y = -16x^2 + 32$ represents a quadratic function.

b. Generally, what is the domain of this type of equation?

The domain of a quadratic equation is generally all real numbers.

c. Why does the domain need to be restricted?

A quadratic function is not one-to-one; therefore, the inverse of the given function is not a function. Also, since the dependent variable x represents time, the values of x cannot be negative.

d. What is the restricted domain?

A quadratic function of the form $f(x) = a(x - h)^2 + k$ can be restricted to a domain of $x \geq h$ or $x \leq h$ in order for it to be one-to-one. In the function $f(x) = -16x^2 + 32$, $a = -16$, $h = 0$, and $k = 32$. The domain of $f(x)$ can be restricted to $x \geq 0$ or $x \leq 0$. Since the independent variable x represents time, which cannot be negative, the restricted domain should be $x \geq 0$.

e. What is the range of $f(x)$ on the restricted domain?

The range of a quadratic function of the form $y = a(x - h)^2 + k$ has an extreme value at $y = k$. If $a > 0$, then the range is $y \geq k$. If $a < 0$, then the range is $y \leq k$. If a quadratic function is restricted to $x \geq h$ or $x \leq h$, then the range is not changed. Since $a < 0$ for the function $f(x) = -16x^2 + 32$, the range is $y \leq 32$.

f. What is the algebraic equation of the function's inverse?

Rewrite the function in the form "$y =$." Switch x and y, and then solve for y.

$f(x) = -16x^2 + 32$

$y = -16x^2 + 32$

$x = -16y^2 + 32$

$16y^2 + x = 32$

$16y^2 = 32 - x$

$$y^2 = \frac{32 - x}{16}$$

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$$y^2 = 2 - \frac{x}{16}$$
$$y = \pm\sqrt{2 - \frac{x}{16}}$$

Because the restricted domain of the original function is $x \geq 0$, then the range of the inverse function is $y \geq 0$, so we can disregard the negative root.

The notation $f^{-1}(x)$ can be used to write the inverse of $f(x)$.

$$f^{-1}(x) = \sqrt{2 - \frac{x}{16}}$$

g. Identify the independent and dependent quantities of the function's inverse.

The independent quantity x in the original function $f(x)$ was time in seconds, and the dependent quantity of $f(x)$ was distance in feet. The independent and dependent quantities are swapped in the inverse. The independent quantity of $f^{-1}(x)$ is distance in feet, and the dependent quantity x in $f^{-1}(x)$ is time in seconds.

h. What are the domain and range of the function's inverse?

The domain of a function's inverse is the range of the original function, and the range of a function's inverse is the domain of the original function. The domain of $f^{-1}(x)$ is $x \leq 32$, and the range of $f^{-1}(x)$ is $y \geq 0$.

i. After about how many seconds were the keys 4 feet above the ground?

Substitute 4 for x in the inverse function, and then solve.

$$f^{-1}(x) = \sqrt{2 - \frac{x}{16}}$$
$$f^{-1}(4) = \sqrt{2 - \frac{(4)}{16}}$$
$$f^{-1}(4) = \sqrt{2 - 0.25}$$
$$f^{-1}(4) = \sqrt{1.75}$$
$$f^{-1}(4) \approx 1.32$$

The keys reach a height of 4 feet above the ground after approximately 1.32 seconds.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 2.7.1: Finding Inverse Functions

Find the inverse of each function for problems 1–6. State the domain and range of both the function and its inverse. Restrict the domain of the function if needed.

1. $f(x) = -x^2$

2. $f(x) = 5x - 1$

3. $f(x) = -x + 3$

4. $f(x) = x^2 + 7$

5. $f(x) = \frac{1}{4}x - 4$

6. $f(x) = -3x + 8$

continued

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For problems 7–10, state a reasonable domain for the problem statement, find the inverse function $f^{-1}(x)$, and identify the independent and dependent quantities of the inverse.

7. Gary is jogging. The total distance he has traveled, in miles, can be estimated using the function $f(x) = 9x$, where x is his jogging time in hours.

8. Camryn sells computers. She earns commission for her computer sales plus a fixed wage for each day she works. Her daily earnings, in dollars, can be estimated using the function $f(x) = 0.15x + 50$, where x represents her computer sales in dollars.

9. The distance a motorboat travels, in meters, can be estimated using the equation $f(x) = 1.9x^2$, where x is the time in seconds, for $0 \leq x \leq 10$.

10. A rock band is selling tickets to a concert at a theater. The band earns money for each ticket sold, but has to pay some of the earnings to the theater. The total money earned by the band can be estimated using the function $f(x) = 7.5x - 300$, where x is the number of tickets sold.

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Progress Assessment

Circle the letter of the best answer.

1. Which function is the inverse of $f(x)=\frac{1}{4}x+2$?
   a. $f^{-1}(x)=4x+2$
   b. $f^{-1}(x)=4x-8$
   c. $f^{-1}(x)=\frac{1}{4}x-2$
   d. $f^{-1}(x)=4x-2$

2. If $f^{-1}(x)=\frac{x}{10}-\frac{3}{10}$, what is $f(x)$?
   a. $f(x) = 10x - 3$
   b. $f(x) = x - 3$
   c. $f(x) = 10x + 3$
   d. $f(x)=\frac{x}{10}+3$

3. Which function is the inverse of $f(x)=9x^2$ if the domain is $x \geq 0$?
   a. $f^{-1}(x)=9\sqrt{x}$
   b. $f^{-1}(x)=3\sqrt{x}$
   c. $f^{-1}(x)=\frac{1}{3}\sqrt{x}$
   d. $f^{-1}(x)=3x^2$

4. Which function is not one-to-one?
   a. $f(x)=x^2-7$
   b. $f(x) = x - 7$
   c. $f(x) = 7x$
   d. $f(x)=\frac{x}{7}-4$

5. If $f^{-1}(x)=\sqrt{x+8}$, what is $f(x)$?
   a. $f(x)=x^2-8$ if $x \geq 0$
   b. $f(x)=x^2+8$ if $x \geq 0$
   c. $f(x)=x^2+64$ if $x \geq 0$
   d. $f(x) = x + 8$

6. Which function is the inverse of $f(x) = 5x + 15$?
   a. $f^{-1}(x)=x-3$
   b. $f^{-1}(x)=\frac{1}{5}x-15$
   c. $f^{-1}(x)=\frac{1}{5}x-3$
   d. $f^{-1}(x)=5x-15$

continued

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7. If $f^{-1}(x) = -\frac{1}{4}x + 2$, what is $f(x)$?

   a. $f(x) = -4x - 2$

   b. $f(x) = -4x + 8$

   c. $f(x) = 4x + 2$

   d. $f(x) = \frac{1}{4}x + 2$

8. Which function is the inverse of $f(x) = \frac{1}{16}x^2$ if the domain is $x \geq 0$?

   a. $f^{-1}(x) = 16\sqrt{x}$

   b. $f^{-1}(x) = \frac{1}{16}\sqrt{x}$

   c. $f^{-1}(x) = \frac{1}{4}\sqrt{x}$

   d. $f^{-1}(x) = 4\sqrt{x}$

9. Which function is not one-to-one?

   a. $f(x) = -3x + 20$

   b. $f(x) = -x^2$

   c. $f(x) = \frac{4}{5}x$

   d. $f(x) = x - 13$

10. If $f^{-1}(x) = \sqrt{x-1}$, what is $f(x)$?

    a. $f(x) = x^2$ if $x \geq 0$

    b. $f(x) = x^2 + 1$ if $x \geq 0$

    c. $f(x) = x^2 - 1$ if $x \geq 0$

    d. $f(x) = \sqrt{x+1}$ if $x \geq 0$

Use the given information to complete all parts of the problem that follows.

11. A restaurant manager purchases food each morning to make all meals ordered for the day. The manager can estimate the total earnings for the day using the function $f(x) = 8.5x - 500$, where x is the number of meals sold.

    a. What is a reasonable domain of $f(x)$ for the given problem statement? Explain.

    b. Find $f^{-1}(x)$.

    c. What are the independent and dependent quantities in $f^{-1}(x)$?

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Unit 2 Assessment

Circle the letter of the best answer.

1. The path of a marshmallow launched from a slingshot can be described by the equation $f(x) = -x^2 + 4x + 5$, where $f(x)$ is the height of the marshmallow and x is the number of seconds that have passed since the slingshot's band was released. Which of the following points shows the extremum for the function?

   a. (0, 5)
   b. (2, 9)
   c. (0, –1)
   d. (4, 5)

2. The field hockey team is hosting a battle of the bands to raise money for new equipment. In the past, the profit from the battle of the bands could be modeled by the function $P(x) = -20(x - 20)(x - 10)$, where x represents the ticket price in dollars. What is a reasonable domain for this function?

   a. $-10 \le x \le 20$
   b. $-20 \le x \le -10$
   c. $10 \le x \le 20$
   d. $-20 \le x \le 10$

3. What is the average rate of change of the function $f(x) = 6x^2 + 12x - 4$ between $x = -1$ and $x = 1$?

   a. 12
   b. 24
   c. 4
   d. –12

4. The dimensions of a community garden are such that the length is 6 feet shorter than 3 times its width. What expression describes the area of the community garden in terms of its width, w?

   a. $(w + 3)(w + 6)$ ft^2
   b. $w(6w - 3)$ ft^2
   c. $3w(w - 6)$ ft^2
   d. $w(3w - 6)$ ft^2

5. Let $f(x) = 4x^2 + 2x$ and $g(x) = -6x + 7$. Which function represents a new function, $h(x)$, for which $h(x) = (f + g)(x)$?

   a. $h(x) = -2x^2 + 9x$
   b. $h(x) = 4x^2 + 4x + 7$
   c. $h(x) = -2x^2 + 2x + 7$
   d. $h(x) = 4x^2 - 4x + 7$

6. Let $f(x) = x^2 - 3x - 18$ and $g(x) = x + 3$. Which value can the expression $\left(\frac{f}{g}\right)(x)$ never equal?

   a. 6
   b. –6
   c. 0
   d. –9

continued

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7. Which function has a point of inflection at (1, –3)?

   a. $y=\sqrt[3]{x}+3$

   b. $y=\sqrt[3]{x-1}+3$

   c. $y=\sqrt[3]{x-1}-3$

   d. $y=3\sqrt[3]{x+3}-1$

8. What are the coordinates of the minimum point of the graph of $y=|x-4|+10$?

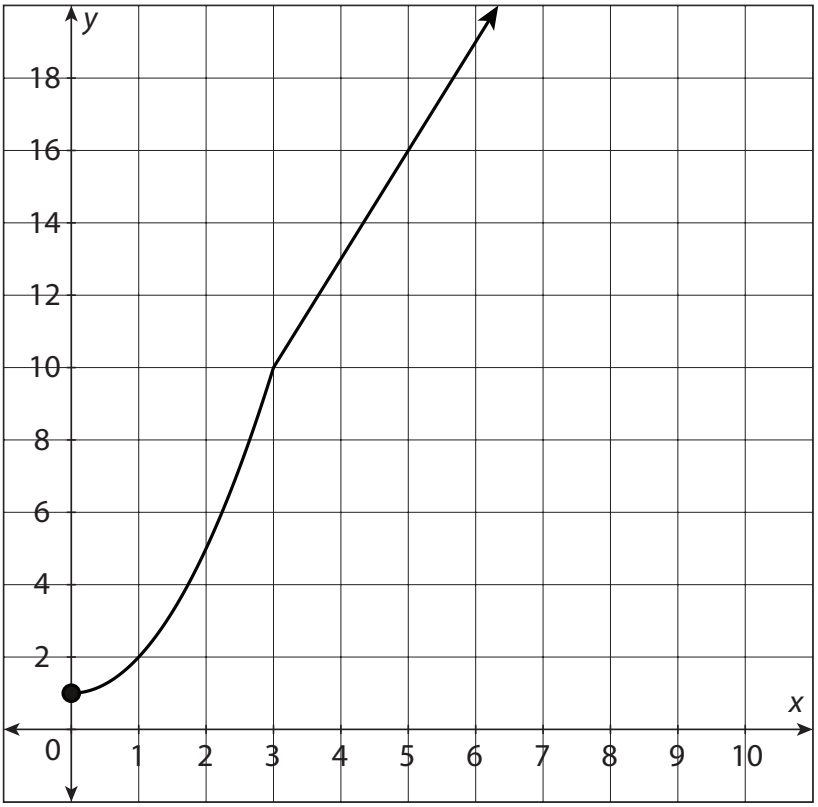
   a. (–4, –10)

   b. (4, 10)

   c. (–10, 4)

   d. (4, –10)

9. Which function represents the graph?



   a. $y=\begin{cases}x^2+1, \text{ if } 0\le x\le 3\\ 3x+1, \text{ if } x>3\end{cases}$

   b. $y=\begin{cases}x+1, \text{ if } 0\le x\le 3\\ 3x+1, \text{ if } x>3\end{cases}$

   c. $y=\begin{cases}x^2, \text{ if } 0\le x\le 3\\ 3x, \text{ if } x>3\end{cases}$

   d. $y=\begin{cases}-x^2+1, \text{ if } 0\le x\le 3\\ -3x+1, \text{ if } x>3\end{cases}$

continued

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10. What is the percent rate of change of y for each unit of x of the function $y = 250(0.81)^x$?

    a. 81% decay
    b. 19% decay
    c. 81% growth
    d. 119% growth

11. If the vertex of $f(x)$ is (3, –5), what is the vertex of $f(x + 3)$?

    a. (3, –8)
    b. (6, –5)
    c. (3, –2)
    d. (0, –5)

12. If the transformation of $f(k \cdot x)$ is applied to $f(x)$, with $0 < k < 1$, which is true about the graph of the function?

    a. The graph of the original function is compressed horizontally by a factor of $\frac{1}{k}$.
    b. The graph of the original function is stretched horizontally by a factor of $\frac{1}{k}$.
    c. The graph of the original function is stretched vertically by a factor of k.
    d. The graph of the original function is compressed vertically by a factor of k.

Use what you have learned about functions to solve each of the following problems.

13. Violet is driving home from school. She can estimate her distance from home, in miles, using the function $f(x) = –35x + 7$, where x is the time in hours since she started driving.

    a. What is a reasonable domain of $f(x)$ for the given problem statement?
    b. Find $f^{-1}(x)$.
    c. What are the independent and dependent quantities in $f^{-1}(x)$?

continued

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14. A seagull drops a crab 144 feet onto the rocks below. The function $f(x) = -16x^2 + 144$ is used to model the height of the crab, with $f(x)$ representing the number of feet x seconds after the crab is dropped.

    a. For what values of x is the function $f(x) = -16x^2 + 144$ increasing?

    b. For what values of x is the function $f(x) = -16x^2 + 144$ decreasing?

    c. What is the vertex and what does it mean in terms of this scenario?

    d. What is a reasonable domain for this scenario?

    e. What is the average rate of change in height between 1 and 2.5 seconds after the crab is dropped?

15. The height of a baseball is given by the equation $f(x) = -16x^2 + 32x + 2$, where $f(x)$ represents the height of the baseball in feet and x represents the time in seconds after it was hit by the batter.

    a. What is the vertex and what does it mean in the context of this scenario?

    b. Determine the equation that represents the axis of symmetry for this function.

    c. Create a graph of this scenario.

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Answer Key

Lesson 1: Analyzing Quadratic Functions

Pre-Assessment, p. U2-1

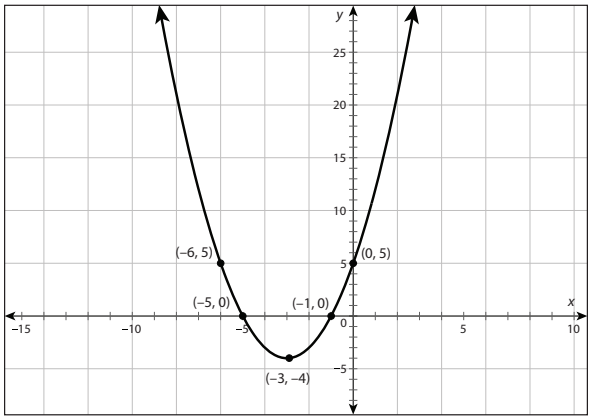
1. d
2. b
3. c
4. a
5. a

Warm-Up 2.1.1, p. U2-4

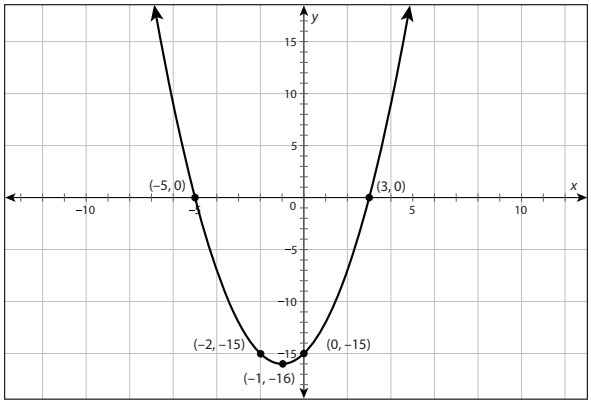
1. $y = -80x + 500$, where x is the number of oil changes in 1 year and y is the total amount spent on car repairs in 1 year.
2. The y-intercept is 500. This is the amount spent on car repairs if the owner does not get any oil changes over the course of a year.
3. The rate of change is the slope of the function, –80. For every oil change performed, he will save $80 on car repairs.

Practice 2.1.1: Graphing Quadratic Functions, p. U2-26

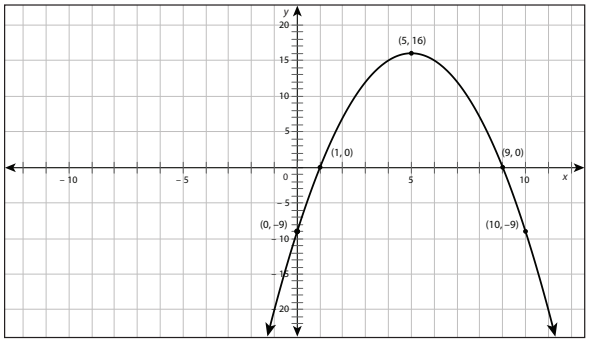
1. y-intercept: (0, 5); vertex: (–3, –4); minimum: –4



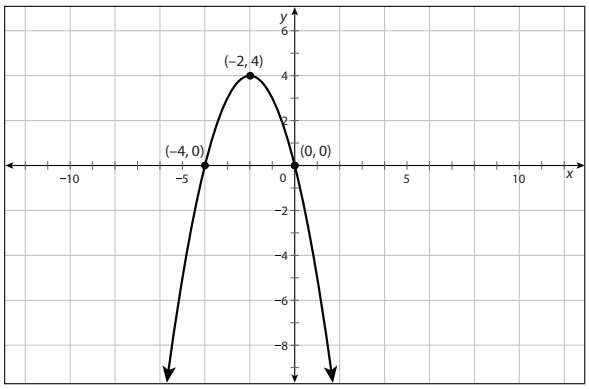
2. y-intercept: (0,–15); vertex: (–1, –16); minimum: –16



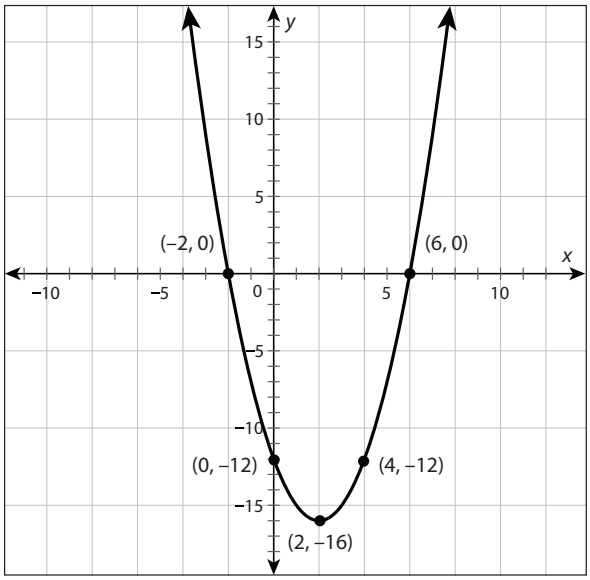
3. y-intercept: (0, –9); vertex: (5, 16); maximum: 16



4. y-intercept: (0, 0); vertex: (–2, 4); maximum: 4



5. y-intercept: (0, –12); vertex: (2, –16); minimum: –16



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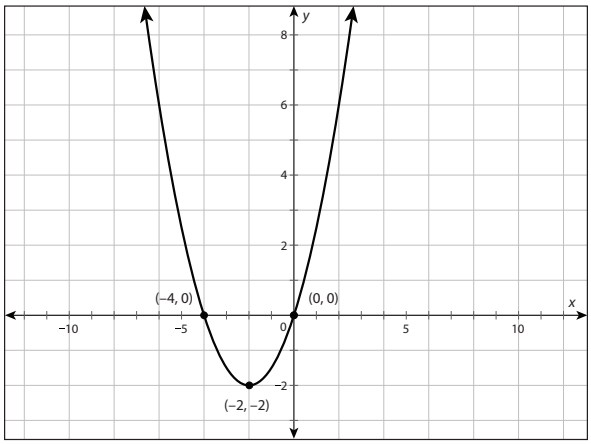
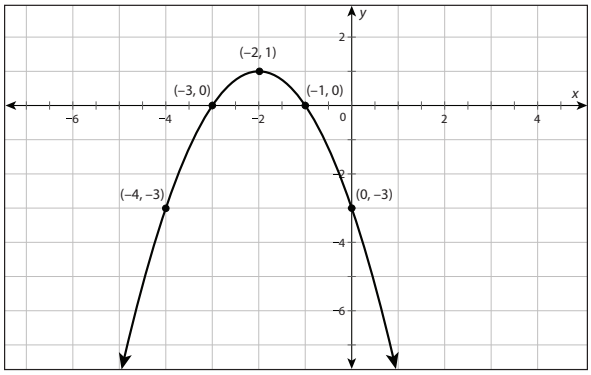
6. *y*-intercept: (0, 0); vertex: (–2, –2); minimum: –2



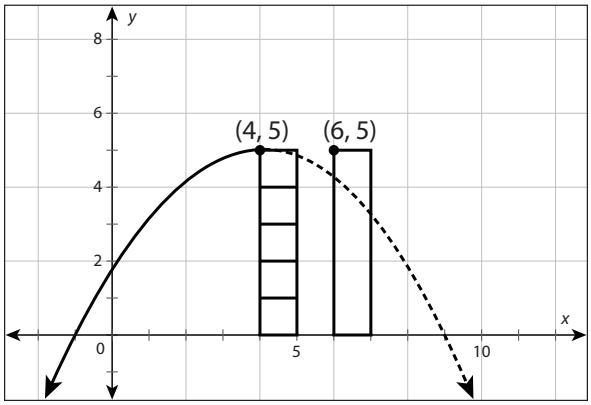
7. *y*-intercept: (0, –3); vertex: (–2, 1); maximum: 1



8. *y*-intercept: (0, 0); vertex: (6, 18); maximum: 18
9. *y*-intercept: (0, 16); vertex: (0.75, 25); maximum: 25
10. *y*-intercept: (0, 16); vertex: (3, 25); maximum: 25

Warm-Up 2.1.2, p. U2-27

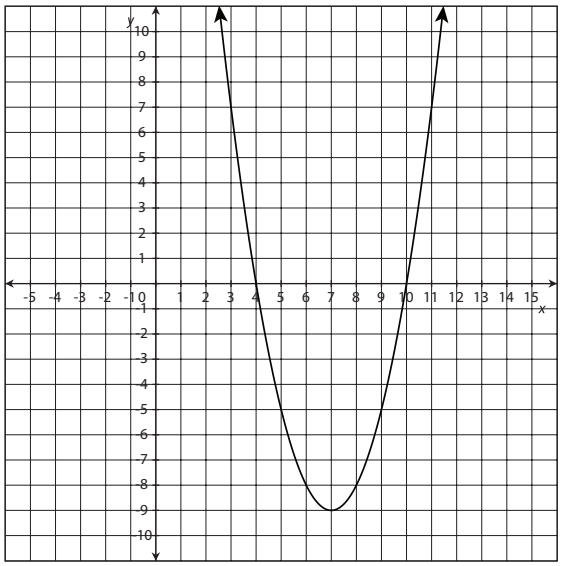
1. The vertex is (4, 5).
2. The second *x*-intercept is 9.
3. The maximum is 5.
4.



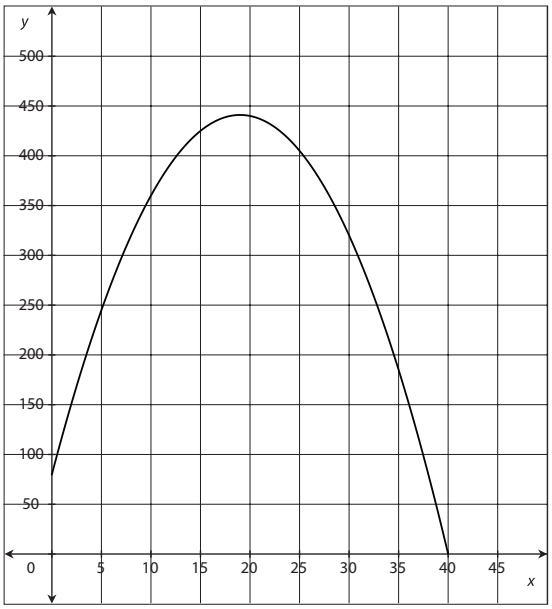
5. No, the water will not clear the boxes. If the boxes were not in the way, the water would still not fill the birdbath because the birdbath is too tall.

Practice 2.1.2: Interpreting Various Forms of Quadratic Functions, p. U2-47

1. a. 8
   b. (3, –1)
   c. minimum
2. a. –2 and 4
   b. 4
   c. $x = 1$
   d. (1, 4.5)
3. a. (1, 10)
   b. maximum
4. (7, –9); after the bird travels for 7 seconds, it reaches its minimum height, which is 9 feet below the surface of the water.



5. –2 and 40; the *x*-intercepts represent the times when the height is 0 miles, which is when the missile is on the ground. The *x*-intercept of –2 does not make sense in the problem because time cannot be negative; 38 seconds.



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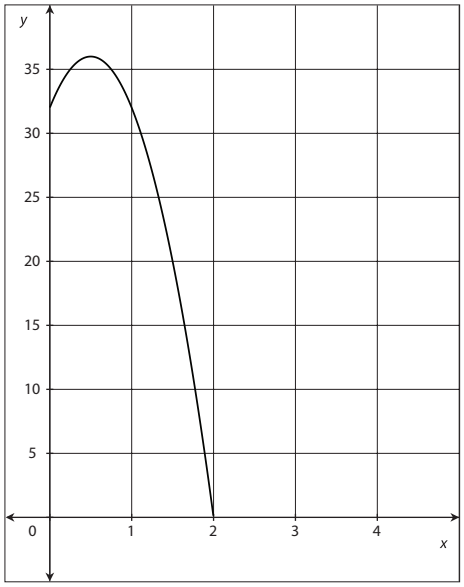
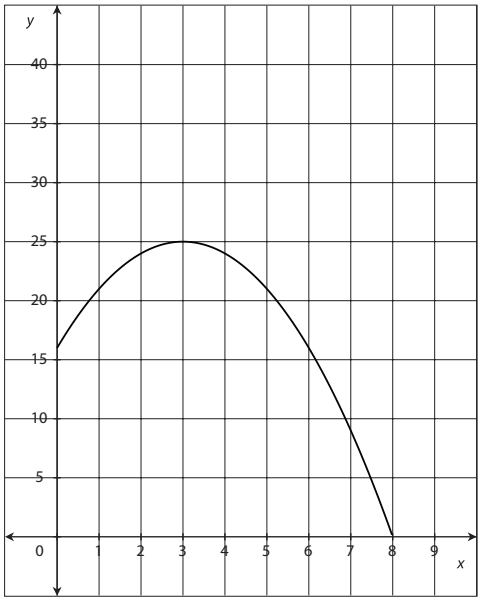
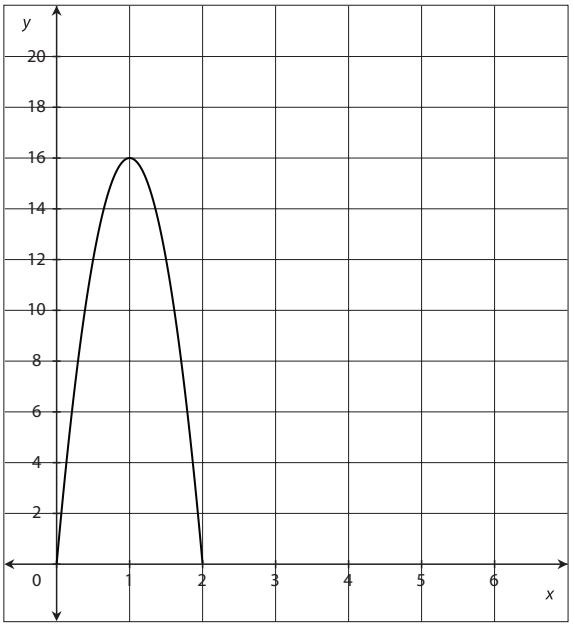
6. –1 and 2; the *x*-intercepts represent the times when the snowboarder is on the ground. The *x*-intercept –1 does not make sense because time cannot be negative; 2 seconds.



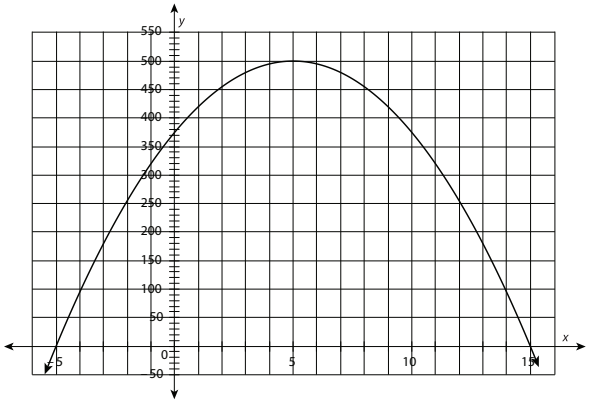
7. (3, 25); the paper airplane will reach a maximum height of 25 feet after traveling 3 feet in the horizontal direction.



8. (1, 16); the golf ball will reach a maximum height of 16 feet 1 second after being hit.



9. –5 and 15; the *x*-intercepts represent the number of dollars in price increase/decrease that would result in no revenue; 5.



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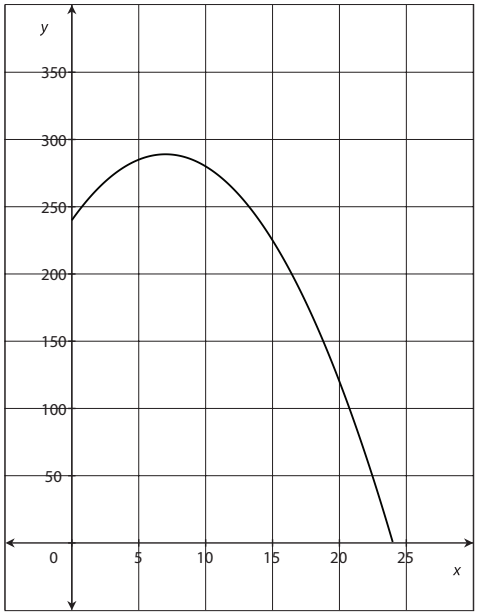
10. (7, 289): the maximum revenue of $289 will occur when the price is reduced by $7.



Progress Assessment, p. U2-49

1. d
2. c
3. d
4. a
5. d
6. b
7. d
8. c
9. a
10. c
11. a. 19 feet
    b. 1.25 seconds
    c. 28 feet

Lesson 2: Interpreting Quadratic Functions

Pre-Assessment, p. U2-51

1. a
2. b
3. c
4. a
5. d

Warm-Up 2.2.1, p. U2-55

1. The x-intercepts are 0 and 8.
2. The equation for the axis of symmetry is $x = 4$. It is midway between the x-intercepts.
3. The vertex is (4, 6).
4. Yes, it will. Sketch the boulder's path using (0, 0), (4, 6), and (8, 0); the path is above the obstacle.
5. No, it will not. If you substitute $x = 7$ into the equation of the parabola, you find that the y-value of the boulder is slightly above the boxes.

Practice 2.2.1: Interpreting Key Features of Quadratic Functions, p. U2-75

1. $x > 0.3$, $x < 0.3$; minimum = –5.3; x-intercepts: –1 and 1.67; y-intercept: –5; neither
2. $x < 1.667$, $x > 1.667$; maximum = 9.333; x-intercepts: –0.097 and 3.431; y-intercept: 1; neither
3. $x > –1$, $x < –1$; minimum = 6; no x-intercepts; y-intercept: 11; neither
4. $x > 1$, $x < 1$; minimum = –13; x-intercepts: –1.55 and 3.55; y-intercept: –11; neither
5. $x = –0.67$; $x > –0.67$, $x < –0.67$
6. $x = 1.125$; $x < 1.125$, $x > 1.125$
7. $x = 0.375$; $x > 0.375$, $x < 0.375$
8. $x = 0.15$; $x > 0.15$, $x < 0.15$
9. The height of the softball increases until it reaches a distance of 25 feet, and then it decreases.
10. The maximum temperature is about 62.5˚.

Warm-Up 2.2.2, p. U2-77

1. $y = 3x + 64$
2. all whole numbers
3. The domain represents the possible number of games that Gina could sell in one 8-hour workday.

Practice 2.2.2: Identifying the Domain of a Quadratic Function, p. U2-88

1. all real numbers or $-\infty < x < \infty$
2. all real numbers or $-\infty < x < \infty$
3. all real numbers or $-\infty < x < \infty$
4. all real numbers or $-\infty < x < \infty$
5. all real numbers; $-\infty < x < \infty$
6. all real numbers; $-\infty < x < \infty$
7. all real numbers; $-\infty < x < \infty$
8. less than 3.9 seconds
9. 9.4 seconds
10. $8.67 \le x \le 28.83$; $x = \$18.75$

Warm-Up 2.2.3, p. U2-91

1. –1/2
2. $G(x) = (–1/2)x + 65$
3. The car's mileage will always be reduced by 1 mile per gallon for every 2 miles per hour the speed of the car increases.

Practice 2.2.3: Identifying the Average Rate of Change, p. U2-104

1. 4
2. 6
3. –4
4. 0
5. –12
6. 7
7. between $x = 0$ and $x = 2$
8. between $x = –2$ and $x = 0$
9. between $x = 0$ and $x = 2$
10. –80 ft/s

Progress Assessment, p. U2-106

1. b
2. c
3. c
4. d
5. d
6. a
7. b
8. b
9. c
10. c
11. a. –1.2 mpg
    b. values of s between 48 and 70 mph
    c. $G(s) = 33$ mpg, at $s = 48$ mph

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Lesson 3: Building Functions

Pre-Assessment, p. U2-110

1. c
2. b
3. c
4. d
5. a

Warm-Up 2.3.1, p. U2-113

1. $t \approx 8.66$ seconds
2. $t \approx 13.86$ seconds
3. $t \approx 5.33$ seconds

Practice 2.3.1: Building Functions from Context, p. U2-131

1. $f(x) = x^2 - 8x - 20$
2. $g(2) = -16$
3. $x(x + 2) = 2915$; 53 and 55 or –53 and –55
4. vertex: (250, 20); y-intercept: 420
5. $f(x) = \frac{4}{625}(x-250)^2 + 20$
6. $f(x) = (24 + 2x)(16 + 2x)$
7. 225 in^2
8. $f(x) = \frac{1}{64}(x-32)(x-80) = \frac{1}{64}(x-56)^2 - 9$
9. $f(0) = 40$; 40 feet
10. $A(x) = x(35 - x) = 35x - x^2$

Warm-Up 2.3.2, p. U2-133

1. $f(x) = 1.5x$
2. $g(x) = 5x$
3. \$3.50
4. $h(x) = (g - f)(x) = g(x) - f(x) = 3.5x$

Practice 2.3.2: Operating on Functions, p. U2-146

1. $h(x) = x^2 + 12x + 35$
2. $u(x) = x^2 - 7x + 10$
3. $m(x) = 22x^2 + 5x - 3$
4. $h(x) = x + 4$; $x \neq -3$ and $h(x) \neq 1$
5. $k(x) = 2x^2 + 11x - 12$
6. $A(x) = x(x + 3)$; $R(x) = (x - 4)(x - 1)$; $d(x) = 8x - 4$
7. $f(x) = 120 - x$; $g(x) = 200 - x$; $A(x) = (120 - x)(200 - x)$
8. $a(x) = 2x + 4$; $b(x) = x$;
   $\frac{1}{2}(a \bullet b)(x) = \frac{1}{2}(2x+4) \bullet x = (x+2) \bullet x = x^2 + 2x$
9. $f(r) = 2\pi r^2$; $g(r) = 2\pi rh$; function for the surface area of a cylinder: $(f + g)(r) = 2\pi r^2 + 2\pi rh$
10. $f(r) + f(r) + f(r) + g(r) = 6 \bullet \pi r^2 + 2\pi rh$

Progress Assessment, p. U2-148

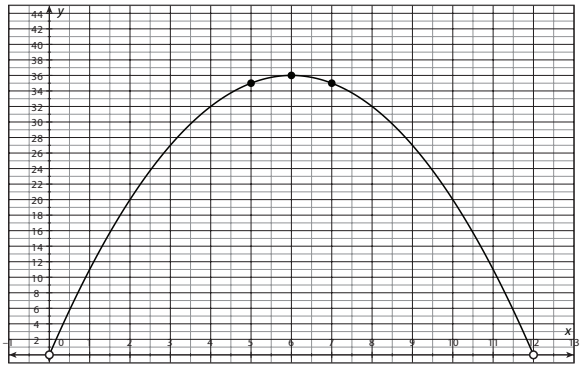
1. d
2. d
3. c
4. a
5. a
6. b
7. a
8. c
9. b
10. d
11. a. $f(x) = x(x - 2) = x^2 - 2x$
    b. $g(x) = (x + 8)(x + 6) = x^2 + 14x + 48$
    c. $h(x) = (g - f)(x) = 16x + 48$

Lesson 4: Graphing Other Functions

Pre-Assessment, p. U2-150

1. c
2. d
3. a
4. d
5. d

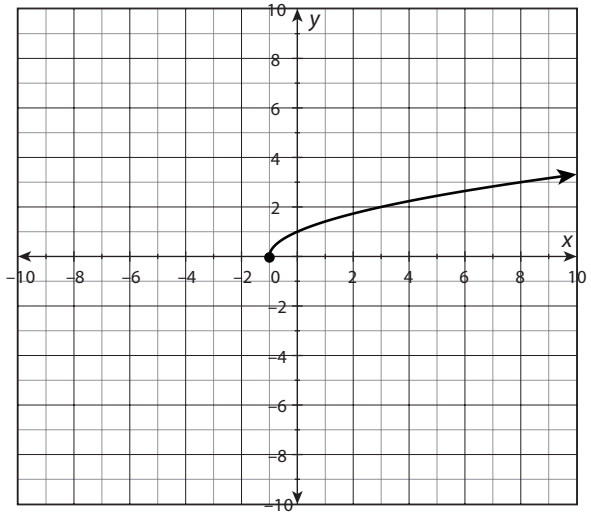
Warm-Up 2.4.1, p. U2-155

1. $y = 12x - x^2$
2. $0 < x < 12$
3. 

4. length = $x = 6$ feet; width = $12 - x = 6$ feet

Practice 2.4.1: Square Root and Cube Root Functions, p. U2-188

1. domain: $x \geq -1$; range: $y \geq 0$; minimum point: (–1, 0)



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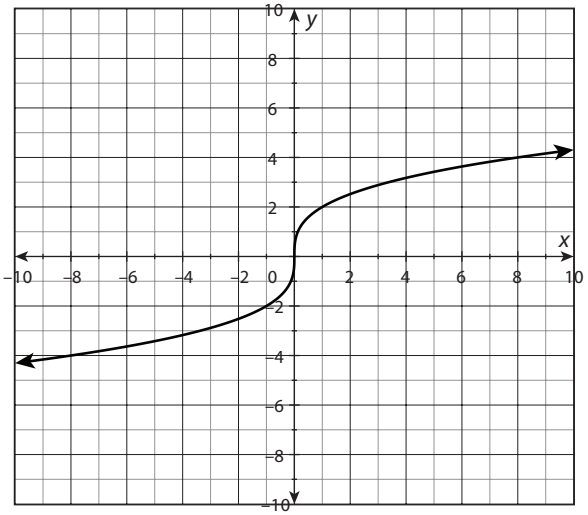
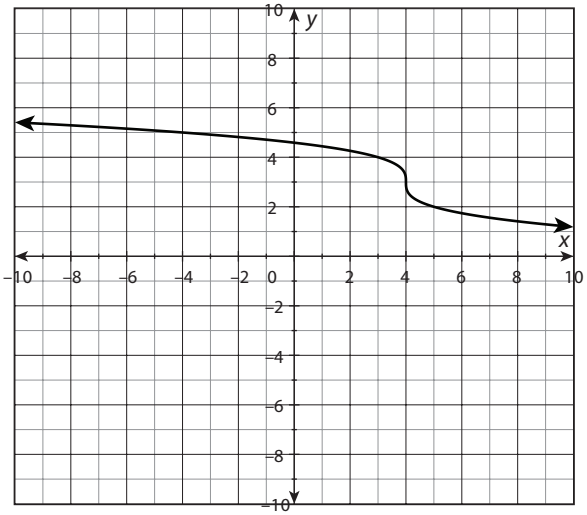
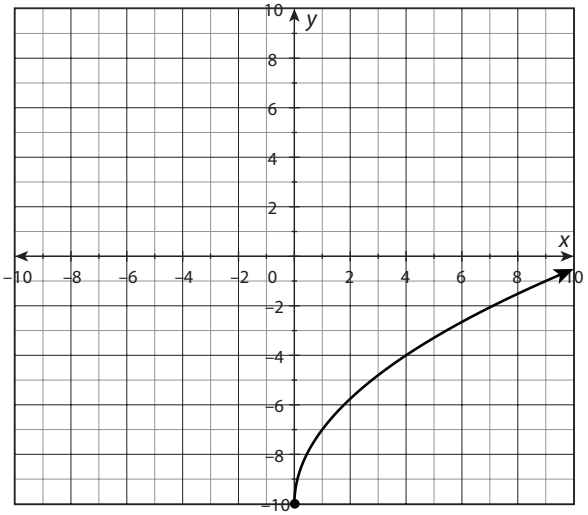
2. domain and range: all real numbers; point of inflection: (0, 0)



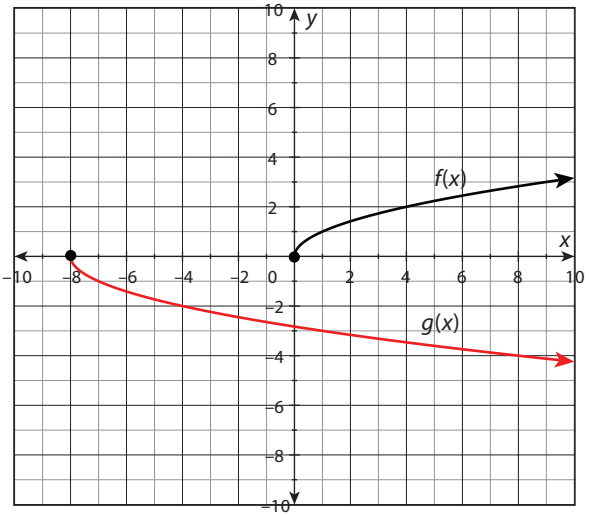
3. domain and range: all real numbers; point of inflection: (4, 3)



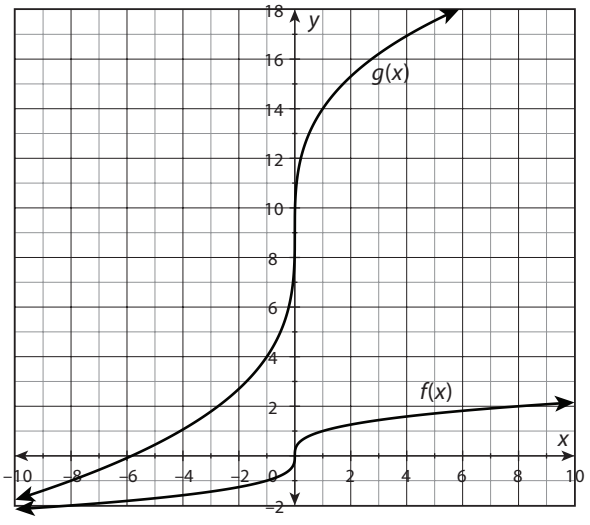
4. domain: $x \geq 0$; range: $y \geq -10$; minimum point: (0, –10)



5. $g(x) = -\sqrt{(x+8)}$ is reflected over the x-axis due to the –1 that is multiplied by the square root, and then shifted 8 units to the left due to the addend of 8 inside the square root symbol. $g(x) = -\sqrt{(x+8)}$ is decreasing and $f(x) = \sqrt{x}$ is increasing. Both functions have the same shape. For $g(x) = -\sqrt{(x+8)}$, the domain is $x \geq -8$, the range is $y \leq 0$, and the maximum point is (–8, 0). For $f(x) = \sqrt{x}$, the domain is $x \geq 0$, the range is $y \geq 0$, and the minimum point is (0, 0).



6. $g(x) = 5\sqrt[3]{x} + 9$ is stretched vertically due to the 5 that is being multiplied by the cube root, and then shifted up 9 units due to the addend of 9 outside the cube root symbol. The domain and range of both functions is all real numbers. $g(x) = 5\sqrt[3]{x} + 9$ has a point of inflection at (0, 9) and $f(x) = \sqrt[3]{x}$ has a point of inflection at (0, 0). The addend of 9 outside the cube root symbol shifts the inflection point of $g(x) = 5\sqrt[3]{x} + 9$ up 9 units.



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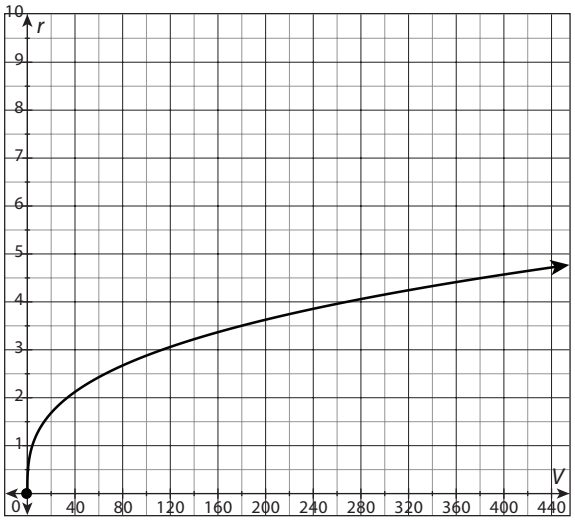
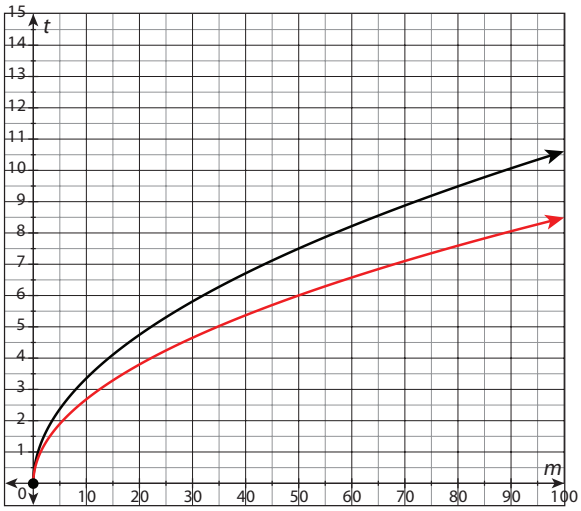
7. A sample graph is shown. Not all x-values are needed as shown, but the graph should be restricted to $V \geq 0$ and $r \geq 0$.



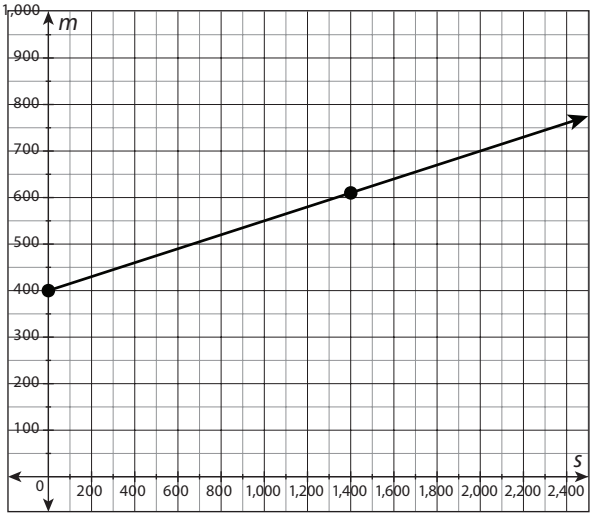
8. estimate: $4.69 \leq r \leq 4.75$
9. Both graphs have a minimum point at (0, 0). The larger factor of the square root in Aurora's function vertically stretches the graph of her kick.



10. Aurora's ball will take more time, as seen by the greater y-value at $x = 90$. It will take Isla's ball approximately 8 seconds to go 90 meters, and it will take Aurora's ball approximately 10 seconds to travel 90 meters.

Warm-Up 2.4.2, p. U2-190

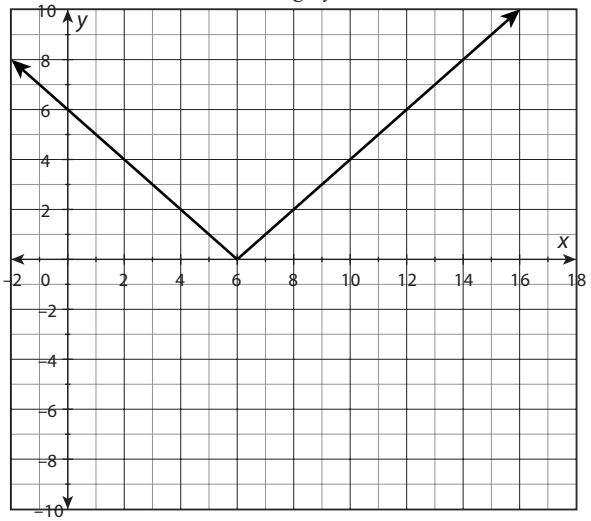
1. \$610
2.



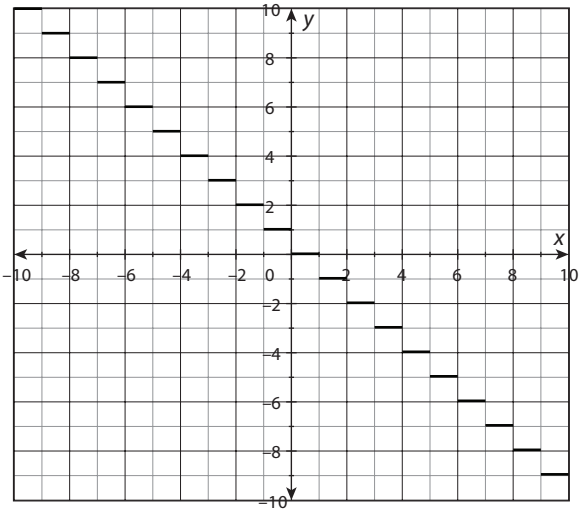
3. \$2,300

Practice 2.4.2: Absolute Value and Step Functions, p. U2-217

1. domain: all real numbers; range: $y \geq 0$; vertex: (6, 0)



2. domain: all real numbers; range: all integers



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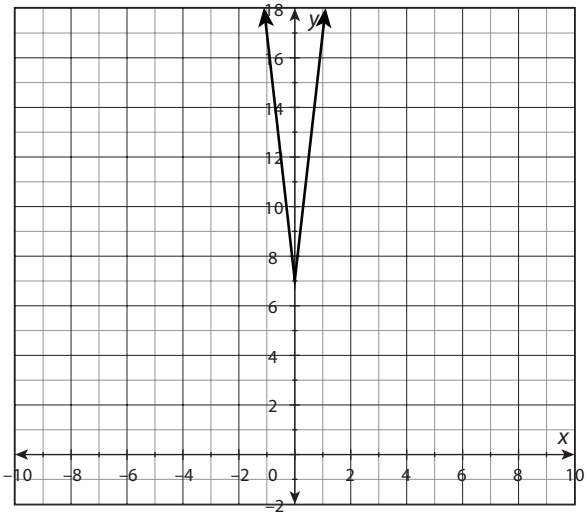
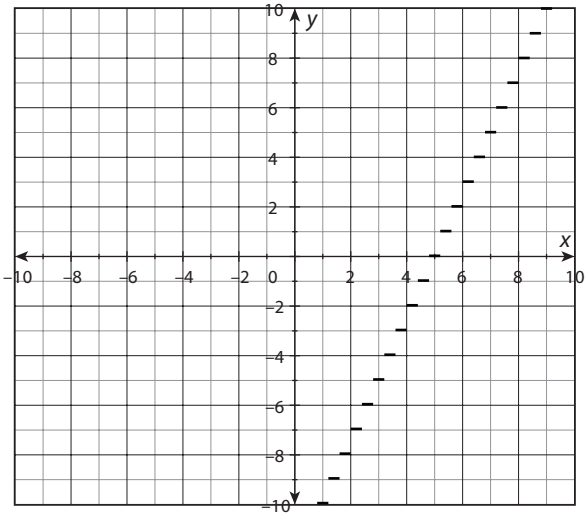
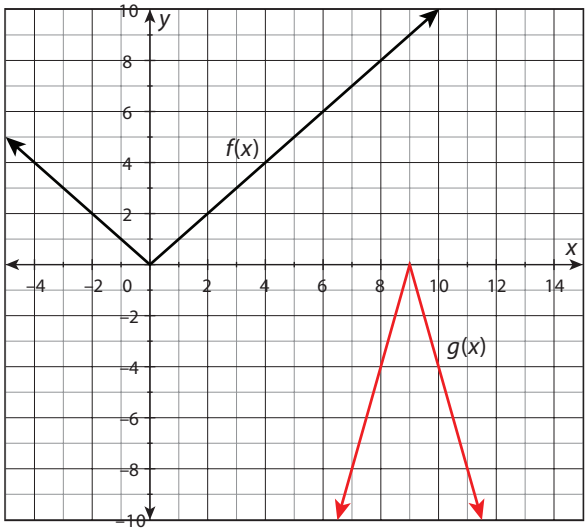
3. domain: all real numbers; range: $y \geq 7$; vertex: (0, 7)



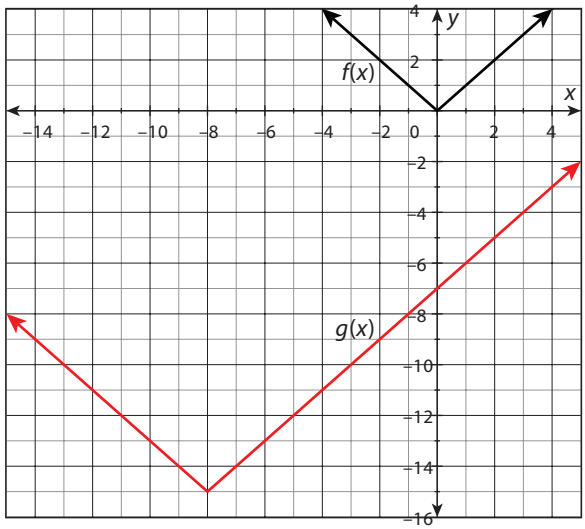
4. domain: all real numbers; range: all integers



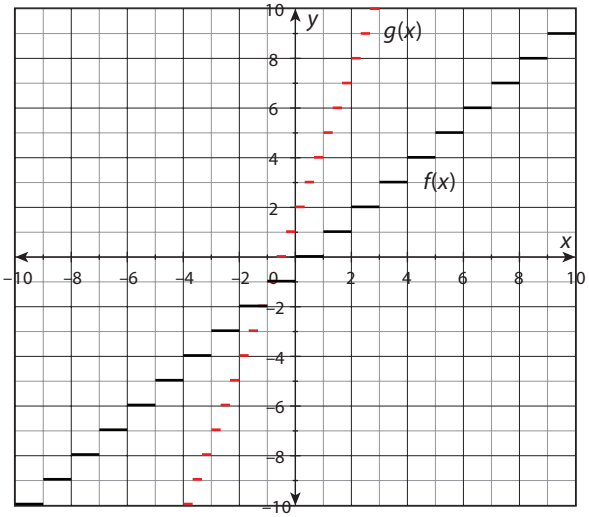
5. The domain of both functions is all real numbers. The range of $f(x) = |x|$ is $y \geq 0$. The range of $g(x) = -4|x - 9|$ is $y \leq 0$. The graph of $g(x) = -4|x - 9|$ is flipped over the x-axis due to the –1 factor of the absolute value. The addend of – 9 shifts the graph 9 units to the right, and shifts the vertex from (0, 0) to (9, 0). The factor of 4 stretches the graph vertically.



6. Both graphs have a domain of all real numbers. The function $g(x) = |x + 8| - 15$ has a range of $y \geq -15$, where $f(x) = |x|$ has a range of $y \geq 0$. The addend of – 15 outside the absolute value in the function $f(x) = \lceil x \rceil$ shifts the graph down 15 units. The addend of + 8 inside the absolute value shifts the graph to the left 8 units. The vertex is shifted from (0, 0) to (–8, –15).



7. Both graphs have a domain of all real numbers and a range of all integers. The steps in the graph of $g(x) = -4|x - 9|$ are shorter due to the factor of 3 inside the greatest integer function. The addend of + 2 inside the greatest integer function shifts the graph left 2/3 unit. Both graphs are increasing.



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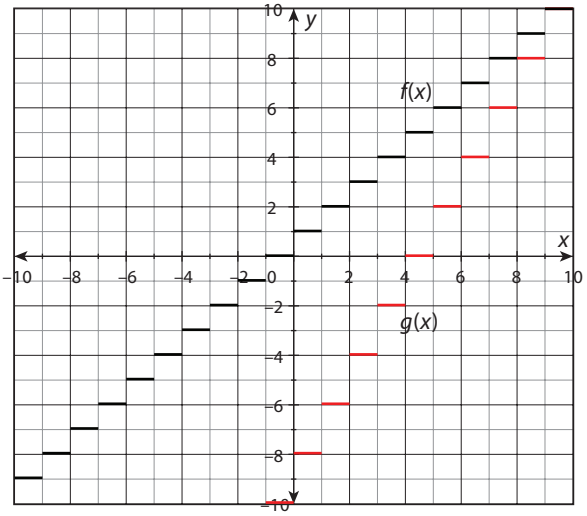
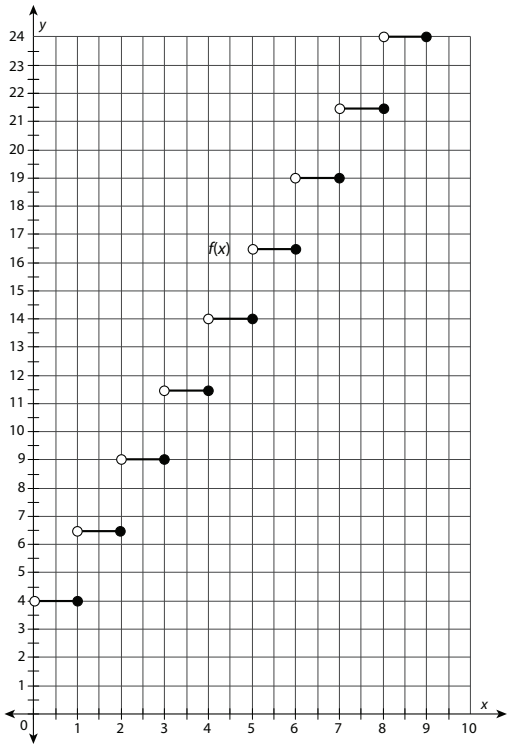
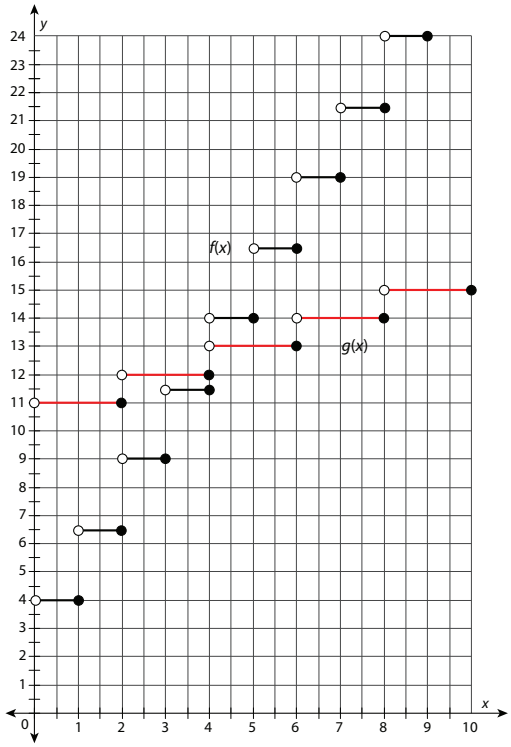
8. Both functions have a domain of all real numbers. The range of $f(x) = \lceil x \rceil$ is all integers, whereas the range of $g(x) = 2\lceil x \rceil - 10$ is multiples of 2, due to the factor of 2 outside the least integer function. Both graphs are increasing. The graph of $g(x) = 2\lceil x \rceil - 10$ is shifted down 10 units due to the addend of – 10 outside the least integer function.



9. The domain is $x > 0$ and the range is $y \geq 4$.

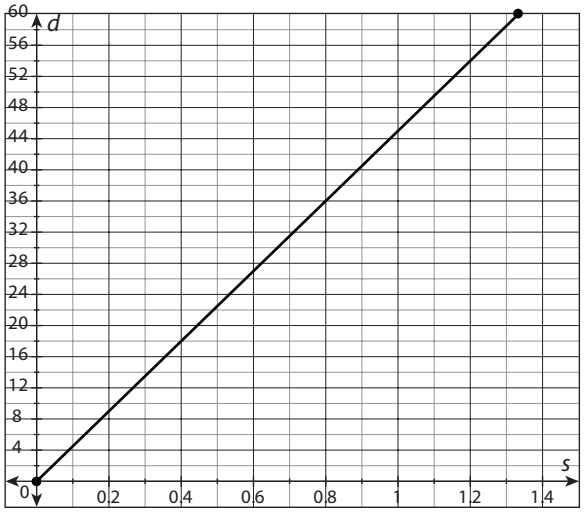


10. The domain of this new function is also $x > 0$, and the range is $y \geq 11$. Both functions are ceiling functions. The prices for the new company are more expensive than those of the original company for packages that weigh less than 4 pounds, but the new company is less expensive for packages that weigh more than 4 pounds. For the original company, the prices increase for every pound, but for the new company, the prices increase at 2-pound intervals.



Warm-Up 2.4.3, p. U2-219

1. 0.67 second
2. A reasonable restricted domain is $0 \leq s \leq 1.33$. A reasonable restricted range is $0 \leq d \leq 60$.



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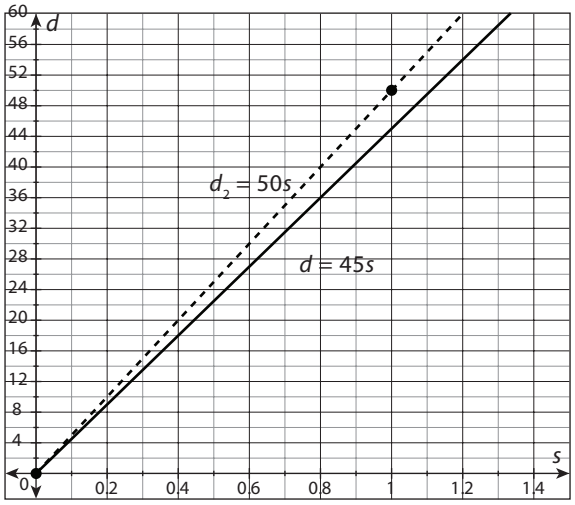
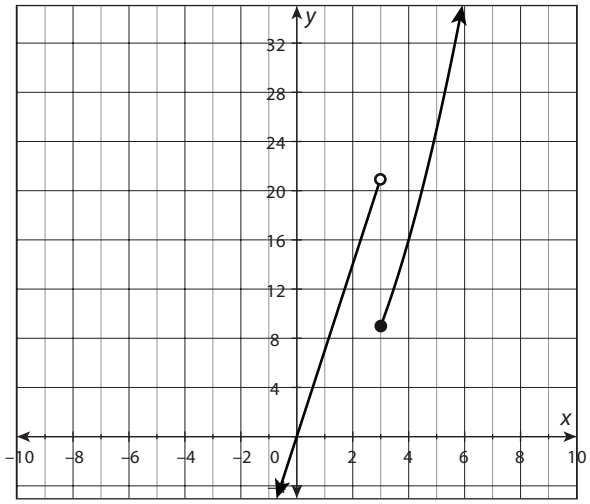
3.



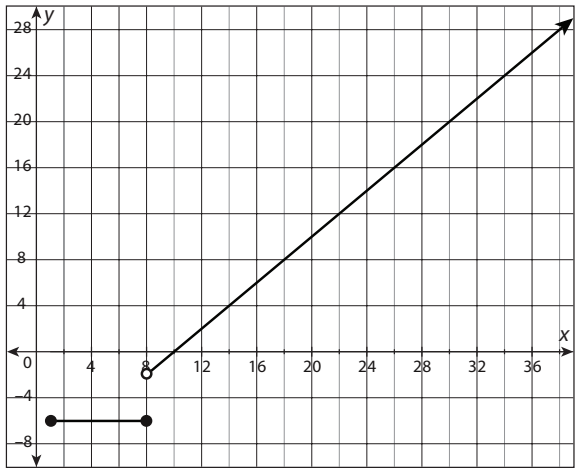
4. Both graphs are linear; we can tell this by looking at the functions, both of which have 1 as the highest power of the variable, s. The graph of the second hockey puck, $d_2 = 50s$, is steeper than the graph of $d = 45s$. In the algebraic representation, the slope of d_2, 50, is greater than the slope of d, 45.

Practice 2.4.3: Piecewise Functions, p. U2-242

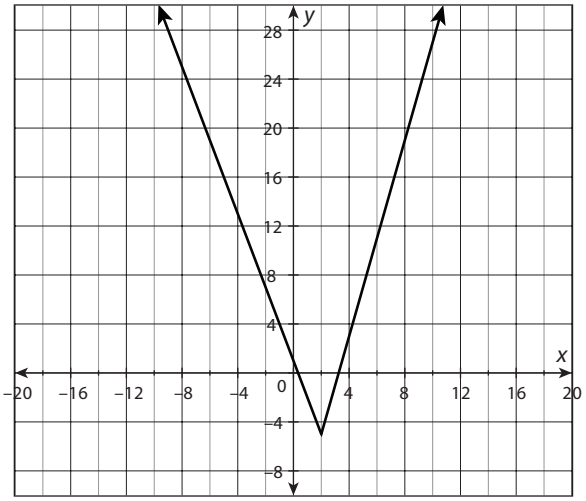
1.



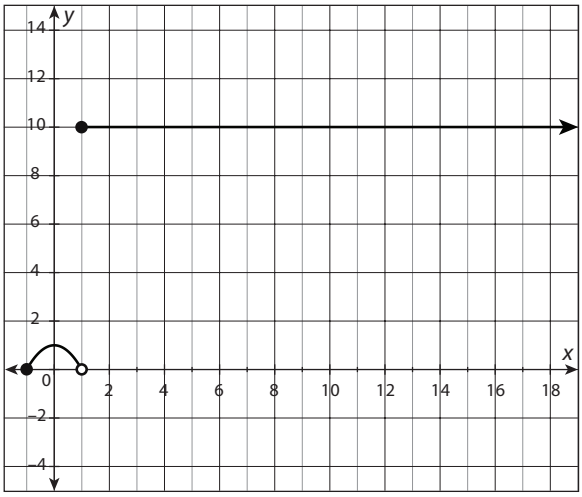
2.



3.



4.



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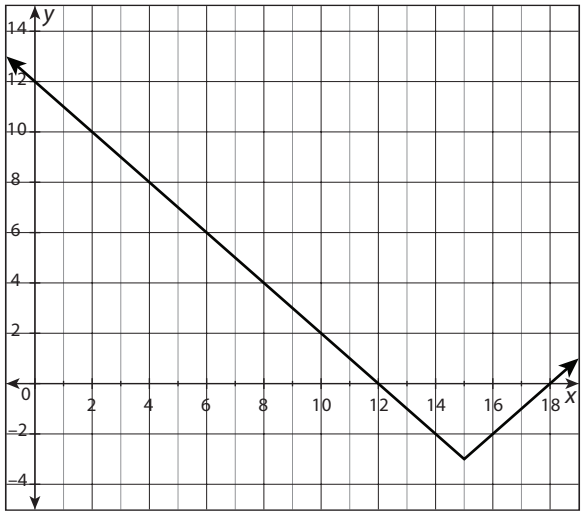
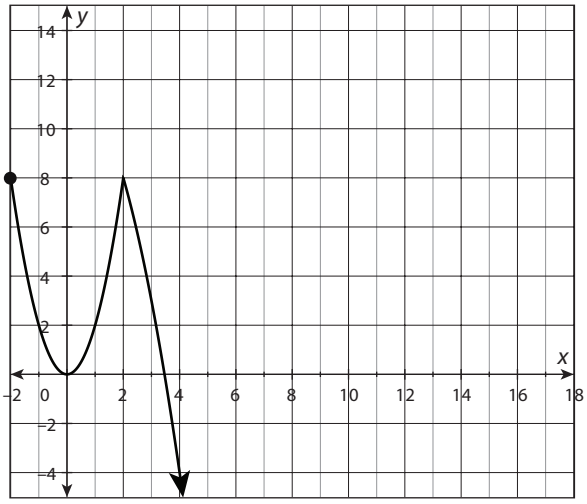
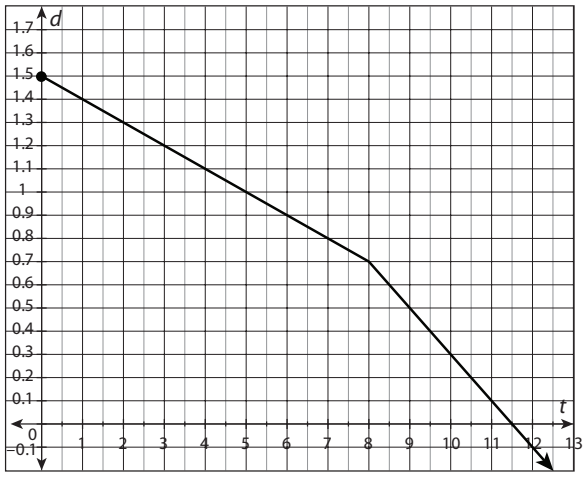
5.



6.

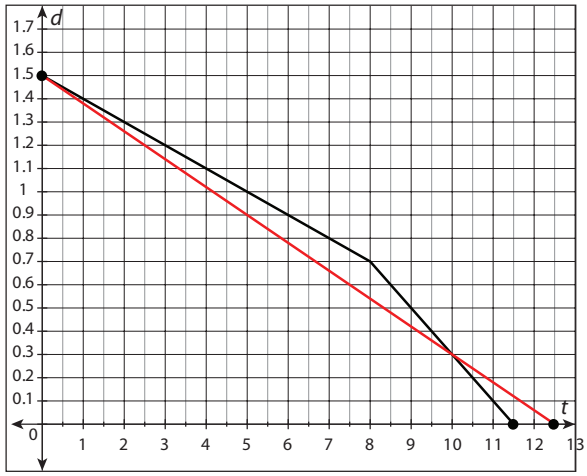


7.



8. The y-value of the graph is Saar's distance from school. When Saar has reached school, his distance from school will be 0. Therefore, when $y = 0$, Saar is at school. Find the x-value where $y = 0$ by looking at the graph. When x is 11.5, or $t = 11.5$ minutes, Saar is at school. It takes Saar 11.5 minutes to reach school.

9.



10. The minimum of both domains is $t = 0$, and both functions have a maximum at (0, 1.5). The maximum of both ranges is 1.5. The slope of the linear function $d_2 = -0.12t + 1.5$ is constant, whereas the slope of the piecewise function changes at $t = 8$. Saar reached school in 11.5 minutes today and he reached school in 12.5 minutes yesterday.

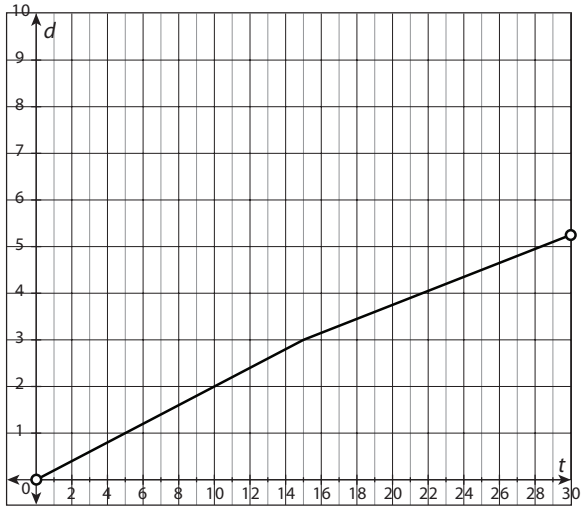
Progress Assessment, p. U2-244

1. b
2. b
3. d
4. a
5. c
6. d
7. a
8. a
9. c
10. b

11. a. $0 < t < 30$

    b. minimum: $d = 0$; maximum: $d = 5.25$

    c.



    d. 4.5 miles

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Lesson 5: Analyzing Functions

Pre-Assessment, p. U2-248

1. c
2. b
3. a
4. c
5. c

Warm-Up 2.5.1, p. U2-254

1. $V = 5{,}832 \text{ in}^3$
2. \$2.92
3. $V = 31{,}023 \text{ in}^3$
4. \$15.51

Practice 2.5.1: Analyzing Exponential Functions, p. U2-266

1. 5%; decay
2. 7%; growth
3. 34.4%; decay
4. 78.4%; decay
5. 32.3%; growth
6. There were 440 deer in 2005.
7. a decrease of 8%
8. The wolf population is decreasing at a rate of approximately 4%. The deer population is decreasing at a higher rate (8%), and is therefore changing faster.
9. \$4,000
10. The yearly rate of change is approximately 1%; it is growing.

Warm-Up 2.5.2, p. U2-268

1. $f(x) = 3(x - 2)^2 - 5$ has a vertex of (2, –5), which is lower than (–6, –4). The first bird descends deeper into the ocean.
2. $g(x) = 1/2(x + 2)^2 - 6$ has a vertex of (–2, –6), which is lower than the vertex of $f(x) = 3x^2 - 12x + 7$, which is (2, –5). The second bird descends deeper into the ocean.
3. The function represented by the table of values has a vertex of (3, 2), which is lower than the vertex (2, 3) of the function $f(x) = 2x^2 - 8x + 11$. The second bird descends closer to the water, but neither bird reaches the surface of the water.

Practice 2.5.2: Comparing Properties of Functions Given in Different Forms, p. U2-285

1. The train arrives first. It is ahead the entire trip, and it arrives just about 22 minutes before the car.
2. The train has its vertex around 19, which is outside the domain of the function. The train arrives in New York City before 19 hours have passed.
3. The car would arrive at 7:27 P.M., the train would arrive at 7:05 P.M., and the plane would arrive at 7:00 P.M. The plane would arrive first.
4. Elmer
5. Fred
6. At $t = 5$ seconds, Elmer and Fred are tied, and at $t \approx 6.5$ seconds, Elmer and George are tied.
7. Fred, because at $t = 7$, Fred is already at 57 feet, whereas Elmer is at 25 feet and George is at 30 feet.
8. a parabola with two x-intercepts with $a > 0$
9. a parabola with no x-intercepts and $a < 0$
10. Day 15

Progress Assessment, p. U2-287

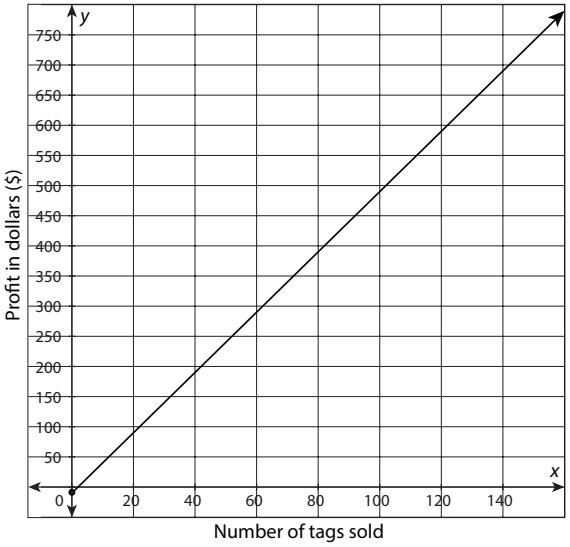
1. d
2. b
3. d
4. a
5. c
6. a
7. a
8. b
9. b
10. b
11. a. \$3,200
    b. 3%
    c. It is growing; the percent rate of change each year is positive.

Lesson 6: Transforming Functions

Pre-Assessment, p. U2-291

1. a
2. d
3. b
4. a
5. c

Warm-Up 2.6.1, p. U2-295

1. $f(x) = 5x - 10$
2. 

3. The revenues would increase by \$2. This would affect the y-intercept, and the graph would move up 2 units.

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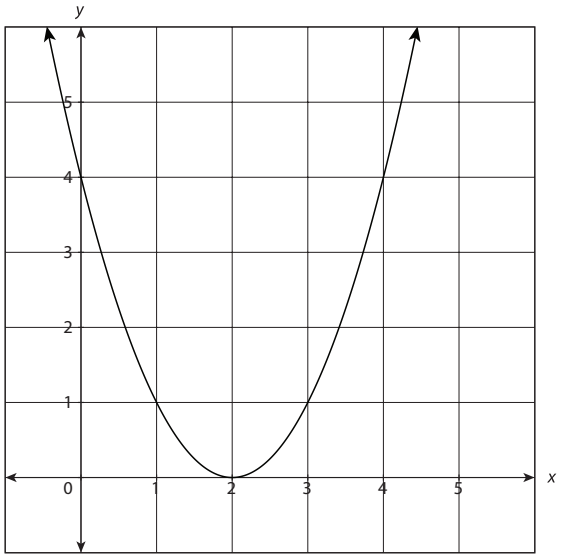
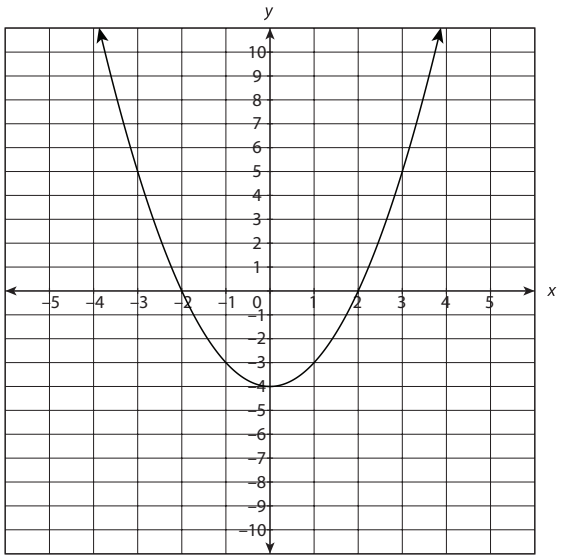
Practice 2.6.1: Replacing $f(x)$ with $f(x) + k$ and $f(x + k)$, p. U2-313

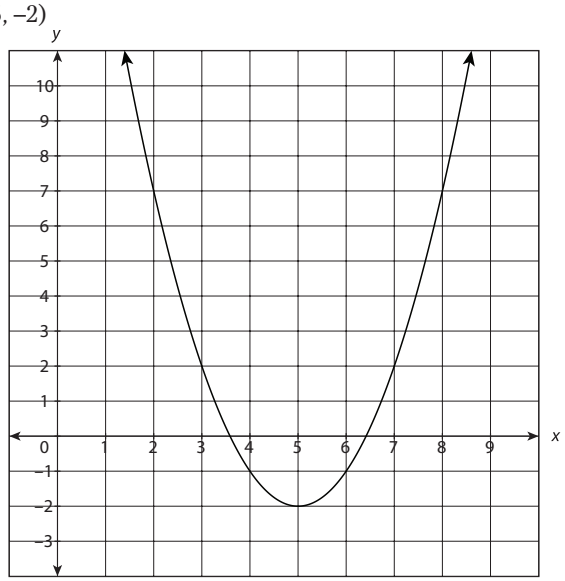
1. $y = (x + 2)^2$
2. $y = x^2 + 3$
3. $y = (x - 5)^2 - 2$
4. (2, 0)



5. (0, –4)



6. (5, –2)



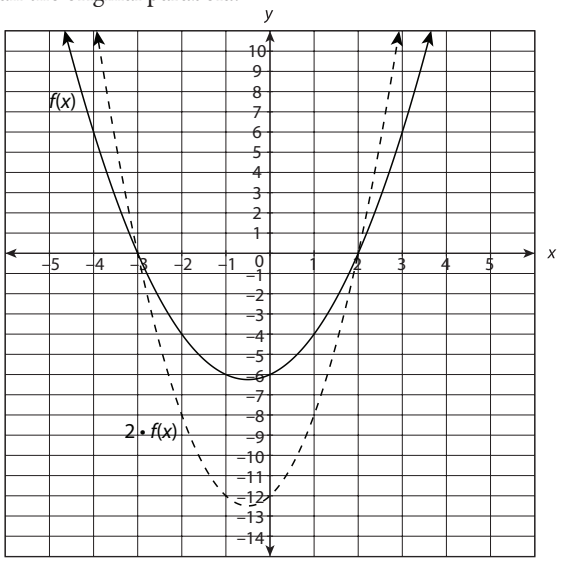
7. $y = -(x + 4)^2$; $k = 4$
8. $f(x) = -0.0009(x - 18)^2 + 0.2088(x - 18)$
9. $f(x) = -0.03x^2 + 1.3x + 16$; the ball will land about 53 feet away.
10. $f(x) = -\frac{1}{8}(x-13)^2 + 10$; yes, it will land in the basket.

Warm-Up 2.6.2, p. U2-315

1. 576 ft^2
2. 1,728 ft^2
3. 1,936 ft^2

Practice 2.6.2: Replacing $f(x)$ with $k \bullet f(x)$ and $f(k \bullet x)$, p. U2-335

1. $2 \bullet f(x) = 2x^2 + 2x - 12$. Since the entire function is being multiplied by 2, the x-intercepts of the two functions are the same. However, the second parabola, $2 \bullet f(x)$, is stretched vertically because each y-coordinate has been multiplied by 2. As a result, the new parabola is narrower than the original parabola.



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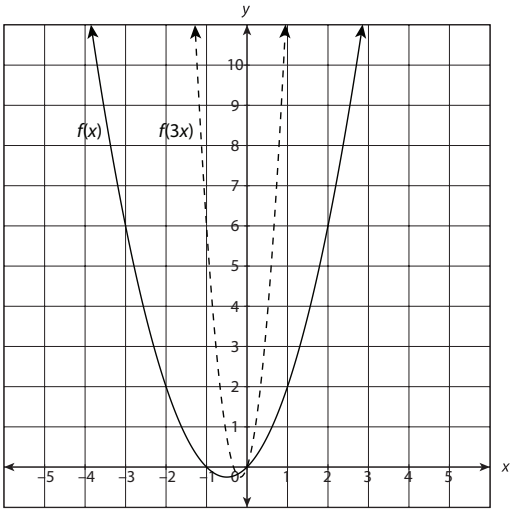
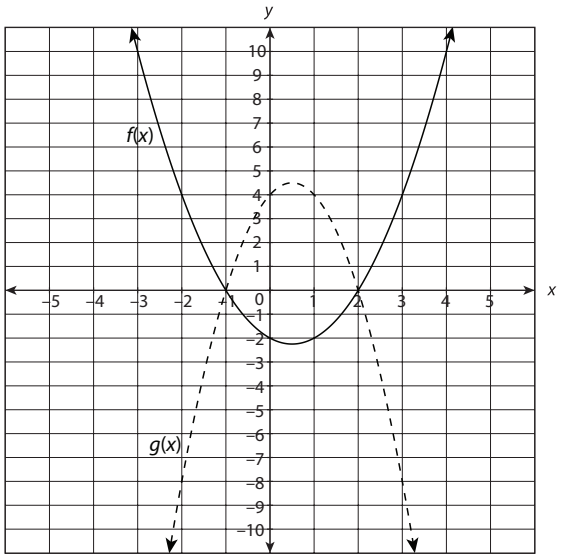
2. $f(3x) = 9x^2 + 3x$. Since the variable x is multiplied by 3, the graph is compressed horizontally by a factor of $\frac{1}{3}$. That is, each x-value is multiplied by $\frac{1}{3}$, so the width of the parabola becomes narrower. The y-intercept remains the same, as do the minimum values of the functions. The x-intercepts change and the interval between them becomes narrower.

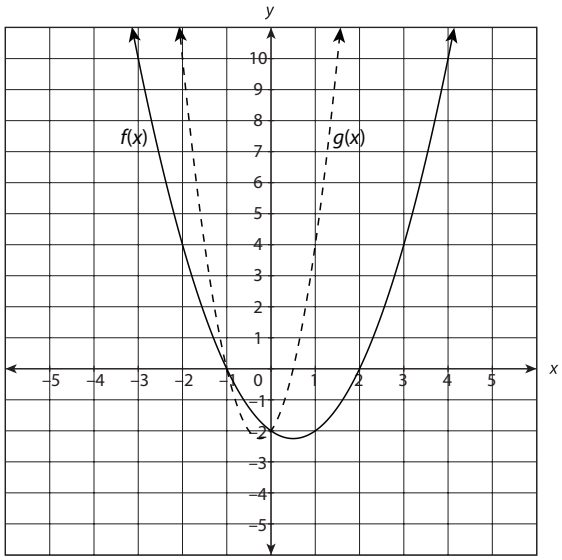


3. Answers may vary. Apply the transformation $k \bullet f(x)$, where $0 < k < 1$. Sample answer: $1/2 \bullet f(x) = g(x) = 0.5x^2 - 0.5x - 1$
4. Answers may vary. Apply the transformation $f(k \bullet x)$, where $k > 1$. Sample answer: $f(2x) = g(x) = 4x^2 + 6x - 4$
5. Graph of $f(x)$ and $g(x)$:



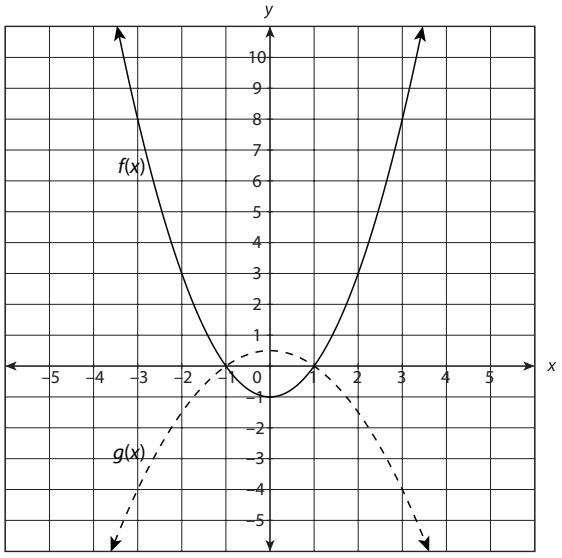
Since $g(x)$ is a multiple of $f(x)$, it will have the same x-intercepts as $f(x)$: –1 and 2. Since the value of k is –2, the graph will be stretched vertically by a factor of 2, and it will reflect over the x-axis because $k < 0$.

6. Graph of $f(x)$ and $g(x)$:



The function $g(x)$ is obtained by multiplying the x-values in $f(x)$ by –2. This means that the y-intercept will be the same, the minimum y-value will be the same, but the x-intercept will change and the interval becomes narrower. The graph of $g(x)$ will be compressed horizontally by a scale factor of 1/2, and the graph will be reflected over the y-axis.

7. Graph of $f(x)$ and $g(x)$:



Since $g(x)$ is a multiple of $f(x)$, it will have the same x-intercepts as $f(x)$: –1 and 1. Since the value of k is $-\frac{1}{2}$, the graph will be compressed vertically by a factor of $\frac{1}{2}$, and it will reflect over the x-axis because $k < 0$.

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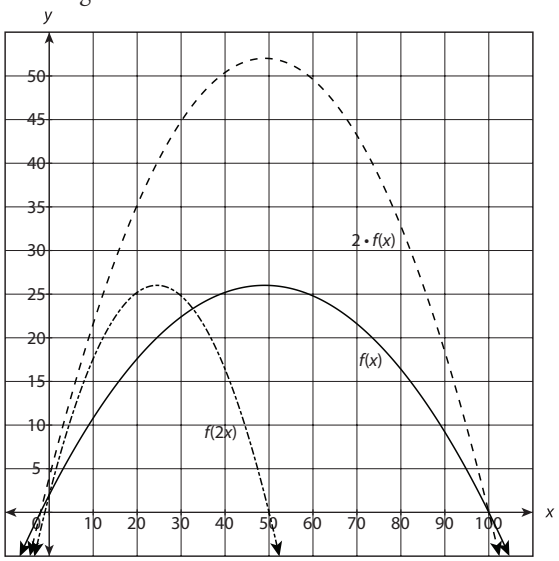
8. If the sides of the original pen were doubled, then the side lengths would be $2x$ and $4x$. The area would be $f(2x) = 2(2x)^2 = 8x^2$. If the farmer were to build a second pen of the same size, the area of the two pens would be $2 \cdot f(x) = 2(2x^2) = 4x^2$. Doubling the side length gives a larger area.
9. $f(x)$ represents the profit, so a doubling of the profit is modeled by $2 \cdot f(x)$.
10. Neither player is correct because neither of the equations leads to a ball going farther than the original ball. Jada's equation, $f(2x) = -0.01(2x)^2 + 0.98(2x) + 2$, leads to a ball going the same height as the original ball but only half as far; Jayla's equation, $2 \cdot f(x) = 2(-0.01x^2 + 0.98x + 2)$, leads to a ball going the same distance as the original ball but twice as high.



Progress Assessment, p. U2-338

1. b
2. b
3. a
4. a
5. d
6. d
7. b
8. b
9. c
10. b
11. a. $f(k \cdot x)$, where $0 < k < 1$ to stretch the parabola horizontally so that the path of the ball extends to reach the hoop.

    b. $k = \frac{19}{20}$

Lesson 7: Finding Inverse Functions

Pre-Assessment, p. U2-343

1. d
2. a
3. b
4. d
5. c

Warm-Up 2.7.1, p. U2-346

1. $220
2. 15 months
3. 6 months

Practice 2.7.1: Finding Inverse Functions, p. U2-360

1. Inverse: $f^{-1}(x) = \sqrt{-x}$; domain of $f(x)$: restricted to $x \geq 0$; range of $f(x)$: $y \leq 0$; domain of $f^{-1}(x)$: $x \leq 0$; range of $f^{-1}(x)$: $y \geq 0$
2. Inverse: $f^{-1}(x) = \frac{x}{5} + \frac{1}{5}$; the domain and range of both $f(x)$ and $f^{-1}(x)$ are all real numbers.
3. Inverse: $f^{-1}(x) = -x + 3$; the domain and range of both $f(x)$ and $f^{-1}(x)$ are all real numbers.
4. Inverse: $f^{-1}(x) = \sqrt{x-7}$; domain of $f(x)$: restricted to $x \geq 0$; range of $f(x)$: $y \geq 7$; domain of $f^{-1}(x)$: $x \geq 7$; range of $f^{-1}(x)$: $y \geq 0$
5. Inverse: $f^{-1}(x) = 4x + 16$; the domain and range of both $f(x)$ and $f^{-1}(x)$ are all real numbers.
6. Inverse: $f^{-1}(x) = -\frac{x}{3} - \frac{8}{3}$; the domain and range of both $f(x)$ and $f^{-1}(x)$ are all real numbers.
7. Domain of $f(x)$: $x \geq 0$; inverse: $f^{-1}(x) = \frac{1}{9}x$; independent quantity of the inverse: distance traveled in miles; dependent quantity of the inverse: jogging time in hours
8. Domain of $f(x)$: $x \geq 0$; inverse: $f^{-1}(x) = 6.67x - 333.33$; independent quantity of the inverse: daily earnings, in dollars; dependent quantity of the inverse: computer sales, in dollars
9. Domain of $f(x)$: $0 \leq x \leq 10$; inverse: $f^{-1}(x) = \sqrt{\frac{x}{1.9}} = 0.725\sqrt{x}$; independent quantity of the inverse: distance traveled in meters; dependent quantity of the inverse: time in seconds
10. Domain of $f(x)$: $x \geq 0$; inverse: $f^{-1}(x) = 0.13x + 40$; independent quantity of the inverse: total money earned in dollars; dependent quantity of the inverse: number of tickets sold

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Progress Assessment, p. U2-362

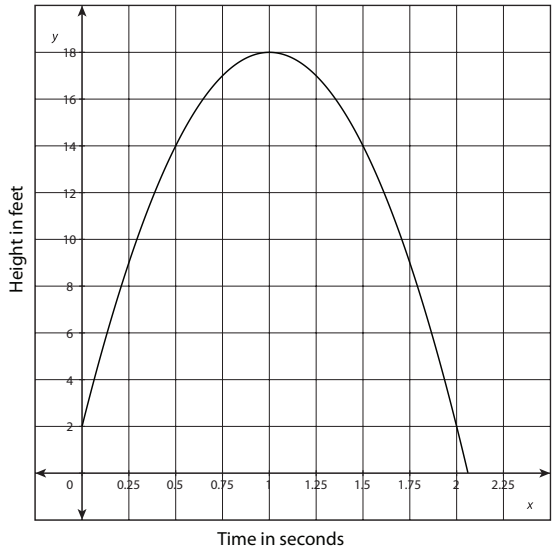
1. b
2. c
3. c
4. a
5. a
6. c
7. b
8. d
9. b
10. b

11. a. x is the number of meals sold; a reasonable domain of $f(x)$ is $x \geq 0$ because the number of meals sold cannot be negative.
    b. $f^{-1}(x) = 0.12x + 58.82$
    c. The independent quantity of $f^{-1}(x)$ is total earnings for the day, and the dependent quantity of $f^{-1}(x)$ is the number of meals sold.

Unit Assessment
p. U2-364

1. b
2. c
3. a
4. d
5. d
6. d
7. c
8. b
9. a
10. b
11. d
12. b

13. a. $x \geq 0$
    b. $f^{-1}(x) = -\frac{x}{35} + \frac{1}{5}$
    c. The independent quantity is the distance from home in miles and the dependent quantity is the time in hours.
14. a. The function is increasing when $x < 0$.
    b. The function is decreasing when $x > 0$.
    c. The vertex is (0, 144) and represents the maximum height of the crab, 144 feet, before it is dropped.
    d. A reasonable domain for this scenario is $0 \leq x \leq 3$ since the crab hits the rock at 3 seconds.
    e. The average rate of change is –56 feet/second.
15. a. The vertex is (1, 18) and represents the maximum height of the baseball, 18 feet, 1 second after it was hit.
    b. The equation that represents the axis of symmetry is $x = 1$.
    c.



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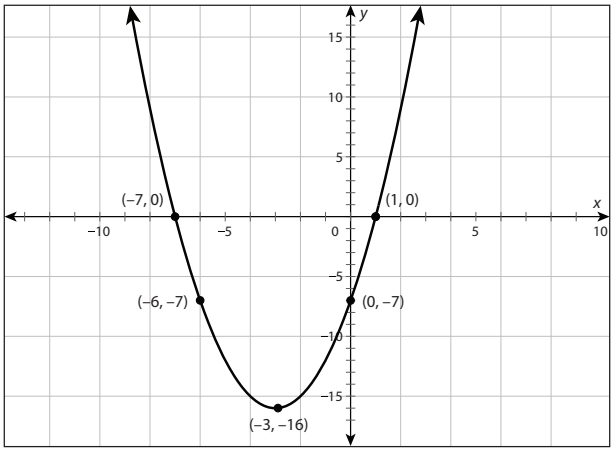
Student Book Answer Key

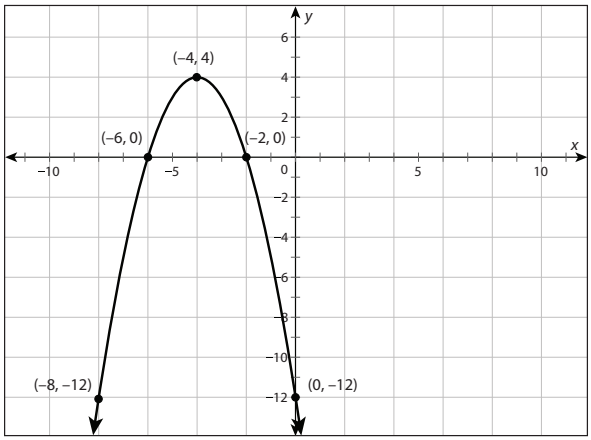
Lesson 1: Analyzing Quadratic Functions

Practice 2.1.1: Graphing Quadratic Functions, p. U2-20

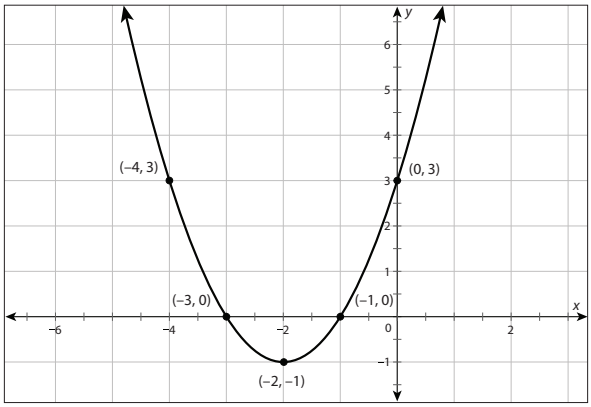
1. *y*-intercept: (0, –7); vertex: (–3, –16); minimum: –16



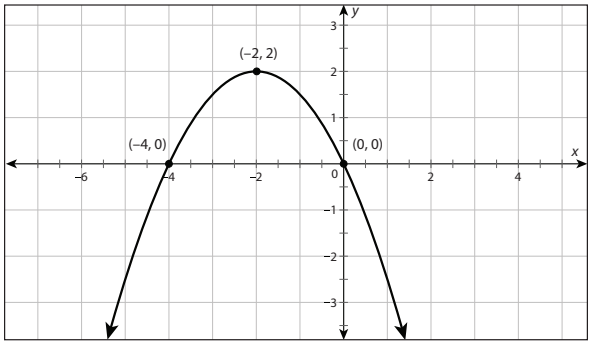
2. *y*-intercept: (0, –12); vertex: (–4, 4); maximum: 4



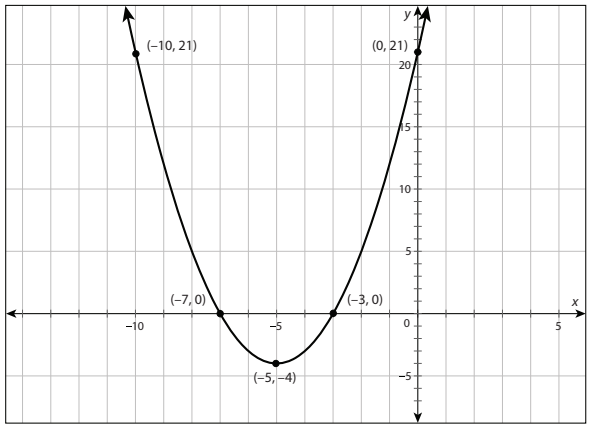
3. *y*-intercept: (0, 3); vertex: (–2, –1); minimum: –1



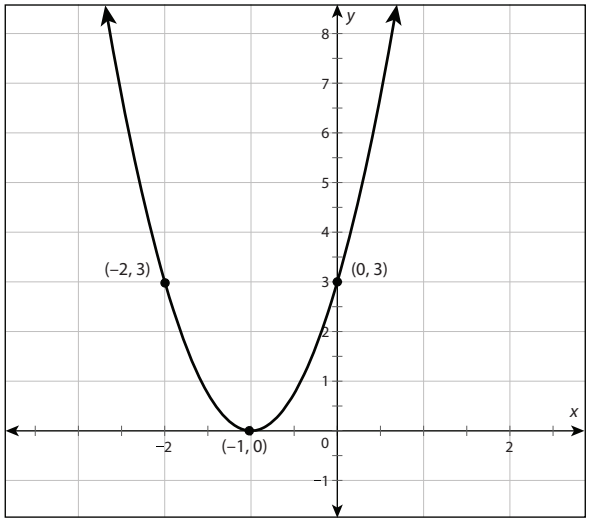
4. *y*-intercept: (0, 0); vertex: (–2, 2); maximum: 2



5. *y*-intercept: (0, 21); vertex: (–5, –4); minimum: –4



6. *y*-intercept: (0, 3); vertex: (–1, 0); minimum: 0



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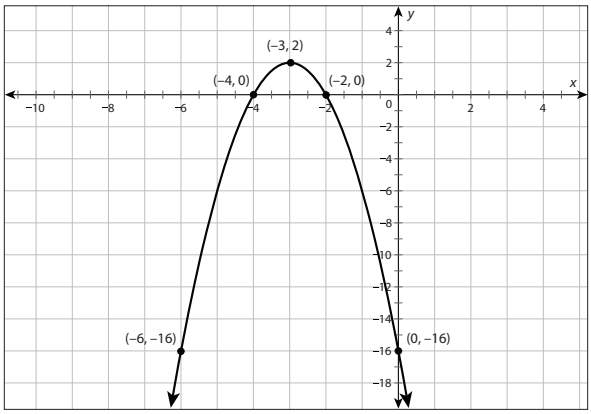
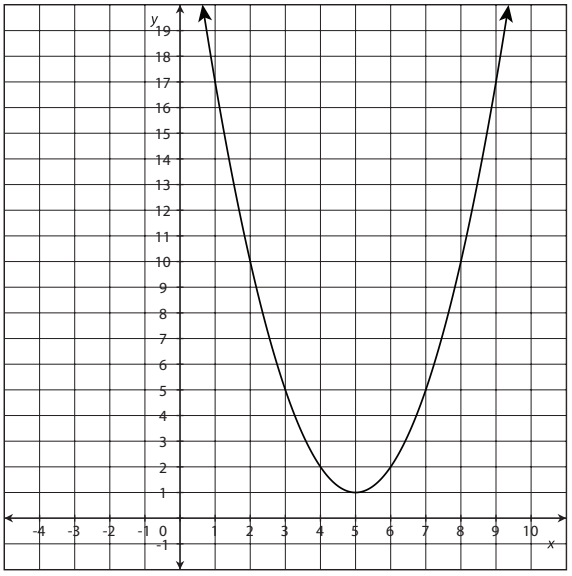
7. *y*-intercept: (0, –16); vertex: (–3, 2); maximum: 2



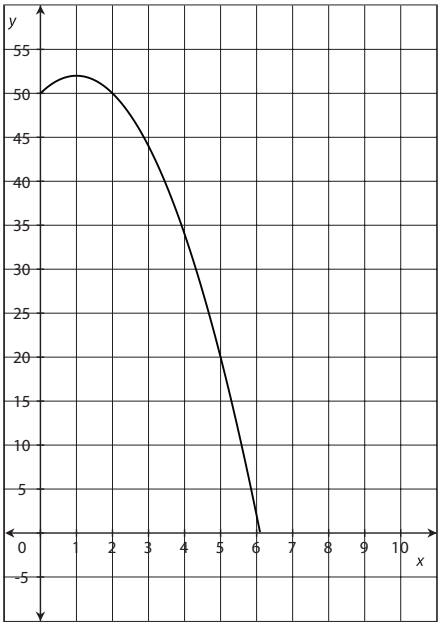
8. *y*-intercept: (0, –4); vertex: (0.625, 2.25); maximum: 2.25
9. *y*-intercept: (0, 750); vertex: (5, 1000); maximum: 1000
10. *y*-intercept: (0, 7); vertex: (3, 16); maximum: 16

Practice 2.1.2: Interpreting Various Forms of Quadratic Functions, p. U2-33

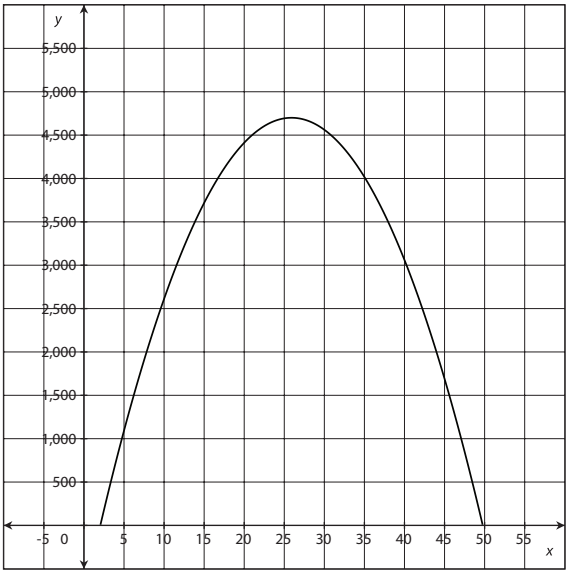
1. a. 12
   b. (4, –4)
   c. minimum
2. a. 3 and –5
   b. 30
   c. $x = -1$
   d. (–1, 32)
3. a. (3, 0)
   b. maximum
4. (5, 1); the butterfly reaches its minimum height of 1 foot above the ground 5 seconds after you see it.



5. (1, 52); the cliff diver reaches a maximum height of 52 feet 1 second after starting the dive; 2 seconds



6. 2 and 50; the *x*-intercepts represent the number of widgets sold that result in no revenue; 26 widgets



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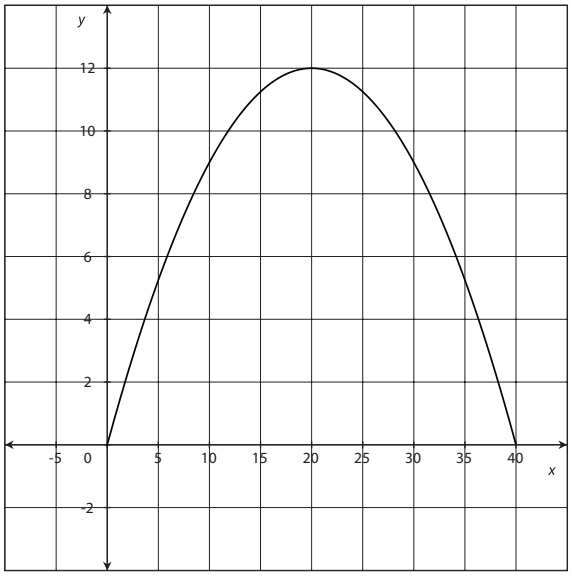
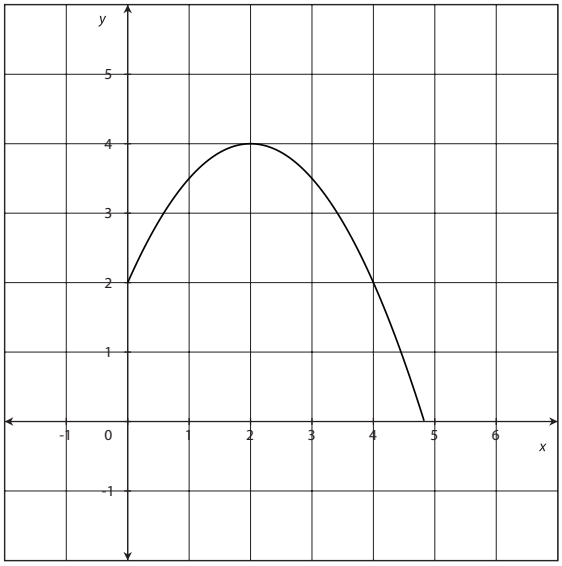
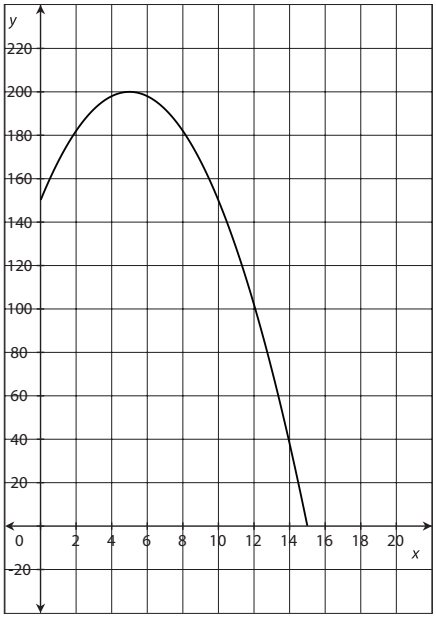
7. (20, 12); the football reaches a maximum height of 12 feet after it has traveled 20 feet in the horizontal direction; 40 feet



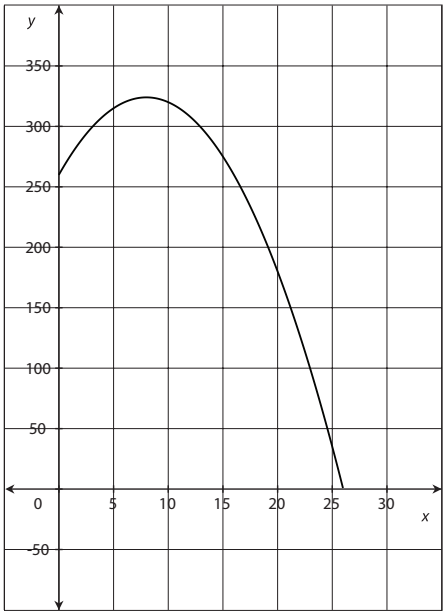
8. (2, 4); the frog reaches its maximum height of 4 feet after 2 seconds; $x = 2$; 4 seconds



9. (5, 200); the revenue reaches its maximum value of $200 when the price is increased by $5; $x = 5$; $10



10. –10 and 26; the x-intercepts represent price decreases that result in no revenue. The x-intercept –10 does not make sense in the context of the problem because a price decrease should not be negative; $x = 8$; $16



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Lesson 2: Interpreting Quadratic Functions

Practice 2.2.1: Interpreting Key Features of Quadratic Functions, p. U2-51

1. $x > 1.5$, $x < 1.5$; minimum = –8.25; x-intercepts: –1.37 and 4.37; y-intercept: –6; neither
2. $x < –2$, $x > –2$; maximum = 11; x-intercepts: –5.32 and 1.32; y-intercept: 7; neither
3. $x < 1$, $x > 1$; maximum = 16; x-intercepts: –1 and 3; y-intercept: 12; neither
4. $x > 0$, $x < 0$; minimum = 0; x-intercept: 0; y-intercept: 0; even
5. $x = –16.3$; $x > –16.3$, $x < –16.3$
6. $x = 2.25$; $x > 2.25$, $x < 2.25$
7. $x = –0.5$; $x < –0.5$, $x > –0.5$
8. $x = –0.02$; $x > –0.02$, $x < –0.02$
9. The height of the football is increasing for the first second and then it is decreasing.
10. The flare will explode at 256 feet.

Practice 2.2.2: Identifying the Domain of a Quadratic Function, p. U2-58

1. all real numbers or $-\infty < x < \infty$
2. all real numbers or $-\infty < x < \infty$
3. all real numbers or $-\infty < x < \infty$
4. all real numbers or $-\infty < x < \infty$
5. all real numbers; $-\infty < x < \infty$
6. all real numbers; $-\infty < x < \infty$
7. all real numbers; $-\infty < x < \infty$
8. less than 10 seconds
9. about 3 seconds
10. 25 increases; 5.5 increases

Practice 2.2.3: Identifying the Average Rate of Change, p. U2-67

1. –16
2. 8
3. –3/2
4. –18
5. –26
6. 2
7. between $x = –1$ and $x = 0$
8. between $x = –2$ and $x = –1$
9. between $x = –1$ and $x = 0$
10. –56 ft/s

Lesson 3: Building Functions

Practice 2.3.1: Building Functions from Context, p. U2-81

1. $f(x) = 3x^2 + 13x + 12$
2. $g(–3) = 39$
3. $x(x + 1) = 5402$; 73 and 74 or –73 and –74
4. vertex: (175, 25); y-intercept: 300
5. $f(x) = \frac{11}{1225}(x-175)^2 + 25$
6. $f(x) = (300 + 4x)(700 + 2x)$
7. 800 by 500 pixels or 400,000 pixels
8. vertex: (60, 40); x-intercepts: 0 and 120
9. $f(x) = -\frac{1}{90}(x-60)^2 + 40$
10. $f(x) = x(200 – 2x) = 200x – 2x^2$

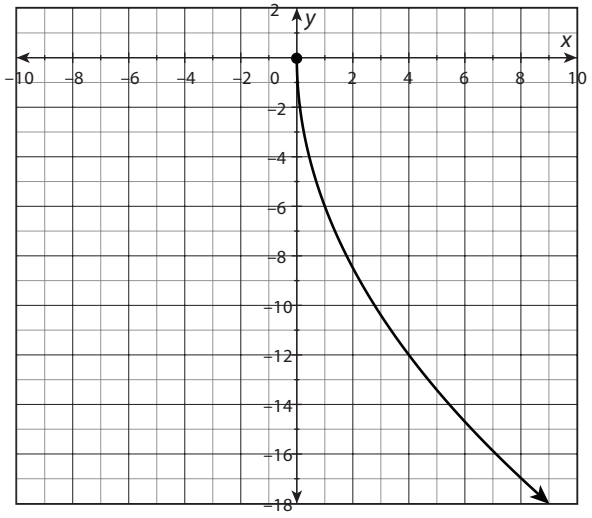
Practice 2.3.2: Operating on Functions, p. U2-90

1. $h(x) = x^2 + 10x + 25$
2. $u(x) = x^2 – 9x + 12$
3. $m(x) = 7x^2 + 30x – 25$
4. $h(x) = x + 7$; $x \neq –6$ and $h(x) \neq 1$
5. $k(x) = 3x^2 + 9x – 20$
6. $r(x) = x + 5$; $x \neq 2$ and $t(x) \neq 7$
7. $A(x) = x(x + 4)$; $R(x) = (x – 6)(x – 2)$; $d(x) = 12x – 12$
8. $a(x) = 3x – 18$; $b(x) = x$; $\frac{1}{2}(a \bullet b)(x) = \frac{1}{2}(3x-18) \bullet x = \frac{3}{2}x^2 - 9x$
9. $f(x) = 200 – x$; $g(x) = 300 – x$; $A(x) = (200 – x)(300 – x)$
10. $f(r) = \pi r^2$ and $g(r) = \pi rs$, so $(f + g)(r) = \pi r^2 + \pi rs$

Lesson 4: Graphing Other Functions

Practice 2.4.1: Square Root and Cube Root Functions, p. U2-117

1. domain: $x \geq 0$; range: $y \leq 0$; maximum point: (0, 0)



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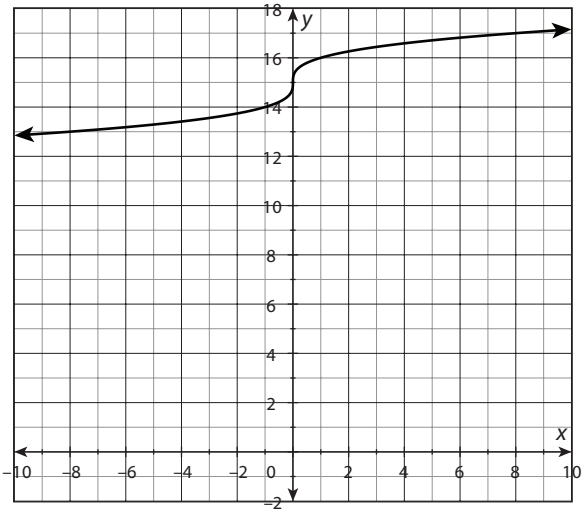
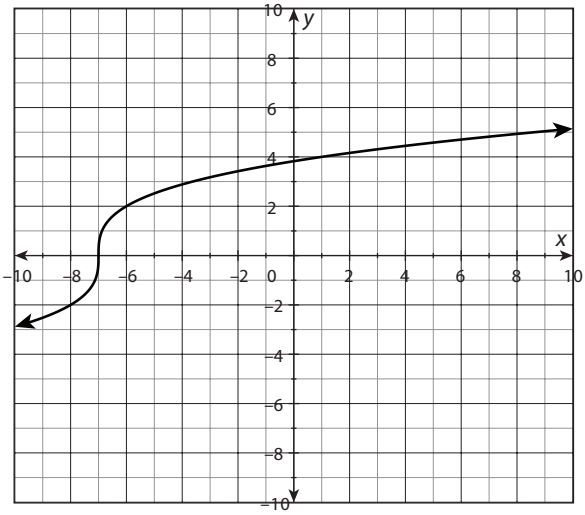
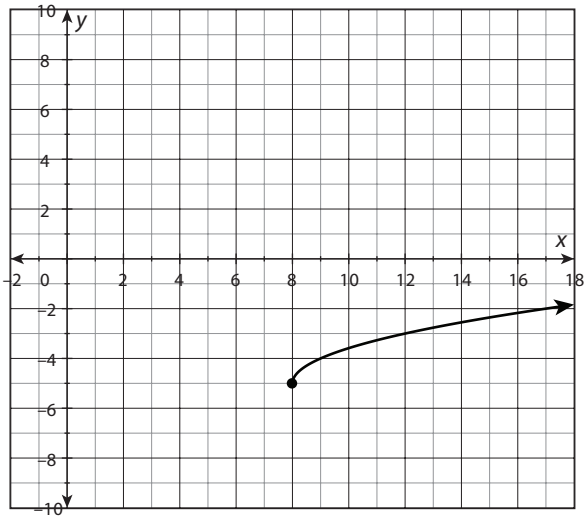
2. domain and range: all real numbers; point of inflection: (0, 15)



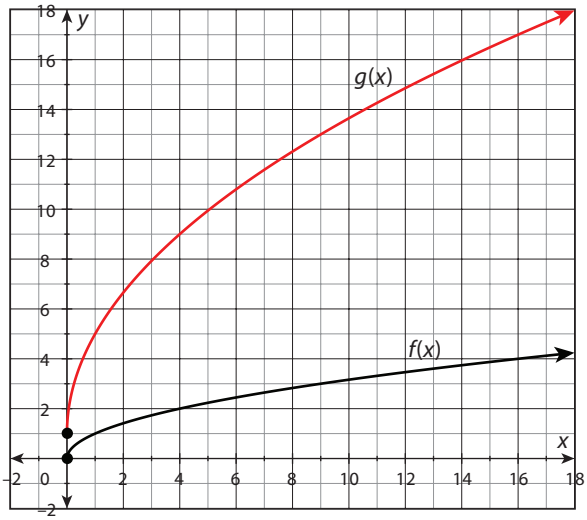
3. domain and range: all real numbers; point of inflection: (–7, 0)



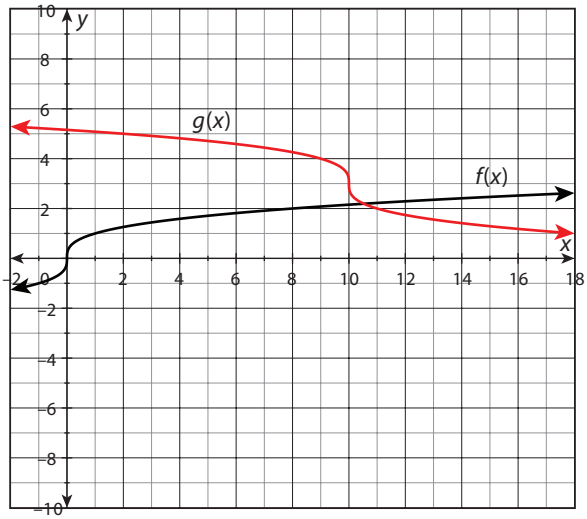
4. domain: $x \geq 8$; range: $y \geq -5$; minimum point: (8, –5)



5. Both have the same domain of $x \geq 0$, but the range of $g(x) = 4\sqrt{x}$ is $y \geq 1$, whereas the range of $f(x) = \sqrt{x}$ is $y \geq 0$. The graph of $g(x) = 4\sqrt{x} + 1$ is shifted up 1 unit due to the addend of 1. The factor of 4 stretches the graph vertically. Both graphs are increasing. The minimum point of $g(x) = 4\sqrt{x} + 1$ is also shifted up 1 unit to (0, 1), whereas the minimum point of $f(x) = \sqrt{x}$ is (0, 0).



6. Both functions have a domain and range of all real numbers. The point of inflection of $g(x) = -\sqrt[3]{(x-10)} + 3$ is (10, 3). The graph has been shifted right 10 units and up 3 units due to the addend of – 10 inside the cube root symbol and the addend of + 3 outside the cube root symbol. The –1 that is multiplied by the cube root reflects the graph so that the first graph is increasing and the second graph is decreasing.



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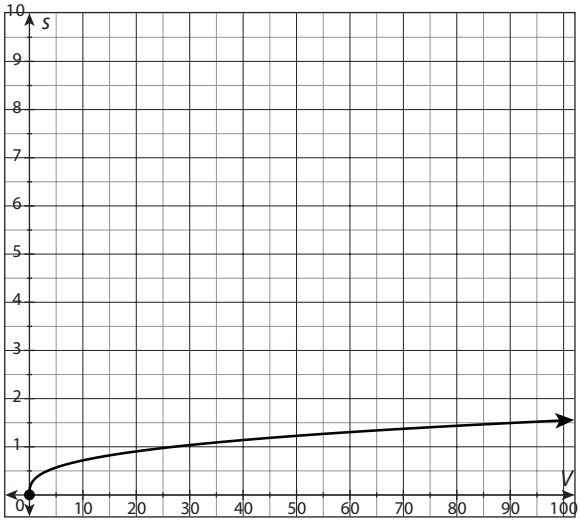
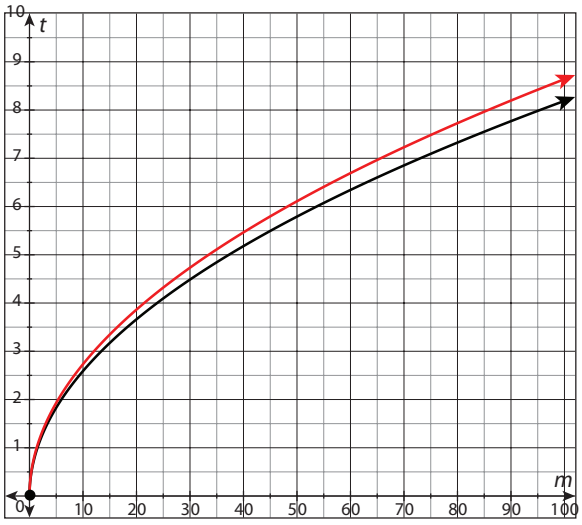
7.



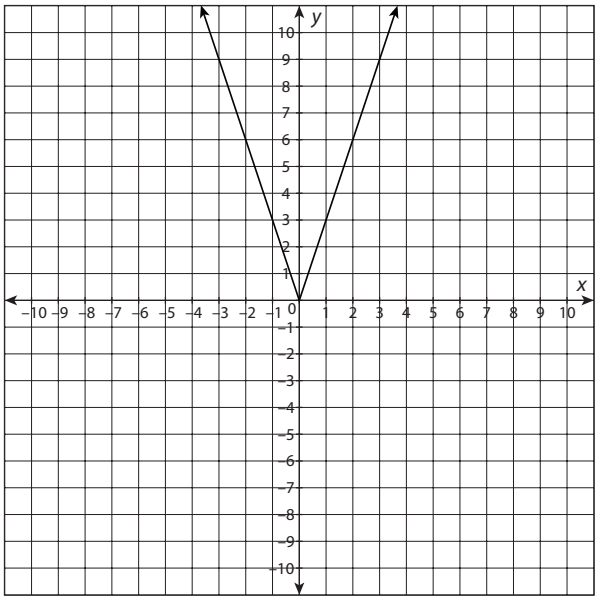
8. The side length is $9s$. For $V = 100$, s is approximately 1.5. The side length is approximately 13.5 inches.
9. The graph of car B's time is above the graph of car A's time except at (0, 0), where they are equal. Both graphs have a domain of $x \geq 0$ and a range of $y \geq 0$, but the greater number being multiplied by the square root in car B's graph stretches this graph vertically.



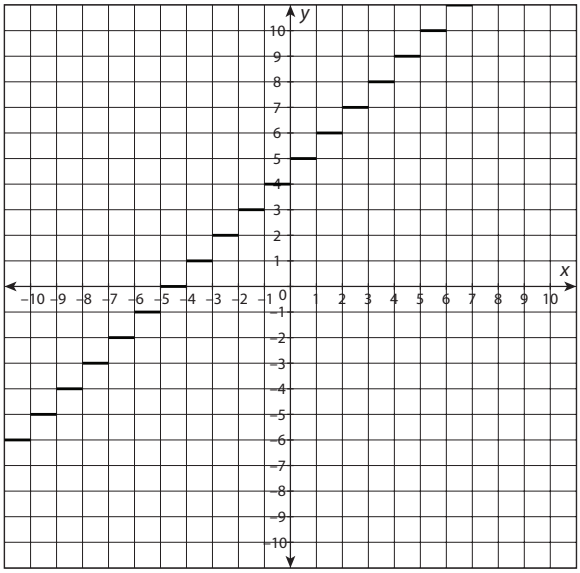
10. Car A's graph is lower than car B's graph at $x = 100$, indicating that it will take less time for car A to travel 100 meters. Car A will take approximately 8.2 seconds to travel 100 meters, whereas car B will take approximately 8.6 seconds.

Practice 2.4.2: Absolute Value and Step Functions, p. U2-137

1. domain: all real numbers; range: $y \geq 0$; vertex: (0, 0)



2. domain: all real numbers; range: all integers



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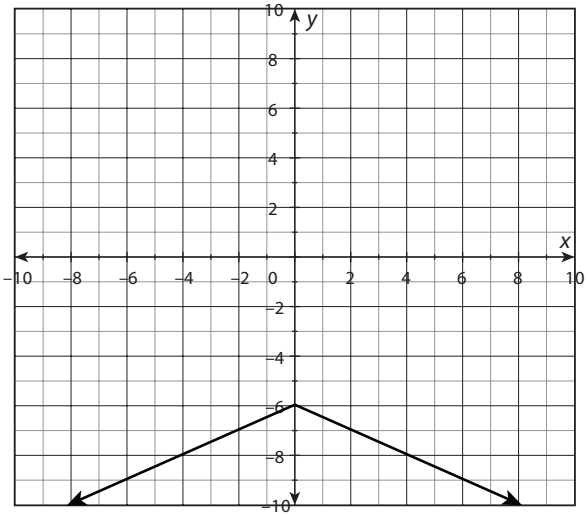
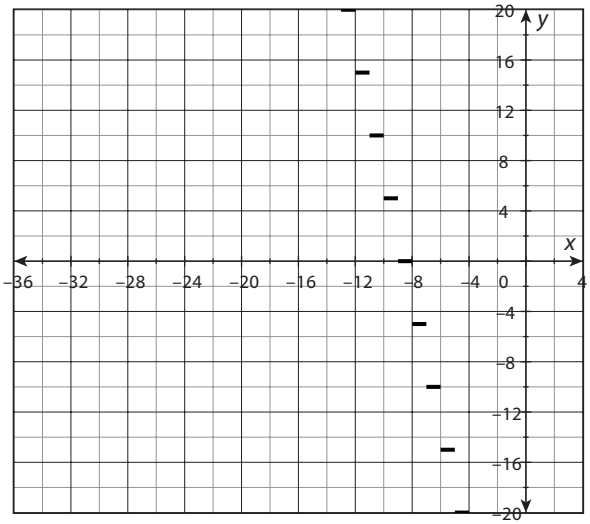
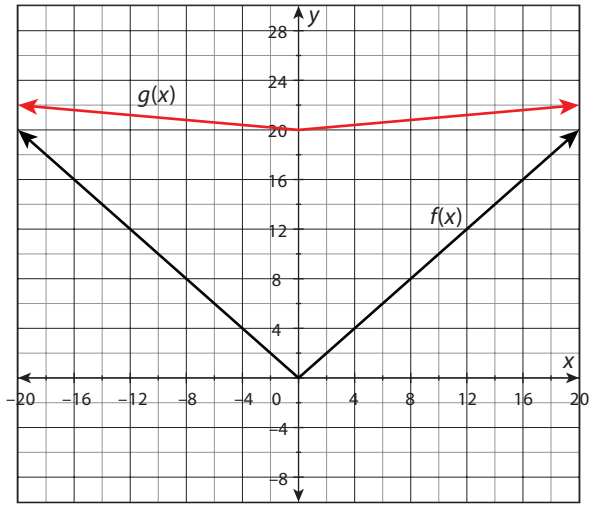
3. domain: all real numbers; range: $y \le -6$; vertex: (0, –6)



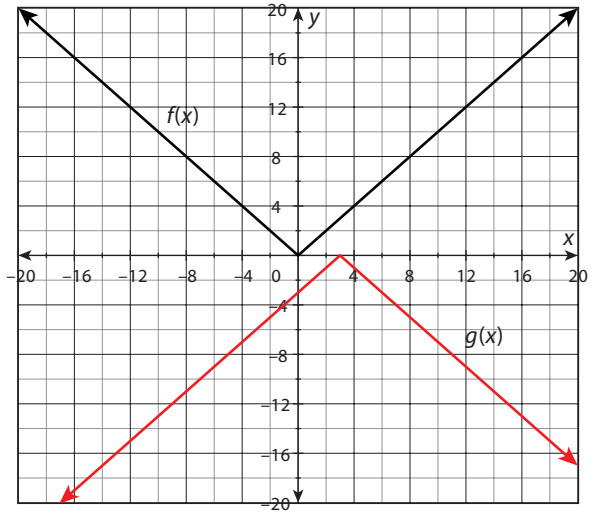
4. domain: all real numbers; range: multiples of 5



5. Both functions have a domain of all real numbers, but the range of $g(x) = 0.1|x| + 20$ is $y \ge 20$, whereas the range of $f(x) = |x|$ is $y \ge 0$. $g(x) = 0.1|x| + 20$ has a vertex, which is a minimum, at (0, 20), whereas the vertex and minimum point of $f(x) = |x|$ is at (0, 0). The factor of 0.10 outside the absolute value symbols has stretched the graph horizontally. Both graphs have a V-shape.



6. The domain of both functions is all real numbers. The range of $g(x) = -|x - 3|$ is $y \le 0$, whereas the range of $f(x) = |x|$ is $y \ge 0$. The graph of $g(x) = -|x - 3|$ is reflected over the *x*-axis due to the factor of –1 outside the absolute value expression. The addend of – 3 inside the absolute value symbols shifts the graph 3 units to the right.



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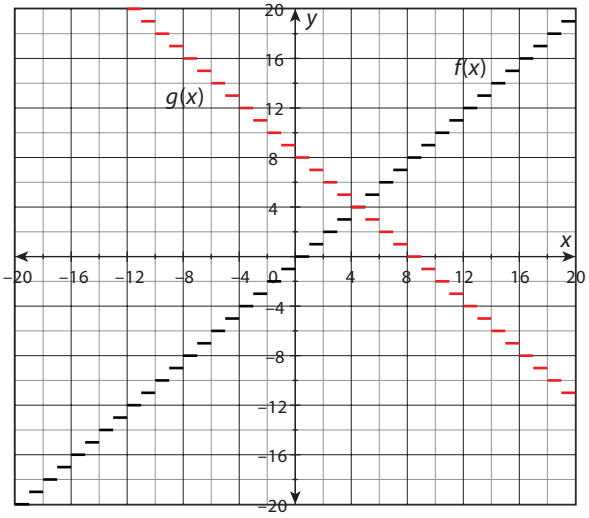
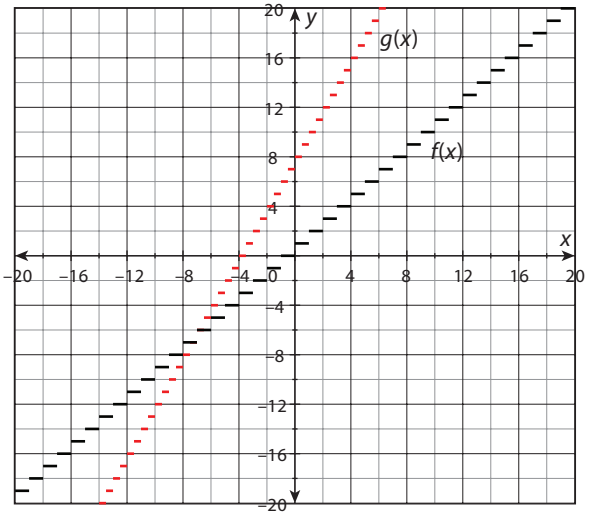
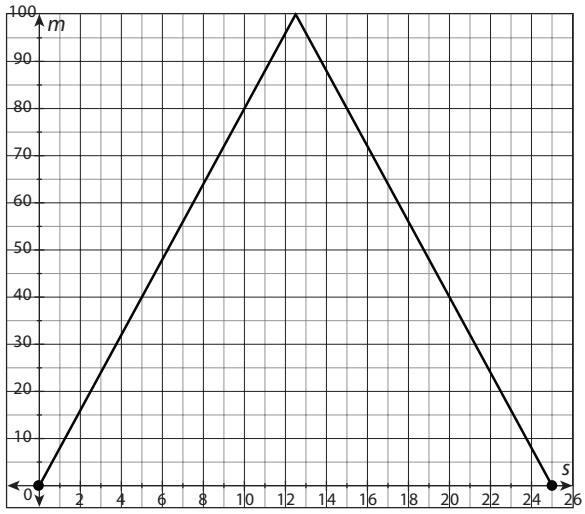
7. The domain of both functions is all real numbers, and the range of both functions is all integers. The steps of $f(x)=\lfloor x \rfloor$ are increasing, whereas the steps of $g(x)=-\lfloor x-8 \rfloor$ are decreasing due to the factor of –1 outside the greatest integer function. The addend of – 8 inside the greatest integer function shifts the graph 8 units to the right.



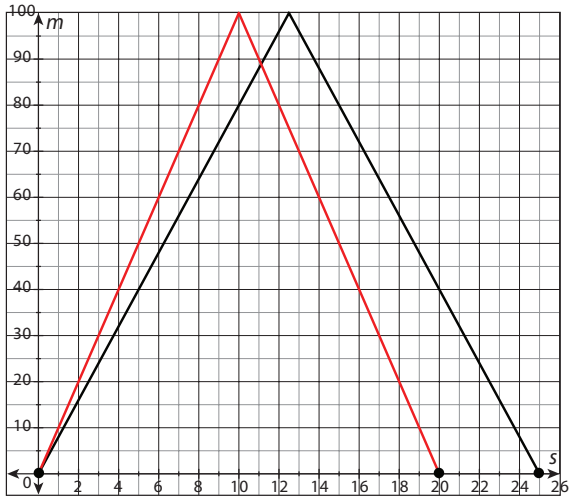
8. The domain of both functions is all real numbers, and the range of both functions is all integers. The steps on the graph of $g(x)=\lceil 2x+7 \rceil$ are shorter due to the factor of 2 inside the least integer function. The addend of + 7 inside the least integer function and the coefficient of 2 shifts the graph 3.5 units to the left. The steps of both functions are increasing.



9.



10. The relay teams start the race at (0, 0) and finish the race when $y = 0$ again. The graph for the first relay team reaches the x-axis at (25, 0), whereas the second team reaches the x-axis at (20, 0). This shows that the first relay team finishes the race at $x = 25$, or 25 seconds after the race starts, whereas the second relay team finishes the race at $x = 20$, or 20 seconds after the race starts. The second team wins.



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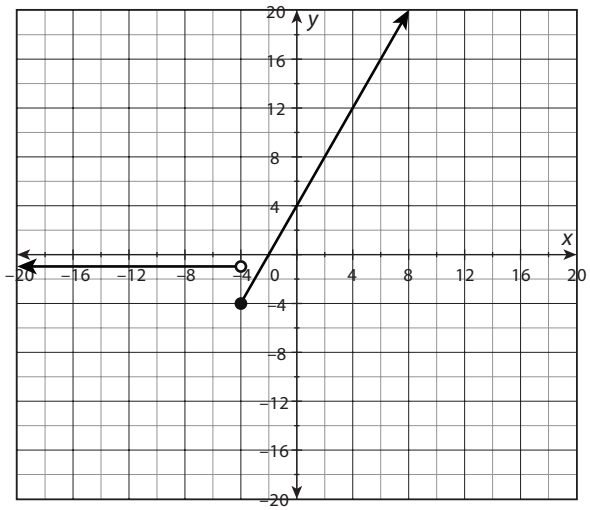
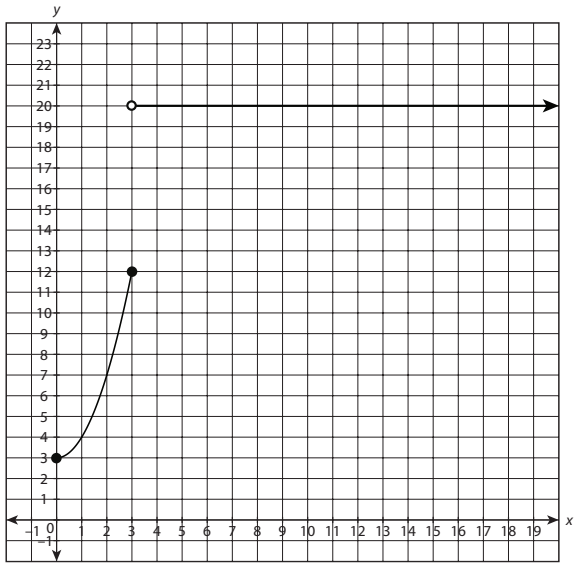
Practice 2.4.3: Piecewise Functions, p. U2-152

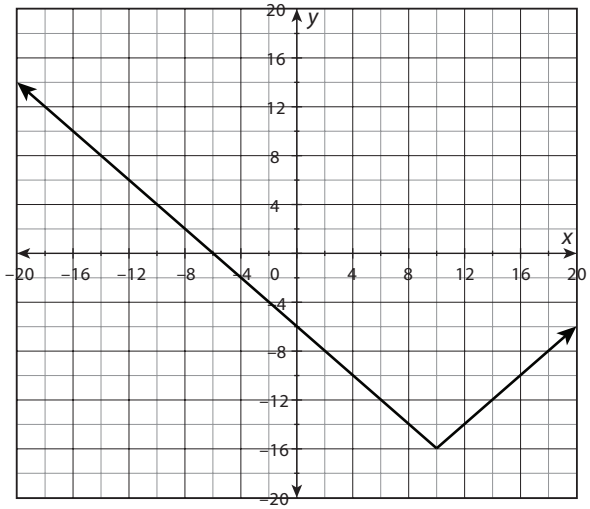
1.



2.



3.



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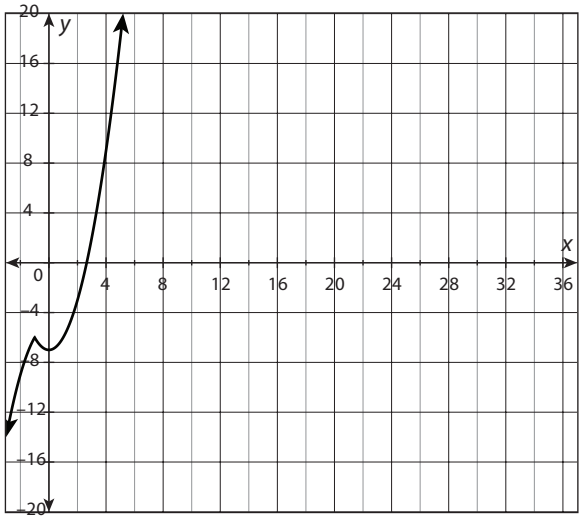
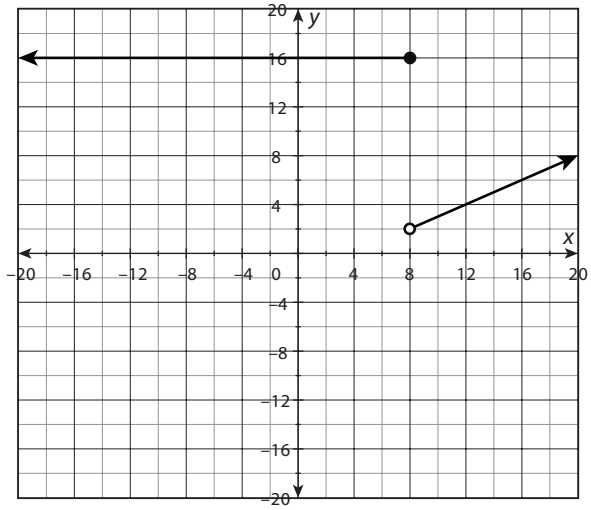


5.



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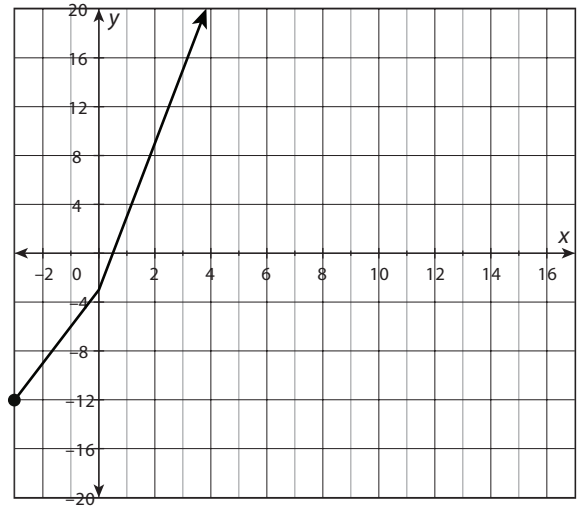

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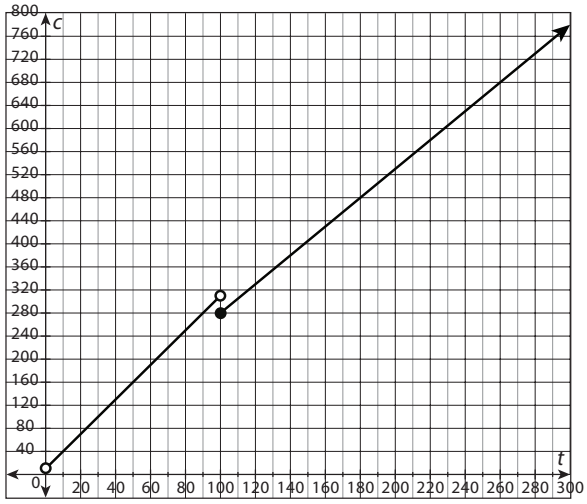
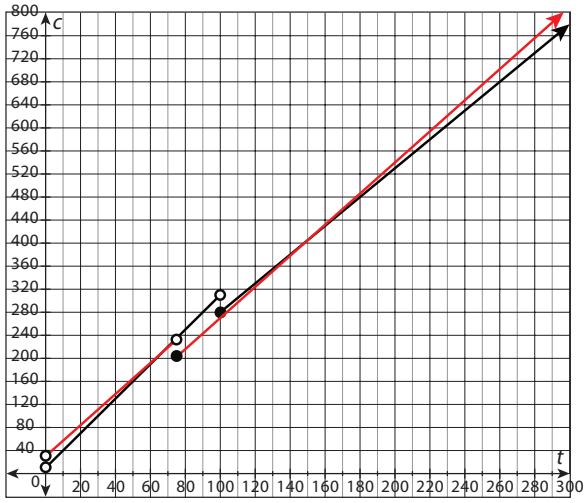
7.



8. The y-value, or total cost, when $x = 90$ is greater than the y-value, or total cost, when $x = 100$. It is cheaper to order 100 shirts than to order 90 shirts. Savita is correct that ordering 100 shirts has a lower total cost than ordering 90 shirts; Kit should order 100 shirts.

9.



10. Answers may vary. Sample response: When the graph of a function is greater than, or appears above, the graph of another function, the total cost c is greater for a given quantity of t. When two graphs cross, the total cost c is equal for the value of t.

Lesson 5: Analyzing Functions

Practice 2.5.1: Analyzing Exponential Functions, p. U2-162

1. 44%; growth
2. 30%; growth
3. 14.3%; decay
4. 31.6%; growth
5. 78.4%; decay
6. $1,800
7. 2% increase
8. The yearly rate of change on the coin collection is a 3% increase, which is a better rate of change than that of the record collection.
9. The app was downloaded 35,000 times in 2010.
10. The yearly rate of change is approximately 1.5%; it is growing.

Practice 2.5.2: Comparing Properties of Functions Given in Different Forms, p. U2-173

1. $f(x) = 0.05x + 5$
2. Nuts and Bolts Company has the lowest price for 200 bolts. 200 bolts would cost $15 at Loose Screws, $7 at Nuts and Bolts Company, and $24 at Nuts for You.
3. The price for 600 bolts would be the lowest at Nuts for You. The table provided does not go up as high as 600 bolts, but the prices go up less and less for every 100 bolts purchased. We can see from the table that 400 bolts would be $27, so the price for 600 bolts is likely a little more than that. The price for 600 bolts at Nuts and Bolts would be $57 and the price for 600 bolts at Loose Screws would be $35.
4. Agatha
5. Bertha and Charlotte are tied.
6. All three are tied at $t = 5$.
7. Bertha and Charlotte. They had the biggest gain from 5 to 6 seconds. This suggests that they will continue to make the larger gains as time progresses and, therefore, will reach 40 feet first.
8. a parabola with two x-intercepts with $a < 0$
9. a parabola with two x-intercepts and $a > 0$
10. in the year 2077

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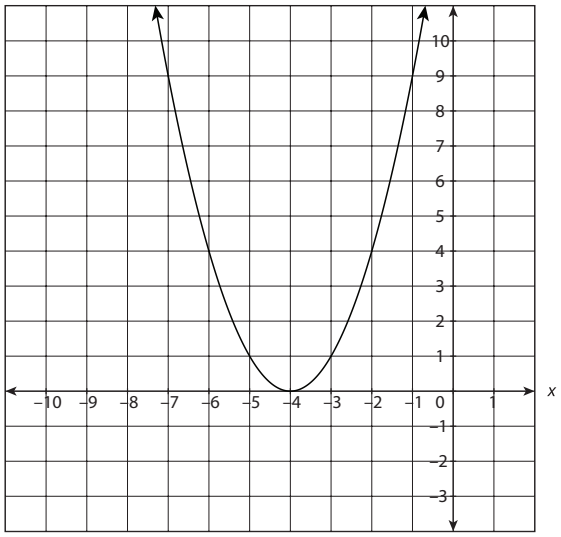
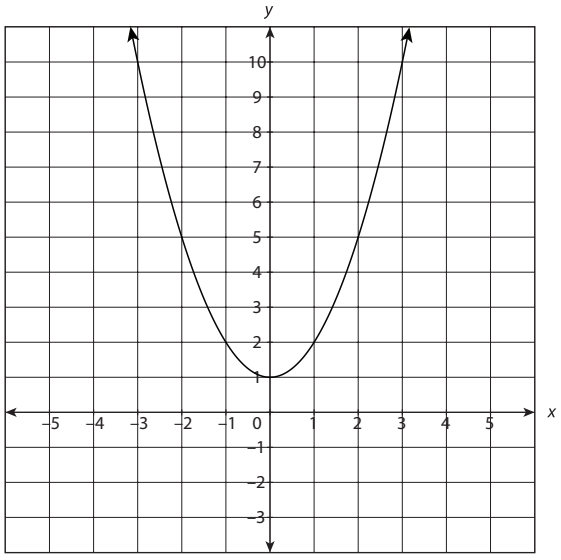
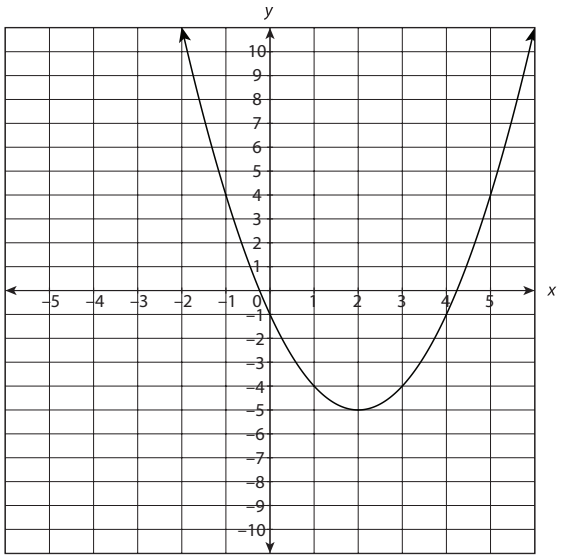
Lesson 6: Transforming Functions

Practice 2.6.1: Replacing $f(x)$ with $f(x) + k$ and $f(x + k)$, p. U2-187

1. $y = (x - 3)^2$
2. $y = x^2 - 4$
3. $y = (x + 6)^2 - 1$
4. 

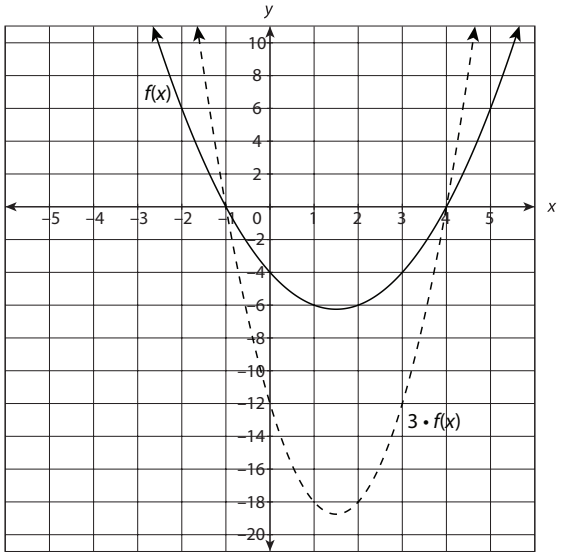
5. 

6. 

7. $y = -(x + 2)^2 + 5$. For the horizontal translation, $k = 2$. For the vertical translation, $k = 5$.
8. $f(x) = -0.0008(x - 12)^2 + 0.24(x - 12)$
9. $f(x) = -0.05x^2 + x + 19$; the paper wad will land about 32 feet away.
10. $f(x) = -0.4\ (x - 9)^2 + 14$; yes, it will hit the target.

Practice 2.6.2: Replacing $f(x)$ with $k \bullet f(x)$ and $f(k \bullet x)$, p. U2-201

1. $3 \bullet f(x) = 3x^2 - 9x - 12$. Since the entire function is being multiplied by 3, the x-intercepts of the two functions are the same (–1 and 4). However, the second parabola, $3 \bullet f(x)$, is stretched vertically because each y-coordinate is multiplied by 3. As a result, the new parabola is narrower than the original parabola.



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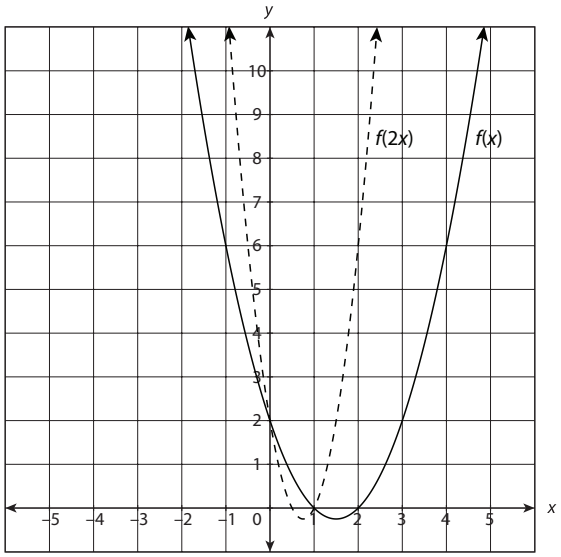
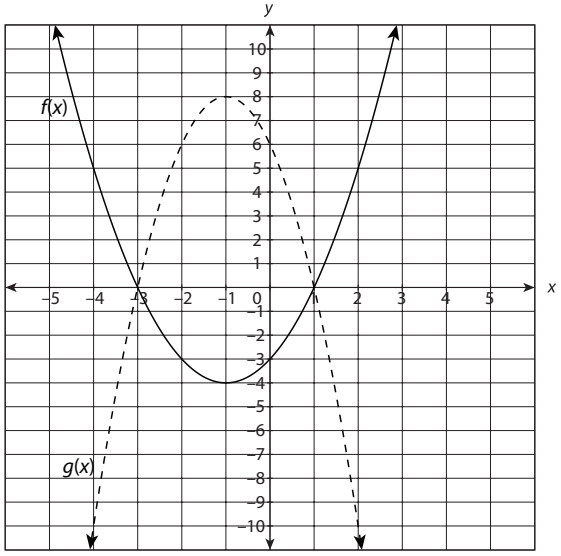
2. $f(2x) = 4x^2 - 6x + 2$. Since the variable x is multiplied by 2, the graph shrinks horizontally by a factor of $\frac{1}{2}$. That is, each x-value is multiplied by $\frac{1}{2}$, so the width of the parabola becomes narrower. The y-intercepts remain the same, as do the minimum values of the functions. The x-intercepts change and the interval becomes narrower.

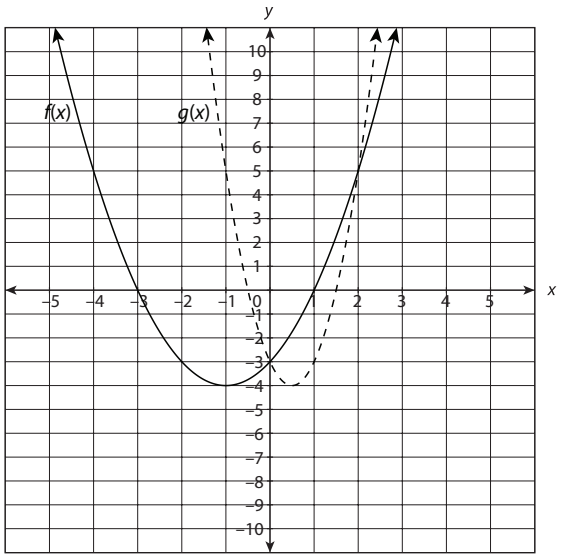


3. Answers may vary. Apply the transformation $k \bullet f(x)$, where $k > 1$. Sample answer: $2 \bullet f(x) = g(x) = 2x^2 - 8x + 6$
4. Answers may vary. Apply the transformation $f(k \bullet x)$, where $k > 1$. Sample answer: $f(2x) = g(x) = 4x^2 - 2x - 1$
5. Graph of $f(x)$ and $g(x)$:



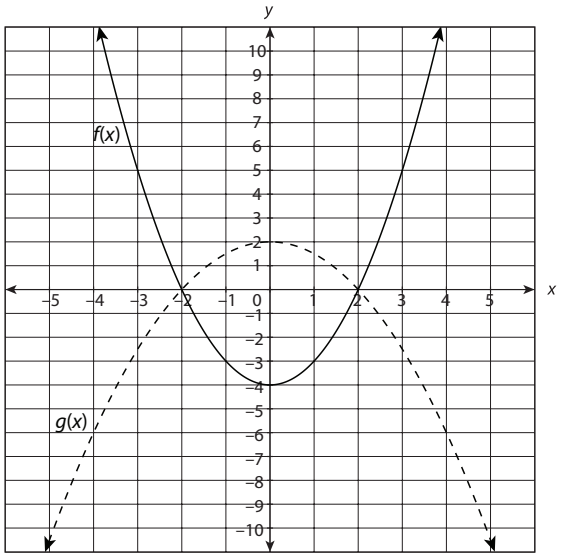
Since $g(x)$ is a multiple of $f(x)$, it will have the same x-intercepts as $f(x)$: –3 and 1. Since the value of k is –2, the graph will be stretched vertically by a factor of 2, and it will reflect over the x-axis because $k < 0$.

6. Graph of $f(x)$ and $g(x)$:



The function $g(x)$ is obtained by multiplying the x-values in $f(x)$ by –2. This means that the y-intercept will be the same, the minimum y-value will be the same, the x-intercepts will change and the interval between them will become narrower. The graph of $g(x)$ will be compressed horizontally by a factor of 1/2, and the graph will be reflected over the y-axis.

7. Graph of $f(x)$ and $g(x)$:



Since $g(x)$ is a multiple of $f(x)$, it will have the same x-intercepts as $f(x)$: –2 and 2. Since the value of k is $-\frac{1}{2}$, the graph will be condensed vertically by a factor of $\frac{1}{2}$, and it will reflect over the x-axis because $k < 0$.

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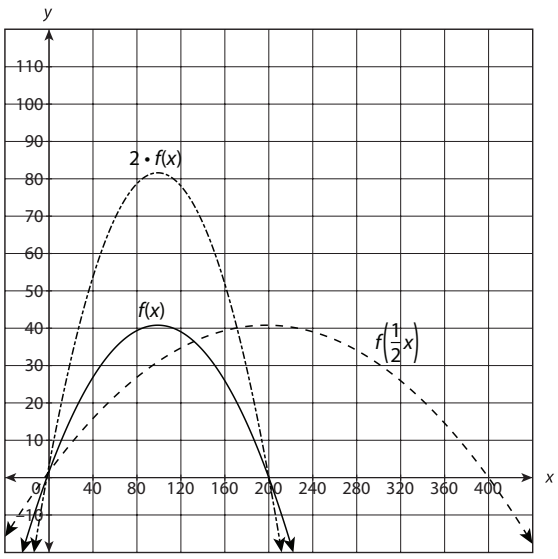
8. The original closet has side lengths of x and $3x$. Let the area of the original closet be $f(x) = 3x^2$. If the sides of the original closet were tripled, then the side lengths would be $3x$ and $9x$. The area would be $f(3x) = 3(3x)^2 = 27x^2$. If the celebrity builds two additional closets of the same size, the area of the three closets would be $3 \bullet f(x) = 3(3x^2) = 9x^2$. Tripling the side length gives a larger area.

9. $f(x)$ represents the profit, so earning half of the profit is modeled by $\frac{1}{2} \bullet f(x)$.

10. Zion is correct because his equation, $f\left(\frac{1}{2}x\right)$, leads to a beanbag going twice as far as the original beanbag. This beanbag reaches the same height as the original beanbag. Zavier's equation, $2 \bullet f(x)$, leads to a beanbag going the same distance as the original beanbag but twice as high.



Lesson 7: Finding Inverse Functions

Practice 2.7.1: Finding Inverse Functions, p. U2-213

1. Inverse: $f^{-1}(x) = -\frac{3}{2}x$; the domain and range of both $f(x)$ and $f^{-1}(x)$ are all real numbers.

2. Inverse: $f^{-1}(x) = \frac{x}{6} - 3$; the domain and range of both $f(x)$ and $f^{-1}(x)$ are all real numbers.

3. Inverse: $f^{-1}(x) = x + \frac{1}{5}$; the domain and range of both $f(x)$ and $f^{-1}(x)$ are all real numbers.

4. Inverse: $f^{-1}(x) = \sqrt{\frac{x}{2}}$; domain of $f(x)$: restricted to $x \geq 0$; range of $f(x)$: $y \geq 0$; domain of $f^{-1}(x)$: $x \geq 0$; range of $f^{-1}(x)$: $y \geq 0$

5. Inverse: $f^{-1}(x) = \sqrt{x+3}$; domain of $f(x)$: restricted to $x \geq 0$; range of $f(x)$: $y \geq -3$; domain of $f^{-1}(x)$: $x \geq -3$; range of $f^{-1}(x)$: $y \geq 0$

6. Inverse: $f^{-1}(x) = -\frac{x}{10} + 4$; the domain and range of both $f(x)$ and $f^{-1}(x)$ are all real numbers.

7. Domain of $f(x)$: $x \geq 0$; inverse: $f^{-1}(x) = 0.4x$; independent quantity of the inverse: total number of pumpkins harvested; dependent quantity of the inverse: number of seeds planted

8. Domain of $f(x)$: $x \geq 0$; inverse: $f^{-1}(x) = 1.1x + 111.1$; independent quantity of the inverse: total dollars Ashton keeps; dependent quantity of the inverse: number of website subscriptions sold

9. Domain of $f(x)$: $x \geq 0$; inverse: $f^{-1}(x) = 0.67x - 133.33$; independent quantity of the inverse: total dollars paid; dependent quantity of the inverse: number of words published

10. Domain of $f(x)$: $0 \leq x \leq 10$; inverse: $f^{-1}(x) = \sqrt{\frac{x}{1.3}} = 0.877\sqrt{x}$; independent quantity of the inverse: distance traveled in meters; dependent quantity of the inverse: time in seconds

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Instruction

Goal: To provide opportunities for students to develop concepts and skills related to graphing quadratic equations and functions

Common Core State Standard

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

Student Activities Overview and Answer Key

Station 1

Students will be given graph paper and a ruler. Students will derive how to find the vertex of the graph of a quadratic function. Then they will find the *x*-intercept and *y*-intercept of the function. They will graph the quadratic function and describe its shape.

Answers

1. $f(x) = x^2 + 6x + 9$
2. $a = 1$; $b = 6$; and $c = 9$
3. $\frac{-b}{2a}$
4. $f\left(\frac{-b}{2a}\right)$
5. $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$; (–3, 0)
6. Use the vertex as the axis of symmetry and a table of values to determine the corresponding *y*-value when $x = 0$.
7. Set $x = 0$ and solve for *y*.
8. (–3, 0) and (0, 9)

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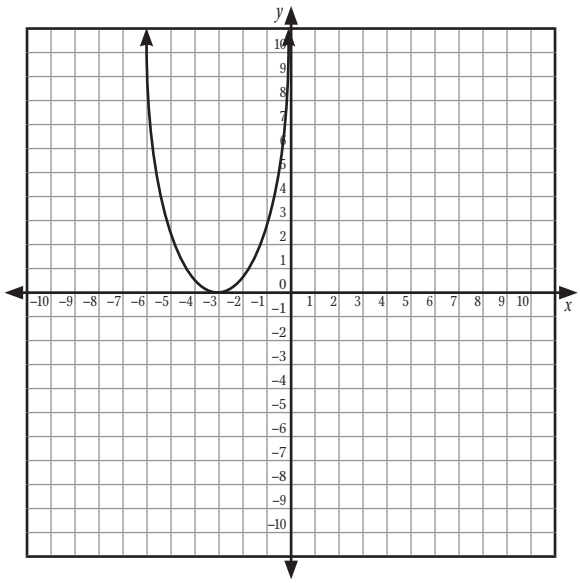
9. 

10. Parabola; the square term gives two x-values for each y-value due to the nature of square roots.

Station 2

Students will be given a graphing calculator. Students will use the graphing calculator to graph two quadratic functions. They will describe and analyze the characteristics of both graphs, including the vertex.

Answers

1. parabola
2. upward
3. x^2, because it has a positive coefficient
4. table of (x, y) values that satisfy the function
5. Find the y-value where $x = 0$. This point is the vertex; (0, 4)
6. parabola
7. upward
8. x^2, because it has a positive coefficient
9. table of (x, y) values for both functions
10. Find the y-value where $x = 0$. This point is the vertex; (0, –4)
11. The value of the constant determines the vertex of the graph.

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Instruction

Station 3

Students will be given a graphing calculator. Students will use the graphing calculator to graph three quadratic functions. The quadratic functions have the same vertex, but varying widths. Students will describe the relationships between the widths of the parabolas.

Answers

1. vertex at (0, 0) because there is no constant
2. $y = x^2$; because the coefficient is 1 versus 3. A smaller coefficient yields a wider parabola.
3. table of x- and y-values for both graphs
4. $Y_2 = 3Y_1$; it shows that the y-value for each x-value of Y_2 is three times as large as Y_1.
5. because it has a smaller coefficient
6. table of x- and y-values for all three graphs
7. $Y_3 = 1/2(Y_1)$; it shows that the y-value for each x-value of Y_3 is half the size of Y_1.
8. $Y_3 = 1/6(Y_2)$; it shows that the y-value for each x-value of Y_3 is one-sixth the size of Y_2.

Station 4

Students will be given graph paper and a ruler. Students will analyze and graph quadratic equations using vertices, x-intercepts, and a table of values. They will determine why certain parabolas open upward while others open downward.

Answers

1. $f(x) = x^2 - x - 6$ and $f(x) = -x^2 + x - 6$

   $a = 1$ $\qquad$ $a = -1$

   $b = -1$ $\qquad$ $b = 1$

   $c = -6$ $\qquad$ $c = -6$

2. For $f(x) = x^2 - x - 6$, the vertex is $(1/2, -25/4)$.

   For $f(x) = -x^2 + x - 6$, the vertex is $(1/2, -23/4)$.

3. (3, 0), (–2, 0)

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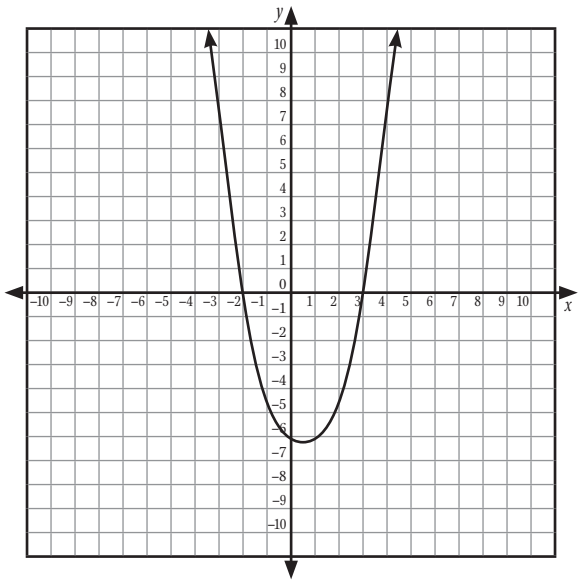
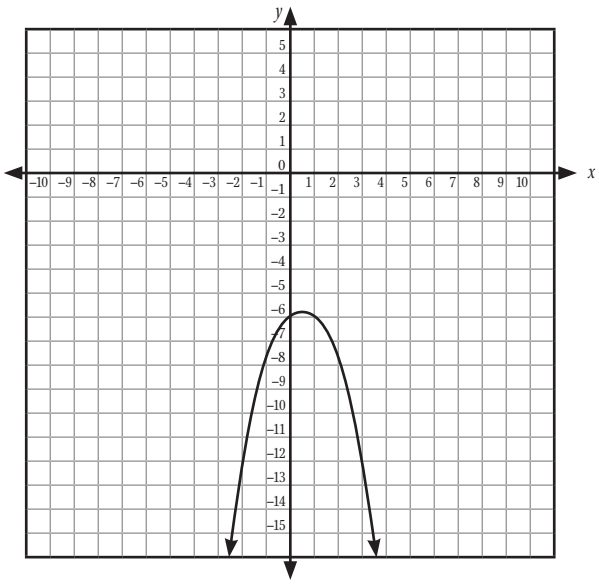
4. 

5. upward, because the coefficient of x^2 is positive

6. Table:

x	$y = f(x)$
–4	–26
0	–6
4	–18

Graph: 

7. downward, because the coefficient of x^2 is negative

8. no, because the parabola opens downward and is below the x-axis

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Instruction

Materials List/Setup

Station 1 graph paper; ruler

Station 2 graphing calculator

Station 3 graphing calculator

Station 4 graph paper; ruler

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. What does the graph of a quadratic equation look like?
2. Why does the graph have this shape?
3. How do you find the vertex of a quadratic equation?
4. Does a quadratic equation with a term of x^2 open upward or downward? Why?
5. Does a quadratic equation with a term of $-x^2$ open upward or downward? Why?
6. How can you find the x-intercepts of a quadratic equation?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. The graph of a quadratic equation looks like a parabola.
2. There are two x-values for each y-value because of the x^2 term.
3. vertex = $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a} \right) \right)$
4. upward, because the coefficient is positive
5. downward, because the coefficient is negative
6. Use the vertex as the axis of symmetry and a table of values to determine the corresponding y-value when $x = 0$.

Possible Misunderstandings/Mistakes

- Not realizing that graphs of quadratic equations are parabolas
- Not using enough data points to construct a parabola for the graph of the quadratic equation
- Ignoring the sign of the coefficient of the x^2 term when constructing a parabola that opens upward or downward

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Station 1

At this station, you will find graph paper and a ruler. Work together to graph the following quadratic equation:

$$y = x^2 + 6x + 9$$

1. Write this quadratic equation as a quadratic function.

2. What are the values of *a*, *b*, and *c* in the quadratic function?

   a = _____

   b = _____

   c = _____

To graph the function, you need the vertex, *x*-intercept, and *y*-intercept.

3. If the *x*-value of the vertex is found by $x = \frac{-6}{2(1)} = -3$, then write this *x* calculation using the general terms *a*, *b*, and/or *c*.

4. If the *y*-value of the vertex is found by $y = f\left(\frac{-6}{2(1)}\right) = f(-3) = 0$, then write this *y* calculation using the general terms *a*, *b*, and/or *c*.

5. Based on problems 3 and 4, how can you find the vertex of the graph for $f(x) = ax^2 + bx + c$?

What is the vertex of the quadratic function $x^2 + 6x + 9 = 0$?

continued

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6. How do you find the x-intercept of a function? (*Hint*: $y = f(x)$)

7. How do you find the y-intercept of a function?

8. What are the intercepts for $y = x^2 + 6x + 9$?

9. On your graph paper, graph the function using the vertex, x-intercept, and y-intercept.

10. What shape is the graph? Why do you think the graph has this shape?

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Station 2

At this station, you will find a graphing calculator. As a group, follow the steps according to your calculator model to graph $y = x^2 + 4$ and $y = x^2 - 4$.

On a TI-83/84:

Step 1: Press [Y=]. At Y_1, type [X,T,θ,n][x^2] [+][4].

Step 2: Press [GRAPH].

On a TI-Nspire:

Step 1: Arrow over to the graphing icon and press [enter]. At *f1*(*x*), enter [*x*], hit the [x^2] key, then type [+][4].

Step 2: Press [enter].

1. What shape is the graph?

2. Does the graph open upward or downward?

3. Which term do you think makes the graph open upward or downward? Explain your reasoning.

On a TI-83/84:

Step 3: Press [2ND], then [GRAPH].

On a TI-Nspire:

Step 3: Press [ctrl], then [T].

4. What information does your calculator show?

5. How can you use this information to find the vertex of the graph?

What is the vertex of the graph?

continued

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On a TI-83/84:

Step 4: Press [Y=]. At Y_2, type [X,T,θ,n][x^2] [–][4].

Step 5: Press [GRAPH].

On a TI-Nspire:

Step 4: Press [ctrl][tab] to go back to the graphing window. Use the touch pad to select ">>" on the bottom left of the screen. At *f2*(*x*), enter [*x*], hit the [x^2] key, then type [–][4].

Step 5: Press [enter].

6. What shape is the graph?

7. Does the graph open upward or downward?

8. Which term do you think makes the graph open upward or downward? Explain your reasoning.

On a TI-83/84:

Step 6: Press [2ND], then [GRAPH].

On a TI-Nspire:

Step 6: Press [ctrl], then [T]. Press [ctrl], then [T] a second time to refresh the screen.

9. What information does your calculator show?

10. How can you use this information to find the vertex of the graph of $y = x^2 - 4$?

What is the vertex of $y = x^2 - 4$?

11. Why do the graphs for $y = x^2 + 4$ and $y = x^2 - 4$ have different vertices?

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Station 3

At this station, you will find a graphing calculator. As a group, follow the steps according to your calculator model to graph $y = x^2$, $y = 3x^2$, and $y = \frac{1}{2}x^2$.

On a TI-83/84:

Step 1: Press [Y=]. At Y_1, type [X,T,θ,n][x^2]. At Y_2, type [3][X,T,θ,n][x^2].

Step 2: Press [GRAPH].

On a TI-Nspire:

Step 1: Arrow over to the graphing icon and press [enter]. At *f1*(*x*), enter [*x*], then hit the [x^2] key. Arrow down. At *f2*(*x*), enter [3][*x*], then hit the [x^2] key.

Step 2: Press [enter].

1. Why do both graphs have the same vertex?

2. Which graph is wider, $y = x^2$ or $y = 3x^2$?

   Why is one graph wider than the other?

On a TI-83/84:

Step 3: Press [2ND], then [GRAPH].

On a TI-Nspire:

Step 3: Press [ctrl], then [T].

3. What information does your calculator show?

4. What is the relationship between Y1 and Y2 in the table?

   How does this relationship relate to $y = x^2$ and $y = 3x^2$?

continued

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On a TI-83/84:

Step 4: Press [Y=]. At Y_3, type [0][.][5] [X,T,θ,n][x^2].

Step 5: Press [GRAPH].

On a TI-Nspire:

Step 4: Press [ctrl][tab] to go back to the graphing window. Use the touch pad to select ">>" on the bottom left of the screen. At *f3*(*x*), enter [0][.][5][*x*], then hit the [x^2] key.

Step 5: Press [enter].

5. Why is the graph of $y = 0.5x^2$ wider than $y = x^2$ and $y = 3x^2$?

On a TI-83/84:

Step 6: Press [2ND], then [GRAPH].

On a TI-Nspire:

Step 6: Press [ctrl], then [T]. Press [ctrl], then [T] a second time to refresh the screen.

6. What information does your calculator show?

7. What is the relationship between Y1 and Y3 in the table?

   How does this relationship relate to $y = x^2$ and $y = 0.5x^2$?

8. What is the relationship between Y2 and Y3 in the table?

   How does this relationship relate to $y = 3x^2$ and $y = 0.5x^2$?

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Station 4

At this station, you will find graph paper and a ruler. Work together to graph the following quadratic equations:

$f(x) = x^2 - x - 6$ and $f(x) = -x^2 + x - 6$

1. What are the values of a, b, and c in each quadratic function?

$f(x) = x^2 - x - 6$	$f(x) = -x^2 + x - 6$
$a =$ _____	$a =$ _____
$b =$ _____	$b =$ _____
$c =$ _____	$c =$ _____

2. Use the information in problem 1 to find the vertex $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$ for each function. Show your work.

3. Find the x-intercepts of $f(x) = x^2 - x - 6$ using factoring. Show your work.

4. On your graph paper, graph $f(x) = x^2 - x - 6$ using its vertex and x-intercepts.

5. Does the parabola open upward or downward? Explain your answer.

continued

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6. Fill out the table below to help you graph $f(x) = -x^2 + x - 6$.

x	$y = f(x)$
–4	
0	
4	

Graph $f(x) = -x^2 + x - 6$ on your graph paper.

7. Does the graph open upward or downward? Explain your answer.

8. Will the graph of $f(x) = -x^2 + x - 6$ have x-intercepts? Why or why not?

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APPENDIX

Problem-Based Task Implementation Guides

Introduction

This addendum contains a collection of Implementation Guides (IGs) designed to support the use of the Problem-Based Tasks found throughout this unit.

Each Implementation Guide includes an overview and suggestions for introducing and facilitating the task. These contain specific recommendations for selected Standards for Mathematical Practice (SMPs) to highlight, guidance on addressing common errors and misconceptions associated with the task, and ideas for monitoring and coaching students as they work through the task. Finally, there is guidance for debriefing the task, with special attention to the SMPs and relevant Key Concepts from the instruction, as well as information on suitable extensions and alternate strategies.

Throughout the Implementation Guides, suggestions are provided in a question/prompt format to support your efforts to elicit student thinking and facilitate classroom discourse.

We developed these Implementation Guides for our Problem-Based Tasks at the request of math educators, and with advice and feedback from mathematics supervisors and teachers. Please let us know if they are helpful. Visit us at www.walch.com, follow us on Twitter (@WalchEd), or e-mail suggestions to customerservice@walch.com.

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Implementation Guide

Problem-Based Task 2.1.1: How High Can a Frog Jump?

Common Core State Standard

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

Task Overview

Focus

How can the path of a fly and a frog be demonstrated? Is it possible for a frog on the ground to catch a fly in the air? Students will compare a quadratic and linear function to see if a frog is able to catch a fly in the air.

This activity will provide practice with:

- graphing quadratic and linear functions
- analyzing the graphs of two functions
- interpreting information from the graph of each function

Introduction

This task is used to explore the concepts of maximum/minimum, intercepts, and vertex of a quadratic function. Students will also be required to graph a linear and quadratic function and interpreting the meaning when the graphs intersect. This task is best to implement after students have mastered the terminology and calculations associated with quadratic functions as well as creating accurate graphs. Graphing may be done using technology, including graphing calculators or online graphing programs.

Begin by reading the problem and clarifying the meaning of the path of the frog jumping and the fly's path of flight. Explain that frogs jump into the air on a quadratic path to catch flies for food. Review the following term:

bank of a creek the edge of a small body of water

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Some students may have difficulty with the explanations for questions a and d. Encourage students to demonstrate their results first algebraically and/or graphically, then draw conclusions from their calculations and/or graphs. If necessary, work with the students to label the points of a graph and/or underline key parts of the calculations.

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- **SMP 2:** Reason abstractly and quantitatively.

  Some students may be able to determine the vertex or whether the graphs intersect with minimal calculations. Allow this method as long as students have a specific justification for their solutions. Help students use the vertex formula for a quadratic to find the maximum value. A general list of rules for a quadratic function may also be used to help students draw conclusions.

- **SMP 4:** Model with mathematics.

  Students are required to sketch a model of the quadratic and linear functions. Some students would prefer to sketch this model first to answer questions a, b, c, and d because this model would help them interpret the results faster. Ask students how the model can be modified for other scenarios (e.g. the frog jumping and/or fly flying on a different path).

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- using the incorrect sign when calculation the *x*-coordinate of the vertex

  Remind students of the formula $-\frac{b}{2a}$ for the *x*-coordinate of the vertex.

- incorrect calculations for the vertex and/or *y*-intercept

  Remind students of the order of operations when completing these calculations.

- graphing $y = 5$ as a vertical line instead of a horizontal line

  Advise students that the equation "*y* = a real number" is a graph of a horizontal line, not a vertical line.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- How can we find the height of the frog before jumping in the air? (**Answer:** Find the *y*-intercept by substituting 0 in for *x* and applying the order of operations.)
- How can we determine if the quadratic function has a maximum or minimum value? (**Answer:** Look at the leading coefficient of the quadratic function; if it is negative the function has a maximum value, and if it is positive the function has a minimum value.)
- How can the vertex of the function be determined? (**Answer:** To find the *x*-coordinate, the formula is $-\frac{b}{2a}$. Once the *x*-coordinate is calculated, this value is substituted into the function to find the *y*-coordinate of the vertex.)

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- How can the graph be sketched? (**Answer:** To graph the quadratic function, substitute in *x*-coordinates that are to the left and right of the vertex. The equation of the line is a horizontal line.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Compare the strategies of different groups. Ask students to share their strategies in determining the solutions and ask why they used these methods.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Key terms referring to the parts of a quadratic function (*y*-intercept, maximum/minimum, and vertex) are all used to help determine the frog's initial height, path of travel, and highest point in the air.
- The graphs of the quadratic and linear functions demonstrate where these two graphs intersect as well as the frog catching the fly at the frog's maximum height.
- A representation of the scenario is shown in words, algebraically, and graphically.

Extending the Task

To extend the task, change any parameters of the task. For example:

- Change the linear function of the fly to a different horizontal line or create a line with a slope.
- Change the quadratic function of the frog to jump higher or shorter as well as jump a shorter or longer distance.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you demonstrate perseverance?" (**Answer:** I kept reviewing the problem to find the information needed. I sketched the graph of both functions first, then answered the rest of the questions.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** Abstract reasoning was used to determine if the frog could catch the fly by not jumping as well as if the frog could catch the fly.) "Which of your strategies represent quantitative reasoning?" (**Answer:** Quantitative reasoning was used to find the ordered pair of the vertex.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I sketched the graph of both functions first, then answered the rest of the questions.)

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Alternate Strategies or Solutions

- Create a table for the quadratic function to sketch an accurate graph:

x	$f(x)$
–1	–4
0	1
1	4
2	5
3	4
4	1
5	–4

- Create a table for the linear function to sketch an accurate graph:

x	$f(x)$
–1	5
0	5
1	5
2	5
3	5
4	5
5	5

These tables can also be compared to determine the maximum height the frog jumps as well as the point of intersection.

Technology

Students can use graphing utilities on online graphing programs.

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Problem-Based Task 2.1.2: Is the Glider Safe?

Common Core State Standards

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

F–IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

Task Overview

Focus

Students will use a quadratic function to analyze the path of a hang glider and determine whether the ride is safe. They will use the key features of a quadratic function to make the decision.

This activity will provide practice with:

- writing a quadratic equation in standard form
- determining the vertex of a parabola from an equation
- interpreting the attributes of a quadratic function in terms of the context of the problem

Introduction

This task should be used to explore a quadratic function which models a real-world situation by analyzing the vertex and intercepts of the function. It is best to implement this task after students have analyzed several quadratic functions and their key features. Graphing may be done using technology, including graphing calculators or online graphing programs.

Begin by reading the problem and clarifying how a hang glider moves through the air. Review the following term:

hang glider an unpowered vehicle used to glide through the air

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Encourage students to create a graph of the function to help make sense of the problem. Discuss various points on the graph and what those points mean in the context of the problem. For example, the point (7, 3) means that after 7 seconds Shin is 3 feet above the safe height.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students more often see negative numbers that do not apply to the problem context, than they do ones that do apply. Use a graph of the function to help students make a valid argument as to why a y-coordinate of -1 makes sense in the context of this problem.

- **SMP 4:** Model with mathematics.

  The path of the hang glider is modeled by the quadratic equation given in the problem statement. Some students may have difficulty understanding how and why a quadratic function models the path of the glider. Allow students to graph the equation and use the graph to aid in understanding the various steps in the solution.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- using 10 instead of -10 for the value of b

  Have students write the equation in the form $y = x^2 + (-10x) + 24$.

- assuming a negative value for the y-coordinate does not make sense in the context of the problem

  Tell students that a safe height occurs when $y \geq 0$, not when $y \geq 6$ because y represents the distance from the safe height, not the distance from the ground.

- assuming the initial height occurs when $y = 0$

  Explain that the initial height occurs when the time is 0, so it occurs when $x = 0$. Remind students that the initial height is not 0. Nor is it 6. The initial height is the height from which Shin jumps.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students say that the minimum height is 0, ask them to describe the path of a hang glider. (**Answer:** A hang glider starts from a height well above 6 feet. Then the glider dips to no more than 6 feet and rises again. The only time the height is 0 is when the glider lands. The landing is not discussed in this model.)
- If students are unsure of where the minimum value occurs with respect to the key features of the function, have them graph the function and describe where they think the minimum value appears graphically. (**Answer:** at the vertex)
- If students say that the value of –1 does not make sense, ask them to describe what a *y*-value of 0 means. (**Answer:** The glider is at the safe height.)
- If students say that *x*-intercepts represent when the glider is at 0 feet, review what *x* and *y* represent in the context of the problem. (**Answer:** *x* represents the number of seconds since Shin jumps, and *y* represents the distance from the safe height. So when *y* is 0, the *x*-values are the times when the glider is at the safe height.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Compare solving the problem algebraically and graphically. Ask students to decide which method they prefer and why. Discuss how one method can be used to solve the problem and the other method can be used to check your work.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Recall that the standard form, or general form, of a quadratic function is written as $f(x) = ax^2 + bx + c$, where a is the coefficient of the quadratic term, b is the coefficient of the linear term, and c is the constant term.
- The vertex of a function can be found by first determining the value of x, $x = \frac{-b}{2a}$, and then finding the corresponding y-value, $y = f\left(\frac{-b}{2a}\right)$.

  Students use the standard form of the function to identify the vertex.
- Recall that the *x*-intercepts of a function are the *x*-values where the function is 0.

  Students use the *x*-intercepts to identify when the glider is at the safe height.

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Extending the Task

To extend the task, have students analyze the jump in more detail. For example:

- Have students draw a graph of the height of the glider with respect to the time. Ask, "How does that graph compare to the graph of the function for the safe height?" Discuss whether the graph accurately represents all phases of the jump.
- Have students write a function that models a safe hang gliding ride, one that does not go below the safe height.
- Show one or more videos of hang gliding from the following website:

  Vimeo.com. "Videos Tagged 'Hang Gliding'."

  http://www.walch.com/rr/07011

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I reread the problem, listing what I knew. I made a graph to help me understand the problem.)
- **For SMP 3, ASK:** "Did you construct viable arguments and did you critique the reasoning of others?" (**Answer:** I determined whether a negative value made sense for the problem and justified my answer.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I graphed the function to help me better understand the problem and what I was asked to find.)

Alternate Strategies or Solutions

This task could first be explored graphically instead of algebraically with students using the graph to answer the questions.

Technology

Students can use a graphing calculator or online software to graph the function.

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Problem-Based Task 2.2.1: One-on-One Basketball

Common Core State Standard

F–IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

Task Overview

Focus

Have you ever wondered the path a basketball travels when shooting it into a basketball hoop? Students will create a quadratic model to determine the path the ball travels. They will also determine when the ball is increasing and decreasing as well as reaching its maximum height.

This activity will provide practice with:

- graphing a quadratic model
- analyzing when the model is increasing and decreasing
- interpreting the maximum height of the object

Introduction

This task should be used to explore or to apply key features of quadratic functions. It is best to implement this task after students learn the various forms of a quadratic function, intercepts, maximum or minimum, increasing or decreasing, concave up or down, and other terms associated with these functions. Graphing may be done using technology, including graphing calculators or online graphing programs.

Begin by reading the problem and clarifying the meaning of path of the basketball. Explain that when shooting a basketball the ball often travels on a parabolic (quadratic) path.

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Some students will have difficulty translating the problem and/or interpreting from a graph when a function is increasing/decreasing and the maximum height. Encourage them to look at the ball height to determine when the object is increasing and decreasing. Encourage students to look at the pattern of the model to help determine when the ball is at its highest point.

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- **SMP 4:** Model with mathematics.

  Students can provide a model for the scenario by first plotting the given points for the graph and from there drawing a parabola. Some students will prefer using a model because it is easier for them to see the information in a visual context. Ask the students how the quadratic model can be modified for different distances.

- **SMP 6:** Attend to precision.

  Some students may struggle with finding the exact values of the maximum height as well as when the graph transitions from increasing to decreasing. Work with students on the exact calculation of the values as well as using a graph to provide accurate estimations.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- misinterpreting identifying when a function is increasing or decreasing

  Remind students that when the graph is going up, the values for y are increasing, and when the graph is going down, the values for y are decreasing.

- incorrectly identifying the model

  Point out to students the model is quadratic since the basketball is shot in the air and must come down.

- incorrectly identifying the axis of symmetry

  Suggest that students review the symmetric coordinates and recalculate the value.

- miscalculating the maximum height

  Remind students that the value can be approximate from the maximum value on the graph or plug in the value from the axis of symmetry into the quadratic model.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students are struggling with determining if there is enough information to draw conclusions, remind them to look at the given table. (**Answer:** Yes, there is enough information since the ball height in feet is increasing, then decreasing.)

- If students are struggling with the best method, remind them of what tool they can use to help them graph. (**Answer:** graphing calculator)

- If students are struggling to determine when the function is increasing and decreasing, remind them of how they can find the axis of symmetry. (**Answer:** find the average of the distance of the shooter at the same height)

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- If students are struggling to determine the maximum height, remind students that there are two options. (**Answer:** use a quadratic model or approximate the maximum height from a graph)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Compare the strategies and explanations used by different students. Ask students to share their information and ask how they came up with these approaches.

Connecting to Key Concepts

Make explicit connections to key concepts:

- A quadratic model displays an object that increases in height, then decreases in height.

  Using symmetric coordinates, the axis of symmetry can be calculated and tell when the graph changes from increasing to decreasing.
- In this task, the maximum value represents the highest point the basketball is when it is shot into the air.

Extending the Task

To extend the task, ask the following questions:

- What is the height of the ball when the ball is 0 feet from the shooter?
- What happens at the ordered pair (14, 9)? (14, 11)
- What is the distance from the shooter when the ball hits the ground?

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I kept reading the problem to determine the information that I needed. I broke down the problem using the key terms (increasing, decreasing, maximum).)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I created a model to help demonstrate information needed to determine increasing and decreasing.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I created an algebraic model for the graph. I extended the table to help create a problem first by finding the maximum height, and then working backward.)

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Alternate Strategies or Solutions

- Students can use a graphing utility to find the exact model and find the exact value of the maximum height.
- Students could find additional points for the table after finding the graph for the model.

Technology

Students can use graphing utilities to help create a quadratic model.

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Problem-Based Task 2.2.2: Window Washers

Common Core State Standard

F–IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

Task Overview

Focus

How can a model be determined from the given information of window washers? What is a reasonable domain for the time a sponge is tossed? Students will take given information from a problem and create a quadratic model. Then they'll take the model and determine the time the sponge reaches a certain height and a reasonable amount of time the sponge is in the air.

This activity will provide practice with:

- analyzing the known and unknown information
- interpreting information to create a quadratic model
- examining the equation to determine the domain of a quadratic model

Introduction

This task should be used to apply the concept of a creating a quadratic model from given information. The domain and the meaning of specific ordered pairs is also discussed in this task. It is best to implement this task after students are able to determine known and unknown information and create a quadratic model, as well as finding values from the model and the domain for a scenario.

Begin by reading the problem and clarifying what information is known and unknown as well as the model needed for the problem. Using the general model, the values can be determined for the model of the wet sponge thrown by the window washer. From there, the time when the object reaches 10 meters and a reasonable domain can be determined from the quadratic model. Review the following term:

window washer a person whose job is to wash windows

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Some students will have difficulty determining the known and unknown information and creating a quadratic model. Encourage the students to identify key statements given in the problem. If necessary, provide various models (linear, quadratic, etc.) and allow the student to choose the appropriate model.

- **SMP 2:** Reason abstractly and quantitatively.

  Some students will be able to find the domain without setting up the quadratic model. Allow this method as long as students can justify their explanations. Help students with the process of taking the verbal information and developing a quadratic model. Some students may use a table to help determine the domain of the model.

- **SMP 6:** Attend to precision.

  Students will explain in their own words on what the meaning of the known and unknown information is in the context of the problem. Students may need to be provided the vertex form of a quadratic equation to assist in solving the problem. The accuracy of the model is necessary to answer questions e, f, g, and h.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- identifying domain with output values instead of input values

  The input values of the equation are the *x*-values instead of the *y*-values.

- using a form of a quadratic model other than vertex form

  Provide students with the vertex form of a quadratic equation or a list for students to choose from.

- incorrectly calculating the number of seconds for the sponge to be at a height of 10 meters

  Advise students to find the vertex first or create a table of values from the model.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- What is the given information in the problem? (**Answer:** the maximum height, the vertex, and the initial height of the sponge)

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- What is the information that is not known? (**Answer:** other heights of the sponge besides the initial and the vertex, as well as the amount of time the sponge is in the air)
- What is the vertex form of a quadratic equation? (**Answer:** The equation is $y = a(x - h)^2 + k$, where (h, k) is the vertex.)
- How can the a-value be located for the model? (**Answer:** Substitute the vertex and the initial height ordered pairs into the equation and solve for a.)
- How can the height of 10 meters be determined when the sponge is falling? (**Answer:** Use a table or the axis of symmetry to find when the sponge is at a height of 10 feet.)
- How is the domain determined for the equation? (**Answer:** The domain is determined from the starting point of the sponge until the sponge lands on the ground.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Compare the strategies and the explanations used by various groups. Ask students to share the strategies they used and ask why they used these methods.

Connecting to Key Concepts

Make explicit connections to key concepts:

- A quadratic model can be created given two points. In this scenario, the vertex form of an equation is used to help determine the model.
- Using the given information, the data is turned into ordered pairs and identify as an initial height and the vertex.
- The applicable domain for this task is the time when the object is thrown into the air until the object hits the ground.

Extending the Task

To extend the task, add the following information. For example:

- A graph of the model is used to demonstrate the path of the thrown sponge.
- Find the range for the height of the sponge.
- Determine other values of the height of the sponge at a specific time.
- Change the initial height and/or vertex to create a new quadratic model.

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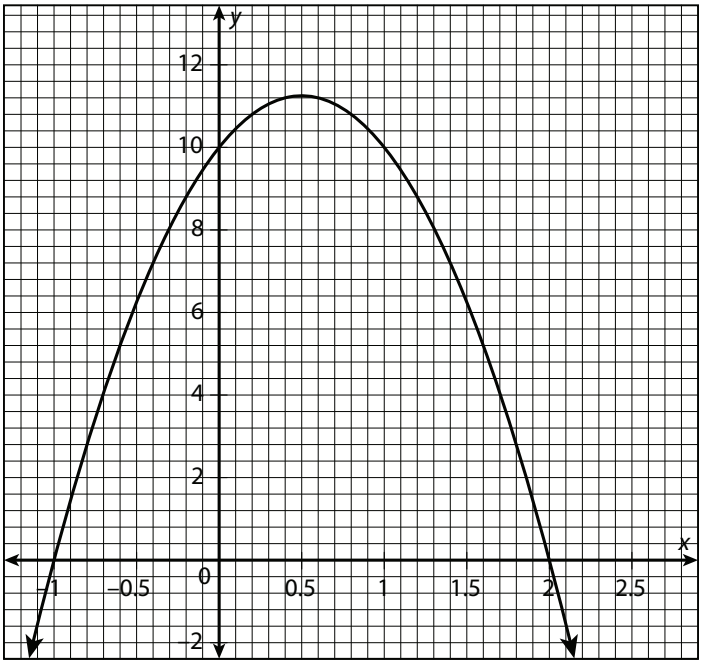
Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you demonstrate perseverance?" (**Answer:** I kept reviewing the problem until I determined the known and unknown information. I focused on one question at a time until the solution or explanation was completed.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I wrote the quadratic model from the given information.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used the vertex to help determine when the sponge would be at 10 feet off the ground. By using the *x*-intercept and *y*-intercept, I was able to find the domain of the equation.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I was able to explain the steps of the problem in order from determining the known information, to the type of model, to finding the correct model, and obtaining information from the model.)

Alternate Strategies or Solutions

A graph of the quadratic model can be created as shown:



Using the graph, questions g and h can be answered visually.

Technology

Students can use scientific calculators and graphing utilities.

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Problem-Based Task 2.2.3: Is the Maximum High Enough?

Common Core State Standard

F–IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

Task Overview

Focus

How does a football travel in the air when kicked? Is the ball kicked high enough to clear the goalpost? Students will determine what information is given from a quadratic model of kicking a ball. From this information, the average rate of change will be determined for different times the ball is in the air.

This activity will provide practice with:

- analyzing the quadratic path traveled by the football
- interpreting the known information from the quadratic function
- examining the average rate of change of the football

Introduction

This task should be used to explore the various properties of a quadratic function as well as calculating the rate of change for specific times.

Begin by reading the problem and clarifying the meaning of the quadratic function and explain the meaning of the key parts. Explain the basics of football and the importance of kicking a field goal for those students not aware. Review the following terms:

field goal	kick the ball through the goal posts for three points
goalpost	extended uprights supported by a cross bar
kicker	a football player who is responsible for kicking
play-offs	played after the regular season to determine a champion
specialist	player that performs a specific role
Super Bowl	championship game for the National Football League

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Some students will have difficulty determining the known and unknown information from the given quadratic model. Encourage students to identify underling key statements given in the problem. If necessary, provide a guide on how to interpret the function.

- **SMP 2:** Reason abstractly and quantitatively.

  Some students would be able to find the average rate of change without any demonstration. Allow this method as long as students can justify their explanations. Help students with the process of taking the verbal information and developing the meaning of the points from the quadratic function.

- **SMP 6:** Attend to precision.

  Students will explain in their own words on what the meaning of the known and unknown information is in the context of the problem. Students may need to be provided the vertex form of a quadratic function to assist in solving the problem. The accuracy of the interpretation is necessary for questions d and e.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- assuming the rate of change is the same for every interval of a quadratic function

  Explain to students the average rate of change is different for the two selected x-values.

- misinterpretation of the values of the quadratic function in vertex form

  Remind students of the general form of a quadratic function and the meaning of each variable.

- incorrectly calculating the rate of change

  Advise students that the rate of change is similar to the slope formula.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- What is the information that is not known? (**Answer:** the maximum height and the rate of change)
- What is the importance of the vertex? (**Answer:** The vertex tells us the time when the object is at the maximum height.)
- How can I tell if the football goes above the goalpost? (**Answer:** If the maximum height is greater than the height of the goalpost, then the ball clears the goalpost.)
- How is the average rate of change calculated for the problem? (**Answer:** The average rate of change is calculated using the ordered pair for the maximum height and the ordered pair when the football hits the ground.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Compare the strategies and explanations used by various groups. Ask students to share the strategies they used and ask why they used these methods.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The average rate of change is unique for each set of x-values used, and the formula is similar to using the slope formula.
- Using a given quadratic function, information about specific points (maximum, intercept(s), etc.) can be obtained.

Extending the Task

To extend the task, add the following information. For example:

- A graph of the model is used to demonstrate the path of the football kicked.
- Graph the equation $y = 17$ along with the quadratic function to determine graphically if the ball would make it over the goalpost.
- Change values in the quadratic function to make the football go over the goalpost.
- Find the average rate of change using different values.

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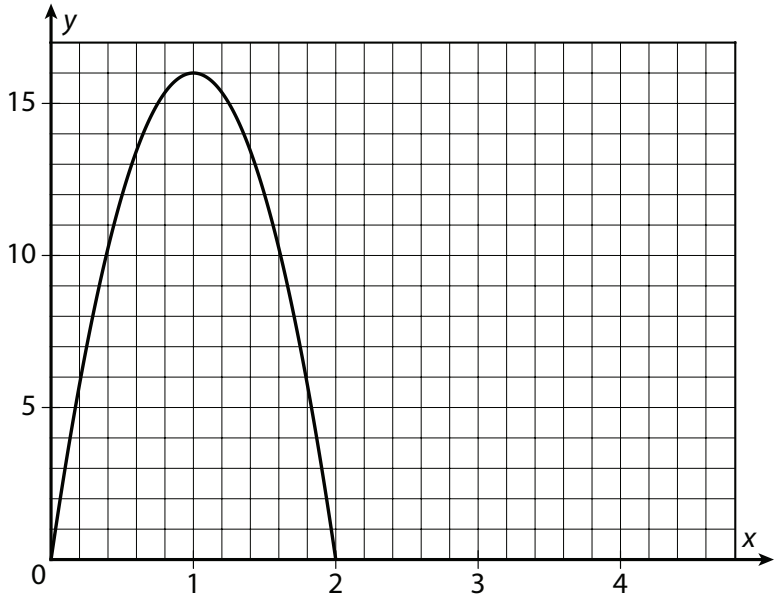
Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you demonstrate perseverance?" **Answer:** I kept reviewing the problem until I determined the known and unknown information. I focused on one question at a time until the solution or explanation was completed.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I determined what information was known and what information needed could be determined from the quadratic function.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used the quadratic model to determine the ordered pairs for the maximum height and when the football hits the ground.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I was able to explain the steps of the problem in order from determining the known information and obtaining information from the given function.)

Alternate Strategies or Solutions

- A graph of the quadratic function and the height of the goalpost can be created as follows:



Using the graph, questions d and e can be answered visually.

Technology

Students can use scientific calculators and graphing utilities (if graphing the function).

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Implementation Guide

Problem-Based Task 2.3.1: To Drill or Not to Drill?

Common Core State Standard

F–BF.1 Write a function that describes a relationship between two quantities.★

a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

Task Overview

Focus

Students will write and manipulate a quadratic function to find the number of oil wells that will produce the most profit in terms of numbers of barrels of oil per day. They will use the key features of the quadratic function to find the number of oil wells.

This activity will provide practice with:

- writing expressions to model real-world situations
- writing quadratic function to model real-world situations
- evaluating functions for a given value
- graphing quadratic functions
- analyzing and interpreting data that represent key features of quadratic functions

Introduction

This task should be used to explore or to apply modeling with quadratic functions. It is best to implement this task after students have learned and practiced modeling situations with quadratic functions. Graphing may be done using technology, including graphing calculators or online graphing programs.

Begin by reading the problem and clarifying the meaning of *profitable.* Review the following terms:

profit	money gained or earned
profitable	to make a profit

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 6:** Attend to precision.

  Some students may be intimidated by the large numbers for b and c in the function. Encourage students to check their work to ensure accurate calculations.

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- **SMP 7:** Look for and make use of structure.

  Students should begin to see a pattern when finding minimum or maximum values of a quadratic function. Have student use graphs of quadratic functions to reinforce the relationship between the vertex of a parabola and its maximum or minimum value.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  The problem is scaffolded for the students, but it is up to the student to work each step correctly. Encourage students to evaluate the reasonableness of each part of the problem as they go along.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- adding the expressions from parts a and b instead of multiplying them.

  Remind students that the number of barrels produced is the product of the number of wells and the number of barrels of oil that can be produced.

- substituting 75 for x instead of 0 to find the number of barrels produced with 75 wells

  Review what x and $f(x)$ represent in terms of the problem situation. x is the number of new wells, not the number of wells that currently exist. Some students will arrive at this number simply by multiplying 945×75. Encourage those students to use the function to check their work.

- not adding the original 75 wells to 120 to find the total number of wells

  Remind students that x represents the number of new wells and that there were already 75 existing wells.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students struggle getting started, tell them that the function that models the situation is a quadratic function. Ask whether they are finding a maximum value of a minimum value. (**Answer:** maximum)

  Ask what the graph of the function will look like. (**Answer:** a parabola that opens downward)

  Ask which point on the parabola represents the maximum value. (**Answer:** the vertex)

- If students have difficulty writing the two linear factors of the quadratic function, ask what the variable represents. (**Answer:** the number of new wells)

  Ask how you can calculate the number of barrels produced. (**Answer:** Multiply the number of wells by the amount of oil each well produces.)

  Lead students to understand that they need to write two expressions, one for the number of wells when x wells are added and one for the amount of oil produced when x wells are added.

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- If students are unclear about how to find the maximum number of barrels that can be produced, have them graph the function. Ask what point represents the maximum amount of oil. (**Answer:** the vertex)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask students to explain how they were able to write the linear factors. Ask what part of the problem statement helped them write them.
- Discuss how students knew the function had a maximum and what that maximum function value represented.

Connecting to Key Concepts

Make explicit connections to key concepts:

- A quadratic expression is the product of two linear factors.

  Students begin solving this problem by writing the two linear factors of a quadratic expression.

- The vertex of a parabola can be found using $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$ from the general form of the quadratic equation $f(x) = ax^2 + bx + c$.

  Students use the standard form of the function to identify the vertex.

- The vertex of a downward-facing (or concave down) parabola represents will be at the top or maximum of the parabola.

  In this task, students are finding a maximum. When the function is graphed, they can see that the parabola opens downward and that the vertex is the maximum.

Extending the Task

To extend the task, have students analyze the function in more detail. For example:

- Have students draw a graph of the function.
- Discuss what the vertex represents.
- Have students choose several points on the parabola and describe what they represent.
- Ask students why this problem is modeled by a quadratic function rather than a linear function.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I checked all of my calculations.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I wrote a quadratic function by first writing the two linear factors that comprise it.)
- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** As I solved the problem, I evaluated the answer to each part to ensure it was reasonable.)

Alternate Strategies or Solutions

This task could first be explored graphically instead of algebraically with students using the graph to find the maximum number of wells.

Technology

Students can use a graphing calculator or online software to graph the function.

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Problem-Based Task 2.3.2: Pushing Envelopes

Common Core State Standard

F–BF.1 Write a function that describes a relationship between two quantities.★

b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*

Task Overview

Focus

Students will build a function to model the amount of paper needed to make note cards and envelopes of different sizes. Students will build the function using operations on functions, specifically addition and multiplication of functions.

This activity will provide practice with:

- writing a function for a real-world situation
- performing arithmetic operations on functions to generate new functions
- analyzing a real-world problem to determine what functions and operations can be used to solve it

Introduction

This task should be used to explore building a function to solve a real-world problem by performing operations on other functions. It is best to implement this task after students have learned the rules for operations with functions.

Begin by reading the problem and reviewing how to find the area of a rectangle. Review the following terms:

area the measure of a two-dimensional region in square units

ratio a comparison of two numbers using division

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Some students may not know how to translate the information in the problem into symbolic representations and function notation. Encourage them to re-read the problem list what they know from the problem statement. Have them define the unknowns in the problem and what each represents.

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- **SMP 6:** Attend to precision.

  Students will use function notation in each step of the solution and need to be precise in their representations of the operations performed on the functions. If necessary, review the operations with functions and how they are represented symbolically.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- defining the functions in part a as $h(x) = 3$ and $w(x) = 5$

  Remind students that a height-to-width ratio of 3 : 5 does not mean the height is 3 and the width is 5. Without a variable there is no way to express different sizes of note cards. Review the problem statement so that students understand the company makes note cards of different sizes. Encourage students to draw a diagram to represent different possible sizes of cards using equivalent ratios.

- expressing $3x \cdot 5x$ as $15x$ or as $8x$

  Review the rules for adding like terms and for multiplying monomials. Show several examples of each operation.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students have difficulty defining the unknown in the problem, ask what words, phrases or sentences in the problem statement help identify the unknown. (**Answer:** "Various sizes," "Regardless of the size of the note card," "Any possible size combinations")
- If students have trouble writing the functions for part a, ask them to name several ratios that are equivalent to 3 : 5 (**Answer:** 6 : 10, 9 : 15). Then ask them to write the numbers in the ratio using factors of 3 and 5 (**Answer:** 3 • 2 : 5 • 2, 3 • 3 : 5 • 3). Use these expressions to help them understand that the expression for the functions are $3x$ and $5x$.
- If students write the functions in part c using different names than in part a, tell them they should write the functions for the envelope in terms of the functions for the note cards.
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Use the final function to find the amount of paper used for various sizes of cards and envelopes. Discuss why this function would be useful to a company that makes stationery and lead students to understand that mathematical functions are actually used in manufacturing and other real-world businesses. Discuss other businesses that might use math equations in their day-to-day operations. Discuss why it is so important to be precise when building functions that will be used in the real world.

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Connecting to Key Concepts

Make explicit connections to key concepts:

- Functions can be added, subtracted, multiplied, or divided.

  Adding and multiplying functions is necessary to solve the problem. Students build a function using operations on functions.

- For two functions $f(x)$ and $g(x)$, the addition of the functions is represented as follows: $(f + g)(x) = f(x) + g(x)$.

  Students perform addition of functions to build the function for the total amount of paper needed for a note card and envelope.

- For two functions $f(x)$ and $g(x)$, the multiplication of the functions is represented as follows: $(f \cdot g)(x) = f(x) \cdot g(x)$.

  Students perform multiplication of functions to build the function for the total amount of paper needed for a note card and envelope.

Extending the Task

To extend the task, change the shape of the note cards and envelopes. For example:

- Change the shape to a square.
- Change the shape to a triangle.
- Change the shape to a circle.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I reread the problem until I understood what I was asked to find. Then I made sure I understood each step in the solution as I went along.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I used the correct symbols and operations when writing and building functions.)

Alternate Strategies or Solutions

This problem can be explored and solved by drawing a diagram to model the cards and envelopes and writing expressions for the area of each. This may lead to a better conceptual understanding of the final function.

Technology

Students do not need technology for this task.

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Problem-Based Task 2.4.1: Auto Acceleration

Common Core State Standard

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

Task Overview

Focus

Auto manufacturers perform many tests to see how good their cars are. Students will evaluate the functions for the Accelerator and the VVI to compare the two models.

This activity will provide practice with:

- modeling the performance of cars with square root functions
- analyzing the domain and range of functions
- graphing square root functions
- comparing the graphs of two square root functions
- relating the algebraic expressions to the graphs of square root functions
- interpreting the model in terms of the context

Introduction

This task should be used to explore comparing square root functions. Graphing may be done using technology, including graphing calculators or online graphing programs. It is best to implement this task as students learn to evaluate the equations and graphs of square root functions.

Begin by reading the problem and clarifying what a car's acceleration is. Have students imagine being in a stopped car and then having the car go to 60 miles per hour. The process of changing the speed is called acceleration. Review the following terms:

manufacturers who makes the cars

accelerate to change speeds

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will consider the model functions and determine what is reasonable for the domain. They will also find the corresponding range. After comparing the graphs of the two cars, they explain how the differences in the functions relate to the differences in the cars.

- **SMP 4:** Model with mathematics.

  Students use an equation to model the acceleration of a car. Then, they use the equation to create a graphical model.

- **SMP 5:** Use appropriate tools strategically.

  Students use grids to graph the equations using points that satisfy the equation. They also use the graphs to help them evaluate the similarities and differences in the functions.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- plotting too few points on a graph to understand the behavior of the function on the graphed domain and range

  Remind students that, for a square root function, they can find the range by substituting the minimum and maximum values of the domain into the function. Using these critical points, plus one more, they can create the graph. Encourage them to find more points if they are not sure.

- incorrectly calculating square roots when identifying points on a graph

  Suggest that students refresh the order of operations. Remind them that the radical acts like parentheses, with it being done first and then being multiplied by the number in front of it.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students struggle to find the domain and range:
  - "What does each variable stand for?" (**Answer**: m is the number of meters a car travels in t seconds)
  - "How do you determine a reasonable domain?" (**Answer**: Since time cannot be negative, both domains are restricted to positive numbers. Since both sample times are less than 10 seconds, 10 seconds is probably a good upper limit on the domain.)
  - "How will you find the range on the restricted domain?" (**Answer**: Substitute in the lower and upper bounds for the domain into the functions to find the corresponding range values.)

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- If students cannot graph the functions:
  - "What does the graph of a square root function look like?" (**Answer**: It is always increasing or always decreasing, and it is a curve.)
  - "What points have you already found for this function?" (**Answer**: Points at the minimum domain value and at the maximum domain value.)
  - "How can you find other points?" (**Answer**: Substitute an *x*-value within the domain to find it corresponding *y*-value.)
  - "How do you make a graph?" (**Answer**: Plot these three points on the grid and connect them.)
- If students don't understand how to compare the functions:
  - "What is the same for both graphs?" (**Answer**: the shape and the minimum point)
  - "What is the different for both graphs?" (**Answer**: One graph is always higher than the other, except for at (0, 0).)
  - "How are these comparisons shown by the algebraic representations?" (**Answer**: The VVI's function has a greater coefficient, which makes it higher on the graph. The positive coefficients for each make both graphs increasing. Since there are no values added or subtracted within the radical, both graphs start at (0, 0).)
- If students are unable to understand how this relates to the cars:
  - "Which car will travel farther than the other in the same amount of time?" (**Answer**: the VVI)
  - "Which car will always be ahead of the other?" (**Answer**: the VVI)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Have students discuss how they came to a conclusion about the similarities and differences in the cars.

Connecting to Key Concepts

Make explicit connections to key concepts:

- A restricted domain of a function is a subset of a function's defined domain. A restricted range of a function is a subset of a function's defined range.

  In this case, the domain and range must be restricted because negative values do not make sense in the context of the problems.

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- The general shape of a square root function is always increasing or always decreasing.

  By finding the minimum and maximum points on the restricted domain, and knowing the shape of the function, the graph can be sketched with only one additional point.

- To graph a square root function, identify the domain and range, plot the minimum and maximum of the function, and plot a few additional points to identify the shape of the function.

  Students complete and compare the graphs of two radical functions using this process.

Extending the Task

To extend the task, change the parameters of the task. For example:

- Change the coefficients for each car.
- Ask students to write equations for cars that accelerate faster or slower than the others.
- Ask students to write equations for cars that do not start at (0, 0).
- Give students equations for other cars. Based on the function equation, ask them to describe the performance compared to other cars.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I determined a reasonable range based on the context of the task.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used the equation to find points to plot so that I could draw an accurate graph.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used the equation of the function as well as its graph to analyze the cars. Both of these are models.)
- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** Rather than try to sketch the graph, I used grid paper so that my graph was more accurate.)

Alternate Strategies or Solutions

- Students can find more than three critical points to graph the function.
- Students graph the functions based on the transformation on the parent function.
- Students can make a table of values to find the points on the graph.

Technology

Students can use a graphing calculator to graph the functions and use those graphs to compare. They can also use a table to find the minimum and maximum values of the range.

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Problem-Based Task 2.4.2: Fancy Wheels

Common Core State Standard

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

Task Overview

Focus

How can a step function be applied to model a real-world scenario? How can the key features of two step functions be compared? In this task, students will determine the domains and ranges of two step functions which represent the costs of renting a luxury limo and a sedan, and they will graph each function and analyze the results to determine which type of car is more expensive to rent.

This activity will provide practice with:

- identifying domains and ranges of step functions
- determining domain and range restrictions in the context of the scenario
- graphing step functions using technology
- comparing graphs of two step functions on the same coordinate plane
- substituting values into step functions
- performing cost analysis of two rental options to determine the least expensive option

Introduction

This task should be used to explore or to apply the skills of graphing step functions expressed symbolically and showing key features of the graph, by hand in simple cases and using technology for more complicated cases. Students should already be familiar with finding domains and ranges of functions and determining restricted domains and ranges for application problems.

Begin by reading the problem and clarifying the meaning of *domain*, *greatest integer function*, *range*, and *step function*.

domain	the set of x-values that are valid for a relation or function
greatest integer function	also known as the floor function; a function represented as $y = \lfloor x \rfloor$. For any input x, the output is the largest integer less than or equal to x; for example, $\lfloor -3 \rfloor = -3$, $\lfloor 2.1 \rfloor = 2$, and $\lfloor -2.1 \rfloor = -3$.
range	the set of y-values for which a function is defined
step function	a function that is a series of disconnected constant functions, and whose graph resembles a staircase

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they recognize that the functions given to represent the costs of renting two different vehicles are step functions, which indicate that the values in the range of the scenario are integers. They will also reason abstractly as they identify the domain and range restrictions on the functions, based on the context of the scenario. They will reason quantitatively as they substitute the domain values in for each function to determine the range values. They will also reason quantitatively as they compare the costs of the two types of rental car options, based on the range values resulting from the same domain values. They will evaluate each step function from the graph of each, in terms of the cost at specific domain values.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled with step functions, specifically greatest integer functions. They will recognize that the two step functions represent the cost of renting a luxury limo and the cost of renting a sedan. They will first determine the domains and ranges of both functions and identify the restrictions on them, based on the context of the scenario. Then they will graph both functions on the same coordinate plane to analyze the graphical representation of each. Once students have calculated the range values of the functions for specific domain values, they will analyze and interpret the results in the context of the scenario and determine which rental car option is the least expensive.

- **SMP 5:** Use appropriate tools strategically.

  Students will use a graphing calculator, or other graphing technology, to graph two step functions on the same coordinate plane so they can compare the key features of each function. They will follow the steps and procedures outlined in the task, and they will enter the expressions for each function, as well as denoting the scale for each axis. Then they will use the graphical displays of each function to analyze the range values at specific domain values, to analyze the cost of each rental car option.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly identifying integers greater than or less than negative values

  Have students write the definitions of a greatest integer function and a least integer function at the top of their papers. Remind them that values for the greatest integer function are rounded down, and values for the least integer function are rounded up. Work through several examples of each function using numerical values.

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- connecting steps on a step function instead of showing the jump between output values

  Review the basic graphs of the greatest integer function and the least integer function. Remind students that since the range values must be integer values, the graph will be disjointed, or broken between integer values on the *y*-axis. Therefore, there is no graph in between the integer values on the *y*-axis.

- not recognizing the restrictions on the domain and range in the context of the scenario

  Remind students to review the given information in the task carefully. Discuss the types of real-life scenarios in which domain values cannot be negative, such as distance and time. Review inequality symbols and work through examples of writing a restricted domain that use phrases such as "no more than" and "at least."

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, review the greatest integer function with them. Ask them, "Why do you think that a greatest integer function is also called a floor function?" (**Answer:** One reason a greatest integer function is also called a floor function is because the output value is the largest integer value less than or equal to the corresponding input value. It is the rounded down value of the input value. The floor terminology implies the bottom of something, or down.)

- After students have read the given information in the task, ask them, "Why do you think a step function, and not a standard linear function, is used in this scenario to represent the two rental car options?" (**Answer:** If a linear function was used to represent the rental car options, then the output values could be decimals. It might be easier for the car rental company to handle costs in whole dollars, instead of having to handle coins.)

- Before students begin the task, ask them to examine both functions and make a prediction. Ask them, "Based on the values represented in the functions, which rental car option do you think will be more expensive, and why?" (**Answer:** Answers may vary. Some students may think the function representing the luxury limo will be the more expensive option because the constant, 10, is greater than the constant on the function for the sedan, which is 2. Some students may think the function representing the sedan will be more expensive because the coefficient with the variable, 0.5, is greater than the coefficient with the variable for the luxury limo, which is 0.2.)

- As students are working through the task, discuss restricted domains and ranges with them. Ask them, "Why is the domain in this scenario restricted? Why does it have an upper bound?" (**Answer:** Since the domain of the scenario represents the number of miles driven by a car, it must be greater than or equal to 0, since distance is not negative. The domain has an upper bound because the problem states that no student will travel more than 25 miles, therefore the greatest domain value possible is 25.)

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- As an extension to the previous question, ask students, "How is the range of the functions determined? Explain." (**Answer:** The range of the functions is determined by substituting the least and greatest domain values into the functions and calculating the output values, or the range values. Since 0 and 25 represent the least and greatest domain values, they are substituted into each function. The output values determine the restricted range for each function.)
- Ensure students' understanding of the graph of a step function. Ask them, "For the function representing the cost to rent the luxury limo, $f(x) = \lfloor 0.2x \rfloor + 10$, why does a domain value of 5 display an open and a closed circle on the graph? Explain in terms of intervals on the graph." (**Answer:** Since the function is a step function, there are disjointed sections of the graph at specific intervals. When the value of 5 is substituted in the function, the result is 11, which is represented on the graph with a closed circle at the point (5, 11). This value is represented with the interval $5 \leq x < 10$. The open circle below, at the point (5, 10), represents the value at the interval of the function $0 \leq x < 5$.)
- Discuss how step functions are similar to linear functions. Ask students, "Compare the formats of the step functions given in the problem to the slope-intercept form of a linear function. How are the formats similar in terms of what the values of each term represent?" (**Answer:** The integer expression in each of the step functions is similar to the expression representing the slope of a line for a linear function. The larger the value of the coefficient with the variable, the greater rate at which the function increases, or rises. The value of the constant for both the step functions and linear function indicate the *y*-intercepts on the graph.)
- Ask students, "Is there a specific number of miles in which the cost of both rental car options is the same? Explain how you know." (**Answer:** By examining the graphs of the two functions, it can be identified that between the domain values of 24 and 25 miles, the cost of both rental cars is the same. The graph shows the overlapping of the steps of the two functions between these values.)
- After analyzing the graphs, and with the exception of the data points between 24 and 25 miles, which rental car option is more expensive? Explain." (**Answer:** The graphs show that although the rate of renting the sedan is higher than the rate of the luxury limo, overall the luxury limo is more expensive. The graph shows that the cost of the sedan is less than the cost of the luxury limo for every possible mile value except between 24 and 25 miles.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask for volunteers to discuss their initial thoughts on how to accomplish the goal of the task. Discuss the given information and what each term in the two step functions represented in the context of the scenario. Ask students to explain their initial predictions of which rental car option was more expensive and why. Discuss how the functions could be graphed by hand in terms of the simplicity of the values. Also discuss any difficulty that may have arisen with using graphing technology to graph the two functions.
- Ask students to explain how the intervals for the steps were created for each function. Also discuss how to write each interval using proper inequality notation. Ask students to explain how the restrictions for the domain and range were determined, based on the context of the scenario. Follow up with the connection between the domains and ranges and how they are indicated on the graphs representing the functions.
- Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task. Compare students' strategies and ways of justifying responses.
- Ask students to share their reasoning process and thoughts about how they determined the restricted domains and ranges for each function. Also ask them to explain how they determined if an open circle or a closed circle was used at the lower and upper bounds of each interval. Discuss the cost analysis for each rental car option, and encourage students to defend their claim that either the luxury limo or the sedan was the better choice.
- Focus on the use of precise mathematical language and clarity, specifically when discussing the integer expression in each step function, as well as how the intervals for each function were displayed on the graph using specific notation. Also focus on the correct terminology when referring to the steps of each function and what each of the domain and range values represent in the steps.

Connecting to Key Concepts

Make explicit connections to key concepts:

- A step function is a function that is a series of disconnected constant functions. Each x-value is assigned only one y-value on the function. The graph of a step function resembles a staircase.

  In this task, students will apply step functions that are used to represent the cost of rental cars in the real-world scenario presented in the task. They will determine the one-to-one correspondence of the x-values to the y-values, and they will observe the staircase look of the functions when graphed.

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- One example of a step function is the greatest integer function, or floor function, represented using $\lfloor \ \rfloor$: $y = \lfloor x \rfloor$

  Students will recognize from the given notation that the type of functions represented in this task are greatest integer functions.

- The domain of the greatest integer function is all real numbers, since the function can be evaluated at any input.

  Students will identify the domain of the greatest integer function represented in the task as all real numbers from 0 to 25. Since the domain represents the number of miles driven, the domain can be represented by rational numbers. They will recognize that the domain is restricted to and upper limit of 25 miles, based on the information given in the task.

- The steps of a greatest integer function are either always increasing or always decreasing.

  Based on the context of the given task, students will recognize that the steps in both greatest integer functions are increasing. They will identify this by the visual representation of the graphs of both functions.

- A graphing calculator can be used to graph the greatest integer function.

  For this task, students will use calculators, or other graphing technology, and graph both step functions on the same coordinate plane to compare them.

Extending the Task

- To extend the task, ask students to rewrite each of the step functions as linear functions. Then, ask them to graph the linear functions on the same graphs as the two step functions that have already been graphed. Ask them to write their observations about the "location" of the linear graphs in comparison to the graphs of the step functions. Ask for volunteers to share their results and their observations. Discuss the types of data values that are represented by the linear functions that are not represented by the step functions. Encourage a discussion about which type of function would be more practical in a real-world context for this type of situation. Discuss any difficulties that students may have experienced with the extended task, and allow students to help each other with error analysis.

- Another option for extending the task is to ask students to work with a partner to create a new real-world scenario involving the least integer function, represented with the notation, $y = \lceil x \rceil$. Ask them to define a least integer function in their own words and use this definition when creating a problem in which items or situations are represented by least integer step functions. Ask for volunteers to share their problems and discuss the application of this type of function.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first recognized that the functions given to represent the costs of renting two different vehicles were step functions, which indicated that the values in the scenario were integers. I also reasoned abstractly as I identified the domain and range restrictions on the functions, based on the context of the scenario.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I reasoned quantitatively as I substituted the domain values in for each function to determine the range values. I also reasoned quantitatively as I compared the costs of the two types of rental options, based on the range values resulting from the same domain values. I evaluated each step function from the graph of each, in terms of the cost at specific domain values.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by first recognizing that the scenario can be modeled with step functions, specifically greatest integer functions. I recognized that the two-step functions represented the cost of renting a luxury limo and the cost of renting a sedan. I first determined the domains and ranges of both functions and identified the restrictions on them, based on the context of the scenario. Then I graphed both functions on the same coordinate plane to analyze the graphical representation of each. Once I calculated the range values of the functions for specific domain values, I analyzed and interpreted the results in the context of the scenario and determined which rental option was the least expensive.)
- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** I used appropriate tools strategically by using a graphing calculator to graph two step functions on the same coordinate plane, so I could compare the key features of each function. I followed the steps and procedures outlined in the task, and I entered the expressions for each function, as well as denoting the scale for each axis. Then I used the graphical displays of each function to analyze the range values at specific domain values, to analyze the cost of each rental car option.)

Alternate Strategies or Solutions

When solving this problem, some students may graph the functions by hand instead of, or in addition to, graphing them on the calculator. If so, encourage them to choose several domain values, with some of them being non-integer values. Discuss the restrictions on the range values and have students explain, using their calculations as evidence, why the range values are always integers in this task. Some students may also determine the pattern of the graph of each function without graphing it on the calculator or calculating many points. Remind students to check the endpoints of each interval to determine the correct notation on the graph.

Technology

Students can use calculators or other graphing technology for this task.

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Problem-Based Task 2.4.3: Stop and Go

Common Core State Standard

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

Task Overview

Focus

How can a scenario which is represented by two different types of functions be shown algebraically and graphically? How can a piecewise function be used to find the output values for a variety of domain values? In this task, students will create graphs of a piecewise function and a quadratic function which represent the time and distance of a car as it accelerates after stopping at a red light. They will compare these functions and analyze them for different domain values of time.

This activity will provide practice with:

- identifying the domain of various functions
- creating a graph of a piecewise function
- evaluating a function given specific domain values
- comparing functions graphically and algebraically
- describing the differences between two functions verbally
- interpreting the differences between functions in the context of a scenario

Introduction

This task should be used to explore or to apply the skill of graphing functions expressed symbolically and comparing graphs of piecewise, linear, and quadratic functions. Students should already be familiar with plotting points in order to graph functions as well as determining the domain of a function.

Begin by reading the problem and clarifying the meaning of *domain, linear function, piecewise function,* and *quadratic function*.

domain	the set of all input values (x-values) that satisfy the given function without restriction
linear function	a function that can be written in the form $f(x) = mx + b$, in which m is the slope, b is the y-intercept, and the graph is a straight line
piecewise function	a function that is defined by two or more expressions on separate portions of the domain
quadratic function	a function that can be written in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$. The graph of any quadratic function is a parabola.

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will be reasoning abstractly as they make sense of the quantities represented in the scenario. They will identify that the independent variable is the time in seconds that the car is traveling, and they will identify that the dependent variable is the total distance the car travels. Students will be reasoning quantitatively as they evaluate the given functions and calculate the total distance the car travels for specific values of time.

- **SMP 4:** Model with mathematics.

  Students will recognize that the scenario can be modeled by the algebraic representation of the given piecewise function and quadratic function, and they will recognize that the scenario can also be modeled graphically. They will analyze the graphic and algebraic representations of the piecewise function representing the total distance traveled if the driver drives at a steady speed after initially accelerating, and they will compare it to the function representing the total distance traveled if the driver continues to accelerate.

- **SMP 6:** Attend to precision.

  Make sure students are using math terms and definitions clearly and accurately to explain their reasoning. Encourage them to analyze the given functions and their domains carefully, and also encourage them to evaluate the functions for the given domain values using order of operations correctly. Strive towards precise mathematical language, such as using the correct terms for domain, linear function, quadratic function, and piecewise function. Also remind students to carefully label the axes on their graph.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- defining one point on a domain with two different expressions, even if the expressions are equivalent at that point

  Remind students that in a piecewise function, it is necessary to closely examine the inequality symbols representing the different domain values, and that a symbol of $<$ or $>$ does not include the value, but a symbol of $\leq$ or $\geq$ does include the value.

- connecting graphs of different components of a piecewise function that should be disconnected

  Remind students that when graphing piecewise functions, be sure to mark each endpoint clearly. If the endpoint is defined for the function, use a closed dot. If the endpoint is undefined for the function, use an open dot.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "When an object accelerates, is this going to be represented by a linear or a non-linear function?" (**Answer:** Since the word "accelerate" means to "go faster," a function which represents acceleration will be a non-linear function, because an object will start off slow and then increase its speed at a non-linear rate.)
- As students are examining the information given in the task, ask them, "In the given piecewise function, which piece of the function will the domain value of $t = 10$ fall under, and why?" (**Answer:** The domain value of $t = 10$ will fall under the first piece of the function, which is $1.34t^2$, because in the domain interval $0 \leq t \leq 10$ the 10 is included because of the less than or equal to inequality symbol, therefore it is defined for this piece of the function. In the other piece of the function, the domain is $t > 10$, so the 10 is not included.)
- As students begin the task, ask them, "How can you determine a reasonable domain for both pieces of the piecewise function in the context of the scenario?" (**Answer:** The domain for both functions must cover all possible domains for both pieces of the piecewise function. The smallest value of the domain is 0, since time cannot be a negative number. The largest value of the domain has to be a value greater than 10 in order to cover the second piece of the function, which has a domain of $t > 10$. In the context of the problem, somewhere around a minute, or 60 seconds, might be a good domain for both pieces: $0 \leq t \leq 60$. However, answers may vary, as $0 \leq t \leq 30$, or $0 \leq t \leq 45$ are also viable domains.)
- As students begin graphing the piecewise function, ask them, "What does the first piece, which is the quadratic function $1.34t^2$, represent in the context of the scenario, and what does the second piece, which is the linear function $26.8t - 134$, represent?" (**Answer:** The quadratic function represents the car's acceleration after the light turns green, and the linear function represents the car driving at a steady speed after it has accelerated and has reached a constant speed.)
- Ask students, "When graphing quadratic and linear functions, what is the minimum number of points that is needed for each function in order to get an appropriate representation of the shape of the graph?" (**Answer:** For a linear function, a minimum of two points is necessary, because the shape of linear function is a straight line. However, for a quadratic function, at least 3 or 4 points should be graphed, since the graph is the shape of a curve.)
- As students graph the piecewise function, ask them, "How is the domain value of $t = 10$ represented on the graph?" (**Answer:** The domain value of $t = 10$ is defined by the first piece of the piecewise function. When this value is substituted into the function $1.34t^2$, the result is 134, so the point (10, 134) is represented by a closed dot on the graph, since 10 is included in the domain.)

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- If students have difficulty understanding the difference between the two functions d_1 and d_2, ask them, "Which function is increasing at a faster rate and why?" (**Answer:** The d_2 function is increasing at a faster rate than the d_1 function because d_2 is a quadratic function, and d_1 is a piecewise function which has a quadratic piece, but from the domain of $t > 10$, the d_1 function is linear, and it is increasing at a steady rate.)
- Ask students, "In the context of the scenario, which function is more practical and realistic to represent the situation, and why?" (**Answer:** The d_1 function is the more practical and realistic function in the context of the scenario, because it is unlikely that a driver will continue to accelerate after a certain number of seconds, as shown by the d_2 function, where the acceleration continues. Most likely, a driver will accelerate until a certain speed is reached, and then he/she will drive at a constant speed.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their processes when graphing the functions, as well as their reasoning when comparing the functions both algebraically and graphically. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Focus on the use of precise mathematical language and clarity, specifically when referring to the terms piecewise function, linear function, quadratic function, and domain.

Connecting to Key Concepts

Make explicit connections to key concepts:

- A piecewise function is a function that is defined by two or more expressions on separate portions of the domain.

  In this task, students are given a piecewise function to represent a scenario in which the total distance a car travels is represented by a quadratic function for one domain interval, and then the distance is represented by a linear function for a second domain interval.

- To graph a piecewise function, graph each portion of the function on its restricted domain.

  Students will create a graph to represent the piecewise function in which the first part of the function is represented by $1.34t^2$, with a domain interval is $0 \le t \le 10$, and the second part of the function is represented by $26.8t - 134$, with a domain interval of $t > 10$.

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- One piece of a piecewise function can be for a single input value, an interval of values, or for infinitely many values.

  In this task, students will see that one piece of the piecewise function is an interval of values, $0 \leq t \leq 10$, which represents all values between 0 and 10, and the other piece of the function contains infinitely many values, as $t > 10$ represents all values of t greater than 10.

Extending the Task

- To extend the task, provide students with an additional scenario such that after driving for 20 seconds, the driver of the car stops at another red light, which lasts for ten seconds, and then the driver accelerates again for another ten seconds before driving at a steady speed. Ask students to create a piecewise function to represent this scenario, and then graph the function on a new graph. Discuss what the piece of the function that represents the driver stopped at a red light would look like.
- Another option for extending the task is to provide students with another piecewise function that represents another driver in the same scenario. However, the second driver has a car which accelerates faster, therefore the original acceleration function could be $1.42t^2$, for example, but the driver's linear function has a smaller slope than the first driver, $24t - 1.42$, for example. Ask students to graph the piecewise function for the second driver and compare it to the first driver's piecewise function.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** When I identified the variables in this scenario, I used abstract reasoning.)

  "Which of your strategies represent quantitative reasoning?" (**Answer:** When I evaluated the functions for specific values of time, I used quantitative reasoning.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I modeled this scenario by first analyzing the given algebraic representation of the piecewise function, as well as the additional quadratic function, and then I created a visual model of both functions by graphing them on the same coordinate plane.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision by using correct math terminology in my explanations and explaining my reasoning in a mathematically correct and precise way. I also carefully and precisely evaluated the functions for specific domain values.)

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Alternate Strategies or Solutions

- Students may choose to create a table of values for each piece of the piecewise function and find the slope of each piece based on the rates of change between consecutive output values in the table. Encourage them to check the rate of change between several consecutive output values, specifically on the quadratic function piece, in order to verify the shape of the graphs.
- Students may choose to graph the functions with a graphing calculator or by hand. Encourage them to carefully choose the intervals for their axes if graphing by hand, and also remind them to be as precise as possible when plotting points.

Technology

Students can use graphing calculators or other graphing technology for this task.

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Implementation Guide

Problem-Based Task 2.5.1: Which Car Is a Better Buy?

Common Core State Standard

F–IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, *and classify them as representing exponential growth or decay.*

Task Overview

Focus

How can exponential decay functions be used to represent a real-life scenario? Which components of functions can be compared in order to determine which function is decreasing at a greater value? In this task, students will analyze two exponential decay functions which represent the value of two different types of cars to determine which car will have a higher value in the future.

This activity will provide practice with:

- identifying quantities/components of exponential functions
- rewriting an exponential expression in a different format
- calculating the rate of change of an exponential function
- determining if an exponential function is a growth or decay function
- comparing two exponential functions
- interpreting the decay rate of two functions in order to determine which yields a higher value in the context of the scenario

Introduction

This task should be used to explore or to apply the skill of writing a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function, as well as comparing quantities in two exponential functions. Students should already be familiar with simplifying exponential expressions and applying order of operations to evaluate expressions.

Begin by reading the problem and clarifying the meaning of *decay rate, exponential decay model,* and *exponential function.*

decay rate	r in the exponential decay model $f(t) = a(1 - r)^t$
exponential decay model	an exponential function, $f(t) = a(1 - r)^t$, where $f(t)$ is the final output value at the end of t time periods, a is the initial value, r is the percent decrease per time period (expressed as a decimal), and t is the number of time periods
exponential function	a function with the general form $f(t) = ab^t$, where a is the initial value, b is the growth or decay factor, t is the time, and $f(t)$ is the final output value

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will be reasoning abstractly as they make sense of the quantities represented in the scenario. They will identify that the independent variable is the number of years, and they will identify that the dependent variable is the total value of the car. They will also make sense of the various quantities contained in the exponential decay model representing the two cars. Students will be reasoning quantitatively as they rewrite an exponential expression using a property of exponents, as well as when they calculate the yearly rate of change for the two cars.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will compare the two exponential decay functions which represent the two different types of cars. They will construct viable arguments about which car will have a higher value in the future, based on the decay rate of each car. Encourage students to discuss their arguments with each other and explain their reasoning if they do not agree with each other.

- **SMP 6:** Attend to precision.

  Make sure students are using math terms and definitions clearly and accurately to explain their reasoning. Encourage them to analyze the quantities of both functions carefully when constructing their reasoning. Strive towards precise mathematical language, such as using the correct terms for decay rate, rate of change, and exponential decay function. Also remind students to perform their calculations for rate of change carefully.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- interpreting the rate r as a constant linear rate

  Remind students that only a linear function has a constant rate of change, whereas an exponential function does not.

- incorrectly applying the order of operations to evaluate the expression ab^t

  Have students write the order of operations at the top of their paper to use as a reference when working through the task. Remind them that the exponent must be evaluated first and then multiplied by the initial value, a.

- identifying $r > 1$ as exponential decay and $r < 1$ as exponential growth

  Remind students that if a function is an exponential decay function, then the value of the decay factor, b, will be between 0 and 1, which will yield a decay rate expressed by $(1 - r)$. An exponential growth function will have a decay factor which is greater than 1, and the growth rate will be expressed by $(1 + r)$.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task:

  Ask them to make a prediction about which car they think will have a higher value in the future. Ask them, "Based on the given equations representing the Roadvana and the Savannah, which car do you think will have a higher value in the future? Explain your prediction." (**Answer:** Some students may predict that the Roadvana will have a higher value because the decimal in its equation, 0.90, is greater than the decimal in the equation for the Savannah, which is 0.86. Some students may predict that the Roadvana will have a higher value because the exponent in its equation, $2t$, is greater than the decimal in the equation for the Savannah, which is t.)

  Ask them, "In the context of the scenario, what will happen to the value of a car over time, and which type of exponential function can be used to represent this type of change?" (**Answer:** The value of a car depreciates, or gets less, over time. Something that decreases in value can be represented by an exponential decay function.)

- As students begin working on the task, ask them, "Why do you think it is helpful to rewrite the equation for the Roadvana as a simplified expression?" (**Answer:** In the equation for the Roadvana, the exponential part is $(0.90)^{2t}$, and in the equation for the Savannah, the exponential part is $(0.86)^t$. In order to be able to compare the quantities of both equations, $(0.90)^{2t}$ can be simplified such that only an exponent of t remains, which is also the exponent in the Savannah equation.)

- If students have difficulty understanding how $(0.90)^{2t}$ simplifies to $(0.81)^t$, ask them, "What property of exponents can be used to rewrite and simplify $(0.90)^{2t}$? Explain." (**Answer:** The Power of a Power Property allows the exponent to be broken down and rewritten such that the base can be raised to the second power: $(0.90)^{2t} = (0.90^2)^t = (0.90 \bullet 0.90)^t = (0.81)^t$.)

- If students have difficulty understanding how to find the rate of change for each car, remind them that the form of an exponential decay model is $f(t) = a(1 - r)^t$. Ask them, "Which value in the two equations for the cars, $v_1 = 22{,}000(0.81)^t$ and $v_2 = 22{,}000(0.86)^t$, represents r? (**Answer:** Since the rate must be a decimal between 0 and 1, the 0.81 and the 0.86 represent r in the exponential decay model.)

- Ask students, "When 1 is subtracted from the base of the exponent for each equation, what does the resulting value represent in the context of the scenario?" (**Answer:** The resulting value, which is changed from a decimal to a percent in the context of the scenario, represents the percent of decrease in value of the car, per year.)

- Ask students, "Once the percent of decrease values have been calculated for both cars, how can these values help Kristin make a decision about which car will have a higher value in the future? Explain." (**Answer:** The Roadvana has a percent of decrease of 19%, and the Savannah has a percent of decrease of 14%. This means that the Roadvana decreases in value faster, so it loses its value quicker than the Savannah. Therefore, the car which will have a higher value in the future is the Savannah.)

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- Ask students, "Based on the exponential decay models, what is the value of the Roadvana after 5 years, to the nearest dollar? What is the value of the Savannah after 5 years, to the nearest dollar? Explain how to determine these values." (**Answer:** In order to determine the value of each car after 5 years, substitute $t = 5$ into the equations for both functions. For the Roadvana: $f(5) = 22{,}000(0.81)^5 = 7{,}670.93 \approx \$7{,}671$. For the Savannah: $f(5) = 22{,}000(0.86)^5 = 10{,}349.39 \approx \$10{,}349$. So, the value of the Roadvana after 5 years is \$7,671, and the value of the Savannah after 5 years is \$10,349.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial predictions about which car they thought would have the higher value in the future. Encourage students to discuss their reasoning for their predictions, as well as how correct or incorrect they were based on the actual results of the task.
- Compare students' strategies and ways of justifying responses. Focus on the use of precise mathematical language and clarity, specifically when referring to calculating the rate of change and the exponential decay models used to represent each car.

Connecting to Key Concepts

Make explicit connections to key concepts:

- An exponential function is a function with the general form $f(t) = ab^t$, where a is the initial value, b is the growth or decay factor, t is the time, and $f(t)$ is the final output value.

  In this task, students are given two exponential functions which represent the value of two different cars. They will see that both cars have an initial value of \$22,000, the Roadvana has a decay factor of 0.81, and the Savannah has a decay factor of 0.86.
- An exponential function can be used to model exponential growth (increase) or decay (decrease) of quantities over time.

  Students will recognize that in the context of the scenario, the value of cars decrease over time, so an exponential decay function will be used to represent the value of each car.
- If $0 < b < 1$, then there is exponential decay, and b is the decay factor, or multiple by which a quantity decreases over time.

  Students will see that the value representing b in both functions is a decimal between 0 and 1, specifically 0.81 and 0.86, which confirms that the scenario is represented by exponential decay.
- A function that represents exponential decay is a decreasing function, in which the dependent variable decreases as the independent variable increases.

  In this task, students will see that as the independent variable increases, which is time in years, the dependent variable decreases, which is the value of the car.

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Extending the Task

- Provide students with a third type of car that has a different initial value, but a higher rate of decrease than the Roadvana or the Savannah. For example, the Carorama has an initial value of $25,500, but its percent of decrease is 22%. Ask students to create an exponential decay model for the Carorama. Then ask students to create a table of values for all three cars for 1 year, 2 years, 3 years, 4 years, and 5 years. Ask them to determine which car has the higher value for each year.
- Another option for extending the task is to ask students to work together and create a list of factors which are involved in determining the value of a car over time. Ask them to consider, "Why do some cars hold their value more than other types of cars? What factors contribute to the value of a car?" Also, ask students to make a list of other types of real-world objects which are represented by exponential decay. Ask for volunteers to share their answers and encourage and guide a discussion about the differences between exponential growth and exponential decay.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I identified the independent and dependent variables in the scenario, as well as when I made sense of the quantities in the exponential decay models.)
- "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning when I rewrote an exponential expression and then calculated the rates of change for the two functions.)
- **For SMP 3, ASK:** "Did you construct viable arguments and did you critique the reasoning of others?" (**Answer:** I constructed viable arguments by comparing the decay rates of the two exponential decay functions, and I was able to base my conclusion about which car has the higher future value based on the percent of decrease of the two functions.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision by using correct math terminology in my explanations and explaining my reasoning in a mathematically correct and precise way.)

Alternate Strategies or Solutions

- Once students have determined the exponential decay functions for each car, they may choose to create a table of values in order to determine which car will have the higher value in the future. It may help students to see the actual values of both cars after 1 year, 2 years, 3 years, etc.
- Some students may choose to graph the table of values for both functions, so they are able to visually compare the two functions. Discuss with students why the value of a car will never be 0, as graphically the curve will keep getting closer and closer to 0, but it will never touch 0 or go below it.

Technology

Students can use scientific calculators for this task.

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Problem-Based Task 2.5.2: On to Washington!

Common Core State Standards

F–IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

F–LE.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.★

Task Overview

Focus

How can a linear and exponential function be compared? What are the similarities and difference between the functions algebraically and graphically? In this task, students will create a table of values and a graph to compare the different types of functions which represent the total distance traveled by three trains in order to determine which train will arrive at a specific destination first.

This activity will provide practice with:

- creating a table of values for different functions
- calculating values using a function
- analyzing a graph of an exponential function
- comparing output values for three different functions
- interpreting data based on the table of values for three functions

Introduction

This task should be used to explore or to apply the skill of comparing properties of functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). Students should already be familiar with identifying and graphing linear, exponential, and quadratic functions.

Begin by reading the problem and clarifying the meaning of *exponential function, linear function,* and *quadratic function.*

exponential function	a function with the general form $f(t) = ab^t$, where a is the initial value, b is the growth or decay factor, t is the time, and $f(t)$ is the final output value
linear function	a function that can be written in the form $f(x) = mx + b$, in which m is the slope, b is the y-intercept, and the graph is a straight line
quadratic function	a function that can be written in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$. The graph of any quadratic function is a parabola.

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will be reasoning abstractly as they make sense of the quantities represented in the scenario. They will identify that the independent variable is the number of hours, and they will identify that the dependent variable is the total distance that the trains travel. They will also make sense of the different functions which represent the distance that the three trains travel. Students will be reasoning quantitatively as they use the functions to calculate the time that the different trains arrive.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will compare a linear function, a quadratic function, and an exponential function in order to draw conclusions about which train will arrive at the desired destination first. They will construct viable arguments about which train will arrive first, as well as which train has the least amount of travel time, based on their analysis of the table of values and the graph of the three functions. Encourage students to discuss their arguments with each other and explain their reasoning if they do not agree with each other.

- **SMP 4:** Model with mathematics.

  Students will recognize that the scenario can be modeled algebraically, graphically, and by a verbal description. They will analyze all three representations of the three functions representing the total distance traveled by three trains. They will compare the output values of the functions in order to make a determination about the time in which each train will arrive at the same destination.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- misidentifying the appropriate type of function (linear, quadratic, or exponential)

  Remind students that linear functions have a constant rate of change and a graph which is a straight line, and quadratic and exponential functions do not have a constant rate of change, and the graphs of those functions is a parabola and a curve, respectively.

- incorrectly determining the rate of change

  Remind students that the rate of change can be found by calculating the slope between two consecutive points. Have students write the slope formula on the top of their papers to use as a reference when calculating the rate of change.

- assuming that the rate of change of a function is linear by only referencing one interval

  Remind students that if the rate of change is linear, then the slope must be the same between any two pairs of consecutive points. If it is not the same between all pairs of points, then the function is non-linear.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task:

  Ask them what they know about Train 1, based only on its verbal description. Ask them, "What can you tell about Train 1 based on the given information, and why do you know this?" (**Answer:** The information states that Train 1 travels nonstop at 66 mph. Since the function that represents this information is continuous, and the rate at which the train travels is constant, the function representing Train 1 is a linear function.)

  Ask, "What formula is used to calculate the rate of change for a function?" (**Answer:** The slope formula, which is $m = \frac{y_2 - y_1}{x_2 - x_1}$, is used to calculate the rate of change for a function.)
  Ask them, "What can you conclude about the function which represents Train 2 based on the information in the table of values?" (**Answer:** The input values in the table of values for Train 2 are consecutive values of time, in increments of one hour. However, the difference in the distance traveled between any two consecutive values, or the output values, is not the same. Therefore, the function is not linear, and instead is either quadratic or exponential.)

- During the task:

  As students are examining the table of values for Train 2 and they are able to conclude that the function is not linear, ask them, "Do you think the function which represents Train 2 is quadratic? What is one way to determine this answer? (**Answer:** A quadratic function has a graph in the shape of a parabola. The table of values can be used to create a graph, and the graph will show a graph in the shape of a curve, and not a parabola, which indicates that the function is exponential.)

  Ask students, "How can the table of values for all three functions be used to identify the type of function for each train?" (**Answer:** The rate of change can be calculated for several pairs of consecutive points for each function. Once the rate of change is found, it can be shown that the first function has the same rate of change between any two points, therefore it is linear. The functions for Trains 2 and 3 do not have the same rate of change between pairs of consecutive points, therefore these functions are exponential.)

  As students create a table of values for the three functions, ask them, "From the table of values, how can you determine which train has the least amount of travel time out of the three trains?" (**Answer:** Since the output values for Train 3 contain 0's for the first three hours, this means that Train 3 does not leave until 12 P.M. The other two trains have values for the distance traveled before this, so Train 3 has the least amount of travel time out of the three trains.)

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- If students are having difficulty with determining how to calculate the time for Train 1's arrival:

  Ask them, "If Train 1 travels at a constant rate of 66 mph, and the total distance is 639 miles, how can you determine the number of hours and minutes that Train 1 travels? Then, how can you use this value to find the time that it arrives?" (**Answer:** To find the total number of hours that it takes Train 1 to travel 639 miles, divide 639 by 66 to get 9.68 hours. Then, to convert 9.68 hours into hours and minutes, set up a proportion for the minutes part and solve for x, which represents the number of minutes: $\frac{68}{100} = \frac{x}{60}$; $x = 40.8$, $x = 41$ minutes. So, to find the time that the train arrives, add 9 hours and 41 minutes to the start time of 10:00 A.M., to get an arrival time of 7:41 P.M.)

  Ask students, "Why is it not possible to determine the exact time that Trains 2 and 3 arrive?" (**Answer:** Since the functions which represent Trains 2 and 3 are not linear, they do not travel at a constant rate, so the exact time of arrival cannot be known.)

  Ask students, "Since the exact arrival times for Trains 2 and 3 cannot be determined, how can an approximate time be estimated?" (**Answer:** Since the total distance of the trip is known, 639 miles, the table of values shows that for Train 2, at 7 P.M., the train had traveled 576 miles, and at 8 P.M., the train had traveled 660 miles. So, it can be estimated that the train arrived at its destination somewhere between 7 and 8 P.M., but closer to 8 since 639 is closer to 660 than it is to 576. For Train 3, at 7 P.M. the train had traveled 520 miles, and at 8 P.M. it had traveled 640 miles. Since 639 is very close to 640, it can be concluded that Train 3 arrived just right before 8 P.M.)

- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their analysis of the different representations of the three functions representing the three trains, as well as their solution paths for calculating their arrival times. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Focus on the use of precise mathematical language and clarity, specifically when referring to the different components and features of a linear, a quadratic, and an exponential function.

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Connecting to Key Concepts

Make explicit connections to key concepts:

- Linear functions have a constant slope.

  In this task, students will identify the function which represents Train 1 as linear, because the rate of change between any two points is the same.

- The rate of change of a linear function will always remain constant; however, the function will either increase, decrease, or remain constant.

  In the context of the scenario, students will see that the linear function which represents Train 1 is increasing, because as the input values increase, the output values also increase.

- The rate of change of an exponential function varies depending on the interval observed.

  In this task, students will observe that the rate of change between the functions for Trains 2 and 3 varies depending on the interval. The rate of change is not the same among different intervals.

- Exponential functions either increase or decrease.

  Students can conclude from the graph of the function for Train 3, and the table of values for the function of Train 2, that both exponential functions increase.

Extending the Task

- To extend the task, ask students to graph each of the three functions, in different colors, in order to compare them visually. Have students determine if and where the three functions intersect, as well as where the graph of each function intersects the y-value of 639 miles. Based on the graphs, ask students to make a case for which train would be preferable to take to Washington, D.C., and ask them to explain their reasoning for their choice.

- Another option for extending the task is to provide students with the scenario of a fourth train, where the function representing Train 4 is given algebraically. For example, Train 4 leaves the station at 11:00 A.M., and it could be represented by the function $d(t) = 70t$. Ask students how this function would compare with the other 3 in regards to which train would arrive at Washington, D.C. first.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I identified the independent and dependent variables in the scenario, as well as when I made sense of the different types of functions representing the three trains.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning when I calculated the arrival time for each train.)

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- **For SMP 3, ASK:** "Did you construct viable arguments and did you critique the reasoning of others?" (**Answer:** I constructed viable arguments by comparing the three functions representing the total distance traveled by the three trains. I used the results of my analysis to draw conclusions about which train arrived in Washington, D.C. first, as well as which train had the least amount of travel time.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I modeled this scenario by first creating a table of values to represent the three functions, as well as analyzing the given verbal descriptions and graphic representations of the functions.)

Alternate Strategies or Solutions

Students may choose to use the table of values for all three functions and graph them on the same graph. If graphing by hand, encourage students to be careful with labeling the axes, as well as when plotting the points. Discuss how the graphs of each function will help identify the type of function, as students will see if each graph forms a line or a curve.

Technology

Students can use scientific calculators and/or graphing calculators, or other graphing technology, for this task.

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Problem-Based Task 2.6.1: The Catch

Common Core State Standard

F–BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Task Overview

Focus

How can you model the path of a football using a quadratic function? Students will write and graph a function that does this using transformations of functions.

This activity will provide practice with:

- writing quadratic equations to model a real-world problem
- writing quadratic equations in vertex form
- identifying key features of a quadratic equation
- graphing a quadratic function
- comparing a quadratic function and a transformation of the function
- interpreting key features of a quadratic equation

Introduction

This task should be used to explore or to apply transformations of quadratic functions using a real-world model. It is best to implement this task after students have graphed horizontal and vertical transformations of quadratic function. Graphing may be done using technology, including graphing calculators or online graphing programs.

Begin by reading the problem and clarifying the meaning of various football terms:

line of scrimmage	the place where two teams meet to start the next play
offense	the team with the ball
quarterback	the player who throws the ball to the receiver
receiver	the player who receives the ball thrown by the quarterback

The following website provides a discussion of how football is played:

- NFL.com. "NFL Beginner's Guide to Football."

  http://www.walch.com/rr/07013

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Some students may have difficulty reasoning that (40, 0) is the second *x*-intercept. Have students reread the problem and review what the *x*- and *y*-axes represent. The *x*-axis represents the distance from the line of scrimmage and the *y*-axis represents the height of the football. There are two times when the height of the football is 0: when the quarterback releases the ball and when the receiver catches it. Remind students that the quarterback releases the ball and the receiver catches the ball at the same height.

- **SMP 4:** Model with mathematics.

  Students should understand the reason they are being asked to model the problem using a quadratic function. Discuss the path of the ball and why a parabola models its path.

- **SMP 7:** Look for and make use of structure.

  Some students may not understand why they are writing the equation of a function that intersects (0, 0) when the ball is thrown from behind the line of scrimmage. Discuss why it is easier to model this path and then transform the function than it is to write the function of the correct path to begin with.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- not accounting for the 5 yards when the quarterback backs away from the line of scrimmage

  Have students draw a diagram that is a number line showing the line of scrimmage (0), the position of the quarterback as he starts to throw (–5) and the place where the receiver catches the ball (35). Have them use the number line to find the total distance.

- measuring the horizontal distance in yards but the height differential in feet

  Remind students that all measures need to be the same unit of measure for the function to be correct. Students can convert all measures to feet or to yards, but because the lines on a football field are measured in yards, encourage students to use that unit of measure.

- placing the vertex at (20, 0) instead of (20, 3)

  Have students again draw a diagram to show the height differential. They can use a vertical number line to label the distances.

- finding $f(x) + 5$ instead of $f(x + 5)$

  Explain that the release point is a horizontal transformation representing the distance the quarterback moves back from the line of scrimmage to throw the ball.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students struggle to find the second x-intercept for part c, ask then what the point (0, 0) represents. (**Answer:** the position of the ball after the quarterback steps back 5 yards and the height of the football relative to 6 feet at the time the ball is thrown)

  Then ask when the ball will have a height of 6 feet again. (**Answer:** when the ball is caught)

  Then ask what distance the ball traveled in that time. (**Answer:** 40 yards)
- If students have difficulty finding a for the equation, ask what a represents. (**Answer:** It tells you which way the parabola opens and how wide the parabola is.)

  Then ask, "How can you find a when you know the vertex of the parabola and one point on the parabola?" (**Answer:** Substitute for x, y, h, and k and then solve for a.)
- If students graph the function incorrectly, ask how you would enter the function to ensure the operations are performed in the correct order? (**Answer:** $y = -\frac{3}{400}(x - 15)^2 + 5$)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Discuss why we solved the problem by first finding the function that intersected the origin and why this function could not be used to model the problem. Ask students to share the strategies they used and discuss other possible ways to solve the problem. Discuss the advantages and disadvantages of solving the problem graphically as opposed to algebraically.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Neither $f(x + k)$ nor $f(x) + k$ will change the shape of the function as long as k is constant.

  Students graph the function $f(x + 5) + 2$ and see that it has the same shape as the function $f(x)$.
- Translations are performed by adding a constant to the independent or dependent variable.

  Students graph a horizontal and vertical translation of a function $f(x)$.
- $f(x + k)$ moves the graph of a function k units to the left if k is positive and k units to the right if k is negative.

  Students graph $f(x + 5)$, which is a horizontal translation of the function $f(x)$.
- $f(x + k)$ moves the graph of a function k units up if k is positive and k units down if k is negative.

  Students graph $f(x + 5) + 2$, which is a vertical translation of the function $f(x + 5)$.

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Extending the Task

To extend the task, have students analyze the graph of the functions $f(x)$, $f(x + 5)$, and $f(x + 5) + 2$. For example:

- Graph the function $f(x)$. Ensure that it intersects the origin and (40,0).
- Graph the function $f(x + 5)$. Confirm that it is a horizontal translation of $f(x)$.
- Graph the function $f(x + 5) + 2$. Confirm that it is a vertical translation of $f(x + 5)$.

Confirm that the shape of the parabola does not change. If students are using graphing software they can move the original function to the transformed function to see that this is true.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I had to figure out what the problem was asking me to find.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I had to figure out how to use the number in the problem to calculate the distances I needed to write the function.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I modeled the path of the football with a quadratic function.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I had to use what I knew about vertex form to write transformations of a quadratic function.)

Alternate Strategies or Solutions

- Perform the vertical translation first.
- Have students graph the function $f(x) = -\frac{3}{400}(x - 20)^2 + 3$ and then move the parabola to translate it 5 units to the left and 2 units up. They can use the graph to find the equation of the function.

Technology

Students can use a graphing calculator or online software to graph the function.

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Problem-Based Task 2.6.2: Fewer Parabolas, Please

Common Core State Standard

F–BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Task Overview

Focus

This task provides students with a real-world example using transformations of functions of the form $f(kx)$. Students will apply transformations to redesign the structure of an arched bridge.

This activity will provide practice with:

- writing quadratic equations to model a real-world problem
- writing equations of transformed quadratic functions
- identifying key features of a quadratic equation
- graphing a quadratic function
- comparing a quadratic function and a transformation of the function
- interpreting key features of a quadratic equation
- analyzing the type of transformation described in a real-world scenario

Introduction

This task should be used to explore or to apply transformations of quadratic functions that change the shape of the parabola. It is best to implement this task after students have graphed transformations that compress or stretch a function. Graphing may be done using technology, including graphing calculators or online graphing programs.

Begin by reading the problem and clarifying which part of the bridge the parabolas form. If possible, show a diagram of a bridge formed this way. Here is one example.

> The Living New Deal. "Mapping the New Deal: Stone Arches in Texas."
>
> http://www.walch.com/rr/07012

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Review the following terms:

civil engineering	a branch of engineering that deals with the design and implementation of bridges, roads, buildings, and other structures in the physical world
critical points	the key features of a function

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students may struggle with the many steps involved in solving this problem. Even though the problem is scaffolded, students must still complete, interpret, and remember each step along the way to the final solution. Suggest that students highlight or underline the parts of each step that they feel will be important as they make their way through each subsequent step.

- **SMP 4:** Model with mathematics.

  Some students process information visually. It may help to have students graph the quadratic functions as they move through the steps to help in understand the problem and solution.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  Ensure that students understand they will use the same information to write the five transformed functions. Steps h and i are repeatedly used to do this.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- using the height of the bridge instead of the height of the arches to find the vertex

  Remind students to use the diagram provided to understand that the vertices of the parabolas represent the height of the arches.

- thinking that this is a $k \cdot f(x)$ problem

  Review the key concepts for vertical and horizontal compressions and stretches. Discuss how changing from seven parabolas to five will change the shape of the parabolas.

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- writing the ratio as 7 : 5 instead of 5 : 7

  Remind students that if $0 < k < 1$, the graph of $f(x)$ will stretch horizontally. We are changing from seven arches to five, so each of the five arches will need to be stretched horizontally to cover the same region as the seven arches. The ratio needs to be between 0 and 1.

- not squaring $\frac{5}{7}$ when finding $f\left(\frac{5}{7}x\right)$

  Discuss the difference between $f(kx)$ and $kf(x)$. Remind students that in general, $f(kx) \neq kf(x)$.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students struggle to find the critical points of the first parabola ask what the vertex of the parabola would be if the diagram were placed in the first quadrant of the coordinate plane. (**Answer:** (0, 75)) Use this same strategy to help students who struggle to find the x-intercept.
- If students try to solve for a before substituting for x in part c, remind them that doing the substitution first will make the equation easier to solve.
- If students confuse vertical compressions with horizontal stretches, review the graphs of each. Ask whether the height of a parabola changes in a vertical compression (**Answer:** yes) in a horizontal stretch (**Answer:** no). This will help students understand that the problem is a horizontal stretch.
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Compare the graphs of the original and transformed parabolas. Discuss the key features of each. Compare strategies for solving the problem and discuss whether students thought a diagram (graph) was helpful in understanding and solving the problem.

Connecting to Key Concepts

Make explicit connections to key concepts:

- In the graph of a function, there are key points of interest that define the graph and represent the characteristics of the function. When a function is transformed, the key points of the graph define the transformation. The key points in the graph of a quadratic function are the vertex and the x-intercepts.

  Students find and analyze the key points of the original functions as well as the transformed functions.

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- In general, multiplying the independent variable in a function by a constant will stretch or shrink the graph of f horizontally. If $0 < k < 1$, the graph of $f(x)$ will stretch horizontally by a factor of $\frac{1}{k}$. A horizontal stretch pulls the parabola and stretches it away from the y-axis.

  The number of arches will change from seven to five, using the same amount of space. This means that each of the transformed functions will be wider than the original functions. Students will calculate k to determine the stretch.

Extending the Task

To extend the task, perform other types of transformations. For example:

- The height of the arch changes.
- The height of the bridge changes.
- The design of the bridge changes so that the parabolas are reflected.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I worked through each step making sure I understood why the step was necessary.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** The transformed equations modeled the arches of the bridge.)
- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** I had to perform the same type of transformation on the equations for all of the five arches.)

Alternate Strategies or Solutions

Students can explore and solve this problem graphically using an online graphing tool.

Technology

Students can use a graphing calculator or online software to graph any or all of the functions representing the arches.

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Problem-Based Task 2.7.1: Falling Keys

Common Core State Standard

F–BF.4 Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *or* $f(x) = (x + 1)/(x - 1)$ *for* $x \neq 1$.

Task Overview

Focus

How can the inverse of a function be written? What are the independent and dependent quantities of a function's inverse? In this task, students will determine the domain and range of a function that represents the distance between an object and the ground, and then they will find the inverse of this function and determine the amount of time that the object was a certain distance off of the ground.

This activity will provide practice with:

- identifying the type of function which represents a scenario
- determining the domain and range of a function and its inverse
- interpreting the restricted domain of a function
- writing the inverse of a function
- evaluating an inverse function for a specific domain value

Introduction

This task should be used to explore or to apply the skill of solving an equation of the form $f(x) = c$ for a simple function f that has an inverse and writing an expression for the inverse. Students should already be familiar with identifying independent and dependent variables, as well as the domain and range of functions. They should also be familiar with applying inverse operations to isolate a variable.

Begin by reading the problem and clarifying the meaning of *function*, *inverse function*, and *one-to-one function*.

function	a relation in which every element of the domain is paired with exactly one element of the range; that is, for every value of x, there is exactly one value of y
inverse function	the function that results from switching the x- and y-variables in a given function
one-to-one function	a function in which each point in a set of points is mapped to exactly one other point

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will analyze the given information and recognize that the type of function which represents the distance between the keys that are dropped from a window and the ground over time can be modeled by a quadratic function. They will also identify the domain and range of the given function, based on the context of the scenario. Finally, they will plan to write the inverse of the function and then evaluate the inverse for a specific distance value in order to find the corresponding time.

- **SMP 2:** Reason abstractly and quantitatively.

  Students will be reasoning abstractly as they make sense of the quantities represented in the scenario. They will identify that the independent variable in the original function is time, in seconds, and they will identify that the dependent variable is the distance, in feet, that the keys are from the ground. They will also determine the independent and dependent variables when the inverse of the function is written. Students will be reasoning quantitatively as they evaluate the inverse function for a specific domain value.

- **SMP 7:** Look for and make use of structure.

  Students will recognize the structure of a quadratic function in terms of its general form, and they will also recognize the quadratic function in the form $f(x) = a(x - h)^2 + k$. They will use the quantities of this form in order to identify the restricted domain and range of the original function and its inverse.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- misinterpreting the superscript in inverse function notation, $f^{-1}(x)$, as a power of $f(x)$

  Have students write "$f^{-1}(x)$ = function notation" at the top of their papers to use as a reference when working through the task. Remind them that when the superscript follows a letter naming a function, then the notation is inverse function notation, and not raising the letter to a power.

- incorrectly performing inverse operations to find the inverse of a function

  Remind students of basic inverse operations: addition/subtraction, multiplication/division, and squaring a number/taking the square root of a number. Remind them to perform the opposite, or inverse, operations in order to isolate a specific variable.

- incorrectly identifying the restricted domain to include $x \leq 0$

  Have students carefully reread the problem and ask them to consider real-life constraints in the context of the scenario. Remind them that time cannot be a negative value.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "What type of function do you think is going to represent an object falling to the ground? In other words, will the object fall at a constant rate or not?" (**Answer:** When an object falls to the ground, the object will start off slowly and then accelerate, due to the force of gravity. The type of function that represents an object falling at a variable rate is a quadratic function.)
- Ask students, "What part of the function allows you to identify it as a quadratic function instead of a linear function?" (**Answer:** The highest power of the given function is 2. The highest power of a linear function is 1. Therefore, the function is quadratic, because the general form of a quadratic function has a 2 as the highest power.)
- Before students begin the task, ask them, "Based on the given function, $f(x) = -16x^2 + 32$, what is the height of the keys before they are dropped from the window? Explain your answer." (**Answer:** Before the keys are dropped, the keys are 32 feet above the ground. If they have not been dropped yet, then the value of x, which represents time in seconds, is 0. When 0 is substituted into the function, the result is 32.)
- If students are having difficulty understanding how to find the domain of the function, ask them, "In the context of this scenario, what do you know about the possible input values for time? How can you find the exact moment when the keys will hit the ground?" (**Answer:** When the keys are dropped from a third-story window, they will hit the ground after a certain number of seconds. This value is found when the height/distance is 0. To find this value, set the function equal to 0 and solve for x: $0 = -16x^2 + 32$; $-16x^2 = -32$; $x^2 = 2$; $x = \sqrt{2} \approx 1.414214$. Therefore, the amount of time it will take the keys to hit the ground is approximately 1.414 seconds.)
- Once students have determined the amount of time it will take for the keys to hit the ground, ask them, "What is the domain of the function if it takes approximately 1.414 seconds for the keys to hit the ground?" (**Answer:** The total amount of time that the keys will be falling is approximately 1.414 seconds. Also, time is a positive quantity; therefore, the domain is $0 \le x \le 1.414$.)
- Ask students, "How can you determine the range of the function?" (**Answer:** The range of a quadratic function of the form $y = a(x - h)^2 + k$ has an extreme value at $y = k$. In the given function, $k = 32$. Since the value of a in the function, which is –16, is less than 0, then the range is $y \le 32$. However, since distance cannot be negative, the range is: $0 \le y \le 32$.)
- If students have difficulty writing the inverse for the function, remind them that the first step is to switch the x and y-variables. Ask them, "What can you use to isolate and solve for y?" (**Answer:** Inverse operations can be used to isolate and solve for y.)

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- Ask students, "When solving for the inverse, how can the fraction $\frac{32-x}{16}$ be simplified?" (**Answer:** The fraction $\frac{32-x}{16}$ can be broken down into two separate fractions: $\frac{32}{16}$ and $-\frac{x}{16}$. The fraction $\frac{32}{16}$ can be simplified to 2, so the original fraction can be simplified to $2-\frac{x}{16}$.)
- Ask students, "How can the inverse function be used to find the number of seconds that the keys were 4 feet above the ground?" (**Answer:** Since the domain and the range of the inverse function are the range and the domain of the original function, the value of 4 is substituted into the inverse function, and the result is approximately 1.32 seconds.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their analysis of the function and its domain and range. Discuss the process of finding the inverse function as well as its domain and range. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Focus on the use of precise mathematical language and clarity, specifically when referring to the domain, range, and inverse of the function.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The inverse of a function, called an inverse function, is the function that results from switching the x and y-variables in a given function.

  In this task, students will determine the inverse of the given function, and they will write the inverse by switching the x and y-variables and then solving for y.

- The inverse of $f(x)$ is written as $f^{-1}(x)$.

  Once students have completed writing the inverse, they will use the correct inverse notation as $f^{-1}(x)=\sqrt{2-\frac{x}{16}}$.

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- The domain of a function's inverse is the range of the function.

  Students will find that the domain of the function's inverse is the range of the function; the range of the original function is $0 \le y \le 32$, so the domain of the function's inverse is $0 \le x \le 32$.

- The range of a function's inverse is the domain of the function.

  Students will find that the range of the function's inverse is the domain of the function; the domain of the original function is $0 \le x \le \sqrt{2}$, so the range of the function's inverse is $0 \le y \le \sqrt{2}$.

Extending the Task

- To extend the task, provide students with a different scenario such as, "The function $f(t) = -4t^2 + 16t + 1.5$ models the path of a football after it is kicked, where t represents the time, in seconds, that the football is in the air, and $f(t)$ represents the height of the football, in meters." Ask students to find the domain, the range, and the inverse of the function. Then, ask them to determine the maximum time that the football is in the air.
- Another option for extending the task is to ask students to first create a table of values, with both positive and negative values, and graph the function from the original scenario. Ask them to graph the entire parabola, but to make the "non-applicable" part of the parabola a dotted line. Discuss the graph and help students understand which quadrant(s) contain the part of the graph that represents the scenario.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I analyzed the given information and identified that the function was a quadratic function, I also determined the domain and range of the function, and I found the inverse of the function.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** When I made sense of the quantities represented in the scenario and I identified the independent and dependent variables, I used abstract reasoning.) "Which of your strategies represent quantitative reasoning?" (**Answer:** When I calculated the value of $x = 4$ into the inverse function in order to find the time when the keys were 4 feet off of the ground, I used quantitative reasoning.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I used the structure of the form of a quadratic equation in order to determine the domain and range, as well as the extreme value.)

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Alternate Strategies or Solutions

Students may choose to create a table of values and graph the original function, in order to verify its parabolic-shaped graph. They may also choose to graph the inverse of the function. Remind them to consider the constraints of the domain and range in the context of the scenario when they graph the function.

Technology

Students can use scientific calculators for this task.

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Common Core State Standards

Mathematics II

Integrated Pathway

Teacher Resource

Unit 3

WALCH
INTEGRATED MATH

This program was developed and reviewed by experienced math educators who have both academic and professional backgrounds in mathematics. This ensures: freedom from mathematical errors, grade level appropriateness, freedom from bias, and freedom from unnecessary language complexity.

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Standards Correlations

Each lesson in this Integrated Pathway: Mathematics II program was written specifically to address the Common Core State Standards. Each lesson lists the standards covered in all the sub-lessons, and each sub-lesson lists the standards addressed in that particular section. In this section, you'll find a comprehensive list mapping the sub-lessons to the CCSS.

Guide to Common Core State Standards Annotation

As you use this program, you will come across a symbol included with the Common Core standards for some of the lessons and activities. These symbols are explained below.

Symbol: ⋆

Denotes: Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (⋆).

From http://www.walch.com/CCSS/00003

Symbol: (+)

Denotes: College and Career Readiness Standards

Advanced mathematics standards that are required in higher-level courses such as advanced statistics may also be included in lower-level courses. These additional standards are denoted by (+). According to the Common Core State Standards Initiative, "the evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6–8."

From http://www.walch.com/CCSS/00004

Connections to Future Courses

This section provides a map between topics introduced in each unit of this course and subsequent courses where each topic is revisited and built upon.

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CCSS INTEGRATED PATHWAY: MATHEMATICS II STANDARDS CORRELATIONS

Unit 3: Expressions and Equations				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Interpreting Structure in Expressions**			
	3.1.1	Identifying Terms, Factors, and Coefficients	A–SSE.1a★	U3-4
	3.1.2	Interpreting Complicated Expressions	A–SSE.1b★	U3-17
Lesson 2	**Creating and Solving Quadratic Equations in One Variable**			
	3.2.1	Taking the Square Root of Both Sides	A–CED.1★ A–REI.4b	U3-37
	3.2.2	Factoring Expressions by the Greatest Common Factor	A–SSE.2	U3-48
	3.2.3	Factoring Expressions with $a = 1$	A–SSE.2	U3-64
	3.2.4	Factoring Expressions with $a > 1$	A–SSE.2	U3-85
	3.2.5	Solving Quadratic Equations by Factoring	A–SSE.2 A–CED.1★ A–REI.4b	U3-107
	3.2.6	Completing the Square	A–SSE.2 A–CED.1★ A–REI.4a A–REI.4b	U3-124
	3.2.7	Applying the Quadratic Formula	A–CED.1★ A–REI.4a A–REI.4b	U3-136
	3.2.8	Solving Quadratic Inequalities	A–SSE.2 A–CED.1★ A–REI.4b	U3-150

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Lesson	Sub-lesson number	Title	Standard(s)	Pages
Lesson 3	**Creating Quadratic Equations in Two or More Variables**			
	3.3.1	Creating and Graphing Equations Using Standard Form	A–SSE.3a★ A–CED.2★	U3-171
	3.3.2	Creating and Graphing Equations Using the *x*-intercepts	A–SSE.3a★ A–CED.2★	U3-199
	3.3.3	Creating and Graphing Equations Using Vertex Form	A–SSE.3b★ A–CED.2★	U3-216
	3.3.4	Rearranging Formulas	A–CED.4★	U3-230
Lesson 4	**Fundamental Theorem of Algebra**			
	3.4.1	Extending Polynomial Identities to Include Complex Numbers	N–CN.8 (+)	U3-249
	3.4.2	Solving Quadratic Equations with Complex Solutions	N–CN.7 N–CN.9 (+)	U3-271
Lesson 5	**Rational Equations**			
	3.5.1	Creating Rational Equations	A–CED.1★	U3-306
	3.5.2	Graphing Rational Equations	A–CED.2★	U3-328
	3.5.3	Creating Rational Inequalities	A–CED.1★	U3-361
Lesson 6	**Writing Exponential Expressions in Equivalent Forms**			
	3.6.1	Writing Exponential Expressions in Equivalent Forms	A–SSE.3c★	U3-410
Lesson 7	**Solving Systems of Equations**			
	3.7.1	Solving Systems Graphically	A–REI.7	U3-442
	3.7.2	Solving Systems Algebraically	A–REI.7	U3-462

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TOPICS FROM FUTURE COURSES: CCSS INTEGRATED PATHWAY MATHEMATICS II

Math II	Topic introduced	Course/unit where addressed	How addressed
Unit 3	Interpreting the structures of quadratic expressions	Math III Unit 2A	interpreting higher-order polynomials
		Math III Unit 2B	interpreting rational expressions
	Quadratic and rational functions in standard form	Math III Unit 2A	applied to polynomial equations in standard form
	Using the Fundamental Theorem of Algebra in solving quadratic equations	Math III Unit 2A	using the Fundamental Theorem of Algebra to solve polynomials of higher degree
	Graphing rational equations	Math III Unit 4B	modeling real-world situations
	Exponential equations including rational exponents and more complex equations in equivalent form	Math IV	graphing logarithmic functions
	Linear-quadratic systems	Math III Unit 2A	applied to linear-rational, linear-polynomial, polynomial-polynomial, and absolute value-polynomial systems
		Math III Unit 2B	extended to linear-rational and linear-radical systems
		Math IV	applied to 3×3 and larger systems of equations using vectors and matrices
	Graphing simple rational functions	Math III Unit 2B	solving rational equations
		Math IV	graphing more complex rational functions and exploring features of the graphs

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Conceptual Activities

Use these interactive open education and/or Desmos resources to build conceptual understanding of mathematical ideas. (*Note*: Activity links will be monitored and repaired or replaced as necessary.)

- Desmos. "Build a Bigger Field."

  http://www.walch.com/ca/01021

  Use quadratic models to optimize the area of a field for a given perimeter.

- Desmos. "Match My Parabola."

  http://www.walch.com/ca/01017

  A series of graphing challenges builds understanding of quadratic functions in various forms and graphing transformations of quadratic functions.

- Desmos. "Penny Circle."

  http://www.walch.com/ca/01023

  Gather data, build a model, and use that model to answer the question, "How many pennies fit in a large circle?"

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. The area of a circle is given by $A = \pi r^2$, for which r is the radius of the circle. What is the value of the coefficient?

   a. 2
   b. r
   c. π
   d. There is no coefficient.

2. What are the factors in the expression $2x^2 + 17x - 12$?

   a. 2 and x^2, 17 and x
   b. 2, 17, and –12
   c. x
   d. This expression does not contain any factors.

3. Which of the following expressions is quadratic?

   a. $x^2 - x^2(x + 2)$
   b. $5 - t$
   c. $s(5s + 4)$
   d. $\sqrt{3x^2 - 2x + 1}$

4. When the expression $(2x + 1)(x - 7)$ is written in the form $ax^2 + bx + c$, what is the value of b?

   a. 2
   b. –13
   c. –14
   d. The expression cannot be written in the form $ax^2 + bx + c$.

5. When the expression $(12 - x)(5x + 6)$ is written in the form $ax^2 + bx + c$, what is the value of the constant term?

   a. 72
   b. 12
   c. 6
   d. There is no constant in the expression.

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Lesson 1: Interpreting Structure in Expressions

Instruction

Common Core State Standards

SMP
1 ✓ 2
3 4 ✓
5 6
7 ✓ 8

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*

Essential Questions

1. How are quadratic expressions and quadratic equations alike? How are they different?
2. How are verbal phrases translated into quadratic expressions?
3. How does changing the value of a coefficient, constant, or variable in an expression change the value of the expression?
4. How are quadratic functions different from linear functions?

WORDS TO KNOW

binomial	a polynomial with two terms
coefficient	the number multiplied by a variable in an algebraic expression
constant term	a term whose value does not change
factor	one of two or more numbers or expressions that are multiplied to produce a product
like terms	terms that contain the same variables raised to the same power
monomial	an expression with one term, consisting of a number, a variable, or the product of a number and variable(s)
polynomial	a monomial or the sum of monomials
quadratic equation	an equation that can be written in the form $ax^2 + bx + c = 0$, where x is the variable, a, b, and c are real numbers, and $a \neq 0$

SRB p. U3-1

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quadratic expression	an algebraic expression that can be written in the form $ax^2 + bx + c$, where x is the variable, a, b, and c are real numbers, and $a \neq 0$
term	a number, a variable, or the product of a number and variable(s)
trinomial	a polynomial with three terms
variable	a letter used to represent a value or unknown quantity that can change or vary

Recommended Resources

- MathIsFun.com. "Algebra—Basic Definitions."

  http://www.walch.com/rr/00098

  This website gives an overview of the important vocabulary for this lesson. Color-coded expressions help users visualize the differences between similar terms.

- Quia. "Rags to Riches: Combining Like Terms."

  http://www.walch.com/rr/00099

  Players combine like terms to simplify expressions in this multiple-choice game modeled on the TV show, *Who Wants to Be a Millionaire?* Players can use up to three hints on their quest to reach the million-dollar question.

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Lesson 3.1.1: Identifying Terms, Factors, and Coefficients

Warm-Up 3.1.1

In order to pay for upkeep of a local highway, the transportation department has set up tollbooths at each of the highway's exits. Drivers are charged a toll of $1.20 for use of the highway, and are then charged an additional $0.04 per mile driven.

1. Write an algebraic expression that can be used to represent the total toll charged if m represents the number of miles driven.

2. What is the toll charged to drive 14 miles on this highway?

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Instruction

Lesson 3.1.1: Identifying Terms, Factors, and Coefficients

Common Core State Standard

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

Warm-Up 3.1.1 Debrief

1. Write an algebraic expression that can be used to represent the total toll charged if m represents the number of miles driven.

   The constant is 1.20, the flat fee for the use of the highway.

   The variable is m, the number of miles.

   The coefficient is 0.04, the charge per mile driven.

   Therefore, the toll can be written as the expression $1.20 + 0.04m$.

2. What is the toll charged to drive 14 miles on this highway?

   Substitute 14 for m and evaluate the expression.

   $1.20 + 0.04(14) = 1.20 + 0.56 = 1.76$

   The toll charged for driving 14 miles on the highway is $1.76.

Connection to the Lesson

- Students will extend their knowledge of linear models in context to interpreting quadratic models in context.
- Students will be asked to identify parts of expressions given in context.
- Students will translate verbal scenarios into quadratic expressions.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- translating verbal expressions into algebraic expressions (6.EE.2a)
- evaluating expressions using the order of operations (5.OA.1)
- adding and subtracting polynomials (A–APR.2)

Introduction

Algebraic expressions are mathematical representations that include numbers, operations, and variables to represent a number or quantity. We know that a **variable** is a letter used to represent a value or unknown quantity that can change or vary. We have seen several linear expressions, such as $2x + 1$, in which the highest power of the variable x is 1. In this lesson, we will look at expressions in which the highest power of the variable is 2.

Key Concepts

- A **quadratic expression** can be written in the form $ax^2 + bx + c$, where x is the variable, a, b, and c are real numbers, and $a \neq 0$. In a quadratic expression, the highest power of the variable is 2.
- Both b and c can be any real number, but a cannot be equal to 0 because quadratic expressions must contain a squared term.
- An example of a quadratic expression is $4x^2 + 6x - 2$. When a quadratic expression is set equal to 0, as in $4x^2 + 6x - 2 = 0$, the resulting equation is called a quadratic equation. A **quadratic equation** is an equation that can be written in the form $ax^2 + bx + c = 0$, where x is the variable, a, b, and c are real numbers, and $a \neq 0$. In a quadratic equation, the highest power of the variable is 2.
- The quadratic expression $4x^2 + 6x - 2$ is made up of many component parts: terms, factors, coefficients, and constant terms.
- A **term** is a number, a variable, or the product of a number and variable(s).
- There are 3 terms in the given expression: $4x^2$, $6x$, and -2.
- A **factor** is one of two or more numbers or expressions that are multiplied to produce a product. In the given expression, the factors of $4x^2$ are 4 and x^2, and the factors of $6x$ are 6 and x.

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Instruction

- The number multiplied by a variable in an algebraic expression is called a **coefficient**. In the given expression, the coefficient of $4x^2$ is 4 and the coefficient of $6x$ is 6.
- The coefficient of the term x is 1 and the coefficient of the term $-x$ is -1; that is, $x = 1x$ and $-x = -1x$.
- A term that does not contain a variable is called a **constant term** because the value of the term does not change. In the given expression, -2 is the constant term.
- Two or more terms that contain the same variables raised to the same power are called **like terms**. Like terms can be combined by adding. Be sure to follow the order of operations when combining like terms.
- The given quadratic expression, $4x^2 + 6x - 2$, has no like terms.
- The quadratic expression $9x^2 - 8x^2 + 2x$ has two like terms: $9x^2$ and $-8x^2$. After simplifying the expression by combining like terms (adding $9x^2$ and $-8x^2$), the result is $x^2 + 2x$.
- A **monomial** is a number, a variable, or the product of a number and variable(s). We can also think of a monomial as an expression containing only one term. $5x^2$ is an example of a monomial.
- A **binomial** is a polynomial with two terms. $6x + 9$ is an example of a binomial.
- A **trinomial** is a polynomial with three terms. $4x^2 + 6x - 2$ is an example of a trinomial.
- A **polynomial** is a monomial or the sum of monomials. A polynomial can have any number of terms. $5x^2$, $6x + 9$, and $4x^2 + 6x - 2$ are all polynomials.

Common Errors/Misconceptions

- not following the order of operations
- incorrectly identifying like terms
- inaccurately combining terms involving subtraction
- incorrectly combining terms by changing exponents

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Instruction

Guided Practice 3.1.1

Example 1

Identify each term, coefficient, and constant term of $6(x - 1) - x(3 - 2x) + 12$. Classify the expression as a monomial, binomial, or trinomial. Determine whether it is a quadratic expression.

1. Simplify the expression.

   The expression can be simplified by following the order of operations and combining like terms.

$6(x - 1) - x(3 - 2x) + 12$	Original expression
$= 6x - 6 - x(3 - 2x) + 12$	Distribute 6 over $(x - 1)$.
$= 6x - 6 - 3x + 2x^2 + 12$	Distribute $-x$ over $(3 - 2x)$.
$= 3x + 6 + 2x^2$	Combine like terms: $6x$ and $-3x$; -6 and 12.
$= 2x^2 + 3x + 6$	Rearrange terms so the powers are in descending order.

2. Identify all terms.

   There are three terms in the expression: $2x^2$, $3x$, and 6.

3. Identify the coefficients of the variable terms.

   The numerical factor in the term $2x^2$ is 2, and the numerical factor in the term $3x$ is 3. Therefore, the coefficients are 2 and 3.

4. Identify any constant terms.

   The number that is not a factor of a variable in the expression is 6; therefore, 6 is the constant term.

SWB p. 9; SRB p. U3-5

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Instruction

5. Classify the expression as a monomial, binomial, or trinomial.

   The polynomial is a trinomial because it has three terms.

6. Determine whether the expression is a quadratic expression.

   The expression is quadratic because the highest power of x in the expression is 2.

Example 2

Translate the verbal expression "take triple the difference of 12 and the square of x, then increase the result by the sum of 3 and x" into an algebraic expression. Identify the terms, coefficients, and constant term(s) of the given expression. Is the expression quadratic?

1. Translate the expression by breaking it down into pieces.

   "The difference of 12 and the square of x" translates to $(12 - x^2)$.

   Triple this expression is $3(12 - x^2)$.

   "The sum of 3 and x" translates to $(3 + x)$.

   Increasing the original expression by this sum translates to $3(12 - x^2) + (3 + x)$.

2. Simplify the expression.

$3(12 - x^2) + (3 + x)$	Expression from the previous step
$= 36 - 3x^2 + (3 + x)$	Distribute 3 over $(12 - x^2)$.
$= -3x^2 + x + 39$	Combine like terms.

   The simplified expression is $-3x^2 + x + 39$.

3. Identify all terms.

   There are three terms in the expression: $-3x^2$, x, and 39.

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Instruction

4. Identify all coefficients.

   The numerical factor in the term $-3x^2$ is -3, and the numerical factor in the term x is 1. Therefore, -3 and 1 are coefficients.

5. Identify any constant terms.

   The number that is not a factor of a variable in the expression is 39; therefore, 39 is the constant term.

6. Determine whether the expression is a quadratic expression.

   The expression is quadratic because the highest power of x is 2.

Example 3

A fence surrounds a park in the shape of a pentagon. The side lengths of the park in feet are given by the expressions $2x^2$, $3x + 1$, $3x + 2$, $4x$, and $5x - 3$. Find an expression for the perimeter of the park. Identify the terms, coefficients, and constant term(s) in your expression. Is the expression quadratic?

1. Find an expression for the perimeter of the park.

   Add like terms to find the perimeter, P.

$P = 2x^2 + (3x + 1) + (3x + 2) + 4x + (5x - 3)$	Set up the equation using the given expressions.
$= 2x^2 + 3x + 3x + 4x + 5x + 1 + 2 - 3$	Reorder like terms.
$= 2x^2 + 15x$	Combine like terms.

   The expression for the park's perimeter is $2x^2 + 15x$.

2. Identify all terms.

   There are two terms in this expression: $2x^2$ and $15x$.

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Instruction

3. Identify all coefficients.

   The numerical factor in the term $2x^2$ is 2, and the numerical factor in the term $15x$ is 15. Therefore, 2 and 15 are coefficients.

4. Identify any constant terms.

   Every number in the expression is multiplied by a variable; therefore, there is no constant term.

5. Determine whether the expression is a quadratic expression.

   The expression is quadratic because the highest power of x is 2.

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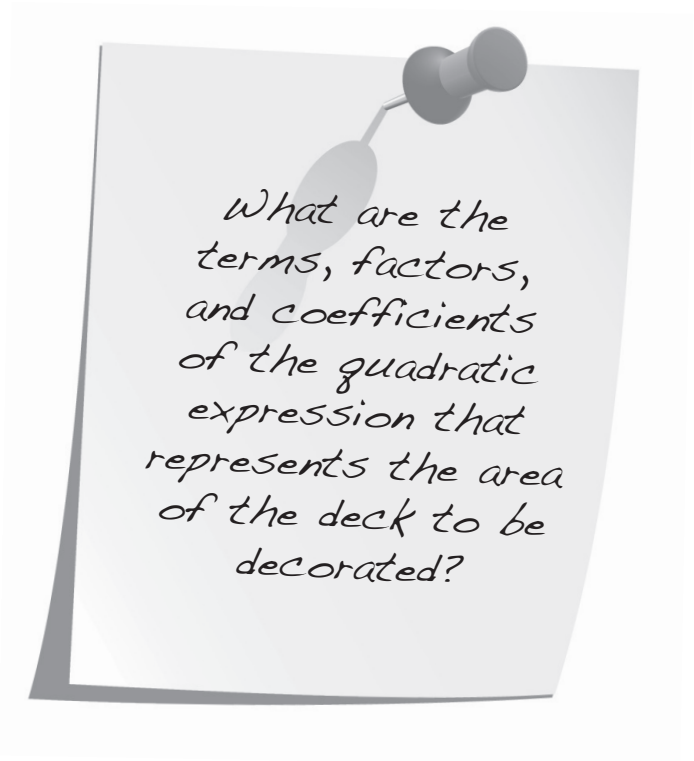
Problem-Based Task 3.1.1: Deck the Deck

SMP	
1 ✓	2
3	4 ✓
5	6
7 ✓	8

Shanna wants to decorate the triangular deck behind her house. The base of the triangle is 10 meters less than the altitude. What are the terms, factors, and coefficients of the quadratic expression that represents the area of the deck to be decorated?



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Problem-Based Task 3.1.1: Deck the Deck

Coaching

a. How can you represent the altitude of the triangle?

b. What algebraic expression represents the base of the triangle in terms of the altitude?

c. What algebraic expression represents the area of the triangle?

d. What units will be used to express the area of the deck?

e. Simplify your expression for the area using the Distributive Property. Is this a quadratic expression?

f. How many terms does the expression from part e include?

g. What are the factors that make up each of the terms?

h. What are the coefficients of each term?

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Instruction

Problem-Based Task 3.1.1: Deck the Deck

Coaching Sample Responses

a. How can you represent the altitude of the triangle?

Let a represent the altitude of the triangle.

b. What algebraic expression represents the base of the triangle in terms of the altitude?

Since the base of the triangle is 10 meters less than the altitude, it can be represented by the expression $a - 10$.

c. What algebraic expression represents the area of the triangle?

An expression for the area of a triangle is $\frac{1}{2}bh$, where b represents the base of the triangle and h represents the altitude (or height) of the triangle.

Substitute the expressions from parts a and b to write the area of this triangle as $\frac{1}{2}a(a-10)$.

d. What units will be used to express the area of the deck?

The lengths of the altitude and base are in meters, so the area of the deck will be expressed in square meters.

e. Simplify your expression for the area using the Distributive Property. Is this a quadratic expression?

After simplifying, the expression becomes $\frac{1}{2}a^2 - 5a$. This is a quadratic expression because the highest power of a is 2.

f. How many terms does the expression from part e include?

There are two terms in the expression $\frac{1}{2}a^2 - 5a$. The terms are $\frac{1}{2}a^2$ and $-5a$.

g. What are the factors that make up each of the terms?

The factors are the quantities that are being multiplied together. The product of $\frac{1}{2}a^2$ is made up of the factors $\frac{1}{2}$ and a^2. The product of $-5a$ is made up of the factors -5 and a.

h. What are the coefficients of each term?

The coefficient of the term $\frac{1}{2}a^2$ is $\frac{1}{2}$. The coefficient of the term $-5a$ is -5.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.1.1: Identifying Terms, Factors, and Coefficients

Use what you know about the components of expressions to complete problems 1–3.

1. Identify the terms, coefficients, constant term, and factors of $16x^2 - 12x + 20$.

2. Simplify the expression $3x^2 + 2(5 - x^2) - 8(x^2 + 9)$ and classify it as a monomial, binomial, or trinomial.

3. Write a quadratic expression that contains two terms, a coefficient of 7, and a constant of 10.

For problems 4 and 5, determine whether each expression is a quadratic expression. Explain your reasoning.

4. $-3x(x + 14)$

5. $8x^2 - 2x(1 + 4x) + 2$

continued

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For problems 6–10, write an algebraic expression. Identify the terms, coefficients, and constant terms of the algebraic expression. Determine whether the expression is quadratic and explain your reasoning.

6. the product of 7 and the square of x, increased by the difference of 5 and x^2

7. half the sum of 12 and x^2 decreased by one-third x

8. the perimeter of a square, which is the product of 4 and the length of its side, s

9. the surface area of a cube, which is the product of 6 and the square of the side length, s

10. the volume of a sphere with radius r, which is four-thirds times the product of π and the cube of the radius

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Lesson 3.1.2: Interpreting Complicated Expressions

Warm-Up 3.1.2

A fence surrounds a park in the shape of a triangle. The side lengths of the park in feet are given by the expressions $8x$, $10x$, and $11x - 2$, where x is the length in feet of one fence panel.

1. Write an expression in terms of x for the perimeter of the park.

2. What is the perimeter of the park if each fence panel is 4 feet long?

3. How does the perimeter change if the length of each fence panel increases to 5 feet?

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Instruction

Lesson 3.1.2: Interpreting Complicated Expressions

Common Core State Standard

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*

Warm-Up 3.1.2 Debrief

1. Write an expression in terms of x for the perimeter of the park.

   Set up an equation using the given expressions. Then combine like terms to find an expression to represent the perimeter, P.

   $P = 8x + 10x + (11x - 2)$

   $= 8x + 10x + 11x - 2$

   $= 29x - 2$

2. What is the perimeter of the park if each fence panel is 4 feet long?

   Substitute 4 for x in the equation for perimeter.

   $P = 29(4) - 2 = 116 - 2 = 114$

   If a fence panel is 4 feet long, the perimeter of the park is 114 feet.

3. How does the perimeter change if the length of each fence panel increases to 5 feet?

   Substitute 5 for x in the equation for perimeter and compare the result to the previous question's answer.

   $P = 29(5) - 2 = 145 - 2 = 143$

   If the length of each fence panel increases to 5 feet, the perimeter of the park increases to 143 feet. Subtracting 114 from 143 indicates this is an increase of 29 feet.

Connection to the Lesson

- Students will interpret the parts of given expressions and determine how changes to the parts affect the expressions.
- Students will evaluate quadratic expressions for given values of the variables.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- evaluating expressions using the order of operations (5.OA.1)
- evaluating expressions for a given value (6.EE.2c)
- identifying parts of an expression (6.EE.2b)

Introduction

Variables change, but constants remain the same. We need to understand how changing the value of a variable in an expression will affect the value of the expression. In this lesson, we will explore these relationships.

Key Concepts

- We can translate verbal expressions into algebraic expressions to analyze how changing the values of coefficients, constants, or variables will affect the value of the expression.
- Changing the value of a variable in an expression will not change the value of a constant, just as changing the value of a constant will not change the value of terms containing variables.
- It is important to always follow the order of operations.
- Simplifying expressions lets us more easily see how different terms interact with one another.
- The Distributive Property is applied when multiplying binomials.
- When presented with a polynomial in factored form, multiply the factors to see if the polynomial is quadratic. The highest power of the variable of a quadratic expression is 2, and quadratic expressions are of the form $ax^2 + bx + c$, where $a \neq 0$.
- Sometimes it is easier to work with the factored form of an expression. For instance, since we know that the product of two positive quantities is always a positive number, it is easier to determine values of a variable that make an expression positive by finding the values that make all of the variable's factors positive.
- We can use similar logic to determine the values that make an expression negative or equal to 0.

Common Errors/Misconceptions

- forgetting to distribute when multiplying binomials
- incorrectly following the order of operations
- incorrectly translating verbal expressions

SRB p. U3-11

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Instruction

Guided Practice 3.1.2

Example 1

Show that $(x + 2)(2x - 1)$ is a quadratic expression by writing it in the form $ax^2 + bx + c$. Identify a, b, and c.

1. Multiply the factors using the Distributive Property.

$(x + 2)(2x - 1)$	Original expression
$= x(2x) + x(-1) + 2(2x) + 2(-1)$	Distribute $(x + 2)$ over $(2x - 1)$.
$= 2x^2 - x + 4x - 2$	Multiply.
$= 2x^2 + 3x - 2$	Combine like terms.

2. Compare the resulting polynomial to $ax^2 + bx + c$.

   Since $2x^2 + 3x - 2$ has the form $ax^2 + bx + c$, where $a \neq 0$, $(x + 2)(2x - 1)$ is a quadratic expression.

3. Identify a, b, and c.

   In the polynomial, $a = 2$, $b = 3$, and $c = -2$.

SWB p. 19; SRB p. U3-12

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Instruction

Example 2

What values of x make the expression $(x + 2)(x - 3)$ positive?

1. Determine the sign possibilities for each factor.

   The expression will be positive if both factors are positive or if both factors are negative.

2. Determine the values of the variable that make both factors positive.

   Solve the inequalities $x + 2 > 0$ and $x - 3 > 0$ for x.

   $x + 2 > 0$ when $x > -2$

   $x - 3 > 0$ when $x > 3$

   Both factors are positive when $x > -2$ and $x > 3$, and the intersection of these two sets is $x > 3$. Therefore, any values of x greater than 3 will make both factors positive.

3. Determine the values of the variable that make both factors negative.

   Solve the inequalities $x + 2 < 0$ and $x - 3 < 0$ for x.

   $x + 2 < 0$ when $x < -2$

   $x - 3 < 0$ when $x < 3$

   Both factors are negative when $x < -2$ and $x < 3$, and the intersection of these two sets is $x < -2$. Therefore, any values of x less than -2 will make both factors negative.

4. Determine the values of x for which the expression is always positive.

   The value of the expression is positive when $x > 3$ or $x < -2$.

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Instruction

Example 3

The length of each side of a square is increased by 2 centimeters. How does the perimeter change? How does the area change?

1. Find an expression for the perimeter of the original square.

   Let x be the length of one side of the square in centimeters.

   Multiply the side length by 4 since a square has four sides of equal length.

   The perimeter of the square is $4x$ centimeters.

2. Find an expression for the perimeter of the new square and compare the new perimeter to the original perimeter.

   One side of the new square is $x + 2$ centimeters.

   The new perimeter is $4(x + 2)$ or $4x + 8$ centimeters.

   Next, subtract the old perimeter from the new perimeter to determine the difference.

   $4x + 8 - 4x = 8$

   The new perimeter is 8 centimeters more than the original perimeter.

3. Find an expression for the area of the original square.

   The area in square centimeters of the original square is x^2.

4. Find an expression for the area of the new square and compare the new area to the original area.

   The new area is $(x + 2)^2$ or $x^2 + 4x + 4$ square centimeters.

   Subtract the old area from the new area to determine the difference.

   $(x^2 + 4x + 4) - x^2 = 4x + 4$

   The new area is $4x + 4$ square centimeters greater than the old area.

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Instruction

Example 4

A car's total stopping distance in feet depends on many factors, but can be approximated by the expression $\frac{11}{10}x+\frac{1}{19}x^2$, where x is the speed of the car in miles per hour. Is this expression quadratic? What effect does doubling the car's speed from 10 mph to 20 mph have on the total stopping distance?

1. Determine whether the expression is quadratic.

   This expression is quadratic because it can be written in the form ax^2+bx+c, with $a=\frac{1}{19}$, $b=\frac{11}{10}$, and $c=0$.

2. Determine the approximate stopping distance of a car traveling 10 mph.

   Substitute 10 for x to find the total stopping distance of a car traveling 10 mph.

   $$\frac{11}{10}(10)+\frac{1}{19}(10)^2=11+\frac{100}{19}\approx 16.26 \text{ feet}$$

   The stopping distance of a car traveling 10 mph is approximately 16.26 feet.

3. Determine the approximate stopping distance of a car traveling 20 mph.

   Substitute 20 for x to find the total stopping distance of a car traveling 20 mph.

   $$\frac{11}{10}(20)+\frac{1}{19}(20)^2=22+\frac{400}{19}\approx 43.05 \text{ feet}$$

   The stopping distance of a car traveling 20 mph is approximately 43.05 feet.

4. Compare the stopping distances for a car traveling 10 mph and a car traveling 20 mph.

   When the speed of the car doubles from 10 mph to 20 mph, the stopping distance almost triples from about 16 feet to about 43 feet.

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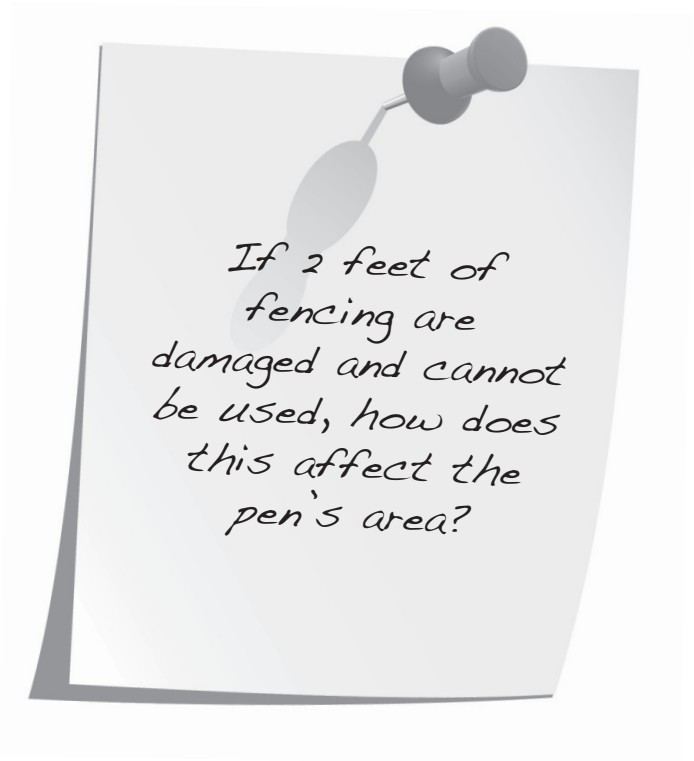
Problem-Based Task 3.1.2: Puppy Pen

SMP	
1 ✓	2
3	4 ✓
5	6
7 ✓	8

Oscar has 20 feet of fencing. He wants to build a rectangular pen for his new puppy. Write an expression for the area of the puppy's pen and show that the expression is quadratic. If 2 feet of fencing are damaged and cannot be used, how does this affect the pen's area?



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Problem-Based Task 3.1.2: Puppy Pen

Coaching

a. Let l be the length of the pen and w be the width of the pen. Write an expression for the perimeter of the rectangular pen.

b. Use what you know about the perimeter of the pen to write an expression for the pen's width in terms of the pen's length, l.

c. How can the area of the pen be expressed in terms of l?

d. Show that the area formula is quadratic.

e. If 2 feet of fencing are damaged, what is the pen's perimeter?

f. How does the change in perimeter affect the expression for width?

g. How will the area change if 2 feet of fencing are damaged?

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Instruction

Problem-Based Task 3.1.2: Puppy Pen

Coaching Sample Responses

a. Let l be the length of the pen and w be the width of the pen. Write an expression for the perimeter of the rectangular pen.

The perimeter of the pen can be expressed as $2l + 2w$ feet.

b. Use what you know about the perimeter of the pen to write an expression for the pen's width in terms of the pen's length, l.

We know that the perimeter is 20 feet. Set the perimeter equal to 20 and solve for w.

$2l + 2w = 20$

$2w = 20 - 2l$

$w = 10 - l$ feet

c. How can the area of the pen be expressed in terms of l?

The area of a rectangle is found by multiplying the length by the width.

$A = l \bullet w = l(10 - l)$ square feet

d. Show that the area formula is quadratic.

Multiply the factors of the expression.

$A = l(10 - l) = 10l - l^2$ square feet

This is a quadratic equation, with $a = -1$, $b = 10$, and $c = 0$.

e. If 2 feet of fencing are damaged, what is the pen's perimeter?

If 2 feet of fencing are damaged, the pen's perimeter will decrease from 20 feet to 18 feet.

f. How does the change in perimeter affect the expression for width?

Substitute 18 for the perimeter to find the new expression for width.

$2l + 2w = 18$

$2w = 18 - 2l$

$w = 9 - l$ feet

If 2 feet of fencing are damaged, the width will change from $10 - l$ feet to $9 - l$ feet.

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g. How will the area change if 2 feet of fencing are damaged?

Substitute the new expression for width into the area formula and multiply the factors.

$A = l(9 - l) = 9l - l^2$ square feet

If 2 feet of fencing are damaged, the area will change from $10l - l^2$ square feet to $9l - l^2$ square feet. The decrease in area is equal to $(10l - l^2) - (9l - l^2)$, which is equal to l, the length of the pen.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.1.2: Interpreting Complicated Expressions

For problems 1 and 2, use what you know about expressions to answer the questions.

1. What values of x make the expression $(x + 7)(x - 10)$ negative?

2. What values of x make the expression $(2x + 1)(x - 3)$ positive?

For problems 3 and 4, show that the expression is a quadratic expression by writing it in the form $ax^2 + bx + c$. Identify a, b, and c.

3. $(3x + 5)(4x - 1)$

4. $3x(2x + 8) + (x - 3)(x + 10)$

For problems 5 and 6, determine whether each expression is a quadratic expression. Explain your reasoning.

5. $(x + 4)(5x - 11)$

6. $(2x^2 + 9)(x - 2)$

continued

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For problems 7–10, translate any verbal expressions into quadratic expressions, and then answer the questions.

7. The property tax in dollars for p acres of property can be found by using the quadratic expression $225p^2$. How does doubling the acreage of property from 4 acres to 8 acres affect the property taxes?

8. The area of an equilateral triangle can be found by multiplying the square of the side length by $\frac{\sqrt{3}}{4}$. How does changing the side length from 2 inches to 10 inches affect the area of the triangle?

9. How does the area of a square change when the side length is halved?

10. The surface area of a cube is the product of 6 and the square of the side length. How does the surface area of a cube change when the side of a cube doubles in length?

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Progress Assessment

Circle the letter of the best answer.

1. The area of a circle is given by $A = \pi r^2$, where r is the radius of the circle. What is the value of the constant term?

   a. 2
   b. r
   c. π
   d. There is no constant term.

2. Classify the expression $x^2 - (1 - x^2) + 2(3x^2)$ when written in the form $ax^2 + bx + c$.

   a. It is a monomial.
   b. It is a binomial.
   c. It is a trinomial.
   d. It is a factor.

3. What is the value of b when $(x + 14)(2x - 3)$ is written in the form $ax^2 + bx + c$?

   a. –3
   b. 2
   c. 14
   d. 25

4. Translate the verbal expression "the difference of the square of x and the sum of nine and x" into an algebraic expression and simplify.

   a. $x^2 - 9 + x$
   b. $x^2 - x - 9$
   c. $9 + x - x^2$
   d. $x^2 + x + 9$

5. What values of x make the expression $(x + 2)(x - 5)$ negative?

   a. $-2 < x < 5$
   b. $x < -2$ and $x > 5$
   c. $5 < x < -2$
   d. $-5 < x < 2$

6. How does the area of a square change when its side lengths double?

   a. The area stays the same.
   b. The area increases by 2 square units.
   c. The area doubles.
   d. The area increases by a factor of 4.

continued

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Assessment

7. Which expression, when simplified, is quadratic?

   a. $(x - 2)(x + 3) - (x - 4)(x + 5)$
   b. $2x(5x + 1) + 3x^2(x - 2)$
   c. $(15x + 32)(17x - 21)$
   d. $x(2x + 1)(x - 13)$

8. Determine the coefficient(s) of the variables in the expression $2x^2 + x - 1$.

   a. 1 and 2
   b. –1 and 2
   c. –1, 1, and 2
   d. 2 only

9. The income in dollars from ticket sales for a concert is represented by $400p - 20p^2$, where p is the ticket price in dollars. How does an increase in the ticket price from $15 to $20 affect the income?

   a. The income decreases by $1,500.
   b. The income decreases by $2,000.
   c. The income increases by $1,500.
   d. The income increases by $2,000.

10. Which verbal expression represents the algebraic expression $2x(x^2 - 4x)$?

    a. the product of 2 and x decreased by the difference of $4x$ and x^2
    b. the difference of the product of $2x$ and x^2 and $4x$
    c. the product of $2x$ and the difference of the square of x and $4x$
    d. the product of $2x$ and the difference of $4x$ and the square of x

Use what you have learned about quadratic expressions to complete the problem that follows.

11. Taja wants to build a rectangular compost bin that has a square base and a volume of 900 cubic centimeters. The height of the compost bin is equal to the quotient of 900 and the square of one side of the base, s.

    a. Write an algebraic expression for the height of Taja's compost bin.

    b. Is this expression quadratic? Explain your reasoning.

    c. How does doubling the side of the base affect the height of the compost bin?

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Solve $2x^2 - 3 = 159$ for x.

   a. $x = 9$
   b. $x = 40.5$
   c. $x = \pm 8.8$
   d. $x = \pm 9$

2. Which factored equation can be used to solve $x^2 + 7x = 30$?

   a. $(x + 7)(x - 30) = 0$
   b. $(x + 7)(x + 30) = 0$
   c. $(x + 10)(x - 3) = 0$
   d. $(x + 3)(x - 10) = 0$

3. Find c so that $x^2 + 3x + c$ is a perfect square trinomial.

   a. $c = \frac{9}{4}$
   b. $c = 3$
   c. $c = \frac{3}{2}$
   d. $c = 9$

4. Solve $4x^2 + 5x - 6 = 0$ for x.

   a. $x = 2$ or $x = 0.75$
   b. $x = 0.75$
   c. $x = -2$
   d. $x = -2$ or $x = 0.75$

5. For what values of x is $x^2 - 2x - 15 < 0$?

   a. $-3 < x < 5$
   b. $-3 < x > 5$
   c. $-5 < x < 3$
   d. $x < -5$ or $x > 3$

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Lesson 2: Creating and Solving Quadratic Equations in One Variable

Instruction

Common Core State Standards

SMP	
1 ✓	2 ✓
3	4 ✓
5 ✓	6 ✓
7 ✓	8

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

A–REI.4 Solve quadratic equations in one variable.

a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Essential Questions

1. How can we solve a quadratic equation by taking square roots?
2. How can we solve a quadratic equation by factoring?
3. How can we solve a quadratic equation by completing the square?
4. How is the quadratic formula derived?
5. How is a quadratic inequality solved?
6. Why is it beneficial to factor a polynomial by the greatest common factor if possible?
7. How is factoring related to the Distributive Property?
8. How can a trinomial be factored when a is greater than 1?
9. Why are there some trinomials that cannot be factored?
10. Why must a quadratic equation be set equal to 0 before it can be solved by factoring?
11. How do you know which factoring method to use when solving a quadratic equation?

SRB p. U3-18

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WORDS TO KNOW

difference of two squares	a pattern for factoring a binomial that consists of two perfect squares that are being subtracted; for example, $x^2 - y^2 = (x + y)(x - y)$
discriminant	an expression whose value indicates the number and types of solutions for a quadratic equation. For a quadratic equation in standard form ($ax^2 + bx + c = 0$), the discriminant is $b^2 - 4ac$.
factor (noun)	one of two or more numbers in an expression that are multiplied to produce a product
factor (verb)	to write an expression as the product of its factors
greatest common factor (GCF)	the largest factor that two or more terms share
interval	the continuous set of all real numbers between two given numbers. The two numbers on the ends of the interval are the endpoints. The endpoints may or may not be included in the interval depending on whether the interval is open, closed, or half-open/half-closed.
irrational number	a real number that cannot be written as $\frac{m}{n}$, where m and n are integers and $n \neq 0$; a non-terminating or non-repeating decimal
leading coefficient	the coefficient of the term with the highest power. For a quadratic equation in standard form ($y = ax^2 + bx + c$), the leading coefficient is a.
perfect square	an expression that is produced by multiplying a value by itself
perfect square trinomial	a trinomial of the form $x^2 + bx + \left(\frac{b}{2}\right)^2$ that can be written as the square of a binomial
prime	a number or an expression that cannot be factored
prime factor	a factor that is a prime number
prime number	a whole number that can only be evenly divided by itself

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Instruction

quadratic equation	an equation that can be written in the form $ax^2 + bx + c = 0$, where x is the variable, a, b, and c are real numbers, and $a \neq 0$
quadratic formula	a formula that states that the solutions of a quadratic equation of the form $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. A quadratic equation can have no real solutions, one real solution, or two real solutions.
quadratic inequality	an inequality that can be written in the form $ax^2 + bx + c < 0$, $ax^2 + bx + c \leq 0$, $ax^2 + bx + c > 0$, or $ax^2 + bx + c \geq 0$
rational number	a real number that can be written as $\frac{m}{n}$, where both m and n are integers and $n \neq 0$; a terminating or repeating decimal
real numbers	the set of all rational and irrational numbers
root(s)	the solution(s) of a quadratic equation
standard form of a quadratic equation	a quadratic equation written as $ax^2 + bx + c = 0$, where a is the coefficient of the quadratic term, b is the coefficient of the linear term, and c is the coefficient of the constant term
Zero Product Property	If the product of two factors is 0, then at least one of the factors is 0.

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Recommended Resources

- Cliffs Notes. "Solving Quadratic Inequalities."

  http://www.walch.com/rr/00101

  This website explains how to solve a quadratic inequality. Users are guided through several examples.

- Illuminations. "Algebra Tiles."

  http://www.walch.com/rr/05004

  This virtual manipulative allows users to use algebra tiles to model various algebraic problems, and includes practice with expanding and factoring expressions.

- Khan Academy. "Solving Quadratic Equations by Square Roots."

  http://www.walch.com/rr/00102

  This video explains how to solve quadratic equations by taking square roots.

- Math Portal. "Quadratic Equation Solver."

  http://www.walch.com/rr/00103

  This interactive website guides users through solving a quadratic equation in standard form. Users input an equation, and then the calculator shows the steps to solve the quadratic by completing the square or by using the quadratic formula.

- MathIsFun.com. "Completing the Square."

  http://www.walch.com/rr/00104

  This website offers the user a tutorial on completing the square.

- National Library of Virtual Manipulatives. "Algebra Tiles."

  http://www.walch.com/rr/00105

  This virtual manipulative shows users how to use algebra tiles to multiply binomials, multiply polynomials using the Distributive Property, factor polynomials, and square binomials. The manipulative requires Java software to run.

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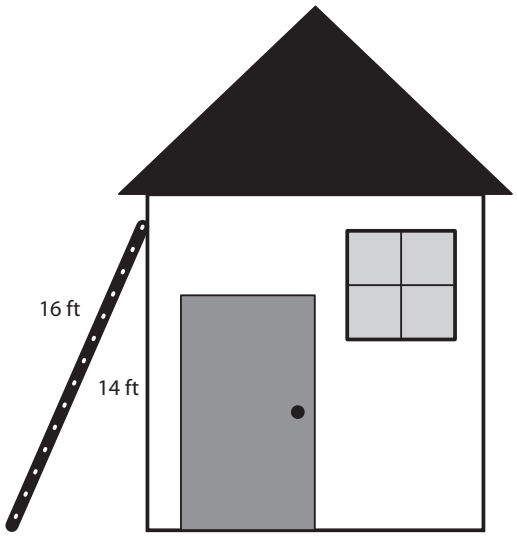
Lesson 3.2.1: Taking the Square Root of Both Sides

Warm-Up 3.2.1

A 16-foot ladder leans against the side of a house. The ladder reaches 14 feet up the side of the house.



1. How far is the base of the ladder from the house?

2. Suppose the ladder is moved 2 feet closer to the house. Now how far up the side of the house does the ladder reach?

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Instruction

Lesson 3.2.1: Taking the Square Root of Both Sides

Common Core State Standards

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

A–REI.4 Solve quadratic equations in one variable.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Warm-Up 3.2.1 Debrief

1. How far is the base of the ladder from the house?

   Use the Pythagorean Theorem to find how far the ladder is from the house. Let a represent the distance from the base of the ladder to the base of the house. Let b represent the base of the house to the point where the ladder reaches the house. Let c represent the length of the ladder.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$a^2 + (14)^2 = (16)^2$	Substitute values for b and c.
$a^2 + 196 = 256$	Simplify.
$a^2 = 60$	Subtract 196 from both sides of the equation.
$a = \sqrt{60} = 2\sqrt{15} \approx 7.75$ feet	Take the principal square root of 60, since all lengths are positive.

   The base of the ladder is about 7.75 feet from the house.

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2. Suppose the ladder is moved 2 feet closer to the house. Now how far up the side of the house does the ladder reach?

   Again, use the Pythagorean Theorem to find how far up the side of the house the ladder reaches.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$(7.75 - 2)^2 + (b)^2 = (16)^2$	Substitute values for a and c.
$5.75^2 + b^2 = 256$	Simplify.
$b^2 \approx 222.94$	Square 5.75 and subtract this value from 256.
$b \approx 14.94$ feet	Take the principal square root of 222.94, since all lengths are positive.

   The ladder reaches about 14.94 feet up the side of the house.

Connection to the Lesson

- Students will use square roots to solve quadratic equations.
- Students will build equations to represent problems in context.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- solving linear equations (8.EE.7b)
- simplifying square roots (N–RN.2)

Introduction

You can determine how far a ladder will extend from the base of a wall by creating a quadratic equation and then taking the square root of both sides. To find the unknown length, you only need to find the positive square root because a negative length would not make sense in this situation. Most problems involving solving equations of the form $x^2 = c$ usually have both a positive square root and a negative square root.

Key Concepts

- The set of **real numbers** is the set of all rational and irrational numbers.
- Real numbers are **rational numbers** when they can be written as $\frac{m}{n}$, where both m and n are integers and $n \neq 0$. Rational numbers can also be written as a terminating or repeating decimal. The real number 0.4 is a rational number because it can be written as the fraction $\frac{2}{5}$.
- Real numbers are irrational when they cannot be written as $\frac{m}{n}$, where m and n are integers and $n \neq 0$. **Irrational numbers** cannot be written as a terminating or repeating decimal. The real number $\sqrt{3}$ is an irrational number because it cannot be written as the ratio of two integers. Other examples of irrational numbers include $\sqrt{2}$ and π.
- A **quadratic equation** is an equation that can be written in the form $ax^2 + bx + c = 0$, where $a \neq 0$.
- Quadratic equations can have no real solutions, one real solution, or two real solutions.
- Quadratic equations that contain only a squared term and a constant can be solved by taking the square root of both sides. These equations can be written in the form $x^2 = c$, where c is a real number.

SRB p. U3-22

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- When we take the square root of both sides of a quadratic equation of the form $x^2 = c$, we need to remember that a number and its opposite have the same square. Therefore, rather than simply taking the principal, or positive, square root, we need to take the positive and negative square root. The solution of the equation $x^2 = c$ is $x = \pm\sqrt{c}$.
- We can use a similar method to solve quadratic equations in the form $(ax + b)^2 = c$.
- The nature of c tells us the number and type of solutions for equations of the form $x^2 = c$.

c	Number and type of solutions
Negative	No real solutions
0	One real, rational solution
Positive and a perfect square	Two real, rational solutions
Positive and not a perfect square	Two real, irrational solutions

Common Errors/Misconceptions

- forgetting to use ± when taking the square root of both sides of an equation, and therefore forgetting that there may be two solutions
- taking the square root before isolating the squared term

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Guided Practice 3.2.1

Example 1

Solve $2x^2 - 5 = 195$ for x.

1. Isolate x^2.

$2x^2 - 5 = 195$	Original equation
$2x^2 = 200$	Add 5 to both sides.
$x^2 = 100$	Divide both sides by 2.

2. Use a square root to find all possible solutions to the equation.

   Take the square root of both sides. Remember that both 10^2 and $(-10)^2$ equal 100, so there are two roots for this equation.

   $x = \pm\sqrt{100} = \pm 10$

   The equation $2x^2 - 5 = 195$ has two solutions, 10 and –10.

Example 2

Solve $4(x + 3)^2 - 10 = -6$ for x.

1. Isolate the squared binomial and take the square root of both sides of the equation.

$4(x + 3)^2 - 10 = -6$	Original equation
$4(x + 3)^2 = 4$	Add 10 to both sides.
$(x + 3)^2 = 1$	Divide both sides by 4.
$x + 3 = \pm\sqrt{1}$	Take the square root of both sides.
$x + 3 = \pm 1$	Simplify.

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2. Isolate x.

$x + 3 = \pm 1$	Equation from the previous step
$x = -3 \pm 1$	Subtract 3 from both sides.

3. Separate the equation into two equations and solve.

$x = -3 + 1 = -2$

$x = -3 - 1 = -4$

The equation $4(x + 3)^2 - 10 = -6$ has two solutions, -2 and -4.

Example 3

Solve $(x - 1)^2 + 15 = -1$ for x.

1. Isolate the squared binomial.

$(x - 1)^2 + 15 = -1$	Original equation
$(x - 1)^2 = -16$	Subtract 15 from both sides.

2. Take the square root of both sides to isolate the binomial.

$(x - 1)^2 = -16$	Equation from the previous step
$x - 1 = \pm\sqrt{-16}$	Take the square root of both sides.

3. Determine the solution, if one exists.

There is a negative number under the radical, so the equation has no real solutions.

The equation $(x - 1)^2 + 15 = -1$ has no real solutions.

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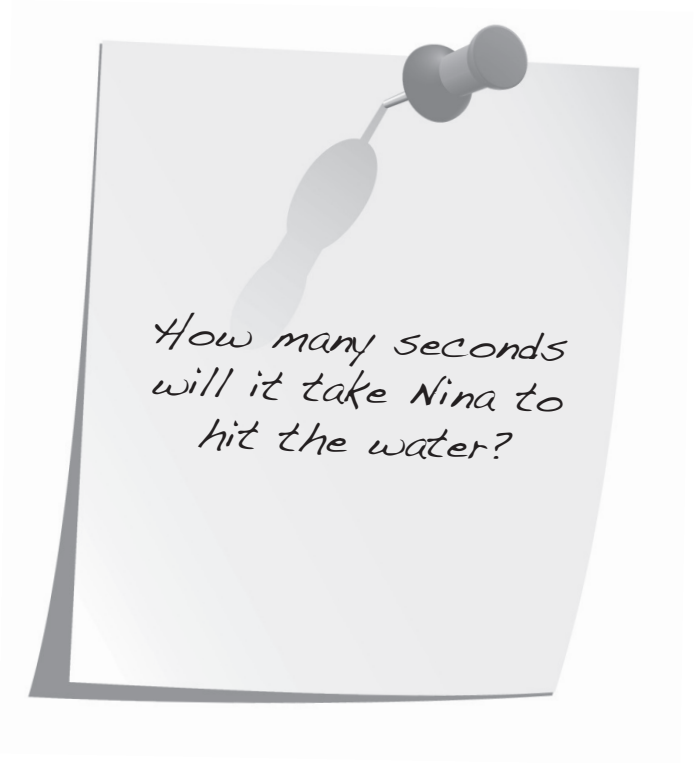
Problem-Based Task 3.2.1: Time to Splash

SMP	
1 ✓	2
3	4 ✓
5	6
7 ✓	8

Nina dives into a pool from a platform 3.75 feet above the water. Her height above the water in feet, x seconds after she jumps, is given by the expression $-5(x - 0.5)^2 + 5$. How many seconds will it take Nina to hit the water?



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Problem-Based Task 3.2.1: Time to Splash

Coaching

a. How high above the water will Nina be when she hits the water?

b. Using your answer from part a, write an equation in terms of x that represents the number of seconds that will have elapsed when Nina hits the water.

c. What is the solution of the equation from part b?

d. Does a negative answer make sense for this problem? Explain your reasoning.

e. Is a positive answer reasonable for this problem? Explain your reasoning.

f. How many seconds will it take Nina to hit the water?

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Instruction

Problem-Based Task 3.2.1: Time to Splash

Coaching Sample Responses

a. How high above the water will Nina be when she hits the water?

When Nina hits the water, she will be 0 feet above the water.

b. Using your answer from part a, write an equation in terms of x that represents the number of seconds that will have elapsed when Nina hits the water.

Set the expression for height equal to 0.

$-5(x - 0.5)^2 + 5 = 0$

c. What is the solution of the equation from part b?

Solve the equation by isolating the squared binomial and taking the square root of both sides.

First, subtract 5 from both sides.

$-5(x - 0.5)^2 = -5$

Next, divide both sides by –5.

$(x - 0.5)^2 = 1$

Take the square root of both sides.

$x - 0.5 = \pm 1$

$x = 0.5 \pm 1$

$x = 1.5$ and $x = -0.5$

d. Does a negative answer make sense for this problem? Explain your reasoning.

The variable x stands for the number of seconds after Nina jumps. A negative answer does not make sense for this problem since time cannot be negative, and Nina will not hit the water before she jumps from the platform.

e. Is a positive answer reasonable for this problem? Explain your reasoning.

The positive answer is 1.5 seconds. It is reasonable that Nina would hit the surface of the water after 1.5 seconds.

f. How many seconds will it take Nina to hit the water?

Nina will hit the water after 1.5 seconds.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.2.1: Taking the Square Root of Both Sides

Solve each equation for x.

1. $x^2 = 81$

2. $x^2 = -25$

3. $x^2 - 5 = 4$

4. $(x + 3)^2 = 1$

5. $(x + 3)^2 + 7 = -2$

6. $4(x - 10)^2 = 25$

Use what you know about square roots to complete problems 7–10. Round to the nearest hundredth, if necessary.

7. When does a quadratic equation in the form $ax^2 + b = c$ have two real, rational solutions?

8. The area of a square with sides of length s is given by s^2. The area of a square is 40 square centimeters. What is the length of one side of the square, rounded to the nearest hundredth?

9. The area of a circle with radius r is given by πr^2. The area of a circle is 60 square millimeters. What is the radius of the circle, rounded to the nearest hundredth?

10. The surface area of a cube with edges of length a is given by $6a^2$. If the surface area of a cube is 200 square inches, what is the length of each edge of the cube, rounded to the nearest hundredth?

SWB p. 33; SRB Practice 3.2.1 p. U3-26

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Lesson 3.2.2: Factoring Expressions by the Greatest Common Factor

Warm-Up 3.2.2

Several graphic artists competing on a reality show must design a new logo for a struggling business. The artists are divided into three groups: Group A, Group B, and Group C. Each group must fit the logo within a rectangular space. To make the task more challenging, each group is assigned specific measurements for one or two of the dimensions of their rectangle—either just the area measurement, the area and the length, or the area and the width. The other dimensions of the rectangle can vary, as long as the assigned measurements are correct. Given the formula for the area of a rectangle, $A = lw$, and the requirement that all dimensions must be in whole numbers, use the information given in each problem to determine the requested information about the rectangles.

1. Group A is assigned an area of 30 square inches. How many different rectangles can be created, and what are the lengths and widths for each rectangle?

2. Group B is assigned a length of 6 inches and an area of $30x$ square inches. What is the width of this rectangle?

3. Group C is assigned a width of $3x$ and an area of $30x^3y$ square inches. What is the length of this rectangle?

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Lesson 3.2.2: Factoring Expressions by the Greatest Common Factor

Common Core State Standard

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

Warm-Up 3.2.2 Debrief

Several graphic artists competing on a reality show must design a new logo for a struggling business. The artists are divided into three groups: Group A, Group B, and Group C. Each group must fit the logo within a rectangular space. To make the task more challenging, each group is assigned specific measurements for one or two of the dimensions of their rectangle—either just the area measurement, the area and the length, or the area and the width. The other dimensions of the rectangle can vary, as long as the assigned measurements are correct. Given the formula for the area of a rectangle, $A = lw$, and the requirement that all dimensions must be in whole numbers, use the information given in each problem to determine the requested information about the rectangles.

1. Group A is assigned an area of 30 square inches. How many different rectangles can be created, and what are the lengths and widths for each rectangle?

   In order to determine the different possibilities, it is necessary to list all of the combinations of numbers that multiply to yield a product of 30.

   - Any number can be written as a product of itself and 1, so 30 • 1 is one possibility.
   - Since 30 is an even number, it can be divided by 2. Therefore, 15 • 2 is another possibility.
   - Since 30 has a 0 in the ones column, it can be divided by both 5 and 10. Two more possibilities are 10 • 3 and 6 • 5.

   There are four possibilities for a rectangle with an area of 30 square inches:

   $l = 30$ in, $w = 1$ in

   $l = 15$ in, $w = 2$ in

   $l = 10$ in, $w = 3$ in

   $l = 6$ in, $w = 5$ in

   Notice that it is also possible to reverse the lengths and widths of each rectangle, but essentially the rectangles are the same.

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2. Group B is assigned a length of 6 inches and an area of $30x$ square inches. What is the width of this rectangle?

   For a rectangle to have a length of 6 inches and an area of $30x$ square inches, it is necessary to determine the missing dimension such that $6 \cdot w = 30x$. What number multiplied by 6 results in 30, or what is 30 divided by 6? The answer is $30 \div 6 = 5$.

   Therefore, the width will be 5 inches, and since the dimensions of the area include an x, the width also needs to include an x.

   The width of this rectangle will be $5x$ inches, such that $6 \cdot 5x = 30x$.

3. Group C is assigned a width of $3x$ and an area of $30x^3y$ square inches. What is the length of this rectangle?

   For a rectangle to have a width of $3x$ inches and an area of $30x^3y$ square inches, it is necessary to determine the missing dimension such that $l \cdot 3x = 30x^3y$.

   First, determine the number that when multiplied by 3 results in 30; in other words, find the result of 30 divided by 3. The answer is $30 \div 3 = 10$. Therefore, the length will have a coefficient of 10.

   Next, determine the variables necessary to obtain an area of $30x^3y$. Since the width ($3x$) already has the variable x, the coefficient of the length (10) will need to be multiplied by x^2. (Recall that by the Product of Powers Property of exponents, $x^3 = x \cdot x^2$.)

   Finally, since the area contains the variable y, the length will also need to be multiplied by y.

   Therefore, the length for this rectangle will be $10x^2y$ inches, such that $10x^2y \cdot 3x = 30x^3y$.

Connection to the Lesson

- Students will identify factors of a monomial and a polynomial.
- Students will identify the greatest common factor of a polynomial and use it to factor the expression.
- Students will apply the Distributive Property in order to verify correct factoring.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- determining the prime factorization of whole numbers (4.OA.4)
- applying the rules of exponents when multiplying them with variables (6.EE.2)
- applying the Distributive Property to multiply a monomial by a polynomial (6.EE.3)

Introduction

A variety of concepts, systems, and processes can be described using polynomial expressions. Some of these include familiar geometrical ideas such as area and perimeter. Other real-world applications involve more complex expressions, like the paths followed by baseballs, or satellites in orbit. **Factoring**, or writing the expressions as the product of their factors, can make the expressions easier to work with and apply to different scenarios. One of the most common forms of factoring is by the greatest common factor, or GCF.

Key Concepts

- Recall that a monomial is an expression with one term, consisting of a number, a variable, or the product of a number and variable(s), whereas a polynomial is an expression with two or more terms, consisting of a number, a variable, or the product of a number and variable(s).
- The factors of a given number or monomial are numbers or expressions which multiply together to equal that given number, or monomial. For example, 6 and 3 are factors of 18, and x and x^2 are factors of x^3. Two possible factors of the monomial $15x^4y$ are $5xy$ and $3x^3$.
- A common factor is a factor that two or more numbers share, or the same number or variable that appears in all terms of a polynomial. You can find common factors by listing the factors for each term and then comparing the results. For example, consider the polynomial $36x^4 + 27x^3y$. One common factor of 36 and 27 is 3, since $3 \cdot 12 = 36$ and $3 \cdot 9 = 27$. As for the variables, x is a common factor found in both terms. Notice that the variable y does not appear in both terms of the polynomial, so it is not a common factor. Thus, a common factor of $36x^4 + 27x^3y$ is $3x$.
- The **greatest common factor (GCF)** of a polynomial is the common factor with the largest constant number and power of each variable. In other words, the GCF is the largest factor that two or more terms share.
- The GCF for the polynomial $36x^4 + 27x^3y$ can be found by first breaking down each term by listing all of its prime factors. **Prime factors** are factors that are **prime numbers**, which are whole numbers that can only be divided evenly by 1 and the number itself.

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Instruction

- For instance, the prime factors of the terms of the polynomial $36x^4 + 27x^3y$ are as follows:

  $36x^4 = 2 \bullet 2 \bullet 3 \bullet 3 \bullet x \bullet x \bullet x \bullet x$

  $27x^3y = 3 \bullet 3 \bullet 3 \bullet x \bullet x \bullet x \bullet y$

- To determine the GCF of this polynomial, determine which factors each term has in common. Each term has two prime factors of 3 in common, and each term also has three factors of x:

  $36x^4 = 2 \bullet 2 \bullet \boxed{3} \bullet \boxed{3} \bullet \boxed{x} \bullet \boxed{x} \bullet \boxed{x} \bullet x$

  $27x^3y = 3 \bullet \boxed{3} \bullet \boxed{3} \bullet \boxed{x} \bullet \boxed{x} \bullet \boxed{x} \bullet y$

- Therefore, the GCF of the polynomial $36x^4 + 27x^3y$ is $3 \bullet 3 \bullet x \bullet x \bullet x = 9x^3$.
- Factoring is the process of rewriting a given expression as the product of its factors. Once the GCF is identified, then the remaining factors are determined by multiplying the GCF by the constants and/or variables necessary to obtain the original term:

  $9x^3 \bullet 4x = 36x^4$

  $9x^3 \bullet 3y = 27x^3y$

- The completed factoring for the polynomial expression $36x^4 + 27x^3y$ is $9x^3(4x + 3y)$.
- To verify that the factors have been correctly identified, apply the Distributive Property to the factored expression. The result should be the original polynomial.
- Distributing $9x^3$ over $(4x + 3y)$ yields $9x^3 \bullet 4x + 9x^3 \bullet 3y = 36x^4 + 27x^3y$, which verifies that the identified factors are correct.
- An expression that cannot be factored is said to be **prime**.

Common Errors/Misconceptions

- misidentifying the greatest common factor
- incorrectly applying a negative sign when factoring or applying the Distributive Property
- incorrectly applying the rules of exponents; e.g., $x^2 \bullet x^3 \neq x^6$
- including factors that are common to some terms but not to all terms

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Guided Practice 3.2.2

Example 1

Factor the polynomial $12x^3 + 30x^2 + 42x$ by finding the greatest common factor (GCF). Verify your results using the Distributive Property.

1. Determine the common factors of the terms' coefficients and variables.

   First factor the coefficients (12, 30, and 42) into their prime factors.

   The prime factors are as follows:

   $12 = 2 \bullet 2 \bullet 3$

   $30 = 2 \bullet 3 \bullet 5$

   $42 = 2 \bullet 3 \bullet 7$

   Each coefficient's factors include 2 and 3, so $2 \bullet 3 = 6$ is common to all three terms.

   Next, determine the greatest power of x common to all three terms. The first term has x^3, the second term has x^2, and the third term has x (or x^1). Expand these terms:

   $x^3 = x \bullet x \bullet x$

   $x^2 = x \bullet x$

   $x^1 = x$

   Only x (or x to the first power) is common to all three terms. Combining this with the greatest numerical factor of 6, the resulting GCF is $6x$.

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Instruction

2. Write expressions for the remaining non-common factors.

   Write out the prime factors of each term:

   $12x^3 = 2 \bullet 2 \bullet 3 \bullet x \bullet x \bullet x$

   $30x^2 = 2 \bullet 3 \bullet 5 \bullet x \bullet x$

   $42x = 2 \bullet 3 \bullet 7 \bullet x$

   Eliminate the common factors of each term and determine which non-common factors are left:

   $12x^3 = \cancel{2} \bullet \boxed{2} \bullet \cancel{3} \bullet \cancel{x} \bullet \boxed{x} \bullet \boxed{x}$; the remaining factors are $2 \bullet x \bullet x$, or $2x^2$

   $30x^2 = \cancel{2} \bullet \cancel{3} \bullet \boxed{5} \bullet \cancel{x} \bullet \boxed{x}$; the remaining factors are $5 \bullet x$, or $5x$

   $42x = \cancel{2} \bullet \cancel{3} \bullet \boxed{7} \bullet \cancel{x}$; the remaining factor is 7

   The expressions for the remaining non-common factors are $2x^2$, $5x$, and 7.

3. Factor the original polynomial by writing it as the product of the GCF and a polynomial whose terms are the expressions found in the previous step.

   The GCF is $6x$. The remaining terms are $2x^2$, $5x$, and 7, which can be summed to produce the polynomial $2x^2 + 5x + 7$.

   The product of the GCF and this polynomial is $6x(2x^2 + 5x + 7)$.

4. Verify the result using the Distributive Property.

   To verify the result, apply the Distributive Property by multiplying $6x$ by each term of the polynomial.

   $6x(2x^2 + 5x + 7)$

   $= 6x \bullet 2x^2 + 6x \bullet 5x + 6x \bullet 7$

   $= 12x^3 + 30x^2 + 42x$

   The result of applying the Distributive Property is the original expression, $12x^3 + 30x^2 - 42x$. Thus, the polynomial is factored correctly and $12x^3 + 30x^2 + 42x = 6x(2x^2 + 5x + 7)$.

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Instruction

Example 2

Factor the polynomial $35xy^4 - 14x^4y^2z + 56x^3y^3z^3$ by finding the GCF. Verify your results using the Distributive Property.

1. Determine the common factors of the coefficients for each term.

   First factor the coefficients (35, –14, and 56) into their prime factors:

   $35 = 5 \bullet 7$

   $-14 = -1 \bullet 2 \bullet 7$

   $56 = 2 \bullet 2 \bullet 2 \bullet 7$

   Note that a negative number, like –14, will always have a negative factor that can be represented by –1.

   The only numerical factor that is common to all three terms is 7.

2. Determine the greatest power of each variable common to all terms, and multiply this by the common factor -from step 1 to complete the GCF.

   The first term has the variables xy^4, the second has variables x^4y^2z, and the third has variables $x^3y^3z^3$. Expand these terms:

   $xy^4 = x \bullet y \bullet y \bullet y \bullet y$

   $x^4y^2z = x \bullet x \bullet x \bullet x \bullet y \bullet y \bullet z$

   $x^3y^3z^3 = x \bullet x \bullet x \bullet y \bullet y \bullet y \bullet z \bullet z \bullet z$

   One factor of x and two factors of y, or y^2, are common to all three terms. Since z is not a factor of the first term, it is not a common factor.

   Multiplying these common variables by the result of step 1 yields a GCF of $7xy^2$.

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Instruction

3. Write expressions for the remaining non-common factors.

   Write out the prime factors of each term:

   $35xy^4 = 5 \cdot 7 \cdot x \cdot y \cdot y \cdot y \cdot y$

   $-14x^4y^2z = -1 \cdot 2 \cdot 7 \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot z$

   $56x^3y^3z^3 = 2 \cdot 2 \cdot 2 \cdot 7 \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot z \cdot z \cdot z$

   Eliminate the common factors of each term and determine which non-common factors are left:

   $35xy^4 = \boxed{5} \cdot \cancel{7} \cdot \cancel{x} \cdot \cancel{y} \cdot \cancel{y} \cdot \boxed{y} \cdot \boxed{y}$

   The remaining factors are $5 \cdot y \cdot y$, or $5y^2$.

   $-14x^4y^2z = \boxed{-1} \cdot \boxed{2} \cdot \cancel{7} \cdot \cancel{x} \cdot \boxed{x} \cdot \boxed{x} \cdot \boxed{x} \cdot \cancel{y} \cdot \cancel{y} \cdot \boxed{z}$

   The remaining factors are $-1 \cdot 2 \cdot x \cdot x \cdot x \cdot z$, which simplifies to $-2x^3z$.

   $56x^3y^3z^3 = \boxed{2} \cdot \boxed{2} \cdot \boxed{2} \cdot \cancel{7} \cdot \cancel{x} \cdot \boxed{x} \cdot \boxed{x} \cdot \cancel{y} \cdot \cancel{y} \cdot \boxed{y} \cdot \boxed{z} \cdot \boxed{z} \cdot \boxed{z}$

   The remaining factors are $2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot y \cdot z \cdot z \cdot z$, which simplifies to $8x^2yz^3$.

   The expressions for the remaining non-common factors are $5y^2$, $-2x^3z$, and $8x^2yz^3$.

4. Factor the original polynomial by writing it as the product of the GCF and a polynomial whose terms are the expressions found in the previous step.

   The GCF is $7xy^2$. The remaining terms are $5y^2$, $-2x^3z$, and $8x^2yz^3$, which can be summed to produce the polynomial $5y^2 - 2x^3z + 8x^2yz^3$.

   The product of the GCF and this polynomial is $7xy^2(5y^2 - 2x^3z + 8x^2yz^3)$.

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5. Verify the result using the Distributive Property.

   Apply the Distributive Property by multiplying $7xy^2$ by each term of the polynomial.

   $7xy^2(5y^2 - 2x^3z + 8x^2yz^3)$

   $= 7xy^2 \bullet 5y^2 - 7xy^2 \bullet 2x^3z + 7xy^2 \bullet 8x^2yz^3)$

   $= 35xy^4 - 14x^4y^2z + 56x^3y^3z^3$

   The result of applying the Distributive Property is the original expression, $35xy^4 - 14x^4y^2z + 56x^3y^3z^3$. Thus, the polynomial is factored correctly and $35xy^4 - 14x^4y^2z + 56x^3y^3z^3 = 7xy^2(5y^2 - 2x^3z + 8x^2yz^3)$.

Example 3

The polynomial $15x^2 - 3x$ represents the area of a rectangular garden plot in square yards, where the length of the garden is equal to the GCF. Determine expressions for the length and width of the garden.

1. Determine the common factors of the terms' coefficients.

   First factor the coefficients (15 and –3) into their prime factors:

   $15 = 3 \bullet 5$

   $-3 = -1 \bullet 3$

   The only numerical factor that is common to both terms is 3.

2. Determine the greatest power of each variable common to all terms, and multiply it by the common factor from step 1 to complete the GCF.

   The first term has the variable x^2, or $x \bullet x$.

   The second term has only x, or x^1. Only x, or x to the first power, is common to both terms.

   Multiplying by the result from step 1 gives a GCF of $3x$.

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3. Write expressions for the remaining non-common factors.

   The original first term, $15x^2$, has factors of 3, 5, x, and x, of which 3 and x are common. The expression from the remaining factors is $5 \cdot x$, or $5x$.

   The original second term, $-3x$, has factors of -1, 3, and x. Ignoring the common factors of 3 and x, the only non-common factor is -1.

4. Write the polynomial whose terms are the expressions found in step 3.

   The terms found in step 3 are $5x$ and -1; therefore, the polynomial is $(5x - 1)$.

5. Identify the required expressions for the length and width.

   The original problem stated that the length of the garden is equal to the GCF, which we found in step 2 to be $3x$. Therefore, the length is $3x$.

   To find the width, use the area formula: $A = lw$.

   The area of the garden in square yards is $15x^2 - 3x$, the original polynomial, and the garden's length is $3x$, so $15x^2 - 3x = 3x \cdot w$.

   The GCF of $3x$ was found by factoring the area into $3x(5x - 1)$.

   Therefore, $(5x - 1)$ is the width of the garden.

   The length of the garden is $3x$ yards, and the width is $(5x - 1)$ yards.

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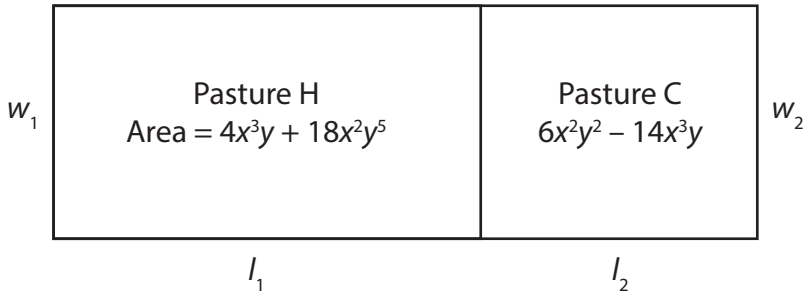
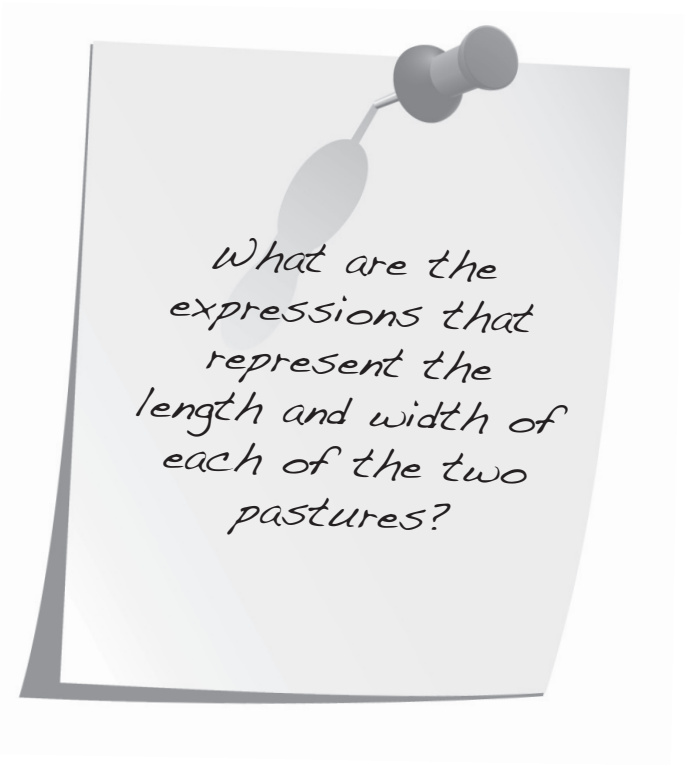
Problem-Based Task 3.2.2: Out to Pasture

SMP	
1 ✓	2 ✓
3	4
5	6
7 ✓	8

A farmer has a long rectangular field, which is divided into two smaller rectangular fields by a fence. The field on the left, Pasture H, is reserved for horses and has an area in square meters represented by the polynomial $4x^3y + 18x^2y^5$. The field's width is equal to the greatest common factor of the polynomial. The field on the right, Pasture C, is reserved for cows. Its area in square meters is $6x^2y^2 - 14x^3y$. What are the expressions that represent the length and width of each of the two pastures?





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Problem-Based Task 3.2.2: Out to Pasture

Coaching

a. What is the greatest common factor of the polynomial representing the area of Pasture H?

b. What is the remaining factor of this polynomial?

c. To what length on the diagram does the answer to part b correspond?

d. On the diagram, what side of Pasture C corresponds to the answer determined in part a?

e. What is the remaining non-common factor of Pasture C's polynomial?

f. To what length on the diagram does the answer to part e correspond?

g. What are the expressions that represent the length and width of each of the two pastures?

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Instruction

Problem-Based Task 3.2.2: Out to Pasture

Coaching Sample Responses

a. What is the greatest common factor of the polynomial representing the area of Pasture H?

List all of the prime factors of the three terms of the polynomial:

$4x^3y = 2 \bullet 2 \bullet x \bullet x \bullet x \bullet y$

$18x^2y^5 = 2 \bullet 3 \bullet 3 \bullet x \bullet x \bullet y \bullet y \bullet y \bullet y \bullet y$

$2x^2y = 2 \bullet x \bullet x \bullet y$

The three terms have 2, x, x, and y in common, so the GCF is $2x^2y$.

The problem states that the GCF is equal to the width, so $2x^2y$ is the width, w_1, of Pasture H.

b. What is the remaining factor of this polynomial?

When the area of Pasture H is factored by the GCF of $2x^2y$, the remaining factors of the polynomial are $2x$ and $9y^4$.

Multiplying the GCF of $2x^2y$ by these terms yields $2x^2y(2x + 9y^4)$, so the other factor is $(2x + 9y^4)$.

c. To what length on the diagram does the answer to part b correspond?

In part a, the width of Pasture H was found to be $2x^2y$. Since $A = lw$ and $4x^3y + 18x^2y^5 = (2x + 9y^4)2x^2y$, the answer to part b corresponds to l_1, the length of Pasture H.

d. On the diagram, what side of Pasture C corresponds to the answer determined in part a?

Since opposite sides of rectangles have equal measures, the width of Pasture H, $2x^2y$, is also w_2, the width of Pasture C.

e. What is the remaining non-common factor of Pasture C's polynomial?

The area of Pasture C is given as $6x^2y^2 - 14x^3y$. Factoring this area by the GCF of $2x^2y$, which is also the width of Pasture C, yields $2x^2y(3y - 7x)$, so the other factor is $(3y - 7x)$.

f. To what length on the diagram does the answer to part e correspond?

In part d, the width of Pasture C was found to be $2x^2y$. Since $A = lw$ and $2x^2y(3y - 7x) = 6x^2y^2 - 14x^3y = (3y - 7x)2x^2y$, the answer to part e corresponds to the remaining unknown side, l_2, which is the length of Pasture C.

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g. What are the expressions that represent the length and width of each of the two pastures?

For Pasture H, the length is $(2x + 9y^4)$ meters, and the width is $2x^2y$ meters.

For Pasture C, the length is $(3y - 7x)$ meters, and the width is $2x^2y$ meters.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.2.2: Factoring Expressions by the Greatest Common Factor

For problems 1–6, factor each expression by the greatest common factor if a common factor exists, or state that the terms have no common factor.

1. $3x^3 + 5x^2$
2. $2x^2y - 8xy^2$
3. $y^4 + 2y^2$
4. $x^2 - 9x + 3$
5. $x^3y^2 - 2x^2y^3 + 5xy^2$
6. $7x - 21x^2y$

Use what you have learned about factoring polynomials by the GCF to complete problems 7 and 8.

7. Christopher has two bags of marbles. The number of marbles in the first bag can be represented by the monomial $45x^2y$, and the number of marbles in the other bag can be represented by $60x^3y^2$. What is the GCF of these two monomials?

8. An equilateral triangle has a perimeter of $(15x^3 + 33y^2)$ feet. What is the length of each side?

Use the following information to complete problems 9 and 10.

> Samuel and Ariana are competing in a speed round for an open position on the math team. To win the spot, each student must factor the same polynomial expression, $12xyz^2 + 16x^2y^2z - 32x^2yz$, by finding the GCF.

9. Samuel's final result was $2xyz(6z + 8xy - 16x)$. Explain his error, if any.

10. Ariana's final result was $4xyz^2(3x + 4xy - 8xyz)$. Explain her error, if any.

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Lesson 3.2.3: Factoring Expressions with $a = 1$

Warm-Up 3.2.3

Erica and her dad want to plant a rectangular vegetable garden in their backyard, and are trying to determine the best size. They need to know how much area will be taken up by the different options they have in mind for the length and width. Use the formula for the area of a rectangle, $A = lw$, to complete the following problems.

1. If the length is $(x + 5)$ feet and the width is $(x + 3)$ feet, what expression represents the area of the garden?

2. If the length is $(x + 4)$ feet and the width is $(x – 4)$ feet, what expression represents the area of the garden?

3. If they want the area to be $(x^2 + 7x)$ square feet with a length of x feet, what does the width need to be?

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Lesson 3.2.3: Factoring Expressions with $a = 1$

Common Core State Standard

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

Warm-Up 3.2.3 Debrief

Erica and her dad want to plant a rectangular vegetable garden in their backyard, and are trying to determine the best size. They need to know how much area will be taken up by the different options they have in mind for the length and width. Use the formula for the area of a rectangle, $A = lw$, to complete the following problems.

1. If the length is $(x + 5)$ feet and the width is $(x + 3)$ feet, what expression represents the area of the garden?

   Since the area of a rectangle is equal to the length times the width, multiply $(x + 5)$ by $(x + 3)$ to find the area.

$A = lw$	Formula for the area of a rectangle
$A = (x + 5)(x + 3)$	Substitute $(x + 5)$ for l and $(x + 3)$ for w.
$A = x^2 + 3x + 5x + 15$	Multiply by distributing.
$A = x^2 + 8x + 15$	Combine like terms.

   The expression representing the area of the garden is $(x^2 + 8x + 15)$ square feet.

2. If the length is $(x + 4)$ feet and the width is $(x - 4)$ feet, what expression represents the area of the garden?

   Again, multiply the length $(x + 4)$ by the width $(x - 4)$ to find the area.

$A = lw$	Formula for the area of a rectangle
$A = (x + 4)(x - 4)$	Substitute $(x + 4)$ for l and $(x - 4)$ for w.
$A = x^2 + 4x - 4x - 16$	Multiply by distributing.
$A = x^2 - 16$	Combine like terms.

   The expression representing the area of the garden is $(x^2 - 16)$ square feet.

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3. If they want the area to be $(x^2 + 7x)$ square feet with a length of x feet, what does the width need to be?

   Substitute the given values for the area and the length into $A = lw$ and solve for the width.

$A = lw$	Formula for the area of a rectangle
$(x^2 + 7x) = (x)w$	Substitute $(x^2 + 7x)$ for A and x for l.
$x(x + 7) = xw$	Factor the polynomial.
$x + 7 = w$	Divide both sides by x.

   The width of the garden would need to be $(x + 7)$ feet.

Connection to the Lesson

- Students will continue to factor expressions by the greatest common factor.
- Students will factor trinomials.
- Students will factor binomials using the difference of two squares.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- multiplying binomials (A–APR.1)
- factoring expressions by the greatest common factor (A–SSE.2)
- factoring monomials and constants (4.OA.4)

Introduction

Knowing how to factor variable expressions is a valuable skill, useful for solving quadratic equations, dividing polynomials, and simplifying radical expressions. Factoring is the opposite of multiplication, and since polynomials are often created by distributing, factoring can often be thought of as reversing distribution to determine the factors that were originally used to create the polynomial. Once an expression has been factored by its greatest common factor, it must be examined to see if it can be factored further.

Key Concepts

Factoring the Difference of Two Squares

- A **perfect square** is an expression that is produced by multiplying a value by itself. When the square root of a perfect square is taken, the result is a whole number or variable.
- Recall that a binomial is a polynomial with two terms.
- A binomial consisting of two perfect squares that are being subtracted is known as the **difference of two squares**. The general form is $x^2 - y^2$.
- Such a binomial can be factored as follows: $x^2 - y^2 = (x + y)(x - y)$. Notice that the square root of the first term becomes the first term in both factors, and the square root of the second term becomes the second term in both factors. Also notice that in the first factor the terms are added, and in the other they are subtracted.
- When distributed, $(x + y)(x - y) = x^2 - y^2$.

Factoring Trinomials when $a = 1$

- Recall that a trinomial is a polynomial with three terms.
- When a trinomial takes the form $ax^2 + bx + c$, the coefficient a is called the **leading coefficient** because it is the coefficient of the term with the highest power.

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Instruction

- When $a = 1$ in a trinomial of the form $ax^2 + bx + c$, you may be able to factor the trinomial by reversing the distribution. To do this, you must determine which set of factors would create the given trinomial when distributed.
- Use the following steps to factor a trinomial when the value of the leading coefficient (a) is 1:

Factoring a Trinomial with a Leading Coefficient of 1

1. Create two sets of parentheses: ( )( )
2. Write the variable as the first term in each set of parentheses.
3. Separately list all the possible sets of factors for the constant term (include both positive and negative numbers).
4. Determine which set of factors has a sum that equals the coefficient of the first-degree term.
5. Write those factors, including their signs, as the second terms in both sets of parentheses.
6. Distribute the factors to verify that they result in the original trinomial.

- Note that not all trinomials can be factored.

Factoring Perfect Square Trinomials

- A **perfect square trinomial** is a trinomial that often takes the form $(ax)^2 \pm 2abx + b^2$. Notice that both the first and last terms are perfect squares, and the middle term is twice the product of the square roots of the first and last terms.
- Perfect square trinomials can be factored using the same process for factoring all trinomials. The trinomial can also be factored as follows: $(ax)^2 \pm 2abx + b^2 = (a \pm b)^2$.
- When factoring any polynomial expression, including perfect square trinomials, always check to see if the expression can be factored by the greatest common factor first.

Common Errors/Misconceptions

- neglecting to factor out common factors first
- neglecting to include the common factor after it has been factored
- not taking the square root when using the difference of two squares pattern
- confusing the signs of the terms when factoring trinomials
- incorrectly applying the rules for factoring trinomials

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Guided Practice 3.2.3

Example 1

Factor $9x^2 - 16$, and then verify your results.

1. Determine any common factors of the given binomial, if common factors exist.

   The prime factors of $9x^2$ are 3, 3, x, and x.

   The prime factors of 16 are 2, 2, 2, and 2.

   There are no factors that are common to both terms.

2. Determine if the given binomial meets the conditions of the difference of two squares.

   Binomials that are the difference of two squares are of the form $x^2 - y^2$.

   In other words, both terms must be perfect squares, and the second term must be subtracted from the first term.

   $9x^2$ is a perfect square because $9x^2$ is the product of two equal factors ($3x \bullet 3x = 9x^2$).

   16 is also a perfect square because 16 is the product of two equal factors ($4 \bullet 4 = 16$).

   Both terms are perfect squares, and they are being subtracted. Thus, this binomial meets the conditions of the difference of two squares.

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Instruction

3. Use the pattern $x^2 - y^2 = (x + y)(x - y)$ to factor the given binomial.

   Since the binomial $9x^2 - 16$ meets the conditions of the difference of two squares, it can be factored using the pattern $x^2 - y^2 = (x + y)(x - y)$.

   Notice that the square root of the first term becomes the first term in both factors and the square root of the second term becomes the second term in both factors. In one factor the terms are added, and in the other they are subtracted.

   Substitute $3x$ and 4 (the square roots of $9x^2$ and 16) for x and y in the difference of two squares formula to factor the binomial.

$x^2 - y^2 = (x + y)(x - y)$	Difference of two squares
$(9x^2 - 16) = (x + y)(x - y)$	Substitute $9x^2 - 16$ for $x^2 - y^2$.
$(9x^2 - 16) = [(3x) + (4)][(3x) - (4)]$	Substitute $3x$ for x and 4 for y.

   The factors of $9x^2 - 16$ are $(3x + 4)$ and $(3x - 4)$.

4. Multiply the factors to verify that they result in the original binomial.

   Factors must always equal the original expression when multiplied.

$(3x + 4)(3x - 4)$	Multiply the factors.
$= 3x(3x) + 3x(-4) + 4(3x) + 4(-4)$	Distribute.
$= 9x^2 - 12x + 12x - 16$	Multiply.
$= 9x^2 - 16$	Combine like terms.

   The product of $(3x + 4)(3x - 4)$ equals the original expression, $9x^2 - 16$.

   Therefore, it is confirmed that $9x^2 - 16$ factors to $(3x + 4)(3x - 4)$.

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Instruction

Example 2

Factor $5y^2 - 45$, and then verify your results.

1. Determine any common factors of the given binomial, if common factors exist.

   The prime factors of $5y^2$ are 5, y, and y.

   The prime factors of 45 are 3, 3, and 5.

   5 is the greatest common factor of these two terms and can be factored out:

   $5y^2 - 45$

   $= 5 \cdot y^2 - 5 \cdot 9$

   $= 5(y^2 - 9)$

   When factored by the greatest common factor, $5y^2 - 45 = 5(y^2 - 9)$.

2. Determine if the binomial $y^2 - 9$ meets the conditions of the difference of two squares.

   Binomials that are the difference of two squares are of the form $x^2 - y^2$.

   y^2 is a perfect square because y^2 is the product of two equal factors ($y \cdot y = y^2$).

   9 is also a perfect square because 9 is the product of two equal factors ($3 \cdot 3 = 9$).

   Both terms are perfect squares, and they are being subtracted. Thus, this binomial meets the conditions of the difference of two squares.

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Instruction

3. Use the pattern $x^2 - y^2 = (x + y)(x - y)$ to factor the binomial.

   Since the binomial term of $5(y^2 - 9)$ meets the conditions of the difference of two squares, it can be factored using the pattern $x^2 - y^2 = (x + y)(x - y)$.

   Substitute y and 3 (the square roots of y^2 and 9) for x and y in the difference of two squares formula to factor the binomial.

$x^2 - y^2 = (x + y)(x - y)$	Difference of two squares
$(y^2 - 9) = (x + y)(x - y)$	Substitute $(y^2 - 9)$ for $x^2 - y^2$.
$(y^2 - 9) = [(y) + (3)][(y) - (3)]$	Substitute y for x and 3 for y.
$(y^2 - 9) = (y + 3)(y - 3)$	

   The factors of $5y^2 - 45$ are 5, $(y + 3)$, and $(y - 3)$.

4. Multiply the factors to verify they result in the original binomial.

   Factors must always equal the original expression when multiplied.

$5(y + 3)(y - 3)$	Multiply the factors.
$= 5(y \bullet y + y \bullet -3 + 3 \bullet y + 3 \bullet -3)$	Distribute the two binomials.
$= 5(y^2 - 3y + 3y - 9)$	Multiply.
$= 5(y^2 - 9)$	Combine like terms.
$= 5 \bullet y^2 + 5 \bullet -9$	Distribute the 5.
$= 5y^2 - 45$	Simplify.

   The product of $5(y + 3)(y - 3)$ equals the original expression, $5y^2 - 45$.

   This confirms that $5y^2 - 45$ factors to $5(y + 3)(y - 3)$.

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Instruction

Example 3

The polynomial expression $x^2 + 7x - 8$ represents the area in square feet of the Bingham family's rectangular backyard. Factor this polynomial to find the expressions that represent the length and the width of the backyard, and then verify your results.

1. Determine any common factors of the given trinomial, if common factors exist.

   The prime factors of x^2 are x and x.

   The prime factors of $7x$ are 7 and x.

   The prime factors of -8 are -2, 2, and 2.

   There are no factors that are common to all three terms.

2. Set up two sets of parentheses and write the variable as the first term in each set of parentheses.

   The variable is x, so x will be the first term in each set of parentheses:

   $(x \quad)(x \quad)$

3. List all of the possible sets of factors for the constant term (include both positive and negative numbers).

   The constant term of $x^2 + 7x - 8$ is -8. Possible sets of factors of -8:

   $(2 \bullet -4) \qquad (-2 \bullet 4) \qquad (1 \bullet -8) \qquad (-1 \bullet 8)$

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Instruction

4. Determine which set of factors has a sum that equals the coefficient of the first-degree term.

   The coefficient of the first-degree term of $x^2 + 7x - 8$ is 7.

   Determine which set of factors has a sum of 7:

   - For $(2 \bullet -4)$, $2 + (-4) = -2$, so this is not the correct set of factors.
   - For $(-2 \bullet 4)$, $-2 + 4 = 2$, so this is not the right set of factors.
   - For $(1 \bullet -8)$, $1 + (-8) = -7$, so this is not the right set of factors.
   - For $(-1 \bullet 8)$, $-1 + 8 = 7$. This is the correct set of factors.

   The right set of factors includes -1 and 8.

5. Write the correct factors (including their signs) as the second terms in the two sets of parentheses.

   In step 2, we set up the parentheses and variables for the factors as $(x \quad)(x \quad)$.

   Write -1 and 8 as the second terms in these two parentheses:

   $(x - 1)(x + 8)$

   The factors of $x^2 + 7x - 8$ are $(x - 1)$ and $(x + 8)$.

6. Distribute the factors to verify that the result is the original trinomial.

   Factors must always equal the original expression when multiplied.

$(x - 1)(x + 8)$	Multiply the factors.
$= x(x) + x(8) + (-1)(x) + (-1)(8)$	Distribute.
$= x^2 + 8x - 1x - 8$	Multiply.
$= x^2 + 7x - 8$	Combine like terms.

   The product of $(x - 1)(x + 8)$ equals the original expression, $x^2 + 7x - 8$.

   This confirms that $x^2 + 7x - 8$ factors to $(x - 1)(x + 8)$.

   Therefore, the expressions that represent the length and width of the backyard are $(x - 1)$ feet and $(x + 8)$ feet.

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Example 4

Factor $2a^2 - 16a + 32$, and then verify your results.

1. Determine any common factors of the given trinomial, if common factors exist.

   The prime factors of $2a^2$ are 2, a, and a.

   The prime factors of $-16a$ are -2, 2, 2, 2, and a.

   The prime factors of 32 are 2, 2, 2, 2, and 2.

   The greatest common factor of all three terms is 2. Thus, 2 must be factored out:

   $2a^2 - 16a + 32$

   $= 2 \bullet a^2 - 2 \bullet 8a + 2 \bullet 16$

   $= 2(a^2 - 8a + 16)$

   When factored by the greatest common factor,
   $2a^2 - 16a + 32 = 2(a^2 - 8a + 16)$.

2. Set up two sets of parentheses and write the variable as the first term in each set of parentheses.

   The variable is a, so a will be the first term in each set of parentheses. Remember to include the 2 that was previously factored:
   $2(a\quad)(a\quad)$.

3. List all of the possible sets of factors for the constant term (include both positive and negative numbers).

   The constant term of the trinomial $a^2 - 8a + 16$ is 16. Possible sets of factors of 16:

$(4 \bullet 4)$	$(2 \bullet 8)$	$(1 \bullet 16)$
$(-4 \bullet -4)$	$(-2 \bullet -8)$	$(-1 \bullet -16)$

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Instruction

4. Determine which set of factors has a sum that equals the coefficient of the first-degree term.

   The coefficient of the first-degree term of $a^2 - 8a + 16$ is –8.

   Determine which set of factors has a sum of –8:

   - For $(4 \cdot 4)$, $4 + 4 = 8$, so this is not the correct set of factors.
   - For $(2 \cdot 8)$, $2 + 8 = 10$, so this is not the right set of factors either.
   - For $(1 \cdot 16)$, $1 + 16 = 17$, so this is not the right set of factors.
   - For $(-4 \cdot -4)$, $-4 + (-4) = -8$. This is the correct set of factors.

   Once the correct set of factors is found, there is no need to continue testing other factors.

   The correct set of factors is –4 and –4.

5. Write the correct factors (including their signs) as the second terms in the two sets of parentheses.

   Previously, we set up the parentheses and variables for the factors as $2(a\quad)(a\quad)$.

   Write –4 and –4 as the second terms in these two parentheses:

   $2(a - 4)(a - 4)$

   This can be written as $2(a - 4)^2$.

   The trinomial $2a^2 - 16a + 32$ factors to $2(a - 4)^2$.

   Notice that since $a^2 - 8a + 16$ is a perfect square trinomial, this could also be factored using the pattern $(ax)^2 \pm 2abx + b^2 = (a \pm b)^2$.

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6. Distribute the factors to verify that they result in the original trinomial.

Factors must always equal the original expression when multiplied.

$2(a-4)^2$	Factors
$= 2(a-4)(a-4)$	Expand the power.
$= 2(a \bullet a + a \bullet -4 + -4 \bullet a + -4 \bullet -4)$	Distribute the binomials.
$= 2(a^2 - 4a - 4a + 16)$	Multiply.
$= 2(a^2 - 8a + 16)$	Combine like terms.
$= 2(a^2) + 2(-8a) + 2(16)$	Distribute the 2.
$= 2a^2 - 16a + 32$	Multiply.

The product of $2(a-4)^2$ equals the original expression, $2a^2 - 16a + 32$.

This confirms that $2a^2 - 16a + 32$ factors to $2(a-4)^2$.

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Example 5

Factor $x^2 - 18x + 81$, and then verify your results.

1. Determine any common factors of the given trinomial, if common factors exist.

   The prime factors of x^2 are $x \cdot x$.

   The prime factors of $-18x$ are $-2 \cdot 3 \cdot 3 \cdot x$.

   The prime factors of 81 are $3 \cdot 3 \cdot 3 \cdot 3$.

   There are no factors that are common to all three terms.

2. Determine if the trinomial is a perfect square trinomial.

   Trinomials that are the difference of two squares are of the form $(ax)^2 \pm 2abx + b^2$.

   In other words, the first and last terms are both perfect squares and the middle term is equal to twice the product of the square roots of the first and last terms.

   For $x^2 - 18x + 81$, both the first term (x^2) and last term (81) are perfect squares since they are both products of two equal factors.

   The square root of x^2 is x and the square root of 81 is 9. Thus, the middle term is twice the product of the square roots of the first and last terms ($2 \cdot x \cdot 9 = 18x$).

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3. Use the pattern for the perfect square trinomial to factor this expression.

   Perfect square trinomials can be factored using the regular process for factoring trinomials. Or, they can be factored using the pattern $(ax)^2 \pm 2abx + b^2 = (a \pm b)^2$.

   For the trinomial $x^2 - 18x + 81$, $a = 1$ (the square root of the coefficient of x^2) and $b = 9$ (the square root of the constant term 81).

   Since the middle term, $-18x$, is negative, the sign within the factored form will also be negative.

   $x^2 - 18x + 81 = (x - 9)^2$

4. Distribute the factors to verify that they result in the original trinomial.

   Factors must always equal the original expression when multiplied.

$(x - 9)^2$	Factors
$= (x - 9)(x - 9)$	Expand the power.
$= (x \bullet x + x \bullet -9 + -9 \bullet x + -9 \bullet -9)$	Distribute the binomials.
$= x^2 - 9x - 9x + 81$	Multiply.
$= x^2 - 18x + 81$	Combine like terms.

   The product of $(x - 9)^2$ equals the original expression, $x^2 - 18x + 81$.

   This confirms that $x^2 - 18x + 81$ factors to $(x - 9)^2$.

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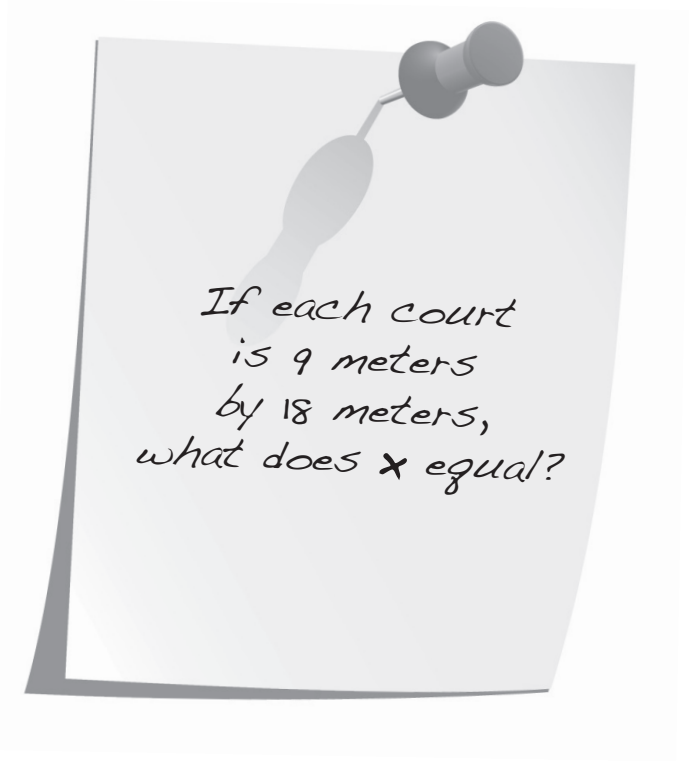
Problem-Based Task 3.2.3: Don't Drop the Ball!

SMP	
1 ✓	2 ✓
3	4
5	6
7 ✓	8

A city is installing two identical rectangular volleyball courts at the local park. The expression $2x^2 + 6x - 36$ represents the combined area of both courts. If each court is 9 meters by 18 meters, what does x equal?



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Problem-Based Task 3.2.3: Don't Drop the Ball!

Coaching

a. What expression represents the area of one of the courts?

b. How can you find the expressions that represent the length and width of one of the courts?

c. What expressions represent the length and width of one volleyball court?

d. Which expression represents 9 meters and which represents 18 meters?

e. How can you determine what x equals?

f. What does x equal?

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Problem-Based Task 3.2.3: Don't Drop the Ball!

Coaching Sample Responses

a. What expression represents the area of one of the courts?

The expression $2x^2 + 6x - 36$ has a common factor of 2. Factor the expression by 2:

$2x^2 + 6x - 36 = 2 \bullet x^2 + 2 \bullet 3x - 2 \bullet 18 = 2(x^2 + 3x - 18)$

Since the expression $2(x^2 + 3x - 18)$ represents the combined area of two identical courts, the area of just one of the courts is $x^2 + 3x - 18$.

The expression $x^2 + 3x - 18$ represents the area of one of the courts.

b. How can you find the expressions that represent the length and width of one of the courts?

$x^2 + 3x - 18$ represents the area of one of the courts, and the area of a rectangle equals the length times the width. When the trinomial is factored, the two factors will represent the length and the width.

c. What expressions represent the length and width of one volleyball court?

Factor $x^2 + 3x - 18$ to find the expressions that represent the length and width of one court.

Write two parentheses and include the variable x as the first term in each factor:

$(x \quad)(x \quad)$

Determine all possible factors for the constant term, –18:

$(1 \bullet -18)$	$(2 \bullet -9)$	$(3 \bullet -6)$
$(-1 \bullet 18)$	$(-2 \bullet 9)$	$(-3 \bullet 6)$

Determine which set of factors has a sum of 3 (the coefficient of the first-degree term).

The set of factors whose sum is 3 is –3 and 6 ($-3 + 6 = 3$).

Write –3 and 6 as the second terms of the factors in parentheses:

$x^2 + 3x - 18 = (x - 3)(x + 6)$

Double-check these factors by distributing:

$(x - 3)(x + 6)$

$= x^2 + 6x - 3x - 18$

$= x^2 + 3x - 18$

The expressions $(x - 3)$ and $(x + 6)$ represent the length and width of one volleyball court.

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d. Which expression represents 9 meters and which represents 18 meters?

The value of $(x + 6)$ will always be greater than the value of $(x – 3)$. Thus, $(x + 6)$ represents the larger value (18 meters) and $(x – 3)$ represents the smaller value (9 meters).

e. How can you determine what x equals?

To determine what x equals, set the expressions equal to the known values they represent and solve for x.

f. What does x equal?

Since $(x + 6)$ represents 18 meters, $x + 6 = 18$. Subtract 6 from both sides to determine $x = 12$.

Double check by repeating this process with the other expression.

Since $(x – 3)$ represents 9 meters, $x – 3 = 9$. Add 3 to both sides to determine $x = 12$.

The variable x equals 12.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.2.3: Factoring Expressions with $a = 1$

For problems 1–7, factor each expression as much as possible. If the expression cannot be factored, write "not factorable."

1. $y^2 - 100$

2. $x^2 - 9x + 14$

3. $x^2 + 16x + 64$

4. $b^2 + 4$

5. $7a^2 - 28$

6. $4x^2 + 32x - 36$

7. $6x^2 - 54y^2$

For problems 8–10, each expression represents the area of a rectangle or square. Factor each expression to find the expressions that represent the length and width of each figure.

8. $(a^2 - 14a + 49)$ square feet

9. $(4x^2 - 25)$ square meters

10. $(y^2 + 3y - 10)$ square inches

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Lesson 3.2.4: Factoring Expressions with $a > 1$

Warm-Up 3.2.4

Rosalind is painting three rectangular pictures. The first, a portrait, will have an area of $(x^2 + 5x + 4)$ square inches. Next, she will paint a landscape with an area of $(x^2 + 10x + 24)$ square inches. Finally, she plans to create a modern art piece with an area of $(x^2 - 16)$ square inches. Use the formula for the area of a rectangle, $A = lw$, to complete the following problems.

1. If the length of the portrait is $(x + 4)$ inches, what is the portrait's width?

2. What expressions represent the length and width of the landscape painting, given that the length is longer than the width?

3. For the modern art piece, how much larger is the width than the length?

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Lesson 3.2.4: Factoring Expressions with $a > 1$

Common Core State Standard

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

Warm-Up 3.2.4 Debrief

Rosalind is painting three rectangular pictures. The first, a portrait, will have an area of $(x^2 + 5x + 4)$ square inches. Next, she will paint a landscape with an area of $(x^2 + 10x + 24)$ square inches. Finally, she plans to create a modern art piece with an area of $(x^2 - 16)$ square inches. Use the formula for the area of a rectangle, $A = lw$, to complete the following problems.

1. If the length of the portrait is $(x + 4)$ inches, what is the portrait's width?

   The portrait will have an area of $(x^2 + 5x + 4)$ square inches.

   Since the area of a rectangle equals its length times its width, and the length is given as $(x + 4)$, factor $x^2 + 5x + 4$ to find the expression that represents the width.

   Write two sets of parentheses and include the variable x as the first term in each factor: $(x \quad)(x \quad)$.

   Determine all possible factors for the constant term, 4:

   $$(2 \bullet 2) \qquad (1 \bullet 4) \qquad (-2 \bullet -2) \qquad (-1 \bullet -4)$$

   Determine which set of factors has a sum of 5 (the coefficient of the first-degree term).

   The set of factors whose sum is 5 is 1 and 4 $(1 + 4 = 5)$.

   Write 1 and 4 as the second terms of the factors:

   $x^2 + 5x + 4 = (x + 1)(x + 4)$

   Double-check these factors by distributing:

   $(x + 1)(x + 4)$

   $= x^2 + 4x + 1x + 4$

   $= x^2 + 5x + 4$

   The factors of the area $x^2 + 5x + 4$ are $(x + 1)$ and $(x + 4)$. Since $(x + 4)$ is the given length, $(x + 1)$ must be the width.

   The width of the portrait is $(x + 1)$ inches.

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2. What expressions represent the length and width of the landscape painting, given that the length is longer than the width?

   The landscape painting will have an area of $(x^2 + 10x + 24)$ square inches.

   Since $A = lw$, factor $x^2 + 10x + 24$ to find the expressions that represent the length and width.

   Write two parentheses and include the variable x as the first term in each factor: $(x \quad)(x \quad)$.

   Determine all possible factors for the constant term, 24:

   $$(1 \bullet 24) \quad (2 \bullet 12) \quad (3 \bullet 8) \quad (4 \bullet 6)$$
   $$(-1 \bullet -24) \quad (-2 \bullet -12) \quad (-3 \bullet -8) \quad (-4 \bullet -6)$$

   Determine which set of factors has a sum of 10 (the coefficient of the first-degree term).

   The set of factors whose sum is 10 is 4 and 6 ($4 + 6 = 10$).

   Write 4 and 6 as the second terms of the factors:

   $x^2 + 10x + 24 = (x + 4)(x + 6)$

   Double-check these factors by distributing:

   $(x + 4)(x + 6)$

   $= x^2 + 6x + 4x + 24$

   $= x^2 + 10x + 24$

   The factors of the area $x^2 + 10x + 24$ are $(x + 4)$ and $(x + 6)$. These expressions represent the length and width of the landscape painting in inches.

3. For the modern art piece, how much larger is the width than the length?

   First, the expressions that represent the length and width of the piece must be found.

   The area of the modern art piece is $(x^2 - 16)$ square inches.

   Since the binomial $x^2 - 16$ meets the conditions of the difference of two squares, it can be factored using the pattern $x^2 - y^2 = (x + y)(x - y)$.

   The square root of x^2 is x; the square root of 16 is 4.

$x^2 - y^2 = (x + y)(x - y)$	Difference of two squares
$(x^2 - 16) = (x + y)(x - y)$	Substitute $x^2 - 16$ for $x^2 - y^2$.
$(x^2 - 16) = [(x) + (4)][(x) - (4)]$	Substitute x for x and 4 for y.
$(x^2 - 16) = (x + 4)(x - 4)$	

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The factors of the area $x^2 - 16$ are $(x + 4)$ and $(x - 4)$. These expressions represent the length and width of the modern art piece in inches.

Since it is stated in the problem that the width is the larger of the two dimensions, the width must be $x + 4$, since $x + 4$ will always be larger than $x - 4$.

To determine how much larger the width is than the length, subtract the two values.

$(x + 4) - (x - 4)$	Subtract the factors.
$= x + 4 - x + 4$	Distribute the negative sign.
$= 8$	Combine like terms.

The width is 8 inches larger than the length.

Connection to the Lesson

- Students will apply their understanding of factoring trinomials to factor more advanced trinomials.

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Prerequisite Skills

This lesson requires the use of the following skills:

- factoring a polynomial by the greatest common factor (A–SSE.2)
- factoring binomials using the difference of two squares pattern (A–SSE.2)
- factoring trinomials with $a = 1$ (A–SSE.3a★)

Introduction

When a trinomial has a leading coefficient of 1, it is relatively straightforward to factor. When the leading coefficient is greater than 1, the process becomes somewhat more complex; however, the basic principle remains the same. Factoring trinomials is essentially reversing the distribution. When the resulting factors are multiplied, their product must equal the original expression.

Key Concepts

Factoring by Grouping

- Sometimes more complex polynomials can be factored by grouping.
- When a polynomial contains four or more terms, a common binomial factor may emerge once terms are grouped together and common factors are factored out of each group. For example, the polynomial $6x^2 + 24x + 5x + 20$ can be grouped into $(6x^2 + 24x) + (5x + 20)$. When the common factors are factored out of each group, the result is $6x(x + 4) + 5(x + 4)$. Since both groups contain the same binomial, $(x + 4)$, the Distributive Property can be used to rewrite this as $(6x + 5)(x + 4)$.
- Recall that the Distributive Property states that $a(b + c) = a \cdot b + a \cdot c = a(b + c)$.

Factoring Trinomials when $a > 1$

- When the leading coefficient of a trinomial is greater than 1, first determine if there are any common factors that can be factored out.

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- If there are no common factors (or if there is still a leading coefficient greater than 1 after the common factor has been factored out), use the following steps to factor a trinomial of the form $ax^2 + bx + c$:

Factoring a Trinomial of the Form $ax^2 + bx + c$

1. List all the possible factors for the product of a times c (including negative factors).
2. Find the set of factors whose sum equals b.
3. Rewrite bx as the sum of two terms, using the factors found in step 2 as the coefficients of the terms.
4. Factor the new polynomial by grouping.

- To determine if a polynomial was factored correctly, distribute the factors. The result must equal the original polynomial.
- Note that not all polynomials can be factored.

Common Errors/Misconceptions

- neglecting to factor out common factors first
- forgetting to consider negative factors
- confusing the signs of factors and terms
- incorrectly rearranging a polynomial in an attempt to factor by grouping

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Guided Practice 3.2.4

Example 1

Factor $5y^2 - 6y + 10y - 12$ by grouping, and then verify your results.

1. Group the terms of the given polynomial according to their common factors, if common factors exist.

   Since $5y^2$ and $10y$ have a common factor of $5y$, group these two terms. Likewise, group $-6y$ and -12 because they have a common factor of -6:

   $5y^2 - 6y + 10y - 12 = (5y^2 + 10y) + (-6y - 12)$

2. Factor out common factors from each set of grouped terms, and rewrite the expression from the previous step.

   Rewrite $5y^2 + 10y$ using the common factor of $5y$.

   $5y^2 + 10y$

   $= 5y \bullet y + 5y \bullet 2$

   $= 5y(y + 2)$

   Rewrite $-6y - 12$ using the common factor of -6.

   $-6y - 12$

   $= -6 \bullet y + (-6) \bullet 2$

   $= -6(y + 2)$

   Therefore, $(5y^2 + 10y) + (-6y - 12) = 5y(y + 2) - 6(y + 2)$.

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3. Use the Distributive Property to rewrite the expression as two factors.

   Since both groups in the expression $5y(y + 2) – 6(y + 2)$ contain the binomial factor $(y + 2)$, this expression can be rewritten using the Distributive Property.

   The Distributive Property states that $ac + bc = (a + b)c$.

   Thus, $5y(y + 2) – 6(y + 2) = (5y – 6)(y + 2)$.

4. Distribute the result of the previous step to confirm that the original polynomial was factored correctly.

   When multiplied, $(5y – 6)(y + 2)$ should equal $5y^2 – 6y + 10y – 12$.

$(5y – 6)(y + 2)$	Factors
$= 5y(y) + 5y(2) + (–6)(y) + (–6)(2)$	Distribute.
$= 5y^2 + 10y – 6y – 12$	Multiply.
$= 5y^2 – 6y + 10y – 12$	Rearrange the terms to match the original expression.

   The product of $(5y – 6)(y + 2)$ equals the original expression, $5y^2 – 6y + 10y – 12$.

   This confirms that $5y^2 – 6y + 10y – 12$ factors to $(5y – 6)(y + 2)$.

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Example 2

Factor $6x^2 - 7x - 5$, and then verify your results.

1. Determine any common factors of the given trinomial, if common factors exist.

   The prime factors of $6x^2$ are 2, 3, x, and x.

   The prime factors of $-7x$ are -7 and x.

   5 is already a prime number.

   There are no factors that are common to all three terms.

2. Multiply a, the coefficient of the second-degree term, by c, the constant.

   Consider the general form $ax^2 + bx + c$.

   Within the trinomial $6x^2 - 7x - 5$, the coefficient of the second-degree term, a, is 6 and the constant, c, is -5.

   Therefore, $a \cdot c$ is equal to $6 \cdot -5$, or -30.

3. List all of the possible factors for the product of a and c.

   The product of a and c is -30.

   Determine the possible sets of factors for -30, including negative factors:

   $(1 \cdot -30)$ $(-1 \cdot 30)$ $(2 \cdot -15)$ $(-2 \cdot 15)$
   $(3 \cdot -10)$ $(-3 \cdot 10)$ $(5 \cdot -6)$ $(-5 \cdot 6)$

4. Determine which set of factors has a sum that equals b, the coefficient of the first-degree term.

   In the trinomial $6x^2 - 7x - 5$, the coefficient of b, the first-degree term, is -7.

   Determine which set of factors has a sum of -7.

   The set of factors whose sum is -7 is 3 and -10 $(3 + (-10) = -7)$.

   The correct set of factors is 3 and -10.

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5. Rewrite bx, the first-degree term, as the sum of two terms.

   Within the trinomial $6x^2 - 7x - 5$, rewrite the first-degree term $-7x$ as $3x - 10x$:

   $6x^2 + 3x - 10x - 5$

6. Group the terms of the new polynomial according to their common factors, if common factors exist.

   Group $6x^2 + 3x - 10x - 5$ so that terms with common factors are together.

   $6x^2$ and $3x$ have a common factor of $3x$, and $-10x$ and -5 have a common factor of -5. Group the terms accordingly:

   $6x^2 + 3x - 10x - 5 = (6x^2 + 3x) + (-10x - 5)$

7. Factor out common factors from each set of grouped terms, and rewrite the expression from the previous step.

   Rewrite $6x^2 + 3x$ using the common factor of $3x$.

   $6x^2 + 3x = 3x \bullet 2x + 3x \bullet 1 = 3x(2x + 1)$

   Rewrite $-10x - 5$ using the common factor of -5.

   $-10x - 5 = -5 \bullet 2x + -5 \bullet 1 = -5(2x + 1)$

   Thus, $(6x^2 + 3x) + (-10x - 5)$ can be rewritten as $3x(2x + 1) - 5(2x + 1)$.

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Instruction

8. Use the Distributive Property to rewrite the expression as two factors.

   Recall that the Distributive Property states that $ac + bc = (a + b)c$.

   $3x(2x + 1) - 5(2x + 1) = (3x - 5)(2x + 1)$

9. Distribute the factors to verify that they result in the original trinomial.

   Factors must always equal the original expression when multiplied.

$(3x - 5)(2x + 1)$	Factors
$= 3x(2x) + 3x(1) + (-5)(2x) + (-5)(1)$	Distribute.
$= 6x^2 + 3x - 10x - 5$	Multiply.
$= 6x^2 - 7x - 5$	Combine like terms.

   The product of $(3x - 5)(2x + 1)$ equals the original expression, $6x^2 - 7x - 5$.

   This confirms that $6x^2 - 7x - 5$ factors to $(3x - 5)(2x + 1)$.

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Example 3

Factor $36x^2 - 54x + 8$, and then verify your results.

1. Determine any common factors of the given trinomial, if common factors exist.

   The prime factors of $36x^2$ are 2, 2, 3, 3x, and x.

   The prime factors of $-54x$ are –1, 2, 3, 3, 3, and x.

   The prime factors of 8 are 2, 2, and 2.

   2 is the greatest common factor of all three terms and can be factored out:

   $36x^2 - 54x + 8$

   $= 2 \bullet 18x^2 + 2 \bullet -27x + 2 \bullet 4$

   $= 2(18x^2 - 27x + 4)$

   When factored by the greatest common factor, $36x^2 - 54x + 8 = 2(18x^2 - 27x + 4)$.

   Next, factor the remaining trinomial.

2. Multiply a, the coefficient of the second-degree term, by c, the constant.

   Consider the general form $ax^2 + bx + c$.

   Within the trinomial $18x^2 - 27x + 4$, the coefficient of the second-degree term, a, is 18 and the constant, c, is 4.

   Therefore, $a \bullet c$ is equal to 18 • 4, or 72.

3. List all of the possible factors for the product of a and c.

   Determine the possible sets of factors for 72, including negative factors:

(1 • 72)	(2 • 36)	(3 • 24)	(4 • 18)
(6 • 12)	(8 • 9)	(–1 • –72)	(–2 • –36)
(–3 • –24)	(–4 • –18)	(–6 • –12)	(–8 • –9)

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Instruction

4. Determine which set of factors has a sum that equals b, the coefficient of the first-degree term.

   In the trinomial $18x^2 - 27x + 4$, the coefficient of the first-degree term, b, is -27.

   Determine which set of factors has a sum of -27.

   The set of factors whose sum is -27 is -3 and -24 $(-3 + (-24) = -27)$.

   The correct set of factors is -3 and -24.

5. Rewrite bx, the first-degree term, as the sum of two terms.

   Within the trinomial $18x^2 - 27x + 4$, rewrite the first-degree term $-27x$ as $-3x - 24x$:

   $18x^2 - 3x - 24x + 4$

6. Group the terms of the new polynomial according to their common factors, if common factors exist.

   Group $18x^2 - 3x - 24x + 4$ so that terms with common factors are grouped together.

   $6x^2$ and $-3x$ have a common factor of $3x$, and $-24x$ and 4 have a common factor of 4.

   Group the terms accordingly:

   $18x^2 - 3x - 24x + 4 = (18x^2 - 3x) + (-24x + 4)$

7. Factor out common factors from each set of grouped terms, and rewrite the expression from the previous step.

   Rewrite $18x^2 - 3x$ using the common factor of $3x$.

   $18x^2 - 3x = 3x \bullet 6x - 3x \bullet 1 = 3x(6x - 1)$

   Rewrite $-24x + 4$ using the common factor of 4.

   $-24x + 4 = 4(-6x) + 4 \bullet 1 = 4(-6x + 1)$

   Therefore, $(18x^2 - 3x) + (-24x + 4)$ can be rewritten as $3x(6x - 1) + 4(-6x + 1)$.

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8. Factor the expressions further to create matching binomial factors.

   The two binomials do not match, but if –1 is factored out of the second binomial, then the binomials will match.

   Factor –1 out of $(-6x + 1)$:

   $4(-6x + 1) = 4(-1 \bullet 6x + -1 \bullet -1) = 4 \bullet -1(6x - 1) = -4(6x - 1)$

   The expression is now $3x(6x - 1) - 4(6x - 1)$.

9. Use the Distributive Property to rewrite the expression as two factors, and then write the result along with the GCF determined in step 1.

   Recall that the Distributive Property states that $ac + bc = (a + b)c$.

   $3x(6x - 1) - 4(6x - 1) = (3x - 4)(6x - 1)$

   Write the factored form of $2(18x^2 - 27x + 4)$, remembering to include the GCF, 2, that was previously factored out.

   The expression $2(18x^2 - 27x + 4)$ factors to $2(3x - 4)(6x - 1)$.

10. Distribute the factors to verify that they result in the original trinomial.

    Factors must always equal the original expression when multiplied.

$2(3x - 4)(6x - 1)$	Factors
$= 2(3x \bullet 6x + 3x \bullet -1 + -4 \bullet 6x + -4 \bullet -1)$	Distribute the binomials.
$= 2(18x^2 - 3x - 24x + 4)$	Multiply.
$= 2(18x^2 - 27x + 4)$	Combine like terms.
$= 2 \bullet 18x^2 + 2 \bullet -27x + 2 \bullet 4$	Distribute the 2.
$= 36x^2 - 54x + 8$	Multiply.

    The product of $2(3x - 4)(6x - 1)$ equals the original expression, $36x^2 - 54x + 8$.

    This confirms that $36x^2 - 54x + 8$ factors to $2(3x - 4)(6x - 1)$.

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Example 4

The polynomial expression $10a^2 + 87a - 27$ represents the area in square yards of a new rectangular playground at the town park. Factor the polynomial to determine the expressions that represent the length and the width of the playground. Verify your results.

1. Determine any common factors of the given trinomial, if common factors exist.

   The prime factors of $10a^2$ are 2, 5, a, and a.

   The prime factors of $87a$ are 3, 29, and a.

   The prime factors of 27 are 3, 3, and 3.

   There are no factors that are common to all three terms.

2. Multiply a, the coefficient of the second-degree term, by c, the constant.

   Consider the general form $ax^2 + bx + c$.

   Within the trinomial $10a^2 + 87a - 27$, a, the coefficient of the second-degree term, is 10. The constant, c, is –27.

   Therefore, $a \cdot c$ is equal to $10 \cdot -27$, or –270.

3. List all of the possible factors for the product of a and c.

   The possible sets of factors for –270, including negative factors, are:

(1 • –270)	(–1 • 270)	(2 • –135)	(–2 • 135)
(3 • –90)	(–3 • 90)	(5 • –54)	(–5 • 54)
(6 • –45)	(–6 • 45)	(9 • –30)	(–9 • 30)
(10 • –27)	(–10 • 27)	(15 • –18)	(–15 • 18)

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4. Determine which set of factors has a sum that equals b, the coefficient of the first-degree term.

   In the trinomial $10a^2 + 87a - 27$, the coefficient of b, the first-degree term, is 87.

   Determine which set of factors has a sum of 87.

   The set of factors whose sum is 87 is –3 and 90 (–3 + 90 = 87).

   The correct set of factors is –3 and 90.

5. Rewrite bx, the first-degree term, as the sum of two terms.

   Within the trinomial $10a^2 + 87a - 27$, rewrite the first-degree term, $87a$, as $-3a + 90a$: $10a^2 - 3a + 90a - 27$.

6. Group the terms of the new polynomial according to their common factors, if common factors exist.

   Group $10a^2 - 3a + 90a - 27$ so that terms with common factors are grouped together.

   $10a^2$ and $90a$ have a common factor of $10a$, and $-3a$ and –27 have a common factor of –3.

   Group the terms accordingly:

   $$10a^2 - 3a + 90a - 27 = (10a^2 + 90a) + (-3a - 27)$$

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7. Factor out common factors from each set of grouped terms, and rewrite the expression from the previous step.

   Rewrite $10a^2 + 90a$ using the common factor of $10a$.

   $10a^2 + 90a = 10a \bullet a + 10a \bullet 9 = 10a(a + 9)$

   Rewrite $-3a - 27$ using the common factor of -3.

   $-3a - 27 = -3 \bullet a + -3 \bullet 9 = -3(a + 9)$

   Thus, $(10a^2 + 90a) + (-3a - 27)$ can be rewritten as $10a(a + 9) - 3(a + 9)$.

8. Use the Distributive Property to rewrite the expression as two factors.

   Recall that the Distributive Property states that $ac + bc = (a + b)c$.

   $10a(a + 9) - 3(a + 9) = (10a - 3)(a + 9)$

9. Distribute the factors to verify that they result in the original trinomial.

   Factors must always equal the original expression when multiplied.

$(10a - 3)(a + 9)$	Factors
$= 10a(a) + 10a(9) + (-3)(a) + (-3)(9)$	Distribute.
$= 10a^2 + 90a - 3a - 27$	Multiply.
$= 10a^2 + 87a - 27$	Combine like terms.

   The product of $(10a - 3)(a + 9)$ equals the original expression, $10a^2 + 87a - 27$.

   This confirms that $10a^2 + 87a - 27$ factors to $(10a - 3)(a + 9)$.

   The expressions that represent the length and width of the playground are $(10a - 3)$ yards and $(a + 9)$ yards.

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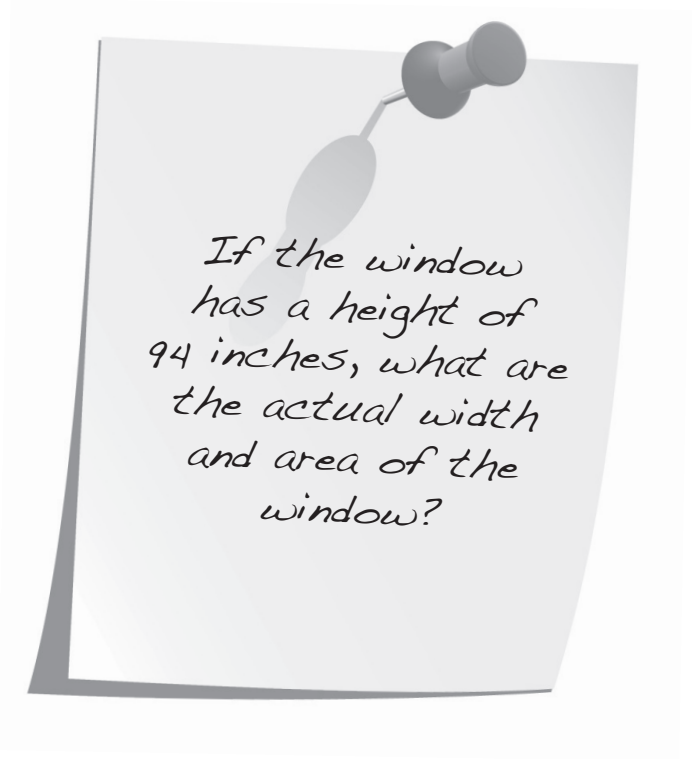
Problem-Based Task 3.2.4: Window Installation

SMP	
1 ✓	2 ✓
3	4 ✓
5	6
7 ✓	8

A construction company is installing a rectangular pane of stained glass at a local museum. The expression $54x^2 + 69x + 20$ represents the area of the window in square inches, and the expression $6x + 5$ represents the width of the window in inches. If the window has a height of 94 inches, what are the actual width and area of the window? Recall that the area of a rectangle is equal to its width times its length (in this case, the height of the window).



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Problem-Based Task 3.2.4: Window Installation

Coaching

a. How can the expression that represents the height of the window be determined?

b. What expression represents the height of the window?

c. What does x equal?

d. What are the width and area of the window?

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Instruction

Problem-Based Task 3.2.4: Window Installation

Coaching Sample Responses

a. How can the expression that represents the height of the window be determined?

The expression $54x^2 + 69x + 20$ represents the area of the window. When this trinomial is factored, the two factors will represent the length (or the height, in this case) and the width. Since $6x + 5$ represents the width, the other expression will represent the height.

b. What expression represents the height of the window?

Factor $54x^2 + 69x + 20$ to find the expressions that represent the height and width of the window.

There are no factors that are common to all three terms.

Multiply the coefficient of the second-degree term, a, by the constant, c.

$a = 54$ and $c = 20$; therefore $54 \bullet 20 = 1080$.

List sets of factors for 1,080 until one is found whose sum is the coefficient of the first-degree term, b, or 69.

$45 \bullet 24 = 1080$ and $45 + 24 = 69$, so this is the correct set of factors.

Rewrite $69x$ as $45x + 24x$:

$54x^2 + 69x + 20 = 54x^2 + 45x + 24x + 20$

Group $54x^2 + 45x + 24x + 20$ so that common factors are grouped together.

Since these terms have many common factors, they can be grouped either as $(54x^2 + 45x) + (24x + 20)$, or as $(54x^2 + 24x) + (45x + 20)$. In this case, both groupings will work.

Let's use the first grouping, $(54x^2 + 45x) + (24x + 20)$.

Factor out the common factors from each group in $(54x^2 + 45x) + (24x + 20)$:

$54x^2 + 45x = 9x \bullet 6x + 9x \bullet 5 = 9x(6x + 5)$

$24x + 20 = 4 \bullet 6x + 4 \bullet 5 = 4(6x + 5)$

Thus, $(54x^2 + 45x) + (24x + 20)$ can be rewritten as $9x(6x + 5) + 4(6x + 5)$.

Use the Distributive Property to rewrite the expression as two factors:

$9x(6x + 5) + 4(6x + 5) = (9x + 4)(6x + 5)$

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Distribute the factors to verify that they result in the original trinomial.

$(9x + 4)(6x + 5)$

$= 9x \bullet 6x + 9x \bullet 5 + 4 \bullet 6x + 4 \bullet 5$

$= 54x^2 + 45x + 24x + 20$

$= 54x^2 + 69x + 20$

The product of the factors equals the original expression.

This confirms that $54x^2 + 69x + 20$ factors to $(9x + 4)(6x + 5)$.

Since $6x + 5$ represents the width, $9x + 4$ must represent the height of the window.

c. What does x equal?

The window has a height of 94 inches, and the expression $9x + 4$ represents the height of the window. Create an equation and solve it for x.

$9x + 4 = 94$

$9x = 90$

$x = 10$

The value of x is 10 inches.

d. What are the width and area of the window?

The area of the window equals $54x^2 + 69x + 20$, the width equals $6x + 5$, and $x = 10$. Substitute 10 for x and evaluate the expressions:

$54x^2 + 69x + 20$	$6x + 5$
$= 54(10)^2 + 69(10) + 20$	$= 6(10) + 5$
$= 54(100) + 69(10) + 20$	$= 60 + 5$
$= 5400 + 690 + 20$	$= 65$
$= 6110$	

The area of the window is 6,110 square inches. The width is 65 inches.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.2.4: Factoring Expressions with $a > 1$

For problems 1–3, factor each expression by grouping.

1. $15x^2 - 5x + 21x - 7$

2. $3n^2 + 2n + 24n + 16$

3. $4y^2 - 18y - 14y + 63$

For problems 4 –7, factor each trinomial.

4. $3x^2 + 11x - 20$

5. $3a^2 + 39a + 108$

6. $25n^2 + 20n + 4$

7. $12y^2 - 46y + 14$

For problems 8–10, each expression represents the area of a rectangle or square. Factor each polynomial to find the expressions that represent the length and width of each figure.

8. $(9x^2 + 42x + 49)$ square inches

9. $(5x^2 + 14x - 3)$ square feet

10. $(6x^2 + 17x - 14)$ square meters

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Lesson 3.2.5: Solving Quadratic Equations by Factoring

Warm-Up 3.2.5

The Jacksons just had an in-ground pool built in their backyard. The pool is 5 feet deep and 12 feet wide. The expression $x + 5$ represents the pool's length and the expression $5x^2 - 15x - 200$ represents its volume. Use the fact that the volume equals the length times the width times the height (or depth) to answer the following questions.

1. What expression represents the width of the pool?

2. What is the value of x?

3. What is the actual length of the pool in feet?

4. What is the actual volume of the pool in cubic feet?

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Lesson 3.2.5: Solving Quadratic Equations by Factoring

Common Core State Standards

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.*

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

A–REI.4 Solve quadratic equations in one variable.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Warm-Up 3.2.5 Debrief

The Jacksons just had an in-ground pool built in their backyard. The pool is 5 feet deep and 12 feet wide. The expression $x + 5$ represents the pool's length and the expression $5x^2 – 15x – 200$ represents its volume. Use the fact that the volume equals the length times the width times the height (or depth) to answer the following questions.

1. What expression represents the width of the pool?

   The volume of the pool is the product of the length, width, and height (depth).

   The length is stated to be $x + 5$ and the volume is $5x^2 – 15x – 200$.

   To find the expression that represents the width of the pool, first factor $5x^2 – 15x – 200$.

   Each term in the expression has a factor of 5; therefore, $5x^2 – 15x – 200 = 5(x^2) + 5(–3x) + 5(–40)$ or $5(x^2 – 3x – 40)$.

   To factor $x^2 – 3x – 40$, first determine which set of factors of –40 has a sum of –3. The factors –8 and 5 have a sum of –3 and a product of –40.

   Write x as the first term in each factor and –8 and 5 as the second terms:

   $5(x^2 – 3x – 8) = 5(x – 8)(x + 5)$

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The factors 5, $x - 8$, and $x + 5$ represent the length, width, and height (depth) of the pool, but not in that order. Based on the information from the problem, we know that 5 is the depth and $x + 5$ represents the length, so $x - 8$ must represent the width.

The expression $x - 8$ represents the width of the pool.

2. What is the value of x?

   The expression $x - 8$ represents the width of the pool, and the pool is 12 feet wide. Create and solve an equation to find the value of x.

$x - 8 = 12$	Set $x - 8$ equal to 12.
$x = 20$	Add 8 to both sides.

   The value of x is 20 feet.

3. What is the actual length of the pool in feet?

   The expression $x + 5$ represents the length of the pool, and $x = 20$. Substitute and solve to find the length of the pool.

   $x + 5 = (20) + 5 = 25$

   The length of the pool is 25 feet.

4. What is the actual volume of the pool in cubic feet?

   The expression $5x^2 - 15x - 200$ represents the volume of the pool, and $x = 20$. Substitute 20 for x to find the volume of the pool, and then solve.

$5x^2 - 15x - 200$	Volume of the pool
$= 5(20)^2 - 15(20) - 200$	Substitute 20 for x.
$= 5(400) - 15(20) - 200$	Evaluate the power.
$= 2000 - 300 - 200$	Multiply.
$= 1500$	Subtract.

   The volume of the pool is 1,500 cubic feet.

Connection to the Lesson

- Students will factor expressions in order to solve quadratic equations.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- factoring by the greatest common factor (A–SSE.2)
- factoring using the difference of two squares (A–SSE.2)
- factoring trinomials (A–SSE.3a★)
- solving one-step and two-step equations (8.EE.7)

Introduction

Equations containing a squared variable can sometimes be solved by taking the square root of both sides. If the equation also contains a first-degree term, this method will not work; however, factoring can often be used to solve the equation.

Key Concepts

- Recall that if a quadratic equation has no first-degree term (for example, $2x^2 - 9 = 9$), then it can be solved by taking the square root of both sides.
- If the quadratic equation does have a first-degree term (for example, $x^2 - 2x + 8 = 1$), then it may be able to be solved by the factoring method.
- To solve a quadratic equation by factoring, use the following steps:

Solving a Quadratic Equation by Factoring

1. Arrange the equation in standard form. Recall that the **standard form of a quadratic equation** is $ax^2 + bx + c = 0$. Notice that the equation must equal 0.
2. Factor the polynomial.
3. Set each factor equal to 0.
4. Solve each equation for the variable.
5. Verify that these values are solutions by substituting them back into the original equation.

- Recall that a quadratic equation will often have two solutions.

SRB p. U3-63

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Instruction

- Solving quadratic equations by factoring works because of the **Zero Product Property**, which states that if the product of two factors is 0, then at least one of the factors is equal to 0. For a factored equation such as $(x + 3)(x – 6) = 0$, if either of the binomials is equal to 0, then the entire left side of the equation will equal 0. Thus, the factors can each be set equal to 0 and solved for x.
- Note that since not all polynomials can be factored, this method will not work for all quadratic equations.

Common Errors/Misconceptions

- not setting the equation equal to 0 before factoring
- not solving the equations that result from setting the factors equal to 0 (for example, for $(x + 4)(x – 1) = 0$, incorrectly stating that the answers are $x = 4$ and $x = –1$)
- factoring incorrectly

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Instruction

Guided Practice 3.2.5

Example 1

Solve $x^2 + 3x = 0$ for x.

1. Write the equation in standard form, $ax^2 + bx + c = 0$, and determine values for a, b, and c.

   In this case, the equation is already in standard form because the polynomial on the left, $x^2 + 3x$, is set equal to 0.

   For $x^2 + 3x = 0$, there is an understood coefficient of 1 for the x^2 term, so $a = 1$ and $b = 3$. There is no constant, so $c = 0$.

2. Factor the polynomial and rewrite the equation.

   First, determine if $x^2 + 3x$ has any common factors.

   Both terms have a factor of x; therefore,
   $x^2 + 3x = (x \bullet x) + (x \bullet 3) = x(x + 3)$.

   The polynomial cannot be factored any further.

   The factors of $x^2 + 3x$ are x and $(x + 3)$.

   The equation $x^2 + 3x = 0$ can be rewritten as $x(x + 3) = 0$.

3. Set each factor equal to 0 and solve for x.

   Once the polynomial has been factored, set each factor equal to 0 and then solve for x.

   $x = 0$ $\qquad$ $x + 3 = 0$

   $x = -3$

   The solutions to the equation $x^2 + 3x = 0$ are $x = 0$ and $x = -3$.

SWB p. 69; SRB p. U3-64

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Instruction

4. Check the solutions by substituting them into the original equation.

   Check that a solution is correct by substituting it back into the original equation and solving the equation to ensure that it results in a true statement.

   Check $x = 0$:

$x^2 + 3x = 0$	Original equation
$(0)^2 + 3(0) = 0$	Substitute 0 for x.
$0 + 0 = 0$	Evaluate the power and multiply.
$0 = 0$	Add.

   Because $x = 0$ makes the equation true, it is a solution to the equation.

   Check $x = -3$:

$x^2 + 3x = 0$	Original equation
$(-3)^2 + 3(-3) = 0$	Substitute -3 for x.
$9 + (-9) = 0$	Evaluate the power and multiply.
$0 = 0$	Add.

   Because $x = -3$ makes the equation true, it is also a solution.

   The solutions to the equation $x^2 + 3x = 0$ are $x = 0$ and $x = -3$.

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Example 2

Solve $4x^2 - 26x = 14$ for x.

1. Write the equation in standard form, $ax^2 + bx + c = 0$, and determine values for a, b, and c.

   The right side of the equation must equal 0. To get 0 on the right side, subtract 14 from both sides: $4x^2 - 26x - 14 = 0$.

   The equation is now in standard form because the polynomial on the left, $4x^2 - 26x - 14$, is set equal to 0.

   For $4x^2 - 26x - 14 = 0$, $a = 4$, $b = -26$, and $c = -14$.

2. Factor the polynomial and rewrite the equation.

   First, determine if $4x^2 - 26x - 14$ has any common factors.

   All three terms have a factor of 2; therefore, $4x^2 - 26x - 14 = 2 \bullet 2x^2 - 2 \bullet 13x - 2 \bullet 7 = 2(2x^2 - 13x - 7)$.

   Next, factor the trinomial $2x^2 - 13x - 7$.

   The product of 2 (the coefficient of the second-degree term) and –7 (the constant) is –14. Find the factors of –14 that add up to –13, the coefficient of the first-degree term.

   –14 and 1 are factors of –14 that have a sum of –13.

   Rewrite $-13x$ as $-14x + 1x$:

   $2x^2 - 13x - 7 = 2x^2 - 14x + 1x - 7$

   Factor $2x^2 - 14x + 1x - 7$ by grouping.

$2x^2 - 14x + 1x - 7$	Expression
$= (2x^2 - 14x) + (x - 7)$	Group $2x^2$ and $-14x$ because they have a common factor of $2x$.
$= 2x(x - 7) + (x - 7)$	Factor out $2x$ from $2x^2 - 14x$.
$= (2x + 1)(x - 7)$	Use the Distributive Property to rewrite the expression as two factors.

   The factors of $4x^2 - 26x - 14$ are 2, $(2x + 1)$, and $(x - 7)$.

   The equation $4x^2 - 26x - 14 = 0$ can be rewritten as $2(2x + 1)(x - 7) = 0$.

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Instruction

3. Set each factor equal to 0 and solve for x.

   Once the polynomial has been factored, set each factor equal to 0.

   The factor 2 is a constant and can be disregarded because 2 cannot equal 0. Furthermore, if both sides of the equation were divided by 2, the 2 would disappear since 0 divided by 2 is 0.

   Only the factors that contain a variable can be set equal to 0 and solved. Set both of these factors equal to 0 and solve:

   $2x + 1 = 0$ $\qquad$ $x - 7 = 0$

   $2x = -1$ $\qquad$ $x = 7$

   $x = -\frac{1}{2}$

   The solutions to the equation $4x^2 - 26x = 14$ are $x = -\frac{1}{2}$ and $x = 7$.

4. Check the solutions by substituting them into the original equation.

   Check that a solution is correct by substituting it back into the original equation and solving the equation to ensure that it results in a true statement.

   Check $x = -\frac{1}{2}$:

   $4x^2 - 26x = 14$ $\qquad$ Original equation

   $4\left(-\frac{1}{2}\right)^2 - 26\left(-\frac{1}{2}\right) = 14$ $\qquad$ Substitute $-\frac{1}{2}$ for x.

   $4\left(\frac{1}{4}\right) - 26\left(-\frac{1}{2}\right) = 14$ $\qquad$ Evaluate the power.

   $1 + 13 = 14$ $\qquad$ Multiply.

   $14 = 14$ $\qquad$ Add.

   Because $x = -\frac{1}{2}$ makes the equation true, it is a solution to the equation.

(*continued*)

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Check $x = 7$:

$4x^2 - 26x = 14$	Original equation
$4(7)^2 - 26(7) = 14$	Substitute 7 for x.
$4(49) - 26(7) = 14$	Evaluate the power.
$196 - 182 = 14$	Multiply.
$14 = 14$	Subtract.

Because $x = 7$ makes the equation true, it is also a solution.

The solutions to the equation $4x^2 - 26x = 14$ are $x = -\frac{1}{2}$ and $x = 7$.

Example 3

The area of the Baker family's rectangular driveway is 324 square feet. The length is 3 feet larger than twice the width. What are the length and width of the driveway?

1. Write expressions that represent the length and width of the driveway.

   Let x represent the width of the driveway.

   The length is 3 feet larger than twice the width. Thus, the length must equal $2x + 3$.

   x represents the width of the driveway and $2x + 3$ represents the length.

2. Write an equation for the area of the driveway.

   Use the fact that the area of a rectangle equals the length times the width to write an equation.

$A = lw$	Formula for the area of a rectangle
$(324) = (2x + 3)(x)$	Substitute 324 for the area, $(2x + 3)$ for the length, and x for the width.

   The equation representing the area of the driveway is $324 = (2x + 3)(x)$.

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Instruction

3. Write the equation in standard form, $ax^2 + bx + c = 0$.

   Rearrange the equation so that the polynomial is in standard form, set equal to 0.

$324 = (2x + 3)(x)$	Original equation
$(2x + 3)(x) = 324$	Apply the Symmetric Property of Equality to switch the left and right sides.
$2x^2 + 3x = 324$	Distribute the x.
$2x^2 + 3x - 324 = 0$	Subtract 324 from both sides.

   The equation $2x^2 + 3x - 324 = 0$ is now in standard form and is ready to be factored.

4. Factor the polynomial and rewrite the equation.

   First, determine if $2x^2 + 3x - 324$ has any common factors.

   There are no factors that are common to all three terms.

   Next, factor the trinomial $2x^2 + 3x - 324$.

   The product of 2 (the coefficient of the second-degree term) and –324 (the constant) is –648. Find the factors of –648 that add up to 3, the coefficient of the first-degree term.

   27 and –24 are factors of –648 that have a sum of 3.

   Rewrite $3x$ as $27x - 24x$:

   $2x^2 + 3x - 324 = 2x^2 + 27x - 24x - 324$

   Factor $2x^2 + 27x - 24x - 324$ by grouping. Note that $2x^2$ and $-24x$ have a common factor of $2x$, and $27x$ and –324 have a common factor of 27.

$2x^2 + 27x - 24x - 324$	Expression
$= (2x^2 - 24x) + (27x - 324)$	Group according to common factors.
$= 2x(x - 12) + 27(x - 12)$	Factor out $2x$ from $2x^2 - 24x$ and 27 from $27x - 324$.
$= (2x + 27)(x - 12)$	Use the Distributive Property to rewrite the expression as two factors.

   The factors of $2x^2 + 3x - 324$ are $(2x + 27)$ and $(x - 12)$.

   The equation $2x^2 + 3x - 324 = 0$ can be rewritten as $(2x + 27)(x - 12) = 0$.

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Instruction

5. Set each factor equal to 0 and solve for x.

Once the polynomial has been factored, set each factor equal to 0.

The first factor is $(2x + 27)$. Set this equal to 0: $2x + 27 = 0$

The second factor is $(x - 12)$. Set this equal to 0: $x - 12 = 0$

Next, solve each equation for x.

$2x + 27 = 0$	$x - 12 = 0$
$2x = -27$	$x = 1$
$x = -\frac{27}{2}$	

The solutions to the equation are $x = -\frac{27}{2}$ and $x = 12$.

6. Determine which solution for x makes sense for the situation.

Since x represents the width of the driveway, x must be positive (the width of the driveway cannot be negative). Thus, $x = -\frac{27}{2}$ does not make sense as a solution, so it can be disregarded. Therefore, $x = 12$ is the only solution to the equation given the context of the scenario.

7. Find the length and the width of the driveway.

Since x represents the width of the driveway and $x = 12$, the width of the driveway is 12 feet.

Since $2x + 3$ represents the length of the driveway, substitute 12 for x and solve to find the length:

$2x + 3$	Length of the driveway
$= 2(12) + 3$	Substitute 12 for x.
$= 24 + 3$	Multiply.
$= 27$	Add.

The length of the driveway is 27 feet.

The driveway is 27 feet long and 12 feet wide.

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Problem-Based Task 3.2.5: Lost Sunglasses

SMP	
1 ✓	2 ✓
3	4 ✓
5	6
7 ✓	8

Alyson lost her sunglasses while riding a roller coaster. The coaster was 144 feet high and traveling uphill at a rate of 40 feet per second when the sunglasses fell. The equation for the sunglasses' height (h) above the ground after t seconds is $h = -16t^2 + 40t + 144$. How long did it take Alyson's sunglasses to reach the ground?

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Problem-Based Task 3.2.5: Lost Sunglasses

Coaching

a. What was the value of h when the sunglasses hit the ground?

b. What equation can be used to find the time at this instance?

c. What are the values of t in this equation?

d. Which value of t makes sense in this situation?

e. How long did it take Alyson's sunglasses to reach the ground?

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Instruction

Problem-Based Task 3.2.5: Lost Sunglasses

Coaching Sample Responses

a. What was the value of h when the sunglasses hit the ground?

When the sunglasses hit the ground their distance above the ground was 0, so the value of h is 0.

b. What equation can be used to find the time at this instance?

Substitute 0 for h in the equation $h = -16t^2 + 40t + 144$:

$0 = -16t^2 + 40t + 144$

The equation $0 = -16t^2 + 40t + 144$ can be used to find the time, t, when the glasses hit the ground.

c. What are the values of t in this equation?

Solve $0 = -16t^2 + 40t + 144$ for t.

First, to arrange the equation in standard form, switch the sides of the equation:

$-16t^2 + 40t + 144 = 0$

The trinomial can be factored as is, but it may be easier to rearrange it so that the ax^2 term is positive. To do this, divide every term on both sides by –1:

$16t^2 - 40t - 144 = 0$

Next, factor the trinomial $16t^2 - 40t - 144$.

All three terms have a common factor of 8. Factor this out:

$16t^2 - 40t - 144 = 8(2t^2) + 8(-5t) + 8(18) = 8(2t^2 - 5t - 18)$

To factor $2t^2 - 5t - 18$, first multiply 2 (the coefficient of the second-degree term) by –18 (the coefficient of the constant) to get –36. Find the factors of –36 that have a sum of –5, the coefficient of the first-degree term.

The factors –9 and 4 are factors of –36 whose sum is –5.

Replace $-5t$ with $-9t + 4t$.

$2t^2 - 5t - 18 = 2t^2 - 9t + 4t - 18$

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Factor $2t^2 - 9t + 4t - 18$ by grouping.

$2t^2 - 9t + 4t - 18$

$= (2t^2 + 4t) + (-9t - 18)$

$= 2t(t + 2) - 9(t + 2)$

$= (2t - 9)(t + 2)$

The factors of $16t^2 - 40t - 144$ are 2, $(2t - 9)$, and $(t + 2)$.

The equation $16t^2 - 40t - 144$ can be rewritten as $2(2t - 9)(t + 2) = 0$.

Set each factor equal to 0 and solve.

The constant factor 2 can be disregarded since it does not contain a variable, and 2 cannot equal 0.

$2t - 9 = 0$ $\qquad$ $t + 2 = 0$

$2t = 9$ $\qquad$ $t = -2$

$t = 4.5$

The values of t are 4.5 and –2.

d. Which value of t makes sense in this situation?

Since the problem is asking how long it took for the sunglasses to reach the ground, the time must be positive. Thus, 4.5 is the only value that makes sense.

e. How long did it take Alyson's sunglasses to reach the ground?

The sunglasses took 4.5 seconds to reach the ground.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.2.5: Solving Quadratic Equations by Factoring

For problems 1–7, solve each quadratic equation by factoring.

1. $x^2 - 2x - 48 = 0$

2. $2y^2 + 9y = 35$

3. $5n^2 - 9n = 0$

4. $2x^2 - 32 = 0$

5. $3y^2 - 24y = -45$

6. $60a^2 - 190a = 70$

7. $(x + 4)(x - 8) = 28$

For problems 8–10, each given equation represents the height (h) of an object above the ground after it has traveled in the air for t seconds. Solve each problem using the provided information.

8. A child throws a water balloon down out of a window. Substitute 0 for h into the equation $h = -16t^2 - 10t + 6$ to determine how many seconds it takes for the water balloon to reach the ground.

9. A person tosses a coin down from a balcony into a fountain below. Substitute 12 for h into the equation $h = -5t^2 - 2t + 36$ to determine how many seconds it will take before the coin passes a sign that is 12 feet above the ground.

10. A boater launches a firework up into the air. Substitute 125 for h into the equation $h = -5t^2 + 50t$ to determine how many seconds it will take before the firework reaches its maximum height of 125 meters and explodes.

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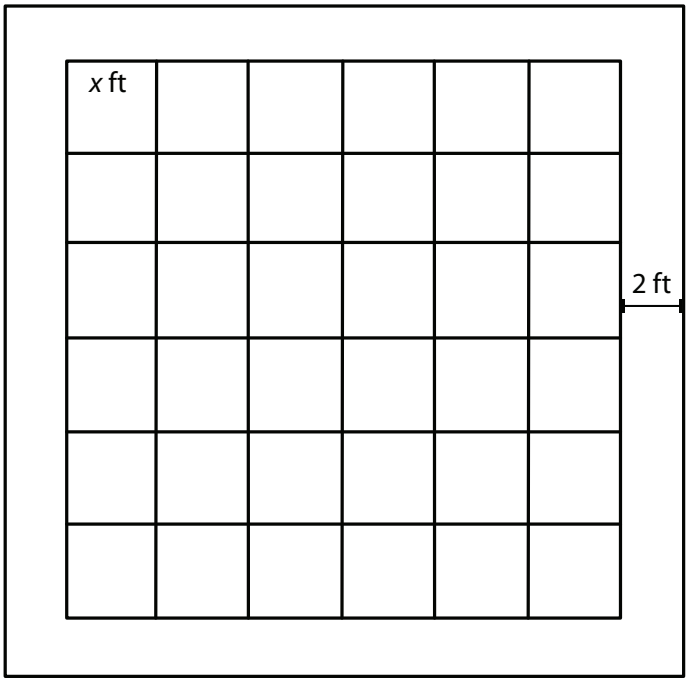
Lesson 3.2.6: Completing the Square

Warm-Up 3.2.6

Pavers are factory-made concrete tiles used in landscaping. Each side of a square patio is 6 pavers long. The length of one side of the patio can be represented by the term $6x$, where x is the length of one paver in feet.



1. Write an expression for the area of the patio covered by pavers.

2. A 2-foot gravel border surrounds the patio. Write an expression for the total area of the patio including the gravel border.

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Lesson 3.2.6: Completing the Square

Common Core State Standards

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

A–REI.4 Solve quadratic equations in one variable.

a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Warm-Up 3.2.6 Debrief

1. Write an expression for the area of the patio covered by pavers.

   $A = \text{side}^2$

   $A = (6x)^2$

   $A = (6x)(6x) = 36x^2$

   The area of the patio covered in pavers, in square feet, is $36x^2$.

2. A 2-foot gravel border surrounds the patio. Write an expression for the total area of the patio including the gravel border.

   $A = \text{side}^2$

   $A = (6x + 4)^2$

   $A = (6x + 4)(6x + 4) = 36x^2 + 48x + 16$

   The total area of the patio in square feet is $36x^2 + 48x + 16$.

Connection to the Lesson

- Students will learn how to create an expression that is a perfect square trinomial.
- Students will extend the idea of squaring a binomial to recognizing perfect square trinomials.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- multiplying polynomials (A–APR.1)
- writing quadratic equations in standard form (A–CED.1★)

Introduction

A trinomial of the form $x^2 + bx + \left(\frac{b}{2}\right)^2$ can be written as the square of a binomial, $\left(x + \frac{b}{2}\right)^2$, and is called a perfect square trinomial. We can solve quadratic equations by transforming the quadratic expression into a perfect square trinomial and taking square roots to solve.

Key Concepts

- When the binomial $(x + a)$ is squared, the resulting perfect square trinomial is $x^2 + 2ax + a^2$.
- When the binomial $(ax + b)$ is squared, the resulting perfect square trinomial is $a^2x^2 + 2abx + b^2$.

Completing the Square to Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$

1. Make sure the equation is in standard form, $ax^2 + bx + c = 0$.
2. Subtract c from both sides of the equation.
3. If a is not equal to 1, divide each term by a to get a leading coefficient of 1.
4. Add the square of one-half of b to both sides to complete the square.
5. Express the perfect square trinomial as the square of a binomial.
6. Solve by taking the square root of both sides of the equation.

Common Errors/Misconceptions

- neglecting to subtract the c term from both sides of the equation
- not isolating x after squaring both sides
- forgetting that, when taking the square root, both the positive and negative roots must be considered (±)

SRB p. U3-72

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Instruction

Guided Practice 3.2.6

Example 1

Solve $x^2 - 8x + 16 = 4$.

1. Determine if $x^2 - 8x + 16$ is a perfect square trinomial.

   Take one-half of the value of b and then square the result. If this is equal to the value of c, then the expression is a perfect square trinomial.

   $$\left(\frac{b}{2}\right)^2 = \left(\frac{-8}{2}\right)^2 = 16$$

   $x^2 - 8x + 16$ is a perfect square trinomial because the square of one-half of -8 is 16.

2. Write the left side of the equation as a binomial squared.

   One-half of b is -4, so the left side of the equation can be written as $(x - 4)^2$.

   $(x - 4)^2 = 4$

3. Take the square root of both sides of the equation to solve for x.

$(x - 4)^2 = 4$	Perfect square trinomial
$x - 4 = \pm 2$	Take the square root of both sides.
$x = 4 \pm 2$	Add 4 to both sides.
$x = 4 + 2 = 6$ or $x = 4 - 2 = 2$	Separate the result into two equations and solve for x.

4. Determine the solution(s).

   The equation has two solutions, $x = 2$ and $x = 6$.

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Example 2

Solve $x^2 + 6x + 4 = 0$ by completing the square.

1. Determine if $x^2 + 6x + 4$ is a perfect square trinomial.

   Take one-half of the value of b and then square the result. If this is equal to the value of c, then the expression is a perfect square trinomial.

   $$\left(\frac{b}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = 9$$

   $x^2 + 6x + 4$ is not a perfect square trinomial because the square of one-half of 6 is not 4.

2. Complete the square.

$x^2 + 6x + 4 = 0$	Original equation
$x^2 + 6x = -4$	Subtract 4 from both sides.
$x^2 + 6x + 3^2 = -4 + 3^2$	Add the square of one-half of b to both sides of the equation to complete the square.
$x^2 + 6x + 9 = 5$	Simplify.

3. Express the perfect square trinomial as the square of a binomial.

   One-half of b is 3, so the left side of the equation can be written as $(x + 3)^2$.

   $(x + 3)^2 = 5$

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4. Take the square root of both sides of the equation to solve for x.

$(x + 3)^2 = 5$	Equation from the previous step
$x+3=\pm\sqrt{5}$	Take the square root of both sides.
$x=-3\pm\sqrt{5}$	Subtract 3 from both sides.

5. Determine the solution(s).

The equation $x^2 + 6x + 4 = 0$ has two solutions, $x = -3\pm\sqrt{5}$.

Example 3

Solve $5x^2 – 50x – 120 = 0$ by completing the square.

1. Determine if $5x^2 – 50x – 120 = 0$ is a perfect square trinomial.

The leading coefficient is not 1.

First divide both sides of the equation by 5 so that $a = 1$.

$5x^2 – 50x – 120 = 0$	Original equation
$x^2 – 10x – 24 = 0$	Divide both sides by 5.

Now that the leading coefficient is 1, take one-half of the value of b and then square the result. If the expression is equal to the value of c, then the quadratic expression is a perfect square trinomial.

$$\left(\frac{b}{2}\right)^2=\left(\frac{-10}{2}\right)^2=25$$

$x^2 – 10x – 24$ is not a perfect square trinomial because the square of one-half of –10 is 25, not –24.

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Instruction

2. Complete the square.

$x^2 - 10x - 24 = 0$	Equation from the previous step
$x^2 - 10x = 24$	Add 24 to both sides.
$x^2 - 10x + (-5)^2 = 24 + (-5)^2$	Add the square of one-half of b to both sides of the equation to complete the square.
$x^2 - 10x + 25 = 49$	Simplify.

3. Express the perfect square trinomial as the square of a binomial.

One-half of b is -5, so the left side of the equation can be written as $(x - 5)^2$.

$(x - 5)^2 = 49$

4. Isolate x.

$(x - 5)^2 = 49$	Equation
$x - 5 = \pm\sqrt{49} = \pm 7$	Take the square root of both sides.
$x = 5 \pm 7$	Add 5 to both sides.
$x = 5 + 7 = 12$ or $x = 5 - 7 = -2$	Separate the result into two equations and solve for x.

5. Determine the solution(s).

The equation $5x^2 - 50x - 120 = 0$ has two solutions, $x = -2$ and $x = 12$.

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Problem-Based Task 3.2.6: Curve Ball

SMP	
1	2 ✓
3	4 ✓
5	6
7 ✓	8

The height of a baseball in feet after it is thrown is represented by $-16x^2 + 32x + 5$, where x is the time in seconds. After how many seconds will the ball be at a height of 7 feet?

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Problem-Based Task 3.2.6: Curve Ball

Coaching

a. How can you represent the situation with an equation?

b. Can you combine the two constant terms?

c. How can you solve the equation?

d. Do both of your answers make sense?

e. After how many seconds will the ball be at a height of 7 feet?

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Instruction

Problem-Based Task 3.2.6: Curve Ball

Coaching Sample Responses

a. How can you represent the situation with an equation?

Set the expression equal to the given height.

$-16x^2 + 32x + 5 = 7$

b. Can you combine the two constant terms?

Yes, the two constant terms can be combined by subtracting 5 from both sides of the equation.

$-16x^2 + 32x = 2$

c. How can you solve the equation?

The equation can be solved by completing the square.

Divide all terms by –16 so that the coefficient of the leading term is 1.

$$x^2 - 2x = -\frac{1}{8}$$

The square of one-half of b is 1.

Add 1 to both sides of the equation.

$$x^2 - 2x + 1 = \frac{7}{8}$$

Write the left side of the equation as the square of a binomial, then solve by taking the square root of both sides of the equation.

$$(x-1)^2 = \frac{7}{8}$$

$$x - 1 = \pm\sqrt{\frac{7}{8}}$$

$$x = 1 \pm \sqrt{\frac{7}{8}}$$

$x \approx 1.94$ or $x \approx 0.06$

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d. Do both of your answers make sense?

Yes. Both answers make sense because both are positive numbers. The ball will reach 7 feet on its way up and again on its way down.

e. After how many seconds will the ball be at a height of 7 feet?

The ball will be at a height of 7 feet after about 0.06 second and again after about 1.94 seconds.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.2.6: Completing the Square

For problems 1–4, find the value of c so that the expression is a perfect square trinomial.

1. $x^2 + 18x + c$

2. $x^2 - 24x + c$

3. $x^2 + 15x + c$

4. $x^2 + x + c$

Solve problems 5–7 by completing the square.

5. $x^2 + 10x = 0$

6. $x^2 + 12x - 13 = 0$

7. $3x^2 + 2x - 7 = 0$

Use what you know about completing the square to solve problems 8–10. Determine whether your answers are reasonable and explain why or why not.

8. A rectangular porch has an area of 75 square feet. The length of the porch is 4 feet longer than the width. What is the width of the porch?

9. A pet owner throws a tennis ball for his dog to chase. The tennis ball's height in feet after it is thrown upward is given by $-16x^2 + 32x + 4$, where x represents the time in seconds after the ball was thrown. After how many seconds will the ball hit the ground?

10. The fuel economy in miles per gallon of a certain vehicle is given by $-0.01x^2 + 1.2x - 5.8$, where x is the car's speed in miles per hour. For what speed(s) does the car have a fuel economy of 22 miles per gallon?

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Lesson 3.2.7: Applying the Quadratic Formula

Warm-Up 3.2.7

Sadie wants to sew a patch onto her shirt in the shape of an equilateral triangle with sides that are 4 inches long. The area of an equilateral triangle with sides of length s is given by $A = \frac{\sqrt{3}}{4}s^2$.

1. How many square inches of fabric will Sadie need to make her patch?

2. If Sadie decides to make 3 of her friends each a matching patch, how many square inches of fabric will Sadie need for the additional patches?

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Lesson 3.2.7: Applying the Quadratic Formula

Common Core State Standards

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

A–REI.4 Solve quadratic equations in one variable.

a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x – p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Warm-Up 3.2.7 Debrief

1. How many square inches of fabric will Sadie need to make her patch?

   Use the given formula to find the amount of fabric needed.

$A = \frac{\sqrt{3}}{4}s^2$ Formula for the area of an equilateral triangle

$A = \frac{\sqrt{3}}{4}(4)^2$ Substitute 4 for s.

$A = \frac{\sqrt{3}}{4} \bullet 16$ Simplify.

$A = 4\sqrt{3}$

The area of the triangular patch is $4\sqrt{3}$ square inches, or about 6.93 square inches.

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2. If Sadie decides to make 3 of her friends each a matching patch, how many square inches of fabric will Sadie need for the additional patches?

   Multiply the amount of fabric needed for one patch by 3.

   $3 \bullet 4\sqrt{3} = 12\sqrt{3}$

   The amount of fabric needed for the 3 additional patches is $12\sqrt{3}$ square inches, or about 20.78 square inches.

Connection to the Lesson

- Students will use the order of operations in the process of applying the quadratic formula.
- Students will incorporate operations with square roots and squares in order to apply the quadratic formula.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- understanding that real numbers are made up of the sets of rational and irrational numbers (8.NS.1)
- recognizing rational numbers and irrational numbers (8.NS.1)
- applying the order of operations (5.OA.1)

Introduction

Completing the square can be a long process, and not all quadratic expressions can be factored. Rather than completing the square or factoring, we can use a formula that can be derived from the process of completing the square. This formula, called the quadratic formula, can be used to solve any quadratic equation.

Key Concepts

- A quadratic equation when written in standard form, $ax^2 + bx + c = 0$, can be solved for x by using the **quadratic formula**: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- Solutions of quadratic equations are also called **roots**.
- The expression under the radical in the formula, $b^2 - 4ac$, is called the **discriminant**.
- The discriminant tells us the number and type of solutions for the quadratic equation.

Discriminant	Number and type of solutions
Negative	No real solutions
0	One real, rational solution
Positive and a perfect square	Two real, rational solutions
Positive and not a perfect square	Two real, irrational solutions

Common Errors/Misconceptions

- not setting the quadratic equation equal to 0 before determining the values of a, b, and c
- forgetting to use ± when taking the square root to solve for x
- writing the denominator of the quadratic formula as a instead of $2a$
- not following the order of operations
- not fully simplifying solutions

SRB p. U3-78

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Guided Practice 3.2.7

Example 1

Given the standard form of a quadratic equation, $ax^2 + bx + c = 0$, derive the quadratic formula by completing the square.

1. Begin with a quadratic equation in standard form.

   $ax^2 + bx + c = 0$

2. Subtract c from both sides.

   $ax^2 + bx = -c$

3. Divide both sides by a.

   $$x^2 + \frac{b}{a}x = \frac{-c}{a}$$

4. Complete the square.

   Add the square of one-half of the coefficient of the x-term to both sides to complete the square.

   Add $\left(\frac{b}{2a}\right)^2$ to both sides of the equation to complete the square.

   $$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$$

SWB p. 89; SRB p. U3-79

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Instruction

5. Write the left side of the equation as a binomial squared and simplify the right side of the equation.

$$\left(x+\frac{b}{2a}\right)^2=\frac{-c}{a}+\frac{b^2}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{-4ac}{4a^2}+\frac{b^2}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

6. Take the square root of both sides of the equation and simplify the right side.

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

7. Subtract $\frac{b}{2a}$ from both sides of the equation to solve for x.

$$x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

8. Combine the two fractions from the previous step.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

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Example 2

Use the discriminant of $3x^2 - 5x + 1 = 0$ to identify the number and type of solutions.

1. Determine a, b, and c.

   $a = 3$, $b = -5$, and $c = 1$

2. Substitute the values for a, b, and c into the formula for the discriminant, $b^2 - 4ac$.

   $b^2 - 4ac = (-5)^2 - 4(3)(1) = 25 - 12 = 13$

   The discriminant of $3x^2 - 5x + 1 = 0$ is 13.

3. Use what you know about the discriminant to determine the number and type of solutions for the quadratic equation.

   The discriminant, 13, is positive, but it is not a perfect square. Therefore, there will be two real, irrational solutions.

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Instruction

Example 3

Solve $2x^2 - 5x = 12$ using the quadratic formula.

1. Write the quadratic equation in standard form.

   $2x^2 - 5x = 12$ Original equation

   $2x^2 - 5x - 12 = 0$ Subtract 12 from both sides.

2. Determine the values of a, b, and c.

   $a = 2$, $b = -5$, and $c = -12$

3. Substitute the values of a, b, and c into the quadratic formula.

   $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Quadratic formula

   $x = \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-12)}}{2(2)}$ Substitute values for a, b, and c.

   $x = \dfrac{5 \pm \sqrt{25 + 96}}{4}$ Simplify.

   $x = \dfrac{5 \pm \sqrt{121}}{4}$

   $x = \dfrac{5 \pm 11}{4}$

4. Determine the solution(s).

   Since the discriminant, 121, is positive and a perfect square, there are two real, rational solutions.

   Write the fraction from step 3 as two fractions and simplify.

   $x = \dfrac{5+11}{4} = \dfrac{16}{4} = 4$ $\qquad$ $x = \dfrac{5-11}{4} = \dfrac{-6}{4} = \dfrac{-3}{2}$

   $x = 4$ $\qquad$ $x = \dfrac{-3}{2}$

   The solutions to the equation $2x^2 - 5x = 12$ are $x = 4$ and $x = -\dfrac{3}{2}$.

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Example 4

Solve $x^2 = 2x - 1$ using the quadratic formula.

1. Write the quadratic equation in standard form.

$x^2 = 2x - 1$	Original equation
$x^2 - 2x = -1$	Subtract $2x$ from both sides.
$x^2 - 2x + 1 = 0$	Add 1 to both sides.

2. Determine the values of a, b, and c.

   $a = 1$, $b = -2$, and $c = 1$

3. Substitute the values of a, b, and c into the quadratic formula.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Quadratic formula
$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1)}}{2(1)}$	Substitute values for a, b, and c.
$x = \frac{2 \pm \sqrt{4 - 4}}{2}$	Simplify the multiplication under the radical sign.
$x = \frac{2 \pm \sqrt{0}}{2}$	Subtract the values under the radical sign.
$x = \frac{2 \pm 0}{2}$	Simplify the square root.

4. Determine the solutions(s).

   Since the discriminant is 0, there is one real, rational solution.

$$x = \frac{2 \pm 0}{2} = \frac{2}{2} = 1$$

$x = 1$

The solution to the equation $x^2 = 2x - 1$ is $x = 1$.

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Example 5

Solve $5x^2 + 2x + 3 = 0$ using the quadratic formula.

1. The quadratic is already in standard form. Determine the values of a, b, and c.

   $a = 5$, $b = 2$, and $c = 3$

2. Substitute these values into the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic formula

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(5)(3)}}{2(5)}$$ Substitute values for a, b, and c.

$$x = \frac{-2 \pm \sqrt{4 - 60}}{10}$$ Simplify the multiplication under the radical sign.

$$x = \frac{-2 \pm \sqrt{-56}}{10}$$ Subtract the values under the radical sign.

3. Determine the solution(s).

   Since the discriminant is negative, there are no real solutions.

   The equation $5x^2 + 2x + 3 = 0$ has no real solutions.

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Problem-Based Task 3.2.7: Divine Proportion

SMP	
1	2 ✓
3	4
5	6
7 ✓	8

The golden ratio is a special number. It is often represented by the Greek letter *phi*, ϕ. It is often suggested that artists and architects have long used the golden ratio to create work that is pleasing to the human eye. The ancient Greeks called the golden ratio the divine proportion; they believed the number to have mystical properties. One property of the golden ratio is that it is the only positive number that when increased by 1 is equal to its square. Find the golden ratio.

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Problem-Based Task 3.2.7: Divine Proportion

Coaching

a. What equation can you use to solve for the golden ratio in terms of ϕ?

b. What form does your equation need to be in so that the quadratic formula can be used? Write your equation in that form.

c. How can you solve your equation using the quadratic formula?

d. Do both of the solutions satisfy the given property?

e. What is the golden ratio, ϕ?

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Instruction

Problem-Based Task 3.2.7: Divine Proportion

Coaching Sample Responses

a. What equation can you use to solve for the golden ratio in terms of ϕ?

$\phi^2 = \phi + 1$

b. What form does your equation need to be in so that the quadratic formula can be used? Write your equation in that form.

The equation needs to be in standard form.

Move all terms to one side so that the equation can be set equal to 0.

Subtract ϕ and 1 from both sides.

$\phi^2 - \phi - 1 = 0$

c. How can you solve your equation using the quadratic formula?

Find the values of a, b, and c.

$a = 1$, $b = -1$, and $c = -1$

Substitute the values you found into the quadratic formula.

$$\phi = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

Simplify.

$$\phi = \frac{1 \pm \sqrt{1+4}}{2}$$

$$\phi = \frac{1 \pm \sqrt{5}}{2} \approx 1.618 \text{ and } \approx -0.618$$

d. Do both of the solutions satisfy the given property?

No. The golden ratio must be positive because it represents a ratio between distances that are always positive, so the negative solution can be discarded.

e. What is the golden ratio, ϕ?

The golden ratio is $\frac{1+\sqrt{5}}{2}$ or approximately 1.618.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.2.7: Applying the Quadratic Formula

For problems 1 and 2, find the discriminant. Determine the number and type of roots of the equation.

1. $4x^2 + 4x + 1 = 0$

2. $x^2 + 3x = -2x - 6$

For problems 3–6, solve using the quadratic formula.

3. $-2x^2 + 3x + 4 = 0$

4. $16 - 8x - x^2 = 0$

5. $3x^2 + 7x + 12 = 0$

6. $-32x = 2x^2 - x - 51$

For problems 7–10, read each scenario and use the quadratic formula to answer the questions.

7. The height of a softball in meters x seconds after it has been thrown upward is given by $-4.9x^2 + 9x + 1.2$. After how many seconds does the ball hit the ground?

8. A company sells about $20x - x^2$ units each month, where x is the price of one unit. For what price(s) does the company sell 100 units?

9. As part of a science experiment, Carson designs and creates a cushioned egg carrier. He puts an egg inside it, and then drops it from a window to see whether his design can safely cushion the egg and keep it from breaking. The egg's height in feet x seconds after being dropped is given by $27 - 16x^2$. After how many seconds will the egg hit the ground?

10. How does the quadratic formula show the number and type of solutions of a quadratic equation?

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Lesson 3.2.8: Solving Quadratic Inequalities

Warm-Up 3.2.8

A small company manufactures two types of computer games. One puzzle game takes 0.6 hour for assembly, and one racing game takes 0.3 hour for assembly. The company has at most 240 labor hours available for game assembly.

1. Let the number of puzzle games equal p and the number of racing games equal r. Write an inequality to model the situation.

2. If the company's manager decides the workers should assemble 100 puzzle games, how many racing games can be assembled?

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Lesson 3.2.8: Solving Quadratic Inequalities

Common Core State Standards

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

A–REI.4 Solve quadratic equations in one variable.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Warm-Up 3.2.8 Debrief

1. Let the number of puzzle games equal p and the number of racing games equal r. Write an inequality to model the situation.

   $0.6p + 0.3r \leq 240$

2. If the company's manager decides the workers should assemble 100 puzzle games, how many racing games can be assembled?

   $0.6(100) + 0.3r \leq 240$ Substitute 100 for p.

   $60 + 0.3r \leq 240$ Simplify, then solve.

   $0.3r \leq 180$

   $r \leq 600$

   The workers can assemble 600 racing games at the most.

Connection to the Lesson

- Students will extend their knowledge of linear inequalities to solve quadratic inequalities.
- Students will build inequalities to represent problems in context.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- solving linear inequalities (A–REI.3)
- factoring quadratic expressions (A–REI.4a)
- solving quadratic equations (A–REI.4b)
- graphing solutions to inequalities on a number line (6.EE.8)

Introduction

We have looked at a variety of methods to solve quadratic equations. **Quadratic inequalities** can be written in the form $ax^2 + bx + c < 0$, $ax^2 + bx + c \leq 0$, $ax^2 + bx + c > 0$, or $ax^2 + bx + c \geq 0$. The solutions to quadratic inequalities can be written as intervals and graphed on a number line. An **interval** is the continuous set of all real numbers between two given numbers. The two endpoints of an interval may or may not be included in the interval depending on whether the interval is open, closed, or half-open (also called half-closed).

Key Concepts

- If one side of a factored inequality is 0, we can use logic to determine the sign possibilities of the factors.
- To solve a quadratic inequality, set one side equal to 0 if it isn't already, replace the inequality sign with an equal sign, and solve the equation. Then replace the equal sign with the inequality sign to represent the solution.
- Recall that multiplying or dividing both sides of an inequality by a negative number requires reversing the inequality symbol.
- Solutions of quadratic inequalities can be graphed on number lines.
- Use these solutions to create regions on a number line and test points in each region to verify the solution of the inequality.
- The solution of a quadratic inequality can lie in one interval or two intervals.
- Recall that the solution of an inequality graphed on a number line is represented by a darkened line segment.
- The endpoints of the solution interval are represented by either an open dot or a closed dot.

SRB p. U3-86

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Instruction

- Graph the endpoints as an open dot if the original inequality symbol is < or >.
- Graph endpoints as a closed dot if the original inequality symbol is ≤ or ≥.

Common Errors/Misconceptions

- writing inequalities with signs facing in opposite directions (for example, $3 < x > 5$)
- forgetting that answers should be inequalities
- forgetting to reverse the inequality signs when multiplying or dividing by a negative number
- using < or > instead of ≤ or ≥, and vice versa
- improperly representing the sections of the number line

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Guided Practice 3.2.8

Example 1

What is the solution of the inequality $(x - 2)(x + 10) > 0$?

1. Determine the sign possibilities for each factor.

   The expression will be positive when both factors are positive or both factors are negative.

2. Determine when both factors are positive.

   $x - 2 > 0$ when $x > 2$.

   $x + 10 > 0$ when $x > -10$.

   Both factors are positive when $x > 2$ and $x > -10$. The solution is the intersection of these two sets, $x > 2$.

3. Determine when both factors are negative.

   $x - 2$ is negative when $x < 2$.

   $x + 10$ is negative when $x < -10$.

   Both factors are negative when $x < 2$ and $x < -10$. The solution is the intersection of these two sets, $x < -10$.

   $(x - 2)(x + 10) > 0$ when $x > 2$ or $x < -10$.

SWB p. 99; SRB p. U3-87

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Instruction

Example 2

Solve $x^2 + 8x + 7 \leq 0$. Graph the solution on a number line.

1. Replace the inequality sign with an equal sign.

   $x^2 + 8x + 7 = 0$

2. Solve the equation.

   The quadratic expression factors easily.

   Find two numbers whose product is 7 and whose sum is 8.

   The numbers are 1 and 7 because (1)(7) = 7 and 1 + 7 = 8.

   Use these numbers to find the factors.

   The factors are $(x + 1)$ and $(x + 7)$.

   Write the expression as the product of factors.

   $(x + 1)(x + 7) = 0$

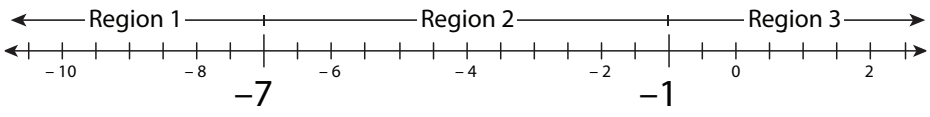
   Use the Zero Product Property to solve.

   $x + 1 = 0$ $\qquad$ $x + 7 = 0$

   $x = -1$ $\qquad$ $x = -7$

3. Use a number line and test points to solve the inequality.

   The sign of the original inequality is ≤. Therefore, we are looking for solutions that make the quadratic expression less than or equal to 0. Plot the solutions to the equation, –7 and –1, on a number line. These points divide the number line into three regions: numbers less than –7; numbers from –7 to –1; and numbers greater than –1.



   Test one number from each region of our number line. Pick any number from the region that is easy to work with.

(*continued*)

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Test $x = -10$ from Region 1 by substituting it into the quadratic expression:

$(-10)^2 + 8(-10) + 7 = 27$

Set up an inequality that compares the result of your test x-value with 0. Refer back to the sign used in the original inequality—in this case, ≤.

$27 \leq 0$ is not true. Therefore, numbers in Region 1 ($x < -7$) are not part of the solution.

Test $x = -2$ from Region 2:

$(-2)^2 + 8(-2) + 7 = 4 - 16 + 7 = -5$

Set up an inequality that compares the result of your test x-value with 0.

$-5 \leq 0$ is true. Therefore, numbers in Region 2 ($-7 \leq x \leq -1$) are solutions.

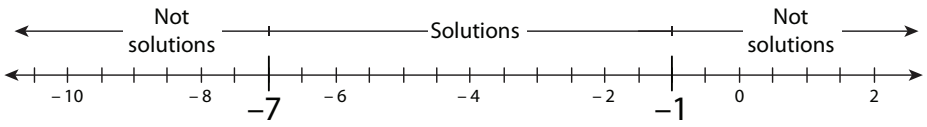
Test $x = 0$ from Region 3:

$(0)^2 + 8(0) + 7 = 7$

Set up an inequality that compares the result of your test x-value with 0.

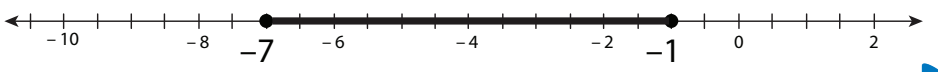
$7 \leq 0$ is not true. Therefore, numbers in Region 3 ($x > -1$) are not solutions.

From the test points, we can see that the expression is less than or equal to 0 when x is between –7 and –1 and equal to –7 and –1.



4. Solve the inequality and graph the solutions on a number line.

   Recall that because the inequality sign is ≤, the circles around the endpoints are filled.



   $x^2 + 8x + 7 \leq 0$ when $-7 \leq x \leq -1$.

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Instruction

Example 3

Solve $4x - 1 > 8 - x^2$. Graph the solution on a number line.

1. Move all terms to the left side of the inequality.

$4x - 1 > 8 - x^2$	Original inequality
$x^2 + 4x - 1 > 8$	Add x^2 to both sides.
$x^2 + 4x - 9 > 0$	Subtract 8 from both sides.

2. Replace the inequality sign with an equal sign.

$x^2 + 4x - 9 = 0$

3. Solve the equation.

The equation cannot be factored over the set of rational numbers.

Solve the equation by completing the square.

$x^2 + 4x - 9 = 0$	Equation from the previous step
$x^2 + 4x = 9$	Add 9 to both sides.
$x^2 + 4x + 2^2 = 9 + 2^2$	Add the square of one-half of b to both sides of the equation to complete the square.
$x^2 + 4x + 4 = 13$	Simplify.
$(x + 2)^2 = 13$	Express the left side as a perfect square trinomial.
$x + 2 = \pm\sqrt{13}$	Take the square root of both sides to isolate the binomial.
$x = -2 \pm \sqrt{13}$	Isolate x by subtracting 2 from both sides.
$x \approx -5.61$ or $x \approx 1.61$	Simplify.

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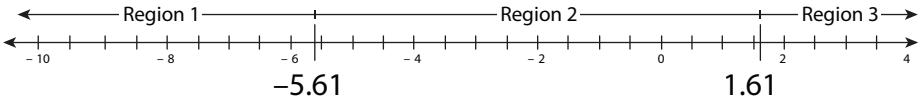
4. Use a number line and test points to solve the inequality.

The sign of the original inequality is >. Therefore, we are looking for solutions make the quadratic expression greater than 0. Plot the end points on a number line. $x = -2-\sqrt{13}$ is approximately –5.61 and $x = -2+\sqrt{13}$ is approximately equal to 1.61.



Test one number from each region of the number line. Pick any number from the region that is easy to work with.

Test $x = -10$ from Region 1 by substituting it into the quadratic expression:

$(-10)^2 + 4(-10) - 9 = 51$

Set up an inequality that compares the result of your test x-value with 0. Refer back to the sign used in the original inequality, in this case >.

$51 > 0$ is true. Therefore, numbers in Region 1 ($x < -5.61$) are solutions.

Test $x = 0$ from Region 2:

$(0)^2 + 4(0) - 9 = -9$

Set up an inequality that compares the result of your test x-value with 0.

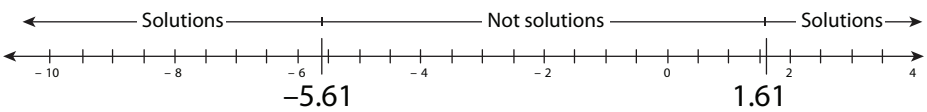
$-9 > 0$ is not true. Therefore, numbers in Region 2 ($-5.61 \leq x \leq 1.61$) are not solutions.

Test $x = 2$ from Region 3:

$(2)^2 + 4(2) - 9 = 3$

Set up an inequality that compares the result of your test x-value with 0.

$3 > 0$ is true. Therefore, numbers in Region 3 ($x > 1.61$) are solutions.



From the test points, we can see that the expression is greater than 0 when x is less than –5.61 or when x is greater than 1.61.

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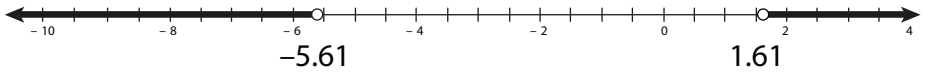
5. Graph the solutions on a number line.

   Recall that because the inequality signs are < and >, the circles around the endpoints are open.



–5.61 1.61

$4x - 1 > 8 - x^2$ when $x < -5.61$ or when $x > 1.61$. Expressed using exact values, $x < -2-\sqrt{13}$ or $x > -2+\sqrt{13}$.

Example 4

What is the solution of $3x^2 + 2x + 2 < 0$?

1. Set the inequality equal to 0 and then find the discriminant.

   Find the discriminant of $3x^2 + 2x + 2 = 0$.

   $a = 3$, $b = 2$, and $c = 2$

$b^2 - 4ac$	Formula for discriminant
$= 2^2 - 4(3)(2)$	Substitute values for a, b, and c.
$= 4 - 24$	Simplify.
$= -20$	

   The discriminant is negative; therefore, the equation has no real solutions.

2. Use the discriminant to solve the inequality.

   Since the equation $3x^2 + 2x + 2 = 0$ has no real solutions, it is not possible to graph a solution on the real number line.

   The inequality $3x^2 + 2x + 2 < 0$ has no real solutions.

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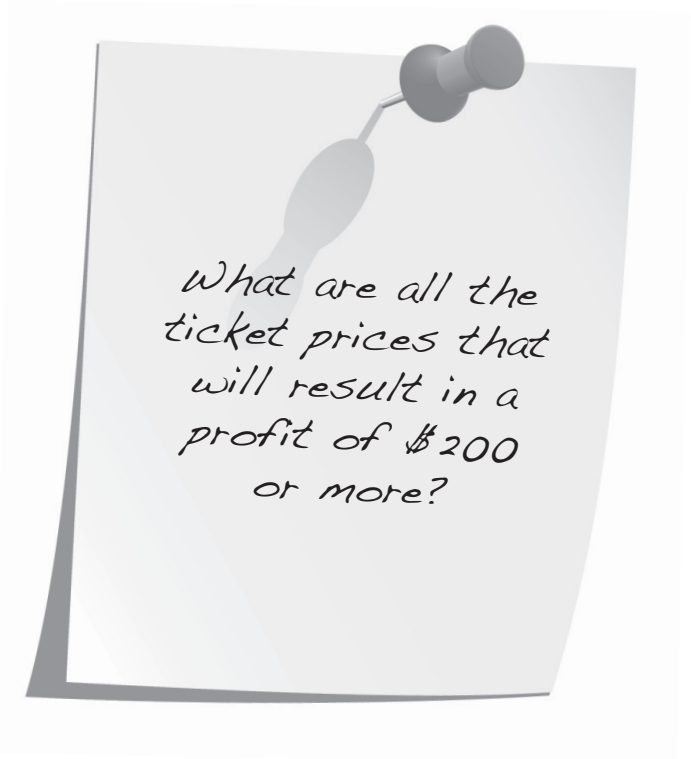
Problem-Based Task 3.2.8: Dancing for Charity

SMP	
1	2 ✓
3	4
5 ✓	6 ✓
7	8

A school is planning to host a dance with all profits going to charity. The amount of profit is found by subtracting the total costs from the total income. The income from ticket sales can be expressed as $200x - 10x^2$, where x is the cost of a ticket. The costs of putting on the dance can be expressed as $500 + 20x$. What are all the ticket prices that will result in a profit of $200 or more?



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Problem-Based Task 3.2.8: Dancing for Charity

Coaching

a. How can profit be expressed in terms of income, I, and costs, C?

b. How can profit be expressed in terms of the cost of a ticket, x?

c. How can your expression for profit in terms of ticket cost be used to write an inequality that represents the situation?

d. How can you solve your inequality for x?

e. Do your answers make sense?

f. What are all the ticket prices that will result in a profit of $200 or more?

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Instruction

Problem-Based Task 3.2.8: Dancing for Charity

Coaching Sample Responses

a. How can profit P be expressed in terms of income, I, and costs, C?

Profit is found by subtracting the total costs from the total income.

$P = I - C$

b. How can profit be expressed in terms of the cost of a ticket, x?

Substitute the expressions given for income and cost.

$P = (200x - 10x^2) - (500 + 20x)$

Simplify the expression.

$P = 200x - 10x^2 - 500 - 20x$

$P = -10x^2 + 180x - 500$

c. How can your expression for profit in terms of ticket cost be used to write an inequality that represents the situation?

The expression for profit must be greater than or equal to 200.

$-10x^2 + 180x - 500 \geq 200$

d. How can you solve your inequality for x?

To set the inequality equal to 0, first subtract 200 from both sides. Rewrite the inequality as an equation to find the boundary points of the inequality. Then, divide both sides by 10 to simplify.

$-10x^2 + 180x - 700 \geq 0$

$-10x^2 + 180x - 700 = 0$

$-x^2 + 18x - 70 = 0$

Use the quadratic formula to solve for x.

$a = -1$, $b = 18$, and $c = -70$

$$x = \frac{-(18) \pm \sqrt{(18)^2 - 4(-1)(-70)}}{2(-1)}$$

$$x = \frac{-18 \pm \sqrt{324 - 280}}{-2}$$

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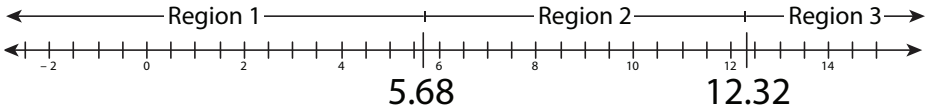
$$x = \frac{-18 \pm \sqrt{44}}{-2}$$

$x \approx 5.68$ or $x \approx 12.32$

Approximate the locations of the boundary points on a number line and test numbers in each region to solve the inequality.



Test $x = 0$ from Region 1 by substituting it into the quadratic expression:

$-10(0)^2 + 180(0) - 700 = -700$

$-700 \geq 0$ is not true. Therefore, numbers in Region 1 ($x < 5.68$) are not solutions.

Test $x = 10$ from Region 2:

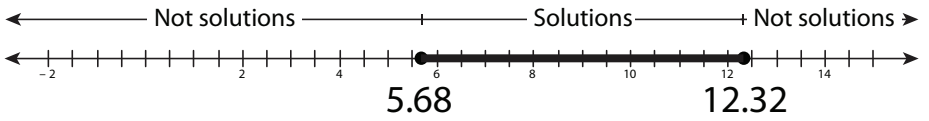
$-10(10)^2 + 180(10) - 700 = 100$

$100 \geq 0$ is true. Therefore, numbers in Region 2 ($5.68 \leq x \leq 12.32$) are solutions.

Test $x = 20$ from Region 3:

$-10(20)^2 + 180(20) - 700 = -1100$

$-1100 \geq 0$ is not true. Therefore, numbers in Region 3 ($x > 12.32$) are not solutions.



$5.68 \leq x \leq 12.32$

e. Do your answers make sense?

Yes. Ticket prices between \$5.68 and \$12.32 seem reasonable.

f. What are all the ticket prices that will result in a profit of at least \$200?

Profit will be at least \$200 for all ticket prices from \$5.68 to \$12.32, inclusive.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.2.8: Solving Quadratic Inequalities

For problems 1–7, solve each quadratic inequality. Graph the solution(s), if any, on a number line.

1. $(x + 3)(x - 2) < 0$

2. $(3x + 4)(2x - 1) \geq 0$

3. $x^2 - 25 \geq 0$

4. $x^2 + x - 12 < 0$

5. $x^2 + 20x - 3 \leq 0$

6. $x^2 + 12 \geq -7x$

7. $4x^2 + 4 < x$

For problems 8–10, solve each inequality using the given information.

8. A flying squirrel jumps from one tree to the next. Its height in feet x seconds into the jump is given by $-16x^2 + 32x + 4$. After how many seconds is the squirrel more than 12 feet above the ground?

9. Milo dives into a pool from a platform. His height above the water in feet x seconds into the dive is given by the expression $-5x^2 + 5x + 3$. After how many seconds is Milo more than 4 feet above the water?

10. Under certain conditions, the stopping distance of a vehicle in feet is given by $0.055x^2 + 1.1x$, where x is the speed of the vehicle in miles per hour. At what speeds is the stopping distance less than 40 feet?

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Progress Assessment

Circle the letter of the best answer.

1. Solve $x^2 = 225$ for x.
   a. $x = \pm 15$
   b. $x = -15$
   c. $x = 15$
   d. $x = \pm 112.5$

2. Solve $(x - 9)^2 + 4 = 85$ for x.
   a. $x = 18$
   b. $x = 0$ or $x = 18$
   c. $x = 9$
   d. $x = \pm 9$

3. Factor $x^2 - 11x + 18$.
   a. $(x - 11)(x + 18)$
   b. $(x - 2)(x - 9)$
   c. $(x + 11)(x - 18)$
   d. $(x - 2)(x + 9)$

4. Which expression is prime over the set of rational numbers?
   a. $x^2 - 12x - 20$
   b. $x^2 + 12x + 20$
   c. $x^2 - 12x + 20$
   d. $x^2 + 9x + 20$

5. Which expression is not a perfect square trinomial?
   a. $x^2 + 12x + 36$
   b. $x^2 - 6x + 9$
   c. $x^2 + 12x - 36$
   d. $x^2 + 6x + 9$

6. Use the discriminant of $-3x^2 + x - 50$ to find the number and type of solutions.
   a. There are two real, rational solutions.
   b. There are two real, irrational solutions.
   c. There are no real solutions.
   d. There is one real, rational solution.

continued

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7. Solve $2x^2 - 5x + 2 = 0$ for x.

   a. $x = 0.5$ or $x = 2$
   b. $x = 0.5$
   c. $x = 2$
   d. $x = -0.5$ or $x = 2$

8. Solve $x^2 + 20x = 44$ for x.

   a. $x = 0$ or $x = -20$
   b. $x = -2$ or $x = 22$
   c. $x = \pm 12$
   d. $x = 2$ or $x = -22$

9. For what values of x is $x^2 - 4x - 5 > 0$?

   a. $-1 < x < 5$
   b. $x < -4$ or $x > 5$
   c. $x < -1$ or $x > 5$
   d. $-5 < x < -4$

10. For what values of x is $x^2 - x + 1 > 0$?

    a. no real solutions
    b. all values of x
    c. $-1 < x < 1$
    d. $x < -1$ or $x > 1$

Use what you have learned about quadratic equations to complete all parts of the following problem.

11. We have studied four different ways to solve quadratic equations: finding square roots, factoring, completing the square, and using the quadratic formula. Solve each quadratic equation that follows using a different method.

    a. $x^2 + 20x + 19 = 0$
    b. $x^2 + 20x + 15 = 0$
    c. $3x^2 + 20x + 7 = 0$
    d. $9x^2 + 15 = 20$

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. What is the vertex of the quadratic function $f(x) = 4x^2 + 16x + 15$?

   a. (–1.5, 0)
   b. (–2, –1)
   c. (0, 15)
   d. (–2.5, 0)

2. What is the y-intercept of the quadratic function $g(x) = –x^2 – 16x – 64$?

   a. –8
   b. –64
   c. 0
   d. 64

3. What is the equation of a quadratic function in standard form, given the zeros $x = \frac{1}{4}$ and $x = –2$ and the point (2, –21)?

   a. $y = -3x^2 - \frac{21}{4}x + \frac{3}{2}$
   b. $y = 3x^2 - \frac{21}{4}x + \frac{3}{2}$
   c. $y = -3x^2 + \frac{21}{4}x + \frac{3}{2}$
   d. $y = 3x^2 - \frac{21}{4}x - \frac{3}{2}$

4. Given the quadratic function $q(x) = –3(x – 2)^2 + 4$, identify the vertex and determine whether it is a minimum or maximum.

   a. (–2, 4); maximum
   b. (–6, 4); minimum
   c. (6, 4); minimum
   d. (2, 4); maximum

5. Solve $7x^2 – 45 = y$ for x in terms of y.

   a. $x = \pm\frac{\sqrt{y-45}}{7}$
   b. $x = \sqrt{\frac{y-45}{7}}$
   c. $x = \pm\sqrt{\frac{y+45}{7}}$
   d. $x = \frac{\sqrt{y+45}}{7}$

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Lesson 3: Creating Quadratic Equations in Two or More Variables

Instruction

Common Core State Standards

SMP	
1 ✓	2 ✓
3	4 ✓
5	6 ✓
7 ✓	8

A–SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

a. Factor a quadratic expression to reveal the zeros of the function it defines.

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

A–CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

A–CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.*★

Essential Questions

1. How are the graphs of quadratic functions related to their equations?
2. How do you rearrange the terms in a formula when solving for the squared term?
3. How do you find the vertex of a quadratic function?
4. What types of real-world situations are modeled by quadratic equations?
5. What do the solutions to quadratic equations represent in terms of real-world situations?

WORDS TO KNOW

axis of symmetry of a parabola	the line through the vertex of a parabola about which the parabola is symmetric. For a quadratic function in the form $f(x) = ax^2 + bx + c$, the equation of the axis of symmetry is $x = \frac{-b}{2a}$.
intercept form of a quadratic function	the factored form of a quadratic equation, written as $f(x) = a(x - p)(x - q)$, where p and q are the zeros of the function

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key features of a quadratic function	characteristics used to describe, draw, and compare quadratic functions. These characteristics are: the x-intercept(s); the y-intercept; values of x for which the function is increasing and decreasing and for which the graph of the function is positive and negative; the function's minimum or maximum; the axis of symmetry; and the end behavior of the graph of the function.
literal equation	an equation that involves two or more variables
maximum	the greatest value of y in the graph of a function
minimum	the least value of y in the graph of a function
parabola	the U-shaped graph of a quadratic function
quadratic function	a function that can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. The graph of a quadratic function is a parabola.
standard form of a quadratic function	a quadratic function written as $f(x) = ax^2 + bx + c$, where a is the coefficient of the quadratic term, b is the coefficient of the linear term, and c is the constant term
vertex form of a quadratic function	a quadratic function written as $f(x) = a(x - h)^2 + k$, where the vertex of the parabola is the point (h, k); the form of a quadratic equation where the vertex can be read directly from the equation
vertex of a parabola	the point at which the curve changes direction; its y-value is either a maximum or a minimum
***x*-intercept**	the x-coordinate of the point where a line or a curve intersects the x-axis
***y*-intercept**	the y-coordinate of the point where a line or a curve intersects the y-axis

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Recommended Resources

- IXL Learning. "Write Linear, Quadratic, and Exponential Functions."

  http://www.walch.com/rr/00106

  This website provides practice and instant feedback on determining linear, exponential, or quadratic equations given a table of data.

- IXL Learning. "Characteristics of Quadratic Functions."

  http://www.walch.com/rr/00107

  Users can practice determining the key features of graphs of quadratic functions, including vertices, *y*-intercepts, maximums, and more. Instant feedback is provided along with explanations for incorrect answers.

- Massey University. "Mathematical Formulae."

  http://www.walch.com/rr/00108

  This website offers basic instructions on rearranging various formulas, including those with squares.

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Lesson 3.3.1: Creating and Graphing Equations Using Standard Form

Warm-Up 3.3.1

The point guard for your school's basketball team shoots the ball from 11.5 feet away from the basket. As the table shows, the height of the ball changes as it nears the basket.

Distance from basket in feet (x)	Ball height in feet (y)
11.5	6
10	10.4
9	11.8
5	14.3
4	13.8
0	10

1. Sketch a graph to show the relationship between the distance the ball is from the basket and the height of the ball. Connect the points with a curve.

2. About how high do you think the ball will be after traveling a horizontal distance of 8 feet from the point guard toward the basket? Explain your reasoning.

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Lesson 3.3.1: Creating and Graphing Equations Using Standard Form

Common Core State Standards

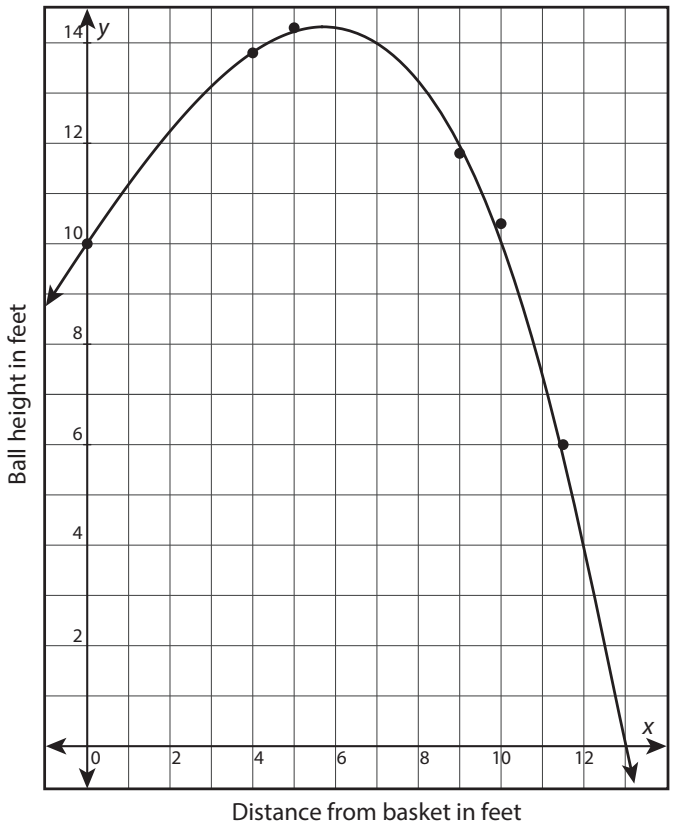
A–SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

a. Factor a quadratic expression to reveal the zeros of the function it defines.

A–CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

Warm-Up 3.3.1 Debrief

1. Sketch a graph to show the relationship between the distance the ball is from the basket and the height of the ball. Connect the points with a curve.



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2. About how high do you think the ball will be after traveling a horizontal distance of 8 feet from the point guard toward the basket? Explain your reasoning.

   Refer to the graph.

   Subtract 8 feet from the starting distance of 11.5 feet to get 3.5 feet.

   When the ball travels 8 horizontal feet from the point guard, it will be 3.5 feet from the basket.

   The corresponding y-value when x is 3.5 is approximately 13.5 feet.

   Therefore, the ball will be about 13.5 feet high when the ball has traveled 8 horizontal feet toward the basket.

Connection to the Lesson

- Students will extend their knowledge of reading graphs to interpreting graphs of quadratic functions.

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Prerequisite Skills

This lesson requires the use of the following skills:

- graphing linear equations (A–CED.2★)
- solving quadratic equations using the quadratic formula (A–REI.4b)
- solving quadratic equations by factoring (A–REI.4b)

Introduction

Imagine the path of a basketball as it leaves a player's hand and swooshes through the net. Or, imagine the path of an Olympic diver as she arcs away from the diving board, slips deep into the water, and resurfaces. Both traveling paths make a U-shaped curve known as a parabola. A **parabola** is the U-shaped graph of a quadratic function, which is an equation with a degree of 2. Formally, a **quadratic function** is a function that can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. Quadratic equations can be used to model varying situations. In this lesson, you will learn about and practice identifying the important points on a parabola and use those points to sketch the corresponding graph.

Key Concepts

- The **standard form of a quadratic function** is $f(x) = ax^2 + bx + c$, where a is the coefficient of the quadratic term, b is the coefficient of the linear term, and c is the constant term.
- The ***x*-intercepts** of the graph of a quadratic function are the x-coordinates of the points at which the graph intersects the x-axis. Not all quadratic functions have x-intercepts.
- The ***y*-intercept** of the graph of a quadratic function is the y-coordinate of the point at which the graph intersects the y-axis.

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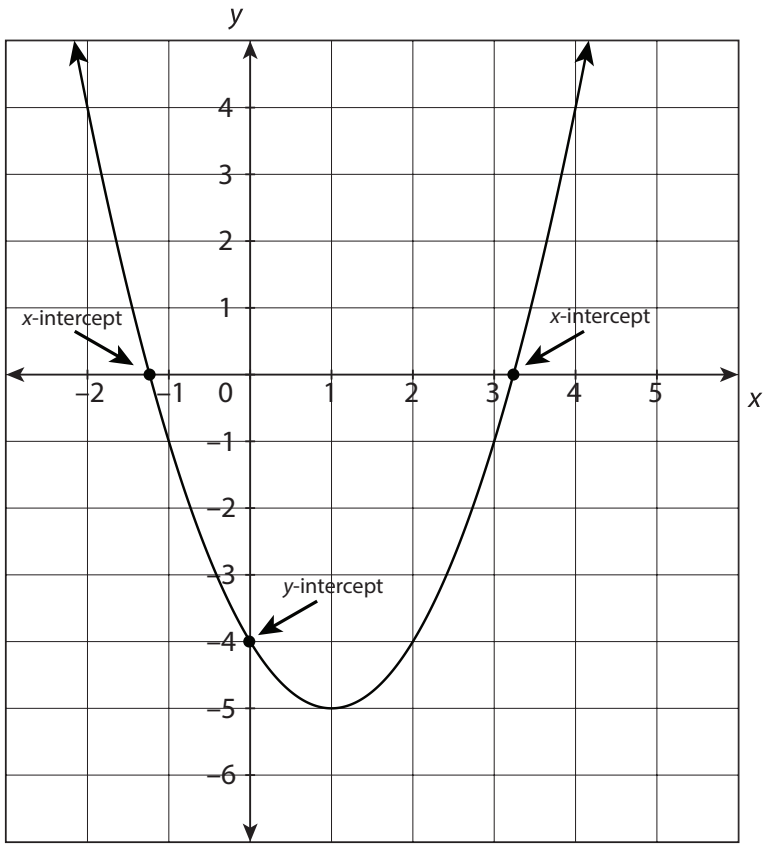
Instruction

- The following graph shows the location of a parabola's *x*- and *y*-intercepts.



- The equation of the graphed parabola is $y = x^2 - 2x - 4$.
- Note that the *y*-intercept of this parabola is –4 and the constant term of its quadratic equation is –4.
- The *y*-intercept of the graph of a quadratic function is the value of *c* of the quadratic function when the equation is written in standard form.

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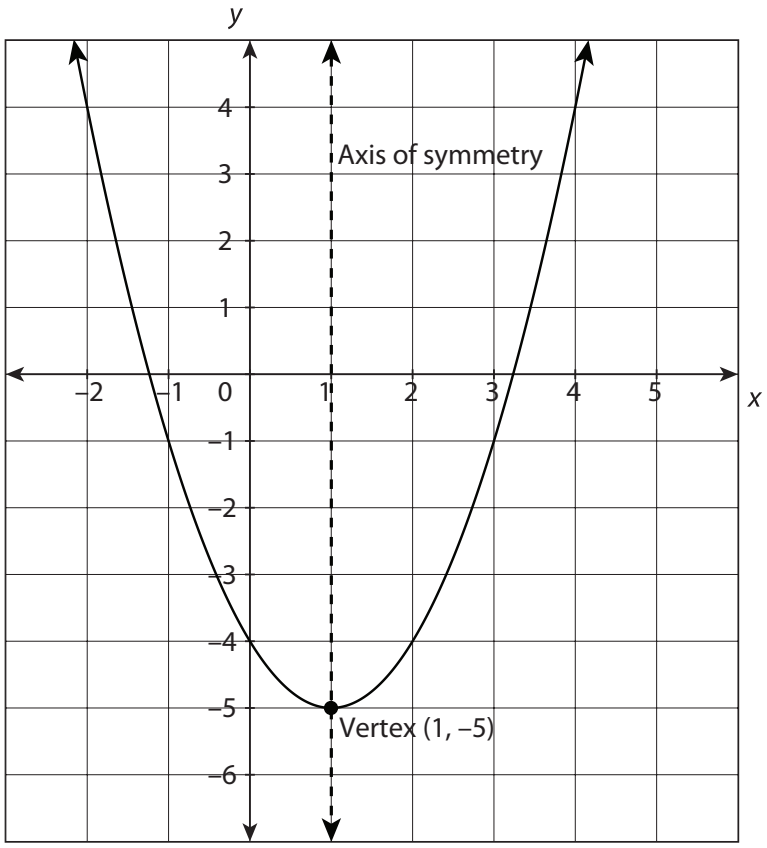
Instruction

- The **axis of symmetry of a parabola** is the line through the vertex of a parabola about which the parabola is symmetric.



- The axis of symmetry extends through the vertex of the graph.
- The **vertex of a parabola** is the point on a parabola where the graph changes direction. The coordinates of the vertex are (h, k), where h is the x-coordinate and k is the y-coordinate.
- The equation of the axis of symmetry of a parabola is $x = \frac{-b}{2a}$.
- In the example graph shown, the equation of the axis of symmetry is $x = 1$ because the vertical line through 1 is the line that cuts the parabola in half.
- The vertex of a quadratic function lies on the axis of symmetry.
- The formula $x = \frac{-b}{2a}$ is not only used to find the equation of the axis of symmetry, but also to find the x-coordinate of the vertex.
- To find the y-coordinate of the vertex, substitute the value of x into the original function. The coordinates of the vertex (h, k) are $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

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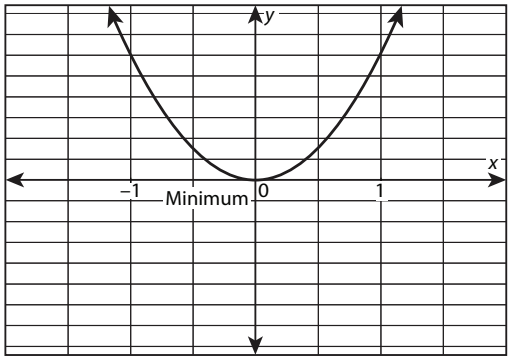
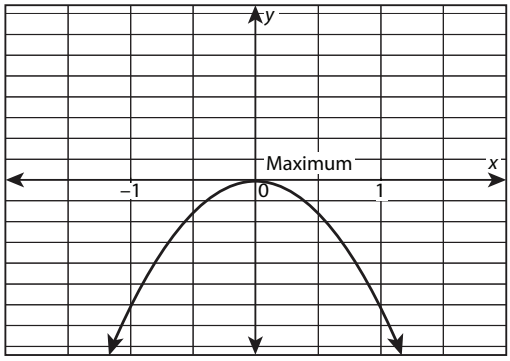
Instruction

- The y-coordinate of the vertex is the maximum or minimum value of a parabola.
- The **maximum** is the greatest y-value of a quadratic function and the **minimum** is the least y-value.
- One might see these terms in real-world situations such as maximizing profit and minimizing cost.
- From the equation of a quadratic function in standard form, you can determine if the function has a maximum or minimum based on the sign of the coefficient of the quadratic term, a.
- If $a > 0$, the parabola opens up and therefore has a minimum value.
- If $a < 0$, the parabola opens down and therefore has a maximum value.

Minimum	Maximum
$y = 3x^2$ $a = 3$ and $a > 0$	$y = -3x^2$ $a = -3$ and $a < 0$
	

- To create the equation of a quadratic function in standard form, one must know some combination of its key features.
- The **key features of a quadratic function** are used to describe, draw, and compare quadratic functions, and include: the x-intercept(s); the y-intercept; the values of x for which the function is increasing and decreasing and for which the graph of the function is positive and negative; its minimum or maximum; the axis of symmetry; and the end behavior of the graph of the function.
- To create the equation of a quadratic function given the vertex and y-intercept, first set the x-coordinate of the vertex, h, equal to $\frac{-b}{2a}$ and solve for b: $b = -2ah$.
- Second, substitute the expression that is equal to b, the y-intercept, and the coordinates of the vertex into the standard form of a quadratic equation, $y = ax^2 + bx + c$.
- Solve the equation for a and then substitute the value of a into the equation you created in the first step to determine the value of b.

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Common Errors/Misconceptions

- forgetting to write an equation for the axis of symmetry of a parabola
- not finding and labeling all important points
- drawing the parabola as a straight line or a series of straight lines instead of a curve

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Guided Practice 3.3.1

Example 1

Find the y-intercept and vertex of the function $f(x) = -2x^2 + 4x + 3$. Determine whether the vertex is a minimum or maximum point on the graph.

1. Determine the y-intercept.

   The function $f(x) = -2x^2 + 4x + 3$ is written in standard form, $f(x) = ax^2 + bx + c$.

   When $x = 0$, the y-intercept equals c, which is 3.

   Verify that 3 is the y-intercept.

$f(x) = -2x^2 + 4x + 3$	Original equation
$f(0) = -2(0)^2 + 4(0) + 3$	Substitute 0 for x.
$f(0) = 3$	Simplify.

   The y-intercept is 3.

2. Determine the vertex of the function.

   $f(x) = -2x^2 + 4x + 3$ is in standard form; therefore, $a = -2$ and $b = 4$.

$x = \frac{-b}{2a}$	The x-coordinate of the vertex
$x = \frac{-4}{2(-2)}$	Substitute –2 for a and 4 for b.
$x = 1$	Simplify.

   The vertex has an x-coordinate of 1.

(*continued*)

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Since h is the x-coordinate of the vertex, or the input value of the vertex, we can find the output value, k, by evaluating the function for $x = 1$.

$f(x) = -2x^2 + 4x + 3$	Original equation
$f(1) = -2(1)^2 + 4(1) + 3$	Substitute 1 for x.
$f(1) = 5$	Simplify.

The y-coordinate of the vertex is 5.

The vertex is the point (1, 5).

3. Use the value of a to determine if the vertex is a maximum or a minimum value.

   Given that $f(x) = -2x^2 + 4x + 3$ is written in standard form, $a = -2$.

   Since $a < 0$, the parabola opens down.

   As such, the vertex (1, 5) is a maximum point.

Example 2

$h(x) = 2x^2 - 11x + 5$ is a quadratic function. Determine the direction in which the function opens, the coordinates of the vertex, the axis of symmetry, the x-intercept(s), if any, and the y-intercept. Use this information to sketch the graph.

1. Determine whether the graph opens up or down.

   $h(x) = 2x^2 - 11x + 5$ is in standard form; therefore, $a = 2$.

   Since $a > 0$, the parabola opens up.

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Instruction

2. Find the vertex and the equation of the axis of symmetry.

$h(x) = 2x^2 - 11x + 5$ is in standard form; therefore, $a = 2$ and $b = -11$.

$x = \frac{-b}{2a}$	The x-coordinate of the vertex
$x = \frac{-(-11)}{2(2)}$	Substitute 2 for a and –11 for b.
$x = 2.75$	Simplify.

The vertex has an x-coordinate of 2.75.

Since the input value is 2.75, find the output value by evaluating the function for $x = 2.75$.

$h(x) = 2x^2 - 11x + 5$	Original equation
$h(2.75) = 2(2.75)^2 - 11(2.75) + 5$	Substitute 2.75 for x.
$h(2.75) = -10.125$	Simplify.

The y-coordinate of the vertex is –10.125.

The vertex is the point (2.75, –10.125).

Since the axis of symmetry is the vertical line through the vertex, the equation of the axis of symmetry is $x = 2.75$.

3. Find the y-intercept.

$h(x) = 2x^2 - 11x + 5$ is in standard form, so the y-intercept is the constant term c, which is 5.

The y-intercept is 5.

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4. Find the x-intercepts, if any exist.

   The x-intercepts occur when $y = 0$.

   Substitute 0 for the output, $h(x)$, and solve.

   This equation is factorable, but if we cannot easily identify the factors, the quadratic formula always works.

   Note both methods.

Solved by factoring:

$h(x) = 2x^2 - 11x + 5$

$0 = 2x^2 - 11x + 5$

$0 = (2x - 1)(x - 5)$

$0 = 2x - 1$ or $0 = x - 5$

$x = 0.5$ or $x = 5$

Solved using the quadratic formula:

$h(x) = 2x^2 - 11x + 5$

$0 = 2x^2 - 11x + 5$

$a = 2$, $b = -11$, and $c = 5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(2)(5)}}{2(2)}$$

$$x = \frac{11 \pm \sqrt{81}}{4}$$

$$x = \frac{11 \pm 9}{4}$$

$$x = \frac{11 - 9}{4} \text{ or } x = \frac{11 + 9}{4}$$

$$x = \frac{2}{4} \text{ or } x = \frac{20}{4}$$

$x = 0.5$ or $x = 5$

The x-intercepts are 0.5 and 5.

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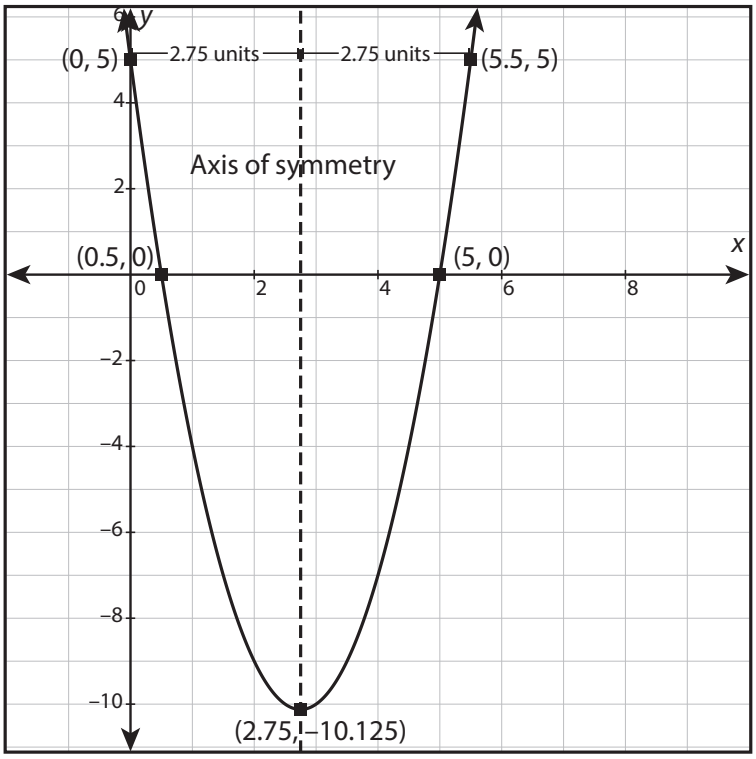
Instruction

5. Plot the points from steps 2–4 and their symmetric points over the axis of symmetry.

   Connect the points with a smooth curve.



Example 3

$R(x) = 2x^2 + 8x + 8$ is a quadratic function. Determine the direction in which the function opens, the vertex, the axis of symmetry, the x-intercept(s), if any, and the y-intercept. Use this information to sketch the graph.

1. Determine whether the graph opens up or down.

   $R(x) = 2x^2 + 8x + 8$ is written in standard form; therefore, $a = 2$.

   Since $a > 0$, the parabola opens up.

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Instruction

2. Find the vertex and the equation of the axis of symmetry.

$R(x) = 2x^2 + 8x + 8$ is in standard form; therefore, $a = 2$ and $b = 8$.

$x = \frac{-b}{2a}$	The x-coordinate of the vertex
$x = \frac{-(8)}{2(2)}$	Substitute 2 for a and 8 for b.
$x = -2$	Simplify.

The vertex has an x-coordinate of –2.

Since the input value is –2, find the output value by evaluating the function for $x = -2$.

$R(x) = 2x^2 + 8x + 8$	Original equation
$R(-2) = 2(-2)^2 + 8(-2) + 8$	Substitute –2 for x.
$R(-2) = 0$	Simplify.

The y-coordinate of the vertex is 0.

The vertex is the point (–2, 0).

Since the axis of symmetry is the vertical line through the vertex, the equation of the axis of symmetry is $x = -2$.

3. Find the y-intercept.

The function, $R(x) = 2x^2 + 8x + 8$, is in standard form, so the y-intercept is the constant term c, which is 8.

The y-intercept is 8.

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4. Find the x-intercepts, if any exist.

   The x-intercepts occur when $y = 0$.

   Substitute 0 for the output, $R(x)$, and solve using the quadratic formula or by factoring.

Solved by factoring:

$R(x) = 2x^2 + 8x + 8$

$0 = 2x^2 + 8x + 8$

$0 = 2(x^2 + 4x + 4)$

$0 = 2(x + 2)^2$

$0 = (x + 2)$

$x = -2$

Solved using the quadratic formula:

$R(x) = 2x^2 + 8x + 8$

$0 = 2x^2 + 8x + 8$

$a = 2$, $b = 8$, and $c = 8$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(8) \pm \sqrt{(8)^2 - 4(2)(8)}}{2(2)}$$

$$x = \frac{-8 \pm 0}{4}$$

$x = -2$

The x-intercept is –2. This is a special case where the vertex is on the x-axis, so there is only one x-intercept.

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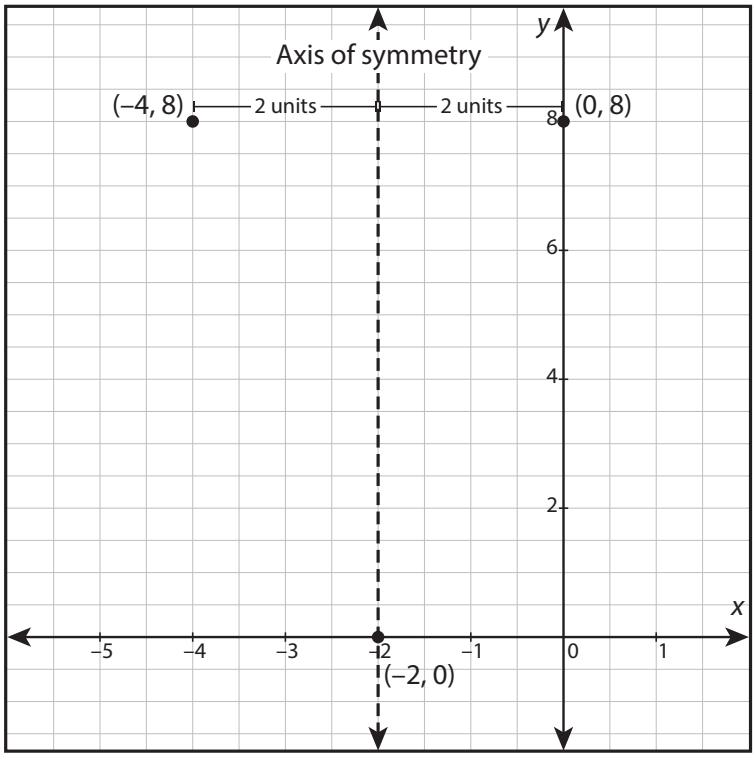
5. Plot the points from steps 2–4 and their symmetric points over the axis of symmetry.



For a more accurate graph, determine an additional pair of symmetric points.

Choose any x-coordinate on the left or right of the axis of symmetry.

Evaluate the function for the chosen value of x to determine the output value.

Let's choose $x = -1$.

$R(x) = 2x^2 + 8x + 8$	Original equation
$R(-1) = 2(-1)^2 + 8(-1) + 8$	Substitute –1 for x.
$R(-1) = 2$	Simplify.

(–1, 2) is an additional point that lies on the parabola.

Plot (–1, 2) on the same graph.

(–1, 2) is 1 unit from the axis of symmetry.

Locate the reflection of (–1, 2) about the axis of symmetry.

(*continued*)

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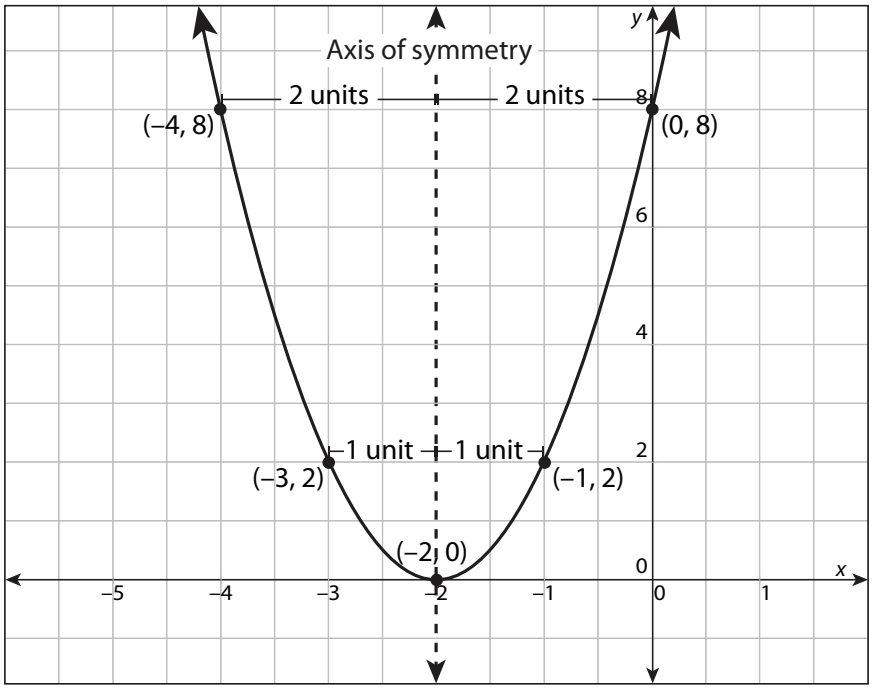
(–3, 2) is also 1 unit from the axis of symmetry and is symmetrical to the original point (–1, 2) with respect to the axis of symmetry.

You can verify that (0, 8) and (–4, 8) are the same distance from the axis of symmetry and are also symmetrical with respect to the axis of symmetry by referring to the graph.



Example 4

$g(x) = -x^2 + 8x - 17$ is a quadratic function. Determine the direction in which the function opens, the vertex, the equation of the axis of symmetry, the x-intercept(s), if any, and the y-intercept. Use this information to sketch the graph.

1. Determine whether the graph opens up or down.

   $g(x) = -x^2 + 8x - 17$ is written in standard form; therefore, $a = -1$.

   Since $a < 0$, the parabola opens down.

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Instruction

2. Find the vertex and the equation of the axis of symmetry.

$g(x) = -x^2 + 8x - 17$ is written in standard form; therefore, $a = -1$ and $b = 8$.

$x = \frac{-b}{2a}$	The x-coordinate of the vertex
$x = \frac{-(8)}{2(-1)}$	Substitute –1 for a and 8 for b.
$x = 4$	Simplify.

The vertex has an x-coordinate of 4.

Since the input value is 4, find the output value by evaluating the function for $x = 4$.

$g(x) = -x^2 + 8x - 17$	Original equation
$g(4) = -(4)^2 + 8(4) - 17$	Substitute 4 for x.
$g(4) = -1$	Simplify.

The y-coordinate of the vertex is –1.

The vertex is the point (4, –1).

Since the axis of symmetry is the vertical line through the vertex, the equation of the axis of symmetry is $x = 4$.

3. Find the y-intercept.

The function $g(x) = -x^2 + 8x - 17$ is in standard form, so the y-intercept is the constant term c, which equals –17.

The y-intercept is –17.

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Instruction

4. Find the x-intercepts, if any exist.

The x-intercepts occur when $y = 0$.

Substitute 0 for the output, $g(x)$, and solve using the quadratic formula since the function is not factorable over the rational numbers.

$g(x) = -x^2 + 8x - 17$	Original equation
$-x^2 + 8x - 17 = 0$	Set the equation equal to 0.

Determine the values of a, b, and c.

$a = -1$, $b = 8$, and $c = -17$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Quadratic formula
$x = \frac{-(8) \pm \sqrt{(8)^2 - 4(-1)(-17)}}{2(-1)}$	Substitute –1 for a, 8 for b, and –17 for c.
$x = \frac{-8 \pm \sqrt{-4}}{-2}$	Simplify.

In this case, the discriminant, –4, is negative, which means there are no real solutions.

This also means that there are no x-intercepts.

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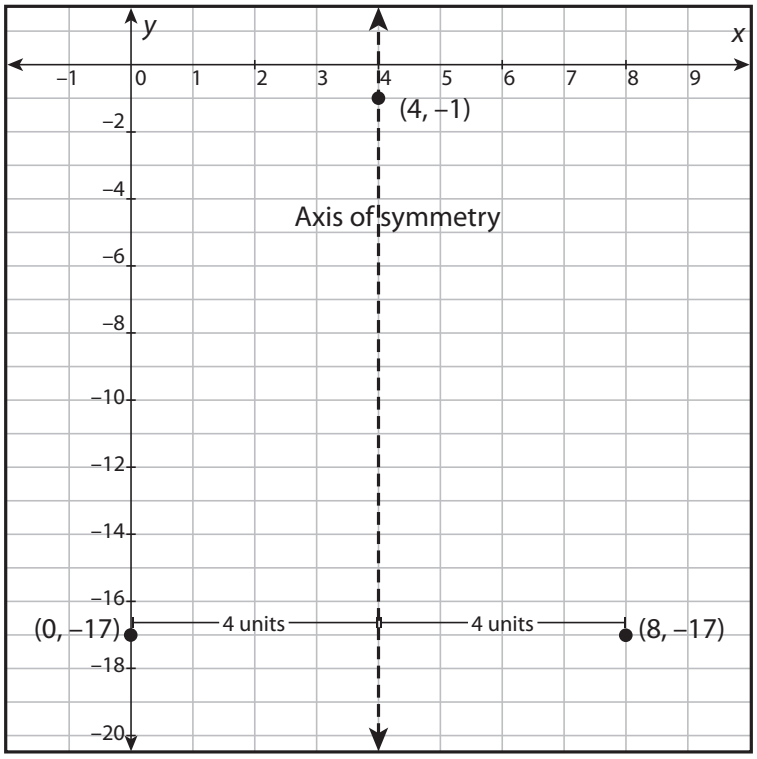
5. Plot the points from steps 2–4 and their symmetric points over the axis of symmetry.



For a more accurate graph, determine an additional pair of symmetric points.

Choose any x-value on the left or right of the axis of symmetry.

Evaluate the function for the chosen value of x to determine the output value.

Let's choose $x = 1$.

$g(x) = -x^2 + 8x - 17$	Original equation
$g(1) = -(1)^2 + 8(1) - 17$	Substitute 1 for x.
$g(1) = -10$	Simplify.

(1, –10) is an additional point on the parabola.

Plot (1, –10) on the same graph.

(1, –10) is 3 units from the axis of symmetry.

Locate the point that is symmetric to the point (1, –10) with respect to the axis of symmetry.

(*continued*)

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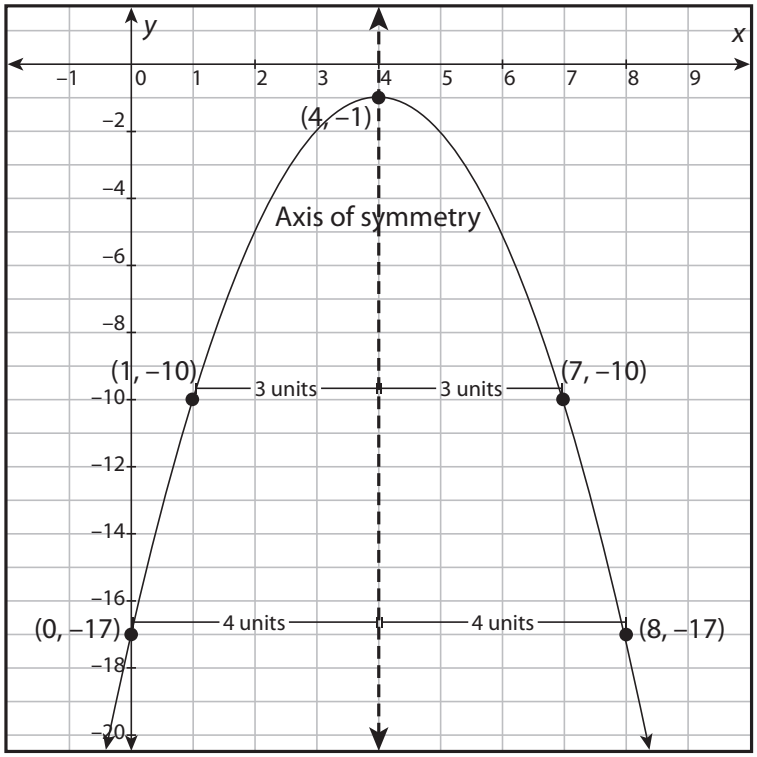
(7, –10) is also 3 units from the axis of symmetry and is symmetrical to the original point (1, –10) with respect to the axis of symmetry.

You can verify that (1, –10) and (7, –10) are the same distance from the axis of symmetry and are also symmetrical by referring to the graph.



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Example 5

Create the equation of a quadratic function given a vertex of (2, –4) and a y-intercept of 4.

1. Write an equation for b in terms of a.

   Set the x-coordinate of the vertex equal to $\frac{-b}{2a}$.

$2 = \frac{-b}{2a}$	Substitute 2 for x.
$4a = -b$	Multiply both sides by $2a$.
$-4a = b$	Multiply both sides by –1.
$b = -4a$	

2. Substitute the expression for b from step 1, the coordinates of the vertex, and the y-intercept into the standard form of a quadratic equation.

$y = ax^2 + bx + c$	Standard form of a quadratic equation
$y = ax^2 + (-4a)x + c$	Substitute $-4a$ for b.
$(-4) = a(2)^2 + (-4a)(2) + c$	Substitute the vertex (2, –4) for x and y.
$(-4) = a(2)^2 + (-4a)(2) + 4$	Substitute the y-intercept of 4 for c.
$-4 = 4a - 8a + 4$	Simplify, then solve for a.
$-8 = -4a$	
$a = 2$	

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Instruction

3. Substitute the value of a into the equation for b from step 1.

$b = -4a$	Equation from step 1
$b = -4(2)$	Substitute 2 for a.
$b = -8$	Simplify.

4. Substitute a, b, and c into the standard form of a quadratic equation.

$y = ax^2 + bx + c$	Standard form of a quadratic equation
$y = 2x^2 - 8x + 4$	Substitute 2 for a, –8 for b, and 4 for c.

The equation of the quadratic function with a vertex of (2, –4) and a y-intercept of 4 is $y = 2x^2 - 8x + 4$.

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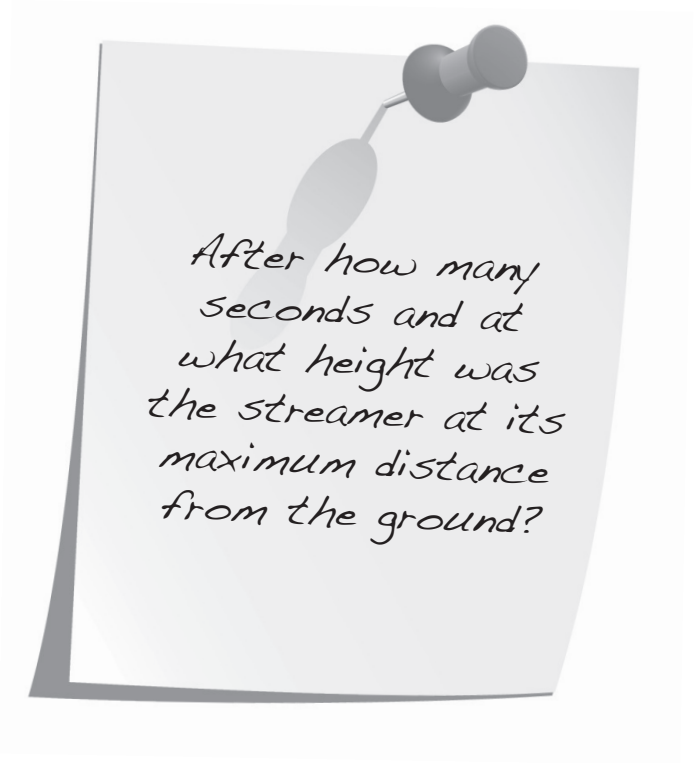
Problem-Based Task 3.3.1: Parabolic Party Streamers

SMP	
1 ✓	2
3	4 ✓
5	6
7 ✓	8

Students in a physics class are testing the pull of gravity at varying heights. With permission of the building manager, each student went to a different floor of a tall office building and tossed a roll of paper streamer up into the air from the window. Another student videotaped the streamers' paths downward so that the class could determine the approximate equations of the parabolas the streamers created as they unraveled. When a streamer was thrown upward from the highest story of the building, the students determined that the distance, in feet, between the streamer and the ground t seconds after the streamer was thrown could be expressed by $h(t) = -16t^2 + 32t + 56$. After how many seconds and at what height was the streamer at its maximum distance from the ground?



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Problem-Based Task 3.3.1: Parabolic Party Streamers

Coaching

a. What is the formula stated in the problem?

b. Is the formula written in standard form?

c. What are the values of a, b, and c?

d. How do you determine when the streamer reached its highest point?

e. After how many seconds was the streamer at its maximum height?

f. What height did the streamer reach at its maximum?

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Instruction

Problem-Based Task 3.3.1: Parabolic Party Streamers

Coaching Sample Responses

a. What is the formula stated in the problem?

$h(t) = -16t^2 + 32t + 56$

b. Is the formula written in standard form?

Yes, the equation $h(t) = -16t^2 + 32t + 56$ is written in the form $f(x) = ax^2 + bx + c$, which is the standard form of a quadratic equation.

c. What are the values of a, b, and c?

The value of a is –16.

The value of b is 32.

The value of c is 56.

d. How do you determine when the streamer reached its highest point?

Identify the y-coordinate of the vertex by using the formula for axis of symmetry to find the input value, t. Then substitute that into the function to find the output value, $h(t)$.

e. After how many seconds was the streamer at its maximum height?

$$t = \frac{-b}{2a}$$
$$t = \frac{-(32)}{2(-16)}$$
$$t = \frac{-32}{-32}$$
$$t = 1$$

The streamer was at its maximum height after 1 second.

f. What height did the streamer reach at its maximum?

$h(t) = -16t^2 + 32t + 56$

$h(1) = -16(1)^2 + 32(1) + 56$

$h(1) = 72$

The streamer reached a maximum height of 72 feet.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.3.1: Creating and Graphing Equations Using Standard Form

Sketch the graph for each of the following quadratic functions.

1. $q(x) = -x^2 - 6x - 8$

2. $f(x) = -3x^2 + 24x - 48$

3. $m(b) = b^2 - 6b + 10$

Find the y-intercept and vertex of the following functions. State whether the vertex is a minimum or maximum point on the graph and explain your reasoning.

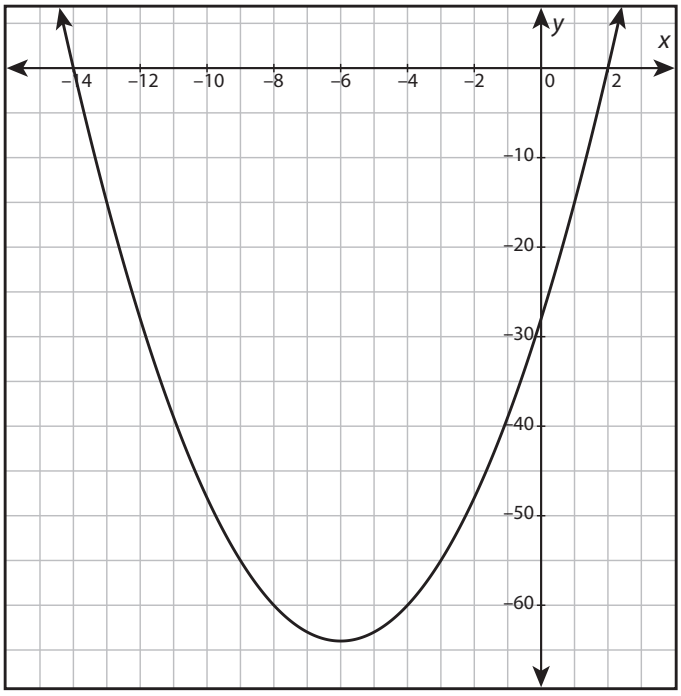
4. $k(h) = h^2 - 4h + 3$

5. $l(d) = d^2 - 6d$

6. $f(x) = -7x^2 - 14x - 6$

Does the following graph represent the given function? Explain your reasoning.

7. $y(x) = x^2 + 12x - 28$



continued

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Use your knowledge of quadratic functions to complete the problems that follow.

8. Create the equation of a quadratic function with a vertex of (–3, 7) and a y-intercept of –2.

9. The path of an arrow shot in the air can be modeled by the function $h(t) = -16t^2 + 144t + 4$, where h is the height, in feet, of the arrow above ground t seconds after it is released. Determine the maximum height that the arrow reaches.

10. The demand, d, for plastic storage containers depends on their price. A retail manager determines that the number of containers she can sell at a price of x dollars each is given by the formula $d(x) = -3x^2 + 220x - 200$. At what price will the demand for the containers be at a maximum?

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Lesson 3.3.2: Creating and Graphing Equations Using the *x*-intercepts

Warm-Up 3.3.2

Peyton is the top car salesperson at her dealership. She makes a fixed salary of $2,500 per month, plus a 0.2% commission on sales.

1. Write an equation to represent Peyton's monthly take-home pay.

2. Graph the equation.

3. What are the *x*- and *y*-intercepts?

4. What do the *x*- and *y*-intercepts mean in terms of Peyton's monthly take-home pay?

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Instruction

Lesson 3.3.2: Creating and Graphing Equations Using the *x*-intercepts

Common Core State Standards

A–SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

a. Factor a quadratic expression to reveal the zeros of the function it defines.

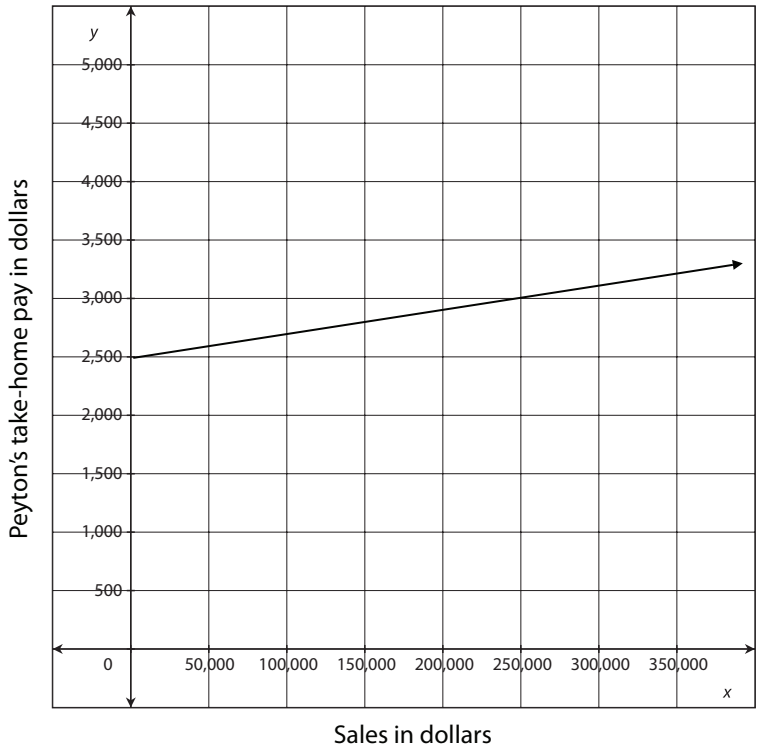
A–CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

Warm-Up 3.3.2 Debrief

1. Write an equation to represent Peyton's monthly take-home pay.

   $y = 0.002x + 2500$, where x is the total dollar amount for cars sold in a month and y is Peyton's monthly take-home pay.

2. Graph the equation.



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3. What are the *x*- and *y*-intercepts?

   To find the *x*-intercept, substitute 0 for *y* and solve for *x*.

$y = 0.002x + 2500$	Equation
$0 = 0.002x + 2500$	Substitute 0 for *y*.
$-0.002x = 2500$	Simplify.
$x = -1{,}250{,}000$	

   The *x*-intercept is –1,250,000.

   To find the *y*-intercept, substitute 0 for *x* and solve for *y*.

$y = 0.002x + 2500$	Equation
$y = 0.002(0) + 2500$	Substitute 0 for *x*.
$y = 2500$	Simplify.

   The *y*-intercept is 2,500.

4. What do the *x*- and *y*-intercepts mean in terms of Peyton's monthly take-home pay?

   The *x*-intercept is the total amount of monthly car sales when Peyton has made $0 in a month. This is impossible since she has a fixed monthly salary of $2,500.

   The *y*-intercept is the total monthly take-home pay when Peyton sells 0 cars.

Connection to the Lesson

- Students will extend their knowledge of the meaning of intercepts in context.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skill:

- solving quadratic equations by factoring (A–REI.4b)

Introduction

The equation of a quadratic function can be written in several different forms. We have practiced using the standard form of a quadratic function to identify key information and graph the function. In this lesson, we will explore writing quadratic functions in intercept form or factored form, graphing a quadratic equation written in intercept form, and creating the equation of a quadratic function given its graph. An equation written in intercept form allows you to easily identify the x-intercepts of a quadratic function.

Key Concepts

- A quadratic function in standard form can be created from the intercept form.
- The **intercept form of a quadratic function** is $f(x) = a(x - p)(x - q)$, where a, p, and q are real numbers, $a \neq 0$, and p and q are the zeros of the function.
- To convert a quadratic equation from intercept form to standard form, distribute the terms of the first set of parentheses over the terms of the second set of parentheses and simplify.
- For example, $f(x) = 3(x - 4)(x - 1)$ becomes $f(x) = 3x^2 - 15x + 12$.
- The y-intercept of a quadratic equation written in intercept form can be found by substituting 0 for x.
- The x-intercept(s) of a quadratic equation written in intercept form can be found by substituting 0 for y.
- The x-intercepts, also known as the zeros of a quadratic function or the roots of a quadratic equation, are the values of p and q.
- The axis of symmetry of a parabola is located halfway between the zeros. To determine its equation, use the formula $x = \frac{p+q}{2}$.
- Recall that the axis of symmetry extends through the vertex of a quadratic function.
- As such, the x-coordinate of the vertex is the constant in the equation of the axis of symmetry.
- If the vertex is the point (h, k), then the equation of the axis of symmetry is $x = h$.
- The y-coordinate of the vertex, written as k, can be found by substituting the value of h into the quadratic equation.

SRB p. U3-119

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Instruction

Common Errors/Misconceptions

- multiplying out the factored form when it is more advantageous not to
- drawing a parabola as a straight line or a series of lines instead of a curve

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Guided Practice 3.3.2

Example 1

Determine the equation of a quadratic function in standard form, given the zeros $x = 2$ and $x = -2$, and the point (0, 3) that lies on the graph of the function.

1. Write the zeros as expressions equal to 0 to determine the factors of the quadratic equation.

   Recall that when solving a quadratic equation in factored form, you take all factors and set them equal to 0 to determine the solution.

   To work backward, we must undo the solving process to find the factors of the equation.

   $x = 2$ $\qquad$ $x = -2$

   $x - 2 = 0$ $\qquad$ $x + 2 = 0$

   The factors of the equation are $(x - 2)$ and $(x + 2)$, so the equation is $f(x) = a(x - 2)(x + 2)$.

2. Use the point (0, 3) to find the value of a.

   The equation $f(x) = a(x - 2)(x + 2)$ is written in intercept form, or $f(x) = a(x - p)(x - q)$.

   Substitute the coordinates of the point (0, 3) into the equation to find a.

$f(x) = a(x - 2)(x + 2)$	Equation
$3 = a(0 - 2)(0 + 2)$	Substitute the point (0, 3) for x and $f(x)$.
$3 = a(-2)(2)$	Simplify, then solve for a.
$3 = -4a$	
$-\frac{3}{4} = a$	

   The value of a is $-\frac{3}{4}$.

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Instruction

3. Write the equation in standard form.

Insert the value found for a into the equation from step 1 and solve.

$f(x) = a(x - 2)(x + 2)$	Equation
$f(x) = -\frac{3}{4}(x-2)(x+2)$	Substitute $-\frac{3}{4}$ for a.
$f(x) = -\frac{3}{4}(x^2 - 4)$	Simplify by multiplying the binomial factors.
$f(x) = -\frac{3}{4}x^2 + 3$	Distribute.

The equation of the quadratic function in standard form, with zeros $x = 2$ and $x = -2$, and the point (0, 3), is $f(x) = -\frac{3}{4}x^2 + 3$.

Example 2

Identify the x-intercepts, if any, the equation of the axis of symmetry, and the vertex of the quadratic function $f(x) = (x + 5)(x + 2)$. Use this information to graph the function.

1. Identify the x-intercepts.

The x-intercepts of the quadratic function are the zeros of the function.

Start by setting $f(x)$ equal to 0 and solving to determine the zeros.

$f(x) = (x + 5)(x + 2)$	Original equation
$0 = (x + 5)(x + 2)$	Set $f(x)$ equal to 0.
$0 = x + 5$ or $0 = x + 2$	Set each factor equal to 0.
$-5 = x$ or $-2 = x$	Solve each equation for x.

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Instruction

2. Determine the equation of the axis of symmetry to find the vertex.

The axis of symmetry is the line that divides the parabola in half.

To determine its equation, find the midpoint of the two zeros and draw the vertical line through that point.

Insert the zeros into the formula $x = \frac{p+q}{2}$ to find the midpoint.

Let $p = -5$ and $q = -2$.

$$x = \frac{-5+(-2)}{2} = \frac{-7}{2} = -3.5$$

The equation of the axis of symmetry is $x = -3.5$.

The axis of symmetry extends through the vertex of the parabola. This means that the vertex (h, k) of the parabola has an x-coordinate of -3.5.

Substitute -3.5 for x in the original equation to determine the y-coordinate of the vertex.

$f(x) = (x + 5)(x + 2)$	Original equation
$f(-3.5) = (-3.5 + 5)(-3.5 + 2)$	Substitute -3.5 for x.
$f(-3.5) = (1.5)(-1.5)$	Simplify.
$f(-3.5) = -2.25$	

The y-coordinate of the vertex is -2.25.

The vertex is the point $(-3.5, -2.25)$.

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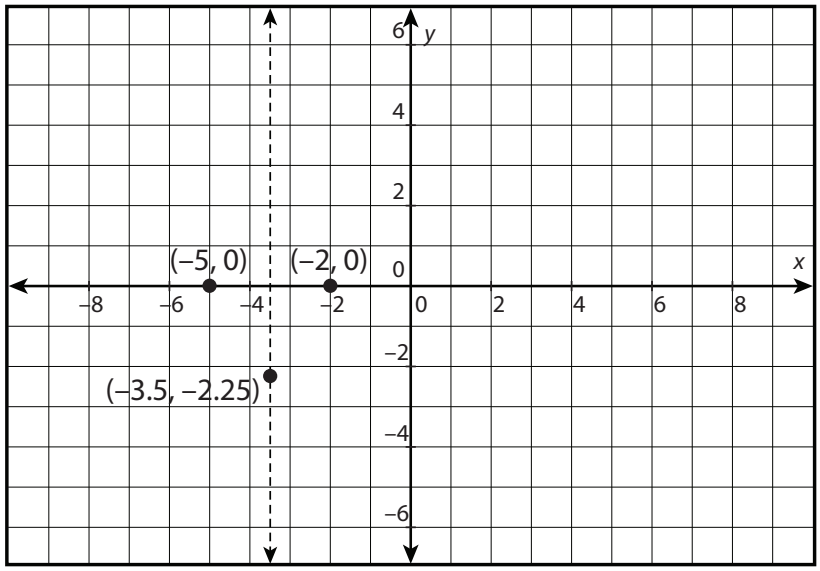
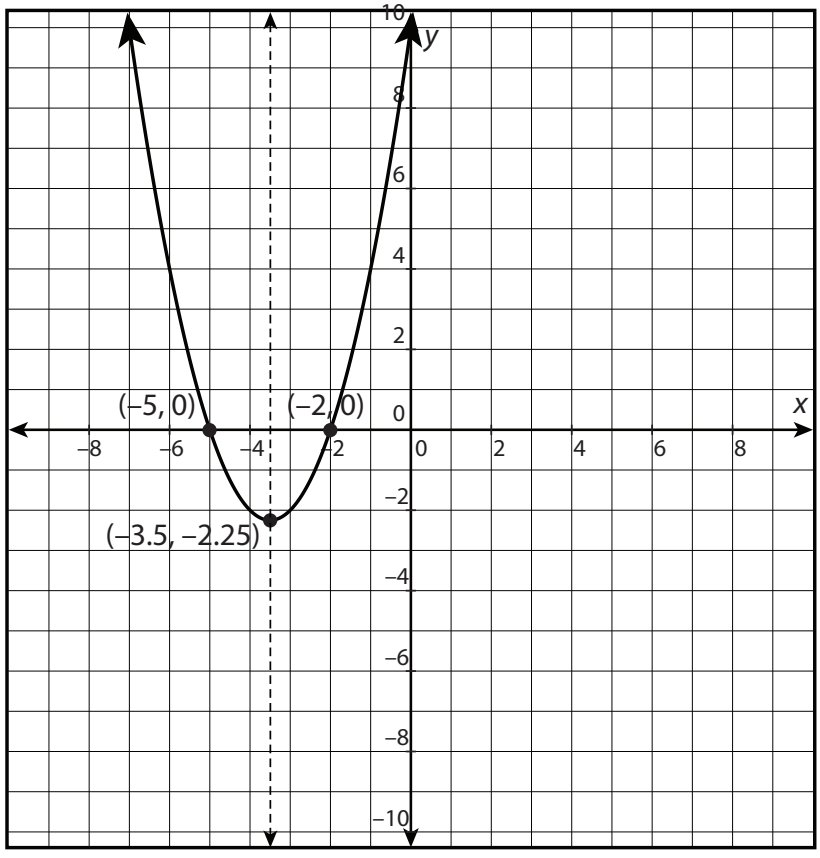
Instruction

3. Graph the zeros, the vertex, and the axis of symmetry of the given equation.



4. Sketch the function based on the zeros and the vertex.



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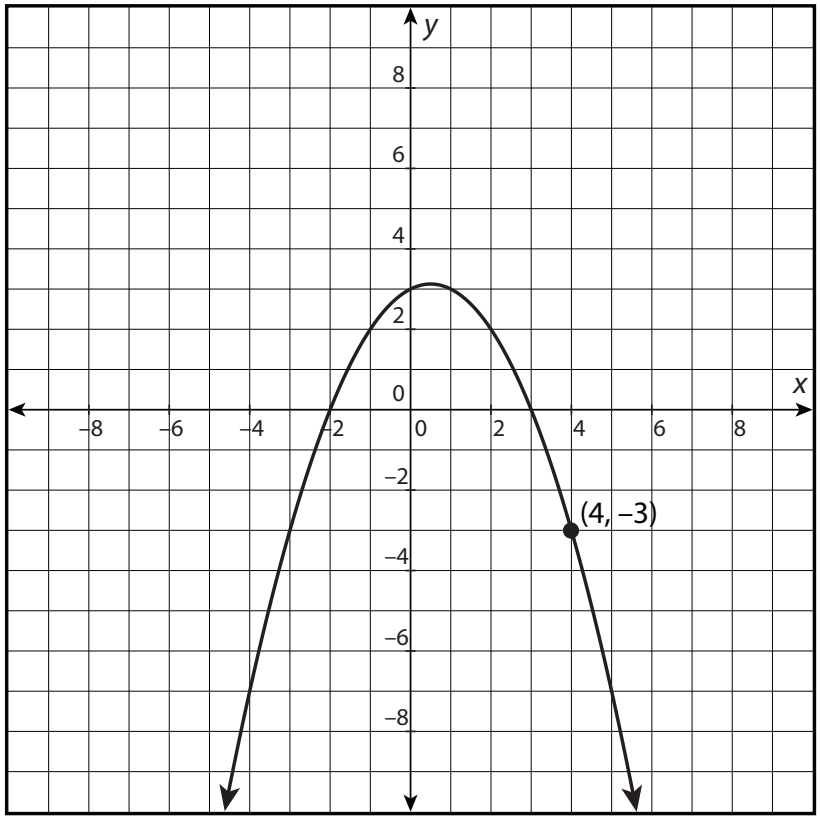
Instruction

Example 3

Use the *x*-intercepts and the graphed point to write the equation of the function in standard form.



1. Find the *x*-intercepts of the graph and write them as zeros of the function.

   Identify the *x*-intercepts and write them as zeros to prepare to write the factors of the equation.

   The *x*-intercepts are –2 and 3.

   Therefore, the zeros are $x = -2$ and $x = 3$.

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Instruction

2. Write the zeros as expressions set equal to 0 to determine the factors of the quadratic equation.

$x = -2$	$x = 3$
$x + 2 = 0$	$x - 3 = 0$

The factors of the quadratic equation are $(x + 2)$ and $(x - 3)$, so the equation is $f(x) = a(x + 2)(x - 3)$.

3. Use the point $(4, -3)$ to find the value of a.

The equation $f(x) = a(x + 2)(x - 3)$ is written in intercept form, or $f(x) = a(x - p)(x - q)$.

Substitute $(4, -3)$ into the equation to find a.

$f(x) = a(x + 2)(x - 3)$	Equation
$-3 = a(4 + 2)(4 - 3)$	Substitute $(4, -3)$ for x and $f(x)$.
$-3 = a(6)(1)$	Simplify.
$-3 = 6a$	
$-\frac{1}{2} = a$	

The value of a is $-\frac{1}{2}$.

4. Write the equation in standard form.

Insert the value found for a into the equation from step 2 and solve.

$f(x) = a(x + 2)(x - 3)$	Equation
$f(x) = -\frac{1}{2}(x+2)(x-3)$	Substitute $-\frac{1}{2}$ for a.
$f(x) = -\frac{1}{2}(x^2 - x - 6)$	Multiply the binomials.
$f(x) = -\frac{1}{2}x^2 + \frac{1}{2}x + 3$	Distribute $-\frac{1}{2}$ over the parentheses.

The standard form of the equation is $f(x) = -\frac{1}{2}x^2 + \frac{1}{2}x + 3$.

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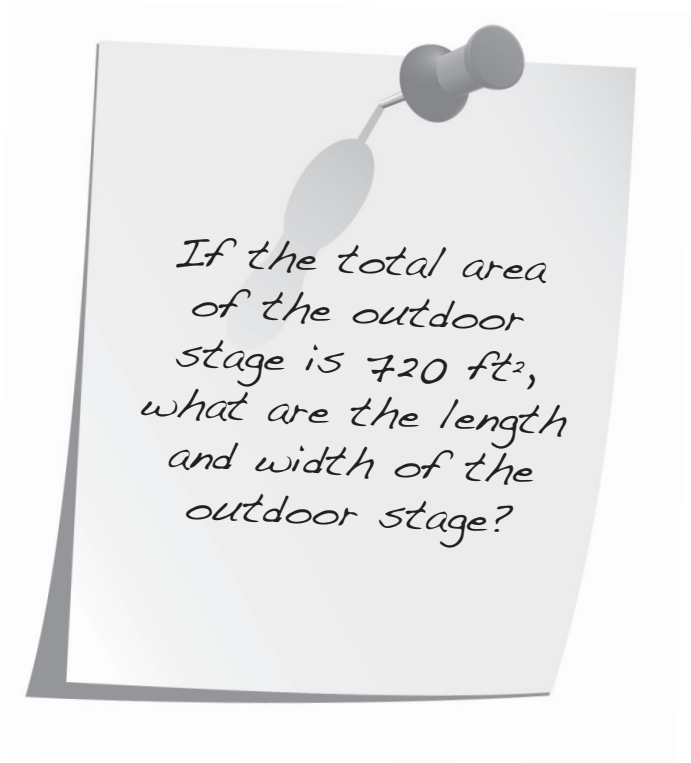
Problem-Based Task 3.3.2: It's Show Time!

SMP	
1	2 ✓
3	4
5	6 ✓
7 ✓	8

The school's drama department is putting on a production. Instead of using the school's indoor stage, the department head decided to build an outdoor stage to accommodate a greater audience. The width of the outdoor stage is 6 feet less than its length. If the total area of the outdoor stage is 720 ft^2, what are the length and width of the outdoor stage?



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Problem-Based Task 3.3.2: It's Show Time!

Coaching

a. What are the unknowns?

b. What would a diagram of this situation look like?

c. What formula do you need to use to find the length and width of the outdoor stage?

d. How can you use the unknowns and the formula from part c to begin to determine the length and width of the outdoor stage?

e. How could you begin to solve for length and width?

f. How can you find the length and width of the outdoor stage?

g. Are all values of x appropriate for this context?

h. What are the length and width of the outdoor stage?

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Instruction

Problem-Based Task 3.3.2: It's Show Time!

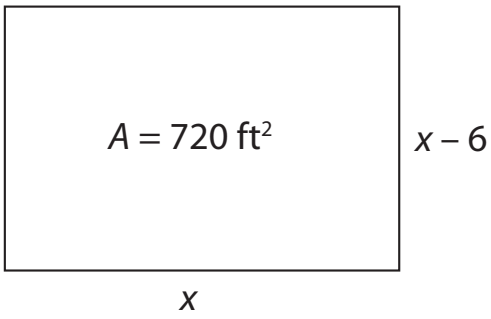
Coaching Sample Responses

a. What are the unknowns?

The unknowns are the length and width of the outdoor stage.

If length is represented by x, then width is represented by $x - 6$.

b. What would a diagram of this situation look like?



c. What formula do you need to find the length and width of the outdoor stage?

You need to use the formula for the area of a rectangle: $A = lw$.

d. How can you use the unknowns and the formula from part c to begin to determine the length and width of the outdoor stage?

Substitute the equations representing the unknowns into the area formula.

$A = lw$

$720 = (x)(x - 6)$

e. How could you begin to solve for length and width?

Rearrange the terms in the equation and set it equal to 0.

$720 = (x)(x - 6)$

$x(x - 6) - 720 = 0$

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f. How can you find the length and width of the outdoor stage?

The equation $x(x - 6) - 720 = 0$ can be factored; therefore, one method for finding the length and width is factoring.

$x(x - 6) - 720 = 0$

$x^2 - 6x - 720 = 0$

$(x - 30)(x + 24) = 0$

$x - 30 = 0$ and $x + 24 = 0$

$x = 30$ and $x = -24$

g. Are all values of x appropriate for this context?

The dimensions of a rectangle are positive, so we can disregard the negative value, –24.

The length is represented by x and the width is represented by $x - 6$.

When $x = 30$, the length is 30 and the width is $30 - 6 = 24$.

Substituting 30 for x results in positive values for both the length and the width.

The only value of x that is appropriate for this context is 30.

h. What are the length and width of the outdoor stage?

The length of the outdoor stage is 30 feet.

The width of the outdoor stage is 6 feet less than the length and thus is 24 feet.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.3.2: Creating and Graphing Equations Using the *x*-intercepts

Identify the *x*-intercepts, if any, of the following quadratic functions. Determine the equation of the axis of symmetry for each parabola.

1. $y = (x - 3)(x + 6)$

2. $f(x) = \left(x - \frac{2}{3}\right)\left(x + \frac{2}{3}\right)$

Determine the equation of each quadratic function in standard form, given the zeros and a point on the graph.

3. $x = -2$; (3, 10)

4. $x = 5$, $x = -12$; (0, –60)

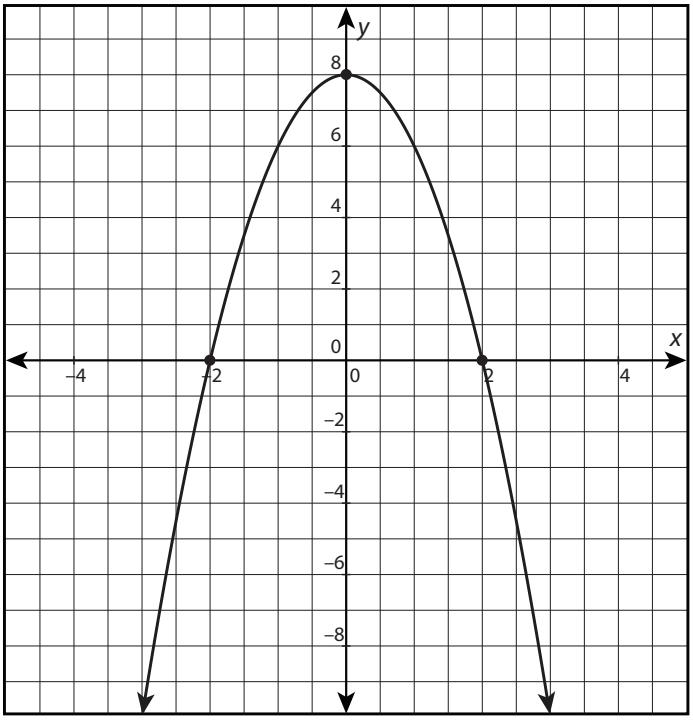
Sketch a graph for each of the following quadratic functions.

5. $y = (x + 3)(x + 1)$

6. $y = (3x - 2)(x - 1)$

Given the graph of a quadratic function, use the intercepts and another point on the graph to write the equation of the function in standard form.

7.



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SWB p. 123; SRB Practice 3.3.2 p. U3-126

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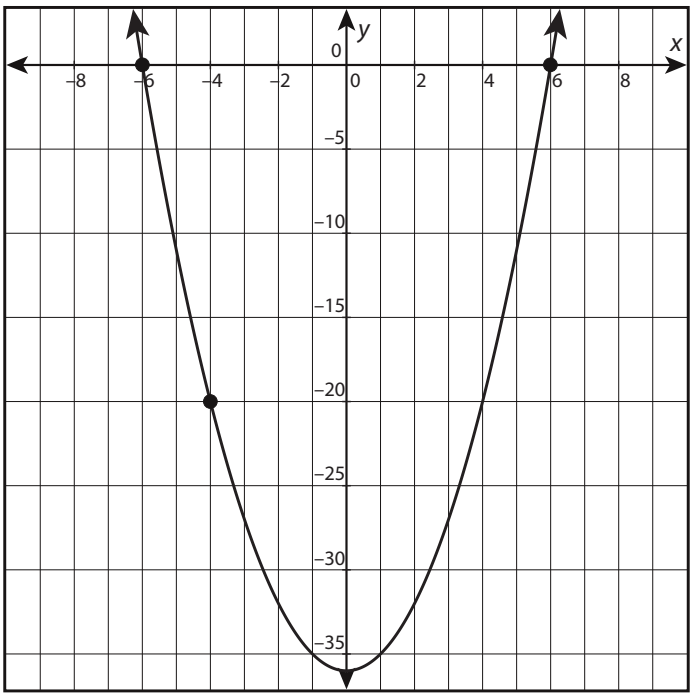
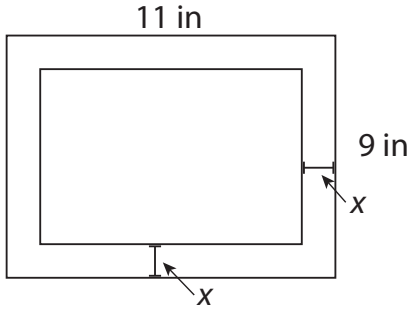


Use the given information to solve the following problems.

9. A family portrait hanging on the wall has a frame with dimensions of 11 inches by 9 inches. The width of the frame is represented by x. What are the dimensions of the portrait if its area is 35 square inches?



10. A bird takes off from the roof of a 250-foot-tall building and flies to the ground below. Its path takes the form of a parabola. The bird's height can be modeled by $h(t) = -t^2 + 15t + 250$, where $h(t)$ is the height of the bird above ground in feet t seconds after leaving the roof. After how many seconds does the bird land on the ground?

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Lesson 3.3.3: Creating and Graphing Equations Using Vertex Form

Warm-Up 3.3.3

An ice-skating rink manager tracks each day's revenue R based on an hourly fee in dollars, x, for skating using the function $R(x) = -480x^2 + 3120x$.

1. What hourly fee will produce the maximum revenue?

2. What is the maximum revenue that the manager should expect?

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Lesson 3.3.3: Creating and Graphing Equations Using Vertex Form

Common Core State Standards

A–SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

A–CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

Warm-Up 3.3.3 Debrief

1. What hourly fee will produce the maximum revenue?

   Find the x-coordinate of the vertex to determine the hourly fee.

   Substitute the values of a and b into the expression for the x-coordinate of the vertex, $\frac{-b}{2a}$, and solve.

   $$x = \frac{-(3120)}{2(-480)} = \frac{-3120}{-960} = 3.25$$

   The hourly fee that will produce the maximum revenue is \$3.25.

2. What is the maximum revenue that the manager should expect?

   Substitute the value of x into the function to find the output.

$R(x) = -480x^2 + 3120x$	Original equation
$R(3.25) = -480(3.25)^2 + 3120(3.25)$	Substitute 3.25 for x.
$R(3.25) = 5070$	Simplify.

   The maximum revenue that the manager should expect is \$5,070.

Connection to the Lesson

- Students will extend their knowledge of the importance of the vertex of a quadratic to writing equations in vertex form.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- identifying the standard form of a quadratic function (F–IF.8a)
- finding the vertex of a quadratic function (F–IF.8a)
- solving quadratic equations by taking the square root of both sides (A–REI.4b)

Introduction

Quadratic functions are used to model various situations. Some situations involve applying the key features of a quadratic function to real-life situations. For example, an investor might want to predict the behavior of a particular mutual fund over time, or an NFL scout might want to determine the maximum height of a ball kicked by a potential football punter.

In this lesson, we will look specifically at the **vertex form of a quadratic function**, $f(x) = a(x - h)^2 + k$, where the vertex is the point (h, k). The vertex can be read directly from the equation.

Key Concepts

- Standard form, intercept form, and vertex form are all quadratic equations written in different forms.
  - Standard form: $f(x) = ax^2 + bx + c$, where a is the coefficient of the quadratic term, b is the coefficient of the linear term, and c is the constant term
  - Intercept form: $f(x) = a(x - p)(x - q)$, where p and q are the zeros of the function
  - Vertex form: $f(x) = a(x - h)^2 + k$, where the vertex of the parabola is the point (h, k)
- To identify the vertex directly from an equation in vertex form, identify h (the x-coordinate of the vertex) and k (the y-coordinate of the vertex).
- Note that the original equation in vertex form has the quantity $x - h$, so if the binomial has a subtraction sign then the x-coordinate of the vertex is h.
- However, if the binomial is written as $x + h$, the x-coordinate of the vertex is $-h$.
- A quadratic function in standard form can be created from vertex form, $f(x) = a(x - h)^2 + k$, where (h, k) is the vertex of the function's graph. To do so, distribute and simplify by combining like terms.
- For example, $f(x) = 3(x - 2)^2 + 4$ becomes $f(x) = 3x^2 - 12x + 16$.

SRB p. U3-129

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Instruction

- A quadratic function in vertex form can be created from standard form, $f(x) = ax^2 + bx + c$. To do so, complete the square, or determine the value of c that would make $ax^2 + bx + c$ a perfect square trinomial.
- To complete the square, take the coefficient of the linear term, divide by the product of 2 and the coefficient of the quadratic term, and square the quotient.

  $ax^2 + bx + c$

  $$a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c$$

  $$a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c$$

- Since the quotient of b and $2a$ is a constant term, we can combine it with the constant c to get the equation $a\left(x + \frac{b}{2a}\right)^2 + k$, where $k = -a\left(\frac{b}{2a}\right)^2 + c$.
- For example, $f(x) = 2x^2 - 12x + 22$ becomes $f(x) = 2(x - 3)^2 + 4$.
- When graphing a quadratic function using vertex form, if the vertex contains the y-intercept, choose two pairs of symmetric points to plot in order to sketch a more accurate graph.

Common Errors/Misconceptions

- forgetting to make sure the coefficient of the quadratic term, x^2, is 1 before completing the square

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Guided Practice 3.3.3

Example 1

Given the quadratic function $f(x)=\frac{1}{2}(x+6)^2-2$, identify the vertex and determine whether it is a minimum or maximum.

1. Identify the vertex.

   The vertex form of a quadratic function is $f(x) = a(x - h)^2 + k$, with the vertex (h, k).

   $f(x)=\frac{1}{2}(x+6)^2-2$ is in vertex form; therefore, the vertex of this function is (–6, –2).

2. Determine if the vertex is a minimum or maximum.

   If $a > 0$, the parabola opens up and thus has a lowest point or a minimum.

   If $a < 0$, the parabola opens down and thus has a highest point or a maximum.

   $f(x)=\frac{1}{2}(x+6)^2-2$ is in vertex form, so $a=\frac{1}{2}$.

   $\frac{1}{2}>0$, so the vertex of the function is a minimum.

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Example 2

Determine the equation of a quadratic function whose vertex is (–4, –8) and that contains the point (–2, –5).

1. Substitute the coordinates of the vertex into $f(x) = a(x - h)^2 + k$.

$f(x) = a(x - h)^2 + k$	Vertex form
$f(x) = a[x - (-4)]^2 + (-8)$	Substitute (–4, –8) for h and k.
$f(x) = a(x + 4)^2 - 8$	Simplify.

2. Substitute the coordinates (–2, –5) into the equation from step 1 and solve for a.

$f(x) = a(x + 4)^2 - 8$	Equation
$-5 = a[(-2) + 4]^2 - 8$	Substitute (–2, –5) for x and $f(x)$.
$-5 = a(2)^2 - 8$	Simplify.
$-5 = 4a - 8$	
$3 = 4a$	
$\frac{3}{4} = a$	

3. Substitute the value of a into the equation from step 1.

$f(x) = a(x + 4)^2 - 8$

$f(x) = \frac{3}{4}(x + 4)^2 - 8$

The equation of the quadratic function whose vertex is (–4, –8) and that contains the point (–2, –5) is $f(x) = \frac{3}{4}(x + 4)^2 - 8$.

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Instruction

Example 3

Convert the function $g(x) = -7x^2 + 14x$ to vertex form.

1. Complete the square to rewrite the function in vertex form.

$g(x) = -7x^2 + 14x$	Original function
$g(x) = -7(x^2 - 2x)$	Factor out –7.
$g(x) = -7(x^2 - 2x + 1 - 1)$	Complete the square.
$g(x) = -7(x^2 - 2x + 1) + 7$	Rewrite the expression in parentheses as a perfect square trinomial.
$g(x) = -7(x - 1)^2 + 7$	Rewrite the trinomial as a binomial squared.

2. Summarize your result.

   The function $g(x) = -7x^2 + 14x$ written in vertex form is $g(x) = -7(x - 1)^2 + 7$.

Example 4

Sketch a graph of the quadratic function $y = (x + 3)^2 - 8$. Label the vertex, the axis of symmetry, the y-intercept, and one pair of symmetric points.

1. Identify the vertex and the equation of the axis of symmetry.

   Given the vertex form of a quadratic function, $f(x) = a(x - h)^2 + k$, the vertex is the point (h, k).

   The vertex of the quadratic function $f(x) = (x + 3)^2 - 8$ is $(-3, -8)$.

   The axis of symmetry extends through the vertex.

   The equation of the axis of symmetry is $x = -3$.

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Instruction

2. Find the y-intercept.

   The parabola crosses the y-axis when $x = 0$.

   Substitute 0 for x to find y.

$y = (x + 3)^2 - 8$	Original equation
$y = (0 + 3)^2 - 8$	Substitute 0 for x.
$y = 3^2 - 8$	Simplify.
$y = 1$	

   The y-intercept is 1.

3. Find an additional point to the left or right of the axis of symmetry.

   Choose an x-value and substitute it into the equation to find the corresponding y-value.

   Typically, choosing $x = 1$ or $x = -1$ is simplest arithmetically, if neither of these numbers is already a part of the vertex or axis of symmetry.

   In this case, let's use $x = 1$.

$y = (x + 3)^2 - 8$	Original equation
$y = (1 + 3)^2 - 8$	Substitute 1 for x.
$y = 4^2 - 8$	Simplify.
$y = 8$	

   The parabola passes through the point (1, 8).

   The point (1, 8) is 4 units to the right of the axis of symmetry, $x = -3$.

   4 units to the left of the axis of symmetry and opposite (1, 8) is the symmetric point (–7, 8).

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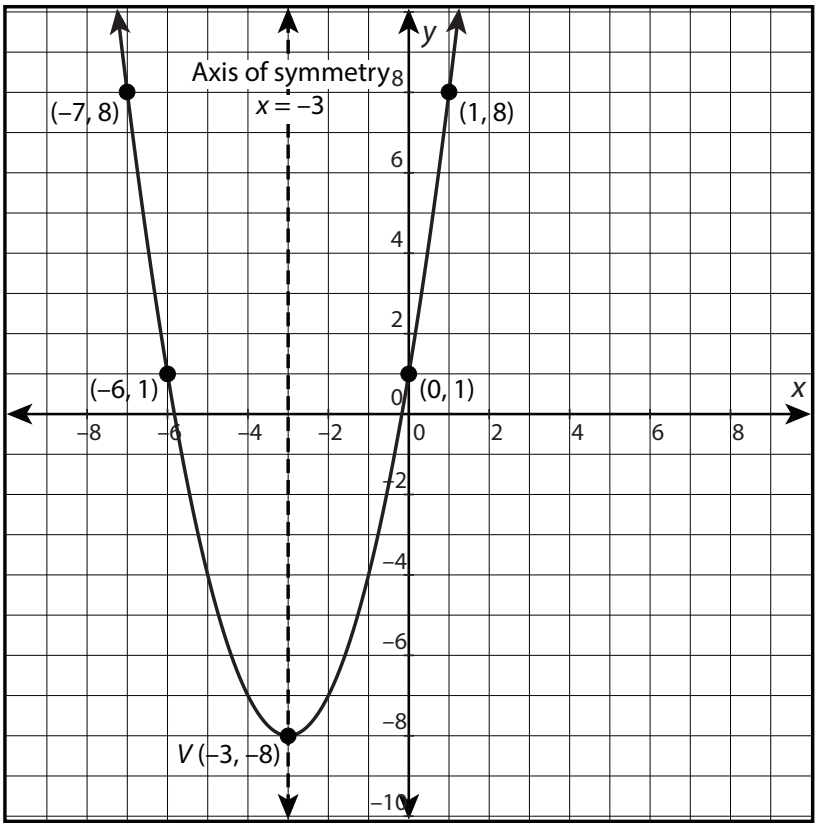
4. Plot the points you found in steps 2 and 3 and their symmetric points over the axis of symmetry.



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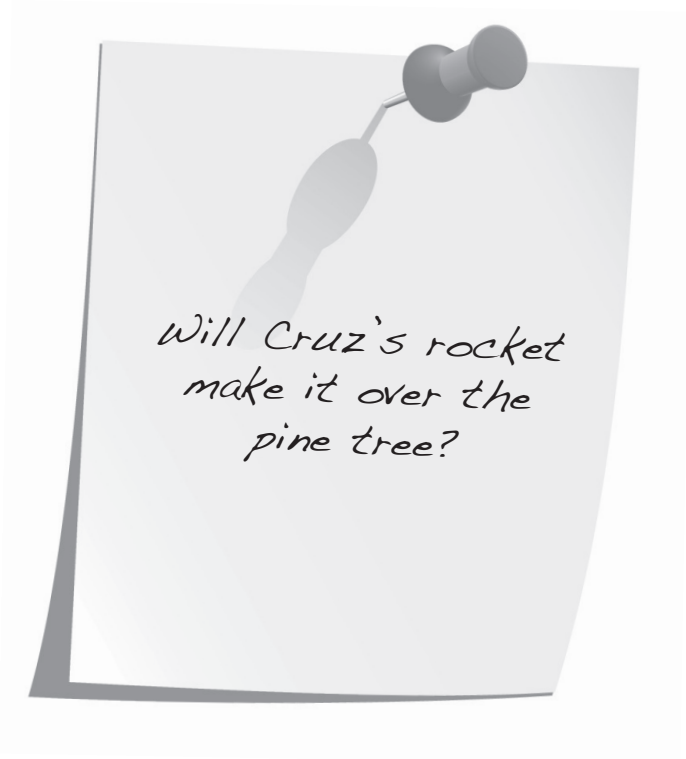
Problem-Based Task 3.3.3: Rocket Heights

SMP	
1 ✓	2
3	4 ✓
5	6 ✓
7	8

The path of a model rocket launched from the ground is parabolic. Cruz built a reusable model rocket using a kit. After launching it a few times from level ground, Cruz determined that the rocket could reach a maximum height of 400 feet after traveling a horizontal distance of 15 feet. On the flight where it reached the maximum height, the rocket landed 30 feet away from the launch site. Cruz's friends asked him to launch the rocket again in an area with several trees. He's worried that the rocket will be destroyed if the trees block its path—especially since a 115-foot-tall pine tree stands just 2 feet from the landing spot. Will Cruz's rocket make it over the pine tree?



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Problem-Based Task 3.3.3: Rocket Heights

Coaching

a. For which quantities do you have a value?

b. Substitute the known values into the vertex form of a quadratic equation.

c. What is still unknown?

d. How could you solve for this?

e. What is the quadratic equation that would allow Cruz to calculate the height of the rocket for any distance?

f. Will Cruz's rocket make it over the pine tree?

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Instruction

Problem-Based Task 3.3.3: Rocket Heights

Coaching Sample Responses

a. For which quantities do you have a value?

The maximum height of the rocket is 400 feet at a horizontal distance of 15 feet from its launch site. This information represents the coordinates of the vertex, (15, 400).

The rocket landed 30 feet from the launch location. This information is one x-intercept, 30.

The second x-intercept is 0, the initial launch location.

b. Substitute the known values into the vertex form of a quadratic equation.

Substitute (15, 400) for (h, k) and (30, 0) for (x, y).

$y = a(x - h)^2 + k$

$0 = a(30 - 15)^2 + 400$

c. What is still unknown?

By looking at the equation in part b, it can be seen that the value of a is still unknown.

d. How could you solve for this?

Use algebra to solve for a.

$0 = a(30 - 15)^2 + 400$

$-400 = a(30 - 15)^2$

$-400 = a(15)^2$

$-400 = 225a$

$a = -\frac{16}{9}$

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e. What is the quadratic equation that would allow Cruz to calculate the height of the rocket for any distance?

Substitute the value of a, $-\frac{16}{9}$, and the coordinates of the vertex, (15, 400), into the vertex form of a quadratic equation, $f(x) = a(x - h)^2 + k$.

$$f(x)=-\frac{16}{9}(x-15)^2+400$$

The quadratic equation that would allow Cruz to calculate the height of the rocket is $f(x)=-\frac{16}{9}(x-15)^2+400$.

f. Will Cruz's rocket make it over the pine tree?

A horizontal distance of 2 feet from the landing spot is 28 feet from the launch site.

Substitute 28 for x to determine how high the rocket will be when it has traveled a horizontal distance of 28 feet from the launch site.

$$f(x)=-\frac{16}{9}(x-15)^2+400$$

$$f(28)=-\frac{16}{9}(28-15)^2+400$$

$$f(28)=-\frac{16}{9}(13)^2+400$$

$$f(28)=-\frac{16}{9}(169)+400$$

$$f(28)=99\frac{5}{9}$$

The height of the rocket when it is a horizontal distance of 2 feet from the landing spot will be approximately 99.6 feet. The rocket will not make it over the tree that stands 115 feet tall.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.3.3: Creating and Graphing Equations Using Vertex Form

For each of the following quadratic functions, identify the vertex, state whether the vertex is a minimum or a maximum, and explain your reasoning.

1. $y = -2(x + 3)^2 - 3$

2. $f(x) = (x - 6)^2 + 6$

Determine the equation of a quadratic function that satisfies the given criteria.

3. The coordinates of the vertex are (–2.5, –3) and it contains the point (–1, –1).

4. The coordinates of the vertex are (2, 10) and it contains the point (1, 8).

Convert each quadratic function given in standard form to vertex form.

5. $f(x) = x^2 + 6x + 9$

6. $g(x) = 3x^2 - 12x + 9$

Sketch a graph of each quadratic function. Label the vertex, the axis of symmetry, the *y*-intercept, and one pair of symmetric points.

7. $y = -2x^2 + 8$

8. $t(x) = -2(x - 2)^2 + 5$

Create the equation of a quadratic function with the given characteristics.

9. The vertex is at (24, 100) and the *x*-intercept is 48.

10. The vertex is at (3, 18) and the *x*-intercept is 6.

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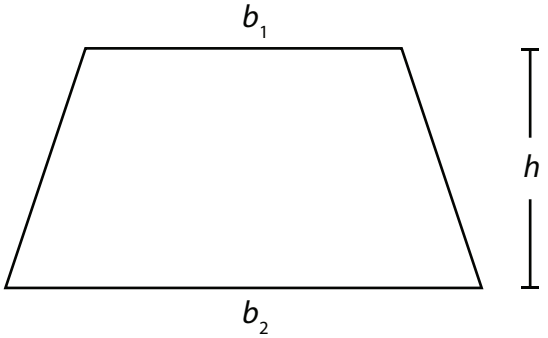
Lesson 3.3.4: Rearranging Formulas

Warm-Up 3.3.4

The walls of a greenhouse are shaped like a trapezoid. The area of a trapezoid is expressed by the formula $A=\frac{1}{2}(b_1+b_2)h$, where A is the area of the trapezoid, b_1 and b_2 are the lengths of the bases of the trapezoid, and h is the height.



1. If you know the area and the length of the two bases of one greenhouse wall, how could you find the height of this wall?

2. What is the formula that expresses the height of the wall?

3. If the two bases measure 30 feet and 34 feet respectively, and the area of the wall is 960 ft^2, what is the wall's actual height?

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Lesson 3.3.4: Rearranging Formulas

Common Core State Standard

A–CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.*★

Warm-Up 3.3.4 Debrief

1. If you know the area and the length of the two bases of one greenhouse wall, how could you find the height of this wall?

   Rearrange the formula so that h is isolated.

2. What is the formula that expresses the height of the wall?

   $A=\frac{1}{2}(b_1+b_2)h$ Formula for the area of a trapezoid

   $2A=(b_1+b_2)h$ Multiply both sides by 2.

   $\frac{2A}{b_1+b_2}=h$ Divide both sides by the sum of the bases.

   The formula that expresses the height of the wall is $\frac{2A}{b_1+b_2}=h$.

3. If the two bases measure 30 feet and 34 feet respectively, and the area of the wall is 960 ft^2, what is the wall's actual height?

   $\frac{2A}{b_1+b_2}=h$ Formula for the height of the wall

   $h=\frac{2(960)}{30+34}$ Substitute the lengths of the bases and the area of the wall for A, b_1, and b_2.

   $h=\frac{1920}{64}$ Simplify.

   $h = 30$

   The wall is 30 feet tall.

Connection to the Lesson

- Students will extend their knowledge of rearranging literal equations and formulas to that of solving for a squared term.

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Prerequisite Skills

This lesson requires the use of the following skill:

- solving quadratic equations (A–REI.4b)

Introduction

The formula for finding the area of a trapezoid, $A=\frac{1}{2}\left(b_1+b_2\right)h$, is a formula because it involves two or more variables that have specific and special relationships with the other variables. In this case, the formula involves four variables: b_1, a base of the trapezoid; b_2, the other base of the trapezoid; h, the height of the trapezoid; and A, the area of the trapezoid. When you rearrange the formula, you are changing the focus of the formula. In its given form, the focus is A, the area. However, when it is necessary to find the height, h becomes the focus of the formula and thus is isolated using algebraic properties. In this lesson, you will rearrange **literal equations** (equations that involve two or more variables) and formulas with a degree of 2.

Key Concepts

- Literal equations and formulas contain equal signs.
- Just as in any other equation, we must apply the properties of equality to maintain a balance when changing the focus of an equation or formula. That is, if you subtract a value from one side of the equation, you must do the same to the other side, and so on.
- When you change the focus of a literal equation or formula, you are isolating the variable in question.
- To isolate a variable that is squared, perform the inverse operation by taking the square root of both sides of the equation.
- When you take the square root of a real number there are two solutions: one is positive and the other is negative.
- For example, in the quadratic equation $x^2 = 4$, there are two solutions, $x = 2$ and $x = –2$, because $(2)^2 = 4$ and $(–2)^2 = 4$.

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- Since taking a square root involves finding the number that you can multiply by itself to result in the square, we must take into account both the positive and negative roots.
- That is, $\sqrt{a^2} = \pm a$.
- When solving for a squared term of a formula, the focus is important in determining whether there are two solutions.
- If the focus is any quantity that would never be negative in real life, such as distance, time, or population, it is appropriate to disregard the negative root.
- When solving for a variable in a multi-step equation, first isolate the term containing the variable using subtraction or addition. Then determine which operations are applied to the variable, and undo them in reverse order.

Common Errors/Misconceptions

- forgetting to use the inverse operations in the correct order
- forgetting that there are likely two solutions (one positive and one negative) when solving for a squared term

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Guided Practice 3.3.4

Example 1

Solve the equation $x^2 + y^2 = 100$ for y.

1. Isolate y.

   Begin by subtracting x^2 from both sides.

$x^2 + y^2 = 100$	Original equation
$y^2 = 100 - x^2$	Subtract x^2 from both sides.
$\sqrt{y^2} = \sqrt{100 - x^2}$	Take the square root of both sides.
$y = \pm\sqrt{100 - x^2}$	Simplify, remembering that the result could be positive or negative.

2. Summarize your result.

   The formula $x^2 + y^2 = 100$ can be rewritten as $y = \pm\sqrt{100 - x^2}$.

Example 2

Solve $y = 3(x - 7)^2 + 8$ for x in terms of y.

1. Isolate x.

$y = 3(x - 7)^2 + 8$	Given equation
$y - 8 = 3(x - 7)^2$	Subtract 8 from both sides.
$\frac{y-8}{3} = \frac{3(x-7)^2}{3}$	Divide both sides by 3.
$\frac{y-8}{3} = (x-7)^2$	Simplify.
$\sqrt{\frac{y-8}{3}} = \sqrt{(x-7)^2}$	Take the square root of both sides.
$\pm\sqrt{\frac{y-8}{3}} = x - 7$	Simplify.
$7 \pm \sqrt{\frac{y-8}{3}} = x$	Add 7 to both sides.

SWB p. 139; SRB p. U3-139

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2. Summarize your result.

   The equation $y = 3(x - 7)^2 + 8$ solved for x is $x = 7 \pm \sqrt{\frac{y-8}{3}}$.

Example 3

The formula for the area of a square is $A = s^2$, where s is the length of a side of the square. Solve the formula for s.

1. Isolate s.

$A = s^2$	Formula for the area of a square
$\sqrt{A} = \sqrt{s^2}$	Take the square root of both sides.
$\pm\sqrt{A} = s$	Simplify.

2. Summarize your result.

   In this case, the length of a square cannot be negative, so the negative root is disregarded.

   The equation $A = s^2$ solved for s is $s = \sqrt{A}$.

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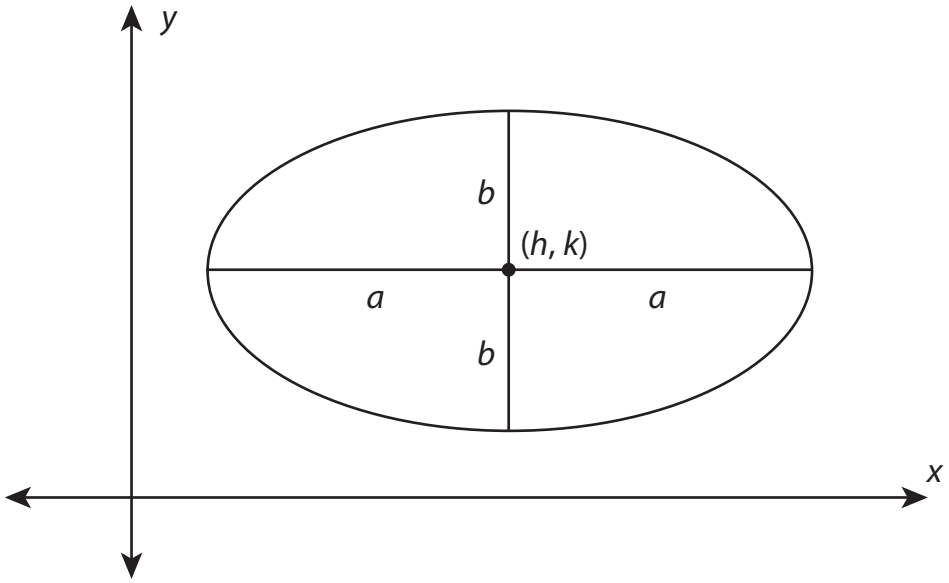
Example 4

The equation to graph an ellipse is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, where (h, k) is the center of the ellipse, a is the number of units right and left of the center on the ellipse, and b is the number of units up and down from the center on the ellipse. Solve the formula for a.



1. Isolate a.

$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	Equation of an ellipse
$\frac{(x-h)^2}{a^2}=1-\frac{(y-k)^2}{b^2}$	Subtract the fraction $\frac{(y-k)^2}{b^2}$ from both sides.
$a^2\left[\frac{(x-h)^2}{a^2}\right]=a^2\left[1-\frac{(y-k)^2}{b^2}\right]$	Multiply both sides by a^2.
$(x-h)^2=a^2\left[1-\frac{(y-k)^2}{b^2}\right]$	Simplify.

(*continued*)

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$$\frac{(x-h)^2}{1-\frac{(y-k)^2}{b^2}} = \frac{a^2\left[1-\frac{(y-k)^2}{b^2}\right]}{1-\frac{(y-k)^2}{b^2}}$$ Divide each side by $\left[1-\frac{(y-k)^2}{b^2}\right]$.

$$\frac{(x-h)^2}{1-\frac{(y-k)^2}{b^2}} = a^2$$ Simplify.

$$\sqrt{\frac{(x-h)^2}{1-\frac{(y-k)^2}{b^2}}} = \sqrt{a^2}$$ Take the square root of both sides.

$$\pm\sqrt{\frac{(x-h)^2}{\frac{b^2}{b^2}-\frac{(y-k)^2}{b^2}}} = a$$ Rewrite 1 as $\frac{b^2}{b^2}$, then simplify $\sqrt{a^2}$.

$$\pm\sqrt{\frac{(x-h)^2}{\frac{b^2-(y-k)^2}{b^2}}} = a$$ Subtract the terms in the denominator.

$$\pm\sqrt{\frac{b^2(x-h)^2}{b^2-(y-k)^2}} = a$$ Simplify the complex fraction.

2. Summarize your result.

Since a is the horizontal distance from the center of the ellipse, this is another situation where we disregard the negative.

The equation $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ solved for a is

$$a=\sqrt{\frac{b^2(x-h)^2}{b^2-(y-k)^2}}\ .$$

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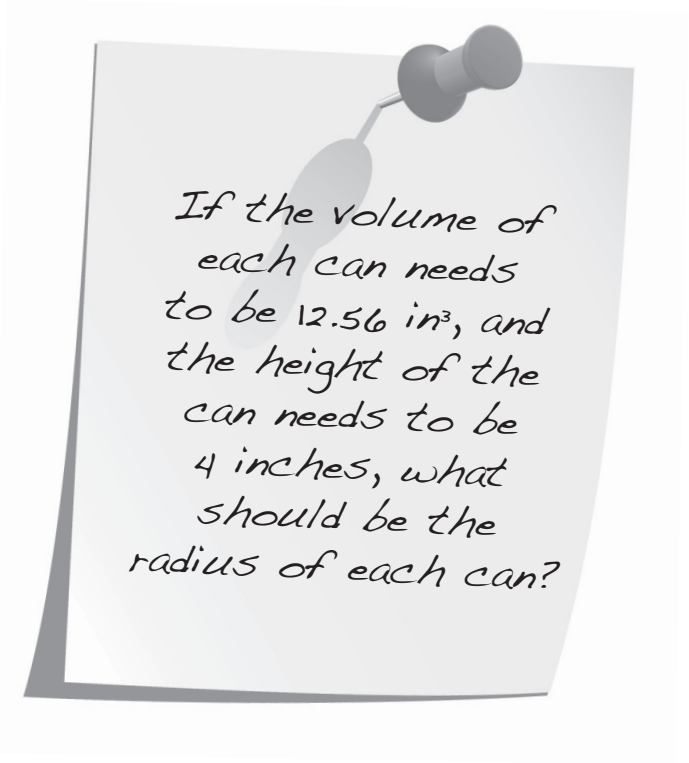
Problem-Based Task 3.3.4: Selecting a Cylinder Size

SMP	
1	2 ✓
3	4 ✓
5	6
7 ✓	8

A local soda company is investigating several different models for a cylindrical, recyclable can for its organic root beer. The formula for the volume of a cylinder is $V = \pi r^2 h$, where πr^2 is the area of the cylinder's base and h is the height of the cylinder. If the volume of each can needs to be 12.56 in^3, and the height of the can needs to be 4 inches, what should be the radius of each can?



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Problem-Based Task 3.3.4: Selecting a Cylinder Size

Coaching

a. What do each of the variables in the formula $V = \pi r^2 h$ represent?

b. What is the focus of the given formula? Explain.

c. What should the new focus be? Explain.

d. Solve the formula for the unknown variable.

e. Is the negative disregarded in this situation? Explain.

f. What values are given in the problem?

g. What should be the radius of each can if the volume needs to be 12.56 in^3 and the height needs to be 4 inches?

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Instruction

Problem-Based Task 3.3.4: Selecting a Cylinder Size

Coaching Sample Responses

a. What do each of the variables in the formula $V = \pi r^2 h$ represent?

V is the volume of the can, r is the radius of the can's base, and h is the height of the can.

b. What is the focus of the given formula? Explain.

The focus of the given formula is the volume because V is isolated.

c. What should the new focus be? Explain.

The new focus should be the radius, r, because this is the value the company is trying to determine.

d. Solve the formula for the unknown variable.

Solve the formula for the radius, r.

$$\frac{V}{\pi h} = \frac{\pi r^2 h}{\pi h}$$

$$\sqrt{\frac{V}{\pi h}} = \sqrt{r^2}$$

$$r = \pm\sqrt{\frac{V}{\pi h}}$$

e. Is the negative disregarded in this situation? Explain.

Yes, because the radius is a distance. Therefore, the equation is $r = \sqrt{\frac{V}{\pi h}}$.

f. What values are given in the problem?

The volume needs to be 12.56 in^3 and the height needs to be 4 inches.

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g. What should be the radius of each can if the volume needs to be 12.56 in^3 and the height needs to be 4 inches?

Substitute 12.56 for V and 4 for h, then solve the equation for r.

$$r = \sqrt{\frac{V}{\pi h}}$$
$$r = \sqrt{\frac{(12.56)}{\pi(4)}}$$
$$r = \sqrt{\frac{12.56}{4\pi}}$$
$$r \approx \sqrt{1}$$
$$r \approx 1$$

The radius of each can should be approximately 1 inch.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.3.4: Rearranging Formulas

For problems 1–4, solve each equation for x in terms of y.

1. $x^2 + y^2 = 25$

2. $y = (x + 3)^2$

3. $(x+7)^2 = \frac{49y^2}{16}$

4. $2(x + 1)^2 - 90 = y$

Problems 5–10 each describe a different mathematical formula. Solve each problem for the requested variable.

5. The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height. Solve for the radius of the base.

6. The formula for the area of a circle is $A = \pi r^2$, where A is the area and r is the radius. Solve for the radius.

7. The formula $a^2 + b^2 = c^2$ is the Pythagorean Theorem, where c is the length of the hypotenuse of a right triangle and a and b are the lengths of the two legs. Rearrange this formula so that the focus is b.

8. Kinetic energy is the energy of a moving object. The formula to determine the kinetic energy of an object is $E_k = \frac{1}{2}mv^2$, where E_k is the kinetic energy, m is the mass of the object, and v is the object's speed. Solve this formula for the object's speed, assuming that kinetic energy is positive.

9. The vertex form of a quadratic is $y = a(x - h)^2 + k$. Solve this equation for x.

10. The formula to graph an ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, where (h, k) is the center of the ellipse, a is the number of units right or left from the center, and b is the number of units up or down from the center. Solve the formula for b.

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Progress Assessment

Circle the letter of the best answer.

1. What is the equation of the axis of symmetry for the function $f(x) = -x^2 - 10x - 27$?

   a. $x = -5$
   c. $x = -27$
   b. $x = -2$
   d. $x = 5$

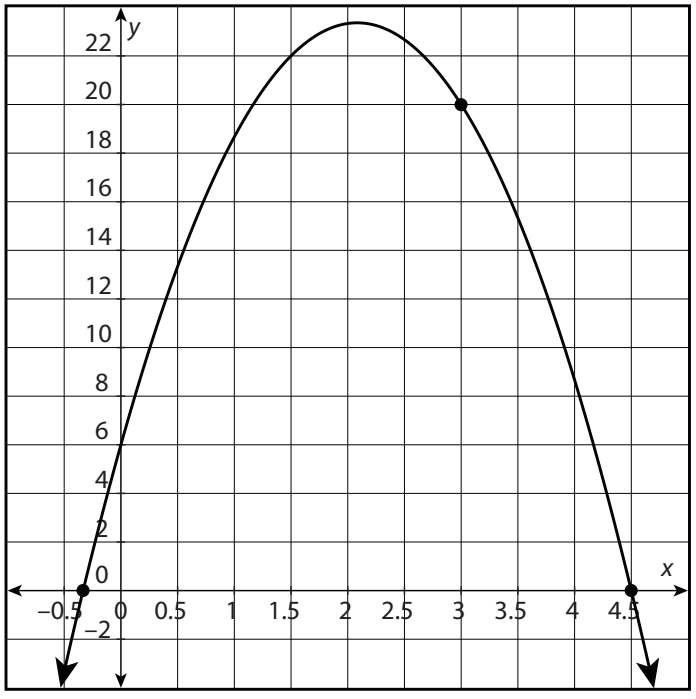
2. What is the y-intercept of the function $f(x) = 4x(3x - 7)$?

   a. -28
   c. 0
   b. 12
   d. -7

3. What is the equation of a quadratic function in standard form, given the zeros $x = -\frac{1}{3}$ and $x = -3$, and the point (1, 16) on the parabola?

   a. $f(x) = -\frac{1}{3}x^2 - 3x$
   c. $f(x) = -\frac{1}{3}x^2 - 3x + 16$
   b. $f(x) = 3x^2 + 10x + 3$
   d. $f(x) = -3x^2 + 10x - 3$

4. What is the standard form of the equation of the graphed function?



   a. $f(x) = -2x^2 - 23x + 6$
   c. $f(x) = -4x^2 + \frac{50}{3}x + 6$
   b. $f(x) = -\frac{9}{2}x^2 + \frac{1}{3}x + 6$
   d. $f(x) = 2x^2 + 23x - 6$

continued

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Assessment

5. What are the zeros of $t(x) = (4x - 16)(-x + 2)$?

   a. $x = 4$ and $x = 2$
   b. $x = 4$ and $x = -2$
   c. $x = 2$
   d. $x = 4$

6. What is the vertex form of the function $f(x) = x^2 + 4$?

   a. $f(x) = (x + 2)^2 + 4$
   b. $f(x) = (x - 0)^2 + 4$
   c. $f(x) = (x + 4)^2$
   d. $f(x) = (x - 4)^2$

7. What is the vertex of the function $f(x) = 3(x - 0.5)^2 + 0.5$?

   a. (–0.5, –0.5)
   b. (1.5, –0.5)
   c. (0.5, 0.5)
   d. (–1.5, 0.5)

8. What is the vertex of the function $f(x) = (x + 6)(x - 4)$?

   a. (6, 12)
   b. (–4, –16)
   c. (–1, –25)
   d. (–6, 4)

9. What is $y = -0.25(x - 6.5)^2 - 20$ solved for x in terms of y?

   a. $x = 6.5 \pm \sqrt{\frac{-y-20}{0.25}}$
   b. $x = -6.5 \pm \sqrt{\frac{y+20}{0.25}}$
   c. $x = 26 \pm \sqrt{\frac{-y-20}{0.25}}$
   d. $x = -26 \pm \sqrt{\frac{y+20}{0.25}}$

10. The formula $a^2 + b^2 = c^2$ is the Pythagorean Theorem, where c is the length of the hypotenuse of a right triangle and a and b are the lengths of the two legs. Rearrange the terms in the formula so that the focus is a.

    a. $a = \sqrt{b^2 - c^2}$
    b. $a = \sqrt{c^2 - b^2}$
    c. $a = \pm\sqrt{c^2 - b^2}$
    d. $a = c - b$

continued

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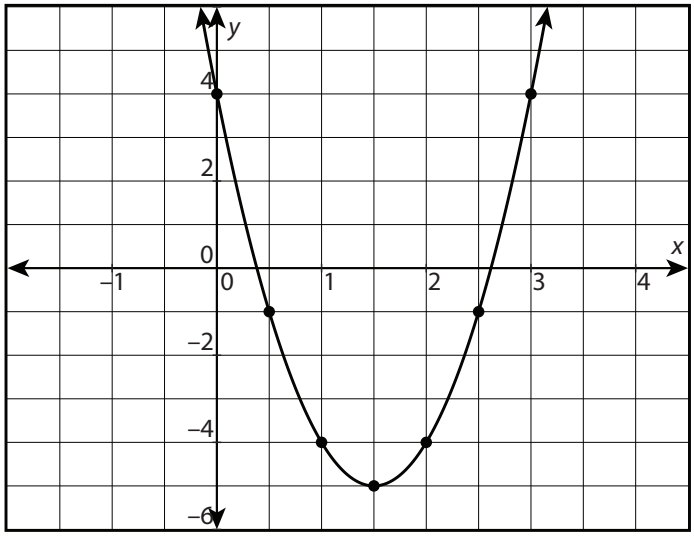
Assessment

Use the graph and the given information to answer the questions that follow.

11. The following graph represents the path of a pelican diving under the water to feed, where y represents the pelican's height in meters above and below the surface of the water and x represents the time in seconds after the pelican begins her descent. The x-axis represents the surface of the water.



a. What are the coordinates of the vertex and the equation of the axis of symmetry?

b. What is the equation that models the path of the pelican's dive?

c. Use the quadratic formula to approximate the x-intercepts. What do the values mean for the pelican's dive?

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Which equation is an identity?

   a. $(x + i)^2 = x^2 - 1$

   b. $(x + 2i)^2 = x^2 + 4xi - 4$

   c. $x(3 - i) = 3x - 3i$

   d. $(1 - 3i) - (1 - 3x) = -3i - 3x$

2. Which expression is equal to $x^2 + 9$ for all real values of x?

   a. $x^2 + (3i)^2$

   b. $(x + 3i)^2$

   c. $(x - 3i)^2$

   d. $(x + 3i)(x - 3i)$

3. Which option describes the solution(s) of the equation $x^2 + 2x + 1 = 0$?

   a. It has a double solution, which is real.

   b. It has a double solution, which is complex.

   c. It has two real solutions that are different.

   d. It has two complex solutions that are different.

4. Which is a solution of the equation $2x^2 + 5 = 0$?

   a. $\frac{\sqrt{5}}{2}$

   b. $\left(\frac{\sqrt{5}}{2}\right)i$

   c. $\frac{\sqrt{10}}{2}$

   d. $\left(\frac{\sqrt{10}}{2}\right)i$

5. Which is a solution of the equation $2x^2 + 4 = 5x$?

   a. $-\frac{5}{4} - \left(\frac{\sqrt{7}}{4}\right)i$

   b. $\frac{5}{4} - \left(\frac{\sqrt{7}}{4}\right)i$

   c. $-\frac{5}{4} - \left(\frac{\sqrt{27}}{4}\right)i$

   d. $\frac{5}{4} - \left(\frac{\sqrt{27}}{4}\right)i$

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Lesson 4: Fundamental Theorem of Algebra

Instruction

Common Core State Standards

SMP	
1	2 ✓
3 ✓	4
5	6
7 ✓	8

N–CN.7 Solve quadratic equations with real coefficients that have complex solutions.

N–CN.8 (+) Extend polynomial identities to the complex numbers. *For example, rewrite $x^2 + 4$ as $(x + 2i)(x – 2i)$.*

N–CN.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Essential Questions

1. What is an identity?
2. How can you factor a quadratic polynomial whose factors include complex numbers?
3. How many solutions does every quadratic equation have and what types of numbers are they?
4. What are the different ways to determine the type of solutions of a quadratic equation?
5. What are the different ways to find the solutions of a quadratic equation?

WORDS TO KNOW

complex conjugates	two complex numbers of the form $a + bi$ and $a – bi$
complex number	a number in the form $a + bi$, where a and b are real numbers, and i is the imaginary unit
corollary	a theorem that accompanies another theorem and is usually easily deduced from the other theorem
Corollary to the Fundamental Theorem of Algebra	If $P(x)$ is a polynomial function of degree $n \geq 1$ with complex coefficients, then the related equation $P(x) = 0$ has exactly n complex solutions (roots), if a double solution is counted as two separate solutions.
degree of a one-variable polynomial	the greatest exponent attached to the variable in the polynomial
double root	two roots that are equal
double solution	two solutions that are equal

SRB p. U3-145

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Instruction

Fundamental Theorem of Algebra	If $P(x)$ is a polynomial function of degree $n \geq 1$ with complex coefficients, then the related equation $P(x) = 0$ has at least one complex solution (root).
identity	an equation that is true regardless of what values are chosen for the variables
imaginary number	any number of the form bi, where b is a non-zero real number and $i = \sqrt{-1}$
imaginary unit, *i*	the letter i, used to represent the non-real value, $i = \sqrt{-1}$
polynomial function	a function whose rule is a one-variable polynomial; $P(x)$ is a polynomial function if $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where n is a nonnegative integer and $a_n \neq 0$
quadratic polynomial in one variable	a one-variable polynomial of degree 2; it can be written in the form $ax^2 + bx + c$, where $a \neq 0$
square root of a negative number	a number defined such that for any positive real number a, $\sqrt{-a} = i\sqrt{a}$
term	a number, a variable, or the product of a number and variable(s)
zeros	the x-values of a function for which the function value is 0

Recommended Resources

- MathIsFun.com. "Fundamental Theorem of Algebra."

  http://www.walch.com/rr/00139

  This website gives an overview of the Fundamental Theorem of Algebra, including examples and polynomials of different degrees.

- MathIsFun.com. "Quadratic Equations."

  http://www.walch.com/rr/00140

  This website gives an overview of how to recognize and solve quadratic equations, including solving by factoring, completing the square, and using the quadratic formula. It includes complex solutions.

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Lesson 3.4.1: Extending Polynomial Identities to Include Complex Numbers

Warm-Up 3.4.1

Mitchell is using algebra tiles to model polynomials.

1. Figures 1, 2, and 3 below represent algebra tiles. For each figure, Mitchell wants to write an expression that represents its area. What expressions should he write?

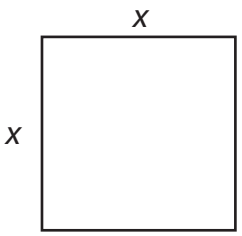


Figure 1

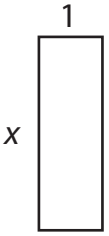


Figure 2

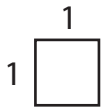
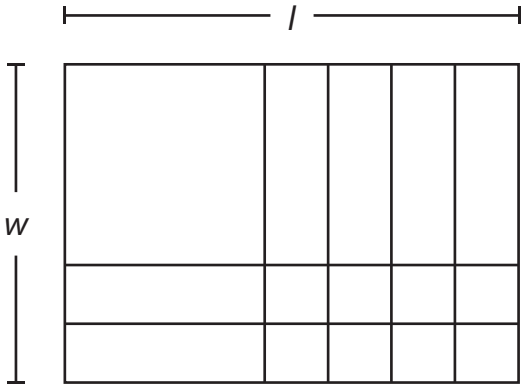


Figure 3

2. Mitchell forms the rectangle diagram below with his algebra tiles. He writes the equation $A = lw$ to represent the area of the rectangle, where l is the length and w is the width. Rewrite the equation $A = lw$, substituting a quadratic trinomial for A and binomials for l and w. Each side of your equation should represent the total area of the diagram. (*Note*: A quadratic trinomial has the form $ax^2 + bx + c$.)



3. The correct equation for problem 2 shows a quadratic trinomial and its factorization. Multiply the binomials to verify that the equation is true.

4. The correct equation for problem 2 is called an identity because it is true for all values of x. Show that the equation is true for $x = 5$, $x = 8$, and one other x-value of your choice.

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Instruction

Lesson 3.4.1: Extending Polynomial Identities to Include Complex Numbers

Common Core State Standard

N–CN.8 (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x – 2i)$.

Warm-Up 3.4.1 Debrief

1. Figures 1, 2, and 3 below represent algebra tiles. For each figure, Mitchell wants to write an expression that represents its area. What expressions should he write?

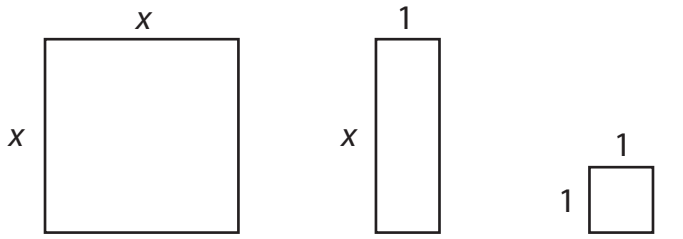
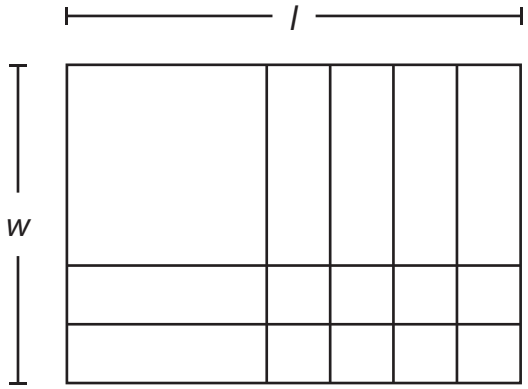


Figure 1 Figure 2 Figure 3

Area of figure 1 = x^2; area of figure 2 = x; area of figure 3 = 1

2. Mitchell forms the rectangle diagram below with his algebra tiles. He writes the equation $A = lw$ to represent the area of the rectangle, where l is the length and w is the width. Rewrite the equation $A = lw$, substituting a quadratic trinomial for A and binomials for l and w. Each side of your equation should represent the total area of the diagram. (*Note*: A quadratic trinomial has the form $ax^2 + bx + c$.)



$A = lw$

$x^2 + 6x + 8 = (x + 4)(x + 2)$

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Instruction

3. The correct equation for problem 2 shows a quadratic trinomial and its factorization. Multiply the binomials to verify that the equation is true.

   $(x + 4)(x + 2) = x^2 + 2x + 4x + 8$

   $(x + 4)(x + 2) = x^2 + 6x + 8$

4. The correct equation for problem 2 is called an identity because it is true for all values of x. Show that the equation is true for $x = 5$, $x = 8$, and one other x-value of your choice.

For $x = 5$:	**For $x = 8$:**
$x^2 + 6x + 8 = (x + 4)(x + 2)$	$x^2 + 6x + 8 = (x + 4)(x + 2)$
$5^2 + 6(5) + 8 = (5 + 4)(5 + 2)$	$8^2 + 6(8) + 8 = (8 + 4)(8 + 2)$
$25 + 30 + 8 = (9)(7)$	$64 + 48 + 8 = (12)(10)$
$63 = 63$	$120 = 120$

   Let a represent one other x-value of your choice.

   For $x = a$:

   $x^2 + 6x + 8 = (x + 4)(x + 2)$

   $(a)^2 + 6(a) + 8 = (a + 4)(a + 2)$

   $a^2 + 6a + 8 = a^2 + 2a + 4a + 8$

   $a^2 + 6a + 8 = a^2 + 6a + 8$

Connection to the Lesson

- Students will study polynomials, including quadratic trinomials and their factorizations.
- Students will study identities.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- adding, subtracting, and multiplying complex numbers (N–CN.2)
- applying the quadratic formula (A–REI.4b)
- simplifying radicals (N–RN.2)
- using solutions of a quadratic equation to write the equation in factored form (F–IF.8a)

Introduction

Identities are commonly used to solve many different types of mathematics problems. In fact, you have already used them to solve real-world problems. In this lesson, you will extend your understanding of polynomial identities to include complex numbers and imaginary numbers.

Key Concepts

- An **identity** is an equation that is true regardless of what values are chosen for the variables.
- Some identities are often used and are well known; others are less well known. The following chart shows some examples of identities.

Identity	True for …
$x + 2 = 2 + x$	This is true for all values of x. This identity illustrates the Commutative Property of Addition.
$a(b + c) = ab + ac$	This is true for all values of a, b, and c. This identity is a statement of the Distributive Property.
$\frac{ab+a}{b+1}=a$	This is true for all values of a and b, except for $b = –1$. The expression $\frac{ab+a}{b+1}$ is not defined for $b = –1$ because if $b = –1$, the denominator is equal to 0. To see that the equation is true provided that $b \neq –1$, note that $\frac{ab+a}{b+1}=\frac{a(b+1)}{(b+1)}=\frac{a\cancel{(b+1)}}{\cancel{(b+1)}}=a.$

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Instruction

- A monomial is a number, a variable, or a product of a number and one or more variables with whole number exponents.
- If a monomial has one or more variables, then the number multiplied by the variable(s) is called a coefficient.
- A polynomial is a monomial or a sum of monomials. The monomials are the terms, numbers, variables, or the product of a number and variable(s) of the polynomial. Examples of polynomials include:

r	This polynomial has 1 term, so it is called a monomial.
$\frac{2}{3}x^5 - x$	This polynomial has 2 terms, so it is called a binomial.
$3x^2 - 5x + 2$	This polynomial has 3 terms, so it is called a trinomial.
$-4x^3y + x^2y^2 - 4xy^3$	This polynomial also has 3 terms, so it is also a trinomial.

- In this lesson, all polynomials will have one variable. The **degree of a one-variable polynomial** is the greatest exponent attached to the variable in the polynomial. For example:
  - The degree of $-5x + 3$ is 1. (Note that $-5x + 3 = -5x^1 + 3$.)
  - The degree of $4x^2 + 8x + 6$ is 2.
  - The degree of $x^3 + 4x^2$ is 3.
- A **quadratic polynomial in one variable** is a one-variable polynomial of degree 2, and can be written in the form $ax^2 + bx + c$, where $a \neq 0$. For example, the polynomial $4x^2 + 8x + 6$ is a quadratic polynomial.
- A quadratic equation is an equation that can be written in the form $ax^2 + bx + c = 0$, where x is the variable, a, b, and c are constants, and $a \neq 0$.
- The quadratic formula states that the solutions of a quadratic equation of the form $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. A quadratic equation in this form can have no real solutions, one real solution, or two real solutions.
- In this lesson, all polynomial coefficients are real numbers, but the variables sometimes represent complex numbers.

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Instruction

- The **imaginary unit *i*** represents the non-real value $i = \sqrt{-1}$. i is the number whose square is –1. We define i so that $i = \sqrt{-1}$ and $i^2 = -1$.
- An **imaginary number** is any number of the form bi, where b is a real number, $i = \sqrt{-1}$, and $b \neq 0$.
- A **complex number** is a number with a real component and an imaginary component. Complex numbers can be written in the form $a + bi$, where a and b are real numbers, and i is the imaginary unit. For example, $5 + 3i$ is a complex number. 5 is the real component and $3i$ is the imaginary component.
- Recall that all rational and irrational numbers are real numbers. Real numbers do not contain an imaginary component.
- The set of complex numbers is formed by two distinct subsets that have no common members: the set of real numbers and the set of imaginary numbers (numbers of the form bi, where b is a real number, $i = \sqrt{-1}$, and $b \neq 0$).
- Recall that if $x^2 = a$, then $x = \pm\sqrt{a}$. For example, if $x^2 = 25$, then $x = 5$ or $x = -5$.
- The **square root of a negative number** is defined such that for any positive real number a, $\sqrt{-a} = i\sqrt{a}$. (Note the use of the negative sign under the radical.)
- For example, $\sqrt{-9} = i\sqrt{9} = i \bullet 3 = 3i$.
- Using p and q as variables, if both p and q are positive, then $\sqrt{p} \bullet \sqrt{q} = \sqrt{pq}$. For example, if $p = 4$ and $q = 9$, then $\sqrt{p} \bullet \sqrt{q} = \sqrt{4} \bullet \sqrt{9} = 2 \bullet 3 = 6$, and $\sqrt{pq} = \sqrt{4 \bullet 9} = \sqrt{36} = 6$.
- But if p and q are both negative, then $\sqrt{p} \bullet \sqrt{q} \neq \sqrt{pq}$. For example, if $p = -4$ and $q = -9$, then $\sqrt{p} \bullet \sqrt{q} = \sqrt{-4} \bullet \sqrt{-9} = i\sqrt{4} \bullet i\sqrt{9} = 2i \bullet 3i = 6i^2 = 6 \bullet (-1) = -6$, but $\sqrt{pq} = \sqrt{(-4)(-9)} = \sqrt{36} = 6$.
- So, to simplify an expression of the form $\sqrt{p} \bullet \sqrt{q}$ when p and q are both negative, write each factor as a product using the imaginary unit i before multiplying.
- Two numbers of the form $a + bi$ and $a - bi$ are called **complex conjugates**.
- The product of two complex conjugates is always a real number, as shown:

$(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2$	Distribute.
$(a + bi)(a - bi) = a^2 - b^2i^2$	Simplify.
$(a + bi)(a - bi) = a^2 - b^2(-1)$	$i^2 = -1$
$(a + bi)(a - bi) = a^2 + b^2$	Simplify.

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Instruction

- Note that $a^2 + b^2$ is the sum of two squares and it is a real number because a and b are real numbers.
- The equation $(a + bi)(a – bi) = a^2 + b^2$ is an identity that shows how to factor the sum of two squares.

Common Errors/Misconceptions

- substituting $\sqrt{pq}$ for $\sqrt{p} \bullet \sqrt{q}$ when p and q are both negative
- neglecting to include factors of i when factoring the sum of two squares

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Instruction

Guided Practice 3.4.1

Example 1

Simplify each expression.

$(3 + 2i) + (-8 + i)$

$(x + 3i)(x - 3i)$

1. Add to simplify $(3 + 2i) + (-8 + i)$.

$(3 + 2i) + (-8 + i) = [3 + (-8)] + (2 + 1)i$	Add the real parts. Add the coefficients of the imaginary parts.
$(3 + 2i) + (-8 + i) = -5 + 3i$	Simplify.

2. Multiply to simplify $(x + 3i)(x - 3i)$.

$(x + 3i)(x - 3i) = x^2 - 3xi + 3xi - 9i^2$	Distribute.
$(x + 3i)(x - 3i) = x^2 - 3xi + 3xi - 9(-1)$	Since $i^2 = -1$, substitute -1 for i^2.
$(x + 3i)(x - 3i) = x^2 + 9$	Simplify.

$(x + 3i)(x - 3i) = x^2 + 9$ is an identity because it is true for all values of x.

$x + 3i$ and $x - 3i$ are complex conjugates because they have the form $a + bi$ and $a - bi$.

SWB p. 149, SRB p. U3-151

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Instruction

Example 2

Write a polynomial identity that shows how to factor each polynomial.

$9c^2 + 100$

$12 + 75y^2$

1. Factor $9c^2 + 100$.

   The polynomial $9c^2 + 100$ is a sum of two squares.

$9c^2 + 100 = (3c)^2 + 10^2$	Write the polynomial in the form $a^2 + b^2$.
$9c^2 + 100 = (3c + 10i)(3c - 10i)$	Apply the identity $a^2 + b^2 = (a + bi)(a - bi)$.

   $9c^2 + 100 = (3c + 10i)(3c - 10i)$ is an identity that shows how to factor $9c^2 + 100$.

2. Factor $12 + 75y^2$.

   To factor $12 + 75y^2$, first factor out the common monomial 3 from both terms. Then apply the identity that shows how to factor the sum of two squares.

$12 + 75y^2 = 3(4 + 25y^2)$	Factor out the common monomial 3 from both terms.
$12 + 75y^2 = 3[2^2 + (5y)^2]$	Write $4 + 25y^2$ in the form $a^2 + b^2$.
$12 + 75y^2 = 3(2 + 5yi)(2 - 5yi)$	Apply the identity for the sum of two squares: $a^2 + b^2 = (a + bi)(a - bi)$.

   $12 + 75y^2 = 3(2 + 5yi)(2 - 5yi)$ is an identity that shows how to factor $12 + 75y^2$.

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Instruction

Example 3

Write a polynomial identity that shows how to factor $x^2 + 3$.

1. Solve for x using the quadratic formula.

   $x^2 + 3$ is not a sum of two squares, nor is there a common monomial. Use the quadratic formula to find the solutions to $x^2 + 3$.

   The quadratic formula is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$x^2 + 3 = 0$	Set the quadratic polynomial equal to 0.
$1x^2 + 0x + 3 = 0$	Write the polynomial in the form $ax^2 + bx + c = 0$.
$x = \dfrac{-0 \pm \sqrt{0^2 - 4(1)(3)}}{2(1)}$	Substitute values into the quadratic formula: $a = 1$, $b = 0$, and $c = 3$.
$x = \dfrac{\pm\sqrt{-12}}{2}$	Simplify.
$x = \dfrac{\pm i\sqrt{12}}{2}$	For any positive real number a, $\sqrt{-a} = i\sqrt{a}$.
$x = \dfrac{\pm i \bullet \sqrt{4 \bullet 3}}{2}$	Factor 12 to show a perfect square factor.
$x = \dfrac{\pm i \bullet \sqrt{4} \bullet \sqrt{3}}{2}$	For any real numbers a and b, $\sqrt{ab} = \sqrt{a} \bullet \sqrt{b}$.
$x = \dfrac{\pm 2i\sqrt{3}}{2}$	Simplify.
$x = \pm i\sqrt{3}$	

   The solutions of the equation $x^2 + 3 = 0$ are $i\sqrt{3}$ and $-i\sqrt{3}$.

   Therefore, the equation can be written in the factored form $\left(x + i\sqrt{3}\right)\left(x - i\sqrt{3}\right) = 0$.

   $x^2 + 3 = \left(x + i\sqrt{3}\right)\left(x - i\sqrt{3}\right)$ is an identity that shows how to factor the polynomial $x^2 + 3$.

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Instruction

2. Check your answer using square roots.

Another method for solving the equation $x^2 + 3 = 0$ is by using a property involving square roots.

$x^2 + 3 = 0$	Set the quadratic polynomial equal to 0.
$x^2 = -3$	Subtract 3 from both sides.
$x = \pm\sqrt{-3}$	Apply the Square Root Property: if $x^2 = a$, then $x = \pm\sqrt{a}$.
$x = \pm i\sqrt{3}$	For anypositive real number a, $\sqrt{-a} = i\sqrt{a}$.

3. Verify the identity $x^2 + 3 = \left(x + i\sqrt{3}\right)\left(x - i\sqrt{3}\right)$ by multiplying.

$\left(x + i\sqrt{3}\right)\left(x - i\sqrt{3}\right) = x^2 - xi\sqrt{3} + xi\sqrt{3} - i^2\left(\sqrt{3}\right)^2$	Distribute.
$\left(x + i\sqrt{3}\right)\left(x - i\sqrt{3}\right) = x^2 - i^2\left(\sqrt{3}\right)^2$	Combine similar terms.
$\left(x + i\sqrt{3}\right)\left(x - i\sqrt{3}\right) = x^2 - (-1)(3)$	Simplify.
$\left(x + i\sqrt{3}\right)\left(x - i\sqrt{3}\right) = x^2 + 3$	

The square root method produces the same result as the quadratic formula.

$x^2 + 3 = \left(x + i\sqrt{3}\right)\left(x - i\sqrt{3}\right)$ is an identity that shows how to factor the polynomial $x^2 + 3$.

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Instruction

Example 4

Write a polynomial identity that shows how to factor the polynomial $3x^2 + 2x + 11$.

1. Solve for x using the quadratic formula.

The quadratic formula is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$3x^2 + 2x + 11 = 0$	Set the quadratic polynomial equal to 0.
$x = \dfrac{-2 \pm \sqrt{2^2 - 4(3)(11)}}{2(3)}$	Substitute values into the quadratic formula: $a = 3$, $b = 2$, and $c = 11$.
$x = \dfrac{-2 \pm \sqrt{4 - 132}}{6}$	Simplify.
$x = \dfrac{-2 \pm \sqrt{-128}}{6}$	
$x = \dfrac{-2 \pm i\sqrt{128}}{6}$	For any positive real number a, $\sqrt{-a} = i\sqrt{a}$.
$x = \dfrac{-2 \pm i\sqrt{64 \bullet 2}}{6}$	Factor 128 to show its greatest perfect square factor.
$x = \dfrac{-2 \pm i \bullet \sqrt{64} \bullet \sqrt{2}}{6}$	For any real numbers a and b, $\sqrt{ab} = \sqrt{a} \bullet \sqrt{b}$.
$x = \dfrac{-2 \pm 8i\sqrt{2}}{6}$	Simplify.
$x = \dfrac{-2}{6} \pm \dfrac{8i\sqrt{2}}{6}$	Write the real and imaginary parts of the complex number.
$x = -\dfrac{1}{3} \pm \dfrac{4\sqrt{2}}{3}i$	Simplify.

The solutions of the equation $3x^2 + 2x + 11 = 0$ are $-\dfrac{1}{3} + \dfrac{4\sqrt{2}}{3}i$ and $-\dfrac{1}{3} - \dfrac{4\sqrt{2}}{3}i$.

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Instruction

2. Use the solutions from step 1 to write the equation in factored form.

If $(x - r_1)(x - r_2) = 0$, then by the Zero Product Property, $x - r_1 = 0$ or $x - r_2 = 0$, and $x = r_1$ or $x = r_2$. That is, r_1 and r_2 are the roots (solutions) of the equation.

Conversely, if r_1 and r_2 are the roots of a quadratic equation, then that equation can be written in the factored form $(x - r_1)(x - r_2) = 0$.

The roots of the equation $3x^2 + 2x + 11 = 0$ are $-\frac{1}{3}+\frac{4\sqrt{2}}{3}i$ and $-\frac{1}{3}-\frac{4\sqrt{2}}{3}i$.

Therefore, the equation can be written in the factored form

$\left[x-\left(-\frac{1}{3}+\frac{4\sqrt{2}}{3}i\right)\right]\left[x-\left(-\frac{1}{3}-\frac{4\sqrt{2}}{3}i\right)\right]=0$, or in the simpler factored form $\left(x+\frac{1}{3}-\frac{4\sqrt{2}}{3}i\right)\left(x+\frac{1}{3}+\frac{4\sqrt{2}}{3}i\right)=0$.

$3x^2 + 2x + 11 = \left(x+\frac{1}{3}-\frac{4\sqrt{2}}{3}i\right)\left(x+\frac{1}{3}+\frac{4\sqrt{2}}{3}i\right)$ is an identity that shows how to factor the polynomial $3x^2 + 2x + 11$.

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Instruction

Example 5

For an alternating current (AC) electrical circuit, $V = I \bullet Z$, where V is the voltage in volts, I is the current in milliamperes, and Z is the impedance in ohms. For circuit 1, find the voltage V. For circuit 2, write an identity that shows two equivalent expressions for the voltage V and explain why it is an identity.

Circuit 1: $I = (3 + 2i)$ milliamperes and $Z = (6 - 3i)$ ohms

Circuit 2: $I = (a + bi)$ milliamperes and $Z = (c + di)$ ohms

1. For circuit 1, find the voltage, V.

$V = I \bullet Z$	Formula for voltage
$V = (3 + 2i)(6 - 3i)$	Substitute the given values for I and Z.
$V = 18 - 9i + 12i - 6i^2$	Distribute.
$V = 18 + 3i - 6(-1)$	Simplify.
$V = 18 + 3i + 6$	
$V = 24 + 3i$	

The voltage for circuit 1 is $(24 + 3i)$ volts.

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Instruction

2. For circuit 2, write an identity that shows two equivalent expressions for the voltage, V, and explain why it is an identity.

$V = I \bullet Z$	Formula for voltage
$V = (a + bi)(c + di)$	Substitute the given expressions for I and Z.
$V = ac + adi + bci + bdi^2$	Distribute.
$V = ac + adi + bci + bd(-1)$	Since $i^2 = -1$, substitute -1 for i^2.
$V = ac + adi + bci - bd$	Simplify.
$V = ac - bd + adi + bci$	Apply the Commutative Property to group real parts together and imaginary parts together.
$V = (ac - bd) + (ad + bc)i$	Apply the Distributive Property to write the expression in complex number form.

$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$, which is an identity because it is true for all values of a, b, c, and d.

Expressed as a product of complex numbers, the voltage is $(a + bi)(c + di)$.

Expressed as a single complex number, the voltage is $(ac - bd) + (ad + bc)i$.

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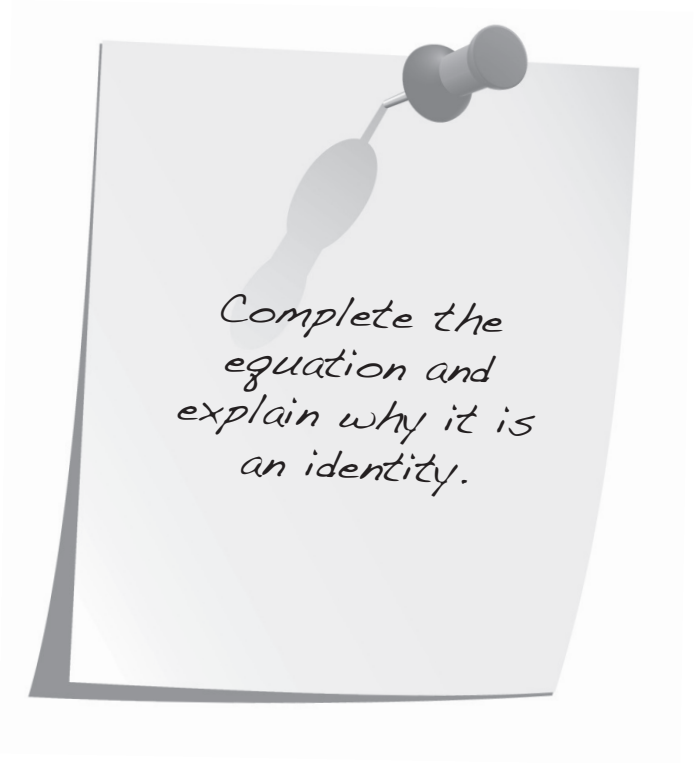
Problem-Based Task 3.4.1: Making a Complex Expression Simple

SMP	
1	2 ✓
3 ✓	4
5	6
7 ✓	8

Kiana is working on a homework problem and has to simplify the expression $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3$. She must then complete the following equation and explain why it is an identity: $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3 \bullet a = ___$.



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Problem-Based Task 3.4.1: Making a Complex Expression Simple

Coaching

a. How can Kiana simplify $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2$ by using the Distributive Property?

b. How can Kiana use her result from part a along with the fact that $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\bullet\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2$ to simplify $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3$?

c. The correct answer to part b is the value of $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3$. How can Kiana use this value to complete the equation $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3\bullet a=$ ___?

d. The correct completed equation in part c is an identity. Why is it an identity?

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Instruction

Problem-Based Task 3.4.1: Making a Complex Expression Simple

Coaching Sample Responses

a. How can Kiana simplify $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2$ by using the Distributive Property?

Factor the expression, and then distribute the first factor over the second factor.

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2=\frac{1}{2}\bullet\frac{1}{2}+\frac{1}{2}\bullet\frac{\sqrt{3}}{2}i+\frac{1}{2}\bullet\frac{\sqrt{3}}{2}i+\frac{\sqrt{3}}{2}i\bullet\frac{\sqrt{3}}{2}i$$

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2=\frac{1}{4}+\frac{\sqrt{3}}{4}i+\frac{\sqrt{3}}{4}i+\frac{3}{4}i^2$$

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2=\frac{1}{4}+\frac{2\sqrt{3}}{4}i+\frac{3}{4}i^2$$

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2=\frac{1}{4}+\frac{\sqrt{3}}{2}i+\frac{3}{4}i^2$$

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2=\frac{1}{4}+\frac{\sqrt{3}}{2}i+\frac{3}{4}(-1)$$

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2=\frac{1}{4}+\frac{\sqrt{3}}{2}i-\frac{3}{4}$$

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

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b. How can Kiana use her result from part a along with the fact that $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\bullet\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2$ to simplify $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3$?

Use the properties of exponents to factor the expression, and then substitute the result from part a for $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2$.

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\bullet\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2$$

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\bullet\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3=\frac{1}{2}\bullet\left(-\frac{1}{2}\right)+\frac{1}{2}\bullet\frac{\sqrt{3}}{2}i+\left(-\frac{1}{2}\right)\bullet\frac{\sqrt{3}}{2}i+\frac{\sqrt{3}}{2}i\bullet\frac{\sqrt{3}}{2}i$$

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3=-\frac{1}{4}+\frac{\sqrt{3}}{4}i-\frac{\sqrt{3}}{4}i+\frac{3}{4}i^2$$

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3=-\frac{1}{4}+\frac{3}{4}i^2$$

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3=-\frac{1}{4}+\frac{3}{4}(-1)$$

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3=-\frac{1}{4}-\frac{3}{4}$$

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3=-1$$

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c. The correct answer to part b is the value of $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3$. How can Kiana use this value to complete the equation: $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3 \bullet a = ___$?

$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3 \bullet a = -1 \bullet a = -a$, so the completed equation is $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3 \bullet a = -a$.

d. The correct completed equation in part c is an identity. Why is it an identity?

The equation $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3 \bullet a = -a$ is an identity because it is true for all values of a.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.4.1: Extending Polynomial Identities to Include Complex Numbers

For problems 1–4, simplify each expression. Identify any identities and complex conjugates in your work.

1. $(7 - i) + (-2 + 4i)$

2. $-x - (5x - i)$

3. $(1 - 3i)(1 + 3i)$

4. $(c + 3di)(c - 3di)$

For problems 5–7, write a polynomial identity that shows how to factor the polynomial.

5. $x^2 + 4$

6. $32y^2 + 18$

7. $x^2 + 4x + 10$

Use the given information to solve problems 8–10.

8. For an alternating current (AC) electrical circuit, $V = I \bullet Z$, where V is the voltage in volts, I is the current in milliamperes, and Z is the impedance in ohms. For circuit 1 below, find the voltage, V. For circuit 2, write an identity that shows two equivalent expressions for the voltage, V, and explain why it is an identity.

   Circuit 1: $I = (4 + 2i)$ milliamperes and $Z = (6 - i)$ ohms

   Circuit 2: $I = (a + bi)$ milliamperes and $Z = (p + qi)$ ohms

continued

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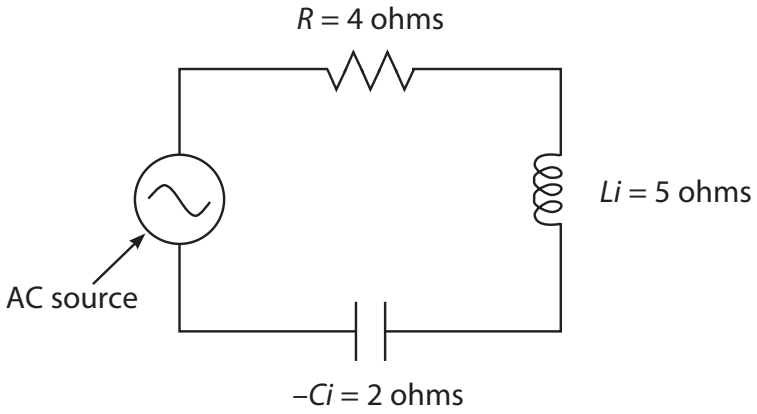
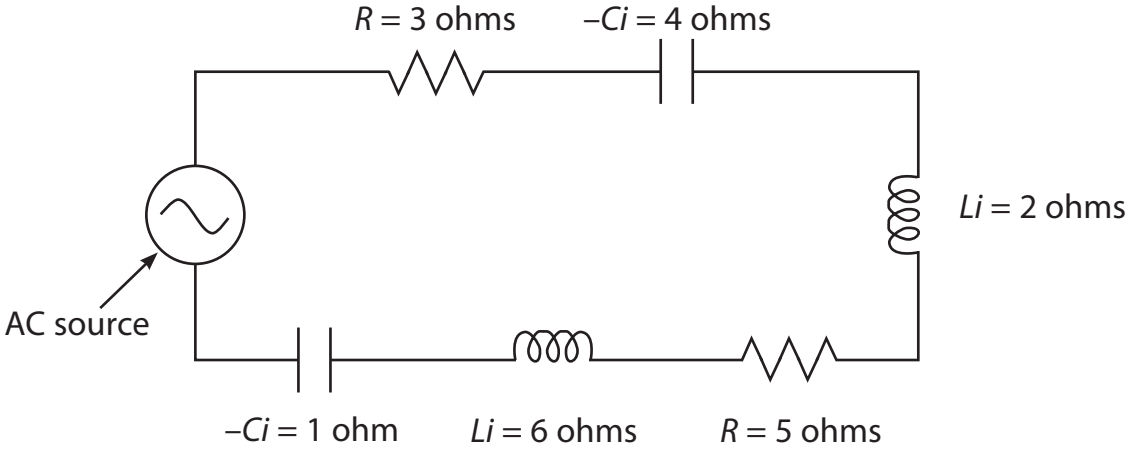
9. Look at the diagram of a circuit below. Resistors (R), inductors (Li), and capacitors (Ci) are circuit components that oppose the flow of current. The total opposition in the circuit is called the total impedance, and is represented by the variable Z. The circuit represented in the diagram below is called a series circuit. Use this diagram and the information that follows it to solve parts a and b.



   a. In a series circuit, the total impedance is the sum of the impedances of the individual components. That is, $Z = R + Li + (–Ci)$. Rewrite the sum $R + Li + (–Ci)$ in complex number form $a + bi$ to complete the following identity: $R + Li + (–Ci) =$ _____.

   b. Find the total impedance of the circuit represented in the diagram.

10. Find the total impedance of the circuit represented in the diagram below if $Z = R + Li + (–Ci)$.



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Lesson 3.4.2: Solving Quadratic Equations with Complex Solutions

Warm-Up 3.4.2

Some students are playing a game. For each expression on card A, they need to choose all the expressions in the same row on card B that are equal to that expression. What are the correct choices?

A	B				
$\sqrt{36}$	6	$\sqrt{9} \bullet \sqrt{4}$	18	$9 \bullet 2$	$3 \bullet 2$	$\sqrt{9 \bullet 4}$
$\sqrt{\frac{16}{25}}$	$\frac{\sqrt{16}}{5}$	$\frac{4}{5}$	$\frac{\sqrt{16}}{\sqrt{25}}$	$\frac{4}{\sqrt{5}}$	$\frac{\sqrt{4}}{\sqrt{5}}$
$\sqrt{50}$	25	$\sqrt{25} \bullet \sqrt{2}$		$5 \bullet 2$	$5\sqrt{2}$
$\sqrt{-9}$	-3	$3i$	$-3i$	$9i$	$-9i$

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Lesson 3.4.2: Solving Quadratic Equations with Complex Solutions

Common Core State Standards

N–CN.7 Solve quadratic equations with real coefficients that have complex solutions.

N–CN.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Warm-Up 3.4.2 Debrief

A	B				
$\sqrt{36}$	6	$\sqrt{9}\bullet\sqrt{4}$	18	$9 \bullet 2$	$3 \bullet 2$	$\sqrt{9\bullet 4}$
$\sqrt{\frac{16}{25}}$	$\frac{\sqrt{16}}{5}$	$\frac{4}{5}$	$\frac{\sqrt{16}}{\sqrt{25}}$	$\frac{4}{\sqrt{5}}$	$\frac{\sqrt{4}}{\sqrt{5}}$
$\sqrt{50}$	25	$\sqrt{25}\bullet\sqrt{2}$	$5 \bullet 2$	$5\sqrt{2}$	
$\sqrt{-9}$	-3	$3i$	$-3i$	$9i$	$-9i$

$\sqrt{36}$ can be written as 6, $\sqrt{9}\bullet\sqrt{4}$, $3 \bullet 2$, and $\sqrt{9\bullet 4}$.

$\sqrt{\frac{16}{25}}$ can be written as $\frac{4}{5}$, $\frac{\sqrt{16}}{\sqrt{25}}$, and $\frac{\sqrt{16}}{5}$.

$\sqrt{50}$ can be written as $\sqrt{25}\bullet\sqrt{2}$ and $5\sqrt{2}$.

$\sqrt{-9}$ can be written as $3i$.

Connection to the Lesson

- Students will use properties of square roots to simplify expressions.
- Students will write square roots of negative numbers using the imaginary unit i.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- simplifying expressions involving square roots of positive and negative numbers (N–RN.2)
- determining perfect square factors of whole numbers (8.EE.2)
- identifying the degree of a polynomial (A–SSE.1a★)
- recognizing the vertex of a quadratic graph as the point containing the function's maximum or minimum (A–SSE.3b★)
- determining the least common denominator of fractions (4.NF.2)
- identifying the x-intercepts of a function $f(x)$ as the real solutions of the equation $f(x) = 0$ (F–IF.4★)
- factoring the sum of two squares using imaginary numbers (N–CN.8 (+))

Introduction

One of the most useful theorems in algebra is the **Fundamental Theorem of Algebra**, which tells us about the number of solutions that a polynomial can have. Sometimes these solutions are real, sometimes they are complex, and sometimes the same solution is repeated more than once. In this lesson, you will learn how to solve quadratic equations that have complex solutions as well as learn how to use the Fundamental Theorem of Algebra.

Key Concepts

- A quadratic polynomial in one variable is a one-variable polynomial of degree 2, and can be written in the form $ax^2 + bx + c$, where $a \neq 0$.
- Recall that a quadratic function is a function that can be written in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$.
- Also, remember that a quadratic equation is an equation that can be written in the form $ax^2 + bx + c = 0$, where $a \neq 0$.

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Instruction

- Review these three examples of quadratic equations and their solutions (roots).

$x^2 + x - 2 = 0$	$x^2 + 6x + 9 = 0$	$x^2 = -4$
$(x + 2)(x - 1) = 0$	$(x + 3)(x + 3) = 0$	$x = \pm\sqrt{-4}$
$x + 2 = 0$ or $x - 1 = 0$	$x + 3 = 0$ or $x + 3 = 0$	$x = \pm i\sqrt{4}$ $x = \pm 2i$
$x = -2$ or $x = 1$	$x = -3$ or $x = -3$	$x = 2i$ or $x = -2i$
Two real solutions (roots)	One real solution (root)	Two imaginary solutions (roots)
They are different.	It is called a double root.	They are complex conjugates: $0 + 2i$ and $0 - 2i$.

- Sometimes an equation has two distinct roots; other times the two roots are the same. This is called a **double root**.
- A **polynomial function** is a function whose rule is a one-variable polynomial. That is, $P(x)$ is a polynomial function if $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where n is a nonnegative integer and $a_n \neq 0$. Polynomial functions in this lesson will have all real coefficients.
- The number of solutions a polynomial will have can be determined using the Fundamental Theorem of Algebra.

Theorem: Fundamental Theorem of Algebra
If $P(x)$ is a polynomial function of degree $n \geq 1$ with complex coefficients, then the related equation $P(x) = 0$ has at least one complex solution (root).

- A **corollary** is a theorem that accompanies another theorem and is usually easily deduced from the other theorem.

Corollary: Corollary to the Fundamental Theorem of Algebra
If $P(x)$ is a polynomial function of degree $n \geq 1$ with complex coefficients, then the related equation $P(x) = 0$ has exactly n complex solutions (roots) if a **double solution**, or two solutions that are equal, is counted as two separate solutions.

- The theorem and corollary apply to polynomials with all real coefficients, since all real numbers are complex numbers.
- A quadratic function is a polynomial function of degree 2, so by the Corollary to the Fundamental Theorem of Algebra, a quadratic function has exactly 2 complex solutions (roots).

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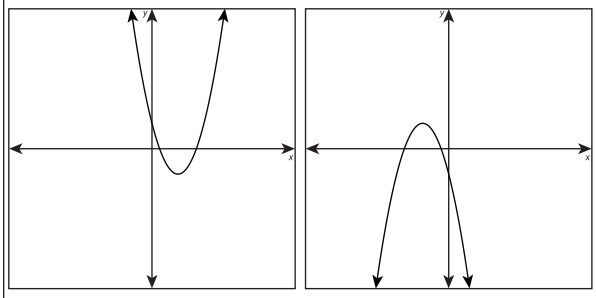
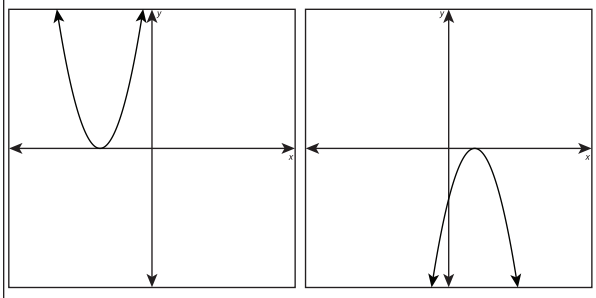
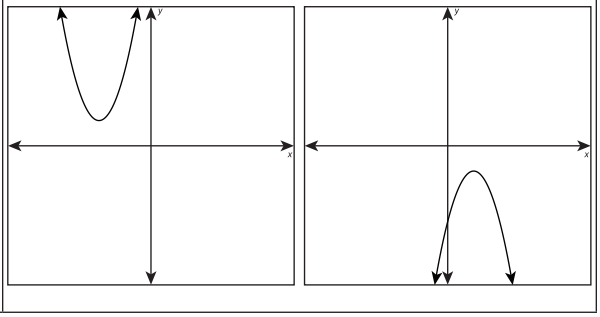
Instruction

- The quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, is used to find those solutions.
- Recall that the expression $b^2 - 4ac$ in the quadratic formula is called the discriminant. The discriminant can be positive, zero, or negative. Each case indicates whether the solutions of $ax^2 + bx + c = 0$ are real or complex, as shown in the following table.

Solutions to Quadratic Equations in the Form $ax^2 + bx + c = 0$

Value of the discriminant	Description of the solutions	Graph of $f(x) = ax^2 + bx + c$
$b^2 - 4ac > 0$	Two real solutions (the x-intercepts); they are different.	
$b^2 - 4ac = 0$	One real solution (the x-intercept); it is a double solution, counted as two solutions by the Fundamental Theorem of Algebra.	
$b^2 - 4ac < 0$	Two complex solutions (no x-intercepts); they are a conjugate pair of the form $a + bi$ and $a - bi$.	

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Instruction

- The following properties involving square roots are important and useful to remember when finding solutions to quadratic equations:

Property	Example
If $x^2 = a$, then $x = \pm\sqrt{a}$.	If $x^2 = 25$, then $x = 5$ or $x = -5$.
For any real numbers a and b with $b \neq 0$, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.	$\sqrt{\frac{36}{9}} = \frac{\sqrt{36}}{\sqrt{9}} = \frac{6}{3} = 2$
For any real numbers a and b, $\sqrt{ab} = \sqrt{a} \bullet \sqrt{b}$.	$\sqrt{9 \bullet 16} = \sqrt{9} \bullet \sqrt{16} = 3 \bullet 4 = 12$
For any positive real number a, $\sqrt{-a} = i\sqrt{a}$.	$\sqrt{-16} = i \bullet \sqrt{16} = i \bullet 4 = 4i$

- The **zeros** of a function $f(x)$ are the x-values of a function for which the function value is 0. The roots or solutions of the equation tell which x-values will make the equation true.

Common Errors/Misconceptions

- using the wrong sign when calculating the discriminant
- writing the value of i^2 as 1 instead of –1

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Guided Practice 3.4.2

Example 1

For each equation, state the number and type of solutions by first finding the discriminant.

$2x^2 + 3x - 5 = 0$

$-0.4x^2 - x = 0$

$3x^2 = 2x - 5$

$9x^2 + 1 = 6x$

1. Find the discriminant of $2x^2 + 3x - 5 = 0$.

   The equation $2x^2 + 3x - 5 = 0$ is in the form $ax^2 + bx + c = 0$ with $a = 2$, $b = 3$, and $c = -5$.

   Use the discriminant formula.

$b^2 - 4ac$	Discriminant formula
$(3)^2 - 4(2)(-5)$	Substitute 2 for a, 3 for b, and –5 for c.
$9 + 40$	Simplify.
49	

   The discriminant of $2x^2 + 3x - 5 = 0$ is 49.

   The discriminant is positive, so the equation has two real solutions.

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Instruction

2. Find the discriminant of $-0.4x^2 - x = 0$.

The equation $-0.4x^2 - x = 0$ is in the form $ax^2 + bx + c = 0$ with $a = -0.4$, $b = -1$, and $c = 0$.

Use the discriminant formula.

$b^2 - 4ac$	Discriminant formula
$(-1)^2 - 4(-0.4)(0)$	Substitute -0.4 for a, -1 for b, and 0 for c.
$(-1)^2 + 0$	Simplify.
1	

The discriminant of $-0.4x^2 - x = 0$ is 1.

The discriminant is positive, so the equation has two real solutions.

3. Find the discriminant of $3x^2 = 2x - 5$.

The equation $3x^2 = 2x - 5$ is not in the form $ax^2 + bx + c = 0$, so begin by writing it in that form.

$3x^2 = 2x - 5$	Original equation
$3x^2 - 2x = -5$	Subtract $2x$ from both sides.
$3x^2 - 2x + 5 = 0$	Add 5 to both sides.

The equation $3x^2 - 2x + 5 = 0$ is in the form $ax^2 + bx + c = 0$, with $a = 3$, $b = -2$, and $c = 5$.

Use the discriminant formula.

$b^2 - 4ac$	Discriminant formula
$(-2)^2 - 4(3)(5)$	Substitute 3 for a, -2 for b, and 5 for c.
$4 - 60$	Simplify.
-56	

The discriminant of $3x^2 = 2x - 5$ is -56.

The discriminant is negative, so the equation has two complex solutions, which together are a complex conjugate pair.

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Instruction

4. Find the discriminant of $9x^2 + 1 = 6x$.

   The equation $9x^2 + 1 = 6x$ is not in the form $ax^2 + bx + c = 0$, so begin by writing it in that form.

$9x^2 + 1 = 6x$	Original equation
$9x^2 - 6x + 1 = 0$	Subtract $6x$ from both sides.

   The equation $9x^2 - 6x + 1 = 0$ is in the form $ax^2 + bx + c = 0$ with $a = 9$, $b = -6$, and $c = 1$.

   Use the discriminant formula.

$b^2 - 4ac$	Discriminant formula
$(-6)^2 - 4(9)(1)$	Substitute 9 for a, –6 for b, and 1 for c.
$36 - 36$	Simplify.
0	

5. The discriminant of $9x^2 + 1 = 6x$ is equal to 0, so the equation has one real solution, which is a double root.

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Example 2

Solve $\frac{1}{2}x^2+\frac{6}{5}=0$ by using a property of square roots. Show that the quadratic formula gives the same solutions. Check your solutions. Explain how the graph of the related function supports the type of solutions.

1. Determine which property of square roots to use.

   There is no x-term, so you can solve the equation by using this property of square roots: If $x^2 = a$, then $x=\pm\sqrt{a}$.

$\frac{1}{2}x^2+\frac{6}{5}=0$	Original equation
$\frac{1}{2}x^2=-\frac{6}{5}$	Subtract $\frac{6}{5}$ from both sides.
$x^2=-\frac{12}{5}$	Multiply both sides by 2.
$x=\pm\sqrt{-\frac{12}{5}}$	Apply this property of square roots: If $x^2 = a$, then $x=\pm\sqrt{a}$.
$x=\pm i\sqrt{\frac{12}{5}}$	For any positive real number a, $\sqrt{-a}=i\sqrt{a}$.

   The solutions to $\frac{1}{2}x^2+\frac{6}{5}=0$ are $\pm i\sqrt{\frac{12}{5}}$.

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Instruction

2. Write the radical in simplest form.

$x = \pm i\dfrac{\sqrt{12}}{\sqrt{5}}$ For any real numbers a and b with $b \neq 0$, $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$.

$x = \pm i\dfrac{\sqrt{12}}{\sqrt{5}} \bullet \dfrac{\sqrt{5}}{\sqrt{5}}$ Multiply by $\dfrac{\sqrt{5}}{\sqrt{5}}$, which is equal to 1.

$x = \pm i\dfrac{\sqrt{12 \bullet 5}}{\sqrt{5 \bullet 5}}$ For any real numbers a and b, $\sqrt{a} \bullet \sqrt{b} = \sqrt{ab}$.

$x = \pm i\dfrac{\sqrt{60}}{\sqrt{25}}$ Simplify.

$x = \pm i\dfrac{\sqrt{60}}{5}$ Simplify the denominator.

$x = \pm i\dfrac{\sqrt{4 \bullet 15}}{5}$ Factor 60 to show a perfect square factor.

$x = \pm i\dfrac{\sqrt{4} \bullet \sqrt{15}}{5}$ For any real numbers a and b, $\sqrt{a} \bullet \sqrt{b} = \sqrt{ab}$.

$x = \pm\dfrac{2\sqrt{15}}{5}i$ Simplify.

The solutions to $\dfrac{1}{2}x^2 + \dfrac{6}{5} = 0$, written in simplest form, are $\dfrac{2\sqrt{15}}{5}i$ and $-\dfrac{2\sqrt{15}}{5}i$.

These solutions are a conjugate pair and can be written as $0 + \dfrac{2\sqrt{15}}{5}i$ and $0 - \dfrac{2\sqrt{15}}{5}i$.

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Instruction

3. Show that using the quadratic formula gives the same solutions.

The quadratic formula states that if $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$\frac{1}{2}x^2 + \frac{6}{5} = 0$	Write the given equation.
$5x^2 + 12 = 0$	Multiply both sides by 10, the least common denominator, to eliminate fractions.
$x = \frac{-0 \pm \sqrt{0^2 - 4(5)(12)}}{2(5)}$	Substitute values for a, b, and c: $a = 5$, $b = 0$, $c = 12$.
$x = \frac{\pm\sqrt{-240}}{10}$	Simplify.
$x = \frac{\pm i\sqrt{240}}{10}$	For any positive real number a, $\sqrt{-a} = i\sqrt{a}$.
$x = \frac{\pm i\sqrt{16 \bullet 15}}{10}$	Factor 240 to show its greatest perfect square factor, 16.
$x = \frac{\pm i\sqrt{16} \bullet \sqrt{15}}{10}$	For any real numbers a and b, $\sqrt{ab} = \sqrt{a} \bullet \sqrt{b}$.
$x = \frac{\pm 4i\sqrt{15}}{10}$	Simplify.
$x = \frac{\pm 2i\sqrt{15}}{5}$	

The solutions to $\frac{1}{2}x^2 + \frac{6}{5} = 0$, found using the quadratic formula, are $\frac{2\sqrt{15}}{5}i$ and $-\frac{2\sqrt{15}}{5}i$, matching those found in step 1.

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Instruction

4. Check the solutions.

Substitute each solution into the original equation, $\frac{1}{2}x^2+\frac{6}{5}=0$.

Check $\frac{2\sqrt{15}}{5}i$:

$$\frac{1}{2}x^2+\frac{6}{5}=0$$

$$\frac{1}{2}\left(\frac{2\sqrt{15}}{5}i\right)^2+\frac{6}{5}=0$$

$$\frac{1}{2}\left(\frac{4\bullet 15}{25}\right)i^2+\frac{6}{5}=0$$

$$\frac{1}{2}\left(\frac{4\bullet 15}{25}\right)(-1)+\frac{6}{5}=0$$

$$-\frac{60}{50}+\frac{6}{5}=0$$

$$-\frac{6}{5}+\frac{6}{5}=0$$

$$0=0$$

Check $-\frac{2\sqrt{15}}{5}i$:

$$\frac{1}{2}x^2+\frac{6}{5}=0$$

$$\frac{1}{2}\left(-\frac{2\sqrt{15}}{5}i\right)^2+\frac{6}{5}=0$$

$$\frac{1}{2}\left(\frac{4\bullet 15}{25}\right)i^2+\frac{6}{5}=0$$

$$\frac{1}{2}\left(\frac{4\bullet 15}{25}\right)(-1)+\frac{6}{5}=0$$

$$-\frac{60}{50}+\frac{6}{5}=0$$

$$-\frac{6}{5}+\frac{6}{5}=0$$

$$0=0$$

5. Graph the related quadratic function $f(x)=\frac{1}{2}x^2+\frac{6}{5}$ on a graphing calculator.

On a TI-83/84:

Step 1: Press [Y=].

Step 2: At Y_1, type in [(][1][÷][2][)][X,T,θ,n][x^2][+][(][6][÷][5][)].

Step 3: Press [WINDOW] to change the viewing window.

Step 4: At Xmin, enter [(–)][9].

Step 5: At Xmax, enter [9].

Step 6: At Xscl, enter [1].

(*continued*)

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Step 7: At Ymin, enter [(–)][6].

Step 8: At Ymax, enter [6].

Step 9: At Yscl, enter [1].

Step 10: Press [GRAPH].

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon and press [enter].

Step 3: Enter the equation [(][1][÷][2][)][×][x^2][+][(][6][÷][5][)] and press [enter].

Step 4: Change the viewing window by pressing [menu], then arrow down to number 4: Window/Zoom, and click the center button of the navigation pad.

Step 5: Choose 1: Window Settings by pressing the center button.

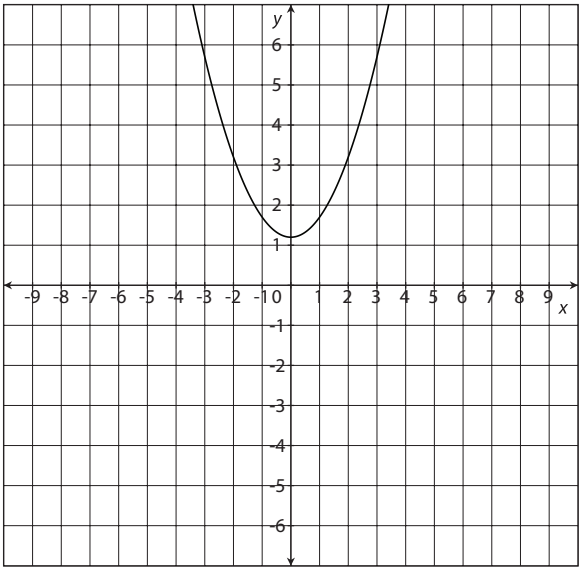
Step 6: Enter in the appropriate XMin value, [–][9], then press [tab].

Step 7: Enter the appropriate XMax value, [9], then press [tab].

Step 8: Leave the XScale set to "auto." Press [tab] twice to navigate to YMin and enter [–][6].

Step 9: Press [tab] to navigate to YMax. Enter [6]. Press [tab] twice to leave YScale set to "auto" and to navigate to "OK."

Step 10: Press [enter].



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Instruction

6. Explain how the graph of the related quadratic function, $f(x)=\frac{1}{2}x^2+\frac{6}{5}$, supports the type of solutions.

The graph of $f(x)=\frac{1}{2}x^2+\frac{6}{5}$ has no x-intercepts, which verifies that the equation $\frac{1}{2}x^2+\frac{6}{5}=0$ has no real solutions.

Example 3

Solve $-x^2 + 6x - 13 = 0$ by using the quadratic formula. Check your solutions. Explain how the graph of the related function supports the type of solutions.

1. Substitute values for the equation into the quadratic formula.

The quadratic formula states that if $ax^2 + bx + c = 0$, then $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

$-x^2 + 6x - 13 = 0$ is written in the form $ax^2 + bx + c = 0$; therefore, $a = -1$, $b = 6$, and $c = -13$.

$-x^2 + 6x - 13 = 0$	Original equation
$x=\frac{-6\pm\sqrt{6^2-4(-1)(-13)}}{2(-1)}$	Substitute values for a, b, and c into the formula: $a = -1$, $b = 6$, $c = -13$.
$x=\frac{-6\pm\sqrt{36-52}}{-2}$	Simplify.
$x=\frac{-6\pm\sqrt{-16}}{-2}$	
$x=\frac{-6\pm i\sqrt{16}}{-2}$	For any positive real number a, $\sqrt{-a}=i\sqrt{a}$.
$x=\frac{-6\pm 4i}{-2}$	Simplify.
$x=\frac{-6}{-2}\pm\frac{4i}{-2}$	
$x=3\pm 2i$	

The solutions of $-x^2 + 6x - 13 = 0$ are $3 + 2i$ and $3 - 2i$.

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Instruction

2. Check the solutions.

Substitute the solutions into the original formula.

Check $3 + 2i$:

$-x^2 + 6x - 13 = 0$

$-(3+2i)^2 + 6(3+2i) - 13 = 0$

$-(3+2i)(3+2i) + 6(3+2i) - 13 = 0$

$-(9+6i+6i+4i^2) + 6(3+2i) - 13 = 0$

$-(9+12i+4i^2) + 6(3+2i) - 13 = 0$

$-9 - 12i - 4i^2 + 18 + 12i - 13 = 0$

$-9 - 12i + 4 + 18 + 12i - 13 = 0$

$-9 + 4 + 18 - 13 - 12i + 12i = 0$

$0 = 0$

Chec $3 - 2i$:

$-x^2 + 6x - 13 = 0$

$-(3-2i)^2 + 6(3-2i) - 13 = 0$

$-(3-2i)(3-2i) + 6(3-2i) - 13 = 0$

$-(9-6i-6i+4i^2) + 6(3-2i) - 13 = 0$

$-(9-12i+4i^2) + 6(3-2i) - 13 = 0$

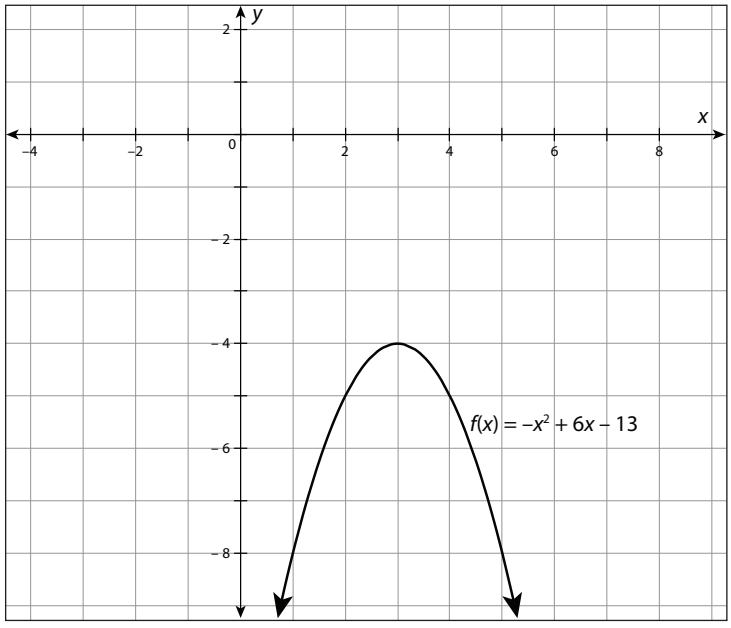
$-9 + 12i - 4i^2 + 18 - 12i - 13 = 0$

$-9 + 12i + 4 + 18 - 12i - 13 = 0$

$-9 + 4 + 18 - 13 + 12i - 12i = 0$

$0 = 0$

3. Explain how the graph of the related quadratic function, $f(x) = -x^2 + 6x - 13$, supports the type of solutions.



The graph of $f(x) = -x^2 + 6x - 13$ has no x-intercepts, which verifies that the equation $-x^2 + 6x - 13 = 0$ has no real solutions, and has two complex solutions.

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Instruction

Example 4

Solve $2x^2 + 9 = -8x$ by using the quadratic formula. Check your solutions. Explain how the graph of the related function supports the type of solutions.

1. Substitute values for the equation into the quadratic formula.

The quadratic formula states that if $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$2x^2 + 9 = -8x$	Original equation
$2x^2 + 8x + 9 = 0$	Add $8x$ to both sides to get the form $ax^2 + bx + c = 0$.
$x = \frac{-8 \pm \sqrt{8^2 - 4(2)(9)}}{2(2)}$	Substitute values for a, b, and c into the formula: $a = 2$, $b = 8$, $c = 9$.
$x = \frac{-8 \pm \sqrt{64 - 72}}{4}$	Simplify.
$x = \frac{-8 \pm \sqrt{-8}}{4}$	
$x = \frac{-8 \pm i\sqrt{8}}{4}$	For any positive real number a, $\sqrt{-a} = i\sqrt{a}$.
$x = \frac{-8 \pm 2i\sqrt{2}}{4}$	Simplify.
$x = \frac{-8}{4} \pm \frac{2i\sqrt{2}}{4}$	
$x = -2 \pm \frac{\sqrt{2}}{2}i$	

The solutions of $2x^2 + 9 = -8x$ are $-2 + \frac{\sqrt{2}}{2}i$ and $-2 - \frac{\sqrt{2}}{2}i$.

2. Check the solutions.

Substitute the solutions into $2x^2 + 8x + 9 = 0$.

(*continued*)

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Check $-2+\frac{\sqrt{2}}{2}i$:

$2x^2+8x+9=0$

$2\left(-2+\frac{\sqrt{2}}{2}i\right)^2+8\left(-2+\frac{\sqrt{2}}{2}i\right)+9=0$

$2\left(-2+\frac{\sqrt{2}}{2}i\right)\left(-2+\frac{\sqrt{2}}{2}i\right)+8\left(-2+\frac{\sqrt{2}}{2}i\right)+9=0$

$2\left(4-i\sqrt{2}-i\sqrt{2}+\frac{2}{4}i^2\right)+8\left(-2+\frac{\sqrt{2}}{2}i\right)+9=0$

$2\left(4-2i\sqrt{2}-\frac{1}{2}\right)+8\left(-2+\frac{\sqrt{2}}{2}i\right)+9=0$

$8-4i\sqrt{2}-1-16+4i\sqrt{2}+9=0$

$8-1-16+9-4i\sqrt{2}+4i\sqrt{2}=0$

$0=0$

Check $-2-\frac{\sqrt{2}}{2}i$:

$2x^2+8x+9=0$

$2\left(-2-\frac{\sqrt{2}}{2}i\right)^2+8\left(-2-\frac{\sqrt{2}}{2}i\right)+9=0$

$2\left(-2-\frac{\sqrt{2}}{2}i\right)\left(-2-\frac{\sqrt{2}}{2}i\right)+8\left(-2-\frac{\sqrt{2}}{2}i\right)+9=0$

$2\left(4+i\sqrt{2}+i\sqrt{2}+\frac{2}{4}i^2\right)+8\left(-2-\frac{\sqrt{2}}{2}i\right)+9=0$

$2\left(4+2i\sqrt{2}-\frac{1}{2}\right)+8\left(-2-\frac{\sqrt{2}}{2}i\right)+9=0$

$8+4i\sqrt{2}-1-16-4i\sqrt{2}+9=0$

$8-1-16+9+4i\sqrt{2}-4i\sqrt{2}=0$

$0=0$

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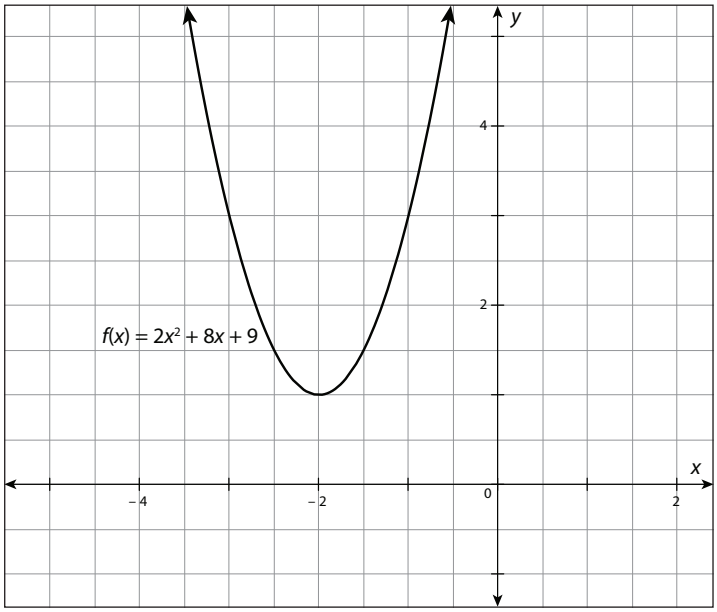
Instruction

3. Explain how the graph of the related quadratic function, $f(x) = 2x^2 + 8x + 9$, supports the type of solutions.



The graph of $f(x) = 2x^2 + 8x + 9$ has no x-intercepts, which verifies that the equation $2x^2 + 8x + 9 = 0$ has no real solutions, and has two complex solutions.

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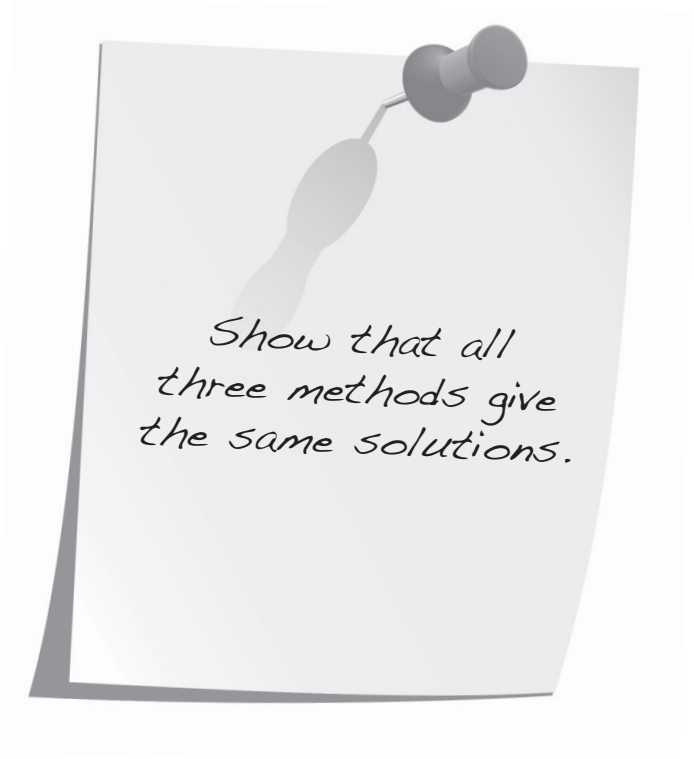
Problem-Based Task 3.4.2: All Roads Lead to the Same Destination

SMP	
1	2 ✓
3 ✓	4
5	6
7 ✓	8

Solve $3x^2 + 5 = 0$ by the following three methods. Show that all three methods give the same solutions.

- Use the quadratic formula.
- Use this property of square roots: If $x^2 = a$, then $x = \pm\sqrt{a}$.
- Show through factoring that $3x^2 + 5$ can be written as a sum of two squares.



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Problem-Based Task 3.4.2: All Roads Lead to the Same Destination

Coaching

a. The quadratic formula states that if $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. What are the values of a, b, and c in the equation $3x^2 + 5 = 0$?

b. How can you solve the equation with the quadratic formula?

c. Recall that if $x^2 = a$, then $x = \pm\sqrt{a}$. How can you solve the equation by using this property of square roots? How can you show that your solutions by this method match the solutions in part b?

d. How can you solve the equation by factoring? (*Hint*: Write the given equation. Then write the equation again, showing that $3x^2 + 5$ is the sum of two squares.)

e. How can you apply this identity that shows how to factor the sum of two squares: $a^2 + b^2 = (a + bi)(a - bi)$?

f. The Zero Product Property allows you to state that if $(a + bi)(a - bi) = 0$, then either $a + bi = 0$ or $a - bi = 0$. How can you use the Zero Product Property to finish solving and then write your solutions in the form that matches the solutions in parts b and c?

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Problem-Based Task 3.4.2: All Roads Lead to the Same Destination

Coaching Sample Responses

a. The quadratic formula states that if $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. What are the values of a, b, and c in the equation $3x^2 + 5 = 0$?

Since the equation is already written in standard form, $a = 3$, $b = 0$, and $c = 5$.

b. How can you solve the equation by the quadratic formula?

$3x^2 + 5 = 0$	Write the given equation.
$x = \frac{-0 \pm \sqrt{0^2 - 4(3)(5)}}{2(3)}$	Substitute values for a, b, and c: $a = 3$, $b = 0$, $c = 5$.
$x = \frac{\pm\sqrt{-60}}{6}$	Simplify.
$x = \frac{\pm i\sqrt{60}}{6}$	For any positive real number a, $\sqrt{-a} = i\sqrt{a}$.
$x = \frac{\pm 2i\sqrt{15}}{6}$	Simplify.
$x = \pm\left(\frac{\sqrt{15}}{3}\right)i$	

c. Recall that if $x^2 = a$, then $x = \pm\sqrt{a}$. How can you solve the equation by using this property of square roots? How can you show that your solutions by this method match the solutions in part b?

$3x^2 + 5 = 0$	Write the given equation.
$3x^2 = -5$	Subtract 5 from both sides.
$x^2 = -\frac{5}{3}$	Divide both sides by 3.
$x = \pm\sqrt{-\frac{5}{3}}$	Apply this property of square roots: If $x^2 = a$, then $x = \pm\sqrt{a}$.

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$x = \pm i\sqrt{\frac{5}{3}}$	For any positive real number a, $\sqrt{-a} = i\sqrt{a}$.
$x = \pm i\frac{\sqrt{5}}{\sqrt{3}}$	For any real numbers a and b with $b \neq 0$, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.
$x = \pm i\frac{\sqrt{5}}{\sqrt{3}} \bullet \frac{\sqrt{3}}{\sqrt{3}}$	Multiply by $\frac{\sqrt{3}}{\sqrt{3}}$, which is equal to 1.
$x = \pm i\frac{\sqrt{5 \bullet 3}}{\sqrt{3 \bullet 3}}$	For any real numbers a and b, $\sqrt{a} \bullet \sqrt{b} = \sqrt{ab}$.
$x = \pm i\frac{\sqrt{15}}{\sqrt{9}}$	Simplify.
$x = \pm i\frac{\sqrt{15}}{3}$	
$x = \pm\left(\frac{\sqrt{15}}{3}\right)i$	

d. How can you solve the equation by factoring? (*Hint*: Write the given equation. Then write the equation again, showing that $3x^2 + 5$ is the sum of two squares.)

$3x^2 + 5 = 0$	Write the given equation.
$\left(x\sqrt{3}\right)^2 + \left(\sqrt{5}\right)^2 = 0$	Show that $3x^2 + 5$ is the sum of two squares.

e. How can you apply this identity that shows how to factor the sum of two squares: $a^2 + b^2 = (a + bi)(a - bi)$?

$3x^2 + 5 = 0$	Given equation
$\left(x\sqrt{3}\right)^2 + \left(\sqrt{5}\right)^2 = 0$	Show that $3x^2 + 5$ is the sum of two squares.
$\left(x\sqrt{3} + i\sqrt{5}\right)\left(x\sqrt{3} - i\sqrt{5}\right) = 0$	Apply the identity $a^2 + b^2 = (a + bi)(a - bi)$.

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Instruction

f. The Zero Product Property allows you to state that if $(a + bi)(a - bi) = 0$, then either $a + bi = 0$ or $a - bi = 0$. How can you use the Zero Product Property to finish solving and then write your solutions in the form that matches the solutions in parts b and c?

Start with the factorization found in part e.

$\left(x\sqrt{3}+i\sqrt{5}\right)\left(x\sqrt{3}-i\sqrt{5}\right)=0$ $\quad a^2 + b^2 = (a + bi)(a - bi)$

$x\sqrt{3}+i\sqrt{5}=0 \text{ or } x\sqrt{3}-i\sqrt{5}=0$ $\quad$ If $(a + bi)(a - bi) = 0$, then either $a + bi = 0$ or $a - bi = 0$.

$x\sqrt{3}=-i\sqrt{5} \text{ or } x\sqrt{3}=i\sqrt{5}$

$x=-\frac{\sqrt{5}}{\sqrt{3}}i \text{ or } x=\frac{\sqrt{5}}{\sqrt{3}}i$

$x=-\frac{\sqrt{5}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}i \text{ or } x=\frac{\sqrt{5}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}i$

$x=-\left(\frac{\sqrt{15}}{3}\right)i \text{ or } x=\left(\frac{\sqrt{15}}{3}\right)i$

$x=\pm\left(\frac{\sqrt{15}}{3}\right)i$

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.4.2: Solving Quadratic Equations with Complex Solutions

For problems 1–4, find the discriminant and state the number and type of solutions.

1. $x^2 - x + 2 = 0$

2. $4x^2 - 4x + 1 = 0$

3. $16x^2 - 1 = 4x$

4. $0.3x^2 = x - 1$

Use the given information to complete problems 5–7.

5. Use what you know about quadratic equations to complete the following.

   a. Explain why every quadratic equation has exactly two solutions (counting a double solution as two solutions).

   b. Explain why every quadratic equation has exactly two *complex* solutions.

continued

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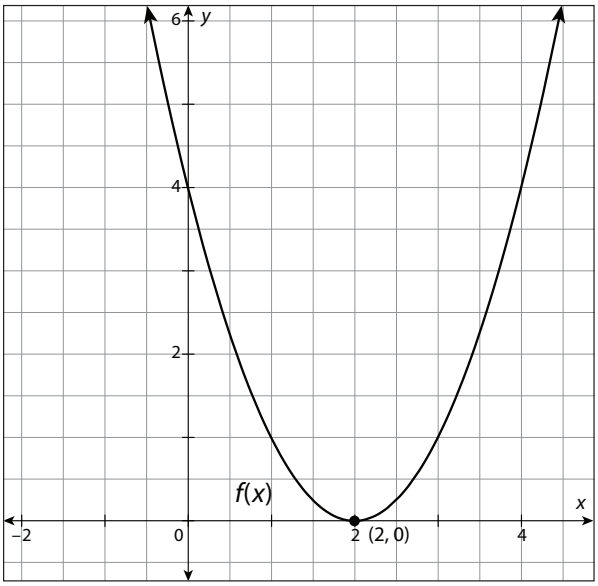
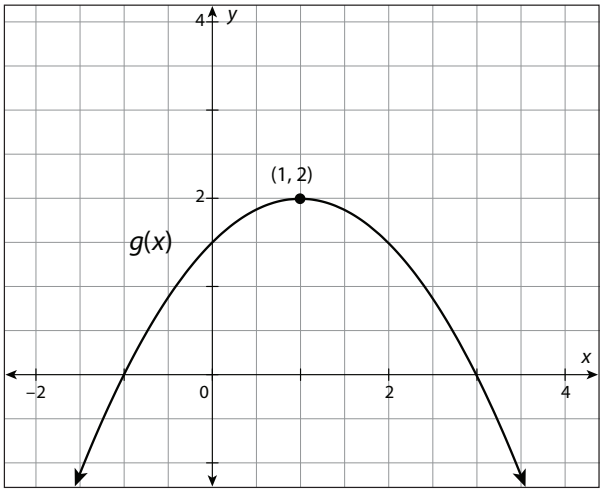
6. The graphs of quadratic functions f and g are shown below, with each vertex labeled. For each function, state the number and type of zeros. For each function, describe what you can conclude about the value of $b^2 - 4ac$, where the rule for the function is in the form $ax^2 + bx + c$.





7. What are the zeros of $f(x) = 2x^2 + 9$?

Solve the equations in problems 8–10.

8. $\frac{1}{3}x^2 + 1 = 0$

9. $x^2 - 10x + 29 = 0$

10. $-3x^2 + 2x = 4$

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Progress Assessment

Circle the letter of the best answer.

1. Which equation is an identity?

   a. $i(x + i) = xi - 1$
   b. $i(x - 2i) = xi - 2$
   c. $(4x - i)(4x + i) = 15$
   d. $(3x + i)(3x + 2i) = 7x + 9i$

2. Which equation is not an identity?

   a. $x^2 + i^2 = (x + i)(x - i)$
   b. $x - i^2 + x = 2x + 1$
   c. $5 - 2i + 3i^2 = 2 - 2i$
   d. $5 + 2i + 3i^2 = 2 + 2i$

3. Which expression is equal to $25p^2 + 81$ for all real values of p?

   a. $(5p - 9i)^2$
   b. $(5p + 9i)^2$
   c. $(5p + 9i)(5p - 9i)$
   d. $(5p + 9i^2)(5p - 9i^2)$

4. Which expression is equal to $5x^2 + 180$ for all real values of x?

   a. $5(x^2 - 36i)$
   b. $5(x^2 + 36i)$
   c. $(5x + 6i)(5x - 6i)$
   d. $5(x + 6i)(x - 6i)$

5. Which expression is equal to $x^2 - 6x + 25$ for all real values of x?

   a. $(x + 3 - 4i)(x + 3 + 4i)$
   b. $(x - 3 - 4i)(x - 3 + 4i)$
   c. $\left[x+3-\left(\frac{\sqrt{94}}{2}\right)i\right]\left[x+3+\left(\frac{\sqrt{94}}{2}\right)i\right]$
   d. $\left[x-3-\left(\frac{\sqrt{94}}{2}\right)i\right]\left[x-3+\left(\frac{\sqrt{94}}{2}\right)i\right]$

6. Which option describes the solution(s) of the equation $2x^2 + 3x + 2 = 0$?

   a. It has a double solution, which is real.
   b. It has a double solution, which is complex.
   c. It has two real solutions that are different.
   d. It has two complex solutions that are different.

continued

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Assessment

7. Which is a solution of the equation $\frac{1}{3}x^2+9=0$?

   a. $-\sqrt{3}$
   b. $-\left(\sqrt{3}\right)i$
   c. $-3\sqrt{3}$
   d. $-\left(3\sqrt{3}\right)i$

8. Which is a solution of the equation $3x^2 + 1 = 2x$?

   a. $-1+\left(\sqrt{2}\right)i$
   b. $1+\left(\sqrt{2}\right)i$
   c. $-\frac{1}{3}+\left(\frac{\sqrt{2}}{3}\right)i$
   d. $\frac{1}{3}+\left(\frac{\sqrt{2}}{3}\right)i$

9. What are the solutions of the equation $2x^2 = 3x - 4$?

   a. $\frac{-3\pm i\sqrt{23}}{4}$
   b. $\frac{3\pm i\sqrt{23}}{4}$
   c. $\frac{-3\pm i\sqrt{29}}{4}$
   d. $\frac{3\pm i\sqrt{29}}{4}$

10. What are the zeros of the function $f(x) = x^2 + 49$?

    a. –7 and 7
    b. $-\sqrt{7}$ and $\sqrt{7}$
    c. $-7i$ and $7i$
    d. $\left(-\sqrt{7}\right)i$ and $\left(\sqrt{7}\right)i$

Use what you have learned in this lesson to complete problem 11.

11. A corollary to the Fundamental Theorem of Algebra states if $P(x)$ is a polynomial function of degree $n \geq 1$ with complex coefficients, then the related equation $P(x) = 0$ has exactly n complex solutions (roots), if a double solution is counted as two separate solutions. Explain how the quadratic formula shows that this corollary is true for all quadratic polynomial functions with real coefficients.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Tariq and Charlene are on a school maintenance crew. They each use a different mower to cut the grass. Tariq can mow the athletic field in 1 hour 20 minutes if he works alone. Tariq and Charlene can mow the field in 30 minutes if they work together. How long does it take Charlene to mow the field if she works alone? Assume both people work at steady rates and their rates are not affected by either working alone or working together.

   a. 40 minutes

   b. 48 minutes

   c. 50 minutes

   d. 55 minutes

2. Micah drove 40 miles in the city, averaging 28 miles per gallon. Then he drove 220 miles on highways. For the entire 260 miles of city and highway driving, he averaged 34 miles per gallon. If x represents his average fuel consumption rate on highways, in miles per gallon, which equation can be solved to find this rate?

   a. $\frac{40}{28}+\frac{220}{x}=\frac{260}{34}$

   b. $\frac{40}{28}+\frac{x}{220}=\frac{34}{260}$

   c. $\frac{28}{40}+\frac{220}{x}=\frac{260}{34}$

   d. $\frac{28}{40}+\frac{x}{220}=\frac{34}{260}$

continued

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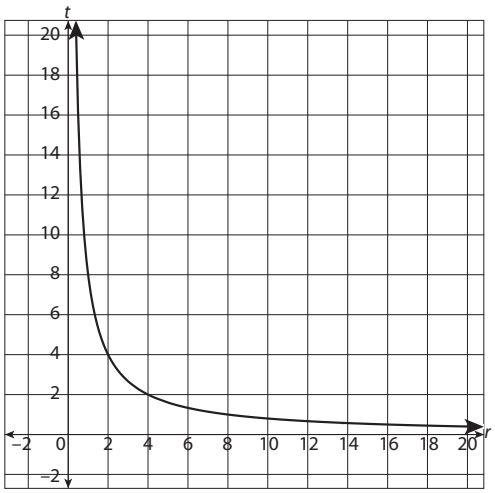
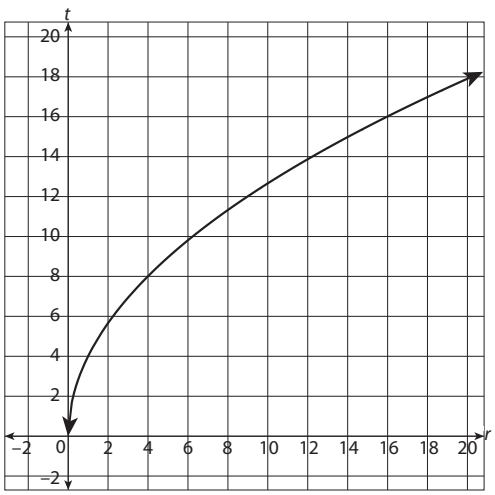
Assessment

3. Carlita is on a 16-kilometer hike. Which graph shows the relationship between r, her average speed in kilometers per hour, and t, the time in hours it will take her to complete the hike?

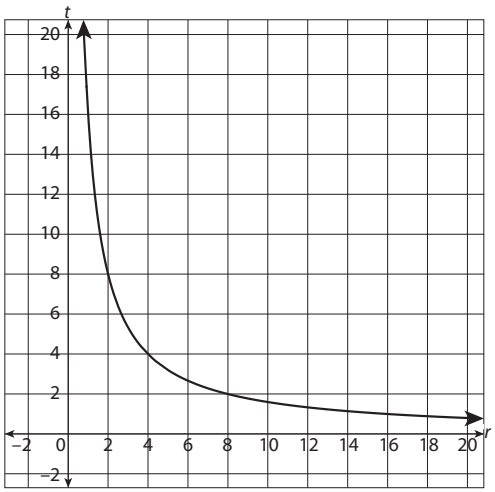
a.



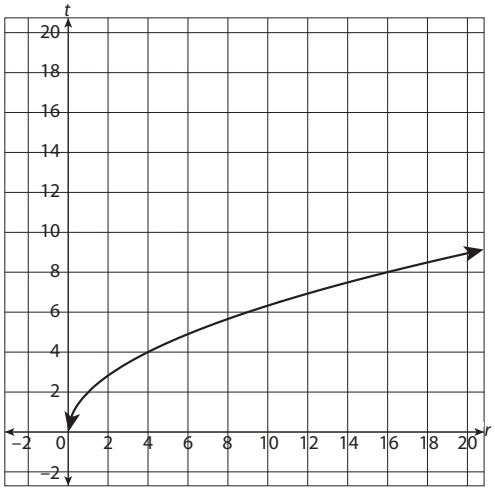
c.



b.



d.



continued

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4. Which option shows the solution set of $\frac{-3x}{x+2} > 3$?
   a. $-2 < x < -1$
   b. $-2 < x < 0$
   c. $x < -2$ or $x > -1$
   d. $x < -2$ or $x > 0$

5. Levi has 15 feet 10 inches of wood molding. He wants to make a rectangular window with an area of 2,000 square inches and use the molding to frame the window. He wrote a rational inequality that satisfies the requirements and then wrote a description of all possible values of w, the width of the window in inches. Which option shows Levi's inequality and description?
   a. $\frac{2000}{w} + w \le 95;\ 47.5 - 2.5\sqrt{41} \le w \le 47.5 + 2.5\sqrt{41}$
   b. $\frac{2000}{w} + w \le 95;\ 47.5 - 2.5\sqrt{281} \le w \le 47.5 + 2.5\sqrt{281}$
   c. $\frac{2000}{w} + w^2 \le 95;\ 47.5 - 2.5\sqrt{41} \le w \le 47.5 + 2.5\sqrt{41}$
   d. $\frac{2000}{w} + w^2 \le 95;\ 47.5 - 2.5\sqrt{281} \le w \le 47.5 + 2.5\sqrt{281}$

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Lesson 5: Rational Equations

Instruction

Common Core State Standards

SMP	
1 ✓	2 ✓
3	4 ✓
5	6 ✓
7 ✓	8

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

A–CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

Essential Questions

1. How can a rational equation in one variable be used to solve a problem?
2. How can a rational equation in two variables represent a relationship between two quantities?
3. How can a rational inequality in one variable be used to solve a problem?

WORDS TO KNOW

asymptote	a line that a function gets closer and closer to as one of the variables increases or decreases without bound
closed interval	an interval that includes its endpoints
critical number of a polynomial inequality	an x-value that makes $f(x) = 0$, where $f(x)$ is a polynomial function and the inequality is written in any of these forms: $f(x) < 0$, $f(x) \leq 0$, $f(x) > 0$, or $f(x) \geq 0$
critical number of a rational inequality	an x-value that makes $f(x) = 0$ or makes $f(x)$ undefined, where $f(x)$ is a rational function and the inequality is written in any of these forms: $f(x) < 0$, $f(x) \leq 0$, $f(x) > 0$, or $f(x) \geq 0$
dependent variable	generally labeled on the y-axis; the quantity that is based on the input values of the independent variable; the output variable of a function
domain	the set of all input values for which a relation or function is defined; the set of x-values that are valid for a relation or function
end behavior	the behavior of the graph as x approaches positive infinity and as x approaches negative infinity

SRB p. U3-181

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Instruction

extraneous solution (extraneous root) of an equation	a solution of an equation that arises during the solving process, but which is not a solution of the original equation
family of functions	a set of functions whose graphs have the same general shape as their parent function. The parent function is the function that represents the family of functions.
half-closed interval	an interval that includes one endpoint but not the other; also called a half-open interval
half-open interval	an interval that includes one endpoint but not the other; also called a half-closed interval
horizontal asymptote	a line defined as follows: The line $y = b$ is a horizontal asymptote of the graph of a function f if $f(x)$ gets closer to b as x either increases or decreases without bound.
independent variable	labeled on the x-axis; the quantity for which values are chosen; the input variable of a function
infinity	going on without bound; represented by the symbol ∞
interval	continuous the set of all real numbers between two given numbers
interval notation	a way of representing an interval using a pair of parentheses, a pair of brackets, or a parenthesis and a bracket
least common denominator (LCD) of fractions	the least common multiple of the denominators of the fractions
least common multiple (LCM) of polynomials	with two or more polynomials, the common multiple of the polynomials that has the least degree and the least positive constant factor
open interval	an interval that does not include its endpoints
parent function	a function that represents a family of functions. The graphs of the functions in the family have the same general shape as the parent function.

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Instruction

quadratic formula — a formula that states that the solutions of a quadratic equation of the form $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

range — the set of all outputs of a relation or function; the set of *y*-values for which a function is defined

rate — a ratio that compares measurements with different kinds of units

ratio — the relation between two quantities; can be expressed in words, fractions, decimals, or as a percentage

rational equation — an equation that includes the ratio of two rational expressions, in which a variable appears in the denominator of at least one rational expression

rational expression — an expression made of the ratio of two polynomials, in which a variable appears in the denominator of a polynomial

rational function — a function that can be written in the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$

rational inequality — an inequality that includes the ratio of two rational expressions, in which a variable appears in the denominator of at least one rational expression

test interval — for a polynomial or rational inequality in *x*, an interval on the *x*-axis formed by one or more critical numbers. The sign of the function on the test interval is the same as the sign of the function value at any *x*-value in the interval.

vertical asymptote — a line defined as follows: The line $x = a$ is a vertical asymptote of the graph of a function f if $f(x)$ either increases or decreases without bound as x gets closer to a.

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Recommended Resources

- MathIsFun.com. "Rational Expressions."

  http://www.walch.com/rr/00141

  This site provides an overview of rational expressions and includes zeros of rational expressions, asymptotes of rational functions, and graphs of rational functions.

- MathIsFun.com. "Solving Rational Inequalities."

  http://www.walch.com/rr/00142

  This website provides an overview of solving rational inequalities, including the use of critical numbers, which it calls "points of interest."

- MathIsFun.com. "Using Rational Expressions."

  http://www.walch.com/rr/00143

  This site shows how to add, subtract, multiply, and divide rational expressions.

- Zona Land Education. "Transform of $f(x) = 1/x$."

  http://www.walch.com/rr/00144

  This interactive applet allows users to transform the parent function $f(x)=\frac{1}{x}$ using sliders and then observe the effects on the graph. Users may choose to show or hide asymptotes. The applet requires Java software to run.

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Lesson 3.5.1: Creating Rational Equations

Warm-Up 3.5.1

Dario, Elliott, and Ivan are in a basketball game. Dario has made 2 baskets in 8 attempts, Elliott has made 1 basket in 5 attempts, and Ivan has made 5 baskets in 14 attempts.

1. Dario's shooting percentage is now 25% because $\frac{2}{8} = \frac{1}{4} = 25\%$. If he makes 4 baskets in his next 4 consecutive attempts, what will be his shooting percentage? Show your work.

2. If Elliott and Ivan also each make 4 baskets in their next 4 consecutive attempts, which player out of all three players will have the highest shooting percentage? Show your work.

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Instruction

Lesson 3.5.1: Creating Rational Equations

Common Core State Standard

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

Warm-Up 3.5.1 Debrief

1. Dario's shooting percentage is now 25% because $\frac{2}{8}=\frac{1}{4}=25\%$. If he makes 4 baskets in his next 4 consecutive attempts, what will be his shooting percentage? Show your work.

$\frac{2}{8}$	Original number of baskets made compared to baskets attempted
$\frac{2+4}{8+4}$	Additional 4 baskets attempted and made
$\frac{6}{12}=50\%$	Simplify.

Dario's shooting percentage will be 50%.

2. If Elliott and Ivan also each make 4 baskets in their next 4 consecutive attempts, which player out of all three players will have the highest shooting percentage? Show your work.

Dario's shooting percentage: $\frac{2+4}{8+4}=\frac{6}{12}=50\%$

Elliott's shooting percentage: $\frac{1+4}{5+4}=\frac{5}{9}\approx 55.6\%$

Ivan's shooting percentage: $\frac{5+4}{14+4}=\frac{9}{18}=50\%$

Elliott will have the highest shooting percentage, at approximately 55.6%.

Connection to the Lesson

- Students will continue to work with ratios similar to the ratio $\frac{\text{successes}}{\text{attempts}}$ and extend their understanding to include rational expressions.

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Prerequisite Skills

This lesson requires the use of the following skills:

- writing percentages as decimals (7.EE.3)
- adding, subtracting, multiplying, and dividing with decimals (6.NS.3)
- determining the least common denominator of fractions (4.NF.2)
- applying the Distributive Property (6.EE.3)
- solving linear one-variable equations (8.EE.7b)
- representing comparisons of quantities as ratios and rates (7.RP.2c)
- writing equivalent fractions, some in simplest form (7.EE.3)
- solving quadratic equations (A–REI.4b)
- factoring quadratic trinomials, including writing them as products of binomials (A–REI.4b)
- multiplying binomials (A–APR.1)

Introduction

There are many types of equations used in solving mathematics. The type of equation you work with depends largely on the problem situation, the information given, and what you are trying to find out. In this lesson, you will work with rational equations, equations that have a variable in a denominator. Creating rational equations to represent problem situations often involves ratios such as $\frac{20 \text{ customers}}{7 \text{ employees}}$, and rates such as $\frac{75 \text{ miles}}{2.5 \text{ gallons}}$. Therefore, solving rational equations frequently uses previously learned skills involving fractions.

Key Concepts

- A **ratio** is a relation between two quantities. A ratio can be expressed in words, fractions, decimals, or as a percentage.
- Examples of ratios include $\frac{2}{5}$, $\frac{15 \text{ successes}}{20 \text{ attempts}}$, and $\frac{66 \text{ students}}{3 \text{ teachers}}$.
- A **rate** is a ratio that compares measurements with different kinds of units.

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Instruction

- Examples of rates include $\frac{3 \text{ feet}}{2 \text{ inches}}$ and $\frac{75 \text{ miles}}{2.5 \text{ gallons}}$.
- The ratio $\frac{\text{successes}}{\text{attempts}}$ appears in various applications, sometimes using different words, but representing the same concept.
- For example, a shooting percentage in basketball can be written as $\frac{7 \text{ successes}}{10 \text{ attempts}}$. If you successfully make 7 shots out of 10 attempts, your shooting percentage is: $\frac{7}{10} = 0.7 = 70\%$.
- Another example is batting average in baseball or softball. If you get 3 hits in 8 at-bats, your batting average is $\frac{3 \text{ successes}}{8 \text{ attempts}} = \frac{3}{8} = 0.375$. Batting averages are typically left as a decimal rather than converted to a percent.
- A **rational expression** is an expression made of the ratio of two polynomials, in which a variable appears in the denominator of a polynomial.
- Examples of rational expressions include $\frac{2x}{3y^2}$, $\frac{12}{x}$, and $\frac{x^2+2x-15}{x-3}$.
- A rational expression is in simplest form when it is expressed as a quotient of polynomials whose greatest common factor is 1.
- To simplify a rational expression, factor either the numerator or the denominator or both so that common factors in the numerator and denominator can be identified, and then cancelled out.
- A **rational equation** is an equation that includes the ratio of two rational expressions, in which a variable appears in the denominator of at least one rational expression.
- Examples of rational equations include $\frac{5+x}{20+x} = 0.90$, $\frac{12}{20} + \frac{15}{x} = 1$, and $\frac{-4}{x^2-2x-3} + 2 = \frac{x-4}{x-3}$.
- The **least common multiple (LCM)** of two or more polynomials is the common multiple that has the least degree and the least positive constant factor.
- The **least common denominator (LCD)** of two or more fractions is the least common multiple of their denominators.

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Instruction

- Given the fractions $\frac{18}{20}$ and $\frac{5}{x}$, the least common multiple of 20 and x is $20x$, so the least common denominator is $20x$.
- Given the fractions $\frac{20}{x}$ and $\frac{4}{x+8}$, the least common multiple of x and $x + 8$ is $x(x + 8)$, so the least common denominator is $x(x + 8)$.
- A **rational function** is a function that can be written in the form $f(x)=\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.
- Examples of rational functions include $f(x)=\frac{5+x}{20+x}$, $f(x)=\frac{-4}{x^2-2x-3}$, and $f(x)=\frac{12}{20}+\frac{15}{x}$.
- $f(x)=\frac{12}{20}+\frac{15}{x}$ can be written in the form $f(x)=\frac{p(x)}{q(x)}$:

  $f(x)=\frac{12}{20}+\frac{15}{x}=\frac{x}{x}\bullet\frac{12}{20}+\frac{20}{20}\bullet\frac{15}{x}=\frac{12x+300}{20x}$
- Sometimes an equation representing a problem situation is naturally related to a rational function.
- For example, the function $f(x)=\frac{5+x}{20+x}$ might represent a shooting percentage in basketball. If you want to find the value of x that makes the shooting percentage 50%, then you could solve the equation $0.50=\frac{5+x}{20+x}$.
- Solutions to quadratic functions of the form $ax^2 + bx + c = 0$ can be determined using the quadratic formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.
- An **extraneous solution**, or **extraneous root,** of an equation is a solution of an equation that arises during the solving process, but which is not a solution of the original equation.
- Extraneous solutions sometimes occur when solving rational equations, so it is important to check all apparent solutions by substituting them into the original equation and determining if they make it true.
- When solving rational equations, restrictions on the variable must be considered. Possible solutions that result in 0 as the denominator must be excluded from the solutions of the rational equation since denominators cannot equal 0.

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Common Errors/Misconceptions

- mistakenly thinking the time to complete a task by two persons or objects working at different rates is the average of the individual times or half the average of the individual times
- neglecting to check apparent solutions to determine if they make the equation true
- using an equation other than the original equation when checking an apparent solution

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Instruction

Guided Practice 3.5.1

Example 1

Simplify the rational expression $\frac{x^2+2x-15}{x-3}$.

1. If possible, determine the factors of the numerator.

   The numerator $x^2 + 2x - 15$ can be written as the product of the two factors $(x - 3)$ and $(x + 5)$.

   Rewrite the fraction with the numerator as a product of the factors.

   $$\frac{x^2+2x-15}{x-3}=\frac{(x-3)(x+5)}{x-3}$$

2. Divide by the common factor.

   The common factor of the fraction is $(x - 3)$.

   $$\frac{x^2+2x-15}{x-3}=\frac{\overset{1}{\cancel{(x-3)}}(x+5)}{\underset{1}{\cancel{x-3}}}$$

3. Simplify the fraction.

   $$\frac{x^2+2x-15}{x-3}=\frac{x+5}{1}=x+5$$

4. Determine if there are any restrictions on the variable.

   The denominator of the original expression is $x - 3$; therefore, x cannot be equal to 3 since this would mean the denominator would result in 0.

   The rational expression $\frac{x^2+2x-15}{x-3}$ simplifies to $x + 5$ where $x \neq 3$.

SWB p. 169, SRB p. 188

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Instruction

Example 2

Josie has made 12 free throws out of 26 attempts in her basketball games this season. If she can raise her free-throw percentage to 60%, she will tie her record from last season. How many consecutive attempts will she have to make in order to reach a free-throw percentage of 60%? Write a function to complete the problem.

1. Assign a variable to represent what you need to find.

   Let x represent the number of consecutive attempts Josie must make.

2. Create a function to represent the situation.

   A free-throw percentage is the ratio $\frac{\text{successes}}{\text{attempts}}$, written in percent form, for shots attempted from the free-throw line.

   Josie has made 12 free throws out of 26 attempts so far.

   If she is successful in x consecutive attempts, then her free-throw percentage will be represented by the following ratio:

   $$\frac{12 \text{ successes} + x \text{ successes}}{26 \text{ attempts} + x \text{ attempts}} = \frac{12+x}{26+x}$$

   The function $f(x)=\frac{12+x}{26+x}$ represents the situation, where x is the number of consecutive successful attempts and the function value $f(x)$ is the equivalent of the free-throw percentage.

3. Use the function to create an equation that can be solved to determine the answer to the problem.

   You need to find how many consecutive attempts Josie must make in order to achieve a free-throw percentage of 60%.

   To do this, you need to find the x-values that make $f(x) = 0.60$.

   Therefore, the equation to be solved is $\frac{12+x}{26+x}=0.60$.

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4. Solve the equation.

$\frac{12+x}{26+x}=0.60$	Equation
$12 + x = 0.60(26 + x)$	Multiply both sides by $26 + x$.
$12 + x = 0.60(26) + 0.60(x)$	Apply the Distributive Property.
$12 + x = 15.6 + 0.60x$	Simplify.
$12 + 0.40x = 15.60$	Subtract $0.60x$ from both sides.
$0.40x = 3.60$	Subtract 12 from both sides.
$x = 9$	Divide both sides by 0.40.

5. Check the apparent solution and answer the question.

$\frac{12+x}{26+x}=0.60$	Equation
$\frac{12+(9)}{26+(9)}=0.60$	Substitute 9 for x.
$\frac{21}{35}=0.60$	
$0.60 = 0.60$	

The answer checks out; therefore, $x = 9$ is the solution of $\frac{12+x}{26+x}=0.60$.

Josie will have to succeed at 9 consecutive attempts in order to raise her free-throw percentage to 60%.

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Example 3

Moe and Marco deliver a regular order to a specialty foods store. Moe can unload the truck in 40 minutes if he works alone. Moe and Marco can unload the truck in 15 minutes if they work together. How many minutes does it take Marco to unload the truck if he works alone? Assume both men work at steady rates and their rates are not affected by either working alone or working together.

1. Assign a variable to represent what you need to find.

   Let x represent the number of minutes it takes Marco to unload the truck if he works alone.

2. Create an equation that represents the situation.

   Moe, working alone, unloads at the rate of $\frac{1}{40}$ of the truck per minute.

   So, in 15 minutes Moe unloads $15 \bullet \frac{1}{40}$ of the truck.

   Marco, working alone, unloads at the rate of $\frac{1}{x}$ of the truck per minute.

   In 15 minutes, Marco unloads $15 \bullet \frac{1}{x}$ of the truck.

   They complete the job working together in 15 minutes.

   Write these elements as an equation, then simplify.

(part of job done by Moe) + (part of job done by Marco) = (one whole job)	Equation
$\left(15 \bullet \frac{1}{40}\right) + \left(15 \bullet \frac{1}{x}\right) = (1)$	Substitute values for each part of the equation.
$\frac{15}{40} + \frac{15}{x} = 1$	Simplify.

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3. Solve the equation.

$\frac{15}{40}+\frac{15}{x}=1$	Equation
$40x\left(\frac{15}{40}+\frac{15}{x}\right)=40x(1)$	Multiply both sides by $40x$, the least common denominator (LCD).
$40x\left(\frac{15}{40}\right)+40x\left(\frac{15}{x}\right)=40x(1)$	Distribute.
$15x + 40 \bullet 15 = 40x$	Simplify.
$15x + 600 = 40x$	
$600 = 25x$	Subtract $15x$ from both sides.
$24 = x$	Divide both sides by 25.

4. Check the apparent solution and answer the question.

$\frac{15}{40}+\frac{15}{x}=1$	Equation
$\frac{15}{40}+\frac{15}{(24)}=1$	Substitute 24 for x.
$\frac{3}{8}+\frac{5}{8}=1$	Simplify each fraction to obtain the LCD, 8.
$\frac{8}{8}=1$	Simplify.
$1 = 1$	

It takes Marco 24 minutes to unload the truck if he works alone. ✓

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Example 4

A new street sweeping machine can sweep a town's streets in 20 fewer hours than it takes the old machine. Working together, the two machines can complete the job in 24 hours. How long does it take each machine to complete the job, working alone? Assume the machines' rates are not affected by whether one machine is being used or both machines are being used.

1. Assign a variable to represent what you need to find.

   Let x represent the number of hours it takes the new machine to complete the job, working alone.

   Therefore, $x + 20$ represents the number of hours it takes the old machine to complete the job, working alone.

2. Create an equation that represents the situation.

   The new machine completes $\frac{1}{x}$ of the job per hour.

   So, in 24 hours the new machine completes $24 \bullet \frac{1}{x}$ of the job.

   The old machine completes $\frac{1}{x+20}$ of the job per hour.

   Thus, in 24 hours the old machine completes $24 \bullet \frac{1}{x+20}$ of the job.

   The machines complete the job working together in 24 hours.

   Write these elements as an equation, then simplify.

(part of job done by new machine) + (part of job done by old machine) = (one whole job)	Equation
$\left(24 \bullet \frac{1}{x}\right) + \left(24 \bullet \frac{1}{x+20}\right) = (1)$	Substitute values for each part of the equation.
$\frac{24}{x} + \frac{24}{x+20} = 1$	Simplify.

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Instruction

3. Solve the equation.

$\frac{24}{x}+\frac{24}{x+20}=1$	Equation
$x(x+20)\left(\frac{24}{x}+\frac{24}{x+20}\right)=x(x+20)(1)$	Multiply both sides by $(x)(x+20)$, the LCD.
$x(x+20)\left(\frac{24}{x}\right)+x(x+20)\left(\frac{24}{x+20}\right)=x(x+20)(1)$	Distribute.
$\frac{x(x+20)24}{x}+\frac{x(x+20)24}{x+20}=x(x+20)(1)$	Simplify.
$\frac{{}^{1}\cancel{x}(x+20)24}{{}_{1}\cancel{x}}+\frac{x\cancel{(x+20)}^{1}24}{\cancel{x+20}_{1}}=x(x+20)(1)$	Divide out common factors.
$(x+20)24+x(24)=x(x+20)$	Simplify.
$24x+480+24x=x^2+20x$	Distribute.
$48x+480=x^2+20x$	Simplify.
$480=x^2-28x$	Subtract $48x$ from both sides.
$0=x^2-28x-480$	Subtract 480 from both sides.

In this example, it is possible to factor the equation, but using the quadratic formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, will work for all quadratic equations.

(*continued*)

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$$x=\frac{-(-28)\pm\sqrt{(-28)^2-4(1)(-480)}}{2(1)}$$ Substitute values for a, b, and c into the quadratic formula.

$$x=\frac{28\pm\sqrt{784+1920}}{2}$$ Simplify.

$$x=\frac{28\pm\sqrt{2704}}{2}$$

$$x=\frac{28\pm52}{2}$$

$x = 40$ or $x = -12$

Since x represents a number of hours, $x = -12$ does not make sense in this situation because time cannot be negative.

Therefore, $x = 40$.

4. Check the apparent solution and answer the question.

$$\frac{24}{x}+\frac{24}{x+20}=1$$ Equation

$$\frac{24}{(40)}+\frac{24}{(40)+20}=1$$ Substitute 40 for x.

$$\frac{24}{40}+\frac{24}{60}=1$$ Simplify.

$$\frac{3}{5}+\frac{2}{5}=1$$ Simplify each fraction to obtain the LCD, 5.

$$\frac{5}{5}=1$$ Simplify.

$1 = 1$

Therefore, $x = 40$ and $x + 20 = 60$.

It takes the new machine 40 hours to complete the job, working alone.

It takes the old machine 60 hours to complete the job, working alone.

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Example 5

Serena drove 54 miles in the city, averaging 24 miles per gallon. Then she drove 251 miles on highways. For the entire 305 miles, she averaged 30 miles per gallon. What was her average fuel consumption rate on highways, in miles per gallon?

1. Assign a variable to represent what you need to find.

   Let x represent Serena's fuel consumption rate on highways, in miles per gallon.

2. Organize the information.

   Recall that to divide by a fraction, you multiply by its reciprocal.

   So, if you divide miles by the rate of miles per gallon, you get gallons:

$$\frac{\text{miles}}{\frac{\text{miles}}{\text{gallons}}} = \text{miles} \bullet \frac{\text{gallons}}{\text{miles}} = \text{gallons}$$

   Use a table to organize the quantities in the problem.

	Distance (mi)	**Fuel consumption rate** $\left(\frac{\text{mi}}{\text{gal}}\right)$	**Fuel consumption (gal)**
City	54	24	$\frac{54}{24}$
Highway	251	x	$\frac{251}{x}$
Overall (Total)	305	30	$\frac{305}{30}$

3. Create an equation that represents the situation.

   (gallons used in city) + (gallons used on highway) = (gallons used overall)

$$\frac{54}{24} + \frac{251}{x} = \frac{305}{30}$$

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4. Solve the equation.

$\frac{54}{24}+\frac{251}{x}=\frac{305}{30}$	Equation
$\frac{9}{4}+\frac{251}{x}=\frac{61}{6}$	Simplify the fractions.
$12x\left(\frac{9}{4}+\frac{251}{x}\right)=12x\left(\frac{61}{6}\right)$	Multiply both sides by $12x$, the LCD.
$12x\left(\frac{9}{4}\right)+12x\left(\frac{251}{x}\right)=12x\left(\frac{61}{6}\right)$	Distribute.
$\frac{12x(9)}{4}+\frac{12x(251)}{x}=\frac{12x(61)}{6}$	Simplify.
$\frac{{}^{3}\cancel{12}x(9)}{{}_{1}\cancel{4}}+\frac{12^{1}\cancel{x}(251)}{{}_{1}\cancel{x}}=\frac{{}^{2}\cancel{12}x(61)}{{}_{1}\cancel{6}}$	Divide out common factors.
$27x + 3012 = 122x$	Simplify.
$3012 = 95x$	Subtract $27x$ from both sides.
$31.7 \approx x$	Divide both sides by 95.

5. Check the apparent approximate solution and answer the question.

$\frac{54}{24}+\frac{251}{x}=\frac{305}{30}$	Equation
$\frac{54}{24}+\frac{251}{(31.7)}=\frac{305}{30}$	Substitute 31.7 for x.
$2.25 + 7.918 \approx 10.167$	Use a calculator to write the fractions as decimals, approximating as needed.
$10.168 \approx 10.167$	

Serena's average fuel consumption rate on highways was approximately 31.7 miles per gallon.

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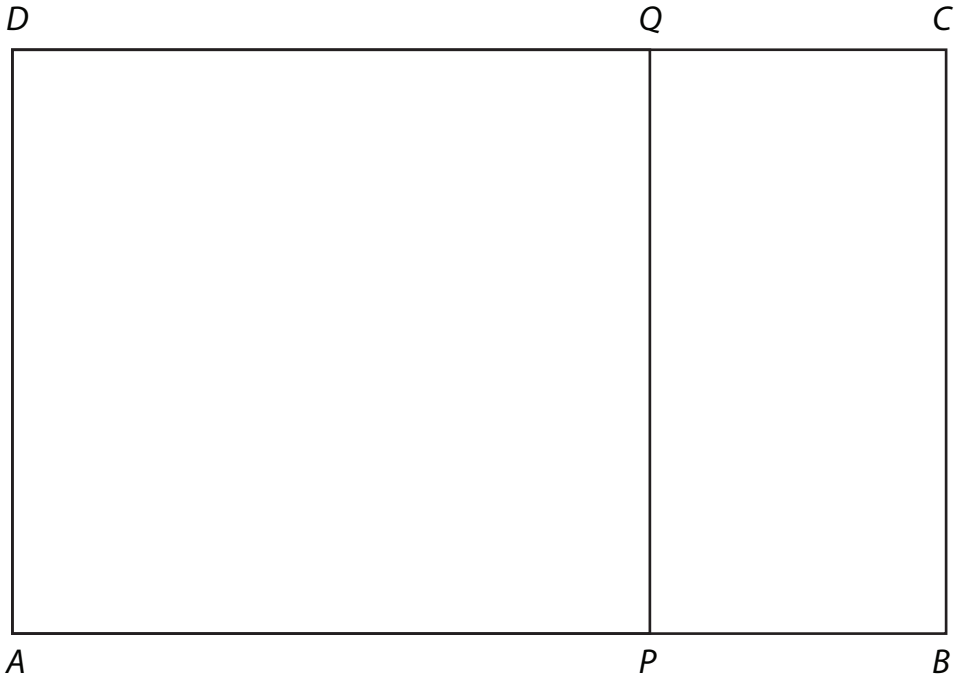
Problem-Based Task 3.5.1: Golden Rectangles and the Golden Ratio

SMP	
1	2 ✓
3	4 ✓
5	6
7 ✓	8

Golden rectangles have been used by artists, architects, and other creative individuals for hundreds of years because of a unique property they all share. A golden rectangle is a rectangle that can be divided into a square and a smaller rectangle that is similar to the original rectangle. In the diagram below, quadrilateral *ABCD* is a golden rectangle, and $\overline{PQ}$ divides it into square *APQD* and rectangle *BCQP*, which is similar to quadrilateral *ABCD*. (Using the similarity symbol, *BCQP* ~ *ABCD*.)



The ratio of length to width in a golden rectangle is called the golden ratio. To find the golden ratio, you can let the width be 1, let the length be *x*, and then find the value of *x* (because $\frac{\text{length}}{\text{width}} = \frac{x}{1} = x$). Let *AD* = 1 and *AB* = *x*. Find the golden ratio.

Find the golden ratio.

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Problem-Based Task 3.5.1: Golden Rectangles and the Golden Ratio

Coaching

a. What is true about corresponding sides of similar polygons?

b. You know that $BCQP \sim ABCD$. How can you complete the proportion $\frac{AB}{BC} = \frac{?}{?}$, using $\overline{AD}$ and $\overline{BP}$?

c. You know $AD = 1$ and $AB = x$. What is the length of $\overline{BC}$? Explain.

d. What is the length of $\overline{AP}$? Explain.

e. What is the length of $\overline{BP}$? Explain your answer and then sketch the given diagram, labeling lengths $\overline{AB}$, $\overline{AD}$, $\overline{AP}$, $\overline{BC}$, and $\overline{BP}$.

f. How can you use the proportion from part b to find the value of x? Show your work and explain your steps.

g. What is a decimal approximation of the golden ratio?

h. What does the decimal approximation of the golden ratio tell you about the shape of any golden rectangle?

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Problem-Based Task 3.5.1: Golden Rectangles and the Golden Ratio

Coaching Sample Responses

a. What is true about corresponding sides of similar polygons?

They are proportional.

b. You know that $BCQP \sim ABCD$. How can you complete the proportion $\frac{AB}{BC} = \frac{?}{?}$, using $\overline{AD}$ and $\overline{BP}$?

$$\frac{AB}{BC} = \frac{AD}{BP}$$

c. You know $AD = 1$ and $AB = x$. What is the length of $\overline{BC}$? Explain.

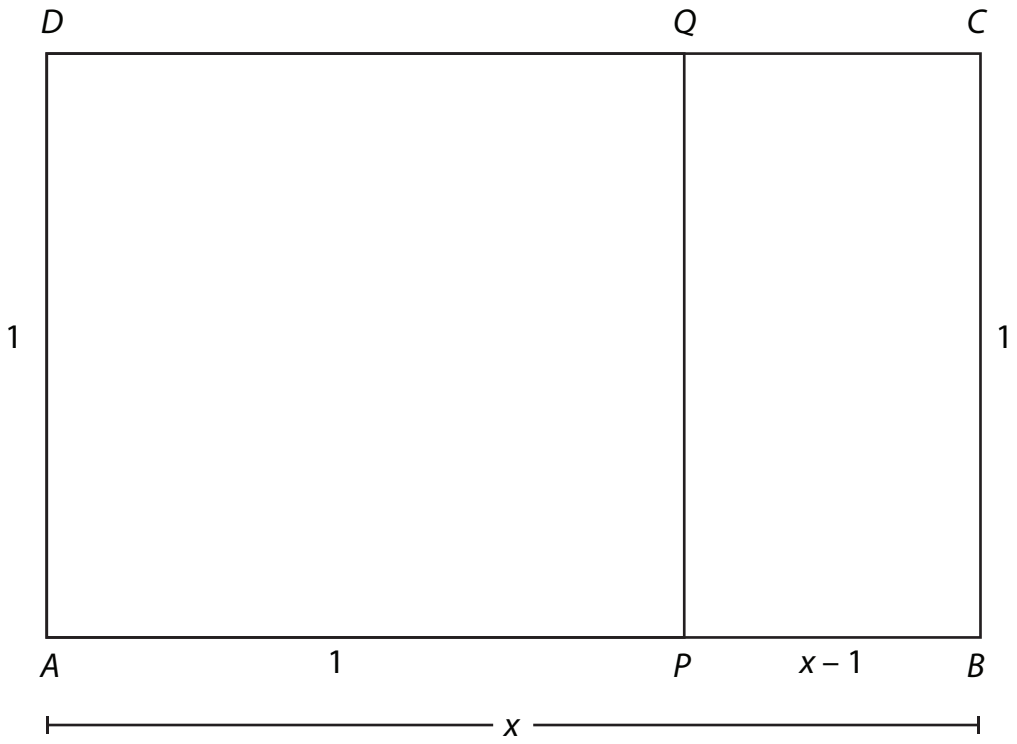
$BC = AD$ because opposite sides of a rectangle are congruent; therefore, $BC = 1$.

d. What is the length of $\overline{AP}$? Explain.

$AD = 1$ and $APQD$ is a square; therefore, $AP = 1$.

e. What is the length of $\overline{BP}$? Explain your answer and then sketch the given diagram, labeling lengths $\overline{AB}$, $\overline{AD}$, $\overline{AP}$, $\overline{BC}$, and $\overline{BP}$.

$AB = x$, $AP = 1$, and $\overline{BP}$ is the length remaining when 1 is subtracted from x; therefore, $BP = x - 1$.



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f. How can you use the proportion from part b to find the value of x? Show your work and explain your steps.

$\frac{AB}{BC}=\frac{AD}{BP}$	Write the proportion from part b.
$\frac{(x)}{(1)}=\frac{(1)}{(x-1)}$	Substitute the values and expressions previously identified.
$(x-1)\left(\frac{x}{1}\right)=(x-1)\left(\frac{1}{x-1}\right)$	Multiply both sides by the LCD.
$(x-1)(x)=1$	Simplify.
$x^2-x=1$	Distribute.
$x^2-x-1=0$	Subtract 1 from both sides.
$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)}$	Use the quadratic formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.
$x=\frac{1\pm\sqrt{1+4}}{2}$	Simplify.
$x=\frac{1+\sqrt{5}}{2}$ or $x=\frac{1-\sqrt{5}}{2}$	

The value of $\frac{1-\sqrt{5}}{2}$ is negative, so it cannot represent the length of a rectangle.

The length of rectangle $ABCD$ is $x=\frac{1+\sqrt{5}}{2}$, which is the golden ratio.

g. What is a decimal approximation of the golden ratio?

$\frac{1+\sqrt{5}}{2}$ is approximately equal to 1.618.

h. What does the decimal approximation of the golden ratio tell you about the shape of any golden rectangle?

The length is about 1.618 times the width.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.5.1: Creating Rational Equations

For problems 1 and 2, simplify each rational expression and state any restrictions on the value of x.

1. $\dfrac{5x+15}{5x^2+5x-30}$

2. $\dfrac{6x-12}{3x^2-3x-6}$

For problems 3 and 4, write a rational function that represents the situation. Use the function to create equations that can be used to solve the problem. Then answer the questions.

3. Alicia has had 16 hits in 46 at-bats in her softball games. How many consecutive hits does she need to raise her batting average to 0.400?

4. Mr. Ortiz is a driving instructor. Of his students, 23 have taken the driver's test, and 17 of them have passed on the first attempt. Mr. Ortiz earns a $300 bonus if 80% of his students pass the test on the first attempt. How many consecutive students must pass on the first attempt in order for Mr. Ortiz to earn the bonus?

For problems 5–10, write a rational equation, solve it, and then answer the question.

5. Juliana and Sarah assemble bicycles at City Wheels Bike Shop. Juliana can assemble a City Cruiser in 20 minutes if she works alone. Juliana and Sarah assembled 16 City Cruisers in 2 hours when they worked together. How many minutes does it take Sarah to assemble one City Cruiser if she works alone? Assume both women work at steady rates and their rates are not affected by either working alone or working together.

6. Pump A can unload an oil tanker in 36 hours 30 minutes if it works alone. Pumps A and B can unload the tanker in 16 hours 30 minutes when they work together. How long does it take pump B to unload the tanker if it works alone?

continued

SWB p. 173, SRB Practice 3.5.1 p. U3-198

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7. Arthur can complete a customer's order in 15 fewer minutes by using copy machine A than by using copy machine B. By using both machines, he can complete the order in 18 minutes. How long does it take Arthur to complete the order by using each machine alone?

8. Natasha and Tobias have a cleaning business. Natasha can clean and wax the floors in an office building in 2 hours 30 minutes less time than it takes Tobias to do the same job. Working together, they can complete the job in 8 hours. How long does it take each person to complete the job, working alone? Assume both people work at steady rates and their rates are not affected by either working alone or working together.

9. Crystal owns a kennel business. The number of dogs she keeps varies. During the last 5 days in June, the dogs consumed dog food at the rate of 4 days per bag. During the last 5 days in June and the first 7 days in July combined, the rate was 5 days per bag. At what rate, in days per bag, did the dogs consume dog food during the first 7 days in July?

10. Ted and Brett went on a two-day backpacking trip in the mountains. On the first day they hiked 10 miles at an average rate of 2.25 miles per hour. On the second day they hiked 12 miles. Their average rate for the entire 22 miles was 2 miles per hour. What was their average rate on the second day?

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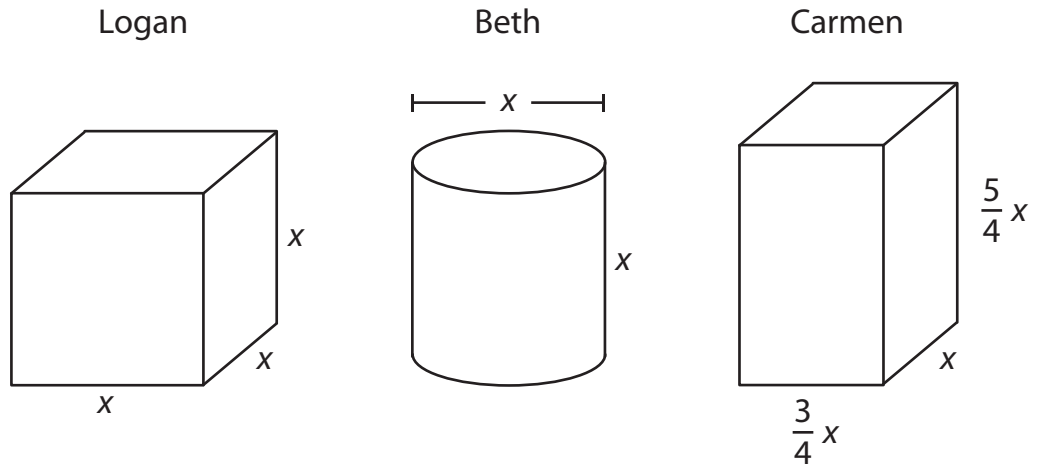
Lesson 3.5.2: Graphing Rational Equations

Warm-Up 3.5.2

Logan, Beth, and Carmen are studying package design technology. The diagram below shows container designs they are working on.



1. Which container has the greatest volume? Explain your reasoning. The volume of a cylinder is $V = \pi r^2 h$, where r is the radius and h is the height.

2. If $x = 10$ inches, what is the approximate total volume of all three containers, rounded to the nearest cubic inch? Show your work.

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Instruction

Lesson 3.5.2: Graphing Rational Equations

Common Core State Standard

A–CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

Warm-Up 3.5.2 Debrief

1. Which container has the greatest volume? Explain your reasoning. The volume of a cylinder is $V = \pi r^2 h$, where r is the radius and h is the height.

   Find the volume of each container by substituting the given values into the appropriate formula for volume.

   Logan's container:

   $V = x \bullet x \bullet x = x^3$

   Beth's container:

   $V = \pi r^2 h$

   $V = \pi\left(\frac{x}{2}\right)^2 x$

   $V = \pi\left(\frac{x^2}{4}\right)x$

   $V = \frac{\pi x^3}{4} \approx \frac{(3.14)x^3}{4}$

   Carmen's container:

   $V = \left(\frac{3}{4}x\right)(x)\left(\frac{5}{4}x\right)$

   $V = \frac{15}{16}x^3$

   The container with the greatest volume is Logan's container, with x^3 cubic units.

2. If $x = 10$ inches, what is the approximate total volume of all three containers, rounded to the nearest cubic inch? Show your work.

   Logan's container:

   $V = x^3 = 10^3 = 1000 \text{ in}^3$

   Beth's container:

   $V = \frac{\pi x^3}{4} = \frac{\pi(10)^3}{4} = \frac{\pi(1000)}{4} \approx 785.4 \text{ in}^3$

   Carmen's container:

   $V = \frac{15}{16}x^3 = \frac{15}{16}(10)^3 = \frac{15}{16}(1000) = 937.5 \text{ in}^3$

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Add the volume of each container to determine the approximate total volume.

$1000 + 785.4 + 937.5 = 2722.9$

Rounded, the approximate total volume of all three containers is 2,723 in^3.

Connection to the Lesson

- Students will create equations involving volumes of solid figures, including a rectangular prism and a cylinder.
- Students will create and solve equations, requiring that they substitute variables for other variables and numbers for variables.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- graphing equations of the form $x = h$ and $y = k$, where h and k are constants (F–IF.7a★)
- evaluating a function $f(x)$, given the x-value and the function rule (F–IF.2)
- plotting points of a function given the function rule (8.F.1)
- identifying the value of a variable that makes a rational expression undefined (F–IF.7d★ (+))
- solving rational equations (A–REI.2)
- writing values as equivalent fractions, decimals, and percentages (7.EE.3)

Introduction

A relationship between two quantities can often be represented by an equation with two variables, where each variable represents a quantity. A graph that represents a relationship between two quantities can be used to help solve problems about those quantities, and is often helpful in understanding the behavior of the function near particular x-values and at extreme x-values. This understanding frequently has an important connection to the problem situation represented by the function.

Key Concepts

- Recall that a rational function is a function that can be written in the form $f(x)=\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.
- Examples of rational functions include $f(x)=\frac{5}{x}$, $f(x)=\frac{500}{\pi x^2}$, and $f(x)=\frac{-4}{x+2}+5$.
- The **domain** of a function is the set of all input values, or **independent variable** values, that are valid for the function. For graphs, the horizontal axis is the independent variable axis.
- When x and y are used in the function rule, x is the independent variable.
- The **range** of a function is the set of all output values, or **dependent variable** values, that are valid for the function.
- For graphs, the vertical axis is the dependent variable axis.
- When x and y are used in the function rule, y is the dependent variable.
- The **end behavior** of a graph is the behavior of the graph as x approaches positive or negative infinity.
- An **asymptote** is a line that a function gets closer and closer to as one of the variables increases or decreases without bound.

SRB p. U3-200

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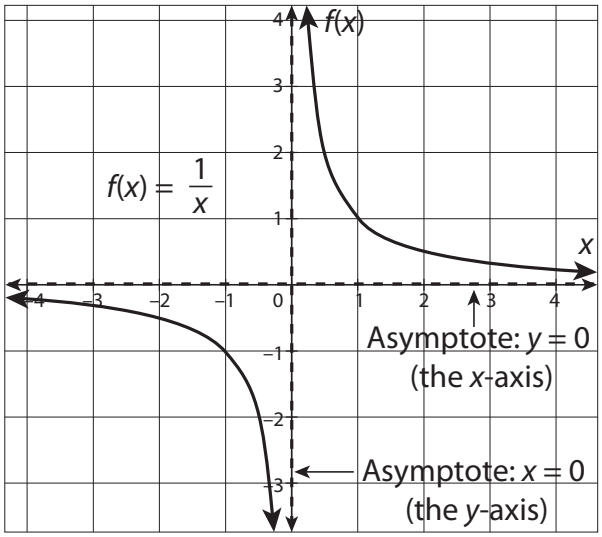
Instruction

- For the graph of a function $f(x)$:
  - The line $x = a$ is a **vertical asymptote** if $f(x)$ either increases or decreases without bound as x gets closer to a.
  - A vertical asymptote occurs at values that make the denominator equal to 0.
  - The line $y = b$ is a **horizontal asymptote** if $f(x)$ gets closer to b as x either increases or decreases without bound.
- A **parent function** is a function that represents a family of functions.
- The functions in a **family of functions** have the same general shape as the parent function.
- The parent function for a family of simple rational functions is $f(x)=\frac{1}{x}$.
- The graph of $f(x)=\frac{1}{x}$ that follows consists of two symmetrical curved sections.
- Its domain is the set of all real numbers except 0 and its range is the set of all real numbers except 0.
- The asymptotes are as follows:
  - The line $x = 0$ (the y-axis) is a vertical asymptote because $f(x)$ decreases without bound as x gets closer to 0 from the left and $f(x)$ increases without bound as x gets closer to 0 from the right.
  - The line $y = 0$ (the x-axis) is a horizontal asymptote because $f(x)$ gets closer to 0 as x either increases or decreases without bound.



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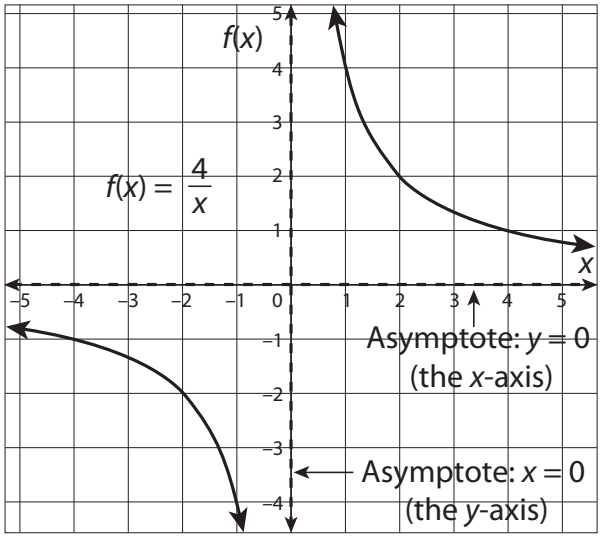
Instruction

- Functions of the form $f(x)=\frac{a}{x}$, with $a \neq 0$, have $f(x)=\frac{1}{x}$ as their parent function. They have the same asymptotes, domain, and range as $f(x)=\frac{1}{x}$. An example is $f(x)=\frac{4}{x}$, shown as follows.



- Functions of the form $f(x)=\frac{a}{x-h}+k$, with $a \neq 0$, also have $f(x)=\frac{1}{x}$ as their parent function.
- They have the following asymptotes, domain, and range:
  - Asymptotes: $x = h$ and $y = k$
  - Domain: all real numbers except h
  - Range: all real numbers except k
- An example is $f(x)=\frac{3}{x-2}-5$, shown as follows.
- It has the following asymptotes, domain, and range:
  - Asymptotes: $x = 2$ and $y = -5$
  - Domain: all real numbers except 2
  - Range: all real numbers except –5

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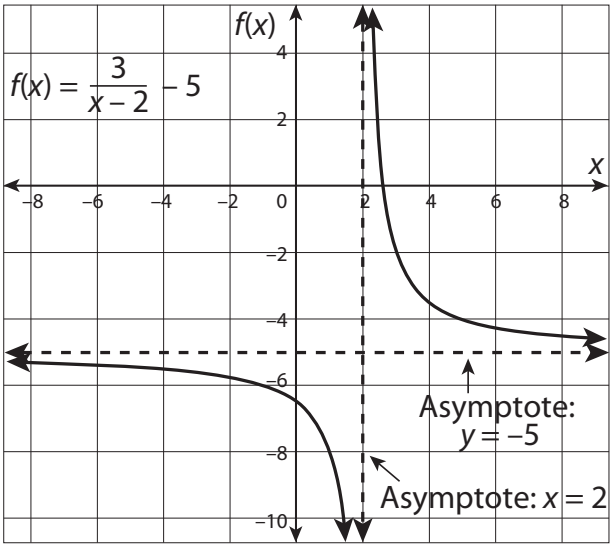
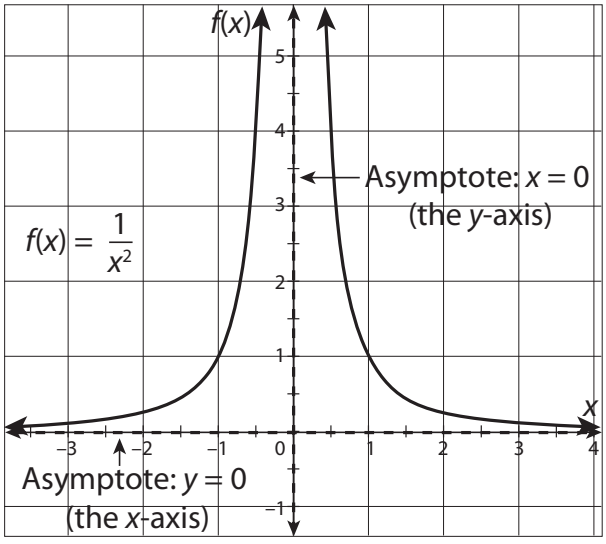


- The function $f(x)=\frac{1}{x^2}$ is another simple rational function.
- It is the parent of a family that includes functions of the form $f(x)=\frac{a}{x^2}$, with $a \neq 0$. These functions have the following asymptotes, domain, and range:
  - Asymptotes: $x = 0$ (the y-axis) and $y = 0$ (the x-axis)
  - Domain: all real numbers except 0
  - Range: all positive real numbers if a is positive; all negative real numbers if a is negative
- The graphs of the parent function $f(x)=\frac{1}{x^2}$ and a family member, $f(x)=\frac{-4}{x^2}$, are shown as follows.



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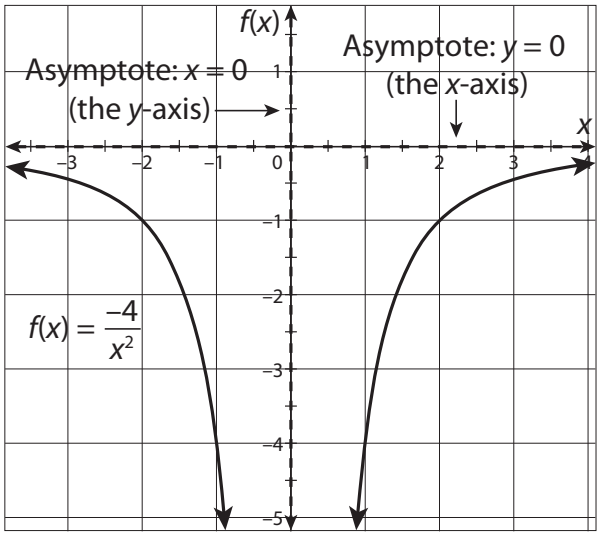

- To graph a rational function, first find any asymptotes and graph using a dashed line.
- Then determine *x*- or *y*-intercepts by first substituting 0 for *y* and solving for *x*, and then substituting 0 for *x* and solving for *y*. Plot any intercepts on the same graph.
- Choose a few more *x*-values and calculate the corresponding *y*-values. Be sure to also choose values to the left and right of the asymptotes.
- Plot all found points and connect using a smooth curve.

Common Errors/Misconceptions

- incorrectly thinking that $x = 0$ represents the *x*-axis
- incorrectly thinking that $y = 0$ represents the *y*-axis
- mistakenly believing that $x = h$, where h is a constant, represents a horizontal line
- mistakenly believing that $y = k$, where k is a constant, represents a vertical line
- sketching a rational function graph incorrectly, showing it "veer away" from its asymptote
- sketching a rational function in the incorrect regions of the graph

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Instruction

Guided Practice 3.5.2

Example 1

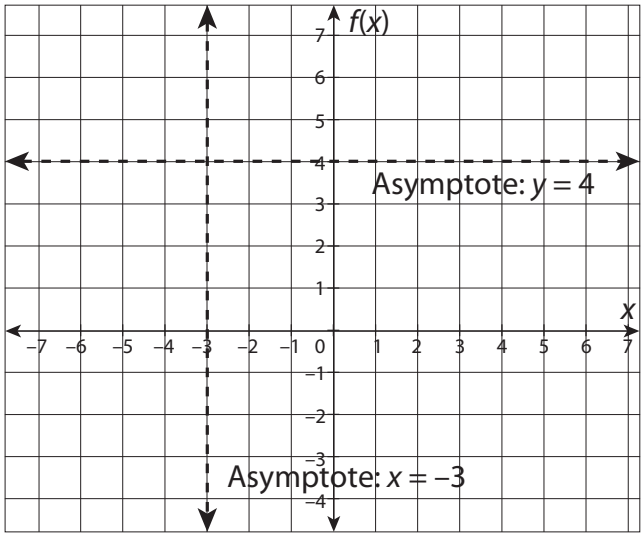
Graph the function $f(x)=\frac{-2}{x+3}+4$. Identify its asymptotes, domain, and range. Describe its end behavior and its behavior near its asymptotes.

1. Identify and draw the asymptotes.

   Functions of the form $f(x)=\frac{a}{x-h}+k$, with $a \neq 0$, have asymptotes $x = h$ and $y = k$.

   $f(x)=\frac{-2}{x+3}+4$ has asymptotes $x = -3$ and $y = 4$.

   Draw the lines $x = -3$ and $y = 4$.



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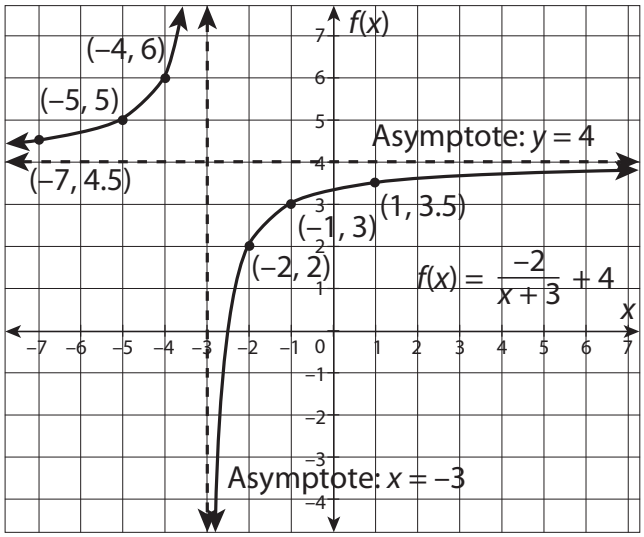
Instruction

2. Plot several points.

   Choose several values of x on both sides of the vertical asymptote to find the corresponding values of y.

   Draw the curved branches of the graph through the plotted points.



   You can also graph the function using a graphing calculator.

3. Follow the steps appropriate to your calculator model.

   On a TI-83/84:

   Step 1: Press [Y=].

   Step 2: At Y_1, enter the equation [(–)][2][÷][(][X,T,θ,n][+][3][)][+][4].

   Step 3: Press [WINDOW] to change the viewing window.

   Step 4: At Xmin, enter [(–)][12].

   Step 5: At Xmax, enter [12].

   Step 6: At Xscl, enter [1].

   Step 7: At Ymin, enter [(–)][8].

   Step 8: At Ymax, enter [8].

   Step 9: At Yscl, enter [1].

   Step 10: Press [GRAPH].

   On a TI-Nspire:

   Step 1: Press the [home] key.

   Step 2: Arrow over to the graphing icon and press [enter].

(*continued*)

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Instruction

Step 3: Enter the equation [(–)][2][÷][(][X][+][3][)][+][4] and press [enter].

Step 4: Change the viewing window by pressing [menu], arrowing down to number 4: Window/Zoom, and clicking the center button of the navigation pad.

Step 5: Choose 1: Window settings by pressing the center button.

Step 6: Enter the appropriate XMin value, [(–)][12].

Step 7: Enter the appropriate XMax value, [12].

Step 8: Leave the XScale set to "Auto." Press [tab] twice to navigate to YMin and enter [(–)][8].

Step 9: Press [tab] to navigate to YMax. Enter [8]. Press [tab] twice to leave YScale set to "auto" and navigate to "OK."

Step 10: Press [enter].

Step 11: Press [menu] and select 2: View and 5: Show Grid.

4. Identify the asymptotes, domain, and range. Describe the end behavior and the behavior near the asymptotes.

We determined the asymptotes to be $x = -3$ and $y = 4$.

The domain is the set of all the input values. In this case, the domain is the set of all real numbers except –3. An x-value can never equal –3 because this would cause the denominator to be 0, which makes the fraction undefined.

The range is the set of all output values. In this case, the range is the set of all real numbers except 4, which is not included because the function can never equal 4. This is because in order for $f(x)$ to equal 4, this would mean that $\frac{-2}{x+3}$ would have to be 0, and a fraction is only equal to 0 if its numerator is equal to 0. The numerator of this fraction will never be 0.

The end behavior of the function can be described in the following way: $f(x)$ approaches 4 as x either increases or decreases without bound.

The behavior near the horizontal asymptote is the same as the end behavior.

Behavior near the vertical asymptote can be described as: $f(x)$ increases without bound as x approaches –3 from the left and $f(x)$ decreases without bound as x approaches –3 from the right.

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Instruction

Example 2

Sang is planning a 40-mile bicycle trip. Complete the following and answer the questions.

- Write a rational equation to represent the relationship between r, his average speed in miles per hour, and t, the time in hours it will take to complete the trip. Make r the independent variable and t the dependent variable.
- Graph the equation.
- How long will the trip take if Sang's average speed is 12 miles per hour?
- What average speed will Sang need to achieve in order to complete the trip in 2.5 hours?
- What is the effect on t as r increases? What is the effect on t as r decreases?

1. Write a rational equation to represent the relationship between r, Sang's average rate of speed in miles per hour, and t, the time in hours that it will take to complete the trip.

   Let d represent distance in miles.

$rt = d$	rate • time = distance
$rt = (40)$	Substitute 40 for d.
$t = \frac{40}{r}$	Divide both sides by r to isolate the dependent variable on one side.

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Instruction

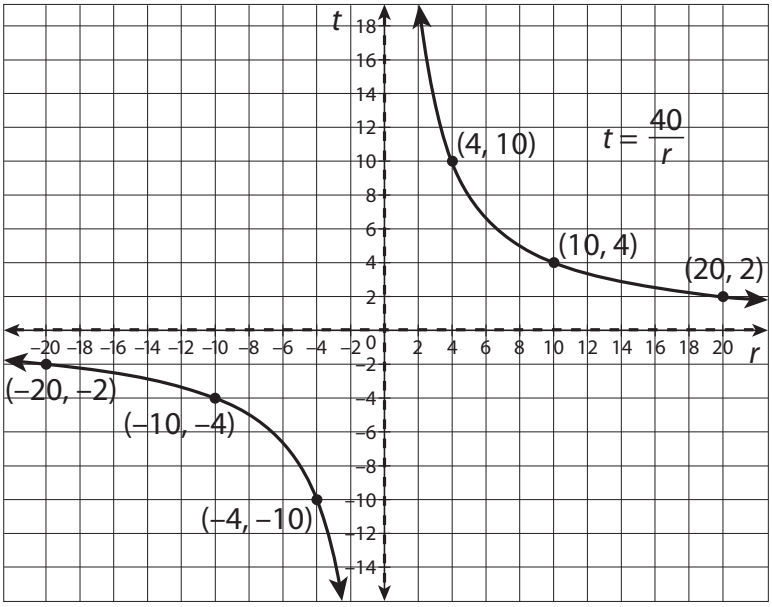
2. Graph the equation.

The equation $t = \frac{40}{r}$ represents a function of the form $f(r) = \frac{a}{r}$, with $a = 40$.

$f(r) = \frac{a}{r}$ is of the same form as $f(x) = \frac{a}{x}$.

The asymptotes are the coordinate axes, and the graph has the same general shape as the graph of the parent function $f(x) = \frac{1}{x}$.

Plot several points to the left and several points to the right of the vertical asymptote, then draw the curved branches of the graph through the plotted points.



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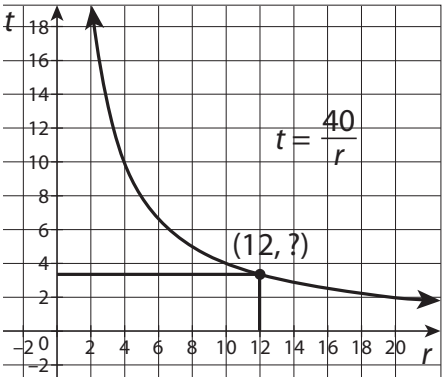
Instruction

3. Determine how long the trip will take at an average speed of 12 miles per hour.

   Because time can only be positive, only positive values make sense for this situation, so consider only the part of the graph in Quadrant I.

   Use the graph to estimate the value of t when $r = 12$.



   When $r = 12$, it appears that t is slightly greater than 3, so a reasonable estimate could be $t \approx 3.2$.

   Calculate the exact value of t.

$t = \frac{40}{r}$	Equation
$t = \frac{40}{(12)}$	Substitute 12 for r.
$t = \frac{10}{3}$ or $3\frac{1}{3}$	Simplify.

   Note that the estimate 3.2 from the graph is reasonable.

   The trip will take $\frac{10}{3}$ hours (simplified to $3\frac{1}{3}$ or approximately 3.33 hours) at an average speed of 12 miles per hour.

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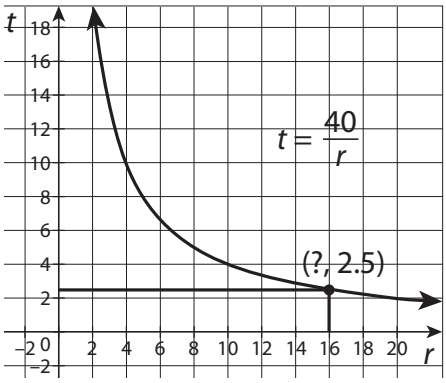
Instruction

4. Determine the average speed needed to complete the trip in 2.5 hours.

Use the graph to estimate the value of r when $t = 2.5$.



When $t = 2.5$, it appears that $r = 16$.

Calculate the exact value of r.

$t = \frac{40}{r}$	Equation
$(2.5) = \frac{40}{r}$	Substitute 2.5 for t.
$2.5r = 40$	Multiply both sides by r.
$r = \frac{40}{2.5}$	Divide both sides by 2.5.
$r = 16$	Divide 40 by 2.5.

Note that the estimate 16 from the graph matches the exact answer.

Sang needs to achieve an average speed of 16 miles per hour to complete the trip in 2.5 hours.

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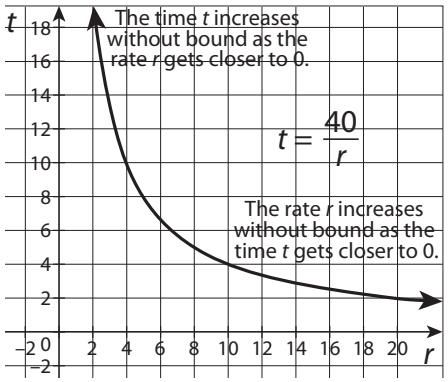
Instruction

5. Determine the effect on t as r increases and the effect on t as r decreases, in the context of the problem situation.

   Use the graph.



   The value of t increases as r gets closer to 0. This means that the time to complete the trip increases as the average speed decreases.

   The value of t gets closer to 0 as r increases. This means that the time to complete the trip decreases as the average speed increases.

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Instruction

Example 3

The newest product at a toy company is a jungle animal puzzle. The startup cost to make the puzzles is $400 and the cost to make each puzzle is $3. Complete the following and answer the questions.

- Write a rational equation to represent the average cost function $A(x)$, where x is the number of puzzles the company makes and $A(x)$ is the average cost per puzzle, in dollars.
- Graph the equation.
- Explain what part of the graph shows the relationship between $A(x)$, the average cost function, and x, the number of puzzles the company makes.
- What is the average cost per puzzle when the company makes 50 puzzles?
- How many puzzles must the company make to have an average cost of $25 per puzzle?
- What is the effect on $A(x)$ as x increases? What is the effect on $A(x)$ as x decreases?

1. Write a rational equation to represent the average cost function $A(x)$, where x is the number of puzzles the company makes and $A(x)$ is the average cost per puzzle.

   Let $C(x)$ represent the total cost, in dollars, to make x puzzles.

   $C(x) = 400 + 3x$ — total cost = $400 startup cost + $3 per puzzle

   $A(x) = \frac{400+3x}{x}$ — The average cost for x items is the total cost divided by x.

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Instruction

2. Graph the function $A(x)$.

The function $A(x)$ from the previous step can be rewritten in an equivalent form that matches one of the forms described earlier.

$A(x)=\dfrac{400+3x}{x}$ Function

$A(x)=\dfrac{400}{x}+\dfrac{3x}{x}$ Rewrite the fraction as a sum of two fractions: $\dfrac{a+b}{c}=\dfrac{a}{c}+\dfrac{b}{c}$.

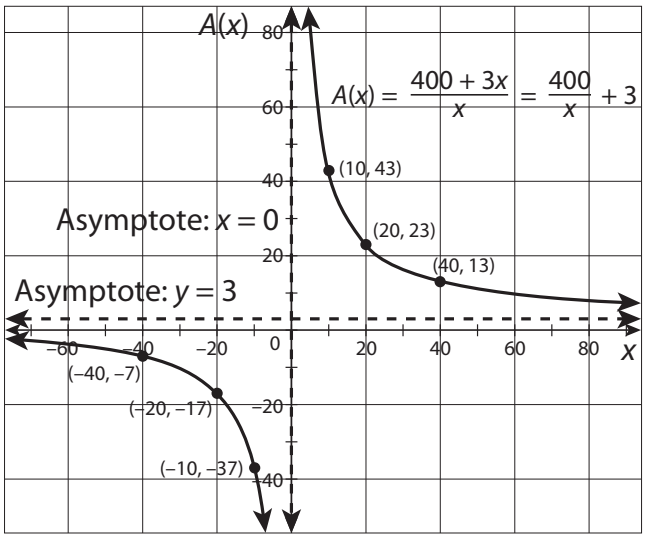
$A(x)=\dfrac{400}{x}+3$ Simplify.

$A(x)=\dfrac{400}{x}+3$, which has the form $f(x)=\dfrac{a}{x-h}+k$, with $a = 400$, $h = 0$, and $k = 3$.

The asymptotes are:

- $x = h$, or in this case $x = 0$
- $y = k$, or in this case $y = 3$

Plot several points to the left of the vertical asymptote and several points to the right of the vertical asymptote. Then draw the graph through the plotted points.



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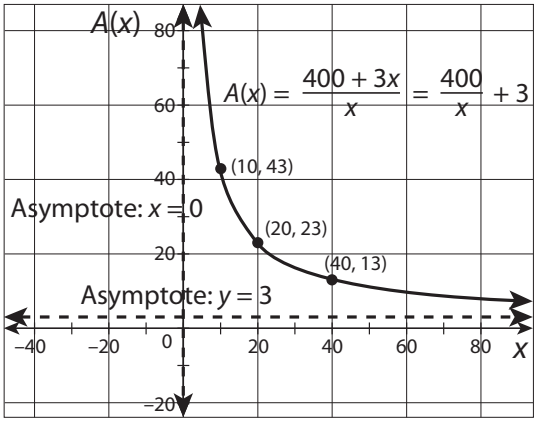
Instruction

3. Explain what part of the graph shows the relationship between $A(x)$, the average cost function, and x, the number of puzzles the company makes.

   The number of puzzles produced can only be a positive number, and the total cost to produce the puzzles can only be positive; therefore, only positive values of x and $A(x)$ make sense in this situation. Only the part of the graph in Quadrant I shows the relationship between $A(x)$, the average cost function, and x, the number of puzzles the company makes.



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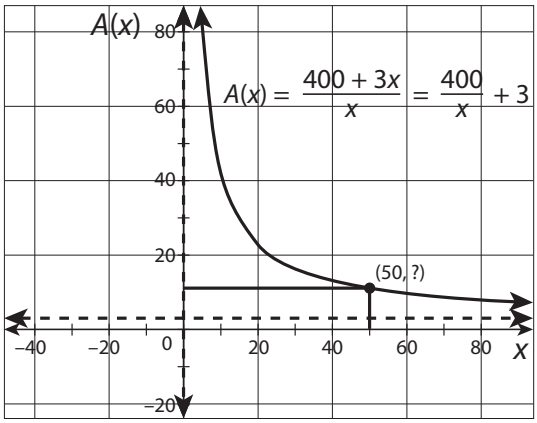
Instruction

4. Find the average cost per puzzle when the company makes 50 puzzles.

Use the graph to estimate $A(50)$.



It appears that $A(50)$ is slightly greater than 10, so a reasonable estimate seems to be $A(50) \approx 11$.

Calculate the exact value of $A(50)$.

$A(x) = \frac{400}{x} + 3$ Equation

$A(50) = \frac{400}{50} + 3$ Substitute 50 for x.

$A(50) = 11$ Simplify.

Note that the estimate 11 from the graph matches the exact value.

The average cost is \$11 per puzzle when the company makes 50 puzzles.

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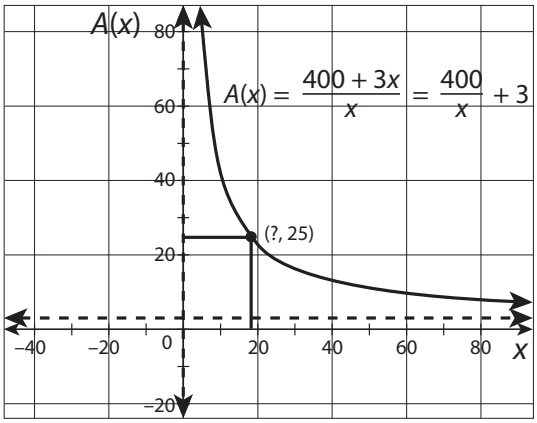
Instruction

5. Find how many puzzles the company must make to have an average cost of $25 per puzzle.

Use the graph to estimate the value of x when $A(x) = 25$.



When $A(x) = 25$, it appears that x is slightly less than 20.

Calculate the exact value of x.

$A(x)=\frac{400}{x}+3$	Write the equation.
$(25)=\frac{400}{x}+3$	Substitute 25 for $A(x)$.
$25x = 400 + 3x$	Multiply both sides by x.
$22x = 400$	Subtract $3x$ from both sides.
$x=\frac{400}{22}$	Divide both sides by 22.
$x=18\frac{2}{11}$	Divide 400 by 22.

The exact solution to the equation is a mixed number, which does not make sense in this context because x represents a number of puzzles, so the best answer is 19.

Note that the estimate "slightly less than 20" is close to the best answer.

The company must make 19 puzzles to make the average cost per puzzle as close as possible to $25.

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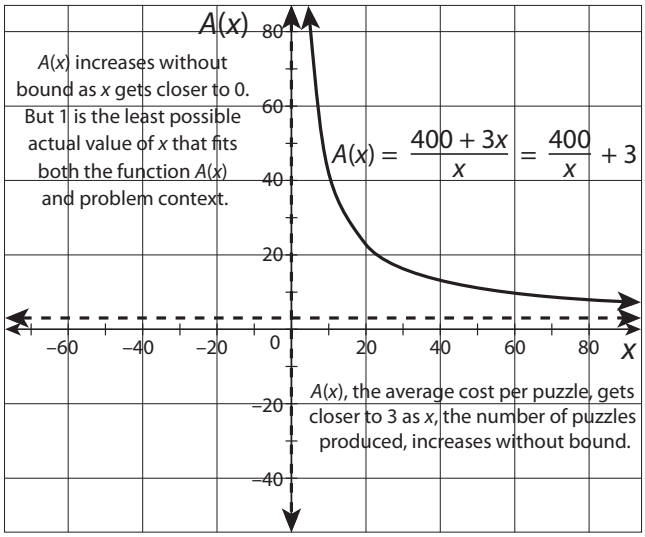
Instruction

6. Determine the effect on $A(x)$ as x increases and the effect on $A(x)$ as x decreases, in the context of the problem situation.

   Use the graph.



   The value of $A(x)$ increases as x gets closer to 0. However, the least possible actual value of x is 0, which represents 0 puzzles produced, and that value of x makes the function rule undefined. Therefore, the least possible actual value of x that fits both the function and the problem context is 1, which represents 1 puzzle produced.

   The value of $A(x)$ gets closer to 3 as x increases, while always being greater than 3. This means that the average cost per puzzle gets closer to \$3, but will always be more than \$3, as more and more puzzles are produced.

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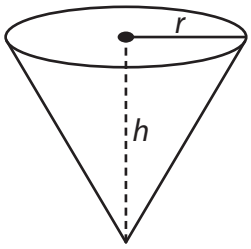
Instruction

Example 4

Michele is a packaging technology consultant. A client wants her to design a cone-shaped cellophane bag that can be used for candy, nuts, party favors, and similar items. The bag must have a volume of 450 cubic inches. Use this information and the following illustration to complete the following.



- Write a rational equation to represent the relationship between *r*, the radius of the base of the cone, and *h*, the height of the cone. Make *r* the independent variable and *h* the dependent variable.
- Graph the equation.
- What height should Michele use if she makes the radius 8 inches?
- What radius should Michele use if she makes the height 12 inches?

1. Write a rational equation to represent the relationship between *r*, the radius of the base of the cone, and *h*, the height of the cone.

$V = \frac{1}{3}\pi r^2 h$	Formula for the volume V of a cone
$(450) = \frac{1}{3}\pi r^2 h$	Substitute 450 for V.
$1350 = \pi r^2 h$	Multiply both sides by 3.
$\frac{1350}{\pi r^2} = h$	Divide both sides by πr^2.

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Instruction

2. Graph the equation.

Functions of the form $f(x)=\frac{a}{x^2}$ were presented earlier. Using r as the independent variable instead of x yields an equivalent form, $f(r)=\frac{a}{r^2}$.

The equation $\frac{1350}{\pi r^2}=h$ can be written in the form $f(r)=\frac{a}{r^2}$, as shown.

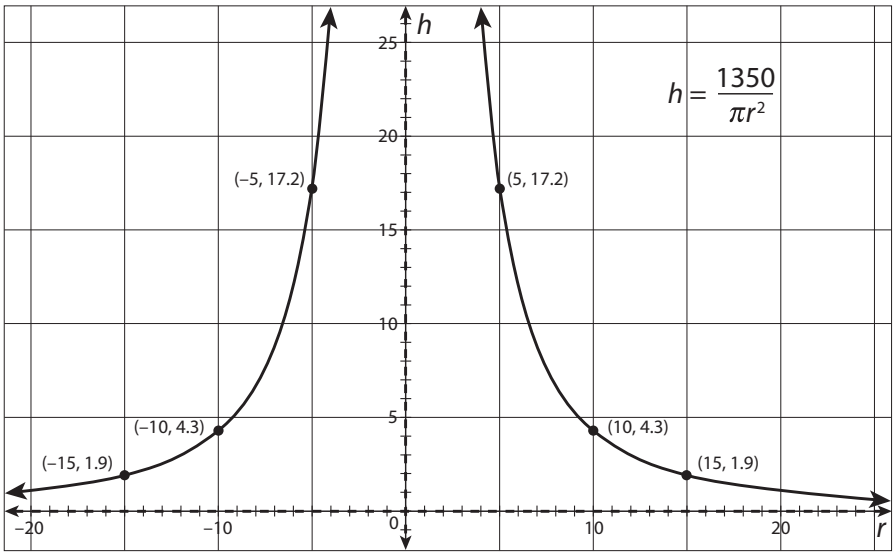
$h=\frac{1350}{\pi r^2}$	Reverse the sides.
$h=\frac{1350}{\pi}\bullet\frac{1}{r^2}$	Use the definition of multiplication of fractions to write a product.
$h=\frac{\frac{1350}{\pi}}{r^2}$	Apply the property $a\bullet\frac{1}{b}=\frac{a}{b}$, where $a=\frac{1350}{\pi}$ and $b = r^2$.

The equation from step 1 has the form $f(r)=\frac{a}{r^2}$, with $a=\frac{1350}{\pi}$.

Therefore, the asymptotes are the coordinate axes and the graph has the same general shape as the graph of the parent function $f(x)=\frac{1}{x^2}$.

Plot several points to the left and several points to the right of the vertical asymptote.

Use a calculator to find approximate h-values for the r-values you substitute into the equation. Then draw the curved branches of the graph through the plotted points.



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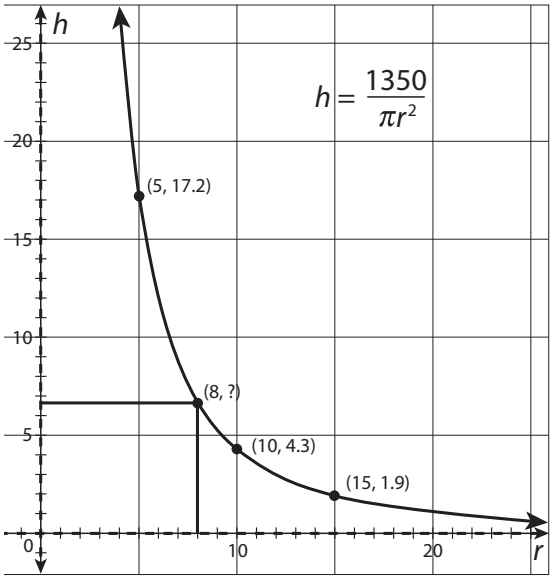
Instruction

3. Determine what height Michele should use if she makes the radius 8 inches.

   The radius, height, and volume of the cone can only be positive; therefore, consider only the part of the graph that applies to the situation.

   Use the graph to estimate the value of h when $r = 8$.



   When $r = 8$, it appears that h is slightly less than 7, so a reasonable estimate seems to be $h \approx 6.8$.

   Calculate the exact value of h.

$$h = \frac{1350}{\pi r^2}$$ Write the equation.

$$h = \frac{1350}{\pi (8)^2}$$ Substitute 8 for r.

$$h \approx 6.71$$

   Note that the estimate 6.8 from the graph is reasonable.

   Michele should use a height of approximately 6.71 inches if she makes the radius 8 inches.

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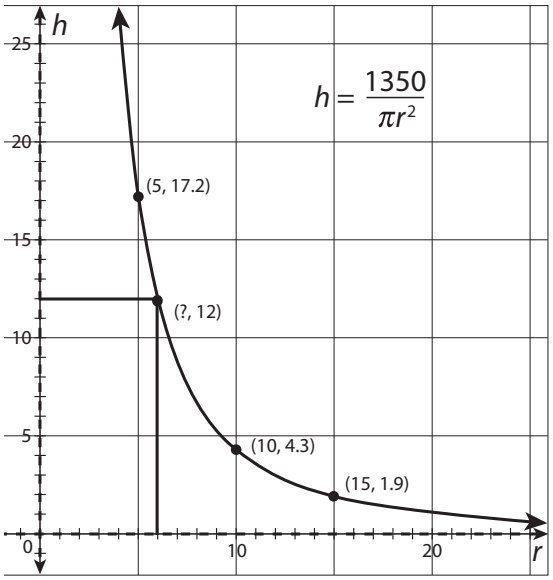
Instruction

4. Determine what radius Michele should use if she makes the height 12 inches.

Use the graph to estimate the value of r when $h = 12$.



When $h = 12$, it appears that r is very close to 6, so a reasonable estimate seems to be $r \approx 6$.

Calculate the exact value of r.

$h = \frac{1350}{\pi r^2}$	Equation
$(12) = \frac{1350}{\pi r^2}$	Substitute 12 for h.
$12\pi r^2 = 1350$	Multiply both sides by πr^2.
$r^2 = \frac{1350}{12\pi}$	Divide both sides by 12π.
$r = \sqrt{\frac{1350}{12\pi}}$	The value of r is the positive square root of the right side.
$r \approx 5.98$	

Note that the estimate 6 from the graph is very close to 5.98.

Michele should use a radius of approximately 5.98 inches if she makes the height 12 inches.

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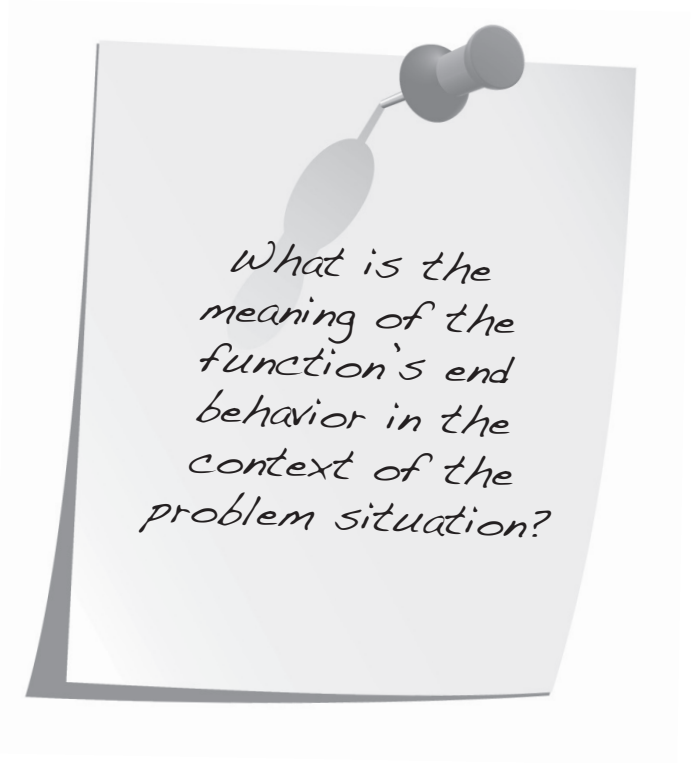
Problem-Based Task 3.5.2: Lacrosse Stats

SMP	
1 ✓	2
3	4 ✓
5	6
7 ✓	8

Latonya is the goaltender on her school's lacrosse team. She blocked 2 shots out of 5 attempted shots on goal in her first game this season, so her save percentage is now 40%. She is interested in knowing how her save percentage will change if she is able to make saves from now on without missing. A save percentage is a ratio, written in percent form. A rational equation is used to represent Latonya's save ratio function $f(x)$, where x is the number of consecutive successful saves she makes from now on. What is the meaning of the function's end behavior in the context of the problem situation?



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Problem-Based Task 3.5.2: Lacrosse Stats

Coaching

a. What is a save ratio, using the terms "saves" and "shots on goal?"

b. What is Latonya's save ratio now? Explain.

c. What will be her save ratio when she makes x consecutive saves?

d. What is the rational function $f(x)$ that represents Latonya's save ratio, where x is the number of consecutive saves she successfully makes from now on?

e. To graph the function, it will help to identify the asymptote(s). What value of a denominator in a rational function rule is associated with a vertical asymptote?

f. What value of x makes the denominator equal to 0?

g. What is the equation of the vertical asymptote?

h. Draw the asymptote and then plot some points on both sides of the asymptote. What are the points on the graph that have the x-values –8, –7, –6, –4, –3, and –2?

i. How can you use the information from parts g and h to help sketch the graph of the function?

j. Sketch the graph, but not very far left of (–8, 2) or very far right of (–2, 0).

k. What are the points on the graph that have x-values –60, –40, –20, 20, 40, and 60? Round to the nearest thousandth if necessary.

l. What value are the y-values getting close to as the x-values get ever farther away from 0 on the left and right (that is, as the x-values decrease and increase without bound)?

m. The function is $f(x)=\frac{x+2}{x+5}$. Is it possible that $f(x) = 1$? Explain.

n. The line $y = b$ is a horizontal asymptote if $f(x)$ gets closer to b as x either increases or decreases without bound. What is the horizontal asymptote for this function?

o. How can you use the information in part n to finish sketching the graph of the function?

p. Finish sketching the graph of the function.

q. What part of the graph is relevant to the situation? Explain.

r. What is the meaning of the function's end behavior in the context of the problem situation?

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Instruction

Problem-Based Task 3.5.2: Lacrosse Stats

Coaching Sample Responses

a. What is a save ratio, using the terms "saves" and "shots on goal?"

The save ratio can be written as $\frac{\text{saves}}{\text{shots on goal}}$.

b. What is Latonya's save ratio now? Explain.

Latonya's save ratio is $\frac{\text{saves}}{\text{shots on goal}} = \frac{2}{5}$. This can be written as 0.4 as well as 40%.

c. What will be her save ratio when she makes x consecutive saves?

Her save ratio can be written as $\frac{2+x}{5+x}$ when she makes x consecutive successful attempts.

d. What is the rational function $f(x)$ that represents Latonya's save ratio, where x is the number of consecutive saves she successfully makes from now on?

The rational equation that represents her save ratio is $f(x) = \frac{2+x}{5+x}$.

e. To graph the function, it will help to identify the asymptote(s). What value of a denominator in a rational function rule is associated with a vertical asymptote?

The value of 0 in the denominator of a rational function is associated with the vertical asymptote.

f. What value of x makes the denominator equal to 0?

Set the denominator equal to 0 and solve for x to find the value that makes the denominator equal to 0.

$5 + x = 0$

$x = -5$

The value of x that makes the denominator equal to 0 is –5.

g. What is the equation of the vertical asymptote?

The equation of the vertical asymptote is $x = -5$.

h. Draw the asymptote and then plot some points on both sides of the asymptote. What are the points on the graph that have the x-values –8, –7, –6, –4, –3, and –2?

By solving for $f(x)$, the points are (–8, 2), (–7, 2.5), (–6, 4), (–4, –2), (–3, –0.5), and (–2, 0).

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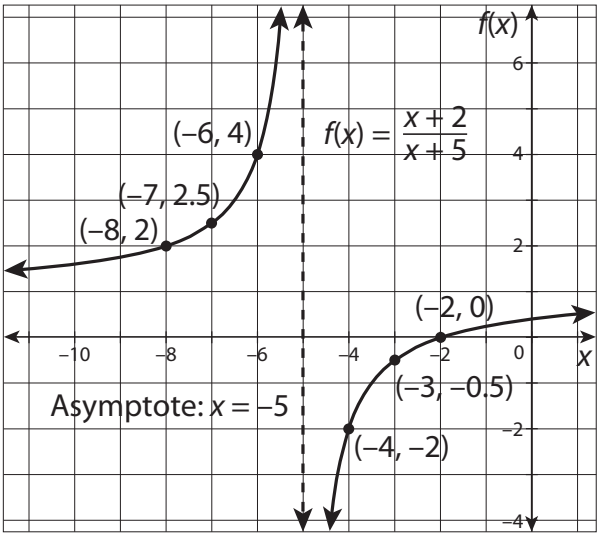
Instruction

i. How can you use the information from parts g and h to help sketch the graph of the function?

Draw the asymptote, plot the points, and then sketch the graph of the function through the plotted points.

j. Sketch the graph, but not very far left of (–8, 2) or very far right of (–2, 0).



k. What are the points on the graph that have x-values –60, –40, –20, 20, 40, and 60? Round to the nearest thousandth if necessary.

Use the equation for $f(x)$ to calculate each point. The points are (–60, 1.055), (–40, 1.086), (–20, 1.2), (20, 0.88), (40, 0.933), and (60, 0.954).

l. What value are the y-values getting close to as the x-values get ever farther away from 0 on the left and right (that is, as the x-values decrease and increase without bound)?

The y-values are getting closer to 1.

m. The function is $f(x)=\frac{x+2}{x+5}$. Is it possible that $f(x) = 1$? Explain.

No, it is not possible. The numerator and denominator are never equal; therefore, there is no value of x that makes $x + 2 = x + 5$.

n. The line $y = b$ is a horizontal asymptote if $f(x)$ gets closer to b as x either increases or decreases without bound. What is the horizontal asymptote for this function?

The horizontal asymptote for this function is $y = 1$.

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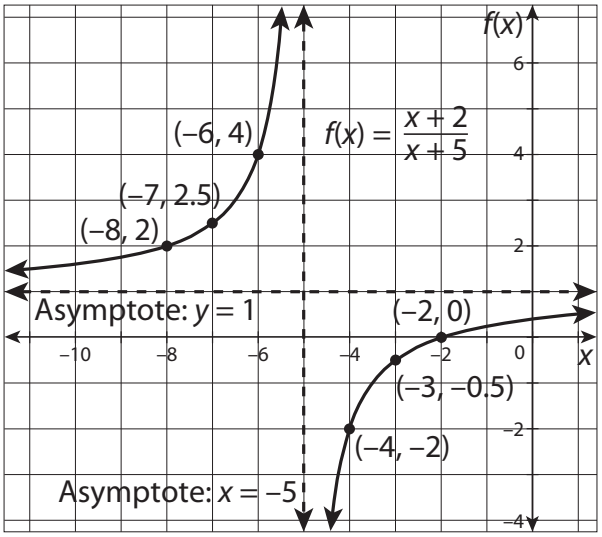
Instruction

o. How can you use the information in part n to finish sketching the graph of the function?

Draw the horizontal asymptote $y = 1$ and extend the curved branches of the function graph to the left and to the right, showing that they get close to the asymptote but do not touch it.

p. Finish sketching the graph of the function.



q. What part of the graph is relevant to the situation? Explain.

The relevant part of the graph is the intersection with the y-axis and all points to the right that have integer x-values. Explanation: x represents the number of consecutive successful saves she makes from now on, so x must be a whole number. That is, x must be in the set {0, 1, 2, 3, ...}.

r. What is the meaning of the function's end behavior in the context of the problem situation?

The description of the end behavior is: $f(x)$ gets closer to 1, but is always less than 1, as x increases without bound. In the context of the problem situation, this means that Latonya's save percentage gets closer to 100%, but is always less than 100%, as she makes even more consecutive successful saves.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.5.2: Graphing Rational Equations

Use what you have learned about graphing rational equations to complete problem 1.

1. Graph the function $f(x)=\frac{-4}{x-1}-2$. Identify the domain, range, and any asymptote(s). Describe the function's end behavior and behavior near its asymptote(s).

Use the information in the following scenario to complete problems 2–4.

> Nora wants to remodel part of her house. She wants a rectangular living room with an area of 360 square feet.

2. Write a rational equation to represent the relationship between w, the width of the living room in feet, and l, the length in feet. Make w the independent variable and l the dependent variable.

3. Graph the equation and label its asymptote(s). Label the points on the equation's graph that have w-coordinates of –30, –20, –12, 12, 20, and 30.

4. Answer the following questions in the context of Nora's problem situation.
   a. What is the meaning, if any, of the six points you labeled on the graph?
   b. What is the effect on l as w increases?
   c. What is the effect on l as w decreases?

Use the information in the following scenario to complete problems 5–7.

> Huan has just started a business for screen printing and selling T-shirts. His startup cost is $600 and his cost to process each shirt is $5.

5. Write a rational equation to represent the average cost function $A(x)$, where x is the number of shirts Huan processes and $A(x)$ is the average processing cost per shirt, in dollars.

continued

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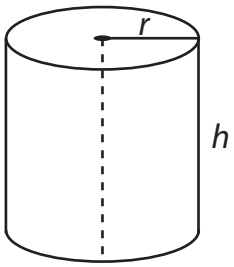
6. Graph the equation and label its asymptote(s). Label the points on the equation's graph that have x-coordinates of –40, –20, –10, 10, 20, and 40.

7. Answer the following questions in the context of Huan's problem situation.
   a. What is the meaning, if any, of the six points you labeled on the graph?
   b. What is the effect on $A(x)$ as x increases?
   c. What is the effect on $A(x)$ as x decreases?

Use the information in the following scenario and the diagram to complete problems 8–10.

Tremaine is a designer for a company that manufactures packaging containers. He needs to design a cylindrical container with a volume of 400 cubic inches. The formula for the volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.



8. Write a rational equation to represent the relationship between r and h for a container that meets Tremaine's requirements. Make r the independent variable and h the dependent variable.

9. Graph the equation and label its asymptote(s). Label the points on the equation's graph that have r-coordinates of –6, –5, –4, –3, 3, 4, 5, and 6. Round h-coordinates to the nearest tenth if rounding is necessary.

10. Tremaine decides that he wants the diameter and height of the container to be equal. Which point on the graph represents the container that matches what he wants? Explain.

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Lesson 3.5.3: Creating Rational Inequalities

Warm-Up 3.5.3

Mr. and Mrs. Barton have just bought several horses. They want to fence in a rectangular horse corral with an area of 2,500 square yards. They will have to choose the dimensions of the rectangle and then have the fence installed. The Bartons have seven grandchildren who are anxious to ride the horses. To investigate some of their options, they assign each grandchild a distance in yards that each child is likely to need for riding. Each distance doubles as a possible length for the corral. Determine the corresponding width and perimeter.

Grandchild	Corral dimensions per grandchild			
	Area (yd^2)	Length (yd)	Width (yd)	Perimeter (yd)
Doug	2,500	20		
Marin	2,500	30		
Cam	2,500	40		
Felicia	2,500	50		
Travis	2,500	60		
Judy	2,500	70		
Lynn	2,500	80		

1. Complete the table. Round decimals to the nearest hundredth if necessary.

2. Whose rectangle will have the lowest fence cost? Explain.

3. Based on the data, what is the best shape to use if you want to minimize the perimeter of a rectangle with a given area?

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Instruction

Lesson 3.5.3: Creating Rational Inequalities

Common Core State Standard

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

Warm-Up 3.5.3 Debrief

1. Complete the table. Round decimals to the nearest hundredth if necessary.

Grandchild	Corral dimensions per grandchild			
	Area (yd^2)	Length (yd)	Width (yd)	Perimeter (yd)
Doug	2,500	20	125	290
Marin	2,500	30	83.33	226.67
Cam	2,500	40	62.5	205
Felicia	2,500	50	50	200
Travis	2,500	60	41.67	203.33
Judy	2,500	70	35.72	211.43
Lynn	2,500	80	31.25	222.5

2. Whose rectangle will have the lowest fence cost? Explain.

   Felicia's rectangle, at a width and length of 50 yards, will have the lowest fence cost because it has the smallest perimeter.

3. Based on the data, what is the best shape to use if you want to minimize the perimeter of a rectangle with a given area?

   Felicia's rectangle is a square, with all sides measuring 50 yards. Based on this data, the best shape to use in order to minimize the perimeter of a rectangle is a square.

Connection to the Lesson

- Students will extend their understanding of area and calculating unknown lengths to making calculations using rational expressions.
- Students will solve problems in which they have a given area and a given maximum perimeter of a rectangle and must determine all possible dimensions of the rectangle.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- identifying the least common denominator of rational expressions (A–APR.6)
- adding and subtracting rational expressions using the least common denominator (A–APR.7 (+))
- recognizing that a rational expression is undefined if it has a denominator equal to 0 (A–APR.6)
- recognizing that a rational function is undefined at $x = a$ if $x = a$ is an asymptote (F–IF.7d★ (+))
- evaluating functions (F–IF.2)
- using the quadratic formula to solve quadratic equations (A–REI.4b)

Introduction

Recall that a rational equation is an equation that includes the ratio of two rational expressions, in which a variable appears in the denominator of at least one rational expression. Similarly, a **rational inequality** is an inequality that includes a ratio of two rational expressions, in which a variable appears in a denominator of at least one rational expression, such as $\frac{4}{x-3} < 2$. The solutions to rational inequalities are written as intervals. An interval is the set of all real numbers between two given numbers. The two numbers on the ends are the endpoints. The endpoints might or might not be included in the interval depending on whether the interval is open, closed, or half-open/half-closed. Sometimes you need to create a rational inequality and then use it to solve a real-world problem in which only positive numbers apply to the situation.

Key Concepts

- Recall that when you multiply or divide both sides of an inequality by a negative number, you must reverse the inequality symbol.
- Consider the inequality $8 < 12$.
  - Multiply both sides by –2:

    $8 < 12$ Original inequality

    $-16 > -24$ Multiply both sides by –2. Reverse the inequality symbol.

SRB p. U3-225

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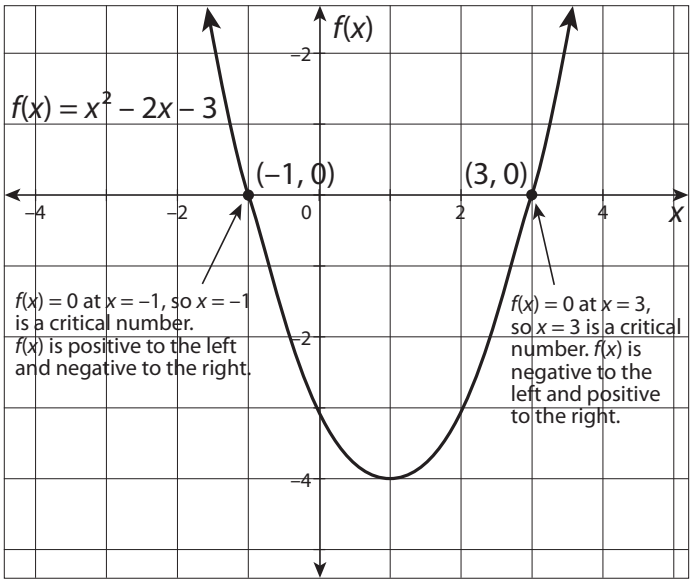
Instruction

- Divide both sides by –4:

  8 < 12 Original inequality

  –2 > –3 Divide both sides by –4. Reverse the inequality symbol.

- To solve an inequality means to find the complete solution set.
- The method of solving inequalities in this lesson involves using critical numbers.
- First consider polynomial inequalities.
- If $f(x)$ is a polynomial function and an inequality is written in any of these forms: $f(x) < 0$, $f(x) \leq 0$, $f(x) > 0$, or $f(x) \geq 0$, then the **critical numbers of the polynomial inequality** are the x-values that make $f(x) = 0$.
- Polynomial functions can change signs only at their critical numbers.
- The critical numbers for the polynomial function $f(x) = x^2 - 2x - 3$ are –1 and 3, as shown in the following graph. Notice that between –1 and 3, the value of $f(x)$ is negative, and outside of this interval, the value of $f(x)$ is positive.



- Now consider rational inequalities.
- If $f(x)$ is a rational function and an inequality is written in any of these forms ($f(x) < 0$, $f(x) \leq 0$, $f(x) > 0$, or $f(x) \geq 0$), then the **critical numbers of the rational inequality** are the x-values that make $f(x) = 0$ or make $f(x)$ undefined.
- Similar to polynomial functions, rational functions can change signs only at their critical numbers.

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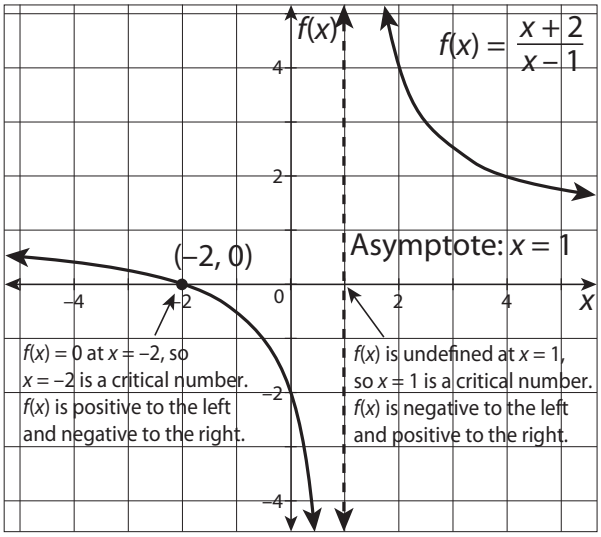
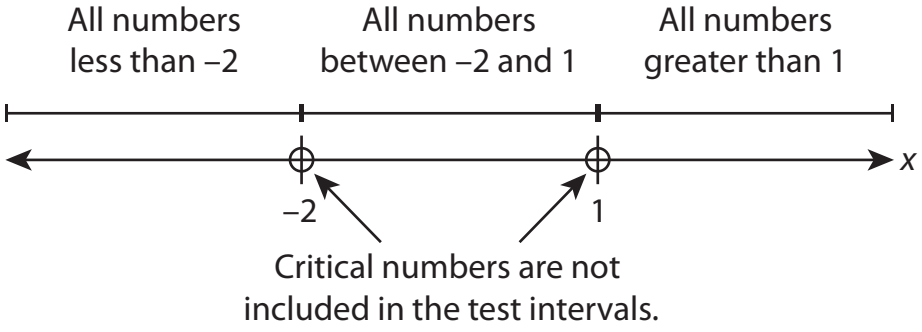
Instruction

- The critical numbers for a rational function are the x-values that make either the numerator or the denominator equal to 0.
- The rational function $f(x)=\frac{x+2}{x-1}$ and its critical numbers are shown in the following graph. Note that $x = -2$ is a critical number because it makes the numerator equal to 0, and $x = 1$ is a critical number because it makes the denominator equal to 0.



- Critical numbers separate the x-axis into **test intervals**, intervals on the x-axis formed by one or more critical numbers.
- The sign of the function on the test interval is the same as the sign of the function value at any x-value in the interval.
- The critical numbers –2 and 1 separate the x-axis into three test intervals, as shown in the following diagram.

Test intervals



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Instruction

- Polynomial functions and rational functions can change signs only at their critical numbers. *Note*: Just because a function has a critical value, it doesn't necessarily change signs. If the sign does change, it will occur at a critical value.
- To determine the sign of all function values within a test interval, you only need to test one function value and determine its sign.
- Consider the function $f(x)=\frac{x+2}{x-1}$.
- If you want to know the sign of the function for all x-values greater than 1, test the function value at any x-value greater than 1, say $x = 3$: $f(3)=\frac{3+2}{3-1}=\frac{5}{2}$. This function value is positive, so the function is positive for all x-values greater than 1.
- A test interval on the x-axis is an example of an interval on a number line.
- In this lesson, intervals are represented in three ways: number line graphs, algebraic descriptions, and interval notation.
- **Interval notation** is a way of representing an interval on a number line using a pair of parentheses, a pair of brackets, or a parenthesis and a bracket.
- An example of an interval on a number line is the set of all real numbers between 2 and 5, but not including 2 or 5.
- The diagram that follows shows the three ways of representing this interval.

Number line graph

$2 < x < 5$ Algebraic description

(2, 5) Interval notation

- *Note*: The interval notation (2, 5) appears to be exactly the same as the ordered pair notation (2, 5).
- The interval notation (2, 5) represents the set of all real numbers between 2 and 5, but not including 2 or 5.
- The ordered pair notation (2, 5) represents an ordered pair of two real numbers; this ordered pair represents a single point in the coordinate plane.
- The context of the problem will always make clear which meaning is intended.

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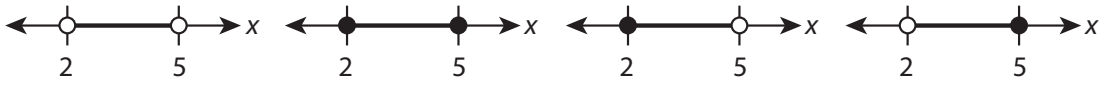
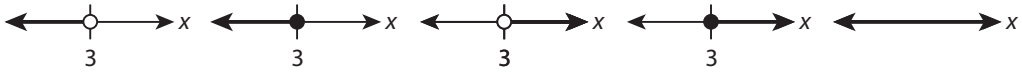
Instruction

- Depending on the type of interval, interval notation can be written with parentheses ( ), brackets [ ], or a combination of both: [ ) or ( ].
- An **open interval** does not include its endpoints.
  - Example: (2, 5) is an open interval because it does not include 2 or 5.
- A **closed interval** includes its endpoints.
  - Example: [2, 5] is a closed interval because it includes 2 and 5.
- A **half-open** (or **half-closed**) **interval** includes one endpoint but not the other.
  - Example: [2, 5) is a half-open (or half-closed) interval because it includes 2, but not 5.
  - Example: (2, 5] is a half-open (or half-closed) interval because it includes 5, but not 2.
- The diagram that follows shows these intervals.



$2 < x < 5$ (2, 5) | $2 \le x \le 5$ [2, 5] | $2 \le x < 5$ [2, 5) | $2 < x \le 5$ (2, 5]

- An interval that extends to **infinity** goes on without bound. Infinity is represented by the symbol ∞.
- The symbols $-\infty$ (negative infinity) and ∞ (positive infinity) are used when representing intervals that are unbounded on either one side or both sides.
- These symbols do not represent any particular real number because given any real number, you can always find a lesser or greater number.
- When an infinity symbol appears in interval notation, it is always written with a parenthesis to indicate an open interval.
- The diagram that follows shows intervals using the infinity symbols.
- The first four intervals are each unbounded on one side; the last interval, all real numbers, is unbounded on both sides.



$x < 3$ $(-\infty, 3)$ | $x \le 3$ $(-\infty, 3]$ | $x > 3$ $(3, \infty)$ | $x \ge 3$ $[3, \infty)$ | All real numbers $(-\infty, \infty)$

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Instruction

- A previous diagram showing critical numbers and test intervals is repeated here, this time including interval notation for each test interval.

Test intervals

All numbers less than –2	All numbers between –2 and 1	All numbers greater than 1
$(-\infty, -2)$	$(-2, 1)$	$(1, \infty)$

Critical numbers are not included in the test intervals.

- Interval notation can be used to represent a union of two intervals.
- The symbol for union is $\cup$.
- The diagram that follows shows a union of two intervals.

$x < 2$ or $x \geq 5$
$(-\infty, 2) \cup [5, \infty)$

Common Errors/Misconceptions

- confusing open interval notation with ordered pair notation
- neglecting to distribute the negative sign when multiplying a binomial by a negative number
- forgetting to change the direction of the inequality symbol when multiplying or dividing by a negative number
- confusing closed and open interval notation

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Instruction

Guided Practice 3.5.3

Example 1

Solve $\frac{-2x+2}{x-4} \leq 0$.

1. Find the critical numbers and use them to identify the test intervals.

   The critical numbers are the x-values that make either the numerator or the denominator equal to 0.

$-2x + 2 = 0$	Set the numerator equal to 0.
$-2x = -2$	Subtract 2 from both sides.
$x = 1$	Divide both sides by –2.

$x - 4 = 0$	Set the denominator equal to 0.
$x = 4$	Add 4 to both sides.

   The critical numbers are $x = 1$ and $x = 4$.

   For the rational inequality $\frac{-2x+2}{x-4} \leq 0$, the related rational function is $f(x) = \frac{-2x+2}{x-4}$.

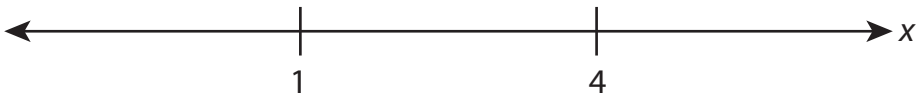
   Verify that $x = 1$ and $x = 4$ are critical numbers:

   $f(1) = \frac{-2(1)+2}{1-4} = \frac{0}{-3} = 0$ and $f(4) = \frac{-2(4)+2}{4-4} = \frac{-6}{0}$, which is undefined.

   The critical numbers 1 and 4 form the test intervals $(-\infty, 1)$, $(1, 4)$, and $(4, \infty)$.

Test intervals

All numbers less than 1 $(-\infty, 1)$	All numbers between 1 and 4 $(1, 4)$	All numbers greater than 4 $(4, \infty)$



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2. Determine the sign of the related rational function in each test interval.

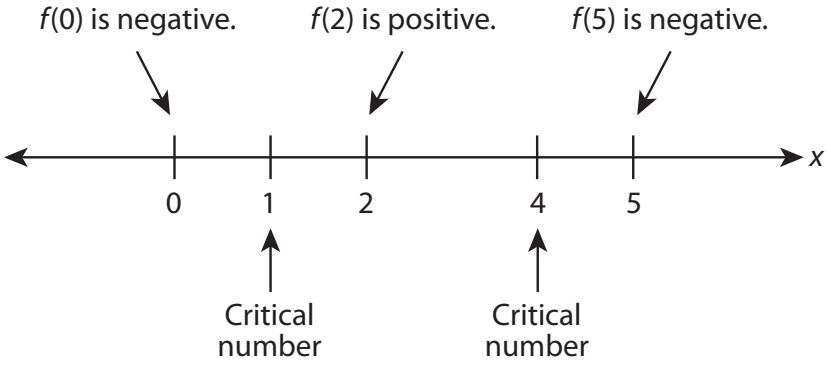
The related rational function is $f(x)=\frac{-2x+2}{x-4}$.

The sign of $f(x)$ does not change within a test interval.

Choose any x-value in each test interval and determine the sign of $f(x)$ at that value; the function will have that sign on the entire test interval.

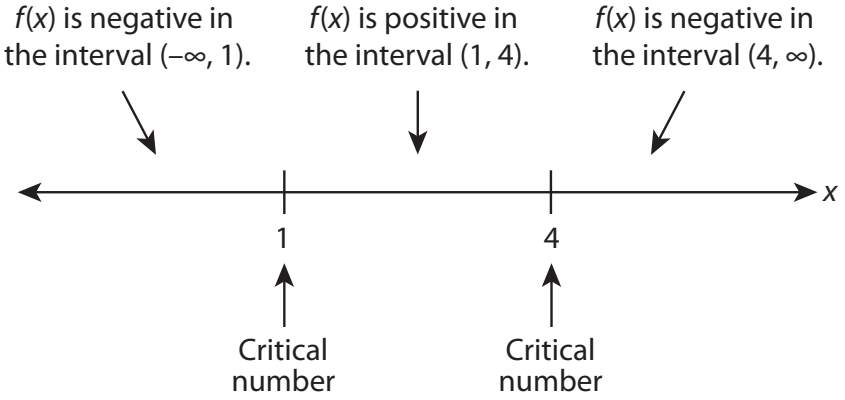
Test intervals at given values of *x*

$f(0)-\frac{-2(0)+2}{0-4}=\frac{2}{-4}=-\frac{1}{2}$ $\quad f(2)-\frac{-2(2)+2}{2-4}=\frac{-2}{-2}=1$ $\quad f(5)-\frac{-2(5)+2}{5-4}=\frac{-8}{1}=-8$



Therefore, the signs of $f(x)$ in the test intervals are as shown below.

Signs of *f*(*x*)



(*continued*)

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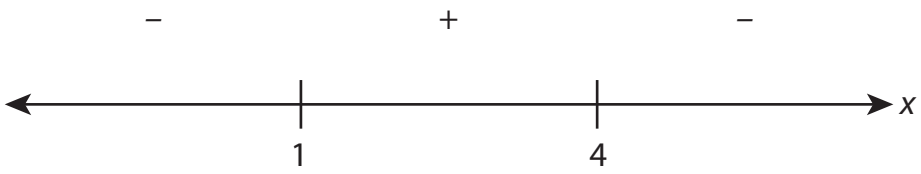
Instruction

The following summary diagram shows the signs of $f(x)$ in the test intervals and the values of $f(x)$ at the critical numbers.

Signs and values of $f(x)$

$$f(x) = \frac{-2x + 2}{x - 4}$$

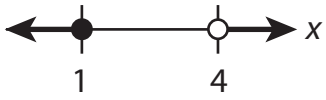


$f(1) = 0$ $f(4)$ is undefined.

3. Use the summary diagram from step 2 to identify the solution set of the inequality.

The diagram shows that $\frac{-2x+2}{x-4} \leq 0$ at all x-values less than 1, at $x = 1$, and at all x-values greater than 4.

The solution set of the inequality is given in three different but equivalent forms below.

$x \leq 1$ or $x > 4$	Algebraic description
$(-\infty, 1] \cup (4, \infty)$	Interval notation
	Number line graph

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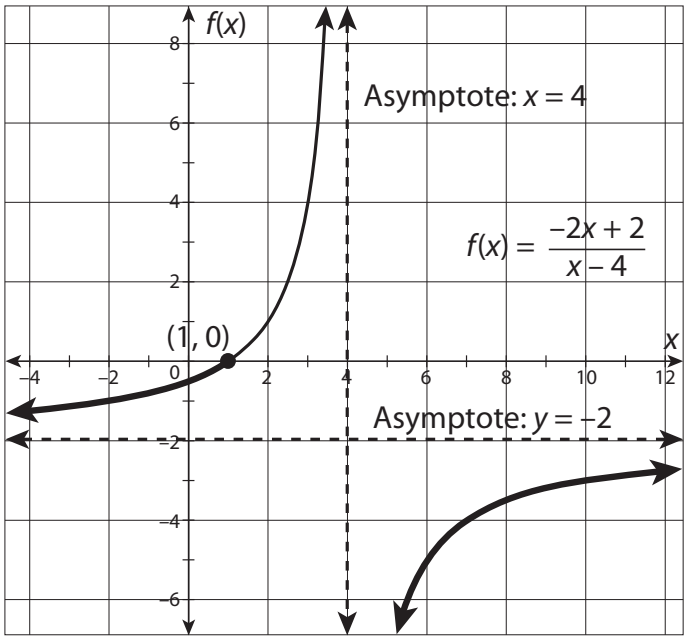
Instruction

4. Connect the solution set of the rational inequality to the graph of its related rational function.

The following diagram shows the graph of the related function $f(x)=\frac{-2x+2}{x-4}$.

The parts of the graph that correspond to the solution set of $\frac{-2x+2}{x-4}\leq 0$ are emphasized.



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Instruction

Example 2

Solve $\frac{4x+3}{x-2}<2$.

1. Write the inequality in an equivalent form with 0 on one side and the related rational function on the other side.

$\frac{4x+3}{x-2}<2$	Original inequality
$\frac{4x+3}{x-2}-2<0$	Subtract 2 from both sides.
$\frac{4x+3}{x-2}+(-2)<0$	Rewrite the subtraction as an addition.
$\frac{4x+3}{x-2}+(-2)\frac{x-2}{x-2}<0$	Multiply –2 by a fraction equal to 1 whose denominator is the least common denominator (LCD).
$\frac{4x+3}{x-2}+\frac{-2(x-2)}{x-2}<0$	Simplify.
$\frac{4x+3}{x-2}+\frac{-2x+4}{x-2}<0$	Distribute.
$\frac{4x+3-2x+4}{x-2}<0$	Add the fractions.
$\frac{2x+7}{x-2}<0$	Simplify.

The inequality $\frac{2x+7}{x-2}<0$ is equivalent to the original inequality.

The related rational function is $f(x)=\frac{2x+7}{x-2}$.

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Instruction

2. Find the critical numbers and use them to identify the test intervals.

The critical numbers are the x-values that make either the numerator or the denominator equal to 0.

$2x + 7 = 0$	Set the numerator equal to 0.
$2x = -7$	Subtract 7 from both sides.
$x = -\frac{7}{2}$	Divide both sides by –2.
$x - 2 = 0$	Set the denominator equal to 0.
$x = 2$	Add 2 to both sides.

The critical numbers are $x = -\frac{7}{2}$ and $x = 2$.

Verify that $x = -\frac{7}{2}$ and $x = 2$ are critical numbers:

$$f\left(-\frac{7}{2}\right) = \frac{2\left(-\frac{7}{2}\right)+7}{-\frac{7}{2}-2} = \frac{-7+7}{-\frac{7}{2}-2} = \frac{0}{-\frac{7}{2}-2} = 0 \text{ and}$$

$$f(2) = \frac{2(2)+7}{2-2} = \frac{11}{0}, \text{ which is undefined.}$$

The critical numbers $-\frac{7}{2}$ and 2 form the test intervals $\left(-\infty, -\frac{7}{2}\right)$, $\left(-\frac{7}{2}, 2\right)$, and $(2, \infty)$.

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Instruction

3. Determine the sign of the related rational function in each test interval.

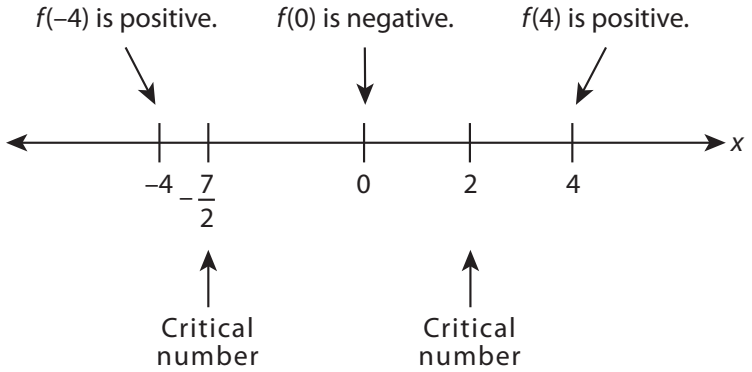
The related rational function is $f(x)=\frac{2x+7}{x-2}$.

The sign of $f(x)$ does not change within a test interval.

Choose any x-value in each test interval and determine the sign of $f(x)$ at that value; the function will have that sign on the entire test interval.

Test intervals at given values of *x*

$f(-4)=\frac{2(-4)+7}{-4-2}=\frac{-1}{-6}=\frac{1}{6}$ $f(0)=\frac{2(0)+7}{0-2}=\frac{7}{-2}=-\frac{7}{2}$ $f(4)=\frac{2(4)+7}{4-2}=\frac{15}{2}=7\frac{1}{2}$



The following summary diagram shows the signs of $f(x)$ in the test intervals and the values of $f(x)$ at the critical numbers.

Signs and values of *f*(*x*)

$$f(x)=\frac{2x+7}{x-2}$$

$f\left(-\frac{7}{2}\right)=0$ $f(2)$ is undefined.

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Instruction

4. Use the summary diagram in step 3 to identify the solution set.

The diagram shows that $\frac{2x+7}{x-2}<0$ at all x-values between $-\frac{7}{2}$ and 2.

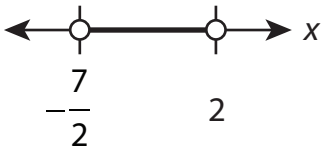
Note: "Between" means not including the end values.

The inequality $\frac{2x+7}{x-2}<0$ is equivalent to the original inequality $\frac{4x+3}{x-2}<2$, so the two inequalities have the same solution set.

The solution set is given in three different but equivalent forms below.

$-\frac{7}{2}<x<2$ — Algebraic description

$\left(-\frac{7}{2}, 2\right)$ — Interval notation



Number line graph

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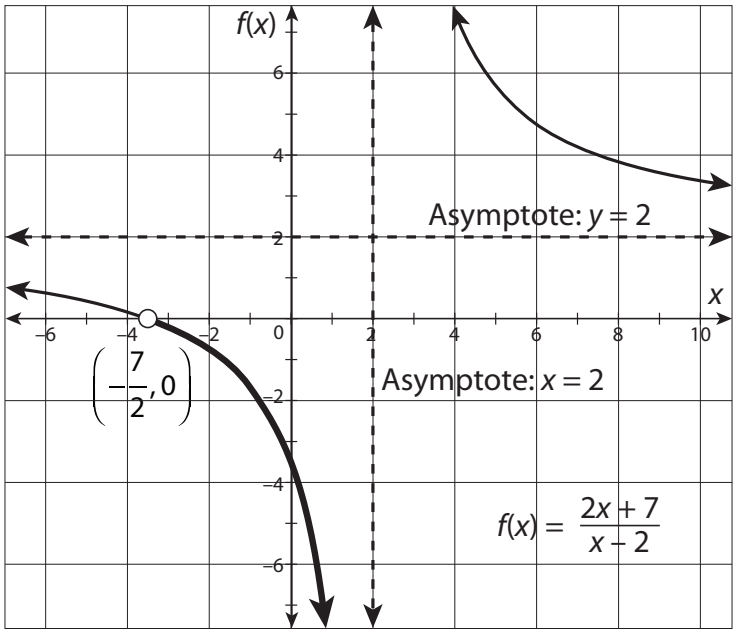
Instruction

5. Connect the solution set of $\frac{2x+7}{x-2}<0$ to the graph of its related rational function.

The following diagram shows the graph of $f(x)=\frac{2x+7}{x-2}$, the related function of the inequality $\frac{2x+7}{x-2}<0$.

The part of the graph that corresponds to the solution set of $\frac{2x+7}{x-2}<0$ is emphasized.



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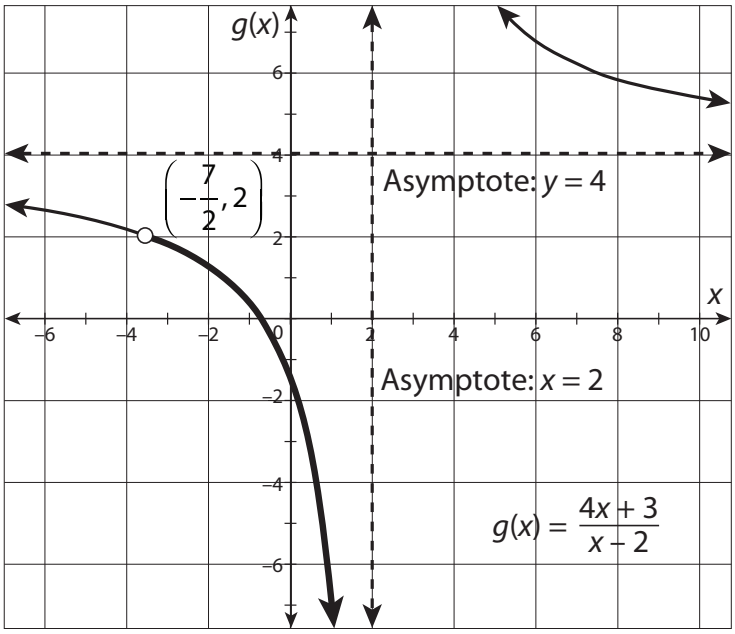
6. The solution set shown in step 4 is the solution set of both $\frac{2x+7}{x-2}<0$ and the original inequality $\frac{4x+3}{x-2}<2$. Connect the solution set to the graph of the function $g(x)=\frac{4x+3}{x-2}$.

The following diagram shows the graph of $g(x)=\frac{4x+3}{x-2}$.

The part of the graph that corresponds to the solution set is emphasized.



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Instruction

Example 3

Ashrita makes and sells leather goods. One of her new products is a belt. Her startup cost for tools and equipment to make the belts is \$160 and it costs her \$10 to make each belt. She sells each belt for \$18. Ashrita wants to earn an average profit of at least \$5 per belt. How many belts could she make and sell to earn an average profit of at least \$5 per belt? What will be her minimum profit if she achieves her goal?

1. Define an average profit function to represent the situation.

   First, define a revenue function $R(x)$ in dollars, a cost function $C(x)$ in dollars, and a profit function $P(x)$ in dollars.

   $R(x) = 18x$ is the revenue taken in if x belts are sold.

   $C(x) = 160 + 10x$ is the cost to make x belts.

   $P(x) = R(x) - C(x)$ is the total profit from making and selling x belts, because profit = revenue – cost.

   Now define an average profit function $P_A(x)$ in dollars per belt.

   $P_A(x) = \frac{P(x)}{x}$ An average is the total divided by the number of items.

   $P_A(x) = \frac{[R(x) - C(x)]}{x}$ Substitute the function rule for $P(x)$.

   $P_A(x) = \frac{[(18x) - (160 + 10x)]}{x}$ Substitute the function rules for $R(x)$ and $C(x)$.

   $P_A(x) = \frac{8x - 160}{x}$ Simplify.

   The average profit function is $P_A(x) = \frac{8x - 160}{x}$, where $P_A(x)$ is dollars of profit per belt and x is the number of belts made and sold.

2. Using the average profit function, write an inequality that can be used to solve the problem.

   Ashrita wants to earn an average profit of at least \$5 per belt, so the inequality is $\frac{8x - 160}{x} \geq 5$.

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Instruction

3. Write the inequality $\frac{8x-160}{x} \geq 5$ in an equivalent form with 0 on one side and the related rational function on the other side of the inequality.

$\frac{8x-160}{x} \geq 5$	Original inequality
$\frac{8x-160}{x} - 5 \geq 0$	Subtract 5 from both sides.
$\frac{8x-160}{x} - 5 \bullet \frac{x}{x} \geq 0$	Multiply –5 by a fraction equal to 1 whose denominator is the LCD.
$\frac{8x-160-5x}{x} \geq 0$	Combine the fractions.
$\frac{3x-160}{x} \geq 0$	Simplify.

The inequality $\frac{3x-160}{x} \geq 0$ is equivalent to the original inequality.

The related rational function is $f(x) = \frac{3x-160}{x}$.

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4. To solve the inequality $\frac{8x-160}{x} \geq 5$, you need to solve the inequality $\frac{3x-160}{x} \geq 0$.

   Begin by finding the critical numbers and using them to identify the test intervals.

$\frac{3x-160}{x} \geq 0$	Inequality
$3x - 160 = 0$	Set the numerator equal to 0.
$3x = 160$	Add 160 to both sides.
$x = 53\frac{1}{3}$	Divide both sides by 3.
$x = 0$	Set the denominator equal to 0; this is already solved.

   The critical numbers are $x = 0$ and $x = 53\frac{1}{3}$; they form the test intervals $(-\infty, 0)$, $\left(0, 53\frac{1}{3}\right)$, and $\left(53\frac{1}{3}, \infty\right)$.

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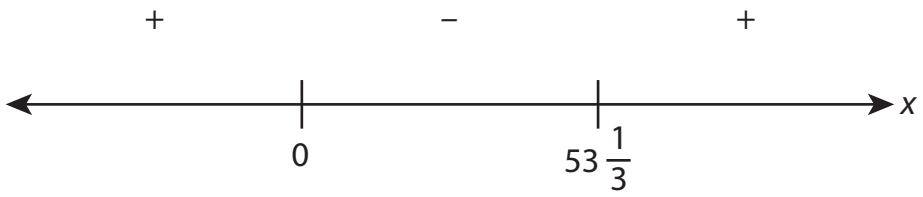
5. Determine the sign of the related rational function in each test interval.

The related rational function is $f(x)=\frac{3x-160}{x}$.

Choose any x-value in each test interval and determine the sign of $f(x)$ at that value; the function will have that sign on the entire test interval.

The following summary diagram shows the signs of $f(x)$ in the test intervals and the values of $f(x)$ at the critical numbers.

$$f(x)=\frac{3x-160}{x}$$



$f(0)$ is undefined. $f\left(53\frac{1}{3}\right)=0$

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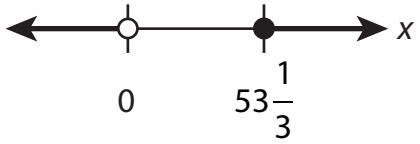
6. Use the summary diagram in step 5 to identify the solution set of the inequality.

   The diagram shows that $\frac{3x-160}{x} \geq 0$ at all x-values less than 0 and at all x-values greater than or equal to $53\frac{1}{3}$.

   The solution set, shown in three forms below, is the solution set of both $\frac{3x-160}{x} \geq 0$ and $\frac{8x-160}{x} \geq 5$ because these two inequalities are equivalent.

   $x < 0$ or $x \geq 53\frac{1}{3}$ — Algebraic description

   $(-\infty, 0) \cup \left[53\frac{1}{3}, \infty\right)$ — Interval notation

   Number line graph

7. Use the solution set of the inequality to identify the answers to the problem.

   You need to find how many belts Ashrita could make and sell to earn an average profit of at least \$5 per belt. Because x represents a number of belts, neither negative numbers nor mixed numbers make sense in this situation.

   Based on the positive solutions of the inequality, it seems that the number of belts is a whole number greater than $53\frac{1}{3}$; that is, any number in the set {54, 55, 56, 57, ...}.

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8. Check the apparent answers that you arrived at in step 7.

For 53 belts, the cost is 160 + 10(53) = 690, the revenue is 18(53) = 954, the profit is 954 – 690 = 264, and the average profit per belt is $\frac{264}{53} \approx 4.98$, which is less than 5.

For 54 belts, the cost is 160 + 10(54) = 700, the revenue is 18(54) = 972, the profit is 972 – 700 = 272, and the average profit per belt is $\frac{272}{54} \approx 5.04$, which is greater than 5.

For 55 belts, the cost is 160 + 10(55) = 710, the revenue is 18(55) = 990, the profit is 990 – 710 = 280, and the average profit per belt is $\frac{280}{55} \approx 5.09$, which is greater than 5.

The checks support the conclusion that to earn an average profit of at least \$5 per belt, 53 belts is not enough, and 54 or more belts is enough.

9. Summarize your conclusions.

To earn an average profit of at least \$5 per belt, Ashrita could make and sell any whole number of belts greater than $53\frac{1}{3}$; that is, any number in the set {54, 55, 56, 57, ...}.

Recall from step 1 that the total profit from making and selling x belts is given by the following:

$$P(x) = R(x) - C(x) = 18x - (160 + 10x) = 8x - 160$$

If she makes and sells 54 belts, her profit will be as follows:

$$P(54) = 8(54) - 160 = 432 - 160 = 272$$

If she achieves her goal, her minimum profit will be \$272.

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10. Make an observation about using a rational inequality for a problem situation.

If you are using a rational inequality to solve a real-world problem in which only positive numbers apply, you might be able to multiply or divide both sides by an expression containing a variable, keeping in mind that you will not be finding the complete solution set of the inequality.

To solve the problem about Ashrita and her belts, you need only find all the positive solutions of the inequality, because x represents a number of belts.

So, the following method can be used, with the understanding that it is valid only for positive values of x.

$\frac{8x-160}{x} \geq 5$	Write the inequality.
$8x - 160 \geq 5x$	Multiply both sides by x. The inequality symbol stays the same because x is positve.
$3x - 160 \geq 0$	Subtract $5x$ from both sides.
$3x \geq 160$	Add 160 to both sides.
$x \geq 53\frac{1}{3}$	Divide both sides by 3.

Note that this method should not be used if you need to find the complete solution set of the inequality.

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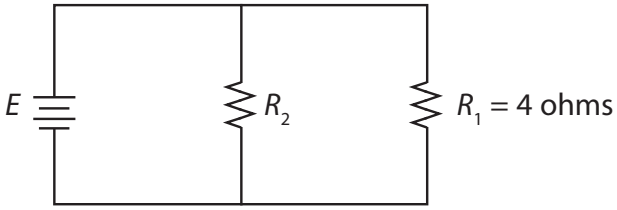
Example 4

The following diagram represents an electrical circuit with an E-volt battery and two resistors of resistances R_1 and R_2, connected in parallel. The total resistance R in the circuit satisfies the equation $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$.



Find the value of R_2 that will make the total resistance R at least 3 ohms.

1. Develop a rational inequality that can be solved for R_2.

$R \geq 3$	The total resistance R must be at least 3 ohms.
$1 \geq \frac{3}{R}$	Divide both sides by R. The inequality symbol stays the same because R is positive.
$\frac{1}{3} \geq \frac{1}{R}$	Divide both sides by 3.
$\frac{1}{3} \geq \left(\frac{1}{R_1}+\frac{1}{R_2}\right)$	Substitute $\frac{1}{R_1}+\frac{1}{R_2}$ for $\frac{1}{R}$.
$\frac{1}{3} \geq \frac{1}{(4)}+\frac{1}{R_2}$	Substitute 4 for R_1. This is a rational inequality that can be solved for R_2.

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2. Find the positive values of R_2 that are solutions of the rational inequality.

$\frac{1}{3} \geq \frac{1}{4} + \frac{1}{R_2}$	Write the rationl inequality developed in step 1.
$12R_2\left(\frac{1}{3}\right) \geq \left(\frac{1}{4} + \frac{1}{R_2}\right)12R_2$	Multiply both sides by $12R_2$, the LCD. The inequality symbol stays the same because $12R_2$ is positive.
$\frac{12R_2}{3} \geq \frac{12R_2}{4} + \frac{12R_2}{R_2}$	Distribute and simplify.
$4R_2 \geq 3R_2 + 12$	Simplify.
$R_2 \geq 12$	Subtract $3R_2$ from both sides.

3. Identify answers to the problem.

Based on the result in step 2, any value of R_2 greater than or equal to 12 ohms will make the total resistance R at least 3 ohms.

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4. Check your answers and then write your final answer to the problem.

Substitute values for R_2 into the equation to check.

Begin by choosing a value less than 12; let's try 11.

If $R_2 = 11$, then $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{4} + \frac{1}{(11)} \approx 0.341$; and if $\frac{1}{R} \approx 0.341$, then $R \approx \frac{1}{0.341} \approx 2.93$, which is less than 3 ohms.

Substitute 12 for R_2.

If $R_2 = 12$, then $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{4} + \frac{1}{(12)} = \frac{3}{12} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$, and if $\frac{1}{R} = \frac{1}{3}$, then $R = 3$, which is at least 3 ohms.

Substitute a value greater than 12 for R_2. Let's try 13.

If $R_2 = 13$, then $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{4} + \frac{1}{(13)} \approx 0.327$; and if $\frac{1}{R} \approx 0.327$, then $R \approx \frac{1}{0.327} \approx 3.06$, which is at least 3 ohms.

The checks support the conclusion that any value of R_2 greater than or equal to 12 ohms will make the value of R at least 3 ohms.

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Example 5

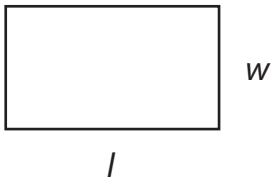
Adam has 100 meters of fencing. He wants to fence in a rectangular region in which he can have a garden that is protected from deer. He wants the region to have an area of 520 square meters. What are all the possible widths that Adam can use? Describe two rectangles that Adam can use and explain why they satisfy the requirements.

1. Write the relevant formulas for a rectangle and draw a sketch.

   For a rectangle with length l and width w:

   $P = 2l + 2w$ represents the perimeter.

   $A = lw$ represents the area.



2. Develop a rational inequality that can be used to find the values of the width w that satisfy the requirements.

   You know:

$A = 520$	The area must be equal to 520 square meters.
$P \leq 100$	The perimeter must be less than or equal to 100 meters because Adam only has 100 meters of fencing.

   Use the equation to obtain an expression for l in terms of w.

$A = 520$	Given
$(lw) = 520$	Substitute lw for A.
$l = \frac{520}{w}$	Divide both sides by w to solve for l in terms of w.

(*continued*)

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Use the inequality to develop a rational inequality.

$P \le 100$	Given
$(2l + 2w) \le 100$	Substitute $2l + 2w$ for P.
$2\left(\frac{520}{w}\right) + 2w \le 100$	Substitute $\frac{520}{w}$ for l.
$\frac{520}{w} + w \le 50$	Divide both sides by 2 to simplify. The inequality symbol stays the same because 2 is positive.

$\frac{520}{w} + w \le 50$ is a rational inequality that represents the situation. It can be used to find the values of w that satisfy the requirements of the problem.

3. Find the positive solutions of the rational inequality.

$\frac{520}{w} + w \le 50$	Write the rational inequality developed in step 2.
$520 + w^2 \le 50w$	Multiply both sides by w to eliminate the denominator. The inequality symbol stays the same because w is positive.
$520 + w^2 - 50w \le 0$	Subtract $50w$ from both sides to get 0 on one side.
$w^2 - 50w + 520 \le 0$	Write the quadratic expression in the form $ax^2 + bx + c$.

Now you have a polynomial inequality, $w^2 - 50w + 520 \le 0$.

To solve it, find the critical numbers and then use test intervals.

(*continued*)

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$w^2 - 50w + 520 = 0$ Set the polynomial equal to 0.

$$w = \frac{-(-50) \pm \sqrt{(-50)^2 - 4(1)(520)}}{2(1)}$$

By the quadratic formula, if $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$w = \frac{50 \pm \sqrt{2500 - 2080}}{2}$$ Simplify.

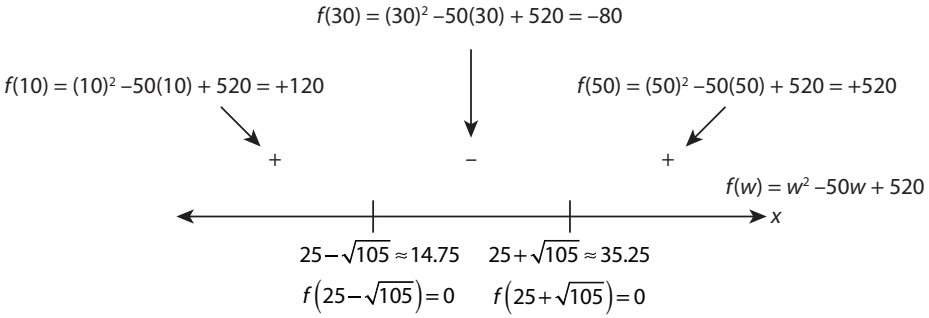
$$w = \frac{50 \pm \sqrt{420}}{2}$$

$$w = \frac{50 \pm 2\sqrt{105}}{2}$$

$$w = 25 \pm \sqrt{105}$$

The quadratic polynomial function $f(w) = w^2 - 50w + 520$ has the value 0 at $w = 25 - \sqrt{105} \approx 14.75$ and $w = 25 + \sqrt{105} \approx 35.25$; these values are the critical numbers.

The following summary diagram shows a sample function value within each test interval, the sign of $f(w)$ in each interval, and the value of $f(w)$ at the critical numbers.



The diagram shows that $w^2 - 50w + 520 \le 0$ at $w = 25 - \sqrt{105}$, at $w = 25 + \sqrt{105}$, and at all w-values between $25 - \sqrt{105}$ and $25 + \sqrt{105}$.

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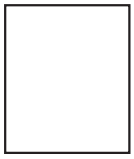
4. Write the answers to the problem.

Adam can use any width w in meters that satisfies $25-\sqrt{105} \le w \le 25+\sqrt{105}$.

The corresponding approximate values are given by $14.75 \le w \le 35.25$.

Two possible rectangles that Adam can use are shown below; one has a width of 20 meters and the other has a width of 25 meters.

$l = \frac{520}{w} = \frac{520}{20} = 26$



$l = \frac{520}{w} = \frac{520}{25} = 20.8$

The requirements are that the area must be 520 square meters and the perimeter must be less than or equal to 100 meters so that Adam has enough fence.

A rectangle with a width of 20 meters and a length of 26 meters has an area of 520 square meters and a perimeter of 92 meters.

A rectangle with a width of 25 meters and a length of 20.8 meters has an area of 520 square meters and a perimeter of 91.6 meters.

Therefore, both rectangles satisfy the requirements.

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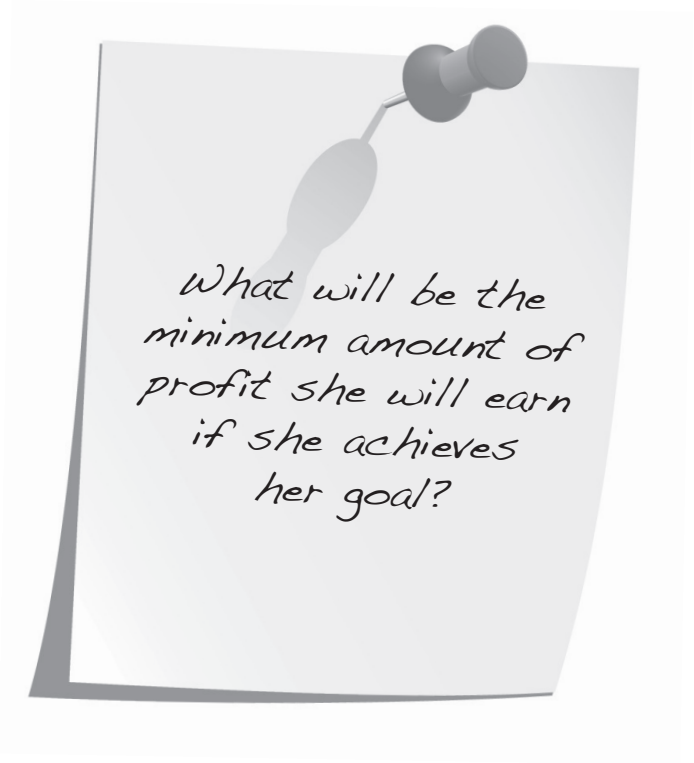
Problem-Based Task 3.5.3: Capping Costs

SMP	
1	2 ✓
3	4 ✓
5	6 ✓
7	8

Verona sells sporting goods. One product she sells is a baseball cap with a team letter of the customer's choice. Her startup cost for the equipment to attach the letters to the caps is $200. It costs her $5 to process each cap, which includes what she pays for the cap. She sells each cap for $15. Verona's goal is for her average profit per cap to exceed her average cost per cap. How many caps could she sell to achieve her goal? What will be the minimum amount of profit she will earn if she achieves her goal?



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Problem-Based Task 3.5.3: Capping Costs

Coaching

a. What are the two quantities you need to compare in order to answer the questions?

b. What is the rule for the function $C(x)$ that represents the cost in dollars to process x caps, including the startup cost?

c. What is the rule for the function $C_A(x)$ that represents the average cost in dollars per cap to process x caps?

d. What is the rule for the function $R(x)$ that represents the revenue in dollars that Verona takes in by selling x caps?

e. What is the rule for the function $P(x)$ that represents the profit in dollars that Verona earns by selling x caps?

f. What is the rule for the function $P_A(x)$ that represents the average profit in dollars per cap that Verona earns by selling x caps?

g. What is Verona's goal?

h. How can you use the answers to parts c and f to write a rational inequality that represents Verona's goal?

i. How can you find the positive solutions of the inequality? Explain your method.

j. How many caps could Verona sell to achieve her goal of having her average profit per cap exceed her average cost per cap?

k. What will be the minimum amount of profit Verona will earn if she achieves her goal? How can you justify your answer?

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Instruction

Problem-Based Task 3.5.3: Capping Costs

Coaching Sample Responses

a. What are the two quantities you need to compare in order to answer the questions?

The two quantities that need to be compared are average profit per cap and average cost per cap.

b. What is the rule for the function $C(x)$ that represents the cost in dollars to process x caps, including the startup cost?

The rule for the function $C(x)$ that represents the cost in dollars to process x caps is $C(x) = 200 + 5x$.

c. What is the rule for the function $C_A(x)$ that represents the average cost in dollars per cap to process x caps?

The rule for the function $C_A(x)$ that represents the average cost in dollars per cap is $C_A(x) = \frac{200+5x}{x}$.

d. What is the rule for the function $R(x)$ that represents the revenue in dollars that Verona takes in by selling x caps?

The rule for the function $R(x)$ that represents the revenue in dollars is $R(x) = 15x$.

e. What is the rule for the function $P(x)$ that represents the profit in dollars that Verona earns by selling x caps?

The rule for the function $P(x)$ that represents the profit in dollars is as follows:

$P(x) = R(x) - C(x)$

$P(x) = 15x - (200 + 5x)$

$P(x) = 10x - 200$

f. What is the rule for the function $P_A(x)$ that represents the average profit in dollars per cap that Verona earns by selling x caps?

The rule for the function $P_A(x)$ that represents the average profit in dollars per cap is $P_A(x) = \frac{10x-200}{x}$.

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g. What is Verona's goal?

Verona's goal is to have her average profit per cap exceed her average cost per cap.

h. How can you use the answers to parts c and f to write a rational inequality that represents Verona's goal?

Verona wants $P_A(x) > C_A(x)$, so the inequality is $\frac{10x-200}{x} > \frac{200+5x}{x}$.

i. How can you find the positive solutions of the inequality? Explain your method.

Solve the rational inequality.

$\frac{10x-200}{x} > \frac{200+5x}{x}$	Write the inequality.
$10x - 200 > 200 + 5x$	Multiply both sides by x. The inequality symbol stays the same because x is a number of caps and is therefore positive.
$5x - 200 > 200$	Subtract $5x$ from both sides.
$5x > 400$	Add 200 to both sides.
$x > 80$	Divide both sides by 5.

j. How many caps could Verona sell to achieve her goal of having her average profit per cap exceed her average cost per cap?

Verona could sell 81 or more caps to achieve her goal.

k. What will be the minimum amount of profit Verona will earn if she achieves her goal? How can you justify your answer?

The minimum amount of profit Verona will earn is \$610. Verona must sell a minimum of 81 caps, and $P(81) = 10(81) - 200 = 610$.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.5.3: Creating Rational Inequalities

Solve the inequalities in problems 1–4. Write each solution set in two forms: as an algebraic description and in interval notation.

1. $\frac{-3x+12}{x+1} \leq 0$

2. $\frac{5x-2}{x+5} > 3$

3. $\frac{x}{2x-5} \geq 2$

4. $\frac{4}{x} > x$

For problems 5 and 6, write a rational inequality that contains the expression for the average profit function, and then answer the questions.

5. Faisal owns and operates Faisal's Fruit Smoothies. His monthly overhead is $1,150. The overhead includes all his costs except the cost of ingredients, which is $1.80 per smoothie. He sells the smoothies for $4.75 each. Faisal wants to earn an average profit of at least $1.00 per smoothie each month. What are all the numbers of smoothies he could make and sell in a month to earn an average profit of at least $1.00 per smoothie? What will be his minimum profit for the month if he achieves his goal?

6. Effie owns and operates Nothing But Cuts, where every customer gets a haircut for $18.50. Effie pays each of her three employees $10.50 per hour. Two of the employees each work 40 hours per week, and the other employee works 30 hours per week. Effie's other costs amount to $475 per week. Effie wants to earn an average profit of at least $2.50 per haircut each week. What are all the numbers of haircuts the staff could give in a week so that Effie will earn an average profit of at least $2.50 per haircut? What will be Effie's minimum profit for the week if she achieves her goal?

continued

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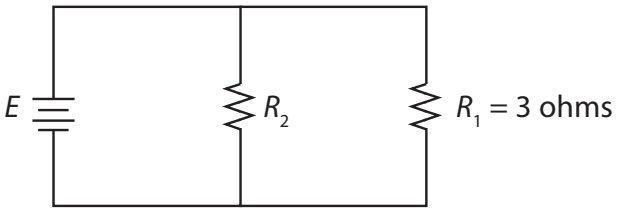
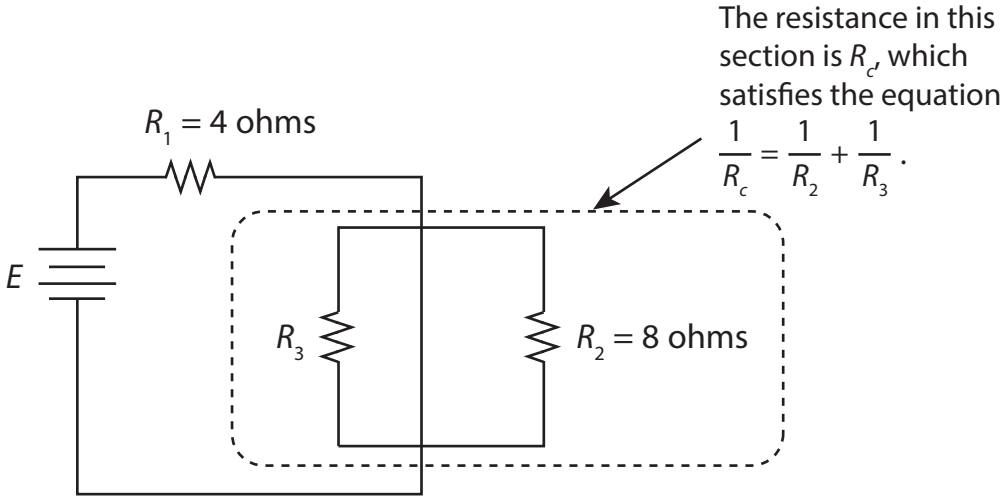
For problems 7 and 8, use the given information and the diagrams to answer the questions.

7. The diagram below represents an electrical circuit with an E-volt battery and two resistors of resistances R_1 and R_2, connected in parallel. The total resistance R in the circuit satisfies the equation $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$. What are all the values of R_2 that will make the total resistance in the circuit at least 2 ohms?



8. The diagram below represents an electrical circuit with an E-volt battery, a resistor of resistance R_1, and two resistors of resistances R_2 and R_3, connected in parallel. The total resistance R in the circuit satisfies the equation $R = R_1 + R_C$, where R_C is as described in the diagram. What are all the values of R_3 that will make the total resistance in the circuit 10 ohms or less?



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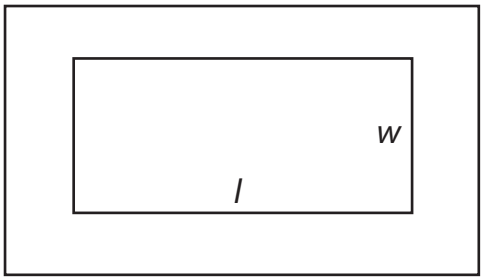
For problems 9 and 10, read each scenario, use the diagram if provided, and answer the questions.

9. Paige has 12 feet 6 inches of wood molding. She wants to use the molding to frame a rectangular section of wall, then fill the area within the molding with a tile mosaic. If she frames an area of 1,200 square inches, what are all possible widths of the wall section? Give your answers in both exact form and approximate form.

10. Clinton wants to construct a rectangular garden with an area of 600 square feet surrounded by a 4-foot wide path, resembling the diagram below. He will use landscape edging to form both rectangles. He has 240 feet of landscape edging. What are all possible widths of the garden? Give your answers in both exact form and approximate form.



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Progress Assessment

Circle the letter of the best answer.

1. Nash has had 13 hits in 52 at-bats in his baseball games. If x represents the number of consecutive hits he needs in order to raise his batting average to 0.350, which equation can be solved to find that number?

   a. $\frac{13}{52+x}=0.350$

   b. $\frac{13+x}{52+x}=0.350$

   c. $\frac{13}{52}+\frac{1}{x}=0.350$

   d. $\frac{1}{13}+\frac{1}{52}+\frac{1}{x}=0.350$

2. Chloe and Daja work in a restaurant. Chloe can prepare for opening in 15 minutes less time than it takes Daja. Chloe and Daja can prepare for opening in 32 minutes if they work together. Approximately how many minutes does it take Chloe to prepare for opening if she works alone? Assume both women work at steady rates and their rates are not affected by either working alone or working together.

   a. 35 minutes

   b. 47 minutes

   c. 57 minutes

   d. 72 minutes

3. Regan took two trains to travel to her destination. She traveled 480 kilometers on the first train, averaging 100 kilometers per hour. Then she traveled 240 kilometers on the second train. Her average rate for the entire 720 kilometers was 80 kilometers per hour. What was her average rate on the second train?

   a. approximately 36 kilometers per hour

   b. approximately 57 kilometers per hour

   c. exactly 60 kilometers per hour

   d. exactly 90 kilometers per hour

4. Which expression is the result of simplifying $\frac{18(x-3)}{6x^2-6x-36}$?

   a. $\frac{3}{x-3}$

   b. $\frac{3}{x+3}$

   c. $\frac{3}{x-2}$

   d. $\frac{3}{x+2}$

continued

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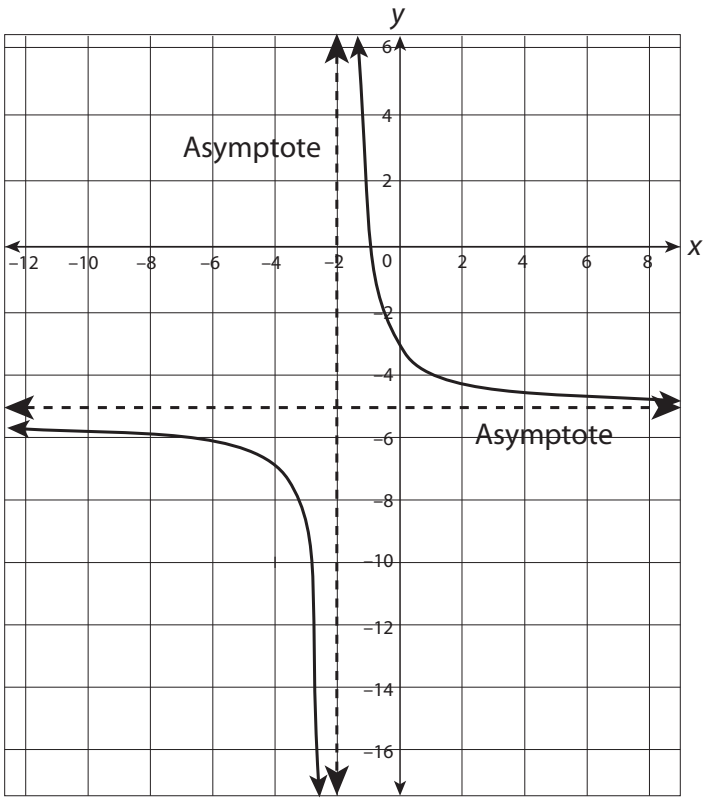
5. Which equation is represented by the graph?



a. $y=\frac{2}{x-4}+5$

b. $y=\frac{2}{x+4}-5$

c. $y=\frac{4}{x-2}+5$

d. $y=\frac{4}{x+2}-5$

6. Kiet makes and sells framed stained glass panels that can be used as wall hangings. The startup cost to make the panels is \$250 and the cost to make each panel is \$15. Which equation represents the average cost function $A(x)$, where x is the number of panels he makes and $A(x)$ is the average cost per panel, in dollars?

a. $A(x)=\frac{250}{x}+15$

b. $A(x)=\frac{x}{250}+15$

c. $A(x)=\frac{15}{x}+250$

d. $A(x)=\frac{x}{15}+250$

continued

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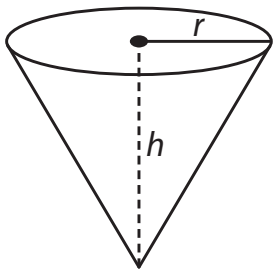
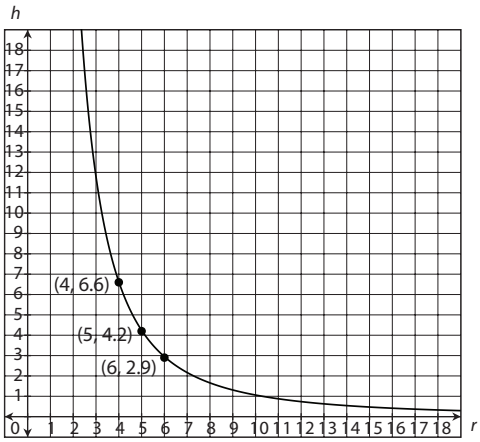
Assessment

7. Mona sells specialty containers. One of her customers needs cone-shaped plastic containers for shipping fresh flower bouquets. Each container must have a volume of 1,000 cubic inches.

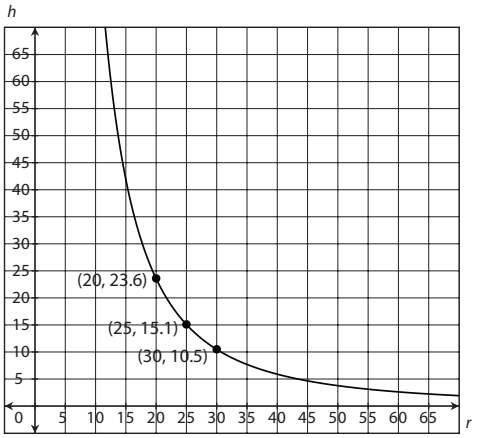


Which graph shows the relationship between r, the radius of the base, and h, the height, of a cone that fits the requirement? The volume formula for a cone is $V = \frac{1}{3}\pi r^2 h$.

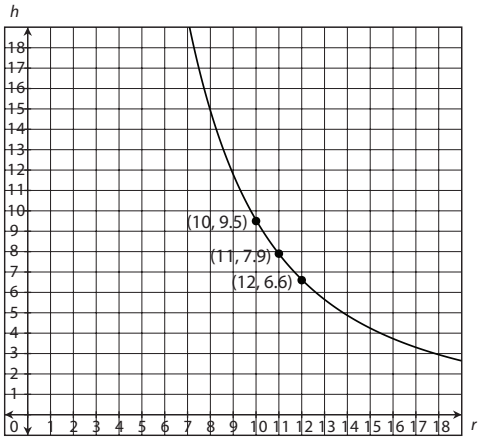
a.



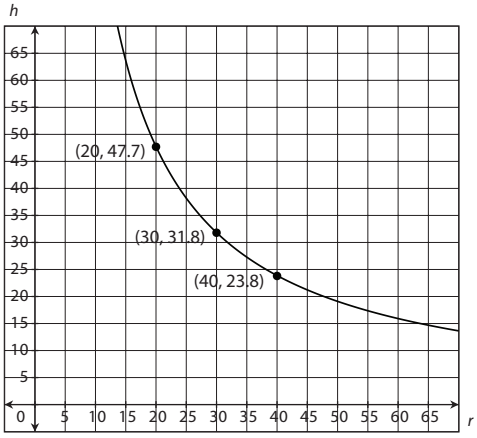
c.



b.



d.



continued

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8. Which option shows the solution set of $\frac{-x+3}{2x+10} \leq 0$?

   a. $-5 < x \leq 3$
   b. $-3 \leq x < 5$
   c. $x < -5$ or $x \geq 3$
   d. $x \leq -3$ or $x > 5$

9. Kadence has a new job providing technical support for customers. On 4 of her first 10 calls, the customers rated her service as outstanding. Kadence wants to be rated outstanding in at least 80% of the calls she takes. She wants to know how many consecutive outstanding ratings she could get to reach her 80% goal. She writes and solves an inequality to find out what she wants to know, letting x represent the number of consecutive outstanding ratings she could get to reach her 80% goal. Which option shows the correct inequality and correct answer?

   a. $\frac{4}{10}x \geq 0.8$; Kadence needs 4 or more consecutive outstanding ratings.
   b. $\frac{4}{10}x \geq 0.8$; Kadence needs 20 or more consecutive outstanding ratings.
   c. $\frac{x+4}{x+10} \geq 0.8$; Kadence needs 4 or more consecutive outstanding ratings.
   d. $\frac{x+4}{x+10} \geq 0.8$; Kadence needs 20 or more consecutive outstanding ratings.

10. Tony has 200 feet of fence. He wants to fence in a rectangular region with an area of 2,200 square feet for his dog. Tony wrote a rational inequality that satisfies the requirements and then wrote a description of all possible values of w, the width of the region in feet. Which option shows Tony's inequality and description?

    a. $\frac{2200}{w} + w \leq 100$; $50 - 10\sqrt{3} \leq w \leq 50 + 10\sqrt{3}$
    b. $\frac{2200}{w} + w \leq 100$; $50 - 5\sqrt{78} \leq w \leq 50 + 5\sqrt{78}$
    c. $\frac{2200}{w} + w^2 \leq 100$; $50 - 10\sqrt{3} \leq w \leq 50 + 10\sqrt{3}$
    d. $\frac{2200}{w} + w^2 \leq 100$; $50 - 5\sqrt{78} \leq w \leq 50 + 5\sqrt{78}$

continued

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Assessment

Use what you have learned about rational equations to answer the following questions.

11. Jay, Lakita, and Oscar work for a landscaping company. Jay can complete a regular job at a customer's residence in 3 hours 20 minutes if he works alone. Lakita can complete the job in 4 hours 10 minutes if she works alone. One day Jay, Lakita, and Oscar are all assigned to the job. Jay and Lakita begin working at 1:00 P.M. Oscar arrives later and begins working at 1:30 P.M. They all complete the job at 2:20 P.M. Complete parts a and b to find how many minutes it takes Oscar to complete the job if he works alone. (Assume they all work at steady rates and their rates are not affected by either working alone or working together.)

    a. Let x represent the number of minutes it takes Oscar to complete the job if he works alone. Write a rational equation to represent the situation.

    b. How many minutes does it take Oscar to complete the job if he works alone?

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. The membership at Fit Yourself Together Gym has increased 40% during the last 8 years. The manager wants to know the annual rate of increase, assuming that the membership increased at the same rate each year. Which option describes how the manager can begin to find the approximate annual rate of increase?
   a. Write $(1+0.05)^1 = \left(1.05^8\right)^{\frac{1}{8}}$ and then evaluate 1.05^8.
   b. Write $(1+0.05)^1 = \left(1.05^{\frac{1}{8}}\right)^8$ and then evaluate $1.05^{\frac{1}{8}}$.
   c. Write $(1+0.40)^1 = \left(1.4^8\right)^{\frac{1}{8}}$ and then evaluate 1.4^8.
   d. Write $(1+0.40)^1 = \left(1.4^{\frac{1}{8}}\right)^8$ and then evaluate $1.4^{\frac{1}{8}}$.

2. Aron bought a new fishing boat for $16,000 in March 1999. The boat was worth $6,400 in March 2009. Assuming a constant annual rate of decrease in value, what was the value of the boat in March 2006?
   a. $8,425
   b. $9,233
   c. $9,280
   d. $11,190

3. Piet's Pizza Shack recorded 572 online orders in June 2012. That represented an increase of 420% compared to the number of online orders in June 2004. Assuming that the number of June online orders increased at the same rate each year, how many online orders were recorded in June 2009?
   a. 204
   b. 308
   c. 333
   d. 399

continued

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4. Marilee has an investment that pays 9% annual interest, compounded annually. Which option shows the best approximation of the equivalent monthly rate, compounded monthly?

   a. 0.28%
   b. 0.72%
   c. 0.75%
   d. 1.33%

5. Sheronda would like to borrow money at 0.6% interest per month, compounded monthly. Her research indicates that rates for loans are often quoted as annual rates. Which of the following is the best approximation for the annual rate, compounded annually, that is equivalent to the rate Sheronda wants?

   a. 7.2%
   b. 7.4%
   c. 8.3%
   d. 8.8%

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Lesson 6: Writing Exponential Expressions in Equivalent Forms

Instruction

Common Core State Standard

SMP	
1	2 ✓
3 ✓	4
5	6 ✓
7	8

A–SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Essential Questions

1. How can exponential expressions be rewritten using reciprocals and properties of exponents?
2. How can rewritten exponential expressions be used to solve problems involving exponential growth and decay?
3. How can an exponential growth model be used to solve problems involving compound interest?

WORDS TO KNOW

compound interest	interest earned on both the initial amount and on previously earned interest
decay factor	$1 - r$ in the exponential decay model $f(t) = a(1 - r)^t$, or b in the exponential function $f(x) = ab^x + k$ if $0 < b < 1$; the multiple by which a quantity decreases over time
decay rate	r in the exponential decay model $f(t) = a(1 - r)^t$
exponential decay	an exponential equation with a base, b, that is between 0 and 1 exclusive (that is, $0 < b < 1$); an example is the formula $y = a(1 - r)^t$, where a is the initial value, $(1 - r)$ is the base (with $0 < r < 1$), t is the variable exponent, and y is the final value
exponential decay model	an exponential function, $f(t) = a(1 - r)^t$, where $f(t)$ is the final output value, a is the initial value, r is the percent decrease per time period (expressed as a decimal), and t is the variable exponent, usually representing time
exponential expression	an expression containing a base and an exponent

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Instruction

exponential function — a function whose independent variable is in the exponent; the general form of its equation is $f(x) = ab^x + k$, where a is the initial value, b is the base, x is the input value, k is the vertical shift, and $f(x)$ is the output

exponential growth — an exponential equation with a base, b, greater than 1 ($b > 1$); an example is the formula $y = a(1 + r)^t$, where a is the initial value, $(1 + r)$ is the base (with $r > 0$), t is the variable exponent, and y is the final value

exponential growth model — an exponential function, $f(t) = a(1 + r)^t$, where $f(t)$ is the final output value, a is the initial value, r is the percent increase per time period (expressed as a decimal), and t is the variable exponent, usually representing time

growth factor — $1 + r$ in the exponential growth model $f(t) = a(1 + r)^t$, or b in the exponential function $f(x) = ab^x + k$ if $b > 1$; the factor by which a quantity increases over time

growth rate — the rate of increase in size per unit of time; r in the exponential growth model $f(t) = a(1 + r)^t$

percent of change — $\frac{\text{amount of change}}{\text{original amount}}$, written as a percent

rational exponent — an exponent of the form $\frac{m}{n}$, where m and n are integers and $n \neq 0$. If m and n are positive integers and a is a real number, then $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$.

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Recommended Resources

- IXL Learning. "Exponential Growth and Decay: Word Problems."

  http://www.walch.com/rr/00145

  This site provides practice in solving exponential growth and decay problems. It includes feedback and worked-out solutions.

- MathIsFun.com. "Compound Interest."

  http://www.walch.com/rr/00146

  This site provides an overview of compound interest and shows how to derive the compound interest formula.

- MathIsFun.com. "Exponential Function Reference."

  http://www.walch.com/rr/00147

  This site is a reference page for the exponential function $f(x) = a^x$ and includes graphs for the cases $0 < a < 1$ and $a > 1$.

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Lesson 3.6.1: Writing Exponential Expressions in Equivalent Forms

Warm-Up 3.6.1

For problem 1, each blank line represents a letter in a missing word. Read each clue given and then write the missing word. Use your answers to complete problem 2.

1.

_ _ _ _ _ _ _ _	3 in the expression x^3
_ _ _ _ _ _ _ _	once per year
_ _ _ _ _ _ _ _	to become less in amount or value
_ _ _ _ _ _ _ _	a type of interest earned; not simple interest
_ _ _ _ _ _	once per year

2. Unscramble the first letters of the words in problem 1 to answer this question: "What is a word that means to diminish, lose value, or fade away?"

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Instruction

Lesson 3.6.1: Writing Exponential Expressions in Equivalent Forms

Common Core State Standard

A–SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Warm-Up 3.6.1 Debrief

1.

E X P O N E N T	3 in the expression x^3
A N N U A L L Y	once per year
D E C R E A S E	to become less in amount or value
C O M P O U N D	a type of interest earned; not simple interest
Y E A R L Y	once per year

2. Unscramble the first letters of the words in problem 1 to answer this question: "What is a word that means to diminish, lose value, or fade away?"

   The word that means to diminish, lose value, or fade away is DECAY.

Connection to the Lesson

- Students will apply each of the words used in the warm-up as they relate to equivalent forms of exponential expressions.

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Prerequisite Skills

This lesson requires the use of the following skills:

- identifying the reciprocal of a number (5.NF.7b)
- evaluating functions (F–IF.2)
- solving simple one-variable equations (8.EE.7b)
- using calculators to evaluate expressions with exponents (no standard)

Introduction

An **exponential expression** is an expression that contains a base and an exponent. Exponential expressions can be used to represent quantities that increase or decrease at the same rate per time period. For example, $1000(1.05)^t$ can be used to represent the number of dollars in a savings account at the end of t years if the initial amount is \$1,000 and the account pays 5% annual interest, compounded annually. This lesson will investigate how to find the approximate equivalent monthly interest rate, compounded monthly, by writing exponential expressions in equivalent forms.

Key Concepts

- Exponential expressions contain a base and an exponent. Examples of exponential expressions include: 3^2, $4x + a^3x^6$, 1.25^x, $(1 + r)^t$, and $\left(1+0.045^{\frac{1}{4}}\right)^{4t}$.
- An **exponential function** is a function whose independent variable is in the exponent. The general form of an exponential function is $f(x) = ab^x + k$, where a is the initial value, b is the base, x is the input value, k is the vertical shift, and $f(x)$ is the final output value.
- Exponential functions are often used to model growth or decay (increase or decrease) of quantities over time, so the variable t (for time) is often used, and we write $f(t) = ab^t + k$. *Note*: Growth and decay models in this lesson do not include a vertical shift, so we use the general form, $f(t) = ab^t$.
- If $b > 1$, then there is exponential growth, and b is the **growth factor**, or the factor by which a quantity increases over time.
- If $0 < b < 1$, then there is exponential decay, and b is the **decay factor**, or the factor by which a quantity decreases over time.
- **Exponential growth**, or increase, can be written as $f(t) = a(1 + r)^t$.

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Instruction

- This form is known as the **exponential growth model**, where $f(t)$ is the final output value, a is the initial value, r is the growth rate per time period (expressed as a decimal), $(1 + r)$ is the base or growth factor, and t is the variable exponent, usually representing the number of time periods.
- The **growth rate**, r, is the rate of increase in size per unit of time, generally expressed as a decimal.
- The following table shows some useful benchmark numbers. It is important to remember that the growth factor is 1 plus the growth rate.

Percent increase	Growth rate (r)	Growth factor ($1 + r$)
1%	0.01	1.01
50%	0.50	1.50
100%	1.00	2.00
200%	2.00	3.00
450%	4.50	5.50

- **Exponential decay**, or decrease, can be written as $f(t) = a(1 - r)^t$.
- This form is known as the **exponential decay model**, where $f(t)$ is the final output value, a is the initial value, r is the **decay rate** per time period (expressed as decimal), $(1 - r)$ is the base or decay factor, and t is the variable exponent, usually representing the number of time periods.
- Calculating **percent of change**, or $\frac{\text{amount of change}}{\text{original amount}}$ written as a percent, is helpful in many exponential growth and decay problems. For instance:
  - A school's enrollment increases from 500 students to 580 students.

    The percent of increase is $\frac{580-500}{500}=\frac{80}{500}=0.16=16\%$.

    The growth rate is 0.16 and the growth factor is 1.16.
  - A town's population increases from 500 people to 1,000 people.

    The percent of increase is $\frac{1000-500}{500}=\frac{500}{500}=1=100\%$.

    The growth rate is 1 and the growth factor is 2. Note that a 100% increase is a doubling of the original amount.

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- Sales at a store increase from \$1,200 one month to \$3,600 the next.

  The percent of increase is $\frac{3600-1200}{1200}=\frac{2400}{1200}=2=200\%$.

  The growth rate is 2 and the growth factor is 3. Note that a 200% increase is a tripling of the original amount.

- The value of a home decreases from \$250,000 one decade to \$200,000 the next.

  The percent of decrease is $\frac{250{,}000-200{,}000}{250{,}000}=\frac{50{,}000}{250{,}000}=0.2=20\%$.

  The decay rate is 0.2 and the decay factor is 1 – 0.2, or 0.8. Note that the decay factor is 1 *minus* the decay rate.

- Exponents are called **rational exponents** when the exponent is of the form $\frac{m}{n}$, where m and n are integers and n does not equal 0.
- If m and n are positive integers and a is a real number, then $a^{\frac{m}{n}}=\left(\sqrt[n]{a}\right)^m=\sqrt[n]{a^m}$. Examples include:

  $8^{\frac{4}{3}}=\left(\sqrt[3]{8}\right)^4=(2)^4=16$

  $25^{\frac{1}{2}}=\left(\sqrt[2]{25}\right)^1=\left(\sqrt{25}\right)^1=\sqrt{25}=5$

- In most practical applications when you need to evaluate an expression that has a rational exponent, it is best to use a calculator to find an approximate value.

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- For example, to approximate $1.2^{\frac{1}{5}}$, follow the steps appropriate to your calculator model:

On a TI-83/84:

Step 1: Enter [1.2].

Step 2: Press [^].

Step 3: Enter [(][1][÷][5][)].

Step 4: Press [ENTER].

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the calculate icon and press [enter].

Step 3: Enter [1.2].

Step 4: Press [^].

Step 5: Enter [(][1][÷][5][)].

Step 6: Press [enter].

- The display will be 1.037137289, which is a very close approximation of $1.2^{\frac{1}{5}}$, or $\sqrt[5]{1.2}$.
- When evaluating expressions, remember to follow the order of operations:
  - Evaluate expressions inside parentheses.
  - Evaluate powers.
  - Multiply and/or divide from left to right.
  - Add and/or subtract from left to right.
- The following property can be useful when working with rational exponents.

Property
Power of a Power Property If a is a real number and m and n are rational numbers, then $(a^m)^n = a^{m \cdot n}$. Examples: $\left(2^4\right)^2 = 2^{4 \cdot 2} = 2^8 = 256$ $\left(1.05^{\frac{1}{3}}\right)^3 = 1.05^{\left(\frac{1}{3} \cdot 3\right)} = 1.05^1 = 1.05$

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Instruction

- In the exponential models $f(t) = a(1 + r)^t$ and $f(t) = a(1 - r)^t$, it is important that the same time period is used for r and t.
- For example, if r is a rate of change that applies to a 5-year time period, then t must represent the number of 5-year time periods.
- If r is a rate of change that applies to a quarter of a year (3 months), then t must represent the number of 3-month time periods.
- When you deposit money in a bank account that pays interest, you are lending money to the bank. Such deposits (and loans in general) are sometimes called investments. When you earn interest on both the initial amount of the deposit (or loan) and on previously earned interest, then you are earning **compound interest**.
- The exponential growth model $f(t) = a(1 + r)^t$ can be used for problems involving compound interest, where r is the interest rate per time period, a is the initial investment, t is the number of time periods, and $f(t)$ is the value of the investment after t time periods. In such cases, we say the interest is compounded once per time period.
- The following table shows the meanings of the terms commonly used to describe compounding periods.

Compounded…	***n* (number of times per year)**
Yearly/annually	1
Semiannually	2
Quarterly	4
Monthly	12
Weekly	52
Daily	365

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Instruction

Common Errors/Misconceptions

- using different time periods for r and t in the exponential models $f(t) = a(1 + r)^t$ and $f(t) = a(1 – r)^t$
- mistakenly dividing an annual interest rate that is compounded annually by 12 to find the approximate equivalent monthly interest rate, compounded monthly
- neglecting to add 1 to a growth rate to obtain a growth factor; for example, not realizing that a 100% increase means doubling the original amount and a 250% increase means multiplying the original amount by 3.5
- neglecting to subtract a decay rate from 1 to obtain a decay factor; for example, not realizing that a 15% decrease means a decay factor of 1 – 0.15 or 0.85
- dividing by the final amount instead of the original amount when finding a percent of change
- incorrectly thinking that a constant *rate* of change means the same *amount* of change each time period

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Instruction

Guided Practice 3.6.1

Example 1

A city's population has increased 40% during the last 5 years and is now 448,000 people. What was the population 5 years ago? Assuming that the population increased at the same rate each year, what was the annual rate of increase?

1. Write the exponential growth model and identify what each variable represents.

   The exponential growth model is $f(t) = a(1 + r)^t$, where $f(t)$ is the amount at the end of t time periods; a is the initial amount; r is the percent increase per time period expressed as a decimal; and t is the number of time periods.

2. Substitute the given values into the growth model.

   In the exponential growth model, the same time period must be used for r and t.

   The given percent increase, 40%, applies to 1 time period of length 5 years.

   The amount of people at the end of 1 time period of 5 years, $f(t)$, is 448,000.

   The increase, r, was 40% in 5 years, or 0.40.

   The number of time periods, t, of length 5 years is 1.

   Substituting these values into the growth model yields the following equation:

   $448{,}000 = a(1 + 0.40)^1$

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Instruction

3. Find a, the initial amount in the exponential growth model.

   a represents the city's population 5 years ago.

$448{,}000 = a(1 + 0.40)^1$	Equation from step 2
$448{,}000 = a(1.4)$	Simplify.
$\frac{448{,}000}{1.4} = a$	Divide both sides by 1.4.
$320{,}000 = a$	Simplify.

   The city's population 5 years ago was 320,000 people.

4. To find the annual rate of increase, determine how to rewrite the exponential expression.

   The expression $448{,}000 = a(1 + 0.40)^1$ represents a 40% increase in 1 time period.

   To write the expression to show the percent increase in each of 5 time periods, the exponent must be 5.

5. Rewrite the exponential expression to reveal the annual rate of increase.

   Write $(1 + 0.40)^1$ so that its exponent is 5.

$(1 + 0.40)^1 = 1.4^1$	Simplify.
$1.4^1 = 1.4^{\left(\frac{1}{5} \cdot 5\right)}$	Write the exponent 1 as the product of 5 and its reciprocal, $\frac{1}{5}$.
$1.4 = \left(1.4^{\frac{1}{5}}\right)^5$	Apply the Power of a Power Property.
$1.4 \approx (1.069610)^5$	Using a calculator, approximate $1.4^{\frac{1}{5}}$.
$1.4 \approx (1 + 0.069610)^5$	Rewrite in the form $(1 + r)^t$.

   The annual rate of increase, expressed as a decimal, is approximately 0.069610.

   Therefore, the annual rate of increase is approximately 6.9610%.

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Instruction

6. Verify the answers shown in steps 3 and 5.

Step 3 shows that the population 5 years ago was 320,000 people.

Step 5 shows that the annual rate of increase was approximately 6.9610%, which has a decimal equivalent of 0.069610.

Use these results to write an exponential growth model for this situation and then verify that the model gives the current population, 448,000 people.

Use 5 time periods of length 1 year each because 6.9610% applies to 1-year time periods.

$f(t) = a(1 + r)^t$	Write the general exponential growth model.
$f(t) = 320{,}000(1 + 0.069610)^t$	Substitute values for the exponential growth model for this situation.
$f(5) = 320{,}000(1 + 0.069610)^5$	Let $t = 5$.
$f(5) = 320{,}000(1.069610)^5$	Simplify.
$f(5) \approx 320{,}000(1.399998)$	Using a calculator, approximate $(1.069610)^5$.
$f(5) \approx 447{,}999$	Multiply.

The function value for $t = 5$ years is very close to 448,000, the given population at the end of 5 years, which verifies the answers shown in steps 3 and 5.

Therefore, the city's population 5 years ago was 320,000 people, and the annual rate of increase was approximately 6.9610%.

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Instruction

Example 2

A financial magazine had 57,500 subscribers on January 1, 2010, which represented an increase of 243% in the 10 years since January 1, 2000. Assuming that the number of subscribers increased at the same rate each year, how many subscribers did the magazine have on January 1, 2004?

1. Find a, the initial amount in the exponential growth model.

   There are 57,500 subscribers at the end of 1 time period of length 10 years.

   The rate of increase for that time period was 243%, which has a decimal equivalent of 2.43.

$f(t) = a(1 + r)^t$	Write the general exponential growth model.
$57{,}500 = a(1 + 2.43)^1$	Substitute the known values.
$57{,}500 = a(3.43)^1$	Simplify.
$57{,}500 = a(3.43)$	Simplify the exponent.
$\frac{57{,}500}{3.43} = a$	Divide both sides by 3.43.
$16{,}764 \approx a$	Simplify.

   The initial amount is 16,764, which is the approximate number of subscribers on January 1, 2000.

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Instruction

2. Find the annual rate of increase.

The number of years is 10, so you need to rewrite $(3.43)^1$ so that its exponent is 10.

$3.43^1 = 3.43^{\left(\frac{1}{10} \cdot 10\right)}$	Write the exponent 1 as the product of 10 and its reciprocal, $\frac{1}{10}$.
$3.43 = \left(3.43^{\frac{1}{10}}\right)^{10}$	Apply the Power of a Power Property.
$3.43 \approx (1.131174)^{10}$	Using a calculator, approximate $3.43^{\frac{1}{10}}$.
$3.43 \approx (1 + 0.131174)^{10}$	Rewrite in the form $(1 + r)^t$.

The annual rate of increase is approximately 13.1174%.

3. Find the number of subscribers on January 1, 2004.

The number of subscribers on January 1, 2000, was approximately 16,764.

The annual rate of increase is approximately 13.1174%.

From January 1, 2000, to January 1, 2004, is a period of 4 years.

$f(t) = a(1 + r)^t$	Write the general exponential growth model.
$f(t) = 16{,}764(1 + 0.131174)^t$	Substitute values for the exponential growth model for this situation.
$f(4) = 16{,}764(1 + 0.131174)^4$	Let $t = 4$.
$f(4) \approx 16{,}764(1.131174)^4$	Simplify.
$f(4) \approx 16{,}764(1.637260)$	Using a calculator, approximate $(1.131174)^4$.
$f(4) \approx 27{,}447$	Multiply.

The magazine had approximately 27,447 subscribers on January 1, 2004.

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Instruction

Example 3

Sajeena paid $20,000 for a new car in April 2004. The car was worth $6,000 in April 2012. Assuming a constant annual rate of decrease in value, what was the annual rate of decrease? What was the value of the car in April 2009? What is the predicted value of the car for April 2018?

1. Find the percent of decrease for the time period April 2004 to April 2012.

   The value of the car decreased from $20,000 to $6,000.

   20,000 – 6000 = 14,000, so the decrease in value was $14,000.

$\text{Percent of change} = \frac{\text{amount of change}}{\text{original amount}}$	Write the formula for percent of change.
$\text{Percent of change} = \frac{14{,}000}{20{,}000}$	Substitute the known amounts.
$\text{Percent of change} = \frac{7}{10}$	Simplify.
$\text{Percent of change} = 0.7$	Write the fraction as a decimal.
$\text{Percent of change} = 70\%$	Write the decimal as a percent.

   The percent of decrease for the time period April 2004 to April 2012 was 70%.

2. Find the annual decay factor.

   From April 2004 to April 2012 is a period of 8 years.

   The value of the car at the end of 1 time period of 8 years, $f(t)$, is $6,000.

   The initial value, a, was $20,000.

   The decrease, r, was 70% in 8 years, or 0.70.

   The number of time periods, t, of length 8 years is 1.

   Substitute these values into the general exponential decay model, $f(t) = a(1 - r)^t$.

   $6000 = 20{,}000(1 - 0.70)^1$

(*continued*)

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Rewrite the equation, simplifying the decay factor: $6000 = 20{,}000(0.3)^1$.

The number of years is 8, so you need to rewrite $(0.3)^1$ so that its exponent is 8.

$0.3^1 = 0.3^{\left(\frac{1}{8} \cdot 8\right)}$ Write the exponent 1 as the product of 8 and its reciprocal, $\frac{1}{8}$.

$0.3^1 = \left(0.3^{\frac{1}{8}}\right)^8$ Apply the Power of a Power Property.

$0.3^1 \approx (0.860281)^8$ Using a calculator, approximate $0.3^{\frac{1}{8}}$.

The annual decay factor is approximately 0.860281.

3. Use the annual decay factor to find the annual rate of decay, or decrease.

In the exponential model $f(t) = a(1 - r)^t$, $1 - r$ is the decay factor and r is the rate of decay.

$1 - r \approx 0.860281$	Use the annual decay factor found in step 2.
$-r \approx 0.860281 - 1$	Subtract 1 from both sides.
$-r \approx -0.139719$	Simplify.
$r \approx 0.139719$	Multiply both sides by –1.

The annual rate of decay, or decrease, is approximately 0.139719, or 13.9719%.

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Instruction

4. Find the value of the car in April 2009.

From April 2004 to April 2009 is a period of 5 years.

$f(t) = a(1 - r)^t$	Write the general exponential decay model.
$f(t) = 20{,}000(1 - 0.139719)^t$	Substitute values for the exponential decay model for this situation.
$f(t) = 20{,}000(0.860281)^t$	Simplify.
$f(5) = 20{,}000(0.860281)^5$	Let $t = 5$.
$f(5) \approx 20{,}000(0.471196)$	Using a calculator, approximate $(0.860281)^5$.
$f(5) \approx 9423.92$	Multiply.

The value of the car in April 2009 was approximately \$9,424.

5. Find the predicted value of the car in April 2018.

From April 2004 to April 2018 is a period of 14 years.

$f(t) = 20{,}000(0.860281)^t$	Write the simplified exponential decay model.
$f(14) = 20{,}000(0.860281)^{14}$	Let $t = 14$.
$f(14) \approx 20{,}000(0.121609)$	Using a calculator, approximate $(0.860281)^{14}$.
$f(14) \approx 2432.18$	Multiply.

The predicted value of the car in April 2018 is approximately \$2,432.

6. Summarize the answers from steps 3, 4, and 5.

The annual rate of decay, or decrease, is approximately 0.139719, or 13.9719%.

The value of the car in April 2009 was approximately \$9,424.

The predicted value of the car for April 2018 is approximately \$2,432.

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Instruction

Example 4

Trina makes two different investments:

- She deposits money into a bank account that pays 4% annual interest, compounded annually.
- She lends money to a start-up company that pays 1.2% quarterly interest, compounded quarterly.

For each investment, what is the approximate equivalent monthly rate, compounded monthly?

1. For the bank deposit, identify an exponential expression you can rewrite to solve the problem, and then determine how to rewrite it.

$f(t) = a(1 + r)^t$	Write the general exponential growth model.
$f(t) = a(1 + 0.04)^t$	The rate of growth is 4% per year; t represents a number of years.
$f(t) = a(1.04)^t$	Simplify.

The exponential expression $(1.04)^t$ represents a growth rate of 4% per year for t years.

But you need to find the growth rate per month.

Since t years = $12t$ months, you need the expression to have an exponent of $12t$.

2. Rewrite the exponential expression $(1.04)^t$ so that its exponent is $12t$.

$1.04^t = 1.04^{\left(\frac{1}{12} \cdot 12t\right)}$	Write the exponent t as $\frac{1}{12} \cdot 12t$ to introduce the expression $12t$.
$1.04^t = \left(1.04^{\frac{1}{12}}\right)^{12t}$	Apply the Power of a Power Property.
$1.04^t \approx (1.003274)^{12t}$	Using a calculator, approximate $1.04^{\frac{1}{12}}$.
$1.04^t \approx (1 + 0.003274)^{12t}$	Rewrite in the form $(1 + r)^t$.

(*continued*)

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The monthly rate of increase is approximately 0.003274, or 0.3274%.

Therefore, for a 4% annual interest rate that is compounded annually, the approximate equivalent monthly interest rate, compounded monthly, is 0.3274%.

3. For the loan, identify an exponential expression you can rewrite to solve the problem and then determine how to rewrite it.

$f(t) = a(1 + r)^t$	Write the general exponential growth model.
$f(t) = a(1 + 0.012)^t$	The rate of growth is 1.2% per quarter; t represents a number of quarters.
$f(t) = a(1.012)^t$	Simplify.

The exponential expression $(1.012)^t$ represents a growth rate of 1.2% per quarter for t quarters.

But you need to find the growth rate per month.

Since t quarters = $3t$ months, you need the expression to have an exponent of $3t$.

4. Rewrite the exponential expression $(1.012)^t$ so that its exponent is $3t$.

$1.012^t = 1.012^{\left(\frac{1}{3} \bullet 3t\right)}$	Write the exponent t as $\frac{1}{3} \bullet 3t$ to introduce the expression $3t$.
$1.012^t = \left(1.012^{\frac{1}{3}}\right)^{3t}$	Apply the Power of a Power Property.
$1.012^t \approx (1.003984)^{3t}$	Using a calculator, approximate $1.012^{\frac{1}{3}}$.
$1.012^t \approx (1 + 0.003984)^{3t}$	Rewrite in the form $(1 + r)^t$.

The monthly rate of increase is approximately 0.003984, or 0.3984%.

Therefore, for a 1.2% quarterly interest rate that is compounded quarterly, the approximate equivalent monthly interest rate, compounded monthly, is 0.3984%.

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Instruction

5. Summarize your answers from steps 2 and 4.

   For the investment with a 4% annual interest rate that is compounded annually, the approximate equivalent monthly interest rate, compounded monthly, is 0.3274%.

   For the investment with a 1.2% quarterly interest rate that is compounded quarterly, the approximate equivalent monthly interest rate, compounded monthly, is 0.3984%.

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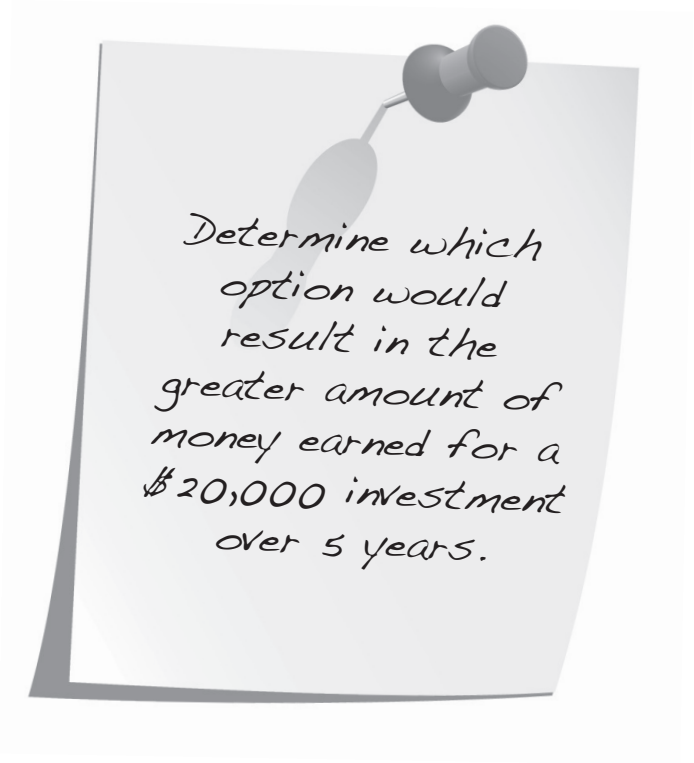
Problem-Based Task 3.6.1: Options of Interest

SMP	
1	2 ✓
3 ✓	4
5	6 ✓
7	8

Devin has money to invest and is considering two investment options:

- Option A: Deposit money in a bank account, which pays 6% annual interest, compounded annually.
- Option B: Lend money to his nephew Hudson, who owns a business and needs capital. Hudson will pay 0.5% monthly interest, compounded monthly.

Use approximate equivalent monthly interest rates, compounded monthly, to determine which option would result in the greater amount of money earned for a $20,000 investment over 5 years.



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Problem-Based Task 3.6.1: Options of Interest

Coaching

a. What is the general exponential growth model?

b. The account in Option A pays 6% annual interest, compounded annually. How can you represent that fact in the exponential growth model, letting t represent a number of years?

c. What is the answer to part b in simplified form?

d. The exponential expression written in part c represents a growth factor per year for t years. How much time is t years, expressed in months?

e. How can you rewrite the exponential expression $(1.06)^t$ so that it represents a growth factor per month for t years, expressed in months?

f. How can the expression in parentheses in the answer to part e be evaluated?

g. What does the exponential expression in part f represent?

h. The given rate for Option B is 0.5% monthly interest, compounded monthly. How can you represent that fact in the exponential growth model, letting t represent a number of months?

i. What is the answer to part h in simplified form?

j. How can you use the general exponential growth model to find the value, rounded to the nearest cent, of $20,000 invested for 5 years at the monthly interest rate, compounded monthly, described in part g?

k. How can you use the general exponential growth model to find the value, rounded to the nearest cent, of $20,000 invested for 5 years at 0.5% monthly interest, compounded monthly?

l. Which option would result in the greater amount of money earned for a $20,000 investment over 5 years?

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Instruction

Problem-Based Task 3.6.1: Options of Interest

Coaching Sample Responses

a. What is the general exponential growth model?

The general exponential growth model is $f(t) = a(1 + r)^t$.

b. The account in Option A pays 6% annual interest, compounded annually. How can you represent that fact in the exponential growth model, letting t represent a number of years?

The annual interest rate is 6%, so $r = 0.06$. The model can be written as $f(t) = a(1 + 0.06)^t$.

c. What is the answer to part b in simplified form?

$f(t) = a(1.06)^t$

d. The exponential expression written in part c represents a growth factor per year for t years. How much time is t years, expressed in months?

There are 12 months per year, so $12t$ represents the number of months in t years.

e. How can you rewrite the exponential expression $(1.06)^t$ so that it represents a growth factor per month for t years, expressed in months?

Rewrite the exponential expression $(1.06)^t$ so that the exponent is $12t$.

To do this, write the exponent t so that it includes the product of 12 and its reciprocal, $\frac{1}{12}$.

$$1.06^t = 1.06^{\frac{1}{12} \cdot 12t} = \left(1.06^{\frac{1}{12}}\right)^{12t}$$

f. How can the expression in parentheses in the answer to part e be evaluated?

Use a calculator to approximate $(1.06)^{\frac{1}{12}}$ and substitute that value into the expression from part e.

$$1.06^t = 1.06^{\frac{1}{12} \cdot 12t} = \left(1.06^{\frac{1}{12}}\right)^{12t} \approx (1.004868)^{12t}$$

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Instruction

g. What does the exponential expression in part f represent?

The exponential expression represents 0.4868% monthly interest, compounded monthly.

h. The given rate for Option B is 0.5% monthly interest, compounded monthly. How can you represent that fact in the exponential growth model, letting t represent a number of months?

The monthly interest is 0.5%, so $r = 0.005$. The model can be written as $f(t) = a(1 + 0.005)^t$.

i. What is the answer to part h in simplified form?

$f(t) = a(1.005)^t$

j. How can you use the general exponential growth model to find the value, rounded to the nearest cent, of $20,000 invested for 5 years at the monthly interest rate, compounded monthly, described in part g?

The value of $20,000 invested for 5 years (60 months) at the monthly interest rate of 0.4868%, compounded monthly, is found using the following expression:

$f(t) = a(1 + r)^t = 20{,}000(1 + 0.004868)^{60} = 20{,}000(1.004868)^{60} \approx \$26{,}765.23$

The value is approximately $26,765.23.

k. How can you use the general exponential growth model to find the value, rounded to the nearest cent, of $20,000 invested for 5 years at 0.5% monthly interest, compounded monthly?

The value of $20,000 invested for 5 years (60 months) at 0.5% monthly interest, compounded monthly, is found using the following expression:

$f(t) = a(1 + r)^t = 20{,}000(1 + 0.005)^{60} = 20{,}000(1.005)^{60} \approx \$26{,}977.00$

The value is approximately $26,977.00.

l. Which option would result in the greater amount of money earned for a $20,000 investment over 5 years?

Option B would result in the greater amount of money earned for a $20,000 investment over 5 years.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.6.1: Writing Exponential Expressions in Equivalent Forms

For problems 1 and 2, read the following scenario and use the information in it to answer the questions.

A town's population has increased 35% during the last 8 years and is now 50,100 people.

1. What was the population 8 years ago?

2. Assuming that the population increased at the same rate each year, what was the annual rate of increase?

For problems 3 and 4, read the following scenario and use the information in it to answer the questions.

Yesinia paid $26,500 for a new car in June 2005. The car was worth $10,600 in June 2011.

3. Assuming a constant annual rate of decrease in value, what was the annual rate of decrease?

4. What was the value of the car in June 2008? What is the predicted value for June 2020?

For problems 5–8, use what you've learned about exponential growth and decay to answer the questions.

5. The Best Service Heating Fuel Company had 1,418 of its customers signed up to pay their bills online as of May 1, 2012, which represented an increase of 185% since May 1, 2005. Assuming that the number of sign-ups increased at the same rate each year, how many Best Service customers were signed up to pay their bills online as of May 1, 2008?

6. A newspaper had 144,800 print subscribers on December 31, 2002. It had 54,300 print subscribers on December 31, 2012. Assuming a constant annual rate of decrease, how many print subscribers did the newspaper have on December 31, 2007? What is the predicted number of print subscribers on December 31, 2020?

continued

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7. The value of Mr. Cooper's house on July 1, 1990, was $80,200. The value on July 1, 2010, was $252,630. Assuming a constant annual rate of increase, what was the value on July 1, 2005? What is the predicted value on July 1, 2020?

8. Jared is comparing two different investments. One is a bank account that pays 3.4% annual interest, compounded annually. The other is a loan that pays 1.25% quarterly interest, compounded quarterly. For each investment, what is the approximate equivalent monthly rate, compounded monthly?

For problems 9 and 10, read the following scenario and use the information in it to answer the questions.

> Anjali has money to invest. She is considering two investment options. One option is to lend money to a start-up software company that will pay 0.4% monthly interest, compounded monthly. The other option is to deposit money in a bank account that pays 3.6% annual interest, compounded annually.

9. For the option of lending to the start-up company, what is the approximate equivalent annual interest rate, compounded annually?

10. Suppose Anjali plans to invest $15,000 for 6 years in one of the options. What would be the value of her investment under each option?

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Progress Assessment

Circle the letter of the best answer.

1. The undergraduate enrollment at a university has increased 40% during the last 9 years. Assuming that the enrollment increased at the same rate each year, which option shows the best approximation of the annual rate of increase?

   a. 3.1%
   b. 3.6%
   c. 3.8%
   d. 4.4%

2. The population of a town has increased 30% during the last 5 years. Rogerio wants to know the annual rate of increase, assuming that the population increased at the same rate each year. Which option shows how Rogerio can begin to find the approximate annual rate of increase?

   a. Write $(1+0.06)^1=\left(1.06^5\right)^{\frac{1}{5}}$ and then evaluate 1.06^5.

   b. Write $(1+0.06)^1=\left(1.06^{\frac{1}{5}}\right)^5$ and then evaluate $1.06^{\frac{1}{5}}$.

   c. Write $(1+0.30)^1=\left(1.3^5\right)^{\frac{1}{5}}$ and then evaluate 1.3^5.

   d. Write $(1+0.30)^1=\left(1.3^{\frac{1}{5}}\right)^5$ and then evaluate $1.3^{\frac{1}{5}}$.

3. London paid $24,000 for a new camper in February 2002. The camper was worth $6,000 in February 2010. Assuming a constant annual rate of decrease in value, what was the value of the camper in February 2008?

   a. $8,485
   b. $10,500
   c. $16,971
   d. $19,342

4. Mr. Alvarez bought a house in May 2003 for $220,000. In May 2012 it was worth $176,000. Assuming a constant annual rate of decrease in value, which function is the best model for the value of the house in dollars, where t is the number of years after 2003?

   a. $f(t)=176{,}000(0.968541)^t$
   b. $f(t)=176{,}000(0.975511)^t$
   c. $f(t)=220{,}000(0.968541)^t$
   d. $f(t)=220{,}000(0.975511)^t$

continued

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Assessment

5. OuttaSite OuttaDoors, a retailer of clothing and recreational equipment, recorded 874 online purchases in August 2011. That represented an increase of 360% compared to the number of online purchases in August 2006. Assuming that the number of August online purchases increased at the same rate each year, how many online purchases were recorded in August 2009?

   a. 350
   b. 475
   c. 524
   d. 600

6. A bookstore had $640,000 in sales revenue in 2000 and $400,000 in 2010. Assuming a constant annual rate of decrease, what is the predicted sales revenue for 2016?

   a. $38,163
   b. $133,239
   c. $256,000
   d. $301,709

7. Owen deposits money in a bank account that pays 6.6% annual interest, compounded annually. Which option shows the best approximation of the equivalent monthly rate, compounded monthly?

   a. 0.53%
   b. 0.55%
   c. 0.68%
   d. 0.74%

8. Mr. Ames lends money to a new business at 1.5% quarterly interest, compounded quarterly. He wants to know the approximate equivalent monthly rate, compounded monthly. Which option shows how he can begin to find the approximate equivalent monthly rate, compounded monthly?

   a. Write $(1+0.015)^t = \left(1.015^3\right)^{\frac{1}{3}t}$ and then evaluate 1.015^3.
   b. Write $(1+0.015)^t = \left(1.015^{\frac{1}{3}}\right)^{3t}$ and then evaluate $1.015^{\frac{1}{3}}$.
   c. Write $(1+0.015)^t = \left(1.015^4\right)^{\frac{1}{4}t}$ and then evaluate 1.015^4.
   d. Write $(1+0.015)^t = \left(1.015^{\frac{1}{4}}\right)^{4t}$ and then evaluate $1.015^{\frac{1}{4}}$.

continued

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Assessment

9. J.T. would like to borrow money at 0.4% interest per month, compounded monthly. He finds advertisements that quote annual rates. Which of the following is the best approximation for the annual rate, compounded annually, that is equivalent to the rate J.T. wants?

   a. 2.8%
   b. 3.3%
   c. 4.9%
   d. 8.4%

10. A house was worth $120,000 on July 1, 1994. It was worth $246,000 on July 1, 2007. The owner pays a property tax of 1.4% of the value of the house. Assuming a constant annual rate of increase in value, how much did the owner pay when she paid the property tax on July 1, 2000?

   a. $1,718
   b. $2,340
   c. $2,494
   d. $3,168

Use what you've learned about exponential growth and decay to solve problem 11.

11. Which of the following interest rate options would result in the greatest amount earned on a given amount invested for a given amount of time? Justify your answer by using approximate equivalent monthly interest rates, compounded monthly.

   - Option A: 3.8% annual interest, compounded annually
   - Option B: 0.95% quarterly interest, compounded quarterly
   - Option C: 0.32% monthly interest, compounded monthly

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Graph the following system to determine its real solution(s), if any exist.

$$\begin{cases} y = x^2 + 2x - 15 \\ y = x - 3 \end{cases}$$

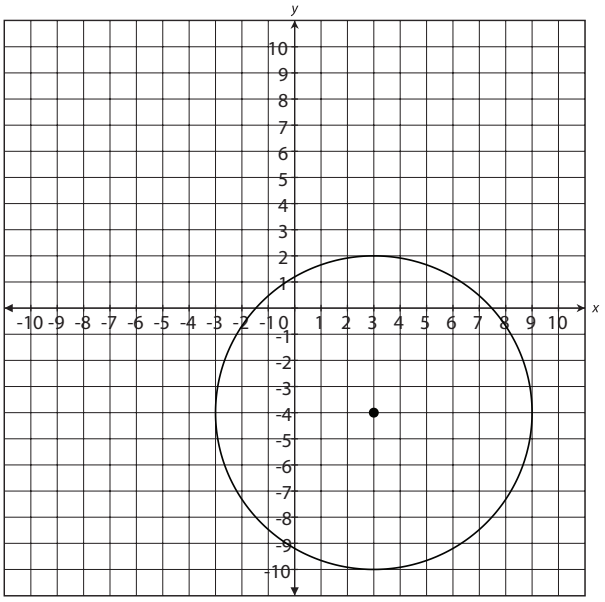
a. (0, 0) and (3, 0)

b. (3, 0) and (–4, –7)

c. (–1, –16)

d. no real solutions

2. Use graphing to approximate the solution(s), if any, to the system that is represented by the circle shown and the line $y = \frac{2}{3}x - 2$.



a. There are no real solutions.

b. There are two real solutions at approximately (–3, –4) and (5.3, 1.5).

c. There is one real solution at approximately (–8, –7.25).

d. There is one real solution at approximately (3, 2).

continued

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Assessment

3. Solve the following system algebraically to determine its real solution(s), if any exist.

$$\begin{cases} y = x^2 + 3x - 10 \\ y = -12.5 \end{cases}$$

a. (0, –12.5)
b. (–12.5, 0)
c. (–5, 0) and (2, 0)
d. no real solutions

4. Solve the following system algebraically to determine its real solution, if one exists.

$$\begin{cases} x^2 + y^2 = 8 \\ y = -x + 4 \end{cases}$$

a. (2, 2)
b. (–2, –2)
c. (–2, 2)
d. no real solutions

5. A baseball is hit so that its height in feet, t seconds after it is hit, can be modeled by the function $h(t) = -16t^2 + 40t + 3$. A bird takes off from its nest 20 feet above the ground at the same time that the batter hits the ball. The bird's flight can be modeled by the function $h(t) = 20 - \frac{3}{2}t$. After how many seconds do the bird and ball first reach the same height? What is that height?

a. 0.75 second and 18.88 feet off the ground
b. 1.5 seconds and 17.75 feet off the ground
c. 0.5 second and 19.24 feet off the ground
d. The bird and the ball will not reach the same height above the ground.

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Lesson 7: Solving Systems of Equations

Instruction

Common Core State Standard

A–REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line $y = –3x$ and the circle $x^2 + y^2 = 3$.*

SMP	
1	2 ✓
3	4 ✓
5	6 ✓
7 ✓	8

Essential Questions

1. How are points of intersection related to the solution of a system of equations?
2. At most, how many solutions can a quadratic-linear system have?
3. Is it possible for a quadratic-linear system to have no real solutions?

WORDS TO KNOW

circle	the set of all points in a plane that are equidistant from a reference point in that plane, the center
point(s) of intersection	the ordered pair(s) where graphed functions intersect on a coordinate plane; these are also the solutions to systems of equations
quadratic formula	a formula that states that the solutions of a quadratic equation of the form $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. A quadratic equation can have no real solutions, one real solution, or two real solutions.
quadratic-linear system	a system of equations in which one equation is quadratic and one is linear
standard form of an equation of a circle	$(x – h)^2 + (y – k)^2 = r^2$, where (h, k) is the center and r is the radius
substitution	the replacement of a variable of an equation by a number or an expression that is known to have the same value
system of equations	a set of equations with the same unknowns

SRB p. U3-279

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Instruction

Recommended Resources

- MathBits.com. "Solving a Linear-Quadratic System."

  http://www.walch.com/rr/00109

  This site describes linear-quadratic systems and demonstrates how to graph them using a TI-83/84 graphing calculator.

- Math Warehouse. "Systems of Linear and Quadratic Equations."

  http://www.walch.com/rr/00110

  This site offers instruction on identifying the systems of linear and quadratic equations graphically, and includes practice problems.

- Oswego City School District Regents Exam Prep Center. "Linear-Quadratic Systems."

  http://www.walch.com/rr/00111

  This site offers a quick and effective review of systems of equations, and also provides examples with answers and step-by-step instructions.

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Lesson 3.7.1: Solving Systems Graphically

Warm-Up 3.7.1

Gabriella is sorting the change in her car so she can buy a snack from the vending machine. She discovers that she has $3.50 in nickels and dimes, and she has 10 more nickels than dimes.

1. Define variables for the unknowns and write a system of equations to represent the situation.

2. Rearrange the terms of the equations so that they are isolated for the same variable.

3. Graph the system.

4. How many nickels does Gabriella have? How many dimes?

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Instruction

Lesson 3.7.1: Solving Systems Graphically

Common Core State Standard

A–REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line $y = –3x$ and the circle $x^2 + y^2 = 3$.*

Warm-Up 3.7.1 Debrief

1. Define variables for the unknowns and write a system of equations to represent the situation.

   n = number of nickels

   d = number of dimes

   There are two quantities in this context: the total number of nickels and dimes, and the dollar amount when the values of the nickels and dimes are taken into account. One equation will represent the total number of nickels and dimes, and the other will represent the dollar amount.

   Gabriella has 10 more nickels than dimes.

   $n = d + 10$

   Gabriella has \$3.50 in nickels and dimes.

   $0.05n + 0.10d = 3.50$

   The system $\begin{cases} n = d + 10 \\ 0.05n + 0.10d = 3.50 \end{cases}$ represents the total number and the value of nickels and dimes Gabriella has.

2. Rearrange the terms of the equations so that they are isolated for the same variable.

   In order to graph the functions, the second equation must be solved to isolate the same variable that is isolated in the first equation. This variable will act as the output value or as "y" when input into the graphing calculator.

   Since n is already isolated in the first equation, isolate the second equation, $0.05n + 0.10d = 3.50$, for n.

$0.05n + 0.10d = 3.50$	Original equation
$0.05n = 3.50 – 0.10d$	Subtract $0.10d$ from both sides of the equation.
$n = 70 – 2d$	Divide both sides by 0.05.

   Choosing to isolate d in both equations would yield the same final results.

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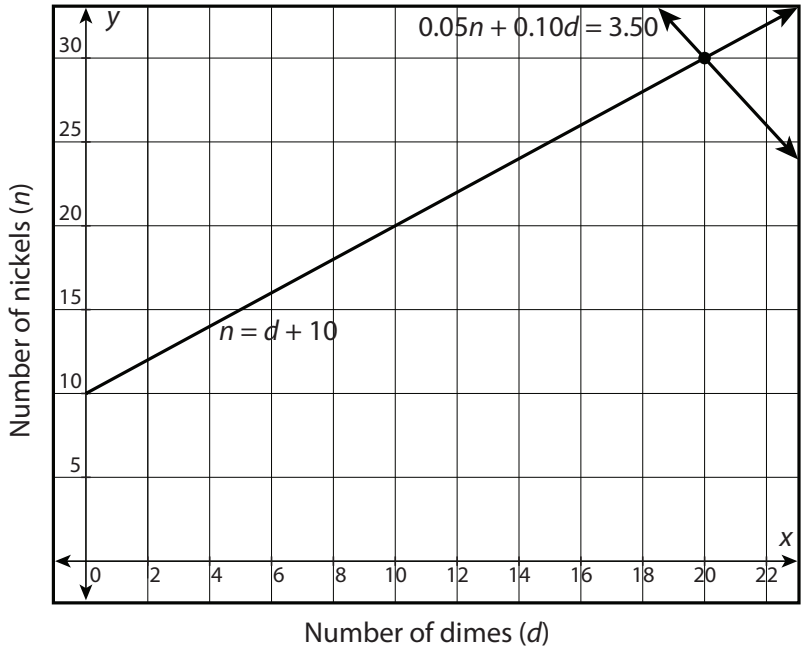
3. Graph the system.



4. How many nickels does Gabriella have? How many dimes?

   Identify the point of intersection.

   The point of intersection is (20, 30).

   This means that Gabriella has 30 nickels and 20 dimes.

Connection to the Lesson

- Students will extend their knowledge of solving linear systems by graphing to that of solving quadratic-linear systems by graphing.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- graphing multiple equations on a graphing calculator (no standard)
- graphing quadratic equations (F–IF.7a★)
- graphing linear equations (A–CED.2★)

Introduction

A **system of equations** is a set of equations with the same unknowns. A **quadratic-linear system** is a system of equations in which one equation is quadratic and one is linear. (*Note*: In this lesson, the graph of a quadratic equation can be a parabola or a circle.) We learned previously how to solve systems of linear equations. In this lesson, we will learn to solve a quadratic-linear system by graphing.

Key Concepts

- A **circle** is the set of all points that are equidistant from a reference point, called the center.
- The equation of a circle is a quadratic equation because the highest exponent is 2.
- The **standard form of an equation of a circle** is $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the center and r is the radius.
- If a circle is centered at (0, 0), then the equation of the circle has the form $x^2 + y^2 = r^2$.
- A quadratic-linear system may have one real solution, two real solutions, or no real solutions.
- The solutions of a quadratic-linear system can be found or approximated by graphing.
- The **points of intersection** of the graphed quadratic and linear equations are the ordered pair(s) where the graphs intersect on a coordinate plane. These are also the solutions to the system.

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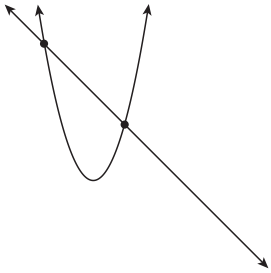
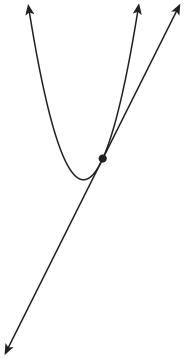
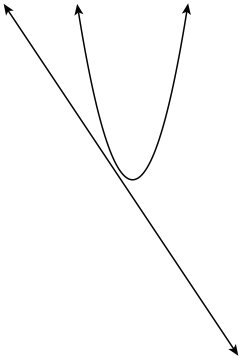
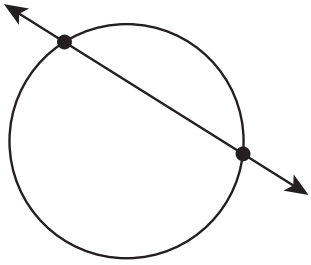
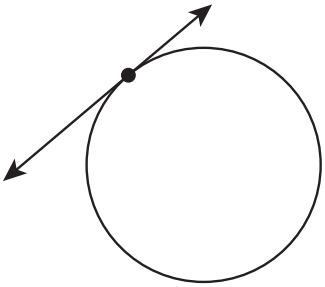
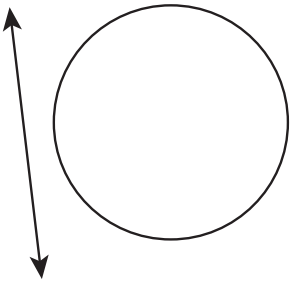
Instruction

Graphed Quadratic-Linear Systems

Two real solutions	One real solution	No real solutions
		
		

- Although the graph of the solution(s) to a real-life problem may have two points of intersection, only one point may be correct for the context. This is because in most real-life scenarios, certain numbers are non-negative, such as units of time and distance.

Common Errors/Misconceptions

- not identifying all points of intersection
- incorrectly entering the equations into the calculator

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Guided Practice 3.7.1

Example 1

Graph the system of equations to determine the real solution(s), if any exist.

$$\begin{cases} y = x^2 + 2x - 2 \\ y = 2x - 2 \end{cases}$$

1. Graph the quadratic function, $y = x^2 + 2x - 2$.

   If graphing by hand, first plot the y-intercept, –2.

   Next, use the equation for the axis of symmetry to determine the x-coordinate of the vertex of the parabola.

   $$x = \frac{-b}{2a} = \frac{-2}{2(1)} = \frac{-2}{2} = -1$$

   Draw the axis of symmetry, $x = -1$.

   Substitute the x-coordinate to determine the corresponding y-coordinate of the vertex.

   $y = (-1)^2 + 2(-1) - 2 = 1 - 2 - 2 = -3$

   Plot the vertex, (–1, –3).

   Finally, because the y-intercept is 1 unit to the right of the axis of symmetry, an additional point on the parabola is 1 unit to the left of the axis of symmetry at (–2, –2). Sketch the curve.

   If using a graphing calculator, follow the directions appropriate to your model.

   On a TI-83/84:

   Step 1: Press [Y=].

   Step 2: At Y_1, type [X, T, θ, n][x^2][+][2][X, T, θ, n][–][2].

   Step 3: Press [GRAPH].

   On a TI-Nspire:

   Step 1: Press [menu] and select 3: Graph Type, then select 1: Function.

(*continued*)

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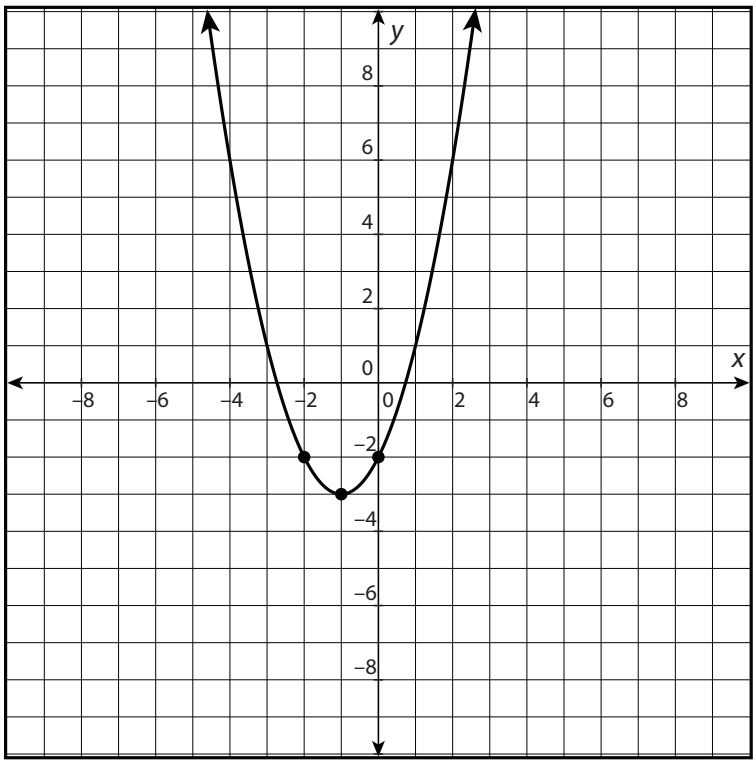
Instruction

Step 2: At *f1*(*x*), enter [*x*], press the [x^2] key, then type [+][2][*x*][–][2] and press [enter] to graph the quadratic function.



2. Graph the linear function, $y = 2x - 2$, on the same grid.

   If graphing by hand, first plot the *y*-intercept, (0, –2).

   Note: Since the two functions have the same *y*-intercept, we have already determined one solution.

   Next, use the slope of 2 to graph additional points in either direction.

   Sketch the line.

   On a TI-83/84:

   Step 1: Press [Y=].

   Step 2: At Y_2, type [2][X, T, θ, n][–][2].

   Step 3: Press [GRAPH].

   On a TI-Nspire:

   Step 1: Press [menu] and select 3: Graph Type, then select 1: Function.

(*continued*)

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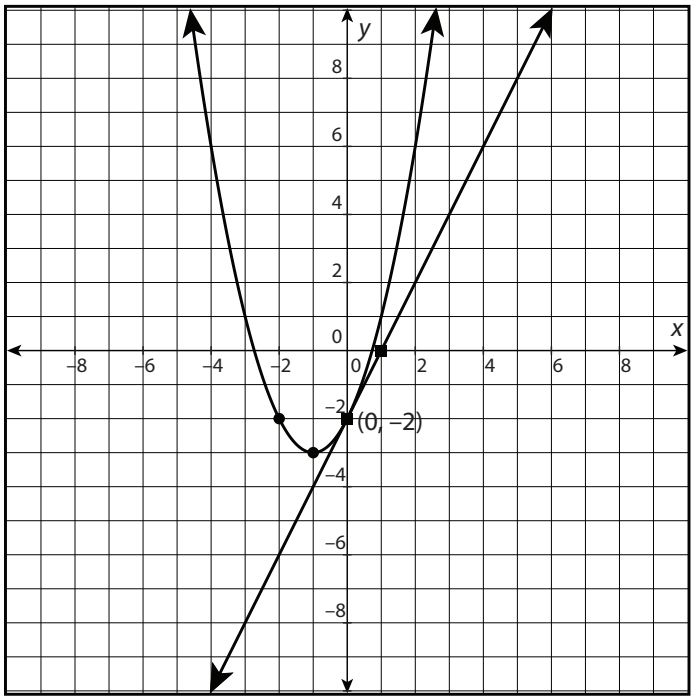
Instruction

Step 2: At *f2*(*x*), enter [2][*x*][–][2] and press [enter] to graph the linear function.



3. Note any intersections that exist as the solution or solutions.

On a TI-83/84:

Step 1: To approximate the intersections, press [2ND][TRACE] and select intersect.

Step 2: Press [ENTER].

Step 3: At the prompt that reads "First curve?", press [ENTER].

Step 4: At the prompt that reads "Second curve?", press [ENTER].

Step 5: At the prompt that reads "Guess?", arrow as close as possible to the first point of intersection and press [ENTER]. Write down the approximate ordered pair.

On a TI-Nspire:

Step 1: Press [menu] and select 7: Points & Lines, then select 3: Intersection Point(s).

Step 2: Select the graphs of both functions.

The only point of intersection is (0, –2).

The solution of $\begin{cases} y = x^2 + 2x - 2 \\ y = 2x - 2 \end{cases}$ is (0, –2).

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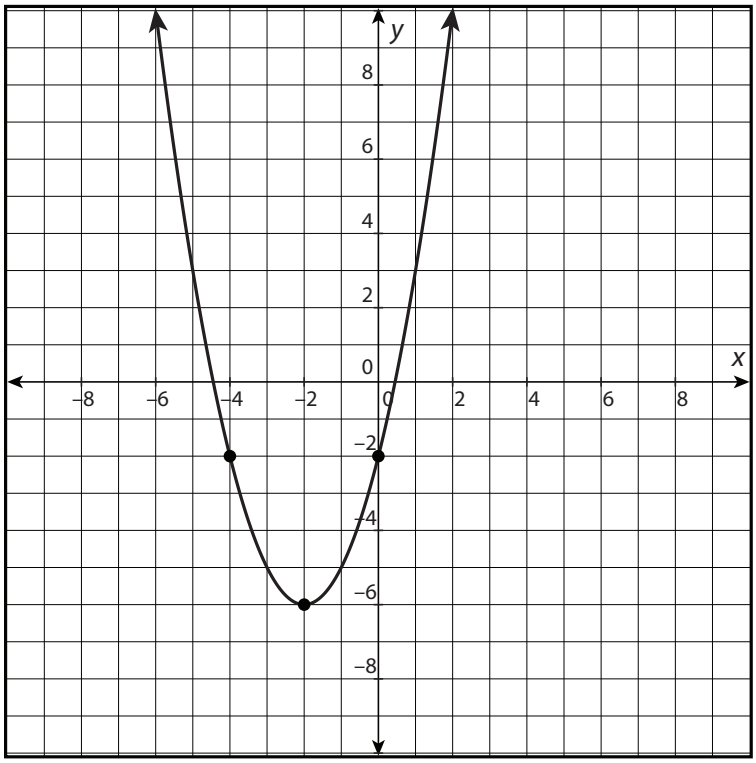
Instruction

Example 2

Graph the system of equations to determine the real solution(s), if any exist.

$$\begin{cases} y = 3x \\ y = x^2 + 4x - 2 \end{cases}$$

1. Graph the quadratic function, $y = x^2 + 4x - 2$, using the methods demonstrated in Example 1.



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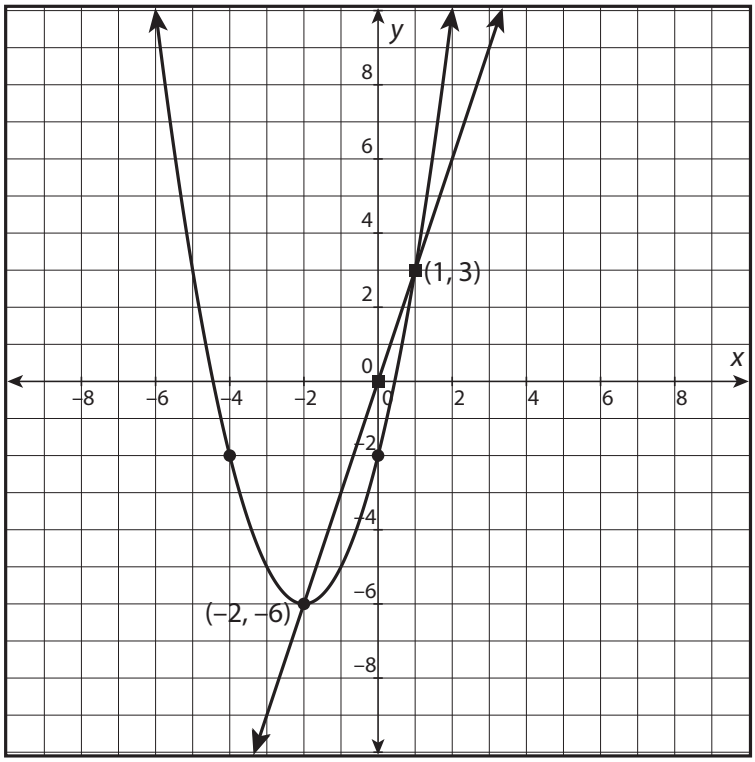
Instruction

2. Graph the linear function, $y = 3x$, on the same grid.



3. Note any intersections that exist as the solution or solutions.

The two points of intersection are (–2, –6) and (1, 3).

The solutions of $\begin{cases} y = 3x \\ y = x^2 + 4x - 2 \end{cases}$ are (–2, –6) and (1, 3).

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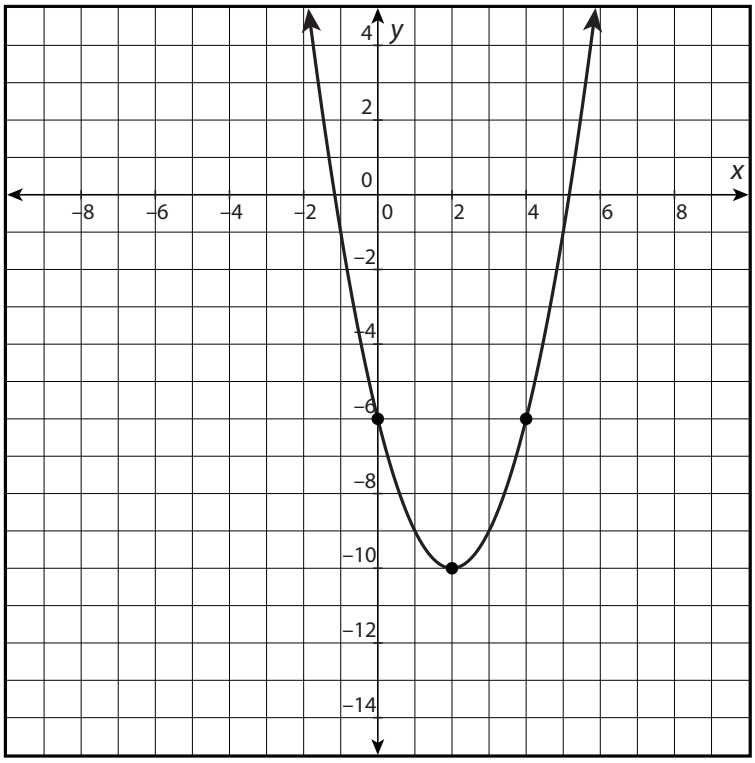
Instruction

Example 3

Graph the system of equations to determine the real solution(s), if any exist.

$$\begin{cases} y = -4x - 10 \\ y = x^2 - 4x - 6 \end{cases}$$

1. Graph the quadratic function, $y = x^2 - 4x - 6$, using the methods demonstrated in Example 1.



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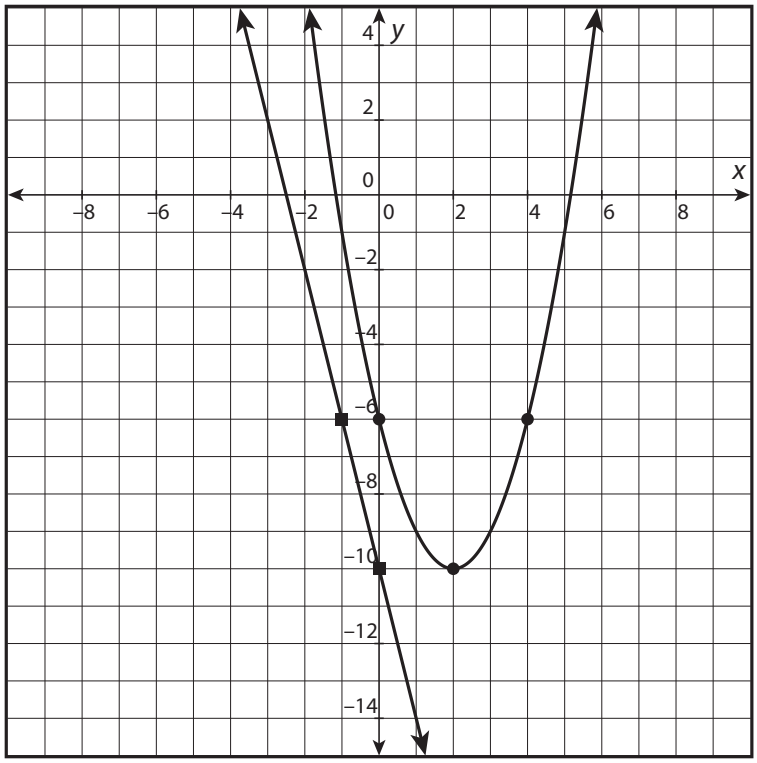
Instruction

2. Graph the linear function, $y = -4x - 10$, on the same grid.



3. Note any intersections that exist as the solution or solutions.

The two functions do not intersect.

The system $\begin{cases} y = -4x - 10 \\ y = x^2 - 4x - 6 \end{cases}$ has no real solutions.

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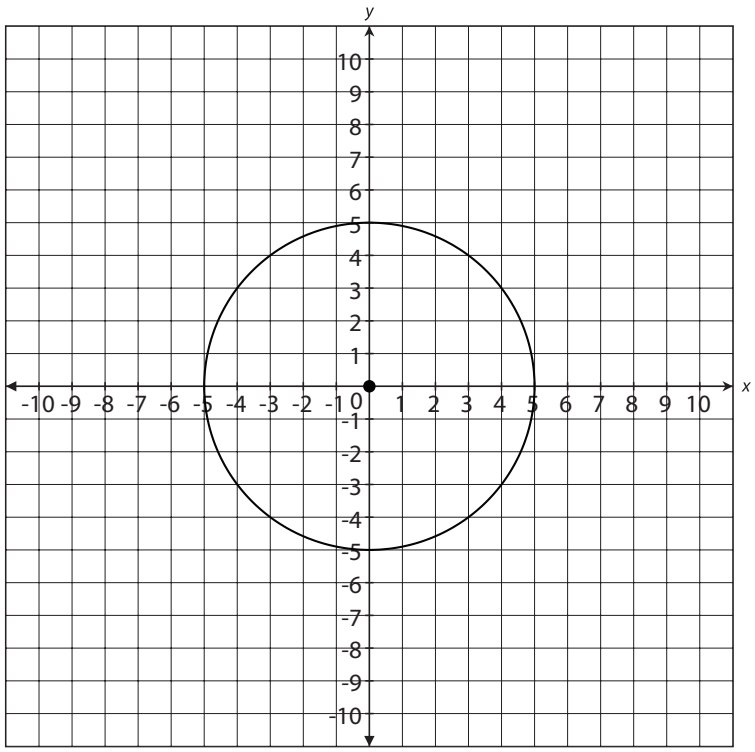
Instruction

Example 4

Solve the system made up of the line with the given equation and the circle shown in the graph.

$$y=\frac{3}{2}x+9$$



1. Graph the line on the same coordinate plane as the circle.

   To graph the line by hand, first graph the *y*-intercept, 9. Then use the slope to plot some more points on the line.

   The equation is already written in slope-intercept form, $y = mx + b$, so $m=\frac{3}{2}$.

   Because the slope is positive, it can be rewritten with two negatives since a negative divided by a negative is a positive.

   $$m=\frac{3}{2}=\frac{-3}{-2}$$

(*continued*)

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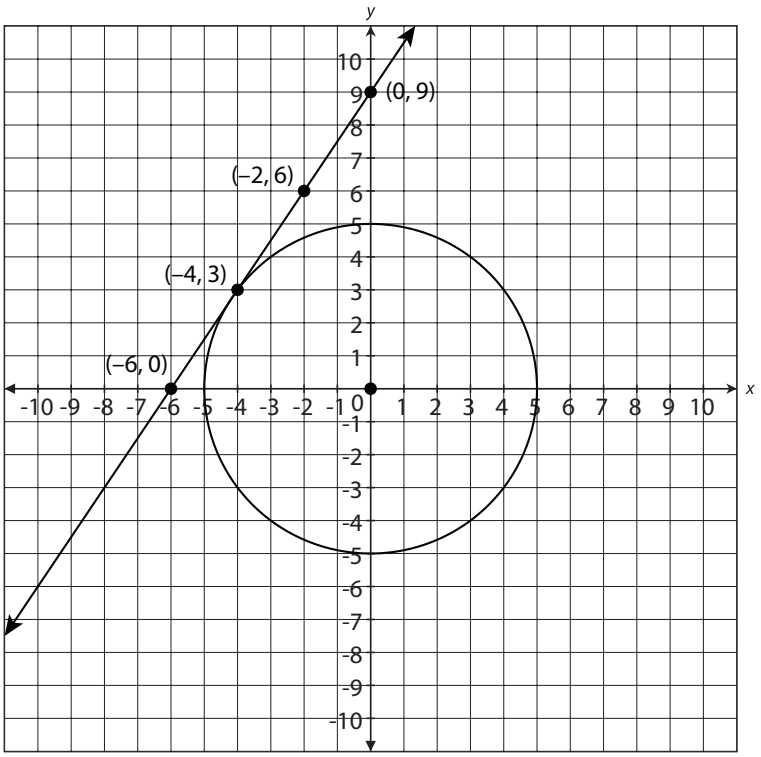
Instruction

Starting from the *y*-intercept, plot the next point by moving down 3 units and to the left 2 units. Another point on the line is located at (–2, 6).

From (–2, 6), plot another point by again moving down 3 units and to the left 2 units to plot point (–4, 3).

From (–4, 3), plot another point by moving down 3 units and to the left 2 units to plot the point (–6, 0).

Use a straightedge to connect the points.



2. Determine the point(s) of intersection, if any.

The graph of the line appears to intersect the circle at one point, (–4, 3).

Therefore, the solution to the system represented by the graphed circle and the linear equation $y=\frac{3}{2}x+9$ is (–4, 3).

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Problem-Based Task 3.7.1: Constant Acceleration and Constant Speed

SMP	
1	2 ✓
3	4 ✓
5	6 ✓
7	8

A car begins at rest and accelerates. Its distance from its starting position in meters, $C(t)$, can be determined as a function of time in seconds, t, by the formula $C(t) = 2t^2$. A second car, 200 meters ahead, is traveling at a constant speed of 20 meters per second. Its distance from the first car's starting point, $C_2(t)$, in meters can be determined as a function of time, t, in seconds by the formula $C_2(t) = 20t + 200$. How long after the first car accelerates will the cars be side by side?

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Problem-Based Task 3.7.1: Constant Acceleration and Constant Speed

Coaching

a. What do the two variables in the problem represent?

b. What do you need to know in order to answer the question posed in the task?

c. How will you determine the solution?

d. How long after the first car accelerates will the cars be side by side?

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Instruction

Problem-Based Task 3.7.1: Constant Acceleration and Constant Speed

Coaching Sample Responses

a. What do the two variables in the problem represent?

The two variables represent time and distance.

b. What do you need to know in order to answer the question posed in the task?

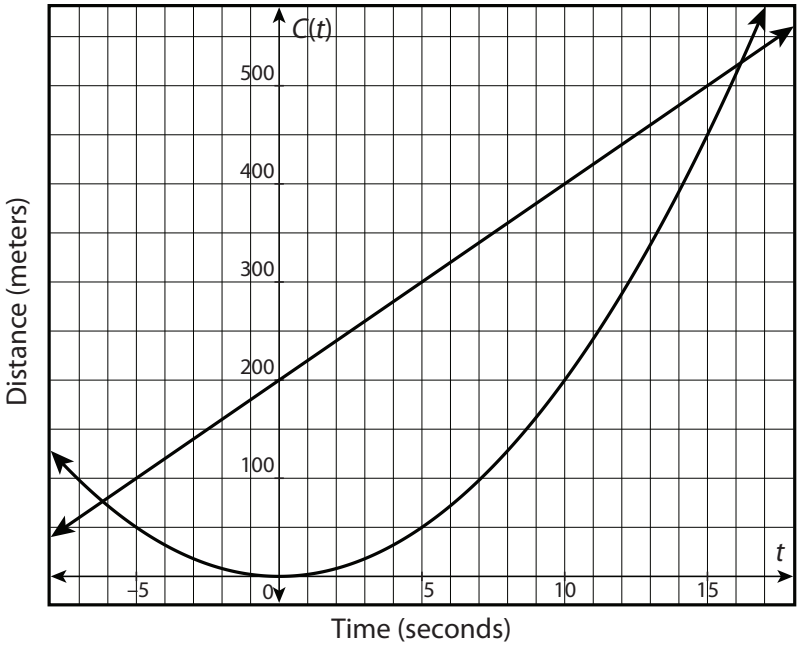
You need to know the time, t, after the first car accelerates until both cars are the same distance from the first car's starting point, $C(t)$.

c. How will you determine the solution?

Graph the functions and find the point(s) of intersection.

d. How long after the first car accelerates will the cars be side by side?

The following graph shows the points of intersection.



There are two points of intersection. However, the cars start moving when $t = 0$, so values of t less than 0 don't apply and we can disregard the solution with negative coordinates.

According to the graph, the cars will be side by side about 16.2 seconds after the first car accelerates.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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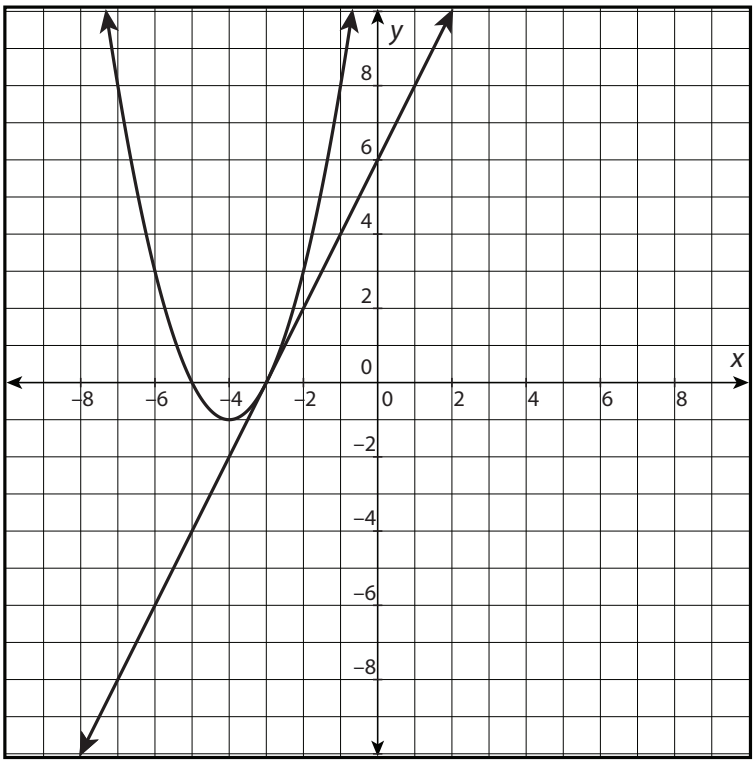
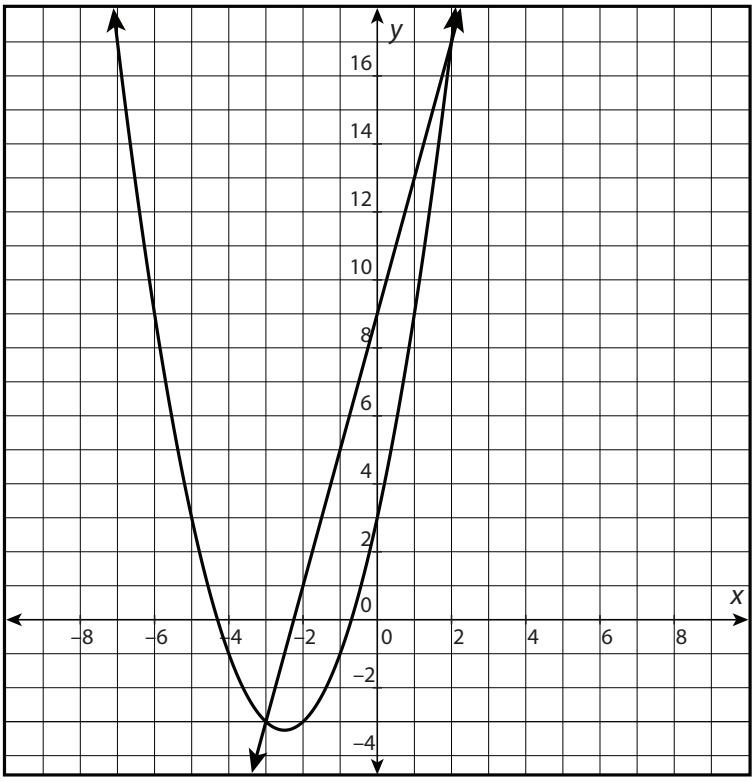
Practice 3.7.1: Solving Systems Graphically

For problems 1 and 2, use the graphs to determine the solution(s) to the system of equations, if any exist.

1.



2.



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SWB p. 217, SRB Practice 3.7.1 p. U3-292

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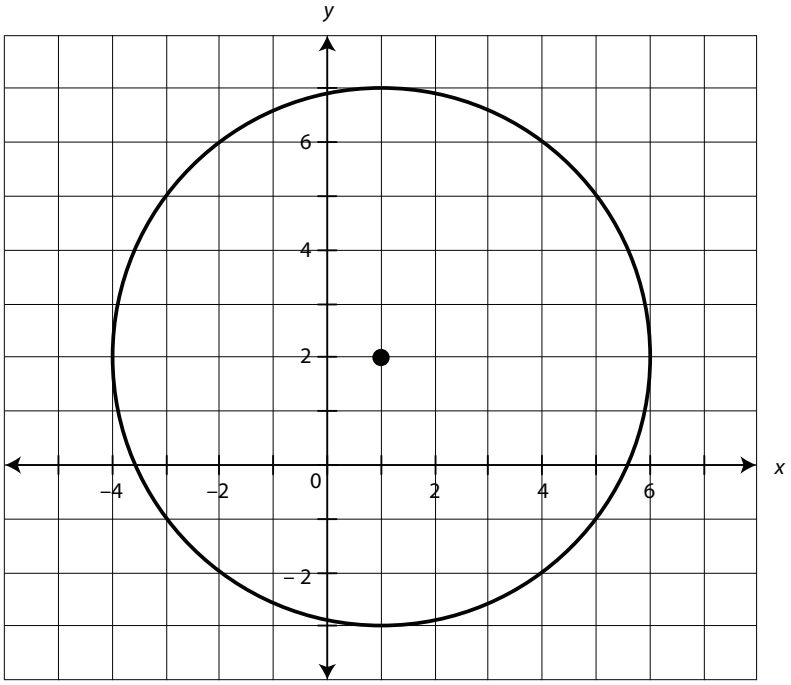
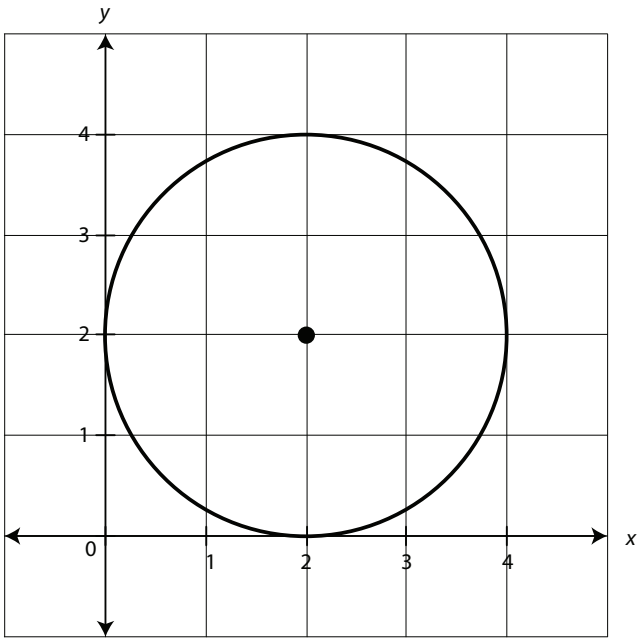
For problems 3 and 4, graph the given line, and then estimate the solutions to the system represented by the circle and the line.

3. $2x + y = 14$



4. $-x + 3y = 6$



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For problems 5–8, graph each system. Determine the real solution(s), if any exist.

5. $\begin{cases} y = x^2 - 12x - 45 \\ y = 3x - 45 \end{cases}$

6. $\begin{cases} y = x^2 - 7x + 10 \\ y = -\frac{5}{6}x + \frac{1}{2} \end{cases}$

7. $\begin{cases} y = x^2 + 5x - 3 \\ y = 2x - 10 \end{cases}$

8. $\begin{cases} y = -4x^2 + 7x + 12 \\ y = 7x + 12 \end{cases}$

For problems 9 and 10, use the given information to create a graph. Then use the graph to answer the questions.

9. The height of a ball thrown straight into the air can be modeled by $h(t) = -16t^2 + 30t + 3.5$, where $h(t)$ is the height of the ball in feet t seconds after the ball is thrown upward. A bird flies in a linear path that intersects the ball's trajectory. The bird's path can be modeled by the function $h(t) = 4.5t + 13.5$, where $h(t)$ is the height of the bird off the ground t seconds since departing its nest. The nest is located on a branch that is 13.5 feet above the ground. If the bird leaves its nest at the same time that the ball is released, how long will it take the bird to cross the path that the ball followed?

10. A car begins at rest and accelerates. Its distance in meters, $C(t)$, can be determined as a function of time in seconds, t, by the formula $C(t) = 3t^2$. A second car, 4 meters ahead, is traveling at a constant speed of 15 meters per second. Its distance, $C_2(t)$, in meters can be determined as a function of time, t, in seconds by the formula $C_2(t) = 15t + 4$. How long after the first car accelerates will the cars be side by side?

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Lesson 3.7.2: Solving Systems Algebraically

Warm-Up 3.7.2

A plane flies 900 miles in 3 hours with a tail wind. It takes the same plane 5 hours to fly 900 miles with a head wind. The system $\begin{cases} 900=(r+w)3 \\ 900=(r-w)5 \end{cases}$ expresses the distance of the plane where r is the speed of the plane in still air and w is the speed of the wind.

1. Use substitution to solve the system of equations.

2. Explain your solution(s).

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Instruction

Lesson 3.7.2: Solving Systems Algebraically

Common Core State Standard

A–REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.*

Warm-Up 3.7.2 Debrief

1. Use substitution to solve the system of equations.

   Simplify the equations by applying the Distributive Property in both equations.

   $900 = (r + w)3$ $\qquad$ $900 = (r - w)5$

   $900 = 3r + 3w$ $\qquad$ $900 = 5r - 5w$

   Isolate a variable in one equation to substitute into the other equation. Let's solve for r in the first equation.

   $900 = 3r + 3w$

   $900 - 3w = 3r$

   $300 - w = r$

   Substitute $300 - w$ for r in the second equation.

   $900 = 5r - 5w$

   $900 = 5(300 - w) - 5w$

   $900 = 1500 - 5w - 5w$

   $-600 = -10w$

   $60 = w$

   Substitute the value of w into the isolated equation to find the value of r.

   $300 - w = r$

   $300 - (60) = r$

   $240 = r$

2. Explain your solution(s).

   The speed of the plane in still air is 240 miles per hour. The speed of the wind is 60 miles per hour.

Connection to the Lesson

- Students will extend their knowledge of solving linear systems using substitution to solving quadratic-linear systems using substitution.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- solving quadratic equations by factoring (A–REI.4b)
- solving quadratic equations by using the quadratic formula (A–REI.4b)

Introduction

Previously, we learned how to solve quadratic-linear systems by graphing and identifying points of intersection. In this lesson, we will focus on solving a quadratic-linear system algebraically. When doing so, substitution is often the best choice. **Substitution** is the replacement of a variable of an equation by a number or an expression that is known to have the same value.

Key Concepts

- When solving a quadratic-linear system, if both equations are written in function form, such as "$y =$" or "$f(x) =$," set the variable expressions of each equation equal to each other.
- When you set the two expressions equal to each other, you are replacing y in one equation with an equivalent expression, thus using the substitution method.
- If one equation is solved for y and the other is not, substitute the expression that is equal to y into the second equation.
- You can solve a quadratic equation by factoring or by using the quadratic formula, a formula that states that the solutions of a quadratic equation of the form $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- Along with the equation for a parabola, the equation of a circle is another form of a second-degree equation, where the greatest exponents of both x and y are 2.
- The standard form of an equation of a circle is $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the center of the circle and r is the radius.
- If the circle is centered at (0, 0), then the equation of the circle has the form $x^2 + y^2 = r^2$.

Common Errors/Misconceptions

- miscalculating signs
- incorrectly distributing coefficients
- incorrectly squaring the binomial after substituting into the equation of the circle

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Instruction

Guided Practice 3.7.2

Example 1

Solve the given system of equations algebraically.

$$\begin{cases} y = x^2 - 11x + 28 \\ y = -3x + 12 \end{cases}$$

1. Since both equations are equal to y, substitute by setting the equations equal to each other.

$-3x + 12 = x^2 - 11x + 28$	Substitute $-3x + 12$ for y in the first equation.

2. Solve the equation either by factoring or by using the quadratic formula.

   Rearrange the terms so that the equation is equal to 0, then solve by factoring.

$-3x + 12 = x^2 - 11x + 28$	Equation
$0 = x^2 - 8x + 16$	Set the equation equal to 0 by adding $3x$ to both sides, and subtracting 12 from both sides.
$0 = (x - 4)^2$	Factor the trinomial.
$x - 4 = 0$	Take the square root of both sides.
$x = 4$	Add 4 to both sides.

   Substitute the value of x into the second equation of the system to find the corresponding y-value.

$y = -3(4) + 12$	Substitute 4 for x.
$y = 0$	

   When $x = 4$, $y = 0$. Therefore, (4, 0) is the solution.

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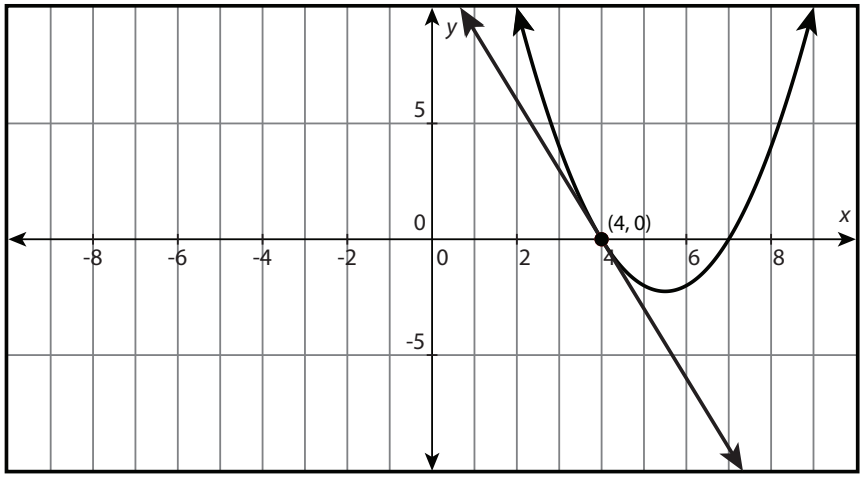
3. Check your solution(s) by graphing.



The graphs of the functions appear to intersect at (4, 0); therefore, (4, 0) checks out as the solution to this system. The solution can also be checked by substituting the coordinates (4, 0) into the given equations and solving to confirm that the results are true statements.

Example 2

Solve the given system of equations algebraically.

$$\begin{cases} y = 2x^2 + 13x + 15 \\ y = x - 1 \end{cases}$$

1. Since both equations are equal to y, substitute by setting the equations equal to each other.

   $x - 1 = 2x^2 + 13x + 15$ Substitute $x - 1$ for y in the first equation.

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2. Solve the equation either by factoring or by using the quadratic formula.

Since the equation can be factored easily, choose this method.

$x - 1 = 2x^2 + 13x + 15$	Equation
$0 = 2x^2 + 12x + 16$	Set the equation equal to 0 by subtracting x from both sides and then adding 1 to both sides.
$0 = 2(x + 2)(x + 4)$	Factor.

Next, set the factors equal to 0 and solve.

$x + 2 = 0$	$x + 4 = 0$
$x = -2$	$x = -4$

Substitute each of the values you found for x into the second equation of the system to find the corresponding y-value.

$y = (-2) - 1$	Substitute –2 for x.
$y = -3$	

$y = (-4) - 1$	Substitute –4 for x.
$y = -5$	

For $x = -2$, $y = -3$, and for $x = -4$, $y = -5$. Therefore, (–2, –3) and (–4, –5) are the solutions to the system.

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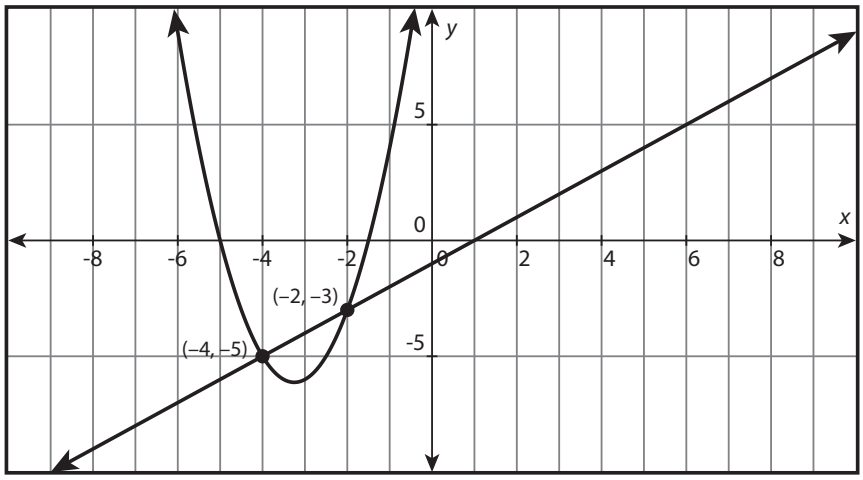
Instruction

3. Check your solution(s) by graphing.



The graphs of the functions appear to intersect at (–2, –3) and (–4, –5); therefore, (–2, –3) and (–4, –5) check out as the solutions to this system. The solutions can also be verified by substituting the coordinates into the original equations and solving to confirm that the results are true statements.

Example 3

Solve the given system of equations algebraically.

$$\begin{cases} y = 7x^2 - 25x - 12 \\ y = -3x - 30 \end{cases}$$

1. Since both equations are equal to y, substitute by setting the equations equal to each other.

$-3x - 30 = 7x^2 - 25x - 12$ Substitute $-3x - 30$ for y in the first equation.

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2. Solve the equation either by factoring or by using the quadratic formula.

The quadratic equation cannot easily be factored, so use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$-3x - 30 = 7x^2 - 25x - 12$	Equation
$0 = 7x^2 - 22x + 18$	Set the equation equal to 0 by adding $3x$ and 30 to both sides; then, apply the quadratic formula.
$a = 7, b = -22, c = 18$	Use the standard form of a quadratic equation, $ax^2 + bx + c = 0$, to find a, b, and c.
$x = \frac{-(-22) \pm \sqrt{(-22)^2 - 4(7)(18)}}{2(7)}$	Substitute values for a, b, and c into the quadratic formula.
$x = \frac{22 \pm \sqrt{-20}}{14}$	Simplify.

Since the value under the square root is negative, there are no real solutions to the system. Simplify to find the complex solutions.

$$x = \frac{22 \pm 2i\sqrt{5}}{14}$$
$$x = \frac{11 \pm i\sqrt{5}}{7}$$

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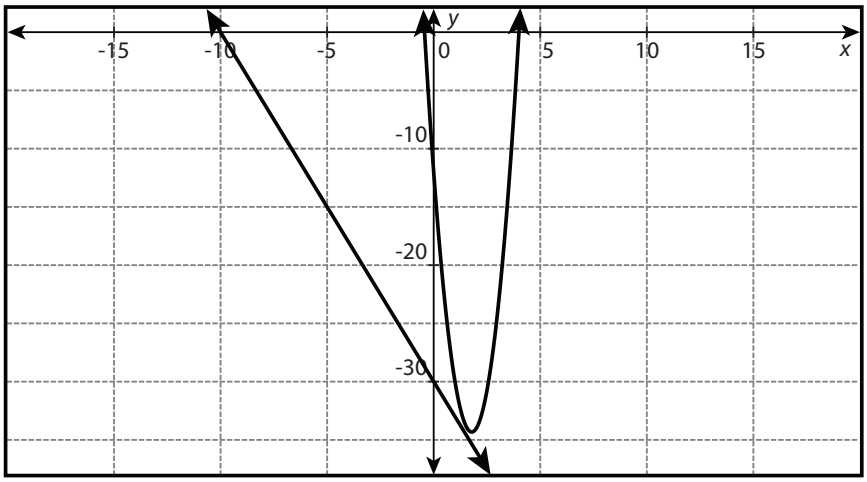
Instruction

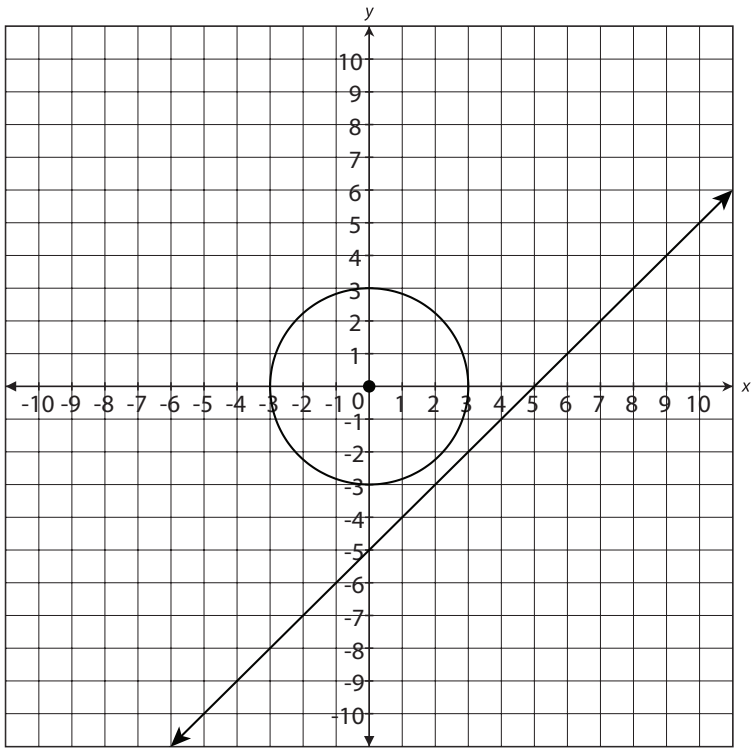
3. Check your solution(s) by graphing.



Since the functions do not intersect, a complex solution is feasible.

Example 4

The system $\begin{cases} y = x - 5 \\ x^2 + y^2 = 9 \end{cases}$ is graphed as shown. Show algebraically that the system has no real solutions. Then, explain what your results mean in terms of the graph.



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Instruction

1. Substitute the expression that is equal to y in the linear equation for the variable y in the circle equation in order to obtain an equation in one variable.

$x^2 + y^2 = 9$	Equation of the circle
$x^2 + (x - 5)^2 = 9$	Substitute $x - 5$ for y.
$x^2 + x^2 - 10x + 25 = 9$	Square the binomial.
$2x^2 - 10x + 25 = 9$	Combine like terms.
$2x^2 - 10x + 16 = 0$	Subtract 9 from both sides so that 0 is on the right side.
$x^2 - 5x + 8 = 0$	Divide both sides by 2 to simplify.

The equation in one variable is $x^2 - 5x + 8 = 0$.

2. Solve the equation found in step 1.

The equation in one variable from step 1 is $x^2 - 5x + 8 = 0$.

This equation is written in the form of a quadratic, $ax^2 + bx + c = 0$, where $a = 1$, $b = -5$, and $c = 8$.

Therefore, we can use the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to solve the equation.

$x = \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(8)}}{2(1)}$	Substitute values for a, b, and c into the quadratic formula.
$x = \dfrac{5 \pm \sqrt{25 - 32}}{2}$	Simplify.
$x = \dfrac{5 \pm \sqrt{-7}}{2} = \dfrac{5 \pm i\sqrt{7}}{2}$	Simplify the radical expression.

The complex solutions to the quadratic equation are $x = \dfrac{5 \pm i\sqrt{7}}{2}$.

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3. Explain what the complex solutions mean in terms of the graph.

   Complex solutions mean that the graphs of the equations in the system do not intersect. There are no real solutions to the system. This can be seen from the graph of the system, which shows no points of intersection.

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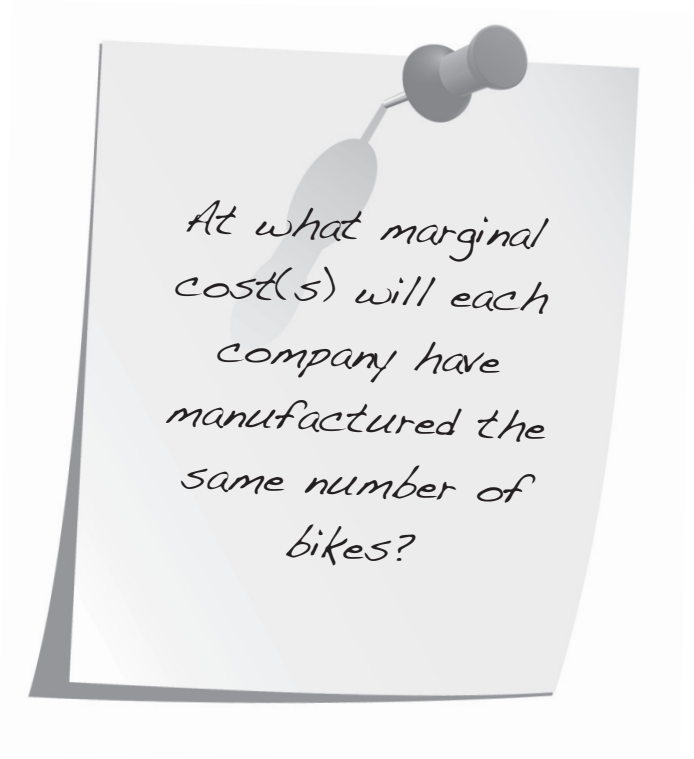
Problem-Based Task 3.7.2: The Business of Bicycles

SMP	
1	2 ✓
3	4 ✓
5	6
7 ✓	8

Nico and Fleur are each designing a fictional bike company for their Business Science class. Nico determines, through research, that the marginal cost of producing a bike is $C_1(x) = 0.0005x^2 + 0.07x + 50$, for which $C_1(x)$ is the marginal cost in dollars per x bikes produced. Fleur determines that the marginal cost for manufacturing her bikes could be modeled by $C_2(x) = \frac{2}{5}x + 62.5$. Assuming each student's fictional company starts "manufacturing" bikes at the same time, at what marginal cost(s) will each company have manufactured the same number of bikes?



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Problem-Based Task 3.7.2: The Business of Bicycles

Coaching

a. What do the variables in the problem represent?

b. What can you do to solve for the unknowns?

c. How will you solve the quadratic equation?

d. Are both solutions useful given the context of the problem? Explain.

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Instruction

Problem-Based Task 3.7.2: The Business of Bicycles

Coaching Sample Responses

a. What do the variables in the problem represent?

The variables are x, the number of bikes produced, and $C_1(x)$, the marginal cost of production.

b. What can you do to solve for the unknowns?

Since both equations are solved for $C(x)$, substitute the expression that is equal to $C(x)$ in one equation for $C(x)$ in the other equation.

$$\frac{2}{5}x + 62.5 = 0.0005x^2 + 0.07x + 50$$

c. How will you solve the quadratic equation?

Rearrange the terms in the equation so that one side equals 0.

Since this equation contains both fractions and decimals, let's rewrite the fraction as a decimal.

$$\frac{2}{5}x + 62.5 = 0.0005x^2 + 0.07x + 50$$

$$0.40x + 62.5 = 0.0005x^2 + 0.07x + 50$$

$$0 = 0.0005x^2 - 0.33x - 12.5$$

Next, solve the quadratic equation using the quadratic formula.

$$x = \frac{-(-0.33) \pm \sqrt{(-0.33)^2 - 4(0.0005)(-12.5)}}{2(0.0005)}$$

$$x = \frac{0.33 \pm \sqrt{0.1339}}{0.001}$$

$x \approx 695.9235$ and $x \approx -35.9235$

Substitute the values of x into either of the given equations to find the corresponding y-values.

$$y = \frac{2}{5}(695.9235) + 62.5$$

$$y \approx 340.8694$$

$$y = \frac{2}{5}(-35.9235) + 62.5$$

$$y \approx 48.1306$$

The solutions are approximately (695.9235, 340.8694) and (–35.9235, 48.1306).

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d. Are both solutions useful given the context of the problem? Explain.

Based on the solutions, when the students have spent $340.87 they will have manufactured about 696 bikes. The second solution, $x \approx -35.9235$, doesn't make sense in this context because the number of bikes manufactured should be a positive number.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 3.7.2: Solving Systems Algebraically

For problems 1–5, solve each system of equations algebraically. Check your solutions graphically.

1. $\begin{cases} y = 5x^2 + 4 \\ y = 4 \end{cases}$

2. $\begin{cases} y = x^2 - 9x \\ y = -8x \end{cases}$

3. $\begin{cases} y = x^2 - 2x - 3 \\ y = -\frac{3}{2}x - 4 \end{cases}$

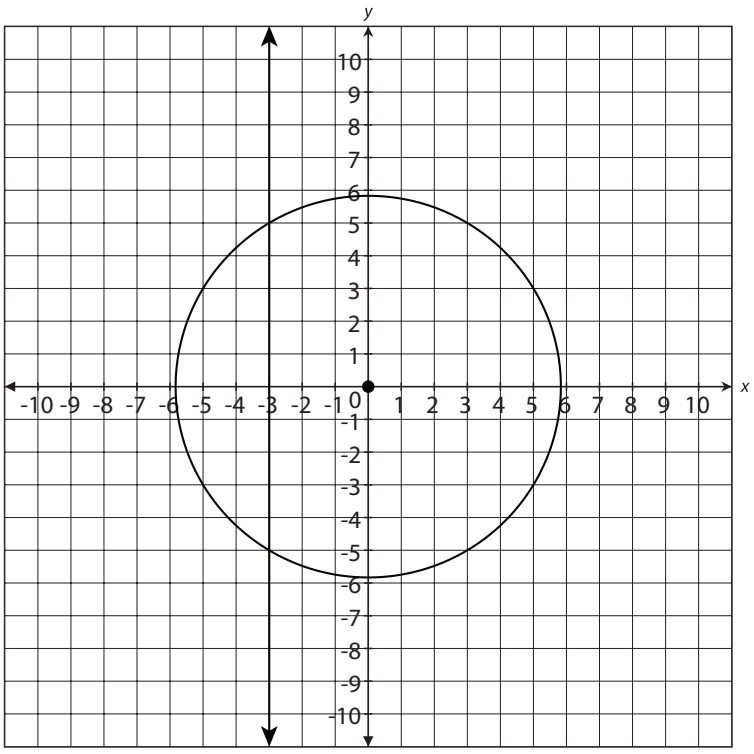
4. $\begin{cases} y = x^2 + x \\ y = 3x - 1 \end{cases}$

5. $\begin{cases} y = 0.25x^2 + 5x - 3.4 \\ y = -4.5x + 7.5 \end{cases}$

For problems 6–8, solve each system of equations algebraically. Verify that your solutions match the provided graph. Show your work.

6.

$\begin{cases} x = -3 \\ x^2 + y^2 = 34 \end{cases}$



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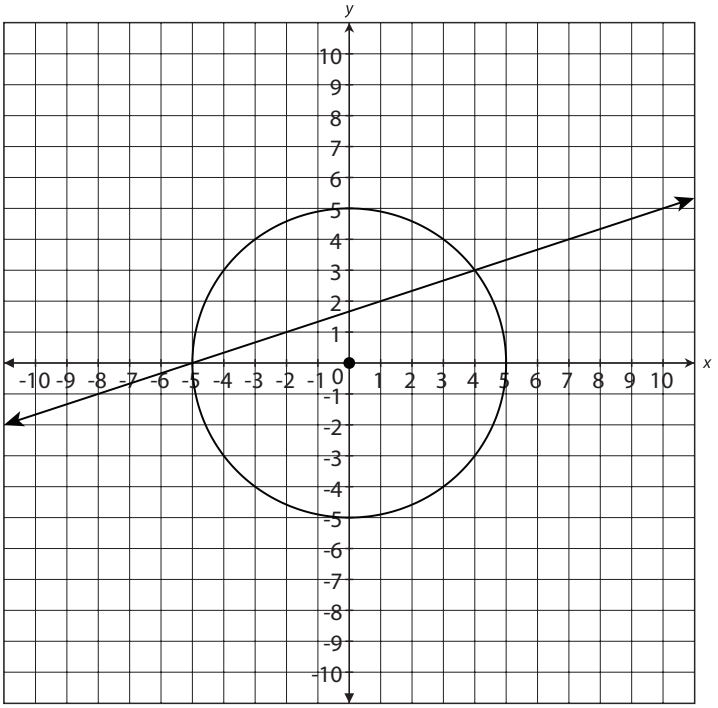
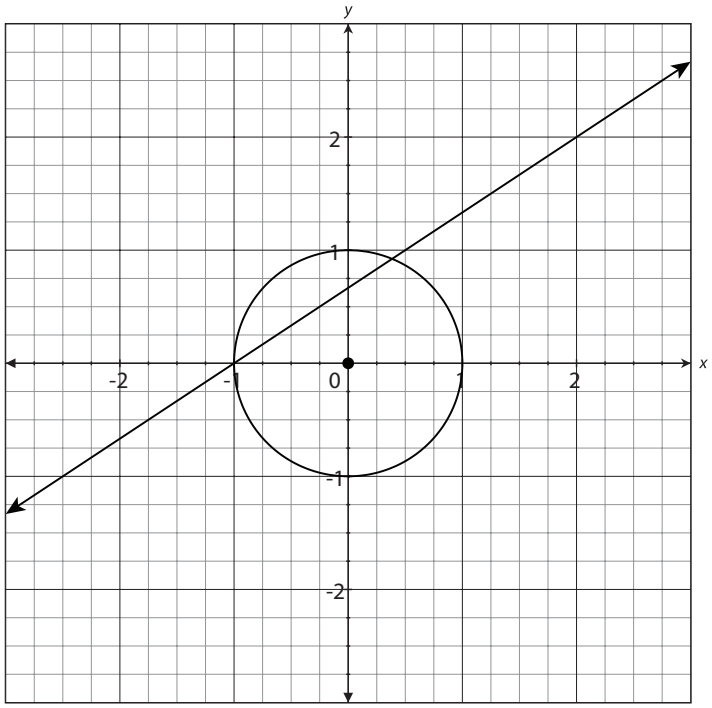
7.

$$\begin{cases} y=\frac{1}{3}x+\frac{5}{3} \\ x^2+y^2=25 \end{cases}$$



8.

$$\begin{cases} y=\frac{2}{3}x+\frac{2}{3} \\ x^2+y^2=1 \end{cases}$$



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For problems 9 and 10, use the given information to write a system of equations, and then solve the system algebraically to answer the questions.

9. A car begins at rest and accelerates. Its distance in meters, $C(t)$, can be determined as a function of time in seconds, t, by the formula $C_1(t) = 4t^2$. A second car, 150 meters ahead, is traveling at a constant speed of 20 meters per second. Its distance, $C_2(t)$, in meters can be determined as a function of time, t, in seconds by the formula $C_2(t) = 20t + 150$. How long after the first car accelerates will the cars be side by side?

10. A year-round ski shop sells both snowboarding and ski equipment. The revenue from selling snowboards can be modeled by the function $R_1(s) = -1.5s^2 + 30s$, for which $R_1(s)$ is the revenue in hundreds of dollars from selling s sets of equipment. The revenue from selling skis can be modeled by the function $R_2(s) = 150s$. Is there ever a time when the revenue from selling skis is the same as the revenue for selling snowboards? Explain.

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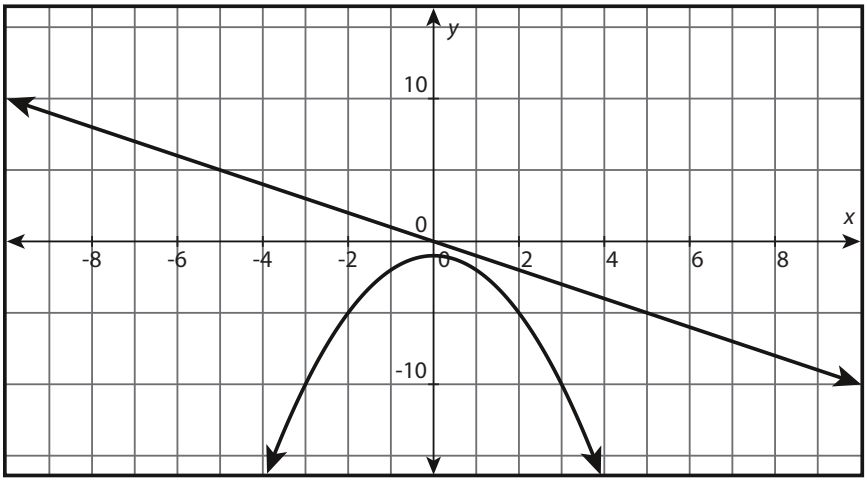
Progress Assessment

Circle the letter of the best answer.

1. The graphs of two equations that make up a system of equations are shown. Use the graph to determine the real solution(s) to the system of equations, if any exist.



   a. (0, 1)

   b. (0, 0)

   c. (–2, –5) and (2, –5)

   d. no real solutions

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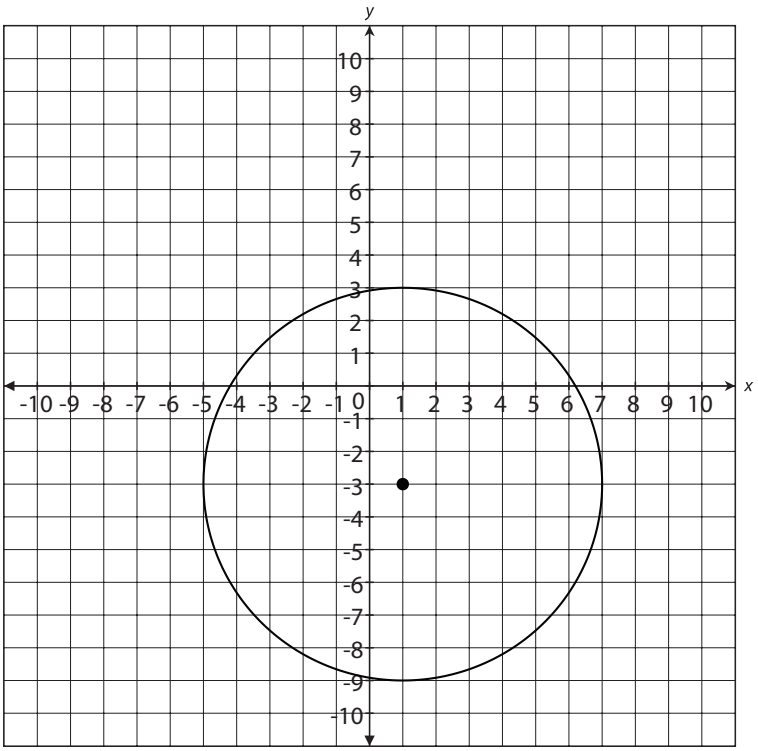
2. Solve the system that is represented by the circle shown and the line with the equation $y = -x + 4$.



a. There are two real solutions at (1, 3) and (7, –3).

b. There is exactly one real solution at (1, 3).

c. There is exactly one real solution at (7, –3).

d. There are no real solutions.

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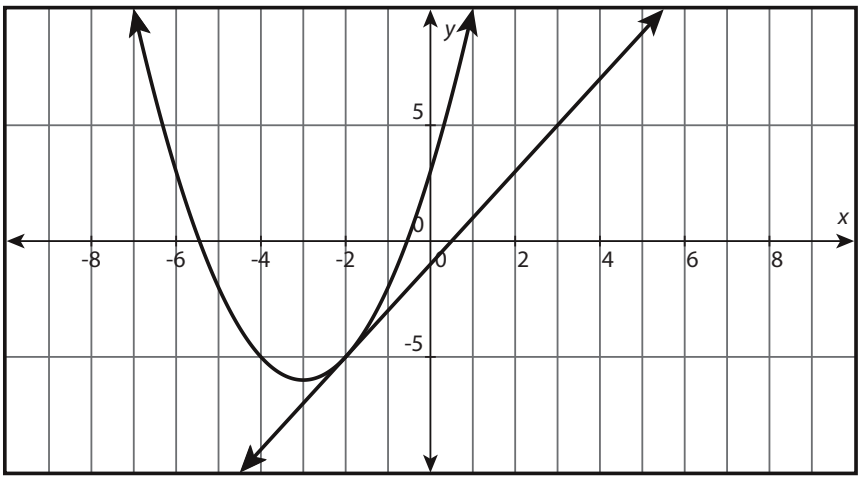
3. Use the graph to determine the real solution(s) to the system of equations, if any exist.



a. (–3, –6)

b. (–5.5, 0) and (–0.5, 0)

c. (–2, –5)

d. no real solutions

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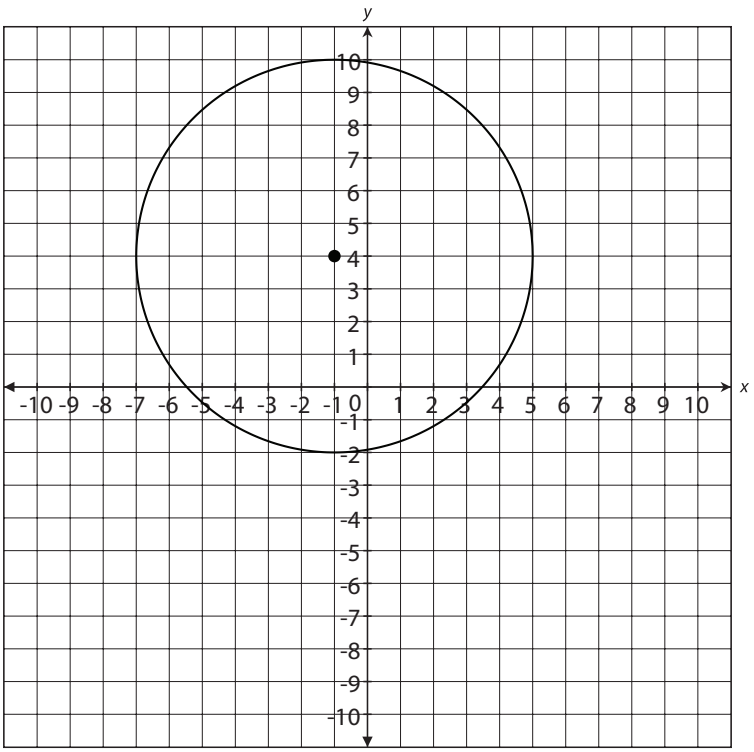
4. Solve the system that is represented by the circle shown and the line with the equation $y=\frac{4}{5}x-4$.



a. There are no real solutions.

b. There is exactly one real solution at (5, 4).

c. There is exactly one real solution at (–7, 4).

d. There are two real solutions at (5, 4) and (–7, 4).

5. Solve the system algebraically.

$$\begin{cases} y=-x^2-12x+43 \\ y=-2x+43 \end{cases}$$

a. (–10, 63) and (0, 43)

b. (0, 43)

c. (6, 79)

d. no real solutions

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6. Solve the system algebraically.

$$\begin{cases} y = x^2 + 5x + 4 \\ y = x \end{cases}$$

a. (–4, 0) and (–1, 0)
b. (0, 0)
c. (–2, –2)
d. no real solutions

7. Solve the system algebraically.

$$\begin{cases} y = \frac{2}{3}x^2 + 1 \\ y = 7 \end{cases}$$

a. $\left(\frac{2}{3}, 1\right)$ and (3, 7)
b. (7, –3) and (7, 3)
c. (–3, 7) and (3, 7)
d. no real solutions

8. Solve the system algebraically.

$$\begin{cases} y = -2x^2 + 2x + 1 \\ y = 2x + 2 \end{cases}$$

a. (0, 1) and (0, 2)
b. (0.5, 1.5)
c. (–0.5, –0.5) and (–0.5, 1)
d. no real solutions

9. Solve the system algebraically.

$$\begin{cases} y = x - 4 \\ x^2 + y^2 = 8 \end{cases}$$

a. There are no real solutions.
b. (2, –2)
c. (–2, –6) and (6, 2)
d. There are two real solutions, and their x-coordinates are $\frac{4+\sqrt{32}}{2}$ and $\frac{4-\sqrt{32}}{2}$.

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10. Solve the system algebraically.

$$\begin{cases} y = x + 4 \\ x^2 + y^2 = 6 \end{cases}$$

a. There are no real solutions.

b. There is exactly one real solution and its x-coordinate is –5.

c. There are two real solutions and their x-coordinates are $\frac{-4+\sqrt{12}}{2}$ and $\frac{-4-\sqrt{12}}{2}$.

d. There are two real solutions and their x-coordinates are $\frac{-4+\sqrt{20}}{2}$ and $\frac{-4-\sqrt{20}}{2}$.

Use what you have learned about solving systems to complete the following problem. Round your answers to the nearest hundredth.

11. A company that makes car parts uses two models to estimate the company's average profit. The first model is $A(p) = 80p - 4p^2$, for which $A(p)$ is the average profit in dollars per part when p parts are produced. The second model is $M(p) = 20p + 150$, for which $M(p)$ is also the average profit in dollars per part when p parts are produced.

a. Use a graph to approximate the coordinates of the point(s) of intersection, if any, where the number of parts described by each model is the same.

b. Use algebra to approximate the coordinates of the point(s) of intersection, if any, where the number of parts described by each model is the same.

c. Explain the solution(s) in parts a and b in relation to the average profit the manufacturing company may receive.

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Unit 3 Assessment

Circle the letter of the best answer.

1. What values of x make the expression $(x - 6)(x + 3)$ positive?

   a. $-3 < x < 6$
   b. $6 < x < -3$
   c. $x < -3$ or $x > 6$
   d. $x > -3$ or $x < 6$

2. Solve $x^2 - 7x = -12$ for x.

   a. $x = 3; x = 4$
   b. $x = -4; x = -3$
   c. $x \approx -1.42; x \approx 8.42$
   d. $x \approx -8.43; x \approx 1.42$

3. Solve $4x^2 + 8x + 3 = 0$ for x.

   a. $x = -8.5; x = -7.5$
   b. $x = 7.5; x = 8.5$
   c. $x = 0.5; x = 1.5$
   d. $x = -1.5; x = -0.5$

4. Solve $x^2 + 9x + 20 > 0$ for x.

   a. $x \geq -5$ or $x \leq -4$
   b. $x \leq -5$ or $x \geq -4$
   c. $x > -5$ or $x < -4$
   d. $x < -5$ or $x > -4$

5. What is the equation of a quadratic function in standard form that has zeros $x = 3$ and $x = 5$ and that passes through the point $(-1, 24)$?

   a. $f(x) = 3x^2 - 4x + 4$
   b. $f(x) = x^2 + 8x + 15$
   c. $f(x) = x^2 - 8x + 15$
   d. $f(x) = 3x^2 - 8x + 15$

6. Solve the equation $y = 0.5(x - 6)^2 + 15$ for x in terms of y.

   a. $x = 13 \pm 2\sqrt{y}$
   b. $x = 6 \pm \sqrt{2y - 30}$
   c. $x = \pm 2\sqrt{2y + 21}$
   d. $x = 6 \pm \sqrt{y - 7.5}$

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7. Solve $3x^2 = 4x - 2$ for x.

   a. $x = \frac{2 \pm \sqrt{10}}{3}$

   b. $x = \frac{-2 \pm \sqrt{10}}{3}$

   c. $x = \frac{2}{3} \pm \left(\frac{\sqrt{2}}{3} \right) i$

   d. $x = -\frac{2}{3} \pm \left(\frac{\sqrt{2}}{3} \right) i$

8. Naya and Scott have a cleaning business. Naya can clean an office suite in 20 minutes less time than it takes Scott to do the same job. Working together, they can complete the job in 90 minutes. Approximately how long does it take Naya to complete the job if she works alone? Assume both people work at steady rates and their rates are not affected by either working alone or working together.

   a. 123 minutes

   b. 148 minutes

   c. 171 minutes

   d. 191 minutes

9. Solve $\frac{-x+6}{4x+8} \leq 0$ for x.

   a. $x < -2$ or $x \geq 6$

   b. $x \leq -6$ or $x > 2$

   c. $-6 < x < 2$

   d. $-2 < x < 6$

10. Car sales were 32% higher nationwide in June 2012 than in June 2008. A team of economists who track car sales want to know the annual rate of increase, assuming that June sales increased at the same rate each year. Which option describes how the team can begin to find the approximate annual rate of increase?

    a. Write $(1+0.08)^1 = \left(1.08^4\right)^{\frac{1}{4}}$ and then evaluate 1.08^4.

    b. Write $(1+0.08)^1 = \left(1.08^{\frac{1}{4}}\right)^4$ and then evaluate $1.08^{\frac{1}{4}}$.

    c. Write $(1+0.32)^1 = \left(1.32^4\right)^{\frac{1}{4}}$ and then evaluate 1.32^4.

    d. Write $(1+0.32)^1 = \left(1.32^{\frac{1}{4}}\right)^4$ and then evaluate $1.32^{\frac{1}{4}}$.

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11. What are the solutions to the system of equations?

$$\begin{cases} y = x^2 - 6x + 8 \\ y = x + 2 \end{cases}$$

a. (1, 6) and (3, 8)

b. (1, 3) and (6, 8)

c. (1, 8) and (3, 6)

d. (4, 0) and (2, 0)

12. What are the values of x in the solution of the following system of equations?

$$\begin{cases} y = -x - 5 \\ x^2 + y^2 = 8 \end{cases}$$

a. $x = \dfrac{-10 \pm 6i}{4}$

b. $x = -4$ and $x = -1$

c. $x = \dfrac{-10 \pm \sqrt{124}}{4}$

d. $x = \dfrac{-10 \pm \sqrt{236}}{4}$

Use what you have learned about expressions and equations to solve each of the following problems.

13. Given the equation $x^2 - 6x = -18$, complete each of the following.

a. State the value of the discriminant.

b. Determine the type and number of solutions of the equation.

c. Solve the equation.

continued

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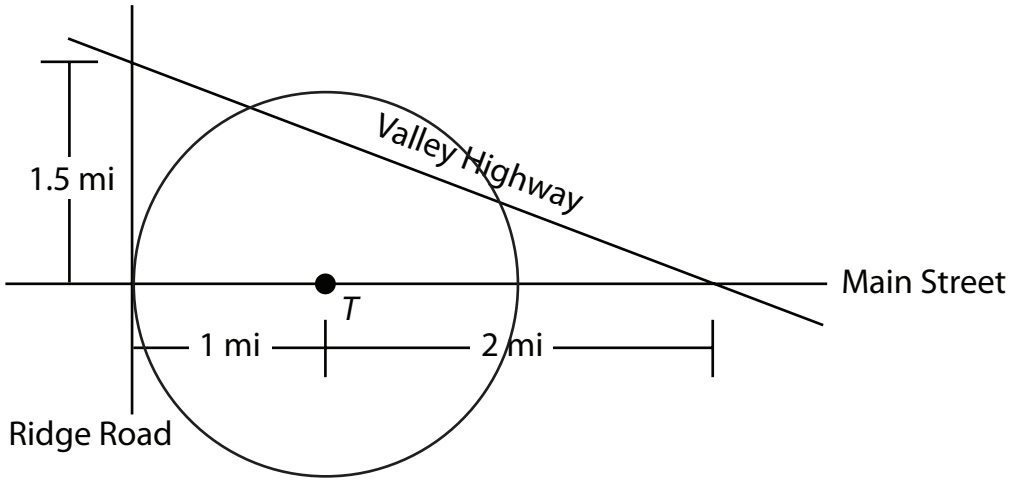
14. An online social networking site has increased its user base 20% during the last 3 months. The site now has 458,256 active users.

    a. How many users did the site have 3 months ago?

    b. Assuming the number of active users increased at the same rate each month, what was the monthly rate of increase?

    c. If the number of active users continues to increase at this rate, how many active users will the site have 5 months from now?

15. A town hall is located at point *T* on Main Street, as shown in the diagram. The town's building regulations state that all new construction within 1 mile of the town hall must conform to a historic building code. A developer wants to know how much of Valley Highway must conform to the code. Complete parts a–d to determine what the developer wants to know.



    a. Set up a coordinate system with *T* at the origin. Use the given dimensions to write coordinates at all points where roads intersect. What does the circle on the diagram represent?

    b. If the equation of the circle is given by $x^2 + y^2 = 1$, what is the second equation of the system?

    c. Solve the system. Show your work.

    d. Determine the length of the portion of Valley Highway that must conform to the code. Use the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

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Answer Key

Lesson 1: Interpreting Structure in Expressions

Pre-Assessment, p. U3-1

1. c
2. a
3. c
4. b
5. a

Warm-Up 3.1.1, p. U3-4

1. The toll can be written as the expression $1.20 + 0.04m$.
2. The toll charged for driving 14 miles on the highway is $1.76.

Practice 3.1.1: Identifying Terms, Factors, and Coefficients, p. U3-15

1. terms: $16x^2$, $-12x$, 20; coefficients: 16, –12; constant term: 20; factors: 16 and x^2; –12 and x
2. $-7x^2 - 62$; binomial
3. $7x^2 + 10$
4. The expression is quadratic because it simplifies to $-3x^2 - 42x$ and can be written in the form $ax^2 + bx + c$, with $c = 0$.
5. The expression is not quadratic because it simplifies to $-2x + 2$. This is a linear expression and cannot be written in the form $ax^2 + bx + c$.
6. Expression: $7x^2 + (5 - x^2) = 6x^2 + 5$; terms: $6x^2$, 5; coefficient: 6; constant term: 5. The expression is quadratic because it can be written in the form $ax^2 + bx + c$.
7. Expression: $(1/2)(12 + x^2) - (1/3)x = (1/2)x^2 - (1/3)x + 6$; terms: $(1/2)x^2$, $(-1/3)x$, 6; coefficients: 1/2, (–1/3); constant term: 6. The expression is quadratic because it can be written in the form $ax^2 + bx + c$.
8. Expression: $4s$; term: $4s$; coefficient: 4; constant term: none. The expression is not quadratic because it cannot be written in the form $ax^2 + bx + c$. It is a linear expression.
9. Expression: $6s^2$; term: $6s^2$; coefficient: 6; constant term: none. The expression is quadratic because it can be written in the form $ax^2 + bx + c$.
10. Expression: $(4/3)\pi r^3$; term: $(4/3)\pi r^3$; coefficient: $(4/3)\pi$; constant term: none. The expression is not quadratic because it cannot be written in the form $ax^2 + bx + c$.

Warm-Up 3.1.2, p. U3-17

1. $29x - 2$
2. If a fence panel is 4 feet long, the perimeter of the park is 114 feet.
3. If the length of each fence panel increases to 5 feet, the perimeter of the park increases to 143 feet. Subtracting 114 from 143 indicates this is an increase of 29 feet.

Practice 3.1.2: Interpreting Complicated Expressions, p. U3-28

1. $-7 < x < 10$
2. $x > 3$ or $x < -1/2$
3. $12x^2 + 17x - 5$; $a = 12$; $b = 17$; $c = -5$
4. $7x^2 + 31x - 30$; $a = 7$; $b = 31$; $c = -30$
5. The expression is quadratic because it can be written in the form $ax^2 + bx + c$.
6. The expression is not quadratic because it cannot be written in the form $ax^2 + bx + c$.
7. The taxes increase from $3,600 to $14,400.
8. The area increases from $\sqrt{3}$ square inches to $25\sqrt{3}$ square inches.
9. The area decreases by a factor of 4.
10. The area increases by a factor of 4.

Progress Assessment, p. U3-30

1. d
2. b
3. d
4. b
5. a
6. d
7. c
8. a
9. a
10. c
11. a. $900/(s^2)$
    b. The expression is not quadratic because it cannot be written in the form $ax^2 + bx + c$.
    c. The height decreases by a factor of 4.

Lesson 2: Creating and Solving Quadratic Equations in One Variable

Pre-Assessment, p. U3-32

1. d
2. c
3. a
4. d
5. a

Warm-Up 3.2.1, p. U3-37

1. The base of the ladder is about 7.75 feet from the house.
2. The ladder reaches about 14.94 feet up the side of the house.

Practice 3.2.1: Taking the Square Root of Both Sides, p. U3-47

1. $x = \pm 9$
2. no real solutions
3. $x = \pm 3$
4. $x = -4$ and $x = -2$
5. no real solutions
6. $x = 12.5$ and $x = 7.5$

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7. A quadratic equation in the form $ax^2 + b = c$ has two real, rational solutions when $\frac{c-b}{a}$ is a positive perfect square.
8. $s = 2\sqrt{10} \approx 6.32$ centimeters
9. $r = \frac{\sqrt{60\pi}}{\pi} \approx 4.37$ millimeters
10. $a = \frac{10\sqrt{3}}{3} \approx 5.77$ inches

Warm-Up 3.2.2, p. U3-48

1. There are four possible sets of dimensions:
   $l = 30$ in, $w = 1$ in
   $l = 15$ in, $w = 2$ in
   $l = 10$ in, $w = 3$ in
   $l = 6$ in, $w = 5$ in
2. $w = 5x$ inches
3. $l = 10x^2y$ inches

Practice 3.2.2: Factoring Expressions by the Greatest Common Factor, p. U3-63

1. $x^2(3x + 5)$
2. $2xy\,(x - 4y)$
3. $y^2(y^2 + 2)$
4. no common factor
5. $xy^2(x^2 - 2xy + 5)$
6. $7x(1 - 3xy)$
7. $15x^2y$
8. $5x^3 + 11y^2$
9. Samuel did not factor out the greatest common number from each term. He factored out a 2, but he should have factored out a 4.
10. Ariana did not factor out the correct common powers of the variables. She factored out xyz^2, when she should have factored out xyz.

Warm-Up 3.2.3, p. U3-64

1. $(x^2 + 8x + 15)$ square feet
2. $(x^2 - 16)$ square feet
3. $(x + 7)$ feet

Practice 3.2.3: Factoring Expressions with $a = 1$, p. U3-84

1. $(y + 10)(y - 10)$
2. $(x - 7)(x - 2)$
3. $(x + 8)^2$
4. not factorable
5. $7(a - 2)(a + 2)$
6. $4(x + 9)(x - 1)$
7. $6(x + 3y)(x - 3y)$
8. $(a - 7)$ feet and $(a - 7)$ feet
9. $(2x + 5)$ meters and $(2x - 5)$ meters
10. $(y + 5)$ inches and $(y - 2)$ inches

Warm-Up 3.2.4, p. U3-85

1. $(x + 1)$ inches
2. $(x + 4)$ inches and $(x + 6)$ inches
3. The width is 8 inches larger than the length.

Practice 3.2.4: Factoring Expressions with $a > 1$, p. U3-106

1. $(5x + 7)(3x - 1)$
2. $(3n + 2)(n + 8)$
3. $(2y - 7)(2y - 9)$
4. $(3x - 4)(x + 5)$
5. $3(a + 4)(a + 9)$
6. $(5n + 2)^2$
7. $2(2y - 7)(3y - 1)$
8. length = $(3x + 7)$ inches; width = $(3x + 7)$ inches
9. length = $(5x - 1)$ feet; width = $(x + 3)$ feet
10. length = $(3x - 2)$ meters; $(2x + 7)$ meters

Warm-Up 3.2.5, p. U3-107

1. $x - 8$
2. $x = 20$ feet
3. 25 feet
4. 1,500 cubic feet

Practice 3.2.5: Solving Quadratic Equations by Factoring, p. U3-123

1. $x = -6, x = 8$
2. $y = \frac{5}{2}, y = -7$
3. $n = 0,\ n = \frac{9}{5}$
4. $x = 4, x = -4$
5. $y = 5, y = 3$
6. $a = \frac{7}{2},\ a = -\frac{1}{3}$
7. $x = 10, x = -6$
8. 0.375 second
9. 2 seconds
10. 5 seconds

Warm-Up 3.2.6, p. U3-124

1. The area of the patio is $(36x^2)$ ft^2.
2. The total area of the patio is $(36x^2 + 48x + 16)$ ft^2.

Practice 3.2.6: Completing the Square, p. U3-135

1. $c = 81$
2. $c = 144$
3. $c = \frac{225}{4}$
4. $c = \frac{1}{4}$
5. $x = -10$ and $x = 0$
6. $x = -13$ or $x = 1$
7. $x = -\frac{1}{3} \pm \frac{\sqrt{22}}{3}$
8. approximately 6.89 feet; you cannot have a negative length of a porch, so only one value of x is reasonable.

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9. after approximately 2.12 seconds; the negative answer represents time before the ball was kicked, so only one value of x is reasonable.
10. approximately 31.36 mph and approximately 88.64 mph

Warm-Up 3.2.7, p. U3-136

1. The area of the triangular patch is $4\sqrt{3}$ square inches, or about 6.93 square inches.
2. The amount of fabric needed for the 3 additional patches is $12\sqrt{3}$ square inches, or about 20.78 square inches.

Practice 3.2.7: Applying the Quadratic Formula, p. U3-149

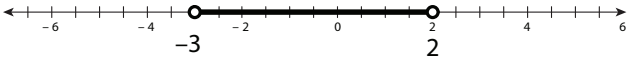
1. 0; one real, rational solution
2. 1; two real, rational solutions
3. $x = \frac{3 \pm \sqrt{41}}{4}$
4. $x = -4 \pm 4\sqrt{2}$
5. no real solutions
6. $x = \frac{3}{2}$ and $x = -17$
7. after approximately 1.96 seconds
8. \$10
9. after approximately 1.30 seconds
10. The discriminant, or the part of the quadratic formula under the radical, tells the number and type of roots of a quadratic equation.

Warm-Up 3.2.8, p. U3-150

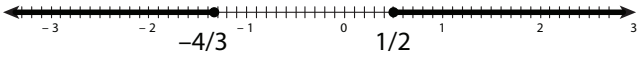
1. $0.6p + 0.3r \leq 240$
2. The workers can assemble 600 racing games at the most.

Practice 3.2.8: Solving Quadratic Inequalities, p. U3-164

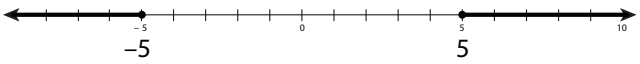
1. $-3 < x < 2$



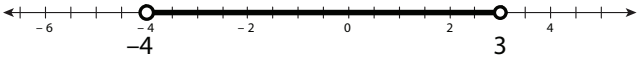
2. $x \leq -\frac{4}{3}$ or $x \geq \frac{1}{2}$



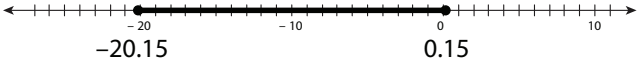
3. $x \leq -5$ or $x \geq 5$



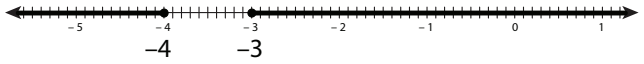
4. $-4 < x < 3$



5. $-20.15 \leq x \leq 0.15$



6. $x \leq -4$ or $x \geq -3$



7. no solution
8. from 0.29 second to 1.71 seconds into the jump
9. from 0.28 second to 0.72 second into the dive
10. at speeds below 18.76 miles per hour

Progress Assessment, p. U3-165

1. a
2. b
3. b
4. a
5. c
6. c
7. a
8. d
9. c
10. b
11. Methods may vary. Recommended methods are listed.
    a. Solve by factoring; $x = -1$ or $x = -19$
    b. Solve by completing the square; $x = -10 \pm \sqrt{85}$
    c. Solve by using the quadratic formula; $x = \frac{-10 \pm \sqrt{79}}{3}$
    d. Solve by taking square roots; $x = \pm \frac{\sqrt{5}}{3}$

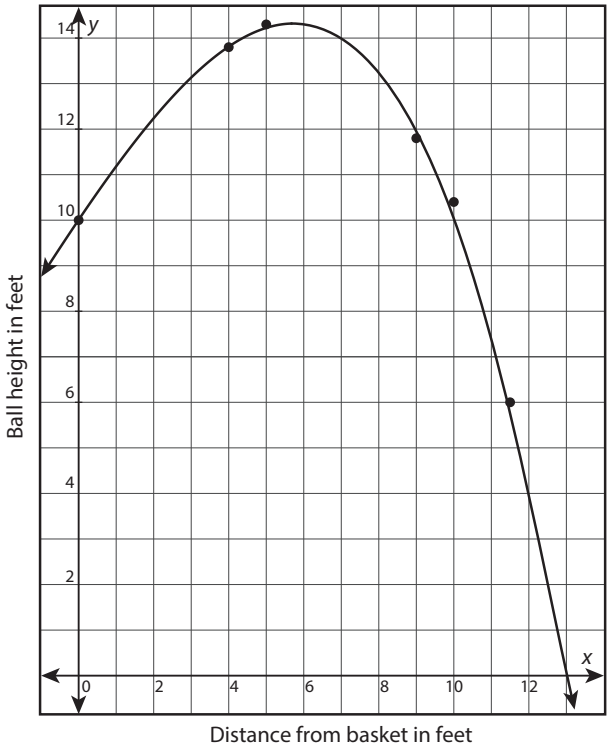
Lesson 3: Creating Quadratic Equations in Two or More Variables

Pre-Assessment, p. U3-167

1. b
2. b
3. a
4. d
5. c

Warm-Up 3.3.1, p. U3-171

1.



2. The ball will be about 13.5 feet high.

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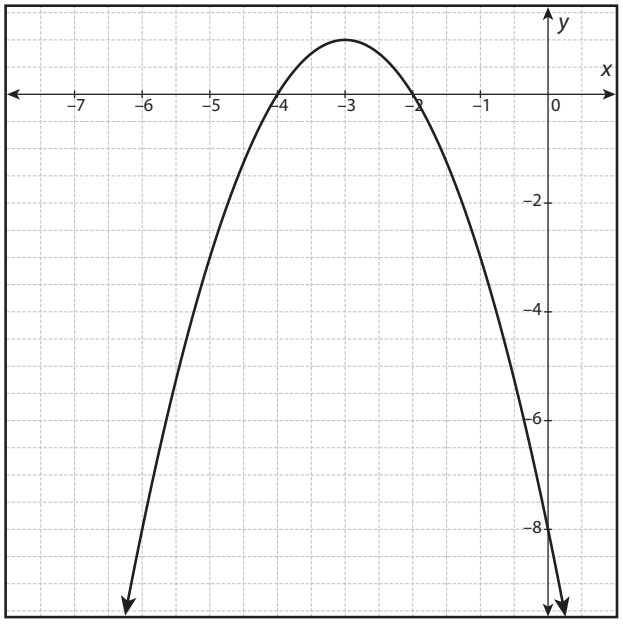
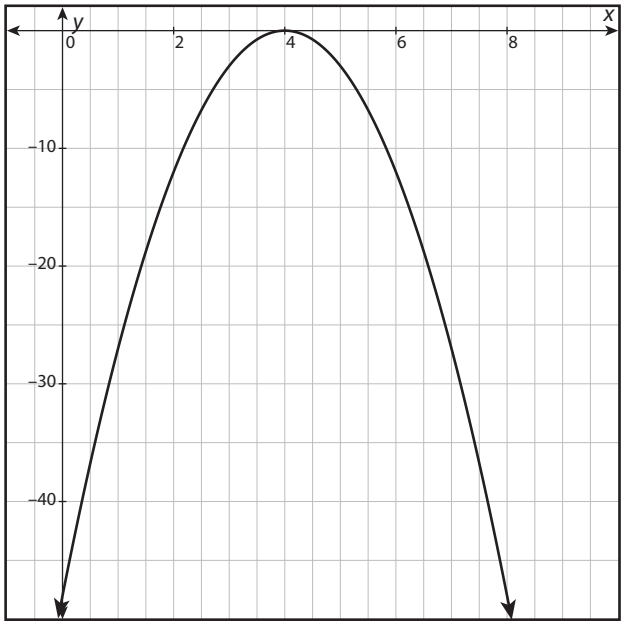
Practice 3.3.1: Creating and Graphing Equations Using Standard Form, p. U3-197

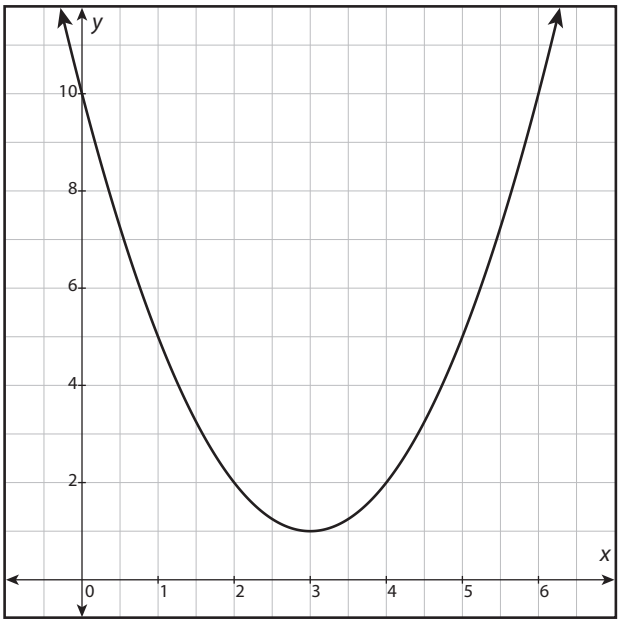
1.



2.



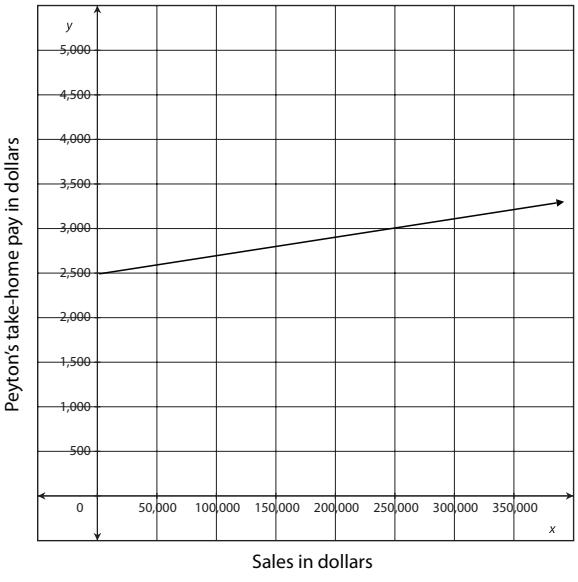
3.



4. *y*-intercept: 3; vertex: (2, –1); the vertex is a minimum because $a > 0$.
5. *y*-intercept: 0; vertex: (3, –9); it has a minimum because $a > 0$.
6. *y*-intercept: –6; vertex: (–1, 1); it has a maximum because $a < 0$.
7. Yes, the *y*-intercept is –28 and the graph opens up because $a > 0$.
8. $y = -x^2 - 6x - 2$
9. 328 feet
10. $36.67

Warm-Up 3.3.2, p. U3-199

1. $y = 0.002x + 2500$
2.



3. *x*-intercept: –1,250,000; *y*-intercept: 2,500

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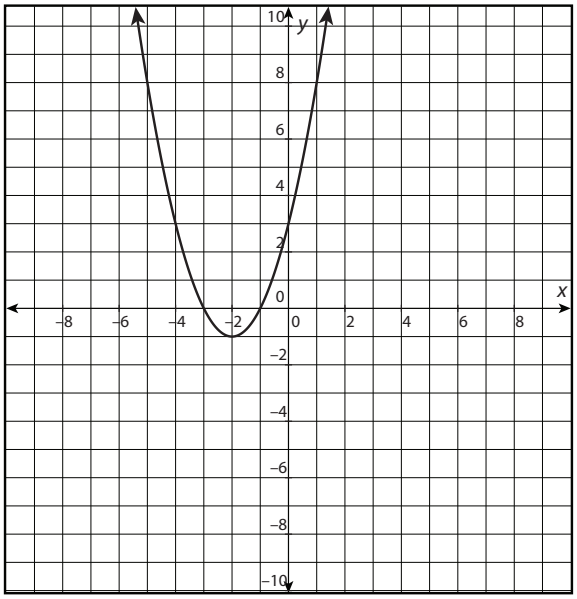
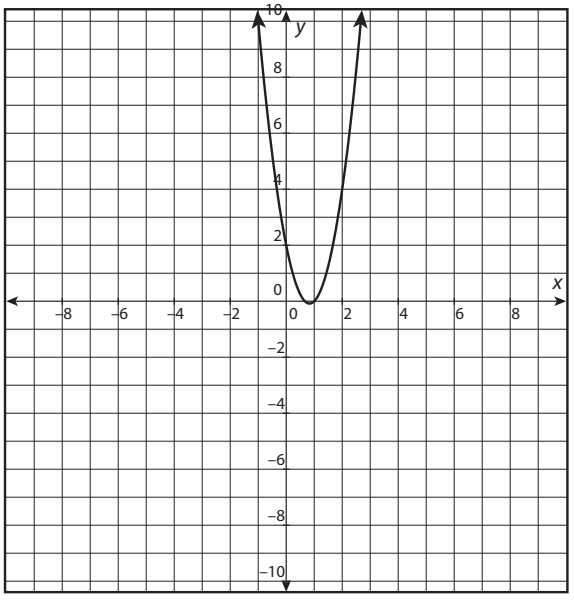
4. The x-intercept is the total amount of monthly car sales when Peyton has made \$0 in a month. This is impossible since she has a fixed monthly salary of \$2,500. The y-intercept is the total monthly take-home pay when Peyton sells 0 cars.

Practice 3.3.2: Creating and Graphing Equations Using the *x*-intercepts, p. U3-214

1. x-intercepts: 3, –6; axis of symmetry: $x = -1.5$
2. x-intercepts: 2/3, –2/3; axis of symmetry: $x = 0$
3. $y = 0.4x^2 + 1.6x + 1.6$
4. $y = x^2 + 7x - 60$
5. 

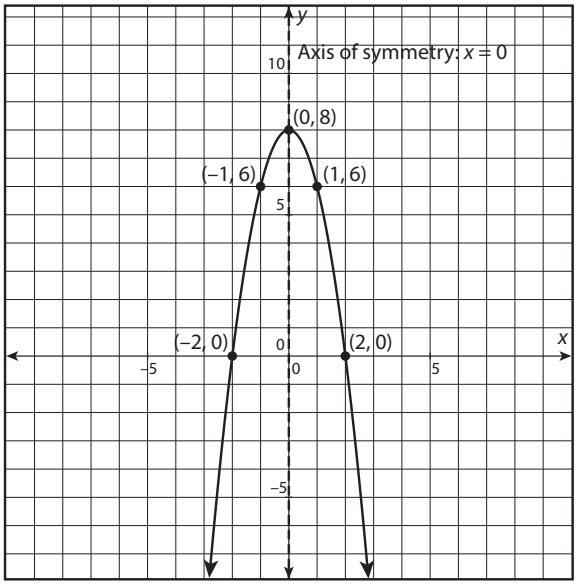
6. 

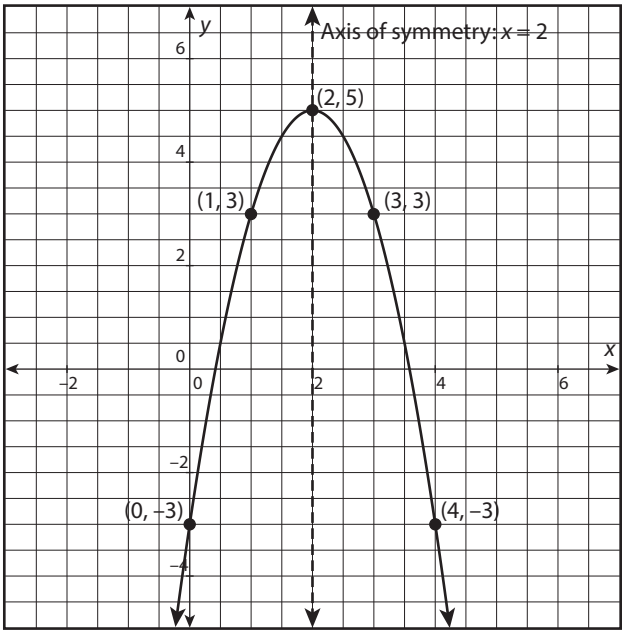
7. $y = -2x^2 + 8$
8. $y = x^2 - 36$
9. 5 inches by 7 inches
10. 25 seconds

Warm-Up 3.3.3, p. U3-216

1. \$3.25
2. \$5,070

Practice 3.3.3: Creating and Graphing Equations Using Vertex Form, p. U3-229

1. Vertex: (–3, –3); it is a maximum because $a < 0$.
2. Vertex: (6, 6); it is a minimum because $a > 0$.
3. $y = 0.89(x + 2.50)^2 - 3$ or $y = \frac{8}{9}\left(x + \frac{5}{2}\right)^2 - 3$
4. $y = -2(x - 2)^2 + 10$
5. $f(x) = (x + 3)^2$
6. $g(x) = 3(x - 2)^2 - 3$
7. 

8. 

9. $y = -\frac{25}{144}(x - 24)^2 + 100$
10. $y = -2(x - 3)^2 + 18$

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Warm-Up 3.3.4, p. U3-230

1. Rearrange the formula so that h is isolated.
2. $h = \dfrac{2A}{b_1 + b_2}$
3. The wall is 30 feet tall.

Practice 3.3.4: Rearranging Formulas, p. U3-242

1. $x = \pm\sqrt{25 - y^2}$
2. $x = -3 \pm \sqrt{y}$
3. $x = -7 \pm \dfrac{7y}{4}$
4. $x = -1 \pm \sqrt{\dfrac{y+90}{2}}$
5. $r = \sqrt{\dfrac{3V}{\pi h}}$
6. $r = \sqrt{\dfrac{A}{\pi}}$
7. $b = \sqrt{c^2 - a^2}$
8. $v = \sqrt{\dfrac{2E_k}{m}}$
9. $x = h \pm \sqrt{\dfrac{y-k}{a}}$
10. $b = \sqrt{\dfrac{a^2(y-k)^2}{a^2 - (x-h)^2}}$

Progress Assessment, p. U3-243

1. a
2. c
3. b
4. c
5. a
6. b
7. c
8. c
9. a
10. b
11. a. vertex: (1.5, –5); axis of symmetry: $x = 1.5$
    b. $y = 4x^2 - 12x + 4$
    c. x-intercepts: 0.38 and 2.62. These represent how much time has elapsed from the start of the pelican's descent, indicating when the pelican enters and exits the water, respectively.

Lesson 4: Fundamental Theorem of Algebra

Pre-Assessment, p. U3-246

1. b
2. d
3. a
4. d
5. b

Warm-Up 3.4.1, p. U3-249

1. area of figure 1 = x^2; area of figure 2 = x; area of figure 3 = 1
2. $A = lw$ is rewritten as $x^2 + 6x + 8 = (x + 4)(x + 2)$.
3. $(x + 4)(x + 2) = x^2 + 2x + 4x + 8 = x^2 + 6x + 8$
4. For $x = 5$:
   $x^2 + 6x + 8 = (x + 4)(x + 2)$
   $5^2 + 6(5) + 8 = (5 + 4)(5 + 2)$
   $25 + 30 + 8 = (9)(7)$
   $63 = 63$

   For $x = 8$:
   $x^2 + 6x + 8 = (x + 4)(x + 2)$
   $8^2 + 6(8) + 8 = (8 + 4)(8 + 2)$
   $64 + 48 + 8 = (12)(10)$
   $120 = 120$

   Let a represent one other x-value of your choice.
   For $x = a$:
   $x^2 + 6x + 8 = (x + 4)(x + 2)$
   $(a)^2 + 6(a) + 8 = (a + 4)(a + 2)$
   $a^2 + 6a + 8 = a^2 + 2a + 4a + 8$
   $a^2 + 6a + 8 = a^2 + 6a + 8$

Practice 3.4.1: Extending Polynomial Identities to Include Complex Numbers, p. U3-269

1. $5 + 3i$
2. $–6x + i$; $–x – (5x – i) = –6x + i$ is an identity.
3. 10; $(1 – 3i)$ and $(1 + 3i)$ are complex conjugates.
4. $c^2 + 9d^2$; $(c + 3di)(c – 3di) = c^2 + 9d^2$ is an identity; $(c + 3di)$ and $(c – 3di)$ are complex conjugates.
5. $x^2 + 4 = (x + 2i)(x – 2i)$
6. $32y^2 + 18 = 2(4y + 3i)(4y – 3i)$
7. $x^2 + 4x + 10 = \left(x + 2 - i\sqrt{6}\right)\left(x + 2 + i\sqrt{6}\right)$
8. Circuit 1: $V = (26 + 8i)$ volts; Circuit 2: $(a + bi)(p + qi) = (ap – bq) + (aq + bp)i$. This is an identity because it is true for all values of a, b, p, and q.
9. a. $R + Li + (–Ci) = R + (L – C)i$
   b. $(4 + 3i)$ ohms
10. $(8 + 3i)$ ohms

Warm-Up 3.4.2, p. U3-271

$\sqrt{36} = 6,\ \sqrt{9} \bullet \sqrt{4}\ ,\ 3 \bullet 2,\ \sqrt{9 \bullet 4}$

$\sqrt{\dfrac{16}{25}} = \dfrac{4}{5},\ \dfrac{\sqrt{16}}{\sqrt{25}},\ \dfrac{\sqrt{16}}{5}$

$\sqrt{50} = \sqrt{25} \bullet \sqrt{2}\ ,\ 5\sqrt{2}$

$\sqrt{-9} = 3i$

Practice 3.4.2: Solving Quadratic Equations with Complex Solutions, p. U3-295

1. discriminant = –7; two complex solutions
2. discriminant = 0; one real double solution
3. discriminant = 80; two real solutions
4. discriminant = –0.2; two complex solutions
5. a. The ± symbol in the expression indicates that there are exactly two solutions (counting a double solution as two solutions).
   b. According to part a, every quadratic equation has exactly two solutions. Since real numbers are a subset of complex numbers, it can be said that every quadratic equation has two complex solutions.
6. $f(x)$ has one real double zero and $b^2 – 4ac = 0$; $g(x)$ has two real zeros and $b^2 – 4ac > 0$
7. $x = \pm\dfrac{3\sqrt{2}}{2}i$
8. $x = \pm i\sqrt{3}$
9. $x = 5 \pm 2i$
10. $x = \dfrac{1}{3} \pm \dfrac{\sqrt{11}}{3}i$

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Progress Assessment, p. U3-297

1. a
2. a
3. c
4. d
5. b
6. d
7. d
8. d
9. b
10. c
11. A quadratic function is a polynomial function with a degree of $n = 2$. The quadratic formula states that the solutions of a quadratic equation in the form $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This expression represents two complex solutions, or a double solution if $b^2 - 4ac = 0$. Therefore, the quadratic formula shows that a quadratic function has exactly n complex solutions (roots), if a double solution is counted as two separate solutions. If the coefficients a and b are real, then they are complex, so that condition in the corollary is satisfied.

Lesson 5: Rational Equations

Pre-Assessment, p. U3-299

1. b
2. a
3. b
4. a
5. a

Warm-Up 3.5.1, p. U3-306

1. 50%; $\frac{2+4}{8+4} = \frac{6}{12} = 50\%$
2. Elliott; approximately 55.6%

Practice 3.5.1: Creating Rational Equations, p. U3-326

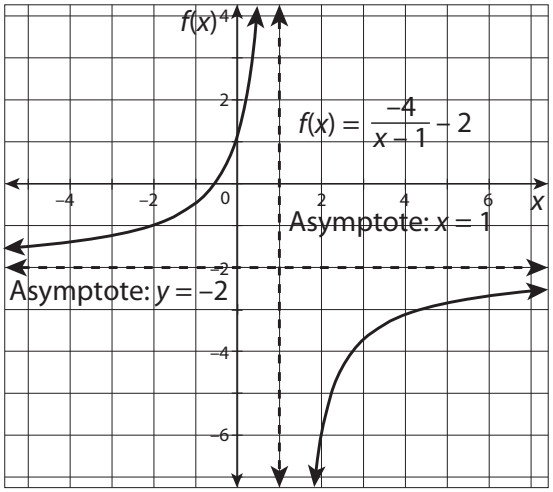
1. $\frac{1}{x-2}$; $x \neq -3$ or 2
2. $\frac{2}{x+1}$; $x \neq -1$ or 2
3. $f(x) = \frac{16+x}{46+x}$; $x = 4$; Alicia needs 4 consecutive hits to reach 0.400.
4. $f(x) = \frac{17+x}{23+x}$; $x = 7$; the next 7 consecutive students must pass on the first attempt in order for Mr. Ortiz to earn the $300 bonus.
5. $\frac{1}{20}(7.5) + \frac{1}{x}(7.5) = 1$; $x = 12$; it takes Sarah 12 minutes to assemble a bike.
6. $\frac{1}{36.5}(16.5) + \frac{1}{x}(16.5) = 1$; $x \approx 30.1$; it takes pump B approximately 30.1 hours.
7. $\frac{1}{x}(18) + \frac{1}{x+15}(18) = 1$; $x = 30$; it takes 30 minutes for copy machine A and 45 minutes for copy machine B.
8. $\frac{1}{x}(8) + \frac{1}{x+2.5}(8) = 1$; $x \approx 14.8$; it takes Natasha approximately 14.8 hours and Tobias approximately 17.3 hours to complete the job working alone.
9. $\frac{1}{4}(5) + \frac{1}{x}(7) = \frac{1}{5}(12)$; $x \approx 6.1$; the dogs consumed food at a rate of approximately 6.1 days per bag.
10. $\frac{1}{2.25}(10) + \frac{1}{x}(12) = \frac{1}{2}(22)$; $x \approx 1.83$; their average rate on the second day was approximately 1.83 miles per hour.

Warm-Up 3.5.2, p. U3-328

1. The container with the greatest volume is Logan's container, with x^3 cubic units.
2. Rounded, the approximate total volume of all three containers is 2,723 in^3.

Practice 3.5.2: Graphing Rational Equations, p. U3-359

1. Graph:



The domain is all real numbers except 1.
The range is all real numbers except –2.
The asymptotes are $x = 1$ and $y = -2$.
End behavior: $f(x)$ gets closer to –2 as x either increases or decreases without bound.
Behavior near the horizontal asymptote is the same as the end behavior.
Behavior near the vertical asymptote: $f(x)$ increases without bound as x gets closer to 1 from the left and $f(x)$ decreases without bound as x gets closer to 1 from the right.

2. $l = \frac{360}{w}$

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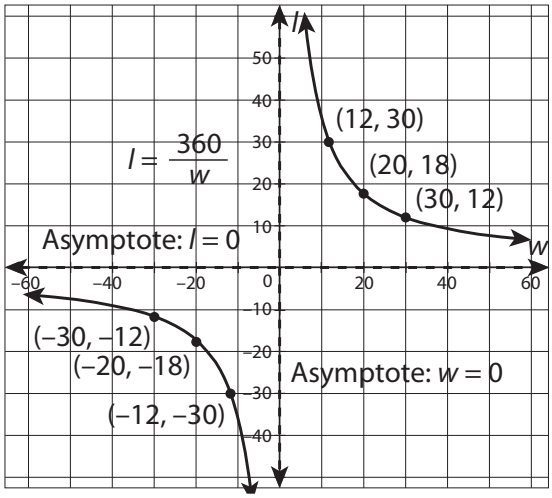
3.



4. a. The points (–30, –12), (–20, –18), and (–12, –30) have no meaning in the context of the problem situation because length and width cannot be negative. Each of the points (12, 30), (20, 18), and (30, 12) represents the width and length of a rectangular room whose area is 360 square feet.

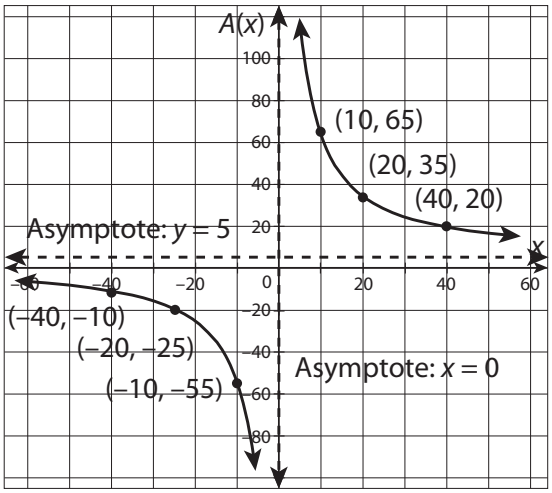
   b. The value of l gets closer to 0 as w increases without bound. This means that the length of the living room decreases as the width increases.

   c. The value of l increases without bound as w gets closer to 0. This means that the length of the living room increases as the width decreases.

5. $A(x) = \frac{600+5x}{x}$ or $\frac{600}{x}+5$

6. Graph:

$$A(x) = \frac{600 + 5x}{x} = \frac{600}{x} + 5$$



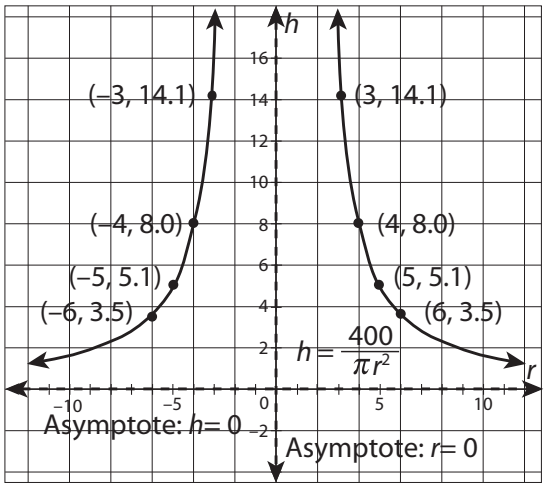
7. a. The points (–40, –10), (–20, –25), and (–10, –55) have no meaning in the context of the problem situation because T-shirts and dollars cannot be negative. Each of the points (10, 65), (20, 35), and (40, 20) represents a number of T-shirts and its associated average cost in dollars per T-shirt.

   b. The value of $A(x)$ increases without bound as x gets closer to 0. However, the least possible actual value of x is 0, which represents 0 T-shirts produced, and that value of x makes the function rule undefined. So, the least possible actual value of x that fits both the function and the problem context is 1, which represents 1 T-shirt produced.

   c. The value of $A(x)$ gets closer to 5 as x increases without bound, while always being greater than 5. This means that the average cost per T-shirt gets closer to \$5, but will always be more than \$5, as more and more T-shirts are produced.

8. $h = \frac{400}{\pi r^2}$

9.



10. (4, 8); Explanation: The point (4, 8) represents a radius of 4 inches and a height of 8 inches. The diameter is twice the radius, so the point (4, 8) represents a diameter of 8 inches and a height of 8 inches.

Warm-Up 3.5.3, p. U3-361

1. Completed table:

Grandchild	Corral dimensions per grandchild			
	Area (yd^2)	Length (yd)	Width (yd)	Perimeter (yd)
Doug	2,500	20	125	290
Marin	2,500	30	83.33	226.67
Cam	2,500	40	62.5	205
Felicia	2,500	50	50	200
Travis	2,500	60	41.67	203.33
Judy	2,500	70	35.72	211.43
Lynn	2500	80	31.25	222.5

2. Felicia's rectangle, at a width and length of 50 yards, will have the lowest fence cost because it has the smallest perimeter.

3. a square

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Practice 3.5.3: Creating Rational Inequalities, p. U3-397

1. $x < -1$ or $x \geq 4$; $(-\infty, -1) \cup [4, \infty)$
2. $x < -5$ or $x > \frac{17}{2}$; $(-\infty, -5) \cup \left(\frac{17}{2}, \infty\right)$
3. $\frac{5}{2} < x \leq \frac{10}{3}$; $\left(\frac{5}{2}, \frac{10}{3}\right]$
4. $x < -2$ or $0 < x < 2$; $(-\infty, -2) \cup (0, 2)$
5. $\frac{2.95x - 1150}{x} \geq 1.00$; 590 or more smoothies per month; \$590.50 minimum profit per month if he achieves his goal
6. $\frac{18.50x - 1630}{x} \geq 2.50$; 102 or more haircuts per week; \$257 minimum profit per week if she achieves her goal
7. $R_2 \geq 6$ ohms
8. 0 ohms $< R_3 \leq 24$ ohms
9. $37.5 - 2.5\sqrt{33} \leq w \leq 37.5 + 2.5\sqrt{33}$; approximate values (in inches): $23.14 \leq w \leq 51.86$
10. $26 - 2\sqrt{19} \leq w \leq 26 + 2\sqrt{19}$; approximate values (in feet): $17.28 \leq w \leq 34.72$

Progress Assessment, p. U3-400

1. b
2. c
3. b
4. d
5. d
6. a
7. b
8. c
9. d
10. a
11. a. $\frac{1}{200}(80) + \frac{1}{250}(80) + \frac{1}{x}(50) = 1$
    b. $x \approx 179$; it takes Oscar approximately 179 minutes, or 2 hours 59 minutes, to complete the job if he works alone.

Lesson 6: Writing Exponential Expressions in Equivalent Forms

Pre-Assessment, p. U3-405

1. d
2. a
3. b
4. b
5. b

Warm-Up 3.6.1, p. U3-410

1. exponent, annually, decrease, compound, yearly
2. decay

Practice 3.6.1: Writing Exponential Expressions in Equivalent Forms, p. U3-433

1. 37,111 people
2. 3.8226%
3. 14.1626%
4. June 2008: \$16,760; June 2020: \$2,682
5. 780 customers
6. December 31, 2007: 88,672 subscribers; December 31, 2020: 24,776 subscribers
7. July 1, 2005: \$189,630; July 1, 2020: \$448,374
8. bank deposit: 0.2790%; loan: 0.4149%
9. 4.9070%
10. value of loan: \$19,995; value of deposit: \$18,546

Progress Assessment, p. U3-435

1. c
2. d
3. a
4. d
5. b
6. d
7. a
8. b
9. c
10. b
11. Option C would result in the greatest amount earned because it has the highest monthly interest rate, compounded monthly, described as follows:
    - Option A's rate is 3.8% annual interest, compounded annually. This is approximately equivalent to 0.3113% monthly interest, compounded monthly, because $1.038^t = \left(1.038^{\frac{1}{12}}\right)^{12t} \approx (1.003113)^{12t}$, and t years is equal to $12t$ months.
    - Option B's rate is 0.95% quarterly interest, compounded quarterly. This is approximately equivalent to 0.3157% monthly interest, compounded monthly, because $1.0095^t = \left(1.0095^{\frac{1}{3}}\right)^{3t} \approx (1.003157)^{3t}$, and t quarters is equal to $3t$ months.
    - Option C's rate is 0.32% monthly interest, compounded monthly, as provided in the problem.

Lesson 7: Solving Systems of Equations

Pre-Assessment, p. U3-438

1. b
2. b
3. d
4. a
5. c

Warm-Up 3.7.1, p. U3-442

1. The system $\begin{cases} n = d + 10 \\ 0.05n + 0.10d = 3.50 \end{cases}$ represents the total number of nickels and dimes Gabriella has.
2. $n = 70 - 2d$

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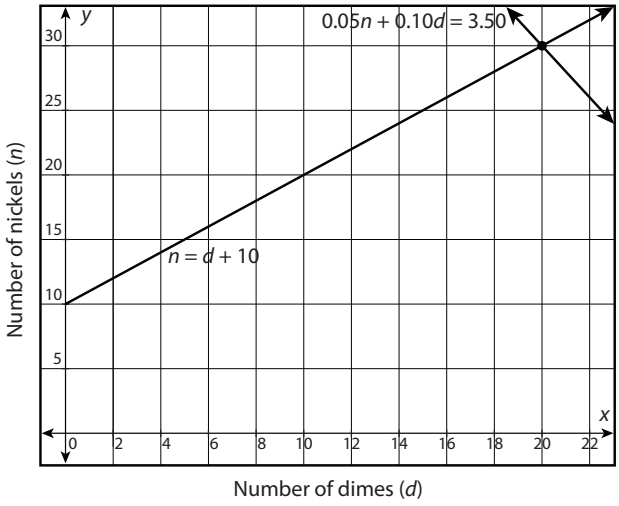
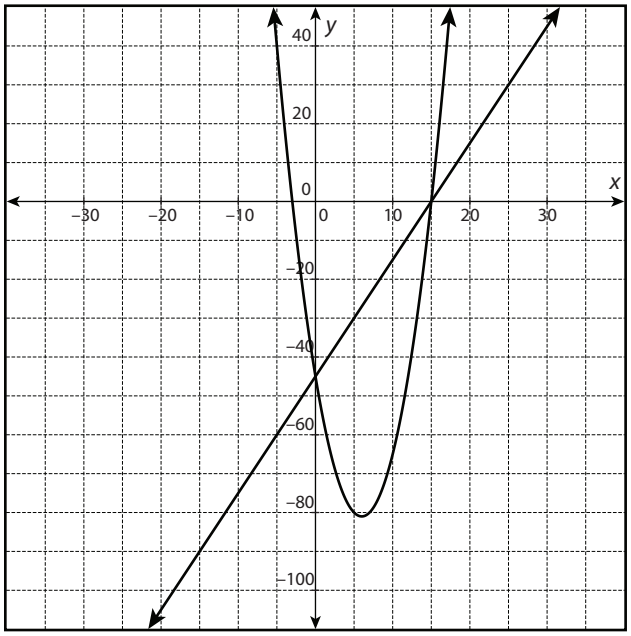
3. Graph:



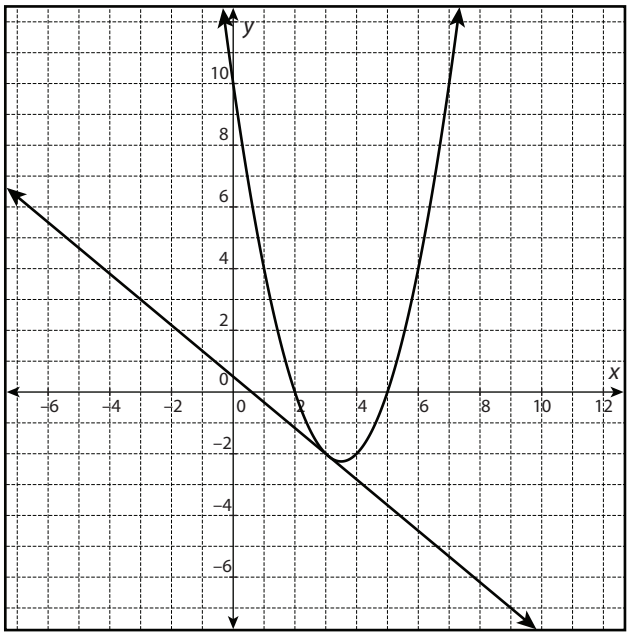
4. Gabriella has 30 nickels and 20 dimes.

Practice 3.7.1: Solving Systems Graphically, p. U3-459

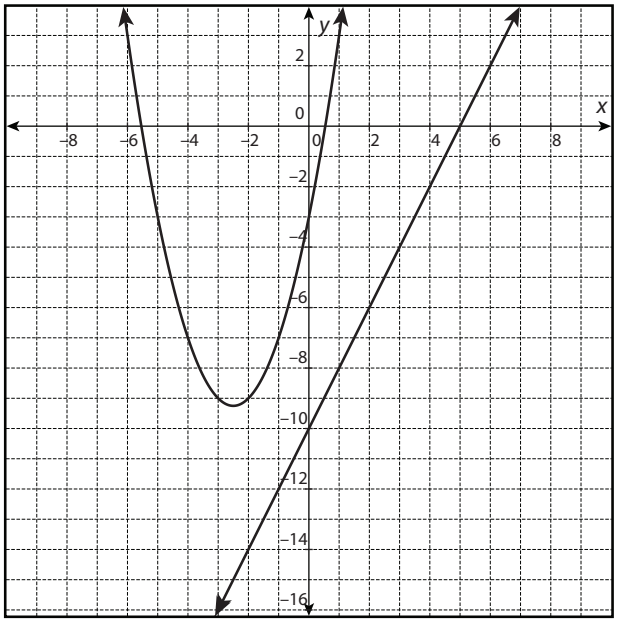
1. (–3, 0)
2. (–3, –3) and (2, 17)
3. (6, 2) and (4, 6)
4. (0, 2) and (3.6, 3.2)
5. (0, –45) and (15, 0)



6. (3, –2)



7. no real solutions



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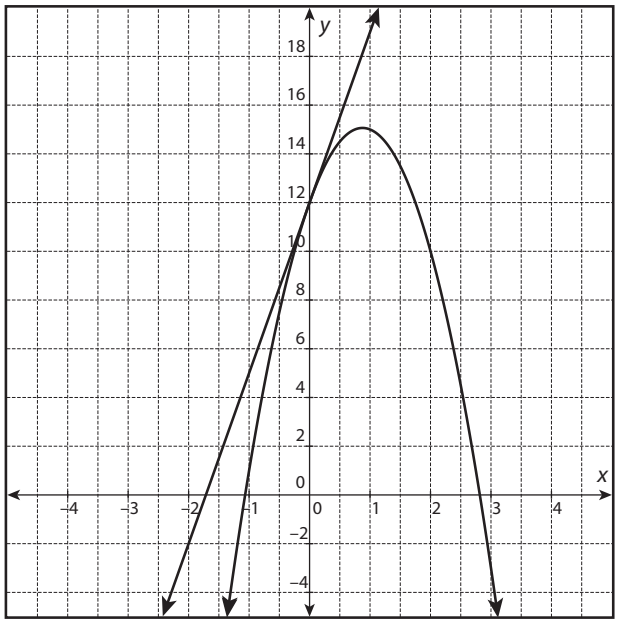
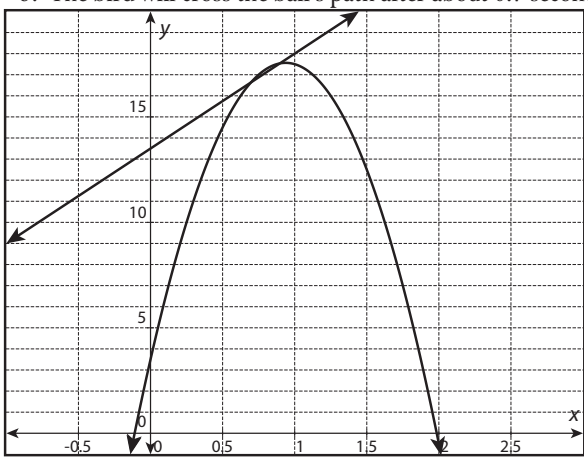
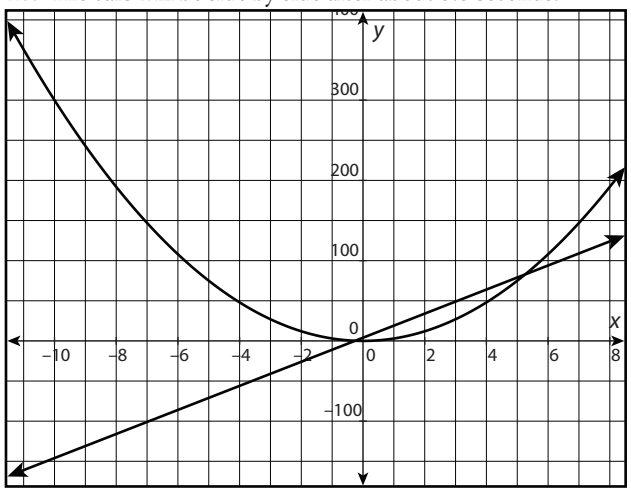
8. (0, 12)



9. The bird will cross the ball's path after about 0.7 second.



10. The cars will be side by side after about 5.3 seconds.



Warm-Up 3.7.2, p. U3-462

1. $w = 60$; $r = 240$
2. The speed of the plane in still air is 240 miles per hour. The speed of the wind is 60 miles per hour.

Practice 3.7.2: Solving Systems Algebraically, p. U3-477

1. (0, 4)
2. (0, 0) and (1, –8)
3. $\left(\frac{1+i\sqrt{15}}{4}, \frac{-3-3i\sqrt{15}}{8}\right)$ and $\left(\frac{1+i\sqrt{15}}{4}, \frac{-3+3i\sqrt{15}}{8}\right)$
4. (1, 2)
5. (1.11, 2.48) and (–39.11, 183.52)
6. (–3, –5) and (–3, 5)
7. (4, 3) and (–5, 0)
8. (–1, 0) and (5/13, 12/13)
9. 9.11 seconds
10. The only solution is negative for both values, so there is never a time when revenues for both skis and snowboards are the same.

Progress Assessment, p. U3-480

1. d
2. a
3. c
4. a
5. a
6. c
7. c
8. d
9. b
10. a
11. a. (3, 213) and (12, 387)
    b. (3.1699, 213.3975) and (11.8301, 386.6025)
    c. After about 3 parts are produced, the profit is $213.40. After about 12 parts are produced, the profit is $386.60.

Unit Assessment p. U3-486

1. c
2. a
3. d
4. d
5. c
6. b
7. c
8. c
9. a
10. d
11. b
12. a
13. a. –36
    b. two complex solutions
    c. $x = 3 \pm 3i$
14. a. 381,880 users
    b. 6.27%
    c. 620,979 users

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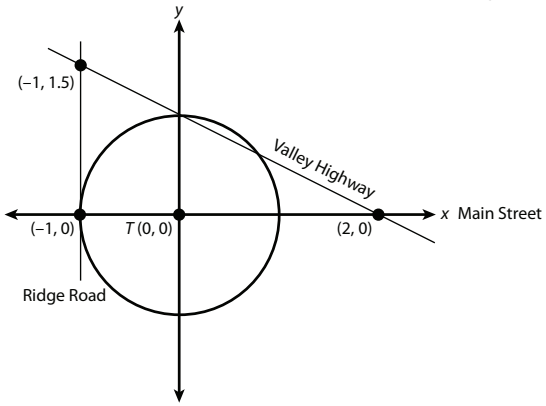
15. a. Coordinates: (–1, 1.5); (–1, 0); (2, 0). The circle on the diagram represents the 1-mile radius around the town hall that must conform to the historic building code.



b. The system is $\begin{cases} y=-\frac{1}{2}x+1 \\ x^2+y^2=1 \end{cases}$. For Valley Highway, the slope is $\frac{1.5-0}{-1-2}=\frac{1.5}{-3}=-\frac{1}{2}$, so the equation has the form $y=-\frac{1}{2}x+b$. The point (2, 0) is on the line, so $0=-\frac{1}{2}(2)+b$, $b = 1$, and the equation is $y=-\frac{1}{2}x+1$.

c. The solutions are (0, 1) and (0.8, 0.6).

d. approximately 0.89 mile

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Student Book Answer Key

Lesson 1: Interpreting Structure in Expressions

Practice 3.1.1: Identifying Terms, Factors, and Coefficients, p. U3-9

1. terms: $30x^2$, $-18x$, 72; coefficients: 30, –18; constant term: 72; factors: 30 and x^2, –18 and x
2. $9x$; monomial
3. $-x^2 + x + 52$ or $x^2 - x + 52$
4. The expression is quadratic because it can be written in the form $ax^2 + bx + c$.
5. The expression is not quadratic because it cannot be written in the form $ax^2 + bx + c$.
6. Expression: $9x - (8 + x^2) = -x^2 + 9x - 8$; terms: $-x^2$, $9x$, –8; coefficients: –1, 9; constant term: –8. The expression is quadratic because it can be written in the form $ax^2 + bx + c$.
7. Expression: $2(2 + x) + (1/2)x^2 = (1/2)x^2 + 2x + 4$; terms: $(1/2)x^2$, $2x$, 4; coefficients: 1/2, 2; constant term: 4. The expression is quadratic because it can be written in the form $ax^2 + bx + c$.
8. Expression: s^2; term: s^2; coefficient: 1; constant term: none. The expression is quadratic because it can be written in the form $ax^2 + bx + c$.
9. Expression: s^3; term: s^3; coefficient: 1; constant term: none. The expression is not quadratic because it cannot be written in the form $ax^2 + bx + c$.
10. Expression: $4\pi r^2$; term: $4\pi r^2$; coefficient: 4π; constant term: none. The expression is quadratic because it can be written in the form $ax^2 + bx + c$.

Practice 3.1.2: Interpreting Complicated Expressions, p. U3-16

1. $-4 < x < 6$
2. $x > 4$ or $x < -\frac{7}{5}$
3. $18x^2 - 21x - 15$; $a = 18$; $b = -21$; $c = -15$
4. $-x^2 + 4x + 14$; $a = -1$; $b = 4$; $c = 14$
5. The expression is quadratic because it can be written in the form $ax^2 + bx + c$.
6. The expression is not quadratic because it cannot be written in the form $ax^2 + bx + c$.
7. The population increases from 50,000 to 68,000.
8. The area increases from 4π square feet to 24π square feet.
9. The area increases by a factor of 9.
10. The area decreases by a factor of 4.

Lesson 2: Creating and Solving Quadratic Equations in One Variable

Practice 3.2.1: Taking the Square Root of Both Sides, p. U3-26

1. $x = \pm 2$
2. no real solutions
3. no real solutions
4. $x = 4 \pm 2\sqrt{10}$
5. $x = -6$
6. $x = 5 \pm \sqrt{7}$
7. A quadratic equation in the form $ax^2 + b = c$ has only one real solution when $c = b$.
8. $s = 7$ inches
9. $r = 2\sqrt{5} \approx 4.47$ units
10. $r = \pm\frac{\sqrt{5\pi}}{\pi} \approx 1.26$ feet

Practice 3.2.2: Factoring Expressions by the Greatest Common Factor, p. U3-35

1. $s(s^2t^2 - 3s + 2t^2)$
2. $2x^3y(2 + x^2y^2)$
3. no common factor
4. $xy(x^3 + x^2y + xy^2 + y^3)$
5. no common factor
6. $x^4(1 + x^{10})$
7. $13x^3y - 6z^2$
8. length: $3x^2$ feet; width: $(x^2 - 4x + 5)$ feet
9. $17x^3 + 24y^2$ cannot be factored because there are no common numbers or common variables between the two terms.
10. $(3a^2b^3 - 4c^2 + 12)$ meters

Practice 3.2.3: Factoring Expressions with $a = 1$, p. U3-49

1. $(n + 6)(n - 6)$
2. $(x - 5)(x - 4)$
3. $(y - 10)^2$
4. not factorable
5. $8(b + 1)(b - 1)$
6. $5(x + 4)(x - 2)$
7. $10(n + 3m)(n - 3m)$
8. $(n - 12)$ feet and $(n - 12)$ feet
9. $(3y + 10)$ meters and $(3y - 10)$ meters
10. $(x + 9)$ inches and $(x - 2)$ inches

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Practice 3.2.4: Factoring Expressions with $a > 1$, p. U3-62

1. $(2x + 3)(4x - 7)$
2. $(8a - 3)(a + 2)$
3. $(2y + 5)(5y - 1)$
4. $(2x - 7)(x - 7)$
5. $(4a - 3)^2$
6. $5(x + 3)(x + 6)$
7. $2(5y - 1)(3y + 4)$
8. length = $(3x + 10)$ inches; width = $(2x + 1)$ inches
9. length = $(3x - 2)$ feet; width = $(x + 8)$ feet
10. length = $(5x - 3)$ meters; width = $(2x - 3)$ meters

Practice 3.2.5: Solving Quadratic Equations by Factoring, p. U3-71

1. $x = 4, x = 9$
2. $y = -2,\ y = -\frac{7}{3}$
3. $b = 0,\ b = \frac{11}{2}$
4. $x = 2, x = -2$
5. $y = -1, y = 8$
6. $n = -\frac{4}{3},\ n = -\frac{1}{2}$
7. $x = 3, x = 8$
8. 1.25 seconds
9. 2 seconds
10. 4 seconds

Practice 3.2.6: Completing the Square, p. U3-77

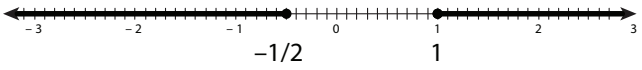
1. $c = 121$
2. $c = 2500$
3. $c = \frac{81}{4}$
4. $c = \frac{4}{25}$
5. $x = 4 \pm \sqrt{14}$
6. $x = -\frac{1}{2} \pm \frac{\sqrt{11}}{2}$
7. $x = 3$ and $x = -7$
8. 8.81 feet; you cannot have a negative length, so only one value of x is reasonable.
9. after about 2.5 seconds; the negative answer represents time before the ball was kicked, so only one value of x is reasonable.
10. approximately 13.49 mph and approximately 61.51 mph

Practice 3.2.7: Applying the Quadratic Formula, p. U3-85

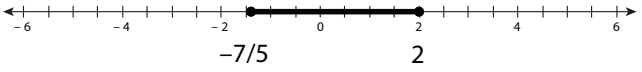
1. 13; two real, irrational solutions
2. –80; no real solutions
3. $x = -1$
4. $x = -\frac{5}{3}$ or $x = -1$
5. no real solutions
6. $x = 1$ or $x = -\frac{12}{7}$
7. after approximately 8.57 seconds
8. approximately \$5.37 and approximately \$18.63
9. after approximately 1.58 seconds
10. No, if a quadratic has two solutions, they must both be real or both be non-real.

Practice 3.2.8: Solving Quadratic Inequalities, p. U3-93

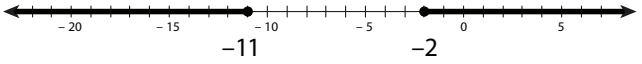
1. $x \le -\frac{1}{2}$ or $x \ge 1$



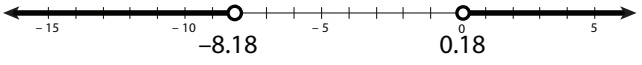
2. $-\frac{7}{5} \le x \le 2$



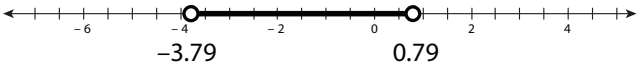
3. no real solutions
4. $x \le -11$ or $x \ge -2$



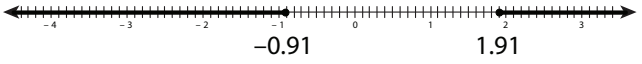
5. $x < -8.18$ or $x > 0.18$



6. $-3.79 < x < 0.79$



7. $x \le -0.91$ or $x \ge 1.91$



8. from 0.12 second to 8.45 seconds after takeoff
9. The balloon is never more than 6 feet above the ground.
10. The time of a complete swing will be greater than 4.49 seconds.

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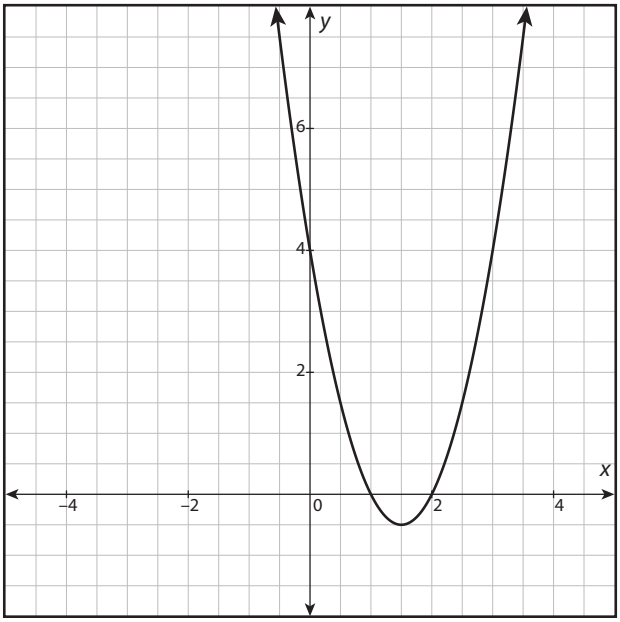
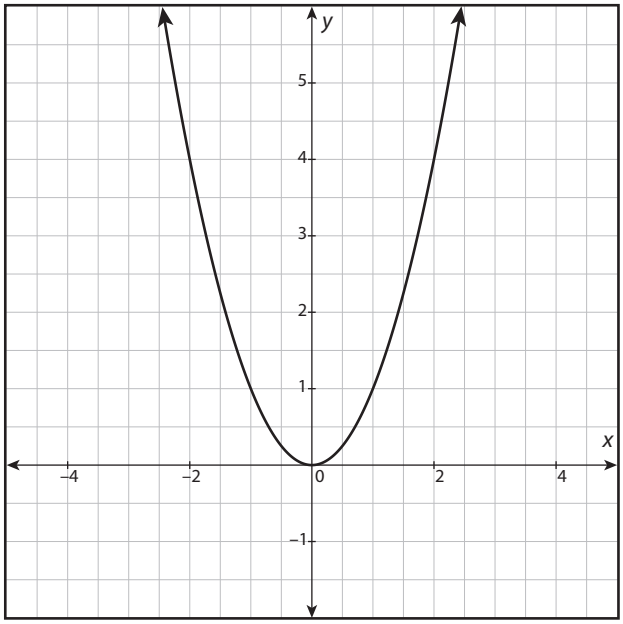
Lesson 3: Creating Quadratic Equations in Two or More Variables

Practice 3.3.1: Creating and Graphing Equations Using Standard Form, p. U3-117

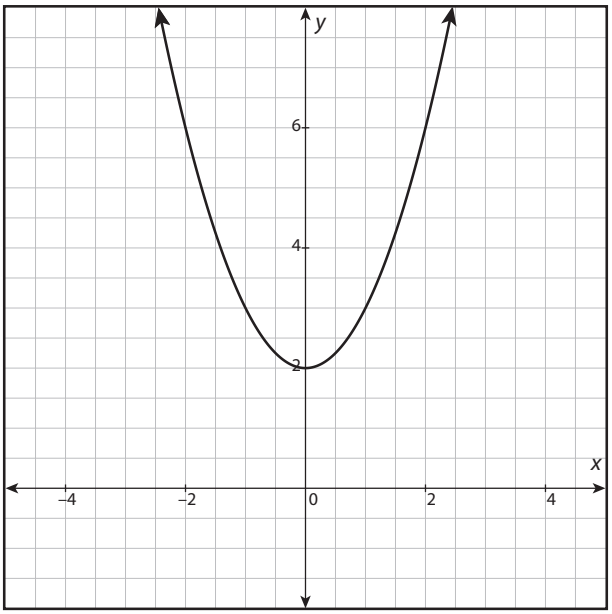
1.



2.



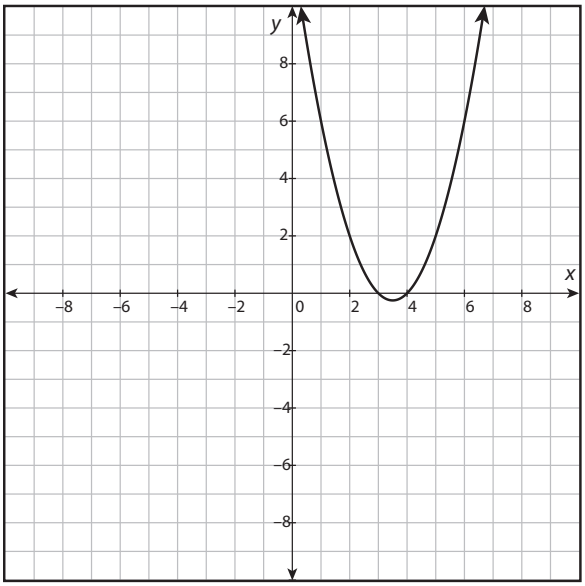
3.



4. y-intercept: 0; vertex: (–1.75, 6.125); the vertex is a maximum because $a < 0$.
5. y-intercept: 7; vertex: (–5, –93); the vertex is a minimum because $a > 0$.
6. y-intercept: 3; vertex: (1, 5); the vertex is a maximum because $a < 0$.
7. No. The y-intercept is –5 because the constant term is –5. However, the graph shows a y-intercept of –10.
8. $y = -3x^2 + 30x - 69$
9. approximately 356 feet
10. 5,000 pairs of socks

Practice 3.3.2: Creating and Graphing Equations Using the *x*-intercepts, p. U3-126

1. x-intercepts: 1/16, 7; axis of symmetry: $x = 3.53$
2. x-intercepts: 3/4, –7/2; axis of symmetry: $x = -1.375$
3. $y = x^2 + 6x + 8$
4. $y = x^2 - 20x + 75$
5.



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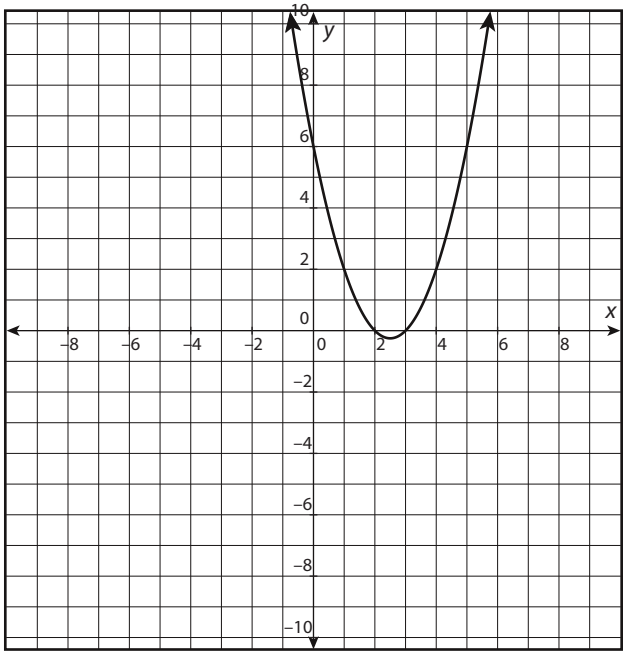
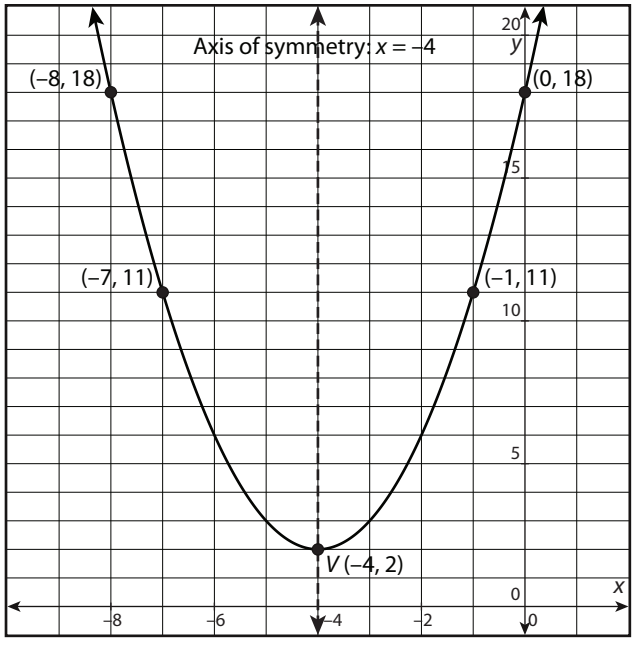
6.



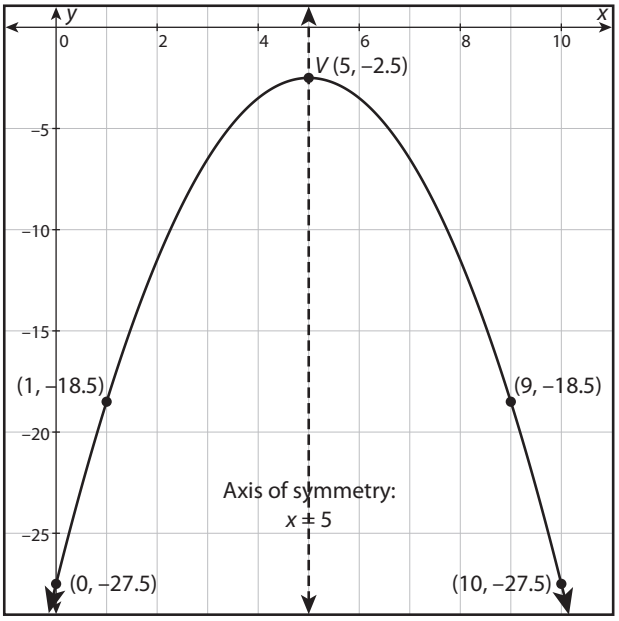
7. $y = 3x^2 + 15x - 18$
8. $y = 5x^2 - 30x + 40$
9. 8 feet
10. 1.25 seconds

Practice 3.3.3: Creating and Graphing Equations Using Vertex Form, p. U3-136

1. Vertex: (–4, 1); it is a maximum because $a < 0$.
2. Vertex: (2, 0); it is a minimum because $a > 0$.
3. $y = 0.96(x - 5.5)^2 - 6$
4. $y = 5(x - 2)^2$
5. $f(x) = (x - 1)^2 - 3$
6. $g(x) = 0.3(x + 2)^2$
7.



8.



9. $y = -2(x - 6)^2 + 72$
10. $y = -3.49(x - 10.5)^2 + 385$

Practice 3.3.4: Rearranging Formulas, p. U3-143

1. $x = \pm\sqrt{360 - 4y^2}$
2. $x = \pm\sqrt{\dfrac{y+9}{16}}$
3. $x = -7 \pm \sqrt{\dfrac{y}{4}}$
4. $x = 6.5 \pm \sqrt{\dfrac{y+20}{2.5}}$
5. $V = \pm\sqrt{PR}$
6. $r = \sqrt{\dfrac{2A}{\theta}}$
7. $c = \sqrt{a^2 + b^2}$
8. $v = \sqrt{ar}$
9. $y = k \pm \sqrt{r^2 - (x-h)^2}$
10. $r = \sqrt{\dfrac{SA}{4\pi}}$

Lesson 4: Fundamental Theorem of Algebra

Practice 3.4.1: Extending Polynomial Identities to Include Complex Numbers, p. U3-159

1. $-4 - 5i$
2. $-2y$; $(y - i) - (3y - i) = -2y$ is an identity.
3. 72; $(6 + 6i)$ and $(6 - 6i)$ are complex conjugates.
4. $(c^2 - 5c) + (5c - 25)i$; $(c - 5)(c + 5i) = (c^2 - 5c) + (5c - 25)i$ is an identity.
5. $x^2 + 100 = (x + 10i)(x - 10i)$
6. $x^2 - x + 2 = \left(x - \dfrac{1}{2} - \dfrac{\sqrt{7}}{2}i\right)\left(x - \dfrac{1}{2} + \dfrac{\sqrt{7}}{2}i\right)$
7. $x^2 + 4x + 14 = \left(x + 2 - i\sqrt{10}\right)\left(x + 2 + i\sqrt{10}\right)$
8. Circuit 1: $V = (44 + 23i)$ volts; Circuit 2: $(c + di)(r + si) = (cr - ds) + (cs + dr)i$. This is an identity because it is true for all values of c, d, r, and s.
9. $(7 - i)$ ohms
10. $(6 - i)$ ohms

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Practice 3.4.2: Solving Quadratic Equations with Complex Solutions, p. U3-179

1. discriminant = 0; one real double solution
2. discriminant = –199; two complex solutions
3. discriminant = 329; two real solutions
4. discriminant = 4; two real solutions
5. a. Solve for the discriminant, $b^2 - 4ac$, to determine the number and type of solutions.
   b. Look at the x-intercepts to determine the number and type of solutions.
   c. If the solutions to a quadratic equation are complex conjugates, then the value of the discriminant is less than 0.
6. $f(x)$ has two complex zeros and $b^2 - 4ac < 0$; $h(x)$ has two real zeros and $b^2 - 4ac > 0$.
7. $x = \frac{1}{4} \pm \frac{3}{4}i$
8. $x = \pm 2i$
9. $x = \pm \frac{\sqrt{15}}{5}i$
10. $x = 1 \pm 3i$

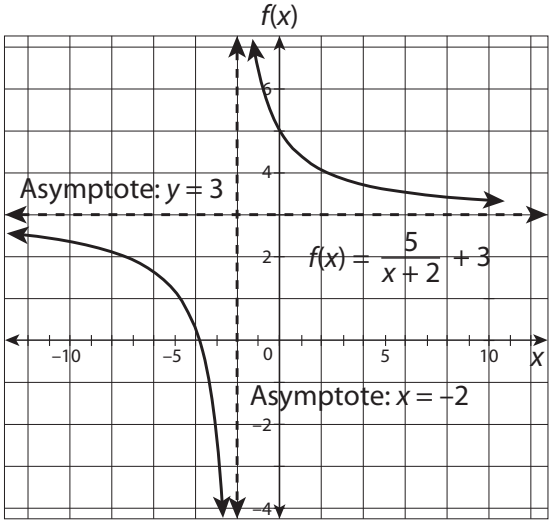
Lesson 5: Rational Equations

Practice 3.5.1: Creating Rational Equations, p. U3-198

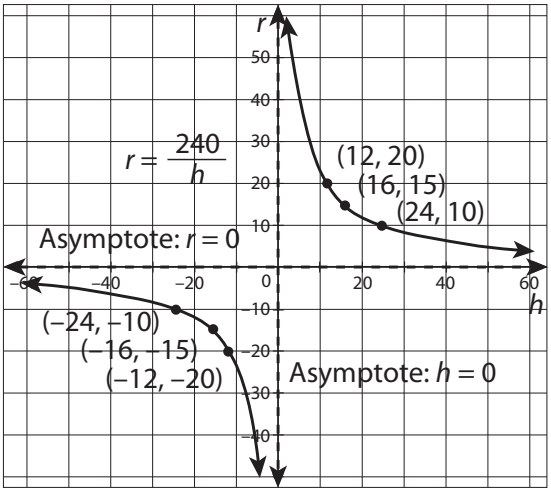
1. $\frac{1}{2(x-3)}$; $x \neq -1$ or 3
2. $2(x + 3)$; $x \neq 3$
3. $f(x) = \frac{20+x}{32+x}$; $x = 8$; Careta needs to put the next 8 consecutive serves in play to reach a serving percentage of 70%.
4. $f(x) = \frac{15+x}{20+x}$; $x = 5$; Pavel needs to attend the next 5 consecutive sessions to be eligible for a service award.
5. $\frac{1}{80}(48) + \frac{1}{x}(48) = 1$; $x = 120$; it takes Ulrich 120 minutes, or 2 hours.
6. $\frac{1}{16}(7) + \frac{1}{x}(7) = 1$; $x \approx 12.4$; it takes approximately 12.4 hours using hose B alone.
7. $\frac{1}{x}(8.75) + \frac{1}{x+6}(8.75) = 1$; $x = 15$; it takes 15 hours for truck A and 21 hours for truck B.
8. $\frac{1}{x}(45) + \frac{1}{x+45}(45) = 1$; $x \approx 73$; it takes Nasir approximately 73 minutes and Danitza approximately 118 minutes to complete the order working alone.
9. $\frac{1}{0.8}(6) + \frac{1}{x}(4) = \frac{1}{0.5}(10)$; $x = 0.32$; the rate of fuel consumption was 0.32 hour per gallon.
10. $\frac{1}{440}(1600) + \frac{1}{x}(1600) = \frac{1}{500}(3200)$; $x \approx 579$; the required average speed on the return flight is approximately 579 mph.

Practice 3.5.2: Graphing Rational Equations, p. U3-223

1. Graph:



The domain is all real numbers except –2.
The range is all real numbers except 3.
The asymptotes are $x = -2$ and $y = 3$.
End behavior: $f(x)$ gets closer to 3 as x either increases or decreases without bound.
Behavior near the horizontal asymptote is the same as the end behavior.
Behavior near the vertical asymptote: $f(x)$ decreases without bound as x gets closer to –2 from the left and $f(x)$ increases without bound as x gets closer to –2 from the right.

2. $r = \frac{240}{h}$
3. 

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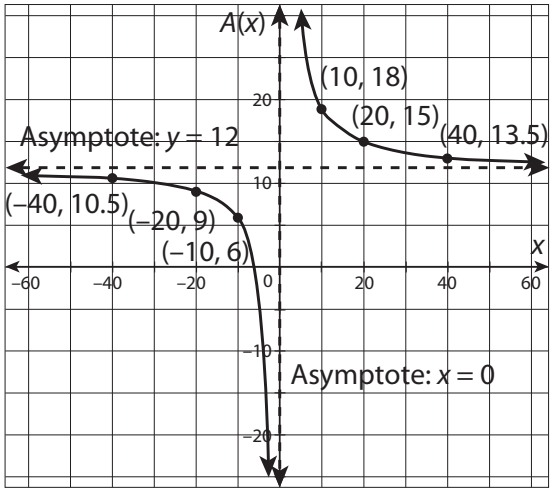
4. a. The points (–24, –10), (–16, –15), and (–12, –20) have no meaning in the context of the problem situation because hourly rate of pay and hours worked cannot be negative. Each of the points (12, 20), (16, 15), and (24, 10) represents a number of hours Fraser could work and its associated rate of pay in dollars per hour that would result in Fraser earning $240.

   b. The value of r gets closer to 0 as h increases without bound. This means that the hourly rate of pay decreases as the number of hours worked increases.

   c. The value of r increases without bound as h gets closer to 0. This means that the hourly rate of pay increases as the number of hours worked decreases.

5. $A(x) = \frac{60+12x}{x}$ or $\frac{60}{x}+12$

6. $$A(x) = \frac{60 + 12x}{x} = \frac{60}{x} + 12$$

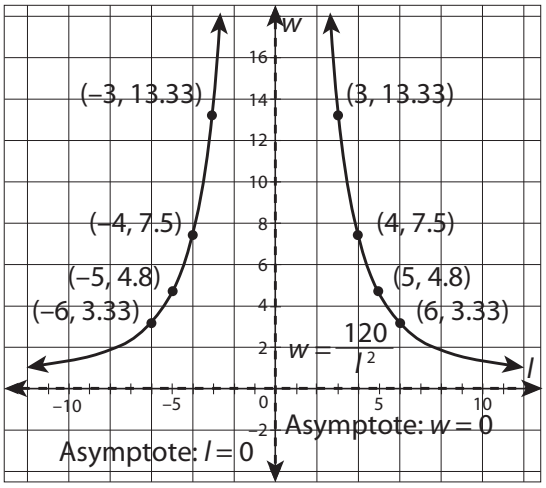


7. a. The points (–40, 10.5), (–20, 9), and (–10, 6) have no meaning in the context of the problem situation because the number of baskets cannot be negative. Each of the points (10, 18), (20, 15), and (40, 13.5) represents a number of gift baskets and its associated average cost in dollars per basket.

   b. The value of $A(x)$ increases without bound as x gets closer to 0. However, the least possible actual value of x is 0, which represents 0 baskets produced, and that value of x makes the function rule undefined. So, the least possible actual value of x that fits both the function and the problem context is 1, which represents 1 basket produced.

   c. The value of $A(x)$ gets closer to 12 as x increases without bound, while always remaining greater than 12. This means that the average cost per basket gets closer to $12, but will always be more than $12, as more and more baskets are produced.

8. $w = \frac{120}{l^2}$

9. 

10. (8, 1.875); Explanation: Georgianna wants the height to be 12 inches. The height is 1.5 times the length, so $l = \frac{12}{1.5} = 8$. The point on the graph with an l-coordinate of 8 has a w-coordinate of $w = \frac{120}{l^2} = \frac{120}{8^2} = \frac{120}{64} = 1.875$.

Practice 3.5.3: Creating Rational Inequalities, p. U3-255

1. $x \le 3$ or $x > 5$; $(-\infty, 3] \cup (5, \infty)$
2. $-4 < x < -\frac{5}{2}$; $\left(-4, -\frac{5}{2}\right)$
3. $x < \frac{7}{3}$ or $x \ge 4$; $\left(-\infty, \frac{7}{3}\right) \cup [4, \infty)$
4. $-5 \le x < 0$ or $x \ge 5$; $[-5, 0) \cup [5, \infty)$
5. $\frac{90x-350}{x} \ge 50$; 9 or more doghouses; $460 minimum profit if he achieves his goal
6. $\frac{9x-1410}{x} \ge 2.25$; 209 or more car washes per week; $471 minimum profit per week if she achieves her goal
7. $R_2 \le 7$ ohms
8. 0 ohms $< R_3 \le 3$ ohms
9. $30-10\sqrt{3} \le w \le 30+10\sqrt{3}$; approximate values (in feet): $12.68 \le w \le 47.32$
10. $25 \le w \le 60$ (Values are in feet.)

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Lesson 6: Writing Exponential Expressions in Equivalent Forms

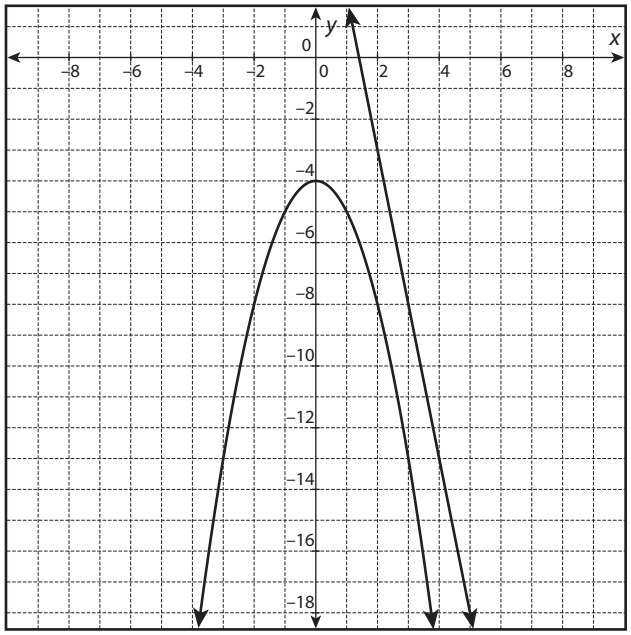
Practice 3.6.1: Writing Exponential Expressions in Equivalent Forms, p. U3-277

1. 1,699 students
2. 2.7970%
3. 18.5552%
4. May 2009: $5,078; May 2020: $531
5. 2,325 homeowners
6. October 15, 2010: 20,973 subscribers; October 15, 2020: 5,947 subscribers
7. December 31, 2001: $95,799; December 31, 2018: $188,089
8. savings account: 0.3073%; loan: 0.5305%
9. 5.5357%
10. value of loan:$15,388; value of deposit: $12,472

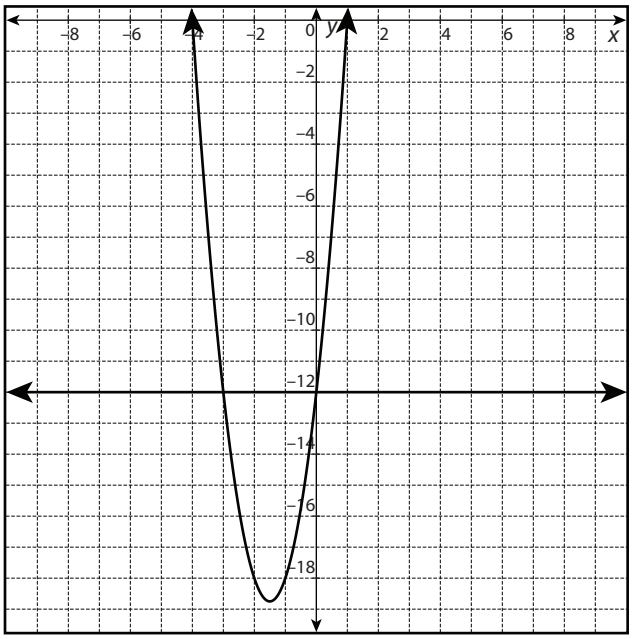
Lesson 7: Solving Systems of Equations

Practice 3.7.1: Solving Systems Graphically, p. U3-292

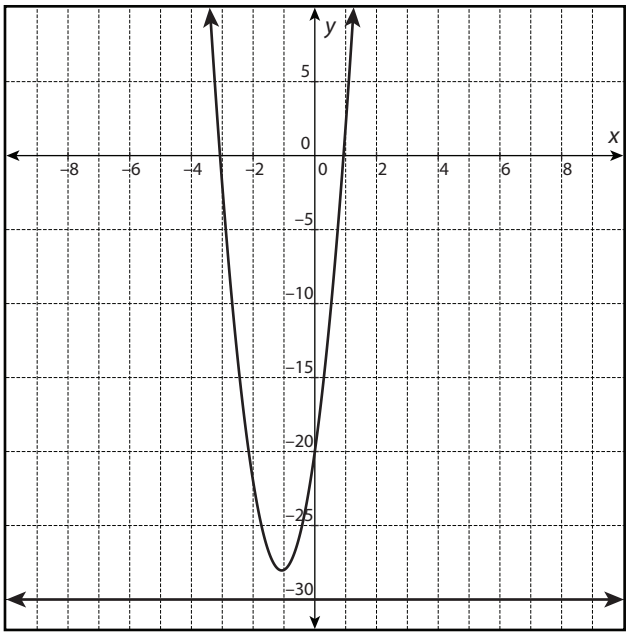
1. no real solutions
2. (–1, 2)
3. (0, –1) and (4.4, 7.8)
4. (0, 3) and (5.4, 1.2)
5. no real solutions



6. (–3, –12) and (0, –12)



7. no real solutions



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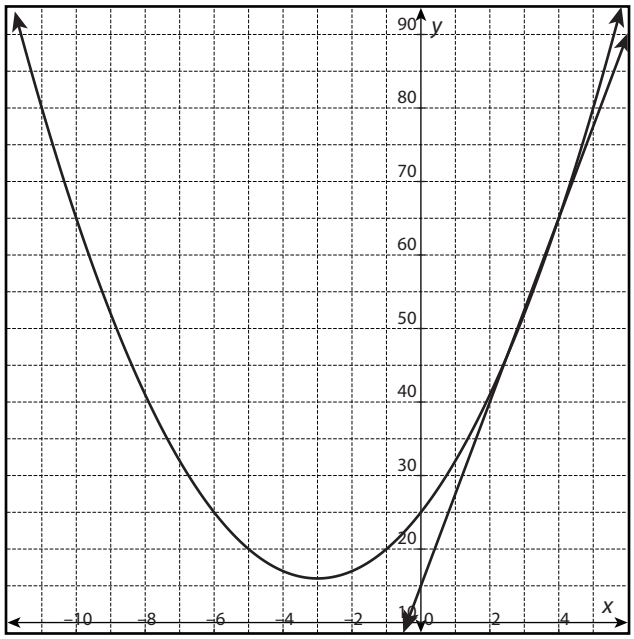
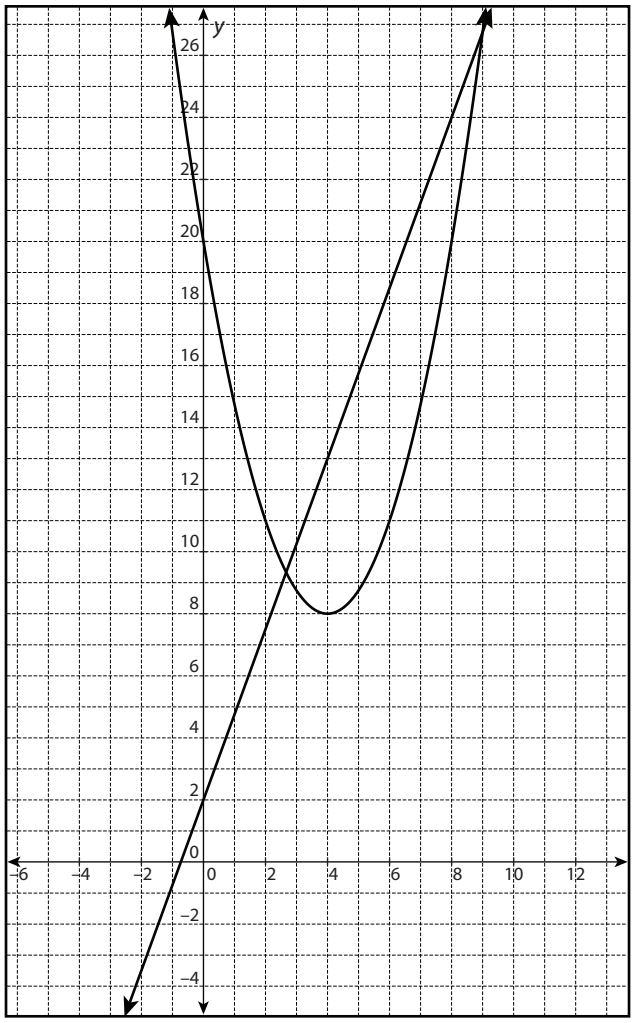
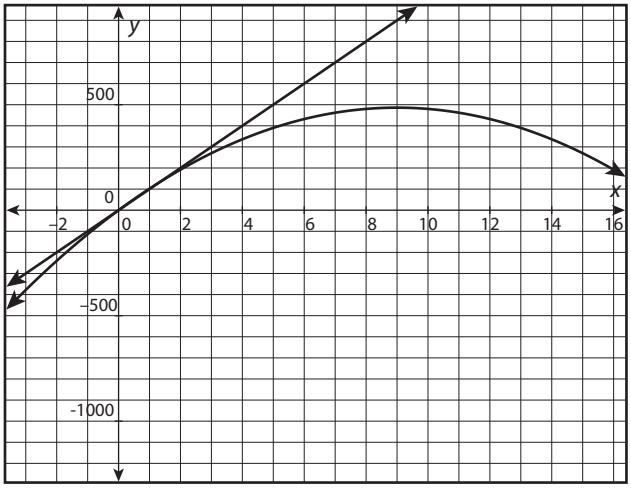
8. (2.5, 46.25), (4, 65)



9. Yes, the stocks were both about $9.33 per share after 2.7 months, and then they were both about $26.75 per share after 9 months.



10. After about 1 day, the number of CDs sold matched the number of deluxe boxed sets sold, at about 133 units sold.



Practice 3.7.2: Solving Systems Algebraically, p. U3-304

1. (0, –5)
2. (1, 9) and (5, 1)
3. $x = \frac{-3 \pm i\sqrt{7}}{4}$
4. (0, 0) and (4, 4)
5. (–0.41, –10.07) and (2.41, 4.07)
6. (–2, 4) and (2, 4)
7. (–4, 4) and (4, –4)
8. (–1, 0) and (3/5, 4/5)
9. After about 1 second, the hawk and the soccer ball will be about 31 feet above the ground.
10. Janetta and Bertram will have the same profit when Janetta charges $2.84 and $9.16. Bertram always makes $500, so in this case setting Janetta's function equal to $500 yields the desired results.

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Instruction

Goal: To provide opportunities for students to practice factoring quadratic equations

Common Core State Standard

A–SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

a. Factor a quadratic expression to reveal the zeros of the function it defines.

Student Activities Overview and Answer Key

Station 1

Given equations in the form $y = ax^2 + bx + c$, where a, b, and c are integers, students work in groups to factor by grouping. Students will use algebra tiles as needed.

Answers

1. $(x + 4)(x + 4)$
2. $(x - 7)(x + 5)$
3. $(x - 1)(x + 9)$
4. $(x - 4)(x + 7)$
5. $(x + 2)(x - 1)$
6. $(x + 3)(x + 6)$
7. $(x - 4)(x + 11)$
8. $(x - 10)(x - 3)$
9. $(x - 5)(x + 5)$

Station 2

Given equations in the form $y = ax^2 + bx + c$, where a, b, and c are real numbers, students factor by grouping.

Answers

1. $(3x + 5)(x - 2)$
2. $(x - 1)(5x + 4)$
3. $\left(x - 8\right)\left(x + 3\right)$
4. $(3x - 1)(x - 2)$
5. $(7x - 3)(x - 2)$
6. $(3x + 4)(5x - 1)$
7. $\left(x + 0.5\right)\left(x + 0.3\right)$
8. $\left(x + 0.2\right)\left(x - 0.12\right)$

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Instruction

Station 3

Given equations in the form $y = ax^2 + bx + c$, where a, b, and c are real numbers, students work in groups to factor by grouping. Students will use algebra tiles as needed.

Answers

1. $4(x+2)(x-2)$
2. $3(x-1)(x-7)$
3. $5(2x+3)(x-6)$
4. $\frac{1}{2}(3x-1)(x-2)$
5. $\frac{2}{5}(x+4)(x-10)$
6. $7(x+3)(3x+1)$
7. $\frac{1}{9}(x-5)(x-2)$
8. $\frac{3}{4}(4x-7)(x+2)$

Station 4

Students are given a set of 12 index cards, each inscribed with one of the following expressions: 6, 4, 5, 1/2 , 3, $(x-5)$, $(x+2)$, $(x-3)$, $(x+7)$, $(x+1)$, $(3x+2)$, and $(2x-1)$. They use grouping to factor a series of equations. Each card appears as a factor at least once in this series of equations. Then students combine their cards in pairs to come up with quadratic equations of their own.

Answers

1. $(x-5)(x+1)$
2. $4(3x+2)(2x-1)$
3. $6(x-3)(3x+2)$
4. $\frac{1}{2}(x+7)(x+2)$
5. $5(2x-1)(x+1)$
6. $3(x-5)(x-3)$
7. Answers will vary. Students should create three quadratic expressions that combine the factors on the cards.

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Instruction

Materials List/Setup

Station 1 algebra tiles

Station 2 none

Station 3 algebra tiles

Station 4 algebra tiles; 12 index cards with the following written on them (one expression per card): 6; 4; 5; $\frac{1}{2}$; 3; $(x - 5)$; $(x + 2)$; $(x - 3)$; $(x + 7)$; $(x + 1)$; $(3x + 2)$; $(2x - 1)$

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. What are factors?
2. What is the Distributive Property? How does it apply to binomial factors?
3. How do you factor a quadratic equation?
4. If you factor a quadratic equation that is set equal to 0, what points on the equation's graph do the factors represent? Why?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. Factors are the quantities that are multiplied to produce a product.
2. The Distributive Property is a property of real numbers that allows the multiplication of a term to a sum of terms. With a pair of binomials that are being multiplied together, the Distributive Property is used twice. Take the first term in the first binomial and multiply it by each term in the second binomial, adding the products. Then take the second term in the first binomial and multiply it by each term in the second binomial, adding all the products together.
3. Rewrite the equation in $y = ax^2 + bx + c$ form, with all x expressions and constants on the same side of the equation. Find the factors of a and the factors of c that combine to create b.
4. They represent the x-intercepts, because those are the points at which $y = 0$.

Possible Misunderstandings/Mistakes

- Incorrectly factoring quadratic expressions
- Incorrectly factoring constants and coefficients
- Not understanding factoring
- Not understanding polynomial factoring
- Not simplifying the equation before factoring
- Making simple arithmetical errors in factoring

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Station 1

Work as a group to factor each equation. Use the algebra tiles if you wish. Show all your work.

1. $y = x^2 + 8x + 16$

2. $y = x^2 - 2x - 35$

3. $y = x^2 + 8x - 9$

4. $y = x^2 + 3x - 28$

5. $y = x^2 + x - 2$

6. $y = x^2 + 9x + 18$

7. $y = x^2 + 7x - 44$

8. $y = x^2 - 13x + 30$

9. $y = x^2 - 25$

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Station 2

Factor each equation. Show all your work. Check your work by using distribution to find the product of your factors.

1. $y = 3x^2 - x - 10$

2. $y = 5x^2 - x - 4$

3. $y = x^2 - 5x - 24$

4. $y = 3x^2 - 7x + 2$

5. $y = 7x^2 - 17x + 6$

6. $y = 15x^2 + 17x - 4$

7. $y = x^2 + 0.8x + 0.15$

8. $y = x^2 + 0.08x - 0.024$

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Station 3

Work as a group to factor each equation. Use the algebra tiles if you wish. Show all your work.

1. $y = 4x^2 - 16$

2. $y = 3x^2 - 24x + 21$

3. $y = 10x^2 - 45x - 90$

4. $y = \frac{3}{2}x^2 - \frac{7}{2}x + 1$

5. $y = \frac{2}{5}x^2 - \frac{12}{5}x - \frac{80}{5}$

6. $y = 21x^2 + 70x + 21$

7. $y = \frac{x^2}{9} - \frac{5}{9}x - \frac{2}{9}x + \frac{10}{9}$

8. $y = 3x^2 + \frac{3}{4}x - \frac{21}{2}$

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Station 4

At this station, you will find algebra tiles and 12 index cards marked with the following expressions:

$6; 4; 5; \frac{1}{2}; 3; (x-5); (x+2); (x-3); (x+7); (x+1); (3x+2); (2x-1)$

Work as a group to factor each equation below, using the index cards provided. You will also use the factors later in the activity. Use the algebra tiles if you wish. Show all your work.

1. $y = x^2 - 4x - 5$

2. $y = 24x^2 + 4x - 8$

3. $y = 18x^2 - 42x - 36$

4. $y = \frac{1}{2}x^2 + \frac{9}{2}x + 7$

5. $y = 10x^2 + 5x - 5$

6. $y = 3x^2 - 24x + 45$

7. Combine your factor cards to form three different quadratic equations. Write your equations below.

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Instruction

Goal: To provide opportunities for students to solve quadratic equations with real and complex solutions

Common Core State Standards

N–CN.7 Solve quadratic equations with real coefficients that have complex solutions.

A–SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

a. Factor a quadratic expression to reveal the zeros of the function it defines.

A–REI.4 Solve quadratic equations in one variable.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Student Activities Overview and Answer Key

Station 1

Students work with a partner to solve the quadratic equations by factoring. Students may use algebra tiles as needed.

Answers

1. $x = -\frac{1}{2}, x = 3$
2. $x = \frac{1}{3}, x = -2$
3. $x = 7, x = 3$
4. $x = -\frac{4}{5}, x = 1$
5. $x = -3, x = 3$
6. $x = 4, x = 2$
7. $x = 2$

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Instruction

Station 2

Students work alone or in groups to solve the quadratic equations by factoring. They should be increasingly comfortable finding real roots independently.

Answers

1. $x = -8, x = \frac{1}{9}$
2. $x = 1, x = \frac{1}{12}$
3. $x = -2, x = 6$
4. $x = 1, x = -\frac{7}{5}$
5. $x = 9$
6. $x = 2, x = 12$
7. $x = -\frac{2}{7}, x = \frac{5}{2}$

Station 3

Students work in groups to solve quadratic equations using the quadratic formula. They will use a graphing calculator. They begin working with complex solutions.

Answers

1. $x = \frac{4 \pm 13.266i}{6}$
2. $x = 3.618, x = 1.382$
3. $x = \frac{3 \pm 22.428i}{32}$
4. $x = \frac{1 \pm 3.873i}{4}$
5. $x = -9.359, x = -0.641$

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Instruction

Station 4

Students work in pairs to solve quadratic equations using factoring and the quadratic formula. They will use a graphing calculator. Students may begin to recognize the relationship between the discriminant and the type of roots.

Answers

1. $x = \frac{5 \pm 3.873i}{20}$
2. $x = \frac{6 \pm 10.392i}{2}$
3. $x = 8$
4. $x = -1.146, x = -7.854$
5. $x = -5, x = -2$
6. $x = -1, x = -2$
7. Students may notice the relationship between the discriminant and the type of roots the equation has. If the discriminant is negative, the roots are complex.

Materials List/Setup

Station 1 algebra tiles

Station 2 none

Station 3 graphing calculator

Station 4 graphing calculator

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. What is the quadratic formula?
2. What is a real number?
3. What is a complex number?
4. What is the value of *i*?
5. What is the relationship among factors, *x*-intercepts, roots, and solutions of a quadratic?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. If $y = ax^2 + bx + c$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
2. A real number is any number that can be expressed as a decimal.
3. A complex number is any number expressed as the sum or difference of a real number and an imaginary number, or $a + bi$.
4. $\sqrt{-1}$
5. A quadratic equation written in factored form shows the factors of the quadratic equation. By taking these factors and setting each of them equal to 0 and solving for *x*, you are finding the roots or the solutions of the quadratic equation. The roots of the equation are the *x*-intercepts, or where the graph crosses the *x*-axis. These are also said to be the solutions. If the equation is in the form "$y =$" or "$f(x) =$", you will be setting that side of the equation equal to 0. This is the equivalent of finding the *x*-intercepts. Use the Zero Product Property to set each factor equal to 0 and solve for the variable.

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Instruction

Possible Misunderstandings/Mistakes

- Incorrectly factoring quadratic expressions
- Incorrectly factoring constants and coefficients
- Not understanding factoring
- Not understanding polynomial factoring
- Making simple arithmetical errors in factoring or in applying the quadratic formula
- Incorrectly applying the quadratic formula
- Rounding inaccurately
- Not setting $ax^2 + bx + c$ equal to 0

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Station 1

Work with a partner to solve the quadratic equations by factoring. Use algebra tiles as needed. Show all your work.

1. $2x^2 = 5x + 3$

2. $3x^2 + 5x = 2$

3. $x^2 - 10x = -21$

4. $5x^2 = x + 4$

5. $x^2 - 9 = 0$

6. $2x^2 = 2(6x - 8)$

7. $7x^2 = 28(x - 1)$

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Station 2

Work alone or with your group to solve the quadratic equations by factoring. Show all your work.

1. $-9x^2 + 8 = 71x$

2. $12x^2 = 13x - 1$

3. $\frac{x^2}{4} - x = 3$

4. $5x^2 = 7 - 2x$

5. $x^2 = -81 + 18x$

6. $2x - x^2 = 24 - 12x$

7. $31x = 14x^2 - 10$

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Station 3

Work with a partner to solve the problem using the quadratic formula. Use the graphing calculator to estimate square roots to the thousandths. Write complex roots in the form of $a \pm bi$.

1. $y = 3x^2 - 4x + 16$

2. $y = x^2 - 5x + 5$

3. $y = 8x^2 - \frac{3}{2}x + 4$

4. $y = 2x^2 - x + 2$

5. $y = \frac{x^2}{2} + 5x + 3$

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Station 4

Work alone or with your group to solve using either factoring or the quadratic formula. Use the graphing calculator to estimate square roots to the thousandths. Write complex roots in the form of $a \pm bi$.

1. $y = 10x^2 - 5x + 1$

2. $y = \frac{x^2}{3} - 2x + 12$

3. $y = x^2 - 16x + 64$

4. $y = x^2 + 9x + 9$

5. $y = x^2 + 7x + 10$

6. $y = 3x^2 + 9x + 6$

7. Do you notice any patterns about the value of $\sqrt{b^2 - 4ac}$ and the roots of the equation? If so, what are they?

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Station Activities Set 3: Quadratic Transformations in Vertex Form

Instruction

Goal: To provide opportunities for students to analyze the relationship between the equation of a parabola and its graph

Common Core State Standards

A–SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

F–IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

F–IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

F–BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Student Activities Overview and Answer Key

Station 1

Given equations in the form $y = x^2 + k$ and $y = (x - h)^2$, where h and k are integers, students graph a series of parabolas, finding the y-intercept and the axis of symmetry. They explore the relationship between the value of h and k and the position of the parabola with respect to the x- and y-axes. Students should also begin to understand the relationship between the equation of the parabola and the axis of symmetry.

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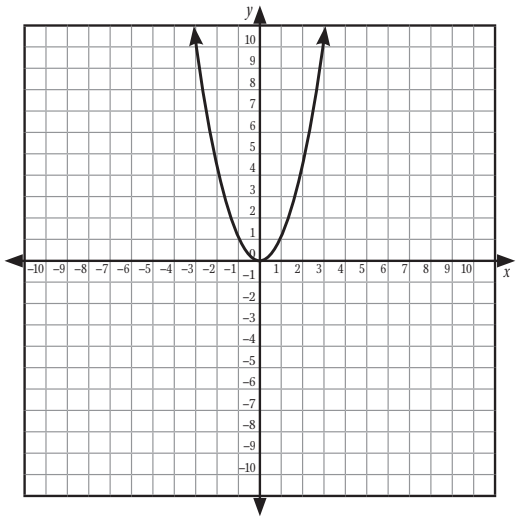
Answers

1.

x	y
0	0
1	1
2	4
3	9
–1	1
–2	4
–3	9

2. 

3. (0, 0)

4. (0, 2)

5. $y = x^2 + 2$

6. $y = x^2 - 5$

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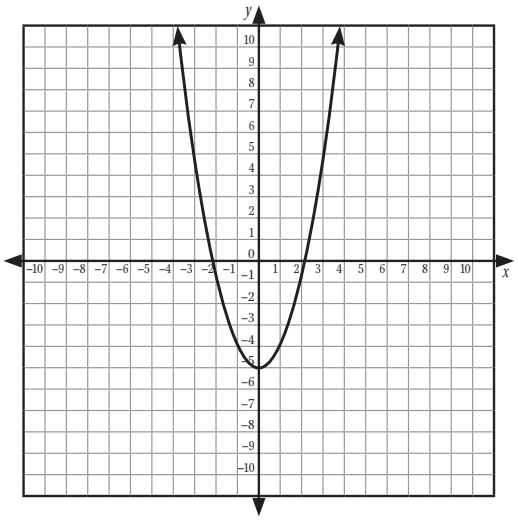
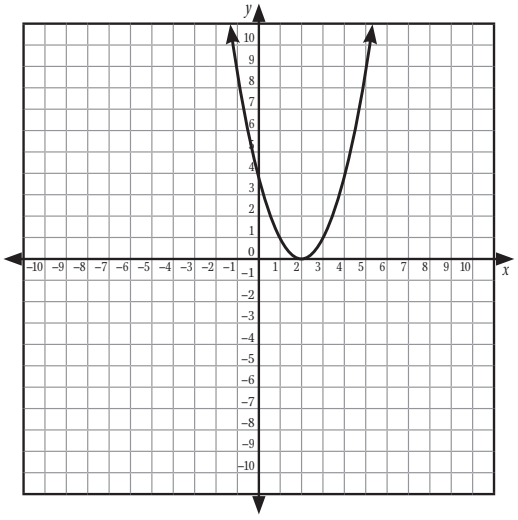
7.



8.



9. $x = 2$

10. $x = -3$

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Instruction

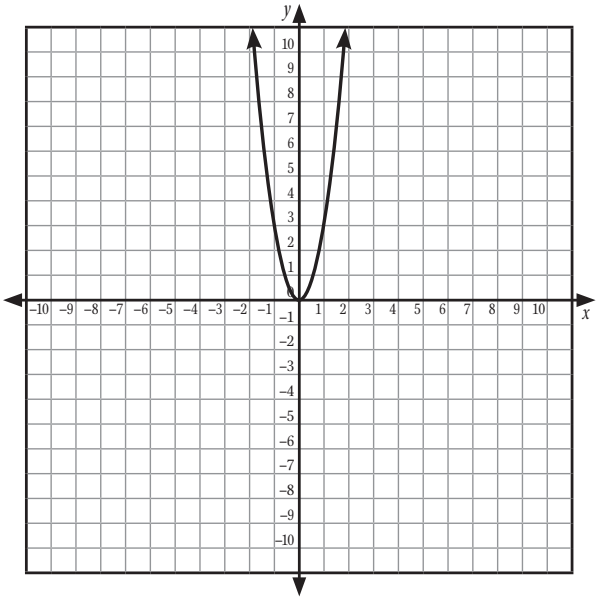
Station 2

Given equations in the form $y = ax^2$, students graph parabolas. Students compare graphs to explore the relationship between the coefficient of x and the width of the parabola.

Answers

1.

x	y
0	0
1	3
2	12
3	27
–1	3
–2	12
–3	27



2. (0, 0)

3. $x = 0$

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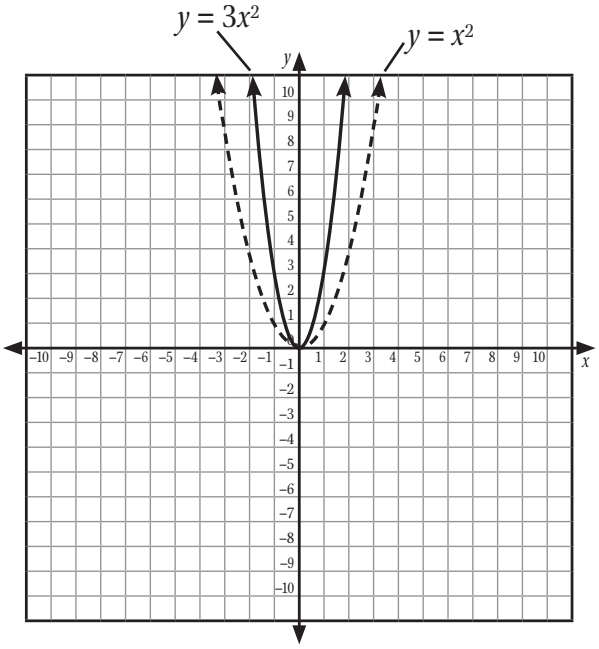
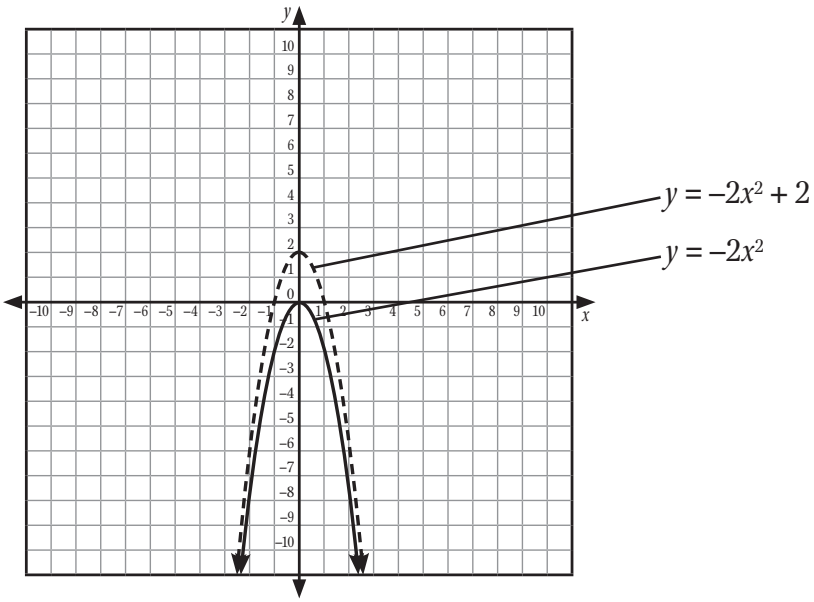
4.



The parabola $3x^2$ is narrower than the parabola x^2.

5–6.



7. (0, 2)
8. The parabola shifts vertically.
9. The parabola changes in width.

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Instruction

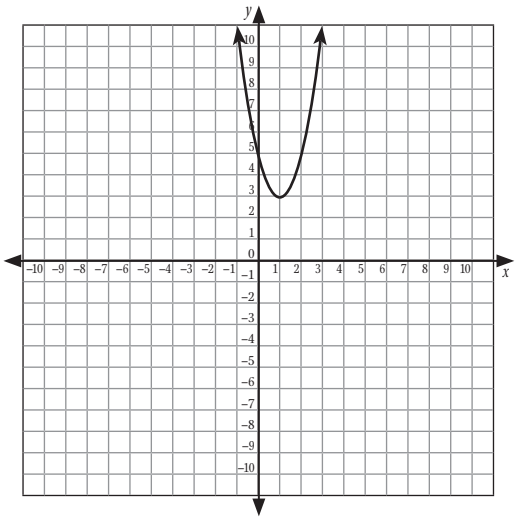
Station 3

Given equations in the form $y = (x - h)^2 + k$, students graph parabolas. Students find the y-intercept and the axis of symmetry from both the graph and the equation, and begin working towards an understanding of the vertex of a parabola.

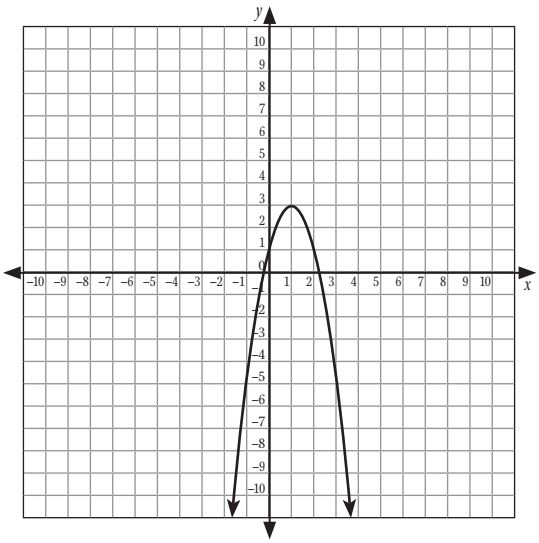
Answers

1.

x	y
0	5
1	3
2	5
3	11
4	21
–1	11
–2	21
–3	35



2. $x = 1$
3. (1, 3)
4. (0, 5)
5. The parabola would open downward but have the same vertex.



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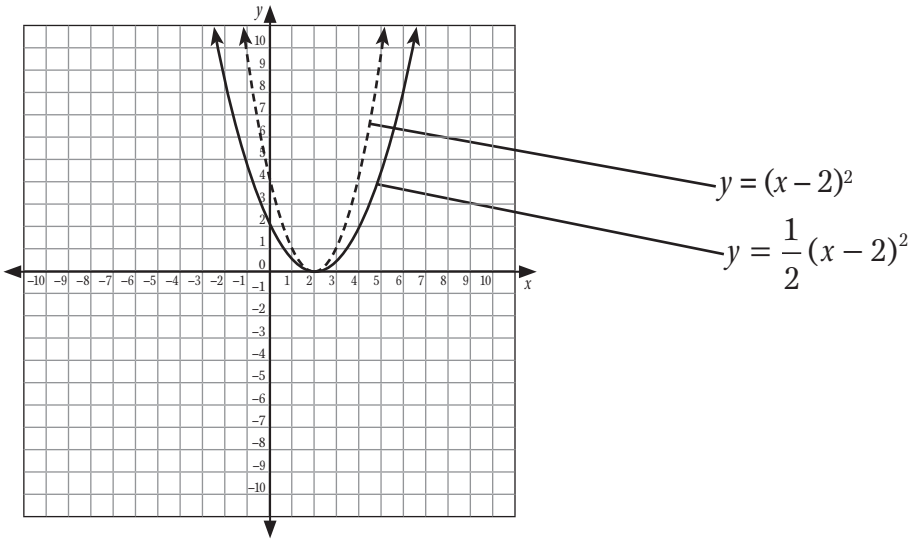
6. (0, 3)
7. (2, 1)
8. It will be wider, because the higher the coefficient of the x^2 term, the narrower the parabola. The coefficient of the second x^2 term is 1, which is higher than ½, the coefficient of the first x^2 term.



9. (0, 17); (2, 5)
10. $y = 3(x - 2)^2 + 10$

    The new y-intercept is (0, 22), so solve for the value of a.

Station 4

Students use two methods (completing the square and finding the midpoint of the x-intercepts) to convert the equations of parabolas from quadratic form to vertex form. They graph to check their work and to understand the correlation between the different forms and the graph. Students should recognize that a parabola's axis of symmetry always runs through its vertex. They should also understand the relationship between the coordinates of the vertex and the equation in vertex form.

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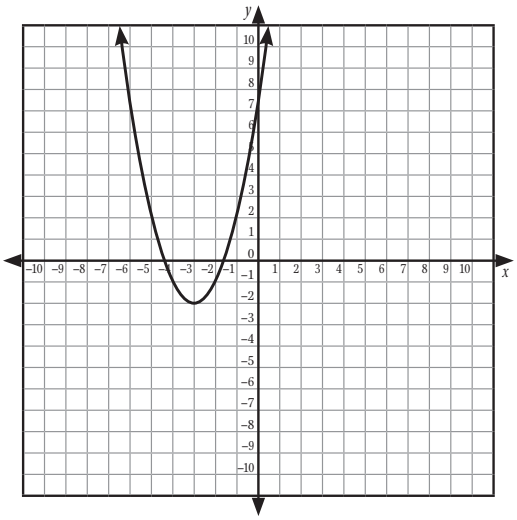
Instruction

Answers

1. 

2. $x = -3$
3. $y = (x + 3)^2 - 2$
4. (–3, –2)
5. (4, 1)
6. $y = \frac{1}{2}(x - 4)^2 + 1$
7. (0, 9)
8. $x = 4$
9. Yes. The parabola opens out from the vertex. The vertex contains the only y-coordinate that is not repeated in the range.
10. Because the parabola is symmetrical, the axis of symmetry will intersect the midpoint of the line between the x-intercepts. The midpoint is at (0, 0). That means the x-coordinate at the vertex must be 0, because the axis of symmetry intersects the vertex. If $x = 0$, $y = -4$, so the coordinates of the vertex are (0, –4).

Materials List/Setup

Station 1 graph paper

Station 2 colored pencils or pens; graph paper

Station 3 colored pencils or pens; graph paper

Station 4 graph paper

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. What is a function's axis of symmetry? Does every parabola have one?
2. What is a *y*-intercept?
3. How do you find the coordinates of a parabola's *y*-intercept?
4. Compare the equations $y = a(x - h)^2 + k$ and $y = ax^2 - 2axh + ah^2 + k$. Do you think they express the same thing? How could you find out?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. An axis of symmetry is the line that divides the graph of the function into two symmetrical halves. Every parabola has an axis of symmetry.
2. A *y*-intercept is the point at which a function crosses the *y*-axis.
3. Set *x* equal to 0 and solve the function for *y*.
4. Yes, they express the same thing. Students should multiply out the equation in vertex form to find the equation in quadratic form.

Possible Misunderstandings/Mistakes

- Incorrectly calculating the value of *y*-coordinates from *x*-coordinates
- Incorrectly graphing parabolas, either from incorrect calculations or from a misunderstanding of graphing itself
- Not understanding the definition of the vertex
- Assuming that the vertex is unrelated to the axis of symmetry
- Incorrectly factoring quadratic equations

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Instruction

- Making simple arithmetical errors in completing the square
- Not understanding the arithmetical manipulations involved in completing the square
- Confusing the y-intercept with the x-intercept
- Confusing the vertex coordinates h and k
- English language learners may struggle with the questions that ask for written explanations. Encourage these students to write out the numeric operations involved and then describe their work out loud.

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Station 1

Work as a group to answer the questions. Construct graphs without the aid of a graphing calculator. Show all your work and label the axes of each graph.

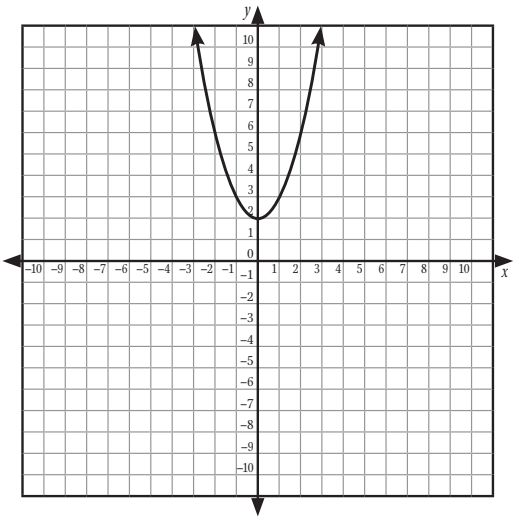
1. Given the equation $y = x^2$, complete the table below with the y coordinates for the following values of x.

x	y
0	
1	
2	
3	
–1	
–2	
–3	

2. Use the coordinates from your table to graph the parabola on graph paper.

3. What are the coordinates for the parabola's y-intercept?

4. Look at the parabola below. What is its y-intercept?



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5. What is the equation for this parabola?

6. How would you write the equation for a similar parabola of y-intercept $(0, -5)$?

7. Graph the parabola from problem 6.

8. Graph the parabola $y = (x - 2)^2$.

9. What is the equation for the axis of symmetry of this parabola?

10. Without graphing, predict the equation for the axis of symmetry of the parabola $y = (x + 3)^2$.

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Station 2

Work with your group to explore the relationship between a quadratic function and its graph.

1. Given the equation $y = 3x^2$, complete the table with the values of y and graph the parabola.

x	y
0	
1	
2	
3	
–1	
–2	
–3	

2. What are the coordinates of this parabola's y-intercept?

3. What is the equation of its axis of symmetry?

4. On the graph from problem 1, draw the parabola $y = x^2$ in a contrasting color. In words, compare the two parabolas.

5. Graph the parabola $y = -2x^2$. Complete the table if you need a reference.

x	y
0	
1	
2	
3	
–1	
–2	
–3	

continued

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6. On the same graph, in a contrasting color, graph the parabola $y = -2x^2 + 2$. Label each parabola.

7. What are the coordinates of the y-intercept of $y = -2x^2 + 2$?

8. What happens to the graph of a parabola when you add a constant to its equation, as in problem 6?

9. What happens to the graph of a parabola when the x^2 expression is given a coefficient other than 1, as in problems 1 and 5? (*Hint*: Compare the parabola $y = 3x^2$ to $y = x^2$.)

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Station 3

Work with your group to answer the following questions.

1. Complete the table for the parabola $y = 2(x - 1)^2 + 3$. Graph the parabola on graph paper.

x	y
0	
1	
2	
3	
4	
–1	
–2	
–3	

2. What is the equation for this parabola's axis of symmetry?

3. What is the vertex?

4. What are the coordinates of this parabola's y-intercept?

5. How would this graph change if the parabola's equation changed to $y = -2(x - 1)^2 + 3$? Graph the new parabola to check your answer.

6. What are the coordinates of the y-intercept of the parabola $y = \frac{1}{2}(x - 2)^2 + 1$?

continued

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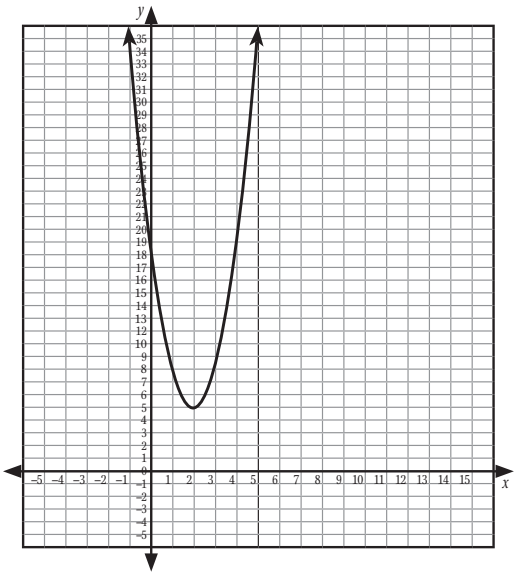
7. What is the vertex?

8. Do you think that the graph of $y = \frac{1}{2}(x-2)^2$ will be wider or narrower than the graph of $y = (x-2)^2$? Why? Graph both parabolas, in contrasting colors, to check your answer.

9. Look at the graph below. The equation for this parabola is $y = 3(x-2)^2 + 5$. What is the vertex? What is the parabola's y-intercept?



10. How would you write the equation for a similar parabola with a y-intercept 5 units higher? Show your work. Write out an explanation in words if necessary.

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Station 4

Work with your group to answer the following questions.

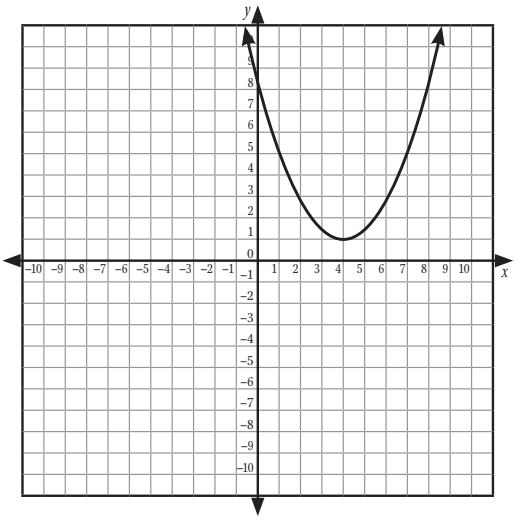
1. Graph the parabola $y = x^2 + 6x + 7$ on graph paper.

2. Give the equation for its axis of symmetry.

3. *Optional*: Complete the square to give the equation for the parabola in vertex form. Show your work.

4. What are the coordinates of the vertex of this parabola?

5. Look at the graph below. What are the coordinates of the vertex of this parabola?



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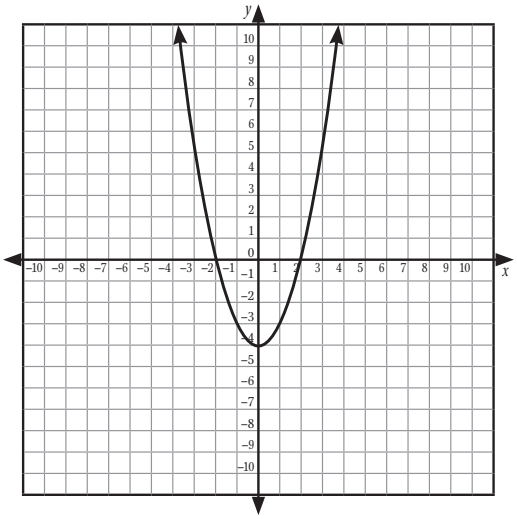
6. The equation for this parabola is $y = \frac{x^2}{2} - 4x + 9$. Use the coordinates found in problem 5 to convert the equation to vertex form. Show your work.

7. What are the coordinates of the *y*-intercept?

8. What is the equation for the axis of symmetry?

9. Does a parabola's axis of symmetry always run through its vertex? Why or why not?

10. Look at the graph below, which shows the parabola $y = x^2 - 4$. The coordinates of the parabola's *x*-intercepts are (2, 0) and (–2, 0). How could you use this information to find the coordinates of the parabola's vertex? Explain, showing your work.



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APPENDIX

Problem-Based Task Implementation Guides

Introduction

This addendum contains a collection of Implementation Guides (IGs) designed to support the use of the Problem-Based Tasks found throughout this unit.

Each Implementation Guide includes an overview and suggestions for introducing and facilitating the task. These contain specific recommendations for selected Standards for Mathematical Practice (SMPs) to highlight, guidance on addressing common errors and misconceptions associated with the task, and ideas for monitoring and coaching students as they work through the task. Finally, there is guidance for debriefing the task, with special attention to the SMPs and relevant Key Concepts from the instruction, as well as information on suitable extensions and alternate strategies.

Throughout the Implementation Guides, suggestions are provided in a question/prompt format to support your efforts to elicit student thinking and facilitate classroom discourse.

We developed these Implementation Guides for our Problem-Based Tasks at the request of math educators, and with advice and feedback from mathematics supervisors and teachers. Please let us know if they are helpful. Visit us at www.walch.com, follow us on Twitter (@WalchEd), or e-mail suggestions to customerservice@walch.com.

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Implementation Guide

Problem-Based Task 3.1.1: Deck the Deck

Common Core State Standard

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression such as terms, factors, and coefficients.

Task Overview

Focus

How can the area of a figure be represented with an algebraic expression? How can the parts of the expression be defined? In this task, students will write an algebraic expression to represent the area of a triangular deck. Then they will identify the terms, factors, and coefficients of the expression.

This activity will provide practice with:

- writing an algebraic expression for the area of a triangle
- substituting expressions into parts of a formula
- labeling units in an area problem
- simplifying an expression using the Distributive Property
- identifying terms, factors, and coefficients in an algebraic expression

Introduction

This task should be used to explore or apply the skill of interpreting parts of an expression such as terms, factors, and coefficients. Students should already be familiar with translating verbal expressions to algebraic expressions, and simplifying expressions using order of operations and the Distributive Property.

Begin by reading the problem and clarifying the meaning of *altitude*, *coefficient*, *factor*, *quadratic expression*, and *term*.

altitude	the perpendicular line from a vertex of a figure to its opposite side; the height of a figure
coefficient	the number multiplied by a variable in an algebraic expression
factor	one of two or more numbers or expressions that when multiplied produce a given product
quadratic expression	an equation that can be written in the form $ax^2 + bx + c = 0$, where x is the variable; a, b, and c are constants; and $a \neq 0$
term	a number, a variable, or the product of a number and variable(s)

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Implementation Guide

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed in this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will analyze the given information and construct an algebraic expression that will represent the area of the deck. They will use their knowledge about order of operations and the Distributive Property to simplify the expression, then they will identify the parts of the expression.

- **SMP 4:** Model with mathematics.

  Students will recognize that the scenario can be modeled by an algebraic expression. They will also recognize that a diagram representing the verbal description can be drawn.

- **SMP 7:** Look for and make use of structure.

  Students will recognize the structure of the formula for the area of a triangle as $\frac{1}{2}bh$ as well as the structure for a quadratic expression. They will use these structures to substitute the expressions for the base and the height of the deck, and they will create a simplified quadratic expression for the area once these expressions are substituted into the formula.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- not following the order of operations

  Have students write the order of operations (parentheses, exponents, multiplication/division, addition/subtraction) on top of their papers to use as a reference when working through the task.

- incorrectly identifying like terms

  Remind students that like terms are two or more terms that contain the same variables raised to the same power.

- inaccurately combining terms involving subtraction

  Remind students to carefully check the signs when combining like terms and distribute the negative sign appropriately.

- incorrectly combining terms by changing exponents

  Remind students that if two or more terms contain the same variable(s), the exponents on the variable(s) must be exactly the same before the terms can be combined.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students have difficulty beginning the task, encourage them to draw a diagram of a triangle and label the base and the altitude. Ask them, "What are the definitions of the base and the altitude of a triangle?" (**Answer**: The base of a triangle is the side that is perpendicular to the altitude of the triangle. The altitude is the triangle's height.)
- As students are writing the algebraic expression to represent the base of the triangle, ask them, "How do you know that subtraction is the operation to use when representing the base of the triangle in terms of the altitude?" (**Answer**: The given information states that the base of the triangle is 10 meters shorter than the altitude. The word "shorter" means the triangle is 10 meters less than, or smaller than, the altitude, so subtraction is used to represent this.)
- Ask students, "What are the units for the area of the deck, and how do you know this?" (**Answer**: The area of a figure is represented in square units, since two components are multiplied together. In the task, the dimensions of the deck are measured in meters, so the units for the area are square meters.)
- If students are having difficulty simplifying the expression for the area using the Distributive Property, ask them to write it out in parts. Ask them, "What are the steps for applying the Distributive Property to $\frac{1}{2}a(a-10)$?" (**Answer:** To use the Distributive Property, multiply $\frac{1}{2}a$ to both the a term and the –10 term. The result is $\frac{1}{2}a \bullet a + \frac{1}{2}a(-10) = \frac{1}{2}a^2 - 5a$.)
- If students have difficulty understanding why $\frac{1}{2}a^2 - 5a$ is a quadratic expression, ask, "What is the general form of a quadratic expression? How can $\frac{1}{2}a^2 - 5a$ be written in this form?" (**Answer:** The general form of a quadratic expression is $ax^2 + bx + c$. The expression $\frac{1}{2}a^2 - 5a$ can be written in this form as $\frac{1}{2}a^2 - 5a + 0$, so the value of c in the expression is 0.)
- If students are having difficulty identifying the factors that make up each of the terms in the expression for the area, encourage them to think of examples of numerical factors, such as $2 \bullet 3$ as factors of 6. Ask them, "What factors are multiplied together to get a product of $\frac{1}{2}a^2$? What factors are multiplied together to get a product of $-5a$?" (**Answer:** To get a product of $\frac{1}{2}a^2$, the factors of $\frac{1}{2}$, a, and a are multiplied together. Another combination of factors is $\frac{1}{2}$ and a. To get a product of $-5a$, the factors of –5 and a are multiplied together.)

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- Ask students, "What different components/items can a 'term' be made of?" (**Answer:** Terms can be composed of a number, a variable, or a product of a number and variable(s).)
- Ask students, "Suppose you were given the monomials $-3x^3y^4z^4$ and $6x^3y^3z^4$. Are these like terms? Explain why or why not." (**Answer:** No, these are not like terms. Although they have the same three variables, the exponents are different on the *y* variables, as the first monomial has a power of 4 on the *y*-variable, and the second monomial has a power of 3 on the *y*-variable.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their plans for constructing the algebraic expression to represent the area of the deck. Discuss the process of identifying the terms, factors, and coefficients of the expression. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Focus on the use of precise mathematical language and clarity, specifically when referring to the altitude, terms, factors, and coefficients of the expression.

Connecting to Key Concepts

Make explicit connections to key concepts:

- A quadratic expression can be written in the form $ax^2 + bx + c$, where x is the variable, and a, b, and c are constants.

  In this task, students will observe that the expression they create to represent the area of the deck, $\frac{1}{2}a^2 - 5a$, can be written in the form of a quadratic expression as $\frac{1}{2}a^2 - 5a + 0$.
- A term is a number, a variable, or the product of a number and variable(s).

  Students will identify that there are two terms in their expression.
- A factor is one of two or more numbers or expressions that, when multiplied, produce a given product.

  Students will identify the factors of each term of the expression.
- The number multiplied by a variable in an algebraic expression is called a coefficient.

  In this task, students will identify the coefficient of $\frac{1}{2}$ in the first term of the expression, and a coefficient of –5 in the second term.

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Extending the Task

- To extend the task, provide students with a different scenario in which they are given binomial expressions to represent the length and width of a rectangle. Ask them to find the expression that represents the area of the rectangle, then ask them to identify the terms, factors, and coefficients of the expression.
- Another option for extending the task is to provide students with a variety of polynomial expressions that have not been simplified. Ask students to work with a partner to simplify the expressions and identify the terms, factors, and coefficients of the expression. Then ask students to create their own polynomial expression, one with at least 5 or 6 terms, and ask their partners to simplify it.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by analyzing the given information and constructing an algebraic expression that represented the area of the deck.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I modeled the scenario by creating an algebraic expression to represent the given verbal description.)
- **For SMP 7, ASK:** "How did you look for, and make use of, structure when solving this problem?" (**Answer:** I used the structure of a quadratic expression to write an expression that represented the area of the deck.)

Alternate Strategies or Solutions

- Students may choose to draw a diagram representing the triangular deck before they begin the process of writing the algebraic expression. Encourage them to carefully label the base and altitude of the triangle.
- Once the algebraic expression is written, students may choose to substitute values into the expression for the altitude, a. Discuss how this can help them understand the validity of the expression, as they will ensure the value of the base is 10 meters less that the value of the altitude.

Technology

No technology is needed for this task.

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Implementation Guide

Problem-Based Task 3.1.2: Puppy Pen

Common Core State Standard

A–SSE.1 Interpret expressions that represent a quantity in terms of its context.★

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.*

Task Overview

Focus

How does changing the perimeter of a rectangle affect its area? Students will use the perimeter of a rectangular pen to model its area with a quadratic expression in one variable. Then they will analyze how changing the perimeter of the rectangular pen affects the area.

This activity will provide practice with:

- interpreting verbal expressions and translating them into algebraic expressions
- analyzing the meaning and impact of each part of an expression
- analyzing how a change to a constant of one expression impacts a related expression
- examining the terms of an expression to determine if it is quadratic
- comparing quadratic expressions to determine how they differ in terms of the context of the problem

Introduction

This task should be used to explore how changing the value of a constant in an expression affects the value of a related expression. It is best to implement this task after students have learned how to translate expressions from verbal statements into symbolic representations, how to solve for a variable in an expression, and how to identify a quadric expression.

Begin by reading the problem and reviewing the formulas for the area and perimeter of a rectangle. Explore the relationship between the two formulas and how the formulas are related because they both can be expressed in terms of the length and the width of the rectangle. Review the following terms: *area* and *perimeter*.

area	the number of square units that are contained within a plane figure
perimeter	the distance around a plane figure

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Some students will not know how to translate the scenario into symbolic representations. Encourage them to break the problem into parts by first connecting the given information with the unknowns of the perimeter formula and then using the resulting perimeter expression to model the area of the rectangle.

- **SMP 4:** Model with mathematics.

  Students will recognize that the perimeter of the pen can be modeled with a linear equation, and that the area of the pen can be modeled with a quadratic equation.

- **SMP 7:** Look for and make use of structure.

  When comparing the two area expressions, students can make use of the fact that the expressions differ only in the linear term, because the quadratic terms are the same. Ask students to compare the expressions by analyzing the coefficients of each term.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- forgetting to simplify expressions

  Remind students that if each term of the numerator of an expression is divisible by the denominator, the expression can be simplified.

- not completely isolating a variable

  Advise students that to express the width of the pen in terms of its length, the w term must have a coefficient of l. Since the coefficient of w is 2, the students must divide both sides of the equation by 2.

- forgetting to distribute to the second term of a binomial

  When multiplying $l(10 - l)$, remind students that the l must by multiplied by both the 10 and the $-l$.

- forgetting to distribute the negative when subtracting the area expressions

  Remind students that when subtracting a binomial from an expression, the negative sign must be distributed to both terms of the binomial.

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- difficulty with determining why the expression is quadratic

  Review that the terms of a quadratic can be in any order. The leading term does not have to have a degree of 2.

- incorrectly comparing the two area expressions

  Remind students how to compare the area of two figures with known constant areas by giving them an example of two areas and asking them to compare them. Emphasize that the subtraction strategy can be used with the area expressions.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students are having trouble getting started, ask them to review the information they are given and make connections using the formulas they know. Ask students, "What is the problem asking?" (**Answer:** How does the area of the pen change when the perimeter is decreased by 2?) "What formula represents the area of the pen?" (**Answer**: $A = lw$.) "What does the 20 feet of fencing represent?" (**Answer**: the perimeter of the pen.) "What formula represents the perimeter of the pen?" (**Answer**: $P = 2l + 2w$.)
- If students are having trouble comparing the areas, remind them that expressing the area in terms of one variable will allow a comparison of the areas when the perimeter changes. Ask students, "How can the given information be used to express the area in terms of one variable?" (**Answer:** Use the perimeter to solve for one variable in terms of the other, then substitute the expression into the area formula.)
- If students are unsure how to write an expression for the pen's width in terms of its length, ask the following questions: "Which variable in the expression do you want to isolate?" (**Answer**: w.) "Which operation is needed to get the $2l$ to the other side of the equation?" (**Answer**: Subtraction.) "Which operation is needed to get the coefficient of w to the other side of the equation?" (**Answer**: Division.)
- If students are struggling with how to express the area of the pen in terms of l, ask the following questions: "What formula represents the area of the pen?" (**Answer**: $A = lw$.) "What does it mean to express the area of the pen in terms of l?" (**Answer**: Write an area expression using only the l variable.) "What expression did you find to represent w?" (**Answer**: $10 - l$.) "How can you use that expression to write the area of the pen in terms of l?" (**Answer**: Substitute $10 - l$ for w in $A = lw$.)
- Once students have the two area expressions, ask them how to compare the areas. (**Answer:** Find the difference of the two expressions.) Ask students to analyze the terms of the expressions and determine the differences. (**Answer:** The expressions only differ in their linear terms.) Ask the students if it is necessary to subtract the entire area expressions to find the difference of the areas. (**Answer:** No; only the linear terms must be subtracted.)

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- When students have determined that the difference of the areas is l, ask them how the solution relates to the context of the problem. (**Answer:** l is the length of the pen, so the areas differ by l square units.) Remind students that the side length labeled l is arbitrary, so the difference of the areas in square units is equal to one of the dimensions of the pen.
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Review how the given information was used with the known perimeter and area formulas to create equations to model the given scenario. Highlight that the area formula is the product of two lengths, and that using the perimeter to solve for one variable in terms of the other allows the area to be expressed in one variable. Discuss how changing the value of the perimeter of the pen affected the area expression. Emphasize that simplifying the perimeter expression allowed for easier manipulation of the area formula. Review the relationship between the structure of the final two area expressions, and explore the relationship between the structure and the difference in the areas.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Changing the value of a variable will not change the value of a constant, just as changing the value of a constant will not change terms containing variables.

  When the perimeter of the pen is decreased by 2 units, the number of square units by which the area is decreased is equal to one of the lengths of the sides of the pen.
- Simplifying expressions lets us more easily see how different terms interact with one another.

  Simplifying the perimeter expression allows the area expression to be more easily manipulated, and therefore easier to differentiate how the area changes when the perimeter is altered.
- When presented with an expression in factored form, multiply the factors to see if the polynomial is a quadratic. Quadratic expressions are of the form $ax^2 + bx + c$, where $a \neq 0$.

  The Distributive Property was applied to expand $l(10 - l) = 10l - l^2$ and $l(9 - l) = 9l - l^2$. Both of these expressions are quadratic, with $a = -1$ and $c = 0$.

Extending the Task

- To extend the task, change either of the parameters. For example, change the initial amount of fencing to another even number, and write a conjecture about how the initial amount of fencing is related to the final answer. Or change the amount of fencing damaged to another even number, and write a conjecture about how the amount of fencing damaged is related to the final answer.
- Another option is to increase the amount of fencing instead of decreasing it. How does the change affect the difference in areas?

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you demonstrate perseverance?" (**Answer:** I reread the problem to find the information I needed. I broke the problem into smaller parts by connecting the perimeter and area formulas to the given information.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I wrote expressions to model the width of the pen in terms of the length ($10 - l$ and $9 - l$), and used these expressions to model the area of the pen with expressions $l(10 - 1)$ and $l(9 - l)$.
- **For SMP 7, ASK:** "How did you make use of structure when solving the problem?" (**Answer:** When the perimeter of the pen is 20 units, I modeled the situation with the expression $10l - l^2$. When the perimeter of the pen is decreased by 2 units, I modeled the situation with the expression $9l - l^2$. I was able to use structure to see that these expressions are the same with the exception of the l terms and determine that the number of square units that the area differs by is equal to the difference of the l terms: $10l - 9l = l$.

Alternate Strategies or Solutions

- Students may try to create various dimensions of a rectangle of perimeter 20 and hypothesize how the area is affected by analyzing the results of various dimensions. For example:

	Length	Width	Perimeter	Area
Original area	5	5	20	25
Decrease perimeter by 2	5	4	18	10

Difference in area: $25 - 20 = 5$

	Length	Width	Perimeter	Area
Original area	3	7	20	21
Decrease perimeter by 2	3	6	18	18

Difference in area: $21 - 18 = 3$

	Length	Width	Perimeter	Area
Original area	8	2	20	16
Decrease perimeter by 2	8	1	18	8

Difference in area: $16 - 8 = 8$

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- After viewing the results of various sets of lengths and widths that fit the parameters of the problem, students may realize the difference of the areas in square units is always equal to one of the dimensions of the pen.
- Another common strategy is to solve for l in terms of w instead of solving for w in terms of l. In this case, the difference of the areas is w. Since the variables l and w are arbitrarily assigned to the side lengths, this solution will help reinforce that the difference of the areas in square units is equal to one of the dimensions of the pen.

Technology

If using the alternative strategy, student can use spreadsheets to calculate the area of each rectangle, and to find the difference of the area for each of the related dimensions.

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Problem-Based Task 3.2.1: Time to Splash

Common Core State Standards

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

A–REI.4 Solve quadratic equations in one variable.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions, and write them as $a \pm bi$ for real numbers a and b.

Task Overview

Focus

How are the height and the amount of time it takes an object to fall related? Students will use a quadratic model to find the amount of time it will take a diver to hit the water after leaving a platform.

This activity will provide practice with:

- using a model
- solving a quadratic equation in one variable
- interpreting the solution to a quadratic equation in terms of a context

Introduction

This task should be used to explore or apply the concept of solving and interpreting the solutions to quadratic equations in one variable. It is best to implement this task after students have learned how to solve quadratic equations in one variable by taking the square root.

Begin by reading the problem and clarifying the meaning of a dive and the path that a diver takes. Explain that the platform is a certain distance, in feet, above the water, and that a diver hits the water when he first enters the water. The time that it takes the diver to leave the platform and enter the water is measured in seconds. Review the following terms: *platform* and *quadratic expression*.

platform	in diving, a type of structure that consists of a vertical rigid "tower" with one or more horizontal boards extending out over a pool
quadratic expression	an algebraic expression that can be written in the form $ax^2 + bx + c$, where x is the variable, a, b, and c are constants, and $a \neq 0$

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Some students will not be able to visualize the given scenario. Encourage students to draw a sketch of the diver and what is happening. Have them label the height of the platform on the sketch.

- **SMP 4:** Model with mathematics.

  While the expression is given to the students, be sure they understand that they need an equation to solve the problem. Help them understand that the expression stands for the diver's height above the water, and that x is the time since the diver jumped off the platform. Then guide them to set the expression equal to 0 to find when the diver hits the water.

- **SMP 7:** Look for and make use of structure.

  Students should think about solving this quadratic equation as they have every other equation that they have solved. Remind them that the objective is to isolate the variable using inverse operations. Discuss that the inverse operation for squaring something is taking the square root.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- identifying two solutions for this scenario

  Explain that the time was 0 when the dive started, so a negative time would not make sense for this model.

- not isolating the squared term before taking the square root

  Remind students that the term that is squared is $(x - 0.5)$, so it must be isolated rather than just the variable.

- incorrectly identifying the solution before completely isolating the variable

  Point out that there are still steps to be taken to isolate the variable and find the solution after taking the square root.

- forgetting to use the ± symbol

  Explain that any time the student writes the radical sign to solve an equation, the ± symbol must be used.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students have difficulty beginning the task, ask the following questions: "What does the given expression represent?" (**Answer**: The diver's height above the water.) "What are you trying to find?" (**Answer**: The amount of time it takes the diver to hit the water.) "What height is the diver above the water when she hits the water?" (**Answer**: 0 feet.)
- If students have difficulty calculating the amount of time it will take Nina to hit the water, ask, "How can you use the given expression to determine the amount of time?" (**Answer**: Set the given expression equal to 0, and solve for x.)
- If students have trouble solving the equation, ask, "What steps are needed to isolate x?" (**Answer**: Subtract 5 from both sides, then divide both sides by 5, take the square root of both sides, and add 0.5 to both sides.)
- If students have difficulty recognizing the need to take the square root, ask, "What is the inverse of squaring?" (**Answer**: Taking the square root.) Remind students that when taking the square root, they must use the ± symbol.
- Once students have solved the equation, ask them to think about whether a negative answer is reasonable. Ask, "What would a negative time represent?" (**Answer**: Before the dive occurred.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Ask students to compare solving quadratic equations to using quadratic models, especially with respect to the number of solutions. Ask them to explain when a negative time might be an appropriate answer.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Quadratic equations that contain only a squared term and a constant can be solved by taking the square root of both sides.

  In this task, the given expression used to write an equation simplifies to an equation that contains only one squared term. Students took the square root of both sides to find the amount of time it took Nina to hit the water.

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- When we take the square root of both sides, we need to remember that a number and its opposite have the same square. Therefore, rather than simply taking the positive square root, we need to take the positive and negative square root. For $x^2 = c$, we find that $x = \pm\sqrt{c}$.

  When taking the square root to solve an equation, there are two possible roots: One positive and one negative. However, in this task, a negative solution did not make sense.

- We can use a similar method to solve quadratic equations in the form $(ax + b)^2 = c$.

  Students had to subtract the constant, 5, from both sides to solve the equation.

Extending the Task

To extend the task, make a graph of the model. For example, have students substitute different values for the time, x, and plot the diver's height at that time. Have students describe the shape of the graph and identify on the graph when the diver hits the water. Have students compare the graphical solution to the algebraic solution.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you demonstrate perseverance?" (**Answer:** I kept reading the problem to understand it, and I worked through many steps to solve a difficult equation.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I made an equation to show when the diver hit the water.)
- **For SMP 7, ASK:** "How did you look for, and make use of, structure when solving this problem?" (**Answer:** I used what I know about solving linear equations and inverse operations to solve a quadratic equation.)

Alternate Strategies or Solutions

- Students could use the expression to create a table showing times and heights to find when the diver hits the water.
- Students could also graph the equations $y = -5(x - 0.5)^2 + 5$ and $y = 0$, and see where they meet.

Technology

Students can use the graphing calculator to create a table of values and find when the height is 0 feet.

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Problem-Based Task 3.2.2: On the Shelf

Common Core State Standards

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

A–REI.4 Solve quadratic equations in one variable.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions, and write them as $a \pm bi$ for real numbers a and b.

Task Overview

How much paint is needed to paint the top of a shelf? What is the area of the triangular shelf? Students will set up and solve equations to determine the area of a shelf being put in a room.

This activity will provide practice with:

- analyzing the scenario
- modeling with an equation
- writing and solving quadratic equations
- interpreting solutions

Introduction

This task should be used to apply the concept of writing and solving quadratic equations. It is best to implement this task after students have learned how to solve quadratic equations by factoring.

Begin by reading the problem and clarifying the dimensions of the shelf. Review the following terms: *area* and *right triangle*.

area the amount of space inside the boundary of a two-dimensional figure

right triangle a triangle with one angle that measures 90°

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Some students may not remember the relationship among the sides of a right triangle. They may need to refresh their understanding of the Pythagorean Theorem. After solving for the legs of a triangle, students must find the area of the triangle.

- **SMP 4:** Model with mathematics.

  Students will set up an equation to find the length of one of the legs of the shelf. They will use the relationship among the legs of the right triangle to write a quadratic equation.

- **SMP 7:** Look for and make use of structure.

  After writing the quadratic equation, students will use what they understand about factoring and the Zero Product Property to find the solution to the equation. Students will understand that they must square the binomial and combine like terms before they can solve the equation. They will also use the structure of multiplying two binomials to square the binomial.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- not multiplying $(x - 2)^2$ before trying to factor the entire expression

  Remind students that quadratic equations must be in standard form before they can be factored.

- not setting the equation equal to 0 before factoring

  To solve a quadratic equation by factoring, the factored side must be equal to 0, because the process requires the application of the Zero Product Property.

- treating the lead coefficient other than 1 as if it were 1

  Ask students to check their factoring by multiplying to see if they get the original equation. Remind them that if there is a common factor for each term, it can be factored out.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students have trouble starting the task, have them create a diagram of the shelf and label the given information.

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- If students have difficulty determining the length of the legs of the triangle, ask them how the shorter leg can be expressed in terms of x if it is 2 inches shorter than the longer leg. (**Answer**: If x is the long leg, then $x - 2$ is the shorter leg.) Then ask, "What formula can you use to show the relationship among the lengths of all the sides of a right triangle?" (**Answer**: The Pythagorean Theorem, or $a^2 + b^2 = c^2$, where a and b are legs of a right triangle and c is the hypotenuse.) "How can the formula be applied here?" (**Answer**: Let a and b be x and $x - 2$, and let c be 10.)
- If students have trouble solving the Pythagorean Theorem for x, ask them how binomials are squared. (**Answer**: Rewrite it as $(x - 2)(x - 2)$, multiply each term in the first binomial by each term in the second binomial, then simplify by combining like terms.)
- If students incorrectly factor the equation, ask, "What is a common factor of every term in the equation? Why would you factor it out?" (**Answer**: 2; factor it out and divide both sides of the equation by it. As a result, the coefficients are lesser, and the equation is still equal to 0.)
- Once students have solved the equation, if they have trouble interpreting the solutions, ask, "What answers can be eliminated? Why?" (**Answer**: No length can be negative, so x cannot equal –6.) "How can you find the length of the other leg?" (**Answer**: Subtract 2 from 8 inches.)
- If students think they are done with the task, remind them of the original question. Ask them what the length of the base and the height of triangle are. (**Answer**: The base and height are 8 and 6 inches, respectively.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Have students describe the steps they took in their solution process. Ask students to discuss differences in their solution methods.

Connecting to Key Concepts

Make explicit connections to key concepts:

- You should always check to see if the terms of an expression have a greatest common factor before attempting to factor further.

  In this task, it is necessary to factor 2 from each term so the leading coefficient is 1, making it easier to factor.

- To factor a trinomial with a leading coefficient of 1 in the form of $x^2 + bx + c$, find two numbers d and e that have a product of c and a sum of b.

  To factor the expression in the equation, find two numbers whose product is –48 and whose sum is –2.

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- When finding d and e, be careful with the signs.

  The factored form will have two binomials, $(x + d)(x + e)$, although the binomial will be written as a number subtracted from x if either is negative.

Extending the Task

To extend the task, have students change the parameters. For example, ask students to change the length of the longest side or the relationship between the lengths of the legs.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I wrote a quadratic equation to model the situation, solved it by factoring, and used the solution to find the length of the two legs. Then I used the lengths of the legs to find the area of the triangle.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I wrote a quadratic equation to model the scenario involving the lengths of the sides of triangles.)
- **For SMP 7, ASK:** "How did you look for, and make use of, structure when solving this problem?" (**Answer:** Because the shelf is a right triangle, I used the Pythagorean Theorem. I could also use the patterns I know, as well as the Zero Product Property, to solve the equation.)

Alternate Strategies or Solutions

- Students could guess and check the legs of the triangle. They know that the longest leg is 10 inches, so the number of guesses would not be unreasonable.
- Students could solve the quadratic equation using any method they know. They may even want to graph it and find the positive x-intercepts.

Technology

- Students could use a graphing calculator to graph the equation and find the zeroes, which would be the possible lengths for the longer leg.
- Students could also use a table or spreadsheet to generate different combinations of leg lengths and determine which would satisfy the problem.

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Problem-Based Task 3.2.3: Curve Ball

Common Core State Standards

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

A–REI.4 Solve quadratic equations in one variable.

a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions, and write them as $a \pm bi$ for real numbers a and b.

Task Overview

Focus

The path of a thrown object can follow the shape of a parabola, which can be modeled by a quadratic equation. How can you find the time a ball will be at a particular height?

This activity will provide practice with:

- modeling a scenario with an equation
- solving a quadratic equation by completing the square
- analyzing the solutions to the quadratic equation
- interpreting the results

Introduction

This task should be used to apply and explore the concept of solving a quadratic equation by completing the square. It is best to implement this task after students have learned how to use completing the square to solve equations.

Begin by reading the problem and clarifying the meaning of the path of a thrown object. Review the following term: *quadratic equation*.

quadratic equation an equation that can be written in the form $ax^2 + bx + c = 0$, where x is the variable, a, b, and c are constants, and $a \neq 0$

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will have to decide if it is reasonable to have two answers. Encourage students to draw the path of the ball and determine whether it is 7 feet high at two different times.

- **SMP 4:** Model with mathematics.

  Some students may not understand how to apply the expression to the find the solution. Explain that the height of the ball is represented by the expression and 7 feet. Ask them to consider how to show these two things are equivalent.

- **SMP 7:** Look for and make use of structure.

  The process of completing the square helps students manipulate an equation to become like equations they have solved before, where a squared term is equal to a constant.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- neglecting to divide by a so the coefficient of x^2 is 1

  Have students create a checklist of the steps for completing the square. To check their steps, they can multiply the perfect square to see if the result is the original equation.

- forgetting that, when taking the square root, both positive and negative roots must be considered

  Remind students that any time they take the square root to solve an equation, they must write ± before it.

- not isolating x after taking the square root of both sides

  Students may stop short of solving the equation. Ask them to identify what they need to know, and what stands for that. That is, they are looking for the time, which is represented by x.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students have difficulty starting the task, ask, "What does the given expression represent?" (**Answer**: the height of the ball.) "What height does the task ask about?" (**Answer**: 7 feet.) "How can you use the given expression and the height of the ball to determine when the ball will reach the specified height?" (**Answer**: Set them equal to each other.)

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- If students have trouble solving the equation, ask, "What must you do with the constants? (**Answer**: Combine them on the opposite side of the equation from the squared term.) "What's the next step in completing the square?" (**Answer**: Get a leading coefficient of 1.) "What do you need to do to the equation so the algebraic expression is a perfect square?" (**Answer**: Divide b by 2, square it, and add it to both sides.) "Once the trinomial is written as a perfect square, what do you do?" (**Answer**: Take the square root of both sides, then isolate x.)
- If students have difficulty interpreting the results, ask, "What does x stand for? (**Answer**: the time since the ball was thrown in seconds.) "What is the shape of the path of the ball?" (**Answer**: A parabola.) "Does it make sense that there are two solutions? Why?" (**Answer**: Yes, because the ball will be at that height on the way up, and again on the way down.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Ask students if they tried other ways to solve the quadratic equation, and whether or not they were successful. Ask them to identify why completing the square is a good choice for this problem.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Quadratic equations can be solved following the steps to completing the square.

  The solution method for this task was completing the square. This method led the student to find the two solutions, both times when the ball was 7 feet high.

- When the binomial $(x + a)$ is squared, the resulting perfect square trinomial is $x^2 + 2ax + a^2$.

  When completing the square, by dividing b by 2, and adding its square to the expression, the student creates a perfect square trinomial.

Extending the Task

To extend the task, have students change the parameters. For example, ask students to find the height of the ball given a time, change the height of the ball, identify the maximum height of the ball, or determine when the ball hits the ground.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I was able to represent the position of a ball by an equation.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I found both times that the ball was 7 feet above the ground.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I wrote an equation to represent the path of a ball over time.)
- **For SMP 7, ASK:** "How did you look for, and make use of, structure when solving this problem?" (**Answer:** I know what a perfect trinomial square looks like, and I used that knowledge to complete the square so I could solve the equation using square roots.)

Alternate Strategies or Solutions

- Students could solve the equation by graphing the equation and seeing for what value of x, $y = 7$.
- Students could use another method to solve the quadratic, such as the quadratic formula.

Technology

Students can use a graphing calculator to find the value of x (time) when y (the height of the ball) is 7. After graphing the equation, they can trace to find this value, or they can also graph the equation $y = 7$, and calculate where the two graphs intersect.

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Problem-Based Task 3.2.4: Divine Proportion

Common Core State Standards

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

A–REI.4 Solve quadratic equations in one variable.

a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions, and write them as $a \pm bi$ for real numbers a and b.

Task Overview

Focus

The golden ratio is often used in design, because it is thought to create objects that are pleasing to the eye. It can also be found in music, the human body, and nature. Students will write and solve a quadratic equation in which the positive root is the golden ratio.

This activity will provide practice with:

- writing quadratic equations
- solving quadratic equations using the quadratic formula
- interpreting the solution to quadratic equations in terms of the context of the problem

Introduction

This task should be used to explore or apply how quadratic equations can be written and solved from a context. This task should be done after students have completed Example 5, so that they have practiced using the formula multiple times.

Begin by reading the problem and discussing the meaning of ratio. Explain how ratios and proportions are related. Explain that phi is simply the name of a letter in the Greek alphabet, and will be used as the variable in the problem. Phi is the variable that represents the golden ratio. Review the following terms: *ratio*, *phi*, and *quadratic equation*.

ratio	a comparison of two quantities by division
***phi* (ϕ)**	the 21st letter of the Greek alphabet, used to represent the golden ratio
quadratic equation	an equation that can be written in the form $ax^2 + bx + c = 0$, where x is the variable, a, b, and c are constants, and $a \neq 0$

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Implementation Guide

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will need to reason abstractly to write the quadratic equation used to find the golden ratio, and reason quantitatively to interpret the roots of the equation.

- **SMP 7:** Look for and make use of structure.

  Students will need to understand that "a positive number increased by 1 is equal to its square" is a statement representing a quadratic equation. Help students translate the verbal statement into a quadratic equation by discussing how to write each part of the statement symbolically.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- interpreting squaring the variable as doubling the variable

  Remind students that in general, $x \bullet x \neq 2 \bullet x$. Squaring a number is not the same as doubling it. For example, $3 \bullet 3 \neq 2 \bullet 3$.

- trying to solve the quadratic equation without first putting it in standard form

  Remind students that the standard form of a quadratic equation is $ax^2 + bx + c = 0$. To use the quadratic formula, the equation must be in standard form.

- using positive values for b and c in the quadratic equation

  Encourage students to rewrite the quadratic equation $\phi^2 - \phi - 1 = 0$ as $\phi^2 + (-\phi) + (-1) = 0$, so they understand that $b = -1$ and $c = -1$.

- writing $\sqrt{1+4}$ as $\sqrt{1} + \sqrt{4}$

  Review properties of radical expressions and remind students that $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students have trouble determining the equation from the information given, ask, "Which sentence in the problem statement can be used to write the equation?" (**Answer**: One property of the golden ratio is that it is the only positive number that, when increased by 1, is equal to its square.)

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- Help students understand that "golden ratio" and "positive number" represent the same unknown, *phi* (ϕ), by asking the following questions: "What is the variable? (**Answer**: ϕ.) "What expression represents a positive number plus 1?" (**Answer**: $\phi + 1$.) "What expression represents the number squared?" (**Answer**: ϕ^2.)
- If students have difficulty writing the equation in standard form, ask them how variables in an equation are isolated. (**Answer**: Use inverse operations.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their plans for writing the quadratic equation to find the solution of phi. Discuss the process of identifying the terms, factors, and coefficients of the equation. Encourage students to discuss any difficulties or confusion they experienced when working through the task.
- Compare students' strategies and ways of justifying responses. Focus on the use of precise mathematical language and clarity, specifically when referring to the solution.
- It is important for students to understand that the *exact* value of the golden ratio is $\frac{1+\sqrt{5}}{2}$, while 1.618 is only an approximation.

Connecting to Key Concepts

Make explicit connections to key concepts:

- A quadratic equation in standard form, $ax^2 + bx + c = 0$, can be solved for x by using the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

  In this task, students must write the quadratic equation in standard form and then use the quadratic formula to determine the roots of the equation in order to find the value of the golden ratio.

Extending the Task

To extend the task, have students investigate the golden ratio further. For example, have students use the ratio to draw golden rectangles of various sizes. Another option is to have students use the Internet to discover examples of how the golden ratio is used in art, packaging, and architecture.

Use the following as an extension to the problem.

- Illuminations. "A Ratio That Glitters: Exploring the Golden Rectangle."

  http://www.walch.com/rr/07010

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I had to analyze the information in the problem to determine what was necessary to solve it.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I determined that one of the roots of the equation did not apply to the golden ratio problem.)
- **For SMP 7, ASK:** "How did you look for, and make use of, structure when solving this problem?" (**Answer:** I wrote the quadratic equation using the information in the problem, and used the quadratic formula to solve the problem.)

Alternate Strategies or Solutions

Allow students to use an online quadratic equation solver to solve the equation or check their work.

Technology

Students can use a calculator to write the golden ratio as a decimal.

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Problem-Based Task 3.2.5: Dancing for Charity

Common Core State Standards

A–SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$, *thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

A–REI.4 Solve quadratic equations in one variable.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions, and write them as $a \pm bi$ for real numbers a and b.

Task Overview

Focus

How does a business calculate its profits? To find an organization's profit, you subtract all its costs from its total income. In this task, students will use expressions that represent the income and cost of a dance to find the profits that will go to a charity.

This activity will provide practice with:

- representing a scenario with an equation
- writing an inequality to solve a problem
- solving a quadratic equation using the quadratic formula
- evaluating the solution for reasonableness

Introduction

This task should be used to apply their understanding of solving a quadratic inequality using the quadratic formula. It is best to implement this task after students have learned how to use the quadratic formula and translate inequalities.

Begin by reading the problem and clarifying the meaning of *charity*, *income*, *costs*, *profit*, and *inequality*. Review the following terms:

charity an organization designed to support a cause or to help others

costs how much an organization pays out

income how much an organization is paid

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inequality	a mathematical sentence that shows the relationship between quantities that may or may not be equivalent. An inequality contains one or more of the following symbols: $<$, $>$, $\leq$, $\geq$, or $\neq$.
profit	the difference in total income and total costs

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed in this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will use expressions rather than actual numbers to represent the cost and income for the dance. Encourage them to write the formula for profit, then substitute the expressions for the income and cost.

- **SMP 5:** Use appropriate tools strategically.

  Students will use a number line to determine the range of ticket prices that result in the desired profit. Encourage students to draw the number line, show the ranges, and use it to record the result of the test for each range.

- **SMP 6:** Attend to precision.

  To apply the quadratic formula, students will need to correctly identify a, b, and c, including the sign on each. Then, during substitution and simplification of the formula, students must carefully watch the signs as they take opposites and subtract products.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- forgetting that answers should be inequalities

  Ask students to highlight the words "or more" in the problem. Discuss what those words mean, and the implication for the solution.

- using the incorrect inequality symbol

  Advise students that another way of thinking of "$200 or more" is at least $200. Discuss whether the profit can actually be equal to $200.

- Improperly representing the sections of the number line

  Draw a number line with two specific points identified. Explain that there are always three possible regions: to the left of the lesser point, between the points, and to the right of the greater point. Guide students to write an inequality for the solution being each region.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students incorrectly write the equation, ask, "How should the expressions that represent total income and total cost be written when substituted into the profit equation?" (**Answer**: Use parentheses to keep the two expressions separate.)
- "To simplify the profit equation, how are the parentheses eliminated?" (**Answer**: Distribute 1 and –1, respectively.)
- If students do not understand why they should write an inequality, ask, "What words in the task translate to an inequality symbol?" (**Answer**: "Or more.") "What inequality symbol should be used?" (**Answer**: ≥.)
- If students struggle to use the quadratic formula, ask, "What can be factored out of each term to simplify the equation?" (**Answer**: 10.) "When the quadratic formula is solved, what does it reveal?" (**Answer**: The values of x.) "In the quadratic formula, what are the values of a, b, and c?" (**Answer**: $a = -1$, $b = 18$, $c = -70$.)
- If students struggle to find both answers, ask, "How can you write two equations for x, rather than using the ± symbol?" (**Answer**: Write one equation using +, and another using –.)
- If students struggle to determine which region represents the ticket price, ask, "How do you find the three regions?" (**Answer**: One is less than 5.68, one is between 5.68 and 12.32, and the third is greater than 12.32.) "How do you choose values to test?" (**Answer**: Choose ones for which the computation is simpler.) "How do you test the values?" (**Answer**: Substitute the values into the quadratic formula. See if the answer is greater than or equal to 0.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Discuss why there is a range of ticket values that will give a profit of at least $200. Talk about what happens when the ticket price gets too high (total sales go down), or if the ticket price is too low (sales don't cover costs).

Connecting to Key Concepts

Make explicit connections to key concepts:

- To solve a quadratic inequality, set the right side equal to 0, replace the inequality symbol with an equal sign, and solve the equation.

  To find the possible ticket values, students change the inequality to an equation and solve using the quadratic formula.

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- Use solutions to create regions on a number line, and test points in each region to solve the inequality.

  Students use a number line, create regions, and test points to see if ticket prices less than $5.68, greater than $12.32, or in between these prices will result in the desired profit.

- Graph endpoints as a closed dot if the original inequality symbol is ≥ or ≤.

  When representing the solution to the range of the ticket prices, which is inclusive, students describe it using the ≤ symbol.

Extending the Task

To extend the task, have students change the parameters. For example, have them change the amount of desired profit, change the expression for the profit, or change the expression for the income.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used expressions to represent the concepts of income, cost, and profit.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I found the range of ticket prices that give a desired profit.)
- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** I used the quadratic formula to solve the equation, and I used a number line to determine the range for the solution to the quadratic inequality.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I used the correct signs when calculating with the quadratic formula. I also determined the possible regions, and tested a point in each for the inequality.)

Alternate Strategies or Solutions

Students could complete the square to solve the problem. Students could also use a graphing calculator to graph the equation, and the equation $y = 200$.They could then use the trace feature to find the points of intersection. The portion that is above the line $y = 200$ would be where profit is at least $200.

Technology

Students could use a spreadsheet to calculate the value of the expression for several values in each region. Students could also use a graphing calculator to graph and check their answers.

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Problem-Based Task 3.3.1: Parabolic Party Streamers

Common Core State Standards

A–CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

A–SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

a. Factor a quadratic expression to reveal the zeros of the function it defines.

Task Overview

Focus

When an object is thrown into the air from a window, how can its path be described? Students will use a quadratic model to find the time and height of a streamer when it is at its maximum distance from the ground.

This activity will provide practice with:

- modeling the situation with a quadratic equation
- identifying the vertex of a parabola
- applying the model to find the maximum distance from the ground

Introduction

This task should be used to explore or to apply the concept of solving quadratic equations. It is best to implement this task after students have learned to graph quadratic equations in standard form. Graphing may be done using technology, including graphing calculators or online graphing programs.

Begin by reading the problem and clarifying the meaning of the students' experiment and how the streamer was thrown. Explain that the streamer went out the window and up before it fell to the ground. The distance between the streamer and the ground is measured in inches and the time it takes to fall to the ground is measured in seconds. Review the following terms: *parabola*, *physics*, *gravity*, *streamers*, and *unraveled*.

gravity	a force that pulls things down towards the center of Earth
parabola	the U-shaped graph of a quadratic equation
physics	the study of the nature and properties of matter and energy
streamers	long narrow paper strips that are used to decorate for parties
unraveled	to come undone, or unrolled

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Implementation Guide

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students may struggle to visualize the path of the streamers. If possible, have students use streamers or toilet paper to model the path. Then, have them sketch the path.

- **SMP 4:** Model with mathematics.

  In order to understand the equation as a mathematical model, students may need to graph the equation so that they can see the path of the streamer. Then, have them identify the point at which the streamer would be the farthest from the ground.

- **SMP 7:** Look for and make use of structure.

  The vertex of a quadratic function that opens down represents the maximum point, or in this case, the maximum height from the ground. Students continue to use the same process for finding the vertex. That is they use the equation $x = \frac{-b}{2a}$ to find its x-coordinate and then substitute that value into the equation to find its y-coordinate.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- not finding the important points on a parabola

  Have students sketch the path of the streamers and identify where the streamer is at its maximum height. Ask them to identify which important point this is.

- finding the x-coordinate of the vertex, but not the y-coordinate

  Remind students that the vertex is a point. The formula $\frac{-b}{2a}$ gives the x-coordinate, but it still needs to be substituted into the equation to find the y-coordinate of the vertex.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students do not understand the given model, ask "What does the given equation model?" (**Answer**: how the height of the streamers changes over time after it was thrown) "What kind of equation is it?" (**Answer**: quadratic) "What is the shape of the graph of this equation?" (**Answer**: a parabola) "What is the task for?" (**Answer**: when the streamers were at the highest point) "What part of a parabola would represent the highest point?" (**Answer**: the vertex)

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- If students cannot find the vertex, ask "What formula can be used to find the *x*-coordinate?" (**Answer**: $x = \frac{-b}{2a}$) "How will you find the *y*-coordinate?" (**Answer**: Substitute the *x*-coordinate into the equation and calculate *y*.)
- If students struggle to identify the answer, ask "What represents the time the streamer was at the highest point?" (**Answer**: The *x*-coordinate of the vertex.)
- "What represents the height at the maximum distance from the ground?" (**Answer**: The *y*-coordinate of the vertex.)
- Challenge students to think about how they would use the model to find the time the streamer is at another height, such as 675 inches above the ground.
- "How would you find the time?" (**Answer**: Substitute 675 for *y* and then solve the equation for *x*.) "Why would there be two answers?" (**Answer**: Because the streamer is at that height on the way up and then again on the way down.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their plan for determining the number of seconds. Discuss the process of identifying the coefficients of the quadratic equation in standard form. Encourage students to discuss any difficulties or confusion they experienced when working through the task.
- Compare students' strategies and ways of justifying responses. Focus on the use of precise mathematical language and clarity, specifically when referring to the maximum height and vertex.
- Ask students whether they need to graph the equation completely to solve this problem. Discuss the importance of the important points in the graph, and ask them to interpret what the other important points would represent in this situation.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The standard form of a quadratic equation is $f(x) = ax^2 + bx + c$.

  The formula for the path of the streamers is given in standard form. Students need not rearrange the formula to any different form.

- The formula $x = \frac{-b}{2a}$ is not only used to find the equation of the axis of symmetry, but also the *x*-coordinate of the vertex.

  To find the highest point, the vertex, of the path, students must use this formula to find the time, *x*, that the distance between the streamers and the ground is the greatest.

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- The vertex gives information about the maximum and minimum value of a quadratic. The maximum is the largest *y*-value of a quadratic.

  The task asks students to find the point when the distance between the streamers and the ground is maximized. Students should recognize that this point is the vertex of the graph.

Extending the Task

To extend the task, further analyze the model. For example, have students identify when the streamers are going up and when they are going down, identify the time that the streamers hit the ground, identify the height of the streamers at other times.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you demonstrate perseverance?" (**Answer:** I didn't just state that the vertex was the maximum point, I used the equation model to find the exact coordinates so that I could answer the question.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used the equation to model the path of the streamers.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I know that the vertex represents the maximum value for a quadratic function with $a < 0$, so I used the formula for the *x*-coordinate to find the time, and then substituted into the equation to find the maximum height.)

Alternate Strategies or Solutions

Students could use a graphing calculator to graph the function $f(x) = -16x^2 + 32x + 673$ and then find the maximum value to identify the vertex.

Technology

Students can use a graphing calculator to graph the function. Students can also use a table on the graphing calculator or a spreadsheet to generate a list of heights at different times and identify the maximum height.

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Problem-Based Task 3.3.2: It's Show Time!

Common Core State Standards

A–CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

A–SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

a. Factor a quadratic expression to reveal the zeros of the function it defines.

Task Overview

Focus

Determining the size of something before it is built is essential, and when that object is rectangular, the size requirement can be described by the length and width and/or the area. Students will use a quadratic equation to find the length and width of a stage that must have a particular area.

This activity will provide practice with:

- modeling a situation with a quadratic equation
- solving a quadratic equation in one variable
- interpreting the solution to a quadratic equation in terms of a context

Introduction

This task should be used to explore or apply the concept of solving a quadratic equation by factoring. It is best to use this task to explore the concept of solving quadratic equations by factoring.

Begin by reading the problem and clarifying the meaning of a stage and its area. Explain that the stage is rectangular, and that the size will be described by its dimensions. Review the following terms:

area the amount of space inside the boundary of a two-dimensional figure

drama a play that centers on a conflict or contrast of character

factoring the act or process of separating an equation, formula, cryptogram, etc., into its component parts

stage in a theater, the platform on which the actors perform

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Implementation Guide

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will define the width of the stage in terms of the length, and use each of these expressions to identify the area. They may choose to use a sketch of the stage to help them with their reasoning. They will then solve the equation to find the actual length and width of the stage.

- **SMP 6:** Attend to precision.

  Students will make sure the stage has the correct area, and also that the relationship between the length and the width holds true. When factoring, they will be sure that terms, when multiplied, actually produce the given product.

- **SMP 7:** Look for and make use of structure.

  To factor a quadratic trinomial, students look for a pair of numbers that give a particular product and sum. Then, they set each factor equal to 0 and solve to find the possible values of the variable. Students also evaluate the possible values of the variable and eliminate those that would not make sense, especially those that give a negative value to a length.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- factoring incorrectly

  Explain that the initial equation looks like it might be factored, but because it is a difference and not a product of two terms, it needs to be written as a trinomial and then factored.

- identifying two solutions for this scenario

  Explain that the variables stand for the length and width of the stage. It does not make sense for a distance to be negative, so the negative solution can be eliminated.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students struggle with writing the equation, ask, "What is it you are being asked to find?" (**Answer**: The length and width of the stage.) "How are the length and width related?" (**Answer**: The width is 6 feet less than the length.) "What expressions could represent the length and width?" (**Answer**: x and $x - 6$.) "What do you know about the area?" (**Answer**: It is 720 feet squared.) "How is area calculated?" (**Answer**: Multiply the length and width.)

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- If students struggle with solving the equation, ask, "To factor this equation, what should you look for?" (**Answer**: Two numbers that multiply to –720 and add to –6.) "What is the next step after factoring the expression?" (**Answer**: Set each factor equal to 0 and solve for *x*.) "Once you find *x*, the length, how do you find the width?" (**Answer**: Subtract 6 from the length.)
- Once students have solved the equation, ask them to think about whether a negative answer is reasonable. "What would a negative distance represent?" (**Answer***:* A distance less than 0 feet.)
- Challenge students to think about whether there is any other combination of length and width that would result in the same area. "Is there another length and width that would give an area of 720 square feet?" (**Answer**: Yes; for example, 10 feet and 72 feet.) "Why did you not find this solution?" (**Answer**: Because 10 is not 6 less than 72.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their plans for constructing the algebraic expression to represent the area of the stage. Discuss the process of identifying the terms, factors, and coefficients of the expression. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Focus on the use of precise mathematical language and clarity, specifically when referring to the area, terms, factors, and coefficients of the expression.
- Discuss with students how they can be sure their solutions are correct.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Standard form, intercept form, and vertex form are equivalent expressions written in different forms.

  While the original equation looks similar to vertex form, students should note that a binomial is not squared, and its coefficient is a variable, not a constant. The equation that they factor is in standard form, and after they factor it, they have found an intercept form where $a = 1$.

- A quadratic function in standard form can be created from the intercept form.

  Once the x is distributed in the original equation, the equation is not in standard form, and it can then be factored to find the intercept form of the equation.

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Extending the Task

- To extend the task, ask students to change the relationship between the length and the width, and to find other sizes of stages.
- Another option is for students to create a new area and then find the length and width of the stage.
- A third option is to research the golden rectangle, and find the length and width that would make a stage with this area fit this ratio.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively?" (**Answer:** I set up an equation for the given constraints.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used a quadratic equation to find the length and width of the stage.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I factored the quadratic equation correctly to find the intercepts.)
- **For SMP 7, ASK:** "How did you look for, and make use of, structure when solving this problem?" (**Answer:** I found two numbers that multiply to –720 and add to –6 to factor the equation. I also eliminated any negative numbers that represented a distance.)

Alternate Strategies or Solutions

- Students can use a graphing calculator to graph the original equation and find the zeroes of the function.
- Students could also use the quadratic formula to solve the equation $x^2 - 6x - 720 = 0$.
- Students could also graph the equation $y = x(x - 6) - 720$ and find the x-intercepts.

Technology

Students can use a graphing calculator to graph the original equation. They can also use a spreadsheet to generate a list of possible lengths and widths that multiply to the area, then search the list to find the pair that has a difference of 6.

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Problem-Based Task 3.3.3: Rocket Heights

Common Core State Standards

A–CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

A–SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

Task Overview

Focus

The path of an object that is launched takes the shape of a parabola. When a model rocket is launched, it has both a vertical and horizontal movement. After it takes off, it goes up and then comes back down.

This activity will provide practice with:

- writing a quadratic equation
- interpreting values that satisfy a quadratic function
- analyzing the solution to a quadratic equation

Introduction

This task should be used to explore or to apply the concept of creating quadratic equations in two variables. It is best to implement this text after students have learned how to find the vertex of a quadratic function. Graphing may be done using technology, including graphing calculators or online graphing programs.

Begin by reading the problem and clarifying the meaning of the launching of a model rocket. Review the following terms:

model rocket	a small rocket of the type usually used by hobbyists
launch	to send up into the air
parabolic	shaped like a parabola, or U
horizontal distance	the linear distance between two points or objects

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Some students will now be able to visualize the path of the model rocket. Encourage them to sketch the patch, labeling key points with data that they discover along the way.

- **SMP 4:** Model with mathematics.

  The quadratic function is used to model the flight path of the model rocket. It tells the height the rocket is at various times after it is launched.

- **SMP 6:** Attend to precision.

  Students may struggle to write the quadratic function using the data points. Have them identify what they know about the function before they write it. For example, the parabola will open down, so $a < 0$. If they think about what the parameters of the function mean, they are less likely to write the equation incorrectly.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- forgetting to write the quadratic equation in vertex form

  Encourage students to record all of the known quantities. Point out that they are given the vertex, which makes writing the equation in vertex form the best choice.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students are unable to identify the values of the known quantities, ask the following questions: "What does the maximum height and horizontal distance tell you?" (**Answer***:* the vertex)
  - "What point represents the vertex?" (**Answer***:* (15, 400))
  - "What do you know about where the rocket landed?" (**Answer***:* 30 feet away)
  - "What point represents the where it landed?" (**Answer***:* (30, 0))
  - "What point represents the initial launch location?" (**Answer***:* (0, 0))

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- If students struggle to write the equation in vertex form, ask the following questions:
  - "What variables represent the vertex?" (**Answer**: (h, k))
  - "What part of the equation do you not know?" (**Answer**: a)
  - "How can you solve for a?" (**Answer**: Substitute the point (30, 0) for x and y, then solve for a.)
  - "What would the quadratic equation be?" (**Answer**: $f(x) = -16/9(x - 15)^2 + 400$.)
- If students do not know how to interpret the equation, ask the following questions:
  - "What are you trying to find out?" (**Answer**: whether the rocket will clear the pine trees)
  - "Where are the trees?" (**Answer**: 2 feet from the landing spot, which is 28 feet from the launch site)
  - "How tall are the trees?" (**Answer**: 115 feet)
  - "How can you find how high will the rocket be when it is 28 feet from the launch site?" (**Answer**: substitute 28 for x in the equation)
  - "How high is the rocket when it is 28 feet from the launch site?" (**Answer**: 99 5/9 feet)
  - "How do you know if it will clear the trees?" (**Answer**: Since the trees reach higher than the flight path, the rocket will not clear them.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Discuss with the students how the quadratic function models the flight of the rocket. Have them elaborate on what else they could find using this model.

Connecting to Key Concepts

Make explicit connections to key concepts:

- A quadratic equation can be written in vertex form.

  The vertex form is $a(x - h)^2 + k$, where the vertex is (h, k).
- When written in vertex form, the vertex is (h, k), and (x, y) represents any point on the graph.

  In this case, the vertex is the highest point (15, 400) and the point that we use to find a is (30, 0).

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Extending the Task

- To extend the task, encourage students to find the height of the rocket at other distances from the launch pad.
- Have students find how far the launch pad should be moved so that the rocket does clear the trees.
- Research the formula that could tell how high the rocket was at times after the launch, instead of horizontal distances. Have students identify what information they need to use that formula and how they would be able to use it.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you demonstrate perseverance?" (**Answer:** I wrote the function to model the situation but then had to analyze it and use the data from the problem to answer the question.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** The quadratic function tells the height of the rocket at any distance from the launch site.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** When solving for the constant in the equation, I was careful to be sure the value was negative since the parabola opens down.)

Alternate Strategies or Solutions

- Students could graph the function and identify the y-coordinate when $x = 28$.
- Students could write the function in standard form, and then substitute 28 for x.

Technology

Students can use a graphing calculator to graph the path of the rocket. Then they could trace the graph to find the height when $x = 28$. They could also use the table function of the graphing calculator to find the height when $x = 28$.

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Problem-Based Task 3.3.4: Selecting a Cylinder Size

Common Core State Standard

A–CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.*★

Task Overview

Focus

Companies not only make products to sell, but they must also package them in a way that catches the consumers' eyes and if an efficient use of resources. Students will use a formula to determine the size of can for the organic root beer.

This activity will provide practice with:

- analyzing a formula
- solving an equation for a variable
- interpreting the results

Introduction

This task should be used to explore or to apply the concept of literal equations and solving quadratic equations in two or more variables and interpreting the solutions. It is best to implement this text after students have learned to isolate the quantity of interest.

Begin by reading the problem and clarifying the meaning the size of a soda can and how it relates to the volume. You may want to review the parts of a cylinder and how to use them to find the volume. Review the following terms:

recyclable after being used, it can be made into a reusable material

organic made from natural ingredients

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will have to reason abstractly as they manipulate an equation that has several variables. They will also need to reason quantitatively when they choose to eliminate the negative solution since a radius is never negative.

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- **SMP 4:** Model with mathematics.

  Help students understand that the equation they are given represents actual measurements for the can. Guide them to understand that they are given two of the measurements, the volume and height, but one more, the radius, still needs to be found. Encourage them to draw a picture of the can and label it so that they can see what the can looks like.

- **SMP 7:** Look for and make use of structure.

  As students solve a literal equation, they use the same processes they would use to solve an equation in one variable. Discuss the benefits of solving a literal equation such as making fewer computational errors in the solution process.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- using the inverse operations in the incorrect order

  Explain that the inverse operations are done in the opposite sequence to the order of operations. Have them to list the order of operations and then check them off as they complete them in reverse order.

- not finding two solutions when solving for a squared term

  Remind students that when they write the radical sign to solve an equation, they must put the ± symbol in front. Then, discuss why, even though the negative solution to this problem is not valid, they should still include the ± symbol.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students do not understand the formula:
  - Ask, "What does V stand for?" (**Answer**: volume) Ask them, "What does r stand for?" (**Answer**: radius) Finally, ask "What does h stand for?" (**Answer**: height)
  - "What is each of these on a can?" (**Answer**: The volume is how much it holds. The radius is halfway across the circular base, and the height is how tall it is.)
- If the students cannot identify the focus of the formula:
  - Ask, "What variable is already isolated?" (**Answer**: V)
  - Next, ask "What variable do you want to be isolated?" (**Answer**: r)

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- If students do not know how to solve the equation for r:
  - Ask, "What will you divide by to isolate r^2?" (**Answer**: πh) Ask,
  - "What do you then isolate r?" (**Answer**: Take the square root of both sides.) Next, ask
  - "What must you put in front of the square root symbol?" (**Answer**: ±)
- If students do not know how to solve the problem:
  - Ask, "What values can you substitute for a variable?" (**Answer**: 12.56 for V and 4 for h) "What value will you use to approximate π?" (**Answer**: 3.14) Next, ask
  - "Why can you disregard the negative solution?" (**Answer**: A radius is a distance, and distance cannot be negative.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Ask students why they may need to solve a literal equation for a different variable than is originally the focus. Ask them to explain some of the benefits of doing so. Then, discuss strategies that they may use to make sure that they rearrange it correctly at every step.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Literal equations and formulas contain equal signs.

  The formula for volume of a cylinder contains an equals sign. It can be rearranged to isolate a different variable.

- To isolate a different variable changes the focus of the formula.

  In this case, the formula for volume of a cylinder is rearranged to isolate r.

- You can apply proper algebraic properties to maintain a balance when changing the focus of an equation or formula.

  To solve the formula for r, take steps to isolate the variable by using the properties of equality, like dividing both sides by πh.

  When solving for a variable in a multi-step equation, determine which operations are applied to the variable, and undo them in reverse order. In this case, you divide and then take the square root.

  To isolate a term that is squared, take the square root of both sides of the equation.

  The final step in solving for r is to square root of both sides.

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- When you take the square root, there are two solutions, one positive and one negative.

  When you take the square root to solve for *r*, you must put a ± sign in front of it.

  If the focus is any quantity that would never be negative in real life, it is appropriate to ignore the negative.

  Since the length of the radius is a distance, it would never be negative. Therefore, you only consider the positive solution.

Extending the Task

To extend the task, ask students to determine the radius for different heights, to determine the radius for different volumes, and to solve for a height given a radius and a volume.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I solved a literal equation with several variables by isolating the desired variable.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I found the radius that a can with a given height and volume would have.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** The formula represents the size of a can of organic root beer.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I used the same procedure to isolate the variable as I would when solving a multi-step equation.)

Alternate Strategies or Solutions

Students could substitute the values for *V* and *h* and then use the process for solving an equation to find the value of *r*.

Technology

Students could use a spreadsheet to find the radius by using a formula with different radii. They would look for the radius that gives the correct volume. They will want to use a calculator to calculate the radius once the variable has been isolated and the values have been substituted. A calculator can also be helpful when finding a square root.

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Problem-Based Task 3.4.1: Making a Complex Expression Simple

Common Core State Standard

N–CN.8 (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x – 2i)$.

Task Overview

Focus

How can the Distributive Property be used to simplify a complex expression? What is necessary to show that an equation is an identity? In this task, students will factor and simplify an expression using the Distributive Property, then they will solve an equation involving the simplified expression in order to verify that a polynomial identity involving complex numbers is created.

This activity will provide practice with:

- applying the Distributive Property and properties of exponents
- factoring an expression
- simplifying an expression involving complex numbers
- substituting quantities into an equation
- interpreting results of a solution of an equation
- explaining the reasons an equation is an identity

Introduction

This task should be used to explore or to apply the skill of extending polynomial identities to complex numbers. Students should already be familiar with factoring and multiplying polynomial expressions, as well as applying the Distributive Property and properties of exponents. They should also be familiar with simplifying expressions involving radicals.

Begin by reading the problem and clarifying the meaning of *Distributive Property* and *identity*.

Distributive Property	property that states that multiplying a sum by a number is the same as multiplying each addend by the number and then adding the products. The Distributive Property says that if a, b, and c are real numbers, then $a \cdot (b + c) = a \cdot b + a \cdot c$.
identity	an equation that is true regardless of what values are chosen for the variables

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they make sense of the goal of the task by first recalling prior knowledge of factoring and simplifying expressions that do not contain complex numbers. They will then make the connection that the same procedure and process can be used when algebraically manipulating an expression when it contains complex numbers. They will apply their results in order to identify that the completed equation is a polynomial identity. They will reason quantitatively as they perform the algebraic steps using factoring, distributing, and properties of exponents in order to reach the desired final equation. They will also reason quantitatively as they substitute values into the identity to verify its validity.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments about the algebraic processes applied to complete the equation. They will describe the procedures of factoring and distributing that were performed on the original complex expression. When presenting their arguments to others, they will explain their results and validate that the completed equation is an identity, by using examples and counterexamples of possible values for a.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure and procedure of factoring the given expression, and they will apply the steps for the Distributive Property. They will use their simplified expression and complete the equation, then they will analyze their results and draw conclusions about the equation, as they discover that a polynomial identity has been created. They will also justify this identity by using the procedure of substituting in values for the variable, a, in order to verify the definition of an identity.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly distributing the first factor over the second factor

  Review the method for multiplying two binomials. Remind students of the process, and have them practice with expressions that do not contain complex numbers.

- forgetting that the value of i^2 is –1

  Remind students that the imaginary unit i represents the non-real value $i = \sqrt{-1}$, and that $i^2 = -1$. Have students write this property of imaginary numbers at the top of their papers.

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- misinterpreting the results of the equation

  Remind students that an identity is an equation that is true regardless of what values are chosen for the variables. Have students perform several trials of values in order to draw the conclusion that the equation is an identity.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "What are some examples of identities that you have used before?" (**Answer**: Answers will vary. Some common examples include binomial identities, such as the difference of squares, $(a + b)(a - b) = a^2 - b^2$, or the square of a binomial, $(a + b)^2 = a^2 + 2ab + b^2$. Students may also list examples such as $x + 1 = 1 + x$.)
- Before students begin the task, discuss the term "identity." Ask them, "How can you prove that an equation is an identity?" (**Answer**: An equation is an identity if it is true for all values that are substituted into the equation. Substitute several values, including positive and negative rational numbers, to test an equation to see if it is an identity.)
- Before students begin the task, ask them, "What is the process for multiplying two binomial expressions?" (**Answer**: When two binomial expressions are multiplied, the first terms of each expression are multiplied, then the first term of the first expression is multiplied by the last term of the second expression. Next, the second term of the first expression is multiplied by the first term of the second expression, and the last terms of each expression are multiplied. Finally, like terms are combined in order to simplify the resulting expression.)
- Ask students, "What property is used when multiplying two binomial expressions?" (**Answer:** The Distributive Property is used when multiplying two binomial expressions, as each term in the first expression is multiplied by each term in the second expression.)
- As students begin the task, ask them, "What properties of exponents can you use to "break down" the original expression, $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$?" (**Answer**: The original expression can be broken down by using the Product of Powers Property, which says that exponents can be added when they have the same base. So, $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 \bullet \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^1$.)

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- As students are working on the task, ask them, "Which property of complex numbers is used to simplify part of the product of the binomial expressions? Which factors does this involve?" (**Answer**: When the two factors $\frac{\sqrt{3}}{2}i$ are multiplied, the result is $\frac{3}{4}i^2$. The property $i^2 = -1$ can be used to simplify the term to $-\frac{3}{4}$.)
- As students are working through the task, ask them, "Why is the Distributive Property used twice in this task?" (**Answer**: The Distributive Property is used twice because the original expression is a cubic expression, so it can be broken down to the product of two factors multiplied by one factor. The resulting expression from the two factors can be applied to the remaining factor, and the Distributive Property can be used again to simplify the expression.)
- Once students have determined that the original expression can be simplified into a value of –1, ask them, "How can you use the simplified value to verify that the completed equation is indeed an identity?" (**Answer**: Because the original expression, $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3$, simplifies to –1, the resulting equation is $-1 \cdot a = -a$. Any value can be substituted in for a and a true statement will occur.)
- Ask students, "What happens to the identity when a negative value for a is substituted in?" (**Answer**: If a negative value is substituted in for a, such as –2, then the equation is $-1 \cdot (-2) = -(-2)$; $2 = 2$. The result is a true statement with positive numbers.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their strategies and processes for factoring a complex expression and applying the Distributive Property to multiply two binomial expressions. Also ask for volunteers to explain how like terms were combined, specifically with the terms involving complex numbers. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and how they were able to draw conclusions, emphasizing their justification that the completed equation resulted in a polynomial identity with complex numbers. Focus on the use of precise mathematical language and clarity, specifically when referring to the terms "complex" and "identity."

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Implementation Guide

Connecting to Key Concepts

Make explicit connections to key concepts:

- An **identity** is an equation that is true regardless of what values are chosen for the variables.

  In this task, students will complete an equation that will create a polynomial identity involving complex numbers. Students will verify that the equation is an identity by substituting in several values.

- The **imaginary unit *i*** represents the non-real value $i = \sqrt{-1}$. *i* is the number whose square is –1. We define *i* so that $i = \sqrt{-1}$ and $i^2 = -1$.

  Students will apply the property $i^2 = -1$ when simplifying and combining like terms in the multiplication of the factors of the original expressions.

- A **complex number** is a number with a real component and an imaginary component. Complex numbers can be written in the form $a + bi$, where *a* and *b* are real numbers and *i* is the imaginary unit. For example, $5 + 3i$ is a complex number: 5 is the real component, and $3i$ is the imaginary component.

  Students will identify the format of the original expression, $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$, as a complex number, where *a* is the real number $\frac{1}{2}$ and *bi* is the imaginary component $\frac{\sqrt{3}}{2}i$.

Extending the Task

- To extend the task, ask students to work together to create their own complex expression, similar to the one given in the task but with different values for $a + bi$. Ask them to explain verbally to their partners the process of factoring the original expression and then applying the Distributive Property. Also ask them to explain how the equation that is created is a polynomial identity. Ask for volunteers to share their problems and processes.
- Another option for extending the task is to ask students to create a variety of polynomial identities with and without complex numbers. Ask them to create a table of values, using a variety of positive and negative rational numbers, to prove their equations are identities. Ask for volunteers to show examples of values that prove their equations result in identities.
- Another option for extending the task is to ask students to work through the task using the same complex expression, but with a power of 4 instead of 3, i.e. $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^4$. Before they simplify the expression, ask students to make predictions about whether or not they think they will get a result of a real number, and whether or not the completed equation will create an identity. Ask for volunteers to share their thoughts and ideas before working through the processes.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I made sense of the goal of the task by first recalling prior knowledge of factoring and simplifying expressions. Then I used the same procedure and processes on the complex expression. I applied my results in order to identify that the completed equation is a polynomial identity.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I reasoned quantitatively when I performed the algebraic steps using factoring and properties of exponents in order to reach the final equation. I also substituted values into the identity to verify its validity.)
- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining the algebraic processes applied to complete the equation. I described the procedures of factoring and distributing that were performed on the original complex expression. I explained all of my results and validated that the completed equation is an identity, by using examples and counterexamples of possible values for a.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** When solving this problem, I made use of structure by applying the same procedure used for factoring and multiplying non-complex binomial expressions, as well as applying the Distributive Property. Then I applied the structure of combining like terms and simplifying.)

Alternate Strategies or Solutions

At the onset of the task, students may choose to use the polynomial identity for a cube of a binomial, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, as opposed to the procedure used in the task of breaking the original expression into factors. Encourage them to check their work by also working through the procedure outlined in the task.

Technology

No technology is needed for this task.

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Implementation Guide

Problem-Based Task 3.4.2: All Roads Lead to the Same Destination

Common Core State Standards

N–CN.7 Solve quadratic equations with real coefficients that have complex solutions.

N–CN.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Task Overview

Focus

How can a quadratic equation be solved with multiple methods? What properties can be applied when factoring a quadratic equation? In this task, students will solve a quadratic equation by using the quadratic formula, using the Square Root Property, and factoring. They will verify that all three methods produce the same solutions.

This activity will provide practice with:

- identifying coefficient values in a quadratic equation
- substituting values into a formula
- applying properties of radicals and properties of square roots
- factoring quadratic equations
- simplifying expressions containing complex and non-complex terms
- interpreting and applying identities involving complex numbers

Introduction

This task should be used to explore or apply the skills of solving quadratic equations that have complex solutions and verifying the Fundamental Theorem of Algebra. Students should already be familiar with factoring and solving quadratic equations, as well as factoring the sum of two squares using imaginary numbers. They should also be familiar with simplifying expressions involving radicals.

Begin by reading the problem and clarifying the meaning of *quadratic formula*, *Square Root Property*, and *Zero Product Property*.

quadratic formula formula which states that if $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Square Root Property property which states that if $x^2 = a$, then $x = \pm\sqrt{a}$

Zero Product Property property which states that if $(a + bi)(a - bi) = 0$, then either $a + bi = 0$ or $a - bi = 0$

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they make sense of the goal of the task by recalling prior knowledge of solving quadratic equations involving non-complex numbers by using the quadratic formula, using properties of square roots, and factoring. They will make the connection that the same procedure and process can be used when algebraically manipulating an expression when it contains complex numbers. They will use their prior knowledge as they work through the three methods of solving a quadratic equation. They will reason quantitatively as they perform the algebraic steps necessary to simplify the quadratic formula, as well as the steps performed in factoring. They will also reason quantitatively as they simplify the radical expressions and rewrite solutions into the proper format.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments about the solutions found when all three solution methods are used. They will describe the procedures and steps used for each method, specifically those used in the quadratic formula, in the Square Root Property, and when factoring after rewriting the equation by creating a sum of two squares. When presenting their arguments to others, students will explain their results and validate that each method resulted in the same solutions, as they illustrate how each method produced solutions that can be displayed in the same formats.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure and procedures involved in solving an equation using the quadratic formula. They will also apply the structure of factoring a quadratic equation. Students will justify their results by showing that each solution method resulted in the same solutions.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- not recognizing that the value of b in the equation is 0

  Have students write the quadratic formula at the top of their papers. Remind them that if a term is missing, the value for the coefficient of that term is 0.

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- not considering both the positive and negative values for solutions of the equation

  Remind students that because the equation in the task is a quadratic equation, there will be two solutions. Also remind them of the format of the quadratic formula, and the Square Root Property, in that the "±" indicates both a positive and negative value for the solution.

- incorrectly applying the properties of complex numbers

  Remind students that when they see a negative value under the radical sign, for any positive real number a, $\sqrt{-a} = i\sqrt{a}$.

- incorrectly rationalizing the denominator

  Work through an example or two to remind students of how to eliminate a radical in the denominator.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them to make predictions about the different solution methods. Ask them, "Which of the three methods do you think will require the most time, and why?" (**Answer**: Answers will vary. Some students may predict that the quadratic formula will require the most time because they might recall that there are several steps involved in the formula once the values are substituted in. Other students may recognize that factoring will take the most time because the equation will need to be rewritten as the sum of two squares first.)

- As students begin to work through the first part of the task with the quadratic formula, ask them, "How can the quadratic formula be used to solve the equation $3x^2 + 5 = 0$, when there are only two terms present on the left side of the equation? What are the values for the coefficients a, b, and c?" (**Answer**: Although there is no bx term in the equation, a value of 0 can be used to represent b in the formula. The value of a is 3, the value of b is 0, and the value of c is 5.)

- As students are working through the steps using the quadratic formula, discuss a step that was left out of the steps in the task. Ask them, "How did the numerator in the expression $x = \frac{\pm i\sqrt{60}}{6}$ simplify to $x = \frac{\pm 2i\sqrt{15}}{6}$?" (**Answer**: The $\sqrt{60}$ part in the numerator was broken down into two factors, one in which is a perfect square: $\sqrt{60} = \sqrt{4 \cdot 15} = \sqrt{4} \cdot \sqrt{15} = 2\sqrt{15}$. Therefore, the numerator $\pm i\sqrt{60} = \pm 2i\sqrt{15}$.)

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- As students are working through the steps using the Square Root Property (if $x^2 = a$, then $x = \pm\sqrt{a}$), review and discuss the term "rationalizing the denominator." Ask them, "Why is it necessary to rationalize the denominator when finding the solutions using this property of square roots, and how is the process of rationalizing the denominator accomplished?" (**Answer:** When the property of square roots is applied, the solutions are left in the format where there is a radical in the denominator: $x = \pm i\frac{\sqrt{5}}{\sqrt{3}}$. In order to eliminate the radical in the denominator, both the numerator and denominator are multiplied by $\sqrt{3}$, and then the expression is simplified: $x = \pm i\frac{\sqrt{5} \bullet \sqrt{3}}{\sqrt{3} \bullet \sqrt{3}} = \pm i\frac{\sqrt{15}}{\sqrt{9}} = \pm i\frac{\sqrt{15}}{3}$.)
- If the students are having difficulty understanding how $3x^2 + 5 = 0$ can be rewritten as the sum of two squares, ask them, "How can each term be rewritten as a factor squared, and then how can you check to be certain each term is rewritten correctly?" (**Answer:** To rewrite each term as a factor squared, think of what makes each part of the term squared. The 3 in $3x^2$ can be rewritten as a factor squared by taking the square root of it, $\sqrt{3}$. The square root of x^2 is x, so this term rewritten as a factor squared is $\left(x\sqrt{3}\right)^2$. Then, the square root of 5 is $\sqrt{5}$, so this term rewritten as a factor squared is $\left(\sqrt{5}\right)^2$. To be certain each term is rewritten correctly, square each term to verify the products are the original terms: $\left(x\sqrt{3}\right)^2 = \left(x\sqrt{3}\right) \bullet \left(x\sqrt{3}\right) = 3x^2$, and $\left(\sqrt{5}\right)^2 = \left(\sqrt{5}\right) \bullet \left(\sqrt{5}\right) = \sqrt{25} = 5$.)
- As students are preparing to apply the identity $a^2 + b^2 = (a + bi)(a - bi)$ to the equation, which has been rewritten as the sum of two squares, $\left(x\sqrt{3}\right)^2 + \left(\sqrt{5}\right)^2 = 0$, ask them, "What are the values of a and b that are substituted into the identity $a^2 + b^2 = (a + bi)(a - bi)$?" (**Answer:** The value of a is $x\sqrt{3}$, and the value of b is $\sqrt{5}$.)
- Once students have completed the steps for solving the equation using all three methods in the task, ask them, "How can you verify that the solutions you found are correct?" (**Answer:** The solutions can be verified by substituting them back into the original equation to ensure that the true statement of 0 = 0 is achieved.)

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- Ask students, "How is the Fundamental Theorem of Algebra applied to this task?" (**Answer:** The Fundamental Theorem of Algebra states that if $P(x)$ is a polynomial function of degree $n \geq 1$ with complex coefficients, then the related equation $P(x) = 0$ has at least one complex solution (root). Because all real numbers are complex numbers, this theorem applies to the original equation in this task. The solutions for the equation are complex solutions.
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their processes for each method of solving the equation, focusing on the properties and procedures applied from step to step in order to reach the solutions. Also ask for volunteers to explain how they were able to ensure that the solutions were written in the same format each time. Ask them to explain rationalizing the denominator, as well as why the solutions are considered complex solutions, and how this demonstrates the Fundamental Theorem of Algebra. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and how they were able to algebraically manipulate the format of the solutions, emphasizing their justification that all three methods resulted in the same complex solutions for the original equation. Focus on the use of precise mathematical language and clarity, specifically when referring to the terminology "rationalizing the denominator" and "complex solutions."

Connecting to Key Concepts

Make explicit connections to key concepts:

- Remember that a quadratic equation is an equation that can be written in the form $ax^2 + bx + c = 0$, where $a \neq 0$.

  In this task, students will identify the given equation, $3x^2 + 5 = 0$, as a quadratic equation because of the form. They will recognize that although the bx term is missing, a value of 0 can be used for this term.

- A quadratic function is a polynomial function of degree 2, so by the Corollary to the Fundamental Theorem of Algebra, a quadratic function has exactly 2 complex solutions (roots).

  Students will apply three methods to solve a quadratic equation, and they will verify that the two solutions to the equation are complex solutions.

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- The quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, is used to find the complex solutions.

  Students will apply the quadratic formula to find the two complex solutions for the original equation, $3x^2 + 5 = 0$.

- The following property involving square roots is important and useful to remember when finding solutions to quadratic equations: if $x^2 = a$, then $x = \pm\sqrt{a}$.

  Students will apply the Square Root Property as one of the solution methods for finding the two solutions for $3x^2 + 5 = 0$.

Extending the Task

- To extend the task, ask students to work together with a partner to create their own quadratic equations. Have the pairs use all three methods to solve these new equations. Ask them to create an equation which has the *bx* term missing, as in the original task, then ask them to create an equation that has all three terms. Have them switch equations with their partners, so that each person is solving new equations. Ask them to explain verbally to their partners the procedures used when working through all three solution methods. Ask for volunteers to share their problems and processes.
- Another option for extending the task is to ask students to create quadratic equations that will *not* have complex solutions. Discuss which components of the original equation must have certain sign values that will result in real solutions, instead of complex solutions. Ask for volunteers to show examples of their equations and explain their thought processes when creating them.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I made sense of the goal of the task by recalling prior knowledge of solving quadratic equations involving non-complex numbers. I did this by using the quadratic formula, using properties of square roots, and factoring. I recognized that the same procedure and process can be used when algebraically manipulating an expression when it contains complex numbers.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I reasoned quantitatively when I performed the algebraic steps necessary to simplify the quadratic formula, as well as the steps performed in factoring. I also used quantitative reasoning when I simplified the radical expressions and rewrote solutions into the proper format.)

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- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining the procedures and steps used when finding the solutions to the original equation, using all three methods. I also explained my results and verified that each method resulted in the same solutions.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** When solving this problem, I made use of structure by applying the procedures and steps involved in solving an equation using the quadratic formula. I also applied structure by simplifying and factoring the equation.)

Alternate Strategies or Solutions

Students may choose to solve the equation using a different order of the three solution methods. They may also choose to organize their work in a different manner than shown in the task. Encourage them to construct whichever organizational system works best for them. As an additional aid, students may choose to graph the equation using a graphing calculator in order to have a visual representation of the quadratic equation.

Technology

No technology is needed for this task, although students may want to use graphing calculators in order to graph the equation.

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Implementation Guide

Problem-Based Task 3.5.1: Golden Rectangles and the Golden Ratio

Common Core State Standard

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

Task Overview

Focus

How can a rational equation be created to represent the ratio of corresponding sides of two rectangles? What properties of quadrilaterals can be applied when determining missing side lengths of a quadrilateral? In this task, students will create and solve a rational equation that represents the ratio of lengths of sides of two rectangles, and they will use the solution of this equation to determine the golden ratio.

This activity will provide practice with:

- creating ratios and proportions
- identifying corresponding sides of rectangles
- substituting values representing side lengths to create a proportion
- simplifying rational expressions
- solving quadratic equations using the quadratic formula
- analyzing solutions of a quadratic equation in the context of the problem
- converting an expression containing a radical to a decimal format

Introduction

This task should be used to explore or to apply the skill of creating rational equations in one variable and using them to solve problems. Students should already be familiar with representing comparisons of quantities as ratios. They should also be familiar with simplifying expressions involving radicals and applying the quadratic formula to solve quadratic equations.

Begin by reading the problem and reviewing and clarifying the meaning of the terms *equivalent ratios*, *golden ratio*, *golden rectangle*, *proportion*, and *similar polygons*.

equivalent ratios	two ratios that are equal
golden ratio	the ratio of the length to width in a golden rectangle
golden rectangle	a rectangle that can be divided into a square and a smaller rectangle that is similar to the original rectangle
proportion	the relationship of one thing to another in terms of size, number, or amount; the ratio
similar polygons	two polygons that have all corresponding angles congruent and all corresponding sides proportional

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they make sense of the goal of the task by recalling prior knowledge of ratios, proportions, properties of quadrilaterals, and solving quadratic equations using the quadratic formula. They will represent the side lengths of two rectangles algebraically, and they will use these lengths to create a proportion. After simplifying the proportion and creating a quadratic equation, they will analyze the solutions of the equation and determine which solution is correct in the context of the problems. They will reason quantitatively as they perform the algebraic steps necessary to simplify the proportion, as well as when they work through the steps of the quadratic formula, as they determine the solutions to the equation.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled by using proportional reasoning involving two similar rectangles. They will sketch the drawings of the two rectangles and use algebraic expressions to represent the lengths of the sides. They will draw conclusions about the ratio of the length to the width of the golden rectangle after solving a quadratic equation created as a result of simplifying the proportion representing the side lengths of both rectangles.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure and procedures involved in solving an equation using the quadratic formula. They will also apply the structure of a proportion by multiplying both sides of the proportion by the LCD. Students will recognize the structure of the side lengths of rectangle as they determine the algebraic expressions representing the lengths.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly matching up the corresponding sides of the newly created rectangle to the original rectangle

  Remind students that the points representing the sides of a rectangle must correspond with the "matching" side of the other rectangle. Have students draw both rectangles with the correct orientation on their papers. Discuss which vertices match on both rectangles.

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- choosing the wrong values for sides *BC* and *AP* of the rectangles

  Review with students the properties of squares and rectangles. Remind them that all four sides of a square have the same length, and that opposite sides of a rectangle have the same length.

- incorrectly identifying the negative solution of the equation as the correct solution for the length of the rectangle

  Remind students to consider the context of the problem and recall that side lengths cannot be negative values.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, review the concept of ratios with them. Ask them, "What are some common ratios that are used in everyday life?" (**Answer**: Answers will vary, but examples may include miles per gallon, miles per hour, words per minute in typing, earned run average, etc.)
- Before students begin the task, ask them, "What is one of the most well-known ratios involving circles? Explain the ratio." (**Answer:** The ratio π (*pi*) is one of the most well-known ratios involving circles. Pi has a value of 3.14 (rounded to the nearest hundredth), and it is the ratio of the length of the circumference of any circle to the length of the diameter of that circle. In other words, the circumference of any circle is always 3.14 (rounded) times larger than the diameter of the circle.)
- As students begin the task, ask them, "What is a way to visually examine the proportional sides of the newly created rectangle to the original rectangle?" (**Answer**: A diagram showing the original rectangle separate from the newly created rectangle can be drawn. The new rectangle will have to be rotated 90 degrees clockwise in order for the corresponding sides to match.)
- If students are having difficulty determining which sides of the new rectangle are proportional to the sides of the original rectangle, remind students that the names of the points representing each corner (vertex) of the sides in the original triangle must match the corresponding points representing the new triangle. Ask them, "List all of the matching corresponding sides of the original rectangle to the new rectangle." (**Answer:** Side *AB* of the original matches with side *BC* of the new, *BC* of the original matches with *CQ* of the new, *CD* of the original matches with *QP* of the new, and *AD* of the original matches with *BP* of the new: $\overline{AB}$ is proportional to $\overline{BC}$, $\overline{BC}$ is proportional to $\overline{CQ}$, $\overline{CD}$ is proportional to $\overline{QP}$, and $\overline{AD}$ is proportional to $\overline{BP}$.)

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- As students are working through the task, ask them, "What properties of squares and rectangles will be applied in this task?" (**Answer:** The properties of the relationship between the side lengths of a square (four congruent sides) and the side lengths of a rectangle (opposite sides are congruent) will be applied in this task.)
- As students are working to find the value of x once they substitute the known values for the side lengths ($\frac{AB}{BC} = \frac{AD}{BP}$; $\frac{(x)}{(1)} = \frac{(1)}{(x-1)}$), ask them, "What is another method that can be used to simplify the proportion, instead of multiplying by the LCD to achieve the same resulting quadratic equation of $x^2 - x - 1 = 0$?" (**Answer:** Cross multiplication can be used to simplify the proportion: $(x)(x-1) = (1)(1)$; $x^2 - x = 1$; $x^2 - x - 1 = 0$.)
- Once students have solved the quadratic equation using the quadratic formula, ask them, "There are two solutions to the equation, so how can you determine which solution is the correct one in the context of this task?" (**Answer:** Solving the equation results in both a positive and a negative value. Because the context of the task involves the length of a side of a rectangle, length cannot be negative. Therefore, the negative solution is discarded, and the positive one is applied instead.)
- Once students have found the value of the length of the rectangle, which is the golden ratio, ask them, "How can you test this value to ensure that it is indeed the golden ratio for any rectangle?" (**Answer:** One way to test the value is to create other rectangles and measure the sides, such that a square is created inside of the original rectangle. Once all the sides are drawn and measured as precisely as possible, then the ratio of the length to the width can be calculated, which should be as close to 1.618 as possible.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their thoughts and processes for setting up the proportion for the similar rectangles, as well as how they determined how to represent the side lengths of the rectangles algebraically. Also ask for volunteers to explain how they analyzed the two solutions of the quadratic equation and determined which one was applicable in the context of the problem. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and how they were able to create the proportion and then create a resulting quadratic equation. Ask them to explain their strategies for choosing the side lengths, and ultimately how they calculated the golden ratio. Focus on the use of precise mathematical language and clarity, specifically when referring to ratios, proportions, and similarity.

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Connecting to Key Concepts

Make explicit connections to key concepts:

- A **ratio** is a relation between two quantities. A ratio can be expressed in words, as fractions, as decimals, or as a percentage.

  In this task, students will determine the ratio between the length and width of a golden rectangle. They will express this ratio in decimal format.

- A **rational equation** is an equation that includes the ratio of two rational expressions, in which a variable appears in the denominator of at least one rational expression.

  Students will create a rational equation that represents the ratio of the sides of the original rectangle and the new rectangle. The variable x appears in the denominator of the second rational expression: $\frac{(x)}{(1)} = \frac{(1)}{(x-1)}$.

- Solutions to quadratic functions of the form $ax^2 + bx + c = 0$ can be determined using the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

  Students will solve the quadratic equation, $x^2 - x - 1 = 0$, using the quadratic formula. They will also analyze the two solutions to determine which one is appropriate for the given context of the task.

- When solving rational equations, restrictions on the variable must be considered.

  Students will analyze the two given solutions of the quadratic equation and determine that only the positive solution is correct, because the variable, x, cannot be negative in the context of this task since it represents the length of the rectangle.

Extending the Task

- To extend the task, ask students to draw their own rectangle. Ask them to carefully measure the side lengths, record the values, and draw a segment within the rectangle that will create a square and an additional rectangle. Next, have students create proportions for the corresponding sides of the two rectangles, using the exact values for each side length. Ask them to calculate the golden ratio, then ask for volunteers to share their rectangles and calculations. Discuss how measurement errors can lead to incorrect values for the golden ratio.

- Another option for extending the task is to ask students to work with a partner and have each person draw a rectangle. Ask them to have their partners follow the same steps as given in the task, and ask them to write down all calculations necessary to find the golden ratio for their rectangle. Ask for volunteers to share their rectangles and explain their thought processes when calculating the golden ratio.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I recalled and applied my prior knowledge of ratios, proportions, properties of quadrilaterals, and solving quadratic equations using the quadratic formula. I also reasoned that the side lengths of the two similar rectangles could be represented with algebraic expressions that were used in a proportion. Then I solved the proportion by creating a quadratic equation.) "Which of your strategies represents quantitative reasoning?" (**Answer:** I used quantitative reasoning as I worked through the steps of the quadratic formula in order to determine the solutions of the equation.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I recognized that this scenario could be modeled using proportional reasoning involving two similar rectangles. I sketched the drawings of the two rectangles and used algebraic expressions to represent the lengths of the sides. I also drew conclusions about the ratio of the length to the width of the golden rectangle after solving a quadratic equation created as a result of simplifying the proportion representing the side lengths of both rectangles.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** When solving this problem, I made use of structure by applying the procedures and steps involved in solving an equation using the quadratic formula. I also applied the structure of a proportion by multiplying both sides of the proportion by the LCD in order to simplify it.)

Alternate Strategies or Solutions

- Students may choose to draw the original rectangle on a coordinate plane, in order to visually recognize the side lengths with whole number units. Remind them to ensure that the line segment drawn in the rectangle creates a square, and remind them to check the four sides of the square to verify that all four sides are equal in length.
- Students may choose to cross multiply instead of multiplying by the LCD in order to simplify the proportion representing the side lengths. Encourage students to apply both methods and verify that their results are the same with each method. Students may also choose to solve the quadratic equation using another method, such as completing the square. Encourage students to ensure their results are the same regardless of the method chosen.

Technology

Students can use scientific calculators in order to determine the decimal value of the golden ratio.

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Implementation Guide

Problem-Based Task 3.5.2: Lacrosse Stats

Common Core State Standard

A–CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

Task Overview

Focus

How can a rational function be created to represent the ratio of two quantities? How is the corresponding rational equation graphed, specifically with asymptotes and appropriate domain values? In this task, students will apply a rational function representing the save percentage of a goaltender on a lacrosse team, graph this function on the coordinate plane, and analyze its behavior.

This activity will provide practice with:

- interpreting verbal information and translating it into an algebraic function
- identifying asymptotes of a function
- writing and solving equations for values of asymptotes
- plotting points on the coordinate plane
- graphing functions on the coordinate plane
- examining a function's behavior from a graph
- analyzing a function's behavior in the context of the problem

Introduction

This task should be used to explore or to apply the skill of creating equations in two or more variables to represent relationships between quantities, as well as graphing equations on coordinate axes with labels and scales. Students should already be familiar with solving and graphing rational equations. They should also be familiar with evaluating functions.

Begin by reading the problem and clarifying the meaning of *asymptote*, *end behavior*, *horizontal asymptote*, and *vertical asymptote*.

asymptote	a line that a function gets closer and closer to as one of the variables increases or decreases without bound
end behavior	the behavior of the graph as x approaches positive or negative infinity
horizontal asymptote	the line $y = b$ which represents if $f(x)$ either increases or decreases without bound as x gets closer to b
vertical asymptote	the line $x = a$ which represents if $f(x)$ either increases or decreases without bound as x gets closer to a

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to apply a rational function which represents a real-world scenario and graph the function on the coordinate plane. They will apply their knowledge of graphing functions, as they plot selected points in order to achieve the shape of the graph. They will plan a solution pathway in order to examine and analyze the behavior of the graph of the function. They will rely on the "picture" of the graph to draw conclusions and make sense of the function representing the save ratio of the lacrosse goaltender.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled by using a visual representation of the scenario, in the form of graphing a function. They will sketch the graph of the function representing a goaltender's save ratio, and they will identify specific features of the graph. From these features, they will draw conclusions about the end behavior of the graph in the context of the problem situation.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure and procedures involved in choosing domain values for a function, and then determining the corresponding range values. They will follow an algebraic process of substituting in values for the domain to get these resulting range values. They will also apply the structure of finding the asymptotes for the graph. They will analyze the graph and make predictions about the shape of the curve.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- Reversing the direction of the lines representing the horizontal and vertical asymptotes

  Have students write "y = # is horizontal" and "x = # is vertical" at the top of their papers. Review that horizontal asymptotes are represented by the equation $y = b$, and vertical asymptotes are represented by the equation $x = a$.

- sketching a rational function graph incorrectly, showing it "veer away" from its asymptote

  Remind students that it is necessary to plot several points in order to accurately determine the graph of the curve. Remind them to carefully plot each point and then extend the graph enough so that it becomes closer and closer to the asymptotes without touching them.

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- incorrectly identifying the relevant part of the graph

  Remind students to consider the context of the problem, such that the number of saves, x, must be a whole number and must be positive. Therefore, the part of the graph in which x is less than 0 should not be considered when analyzing the graph.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, review ratios and the different formats in which ratios can be written. Ask them, "What are the different ways in which ratios can be written? Give an example." (**Answer:** A ratio can be written with a colon in between quantities, it can be written with the word "to" in between quantities, and it can be written as a fraction. Ratios can also be expressed as percentages or decimals. For example, the ratio of the quantity 3 to 4 can be written in the following formats: 3 to 4; 3:4; $\frac{3}{4}$; 0.75; 75%.
- Before students begin the task, ask them, "What other types of percentages are used in sports to represent a player's success? List some examples." **Answer:** Answers will vary, but examples may include a player's free throw percentage in basketball, a baseball player's batting average, a pitcher's earned run average, a football quarterback's rating, etc.)
- If students are having difficulty understanding the concepts of asymptotes, discuss how a function can get closer and closer to an asymptote and never touch it. Ask them, "How is it numerically possible for a function to never reach a specific asymptote value? Give an example using a decimal representation." (**Answer**: Since there can be an infinite number of place values, a decimal number can continue on indefinitely without ever reaching the next greatest whole number. For example, if a function had an asymptote of $y = 2$, then the values for y could be 1.9, 1.91, 1.911, 1.9111, 1.91111, etc.)
- As students are determining the asymptotes for the graph, ask them, "Why is there not a vertical asymptote for the function $f(x) = \frac{1}{x^2 + 1}$?" (**Answer**: No matter what negative or positive value is substituted in for x, that value will be squared, and it will always be positive. Therefore, adding one to it will keep it a positive value. There is not a real number that will make the denominator equal to zero for this function.)

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- As students are working through the task and analyzing the function, $f(x) = \frac{x+2}{x+5}$, ask them, "How do you know that the value of $f(x)$ will never be equal to or greater than 1?" (**Answer**: The value of the function will never be equal to or greater than one because no matter what values you substitute in the function for x, positive or negative, the constant in the numerator, 2, is always less than the constant in the denominator, 5. Testing several positive and negative x-values will verify this.)
- As students are working through the task, ask them, "In the context of the problem, why does x have to be a whole, positive number?" (**Answer**: In the context of the problem, the number of consecutive saves that the goaltender makes is a whole number, as part-saves are not considered. Also, only a positive value is associated with saves, so x has to be a whole, positive number.)
- As students are analyzing the graph of the function and drawing conclusions about its end behavior, ask them, "Even if Laytona were to indefinitely make consecutive saves, why won't her save percentage ever get to 100%?" (**Answer:** Since Tonya's beginning save ratio was less than 100%, it will always remain less than 100%. It can get closer and closer to 100%, but it will never reach that value or go over than value.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their thoughts and processes for choosing points to graph, as well as their processes for identifying the vertical and horizontal asymptotes for the graph. Also ask for volunteers to explain how they analyzed the behavior of the graph and determined which part of the graph was relevant to the context of the problem. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and how they were able to identify the asymptotes and then graph the chosen points. Ask them to explain their conclusions about what the graph showed in terms of Latonya's future save percentage. Focus on the use of precise mathematical language and clarity, specifically when referring to rational functions, asymptotes, and end behavior.

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Connecting to Key Concepts

Make explicit connections to key concepts:

- A **ratio** is a relation between two quantities. A ratio can be expressed in words, fractions, decimals, or as a percentage.

  In this task, students will write the ratio between two quantities as a fraction and a percent. They will also recognize the relationship between the two quantities as described verbally.

- A **rational function** is a function that can be written in the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

  Sometimes an equation representing a problem situation is naturally related to a rational function. Students will recognize that a function representing the goaltender's save ratio can be used to solve for values of consecutive goal shots saved.

- The **end behavior** of a graph is the behavior of the graph as x approaches positive or negative infinity.

  Students will examine and interpret the behavior of the graph of the function as it approaches the horizontal and vertical asymptotes.

- An **asymptote** is a line that a function gets closer and closer to as one of the variables increases or decreases without bound.

  Students will identify the equations for the asymptotes of the function in the task.

- To graph a rational function, first find any asymptotes and graph using a dashed line.

  Students will identify the vertical and horizontal asymptotes, and then they will graph these using dashed lines.

- Choose a few more x-values and calculate the corresponding y-values. Be sure to also choose values to the left and right of the asymptotes.

  After graphing the asymptotes, students will choose and plot several points to the left and right of the asymptotes.

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Extending the Task

- To extend the task, ask students to create their own scenario involving a sports ratio or percentage. For example, a student could create a problem involving a basketball player's free throw percentage. Ask them to use this percentage and create a function representing this percentage, and then graph the function in the same manner as was done in the original task. If students are having difficulty beginning the extended task, discuss how to convert a free throw percentage to a ratio and then create a corresponding function: for example, 75% can be converted to 3 out of 4, and the corresponding function would be $f(x) = \frac{x+3}{x+4}$. Ask for volunteers to share their scenarios and explain their processes for graphing their functions.
- Another option for extending the task is to ask students to work together and create their own rational functions in different formats than the one given in the task. For example, ask them to consider a rational function which contains a quadratic expression in the numerator, as well as one in the denominator. Ask them to make predictions on the number and types of asymptotes, as well as how the graph will be drawn. Ask for volunteers to share their functions and explain their graphs of the functions.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem and demonstrated perseverance by recognizing that the goal of the task was to apply a rational function which represents a real-world scenario and graph the function on the coordinate plane. I applied my knowledge of graphing functions and plotted selected points in order to achieve the shape of the graph. I planned a solution pathway in order to examine and analyze the behavior of the graph of the function.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by using a visual representation of the scenario, in the form of graphing a function. I sketched the graph of the function representing a goaltender's save ratio and identified specific features of the graph. I then used these features, as well as the shape of the graph, to draw conclusions about the end behavior of the graph in the context of the problem situation.)

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- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I made use of structure by applying the procedures involved in choosing domain values for a function, and then determining the corresponding range values. I also used an algebraic process of substituting in values for the domain to get these resulting range values. I also applied the structure of finding the asymptotes for the graph.)

Alternate Strategies or Solutions

- Students may choose to create a table of values for the function which contains more and different points than the ones given in the task. Remind them to ensure that they choose a variety of positive and negative values, so that they can see the entire graph of the function.
- Students may also choose to plot the points before determining the asymptotes for the graph. Encourage them to choose enough points to plot so that they are able to determine the asymptotes, and then algebraically check to make sure that the asymptotes are correct.

Technology

Students can use a scientific calculator and/or a graphing calculator in order to check their work after the task has been completed.

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Problem-Based Task 3.5.3: Capping Costs

Common Core State Standard

A–CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

Task Overview

Focus

How can rational inequalities be created to represent a real-world scenario? What restrictions on the variable must be considered in the context of the problem? In this task, students will first create rational functions, and then they will create a rational inequality representing cost and profit in order to determine the minimum number of baseball caps a business woman must sell in order for her profit to exceed her costs.

This activity will provide practice with:

- interpreting verbal information and translating it into algebraic functions
- representing costs and profit of a business with functions
- simplifying algebraic functions
- writing rational inequalities used to compare cost and profit
- solving rational inequalities
- analyzing inequality solutions in the context of the problem
- drawing conclusions about the results of an inequality

Introduction

This task should be used to explore or to apply the skill of creating equations and inequalities in one variable and using them to solve problems. Students should already be familiar with solving inequalities and simplifying equations. They should also be familiar with evaluating functions.

Begin by reading the problem and clarifying the meaning of *profit*, *rational inequality*, *revenue*, and *startup costs*.

profit	the amount of money earned after the costs have been subtracted
rational inequality	an inequality that includes a ratio of two rational expressions, in which a variable appears in a denominator of at least one rational expression
revenue	the amount of money taken in from the sale of an item
startup costs	the initial amount of money a business spends for the production of a product; a non-recurring cost

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they make sense of the context and goal of the problem and make the connection between the terms profit, cost, and revenue. They will reason that rational functions can be created to represent these terms, and that an inequality can be created using these functions such that a solution can be found for the number of caps needed to reach the goal of profit exceeding costs. They will reason quantitatively as they perform the necessary steps to solve the inequality, as well as substituting the value of the solution of the inequality into the appropriate function.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled by using rational functions and rational inequalities. They will interpret and translate verbal information about the costs and desired profit and write functions and inequalities that model this information. They will solve the inequality and draw conclusions about the amount of baseball caps that must be sold in order to reach the goal of the amount of profit exceeding the costs.

- **SMP 6:** Attend to precision.

  Students will apply the definitions of the business-related terms introduced in the task in order to create rational functions and inequalities representing the context of the problem. They will apply the correct inequality symbols and they will apply function notation in order to accurately represent the problem. Encourage students to use the correct terminology of the terms "profit", "revenue", and "cost" as they relate to the various parts of the task, as well as the correct terminology related to comparing the original quantities in the task (i.e. average profit per cap to average cost per cap).

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- forgetting to place the variable x in the denominator when creating the average cost or average profit function

  Remind students that an average profit, or average cost, function is the function divided by the number of units. In this case, these functions are divided by the number of baseball caps, represented by the variable x.

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- confusing the terms "profit" and "revenue"

  Have students write the definitions of profit and revenue on the top of their papers. Remind them that revenue is the amount of money taken in when a product is sold, and profit is the amount of money made once the costs are subtracted.

- reversing the direction of the inequality used in the problem

  Remind students that the goal is for the amount of profit to be greater than the costs, so the function representing the average profit should be greater than, >, the function representing the average cost.

- incorrectly interpreting the solution of the inequality

  Remind students that the goal of the task is to determine the minimum amount of profit per cap that needs to be made to reach the goal of the amount of profit being greater than the costs. Therefore, the solution represents the smallest number of caps that need to be sold to reach the goal.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "What are some terms that are used when applying inequality symbols to real-world scenarios?" (**Answer:** Answers will vary, but examples may include: minimum, maximum, at most, at least, no more than, no less than, etc.)
- Before students begin the task, discuss the concept of the basic business model with them. Ask them, "If you wanted to be a successful business owner, what must you consider when planning the production of a product?" (**Answer**: The cost it will take to make each item, as well as any costs for equipment, rent, etc. (startup costs) must be considered. Then, the price that the item will be sold for must be considered.)
- Ask students, "Why do you think it is important to model a business plan using functions and inequalities to represent cost, revenue, and profit?" (**Answer:** It is important that a business owner has an equations/functions that models his/her business so he/she will know how many of an item must be sold in order to make a profit.)
- As students are beginning this task, ask them, "What type(s) of constraints are there on the variable, x, in this problem, and why?" (**Answer**: Since x represents the number of baseball caps sold, it has to be a positive, whole number, because a negative number of caps will not be sold, and a decimal part of a cap will not be sold.)

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- As students are creating the function representing the cost to produce x number of caps, ask them, "Why is the $200 used as a constant term and not the variable term in the function? Why is "15x" the variable term?" (**Answer:** The $200 represents the startup costs, which are the costs for the equipment used to produce the caps. This is a one-time amount, so it is not included in the price for each cap produced. Therefore, it is the constant term. The $15 represents the cost per cap, so that is the variable term.)
- As students are analyzing the solution of the inequality, $x > 80$, ask them, "Why does Veronica have to sell 81 or more caps to meet her goal? Why would she not meet her goal if she sold 80 caps? Explain in terms of the inequality symbol used in the problem." (**Answer**: The inequality symbol used in the problem is "greater than", >, and not "greater than or equal to", ≥. Therefore the amount of caps must be more than 80. The next whole number greater than 80 is 81.)
- As students are completing the task, ask them, "What types of real-world issues might arise that would cause a business owner to revise the functions for the business models?" (**Answer:** Answers will vary, but some types of issues might include having to buy new equipment and/or pay a higher rent, or increasing or decreasing the price for which an item is sold. These components would change the coefficients and/or constants used in the functions.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their thoughts and processes for creating the cost, revenue, and profit functions. Specifically discuss the choices for the constants and coefficients of each function and what they represent in the context of the problem. Also ask for volunteers to explain how they interpreted the solution of the inequality and how it applied to the business scenario. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and how they were able to create the required functions from the given verbal information. Ask them to explain their conclusions about the minimum number of baseball caps that Veronica needs to sell to make a profit. Focus on the use of precise mathematical language and clarity, specifically the business terminology, specifically startup costs, costs, revenue, and profit.

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Connecting to Key Concepts

Make explicit connections to key concepts:

- A **rational inequality** is an inequality that includes a ratio of two rational expressions, in which a variable appears in a denominator of at least one rational expression.

  In this task, students will create rational inequalities that stem from rational functions in which the variable "x" is in the denominator of each expression in the inequality.

- To solve an inequality means to find the complete solution set.

  Students will solve a rational inequality and make sense of its solution in the context of the real-world scenario presented in the task.

Extending the Task

- To extend the task, ask students to work with a partner and create their own fictional business. Ask them to create a product they would like to sell, and then have them determine values for their startup costs and cost per item sold. Ask them to also create a goal for the minimum number of items they need to sell in order to make a minimum profit. Have them work through the same steps as in the original task, focusing on the creation of cost, revenue, and profit functions. Ask for volunteers to share their fictional scenarios and explain their functions and inequality solutions.
- Another option for extending the task is to ask students to work through the original task, but with a change in one component, such as the cost it takes to produce each item, or the cost the item sells for. Once they have changed one of the components, ask them to make predictions on how this one change will affect Veronica's goal for the number of caps she must sell in order for her profit to exceed her costs. Then, ask them to determine if other components of the equations will need to change. Ask for volunteers to discuss their predictions and justifications for their choices.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I made sense of the context and goal of the problem and made the connection between the terms profit, cost, and revenue. I reasoned that rational functions could be created to represent these terms, and that an inequality could be created using these functions such that a solution can be found for the number of caps needed to reach the goal of having the profit exceed the costs.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning as I solved the inequality using algebraic steps to simplify it, and I substituted values into the profit function in order to determine the profit made with the minimum number of caps sold.)

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- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I recognized that the scenario used in this task could be modeled by using rational functions and rational inequalities. I interpreted and translate the verbal information about the costs and desired profit, and used that to write functions and inequalities that modeled this information. Then, I solved the inequality and drew conclusions about the amount of baseball caps that must be sold in order to reach the goal of the profit exceeding the costs.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision by applying the definitions of the business-related terms introduced in the task in order to create rational functions and inequalities representing the context of the problem. I applied the correct inequality symbols and function notation in order to accurately represent the problem.)

Alternate Strategies or Solutions

Instead of creating an inequality, students may choose to use trial and error to substitute values into the functions until they find the correct value that will represent the minimum number of caps sold in order to make sure the profit exceeds the cost. Remind students that substituting a variety of values into the functions may be time-consuming, as opposed to creating an inequality with the functions and solving it for x.

Technology

No technology is needed for this task.

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Implementation Guide

Problem-Based Task 3.6.1: Options of Interest

Common Core State Standard

A–SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Task Overview

Focus

How can an exponential expression be rewritten using properties of exponents? How can an exponential growth model be used to represent a real-world scenario? In this task, students will write exponential growth models to represent two investment options, and determine which model yields the most amount of money over 5 years.

This activity will provide practice with:

- identifying quantities/components of an exponential growth model
- rewriting an exponential expression in a different format
- applying properties of exponents to simplify an exponential expression
- simplifying an exponential expression
- calculating the value of an exponential model
- comparing two exponential models to determine which yields a higher value

Introduction

This task should be used to explore or apply the skill of using the properties of exponents to transform expressions for exponential functions and simplify exponential expressions. Students should already be familiar with evaluating functions and using a calculator to evaluate expressions with exponents.

Begin by reading the problem and clarifying the meaning of *compound interest*, *exponential expression*, and *exponential growth model*.

compound interest	interest earned on both the initial amount and on previously earned interest
exponential expression	an expression that contains a base and a power/exponent
exponential growth model	an exponential function, $f(t) = a(1 + r)^t$, where $f(t)$ is the final output value at the end of t time periods, a is the initial value, r is the percent increase per time period (expressed as a whole number or decimal), and t is the number of time periods

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they make sense of the quantities represented in the scenario. They will identify that in the exponential growth model, a is the initial amount invested, r represents the interest rate, and t represents the number of time periods in which the money is invested. They will apply the exponential growth model to determine which investment option yields the most money after 5 years based on the different interest rates and compounding periods. Students will reason quantitatively as they rewrite an exponential expression using a property of exponents, and when they calculate the amount of money earned with each investment option.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will compare the two exponential growth models that represent the two different investment options for Devin. They will construct viable arguments about which model yields the most amount of money earned based on the results of evaluating each model given an initial investment amount, an interest rate, and a time of 5 years. Encourage students to discuss their arguments with each other and explain their reasoning if they do not agree with each other.

- **SMP 6:** Attend to precision.

  Make sure students are using math terms and definitions clearly and accurately to explain their reasoning. Encourage students to analyze each exponential growth model carefully for the two investment options when simplifying the expressions and evaluating the models. Strive toward precise mathematical language, such as using the correct terms for compound interest, exponential expression, and exponential growth model. Remind students to perform their calculations carefully when using their calculators.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- using different time periods for r and t in the exponential model $f(t) = a(1 + r)^t$

  Have students create a list of what each variable in the exponential growth model represents so they can use the list as a reference when working through the task. Remind them to check the time period in which the interest is compounded, and to use the correct number to represent the number of times the interest is compounded per year.

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- mistakenly dividing an annual interest rate that is compounded annually by 12 to find the approximate equivalent monthly interest rate, compounded monthly

  Remind students that it is only necessary to divide by 12 when the compounding rate is monthly.

- neglecting to add 1 to a growth rate to obtain a growth factor

  Remind students that the interest rates, or growth rates, are converted from a percent to a decimal and then added to 1. For example, in this task, the first investment option has an interest rate of 6%, so that is changed to 0.06 and added to 1, and the rate is expressed as 1.06.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them to make a prediction about which investment option will result in the most amount of money earned for a $20,000 investment over 5 years. Ask them, "Based on the given information about each investment option, which do you think will result in the most amount of money? Explain your prediction." (**Answer**: Some students may predict that the first option will yield the most amount of money, because it has a higher interest rate than the second investment option. Some students may predict that the second option will yield the most money, because it is compounded more times in a year than the first option. Encourage students to write down their detailed reasoning for their predictions.) Then ask, "How do you know that an exponential growth model will be used to represent this scenario, as opposed to an exponential decay model?" (**Answer**: An exponential growth model will be used, because the amount of money is increasing over time. When Devin invests his money, he will receive interest that is compounded, which means he will have more money as time goes on.)
- If students are having difficulty writing the interest rates as decimals, ask them, "How will you convert the two interest rates for the two investment options from percents to decimals?" (**Answer**: To convert a percent to a decimal, move the decimal point two places to the left. The first investment option has an interest rate of 6%, so that will be 0.06 as a decimal. The second investment option has an interest rate of 0.5%, so that will be 0.005 as a decimal.)
- If students are having difficulty understanding why "12t" is used to represent the time for the first investment option, ask, "In terms of quantity of time in a year, what does 12 represent?" (**Answer:** There are 12 months in a year, so 12t represents the amount of months.)
- Ask students, "Why can the exponent, t, be written as the product of 12 and its reciprocal?

  Why is it necessary to rewrite the exponent this way in the context of the exponential

  expression?" (**Answer:** The product of any number and its reciprocal is 1, so when the exponent t is rewritten as the product of 12 and its reciprocal, $\frac{1}{12}$, then the Power of a Power Property can be used to rewrite the expression so that the final result yields an exponent of 12t.)

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- After students have found the exponential growth models to represent the two investment options, ask them, "How can you determine, before you actually evaluate each model, which of the two options will yield the most amount of money?" (**Answer:** The amount of initial investment, $20,000, and the number of time periods, 60, is the same for both options. The only difference is the growth rate, which is 1.004868 for the first option, and 1.005 for the second option. Because the second option has the higher growth rate, and the other values are the same, the second option will yield the most amount of money.)
- As students complete the task, ask, "As you are evaluating each model, how do you know how many decimal places to which the final results should be rounded?" (**Answer:** Because the scenario involves money, the final results should be rounded to 2 places past the decimal point, or the hundredths place, which represents cents.)
- Ask students, "How much more money will Devin make after 5 years with the second investment option? How is this amount determined?" (**Answer:** To determine how much more money Devin makes with the second investment option after 5 years, subtract the total amount of money earned with Option A from the total amount earned with Option B: 26,977.00 – 26,765.23 = 211.77. Therefore, Devin earned $211.77 more with Option B.)
- Ask students, "If Devin were to invest the money for 10 years instead of 5, do you think he would earn another $211.77 more than he did at the 5-year mark? In other words, is the rate of change the same from year to year? Why or why not?" (**Answer:** No, the rate of change is not the same from year to year, because the models representing the investment options are exponential growth models. In an exponential function, the rates of change vary from interval to interval.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial predictions about which investment option would result in the most amount of money earned for a $20,000 investment over 5 years. Encourage students to discuss their reasoning for their predictions, as well as how correct or incorrect they were based on the actual results of the task.
- Compare students' strategies and ways of justifying responses. Focus on the use of precise mathematical language and clarity, specifically when referring to writing the exponential growth models as well as using the Power of a Power Property to simplify the exponential expressions.

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Connecting to Key Concepts

Make explicit connections to key concepts:

- Exponential expressions contain a base and an exponent.

  In this task, students will work with exponential expressions and identify that the base is the annual interest rate, or growth rate, and the exponent is the number of time periods.

- Exponential growth, or increase, can be written as $f(t) = a(1 + r)^t$.

  Students will use the formula for exponential growth to create models for the two investment options in this task.

- In the exponential growth model $f(t) = a(1 + r)^t$, $f(t)$ is the final output value at the end of t time periods, a is the initial value, r is the growth rate per time period (expressed as a whole number or decimal), and t is the number of time periods.

  Students will identify the quantities for each investment option, and will substitute these values into the exponential growth model in the appropriate locations to determine the amount of money earned over a period of time.

Extending the Task

- To extend the task, ask students to create a table of values for both investment options. Ask them to use values of 1 year, 2 years, 3 years, 10 years, and 15 years. Discuss how they can verify that the functions used to represent the models are exponential, as there is not a constant rate of change between consecutive input values. Ask students to use graphing calculators or other graphing technology, and to graph the two functions on the same graph. Discuss the shape of the graphs.
- Another option for extending the task is to provide students with a third investment option, Option C. Let this third option be represented by a linear function in which Devin begins with the initial amount of $20,000, and is able to save $600 every 6 months. Ask students to write a function to represent this option, then ask them to compare it to Options B and C to determine which of the three options will result in more money at the end of 5 years.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I identified the quantities represented in the exponential growth models for the investment options. Then I evaluated both models to determine which one yielded the most amount of money after 5 years.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning when I rewrote exponential expressions, then calculated the amount of money earned with each investment option.)

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- **For SMP 3, ASK:** "Did you construct viable arguments, and did you critique the reasoning of others?" (**Answer:** I constructed viable arguments by evaluating each exponential growth model and interpreting the results in the context of the scenario.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision by using correct math terminology in my explanations and explaining my reasoning in a mathematically correct and precise way.)

Alternate Strategies or Solutions

- Once students have determined the exponential growth models for each investment option, they may choose to create a table of values to determine which option will yield the most amount of money after 5 years. Remind students to carefully check their calculators as they input the values.
- Some students may choose to graph the table of values for both exponential growth models so they can visually compare the two functions. Discuss how the shape of the graphs verifies each model is exponential.

Technology

Students can use scientific calculators and graphing calculators, or other graphing technology, for this task.

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Implementation Guide

Problem-Based Task 3.7.1: Constant Acceleration and Constant Speed

Common Core State Standard

A–REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line $y = –3x$ and the circle $x^2 + y^2 = 3$.*

Task Overview

Focus

How does a constant rate of change compare to an increasing rate of change? If the value of a linear function at first exceeds the value of a quadratic function, how long will it take for the value of the quadratic function to equal the value of the linear function? Students will solve a system consisting of a linear equation and a quadratic equation to find how long it takes for the vehicles to be side by side.

This activity will provide practice with:

- solving systems of equations
- comparing constant rates of change to increasing rates of change
- graphing systems consisting of a linear equation and a quadratic function
- interpreting the solutions to systems of equations

Introduction

This task should be used to explore or to apply the connections between algebraic and graphic methods for solving systems of equations. Graphing may be done using technology, including graphing calculators or online graphing programs.

Begin by reading the problem and clarifying the meaning of each equation. Explain that one of the cars is 200 meters behind the other car, and that it is going to catch up to the other car as it accelerates. Review the following terms:

at rest	stationary, or not moving
constant speed	a steady rate of speed, traveling the same number of feet every second
accelerate	to increase speed

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Some students will have trouble making the connection between the equations and the situation. Draw pictures on the board and discuss the situation in terms of local points of reference with which everyone is familiar, such as restaurants and grocery stores. Emphasize that one vehicle is at rest and begins to accelerate at the exact moment when the other vehicle is 200 feet ahead.

- **SMP 4:** Model with mathematics.

  Help the students by plotting the initial values of the each function for them. Ask them which function is going to increase at a constant rate and which has an increasing rate of change.

- **SMP 6:** Attend to precision.

  Students might use a graphic model to arrive at an approximate solution. Encourage students to use both graphic and algebraic methods to solve the system. Solving the system algebraically will give a precise solution.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- assuming that both equations can be represented by a line

  Remind students that one of the equations is quadratic, meaning that the slope, or rate of change, is increasing and the graph will be curved.

- not recognizing that one of the solutions to the system is extraneous

  Point out that the domain for both functions is all real numbers greater than or equal to zero.

- estimating the solution by graphing

  Explain that the scale of the graph is large, so it may not be obvious that the point of intersection does not exactly coincide with the grid lines.

- incorrectly solving the system

  Show students how to find the solution algebraically. The solution is the value of t at which both functions simultaneously have the same value. Set the expressions for the functions equal to each other, and solve for t using the quadratic formula.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students don't understand which equation represents which car, ask them to identify which equation has the greater *y* intercept, and remind them that this is the car that starts out 200 feet in front. (**Answer**: The linear equation, $C_2(t) = 20t + 200$, represents the car in front, and the quadratic equation, $C(t) = 2t^2$, represents the car that starts out at rest.)
- If students don't understand what the independent and dependent variables are, remind them that the facts about both cars are given in the same units, seconds and feet. (**Answer**: The independent variable for both equations is *t*, representing seconds, and the dependent variable for both equations is distance, in feet, with 0 feet representing the starting position of the car at rest.)
- If students don't understand what the domain of the equations is, ask them to identify the value of *t* when the second car is 200 feet ahead of the car at rest. (**Answer**: The domain is all real numbers greater than or equal to 0: $t \geq 0$.)
- If students don't know where to begin with graphing the equations:
    - Provide graph paper and suggest scales for both axes. Also help them identify the initial values (*y* intercepts) for both equations. (**Answer**: Plot the *x* axis from 0 to 20 with a scale of 1, and plot the *y* axis from 0 to 600 with a scale of 40.)
    - Once students have their graphs set up with labeled axes and *y*-intercepts plotted and labeled for both functions, tell them to plot several more points for each equation so they can draw a line and an approximate curve.
    - After they have graphed the equations by hand, have them repeat the exercise using technology, to achieve a more accurate curve. Ask them to use their calculator functions to find the point of intersection, as precisely as possible.
    - Now explain that it is important to validate their answers algebraically. Explain that by setting the expressions for the functions equal to each other and solving for *t*, they can find the value of *t* for which both functions simultaneously have the same value. Remind students how to use the quadratic formula.
    - Encourage students to compare their results from graphing to their algebraic results. Check student answers. The value of *t* is approximately 16.18. If students report a negative value for *t* (namely –6.18) remind them to check the domain.
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

Review the steps outlined earlier in this guide: graphing by hand, graphing with technology, and solving algebraically. Ask students if any of them noticed alternative paths to a solution along the way.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Solving a system consisting of a quadratic and linear equation is just like solving a system of two linear equations, except that the graph may have two points of intersection.
- A system of equations is also known as a system of simultaneous equations. Both equations share the same variables, in the same units, and the graphs of both equations exist in the same space. The solution(s) to the system is the point or points where the functions both have the same value.

Extending the Task

To extend the task, change any of the parameters. For example, change the starting position of the forward car, change the rate of speed of the forward car, or change the rate of acceleration of the rear car.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I asked myself if the first car starts closing the 200-foot gap as soon as it starts accelerating.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I realized that the point where the slope of the curve matches the slope of the line represents the greatest distance between the vehicles, which is greater than 200 feet.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used graphic methods to represent the equations, so as to visualize the solution to the problem.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I carefully solved the problem algebraically after graphing, keeping in mind the limited domain.)

Alternate Strategies or Solutions

Students could estimate the solution to the problem and then create a table of values above and below the estimate to find the approximate answer with reasonable accuracy.

Technology

Students can use graphing calculators to graph the functions and also to compute the values algebraically with the quadratic formula.

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Problem-Based Task 3.7.2: The Business of Bicycles

Common Core State Standard

A–REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line $y = –3x$ and the circle $x^2 + y^2 = 3$.*

Task Overview

Focus

For each additional unit of a product that a company makes, there is a cost that the company incurs. This is called the marginal cost. Students will solve a system of equations, one linear and one quadratic, to find the number of bikes that creates the same marginal cost for two different companies.

This activity will provide practice with:

- solving a system of equations
- solving a quadratic equation using the quadratic formula
- interpreting the results
- analyzing the solutions

Introduction

This task should be used to apply their understanding of solving systems of equations when one equation is linear and the other is quadratic. Graphing may be done using technology, including graphing calculators or online graphing programs.

Begin by reading the problem and clarifying that businesses must spend money in order to make the products they sell. Review the following terms:

fictional	not real, imaginary
manufacturing	making
marginal cost	the cost of making an additional product
producing	making

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students are asked to interpret the question and then combine two models to solve a problem. Remind them that the words "the same" imply that two things are equal. Then, after solving the equations, have them reread the question to see how to answer the question.

- **SMP 4:** Model with mathematics.

  Using equations to model the marginal cost of producing bikes, students will determine the marginal cost when the number of bikes is the same. While the model for each marginal cost is provided for the students, they will determine how to use them to find the number of bikes and then again to find the marginal cost.

- **SMP 7:** Look for and make use of structure.

  After setting the equations equal to each other, students may not know how to solve the equation. Point out that there are some like terms. Remind them that they can rearrange the equation so that one side is equal to zero. Then, ask them to identify what type of equation they have and how they will choose a solution method based on that equation.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- miscalculating signs

  Encourage students to work carefully without neglecting any signs, especially when using the addition property of equality and setting one side of the equation equal to zero. Students also need to pay close attention to signs when using the quadratic formula.

- not answering the question

  While solving the equation will find x, the number of bikes, remind students that the question actually asks for the marginal cost.

- disregarding the context of the problem when identifying a solution

  Ask students to determine if both solutions are valid based on the context of the problem. Encourage them to think about what x stands for and the reasonable values for x as a result.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students are struggling to solve the problem, ask them to review what the variables stand for and what they are seeking. (**Answer**: x is the number of bikes, $C(x)$ is the marginal cost, and the marginal cost when the number of bikes is the same.)
- If students don't know how to set up the equation, ask "What do you want to be the same?" (**Answer**: The number of bikes each company manufactures.) Ask, "How can you show that they are the same?" (**Answer**: Set the equations equal to each other.)
- If students don't know how to solve the equation, ask "How can you set one side equal to 0?" (**Answer**: Rearrange the equation by moving all linear and constant terms to the side with the quadratic term.)
- Ask, "What kind of equation do you have now?" (**Answer**: quadratic) Next, ask students, "What solution strategy will you use to solve the equation? Explain." (**Answer**: I will use the quadratic formula because the coefficients are not integers, and I cannot easily factor it.)
- Once students have solved the equation, ask them to answer the question given in the task, ask "Is a negative number of bikes reasonable?" (**Answer**: No, there is no way to produce a negative number of bikes.) Ask, "How can you find the marginal cost if you know the number of bikes?" (**Answer:** Substitute the number of bikes into either original equation.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Discuss why there are two solutions to the equation but only one answer to the question. Encourage students to debate whether there were any other possible solution methods.

Connecting to Key Concepts

Make explicit connections to key concepts:

- To solve a quadratic-linear system, if both functions are written in function form, set the equations equal to each other.

  To find the number of bikes that result in the same marginal cost, the equations for marginal cost must be set equal to each other. This can be justified by the Substitution Property.

- You can solve by using the quadratic formula.

  Since the coefficients of the quadratic equation are not integers, the quadratic formula is the best solution strategy.

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Extending the Task

To extend the task, change the parameters or question for the problem. For example, change one of the marginal cost equations, identify where the marginal cost of one is twice the other, or find the range of number of bikes for which it is less expensive for one company to produce another bike than the other.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used what I was given to write an equation to represent the situation.) "Which of your strategies represent quantitative reasoning?" (**Answer:** Once I found the number of bikes, I used it to find the marginal cost.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used equations to model the marginal costs of companies.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** Once I wrote the equation, I rearranged it so that it was a quadratic equation set equal to 0, which meant I could use the quadratic formula.)

Alternate Strategies or Solutions

Graph the system to approximate the solutions.

Technology

Students could use the graphing calculator to graph each marginal cost function and then calculate the points of intersection. The *y*-value of the intersection point will be the marginal cost.

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Common Core State Standards

Mathematics II

Integrated Pathway

Teacher Resource

Unit 4

WALCH
INTEGRATED MATH

This program was developed and reviewed by experienced math educators who have both academic and professional backgrounds in mathematics. This ensures: freedom from mathematical errors, grade level appropriateness, freedom from bias, and freedom from unnecessary language complexity.

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Standards Correlations

Each lesson in this Integrated Pathway: Mathematics II program was written specifically to address the Common Core State Standards. Each lesson lists the standards covered in all the sub-lessons, and each sub-lesson lists the standards addressed in that particular section. In this section, you'll find a comprehensive list mapping the sub-lessons to the CCSS.

Guide to Common Core State Standards Annotation

As you use this program, you will come across a symbol included with the Common Core standards for some of the lessons and activities. These symbols are explained below.

Symbol: ⋆

Denotes: Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (⋆).

From http://www.walch.com/CCSS/00003

Symbol: (+)

Denotes: College and Career Readiness Standards

Advanced mathematics standards that are required in higher-level courses such as advanced statistics may also be included in lower-level courses. These additional standards are denoted by (+). According to the Common Core State Standards Initiative, "the evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6–8."

From http://www.walch.com/CCSS/00004

Connections to Future Courses

This section provides a map between topics introduced in each unit of this course and subsequent courses where each topic is revisited and built upon.

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CCSS INTEGRATED PATHWAY: MATHEMATICS II STANDARDS CORRELATIONS

Unit 4: Applications of Probability				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Events**			
	4.1.1	Describing Events	S–CP.1★	U4-6
	4.1.2	The Addition Rule	S–CP.7★	U4-33
	4.1.3	Understanding Independent Events	S–CP.2★	U4-48
Lesson 2	**Conditional Probability**			
	4.2.1	Introducing Conditional Probability	S–CP.3★ S–CP.5★ S–CP.6★	U4-78
	4.2.2	Using Two-Way Frequency Tables	S–CP.4★ S–CP.5★ S–CP.6★	U4-102
	4.2.3	The Multiplication Rule	S–CP.8★ (+)	U4-125
Lesson 3	**Combinatorics**			
	4.3.1	Combinations and Permutations	S–CP.9★ (+)	U4-153
	4.3.2	Probability with Combinatorics	S–CP.9★ (+)	U4-173
Lesson 4	**Making and Analyzing Decisions**			
	4.4.1	Making Decisions	S–MD.6★ (+)	U4-196
	4.4.2	Analyzing Decisions	S–MD.7★ (+)	U4-216

TOPICS FROM FUTURE COURSES: CCSS INTEGRATED PATHWAY MATHEMATICS II

Math II	Topic introduced	Course/unit where addressed	How addressed
Unit 4	Probability	Math III Unit 1	applied to inferences and decision-making in statistics

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Conceptual Activities

Use these interactive open education and/or Desmos resources to build conceptual understanding of mathematical ideas. (*Note*: Activity links will be monitored and repaired or replaced as necessary.)

- Illustrative Mathematics. "The Titanic 3."

  http://www.walch.com/ca/10005

  This task poses an open-ended question that forces students to think about how they can answer it. Students will analyze data given in the two-way table to find meaningful probabilities and increase their understanding of conditional probability and independence.

- Mathematics Assessment Resource Service, University of Nottingham. "Representing Conditional Probabilities 1."

  http://www.walch.com/ca/10007

  In this resource, students will use tables and tree diagrams to represent events as a subset of a sample space and answer questions involving conditional probabilities. This resource includes a diagnostic assessment, a learning task, and sample solutions that students will analyze and evaluate. Some extra materials are required.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Some cars in a parking lot have the following license plate identifications.

   5HEX32 4EAE32 8TAS12 2AAU62 4JSA48 6JBL62

   One of the license plates is chosen at random. Consider the following events.

   V: All the letters of the chosen license plate are vowels.

   E: All the digits of the chosen license plate are even integers.

   How many outcomes are in the event *V* and *E*?

   a. 1
   b. 2
   c. 3
   d. 4

2. Ines is playing a card game with a standard 52-card deck. She wants her first draw to be a heart or a face card. What is the probability that she draws a heart or a face card on her first draw?

   a. $\frac{3}{52}$
   b. $\frac{4}{13}$
   c. $\frac{11}{26}$
   d. $\frac{25}{52}$

3. West End High School has 640 students. All the students responded to a survey about what sights they have visited in Washington, D.C. Some of the results are as follows.

   180 students have visited the National Zoo.

   120 students have visited Capitol Hill.

   220 students have visited the National Zoo or Capitol Hill.

   Based on the survey, what is the probability that a randomly chosen student at West End High School has visited both the National Zoo and Capitol Hill?

   a. $\frac{1}{16}$
   b. $\frac{1}{8}$
   c. $\frac{5}{32}$
   d. $\frac{15}{32}$

continued

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Assessment

4. Arturo has a bag that contains 10 tiles: 3 are red, 2 are white, and 5 are blue. He takes a tile from the bag without looking. He does not replace it. Then he takes another tile from the bag without looking. Consider the following events.

   RF: Arturo takes a red tile first.

   BS: Arturo takes a blue tile second.

   RF and *BS*: Arturo takes a red tile first and a blue tile second.

   Which option describes the events?

   a. The events *RF* and *BS* are dependent, and the probability of the event *RF* and *BS* is $\frac{1}{6}$.

   b. The events *RF* and *BS* are dependent, and the probability of the event *RF* and *BS* is $\frac{3}{20}$.

   c. The events *RF* and *BS* are independent, and the probability of the event *RF* and *BS* is $\frac{1}{6}$.

   d. The events *RF* and *BS* are independent, and the probability of the event *RF* and *BS* is $\frac{3}{20}$.

5. The expression for calculating a baseball player's batting average is $\frac{\text{number of hits}}{\text{number of times at bat}}$. Larry's batting average is 0.250, and he is at bat. Chase's batting average is 0.300, and he will be at bat after Larry. Assume that each player's performance is independent of the other's. What is the probability that Larry or Chase will get a hit, based on their batting averages?

   a. 0.275

   b. 0.300

   c. 0.475

   d. 0.550

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Lesson 1: Events

Instruction

SMP
1 ✓ 2 ✓
3 ✓ 4 ✓
5 6
7 ✓ 8 ✓

Common Core State Standards

S–CP.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").★

S–CP.2 Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent.★

S–CP.7 Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model.★

Essential Questions

1. How can you describe events?
2. How can you determine whether two events are independent?
3. For events *A* and *B*, how can you find the probability of the event "*A* or *B*"?
4. For events *A* and *B*, how can you find the probability of the event "*A* and *B*"?

WORDS TO KNOW

Addition Rule If *A* and *B* are any two events, then the probability of *A* or *B*, denoted $P(A \text{ or } B)$, is given by: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$. Using set notation, the rule is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

complement a set whose elements are not in another set, but are in some universal set being considered. The complement of set *A*, denoted by $\overline{A}$, is the set of elements that are in the universal set, but not in *A*. The event does not occur. The probability of an event not occurring is 1 minus the probability of the event occurring, $P\left(\overline{A}\right) = 1 - P(A)$.

dependent events events that are not independent. The outcome of one event affects the probability of the outcome of another event.

disjoint events events that have no outcomes in common. If *A* and *B* are disjoint events, then they cannot both occur. Disjoint events are also called mutually exclusive events.

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element	an item in a set; also called a member
empty set	a set that has no elements, denoted by $\varnothing$. The empty set is also called the null set.
equal sets	sets with all the same elements
event	an outcome or set of outcomes of an experiment. An event is a subset of the sample space.
experiment	a process or action that has observable results. The results are called outcomes.
independent events	events such that the outcome of one event does not affect the probability of the outcome of another event
intersection	a set whose elements are each in both of two other sets. The intersection of sets A and B, denoted by $A \cap B$, is the set of elements that are in both A and B.
member	an item in a set; also called an element
mutually exclusive events	events that have no outcomes in common. If A and B are mutually exclusive events, then they cannot both occur. Mutually exclusive events are also called disjoint events.
null set	a set that has no elements, denoted by $\varnothing$. The null set is also called the empty set.
outcome	a result of an experiment
probability	a number from 0 to 1 inclusive or a percent from 0% to 100% inclusive that indicates how likely an event is to occur
probability of an event *E*	denoted $P(E)$, and is given by $P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}$ in a uniform probability model
probability model	a mathematical model for observable facts or occurrences that are assumed to be random; a representation of a random phenomenon
relative frequency (of an event)	the number of times an event occurs divided by the number of times an experiment is performed
sample space	the set of all possible outcomes of an experiment
set	a collection or list of items

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subset	a set whose elements are in another set. Set A is a subset of set B, denoted by $A \subset B$, if all the elements of A are also in B.
uniform probability model	a probability model in which all the outcomes of an experiment are assumed to be equally likely
union	a set whose elements are in at least one of two other sets. The union of sets A and B, denoted by $A \cup B$, is the set of elements that are in either A or B or both A and B.
universal set	a set of all elements that are being considered in a particular situation. In a probability experiment, the universal set is the sample space.
Venn diagram	a diagram that shows how two or more sets in a universal set are related

Recommended Resources

- MathIsFun.com. "Mutually Exclusive Events."

  http://www.walch.com/rr/00091

  This site gives an overview of events that are mutually exclusive and events that are not mutually exclusive, including how to calculate probabilities for both cases. The site explains how the Addition Rule is used and includes practice questions.

- MathIsFun.com. "Probability."

  http://www.walch.com/rr/00092

  This site gives an overview of probability, and includes key definitions and practice questions.

- MathIsFun.com. "Probability: Independent Events."

  http://www.walch.com/rr/00093

  This site reviews dependent and independent events, including how to calculate the probability of two or more independent events. It also has practice questions.

- OnlineMathLearning.com. "Complementary Probability."

  http://www.walch.com/rr/00094

  This website offers an interactive worksheet about complementary probabilities and unions. After completing the worksheet, users can click on the button to check their work. The page also offers hints.

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Lesson 4.1.1: Describing Events

Warm-Up 4.1.1

Kendra is playing a card game with a standard 52-card deck. She wants her first draw to be a heart or an ace.

1. How many ways can Kendra draw a heart or an ace?

2. How many ways can Kendra draw a card that is neither a heart nor an ace?

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Lesson 4.1.1: Describing Events

Common Core State Standard

S–CP.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").★

Warm-Up 4.1.1 Debrief

1. How many ways can Kendra draw a heart or an ace?

   You can use a table to show all the cards in the deck. Then identify and count the cards that are either a heart or an ace or both a heart and an ace.

Suit	2	3	4	5	6	7	8	9	10	J	Q	K	A
Spade													✔
Club													✔
Diamond													✔
Heart	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔

   There are 16 ways Kendra can draw a heart or an ace.

2. How many ways can Kendra draw a card that is neither a heart nor an ace?

   Count the cards that are neither hearts nor aces, or subtract the first answer from 52: $52 - 16 = 36$.

   There are 36 ways Kendra can draw a card that is neither a heart nor an ace.

Connection to the Lesson

- Drawing a card from a deck is an experiment and each card in the deck is a possible outcome; students will work with experiments and outcomes.
- All the cards in the deck comprise the sample space of the experiment; students will work with sample spaces.
- When students identify the cards that are hearts or aces, they are identifying an event; students will work with events.

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Prerequisite Skills

This lesson requires the use of the following skills:

- drawing and interpreting Venn diagrams (7.SP.8b)
- drawing and interpreting tree diagrams (7.SP.8b)

Introduction

Probability is a number from 0 to 1 inclusive or a percent from 0% to 100% inclusive that indicates how likely an event is to occur. In the study of probability, an **experiment** is any process or action that has observable results. The results are called **outcomes**. The set of all possible outcomes is called the **sample space**. An **event** is an outcome or set of outcomes of an experiment; therefore, an event is a **subset** of the sample space, meaning a set whose elements are in another set, the sample space. In this lesson, you will learn to describe events as subsets of the sample space.

For example, drawing a card from a deck of cards is an experiment. All the cards in the deck are possible outcomes, and they comprise the sample space. The event that a jack of hearts is drawn is a single outcome; it is a set with one element: {jack of hearts}. An **element** is an item or a **member** in a set. The event that a red number card less than 5 is drawn is a set with eight elements: {ace of hearts, ace of diamonds, 2 of hearts, 2 of diamonds, 3 of hearts, 3 of diamonds, 4 of hearts, 4 of diamonds}. Each event is a subset of the sample space.

Note that the words "experiment" and "event" in probability have very specific meanings that are not always the same as the everyday meanings. In the example above, "drawing a card" describes an experiment, not an event. "Drawing a jack of hearts" and "drawing a red number card less than 5" describe events, and those events are sets of outcomes.

Note: In the strictest sense, it can be argued that "drawing a card" does *describe* an event. However, it describes the special-case event that contains all possible outcomes; in other words, it describes the entire sample space. The important distinction to remember is that an event is a set of one or more outcomes, while the action or process is the experiment.

Key Concepts

- A **set** is a list or collection of items. Set A is a subset of set B, denoted by $A \subset B$, if all the elements of A are also in B. For example, if $B = \{1, 2, 3, x\}$, then some subsets of B are $\{1, 2, 3\}$, $\{2, x\}$, $\{3\}$, and $\{1, 2, 3, x\}$. Note that a set is a subset of itself.
- An **empty set** is a set that has no elements. An empty set is denoted by $\varnothing$. An empty set is also called a **null set**.

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- **Equal sets** are sets with all the same elements. For example, consider sets A, B, and C as follows:
  - A is the set of integers between 0 and 6 that are not odd.
  - B is the set of even positive factors of 4.
  - $C = \{2, 4\}$
- $A = B = C$ because they all have the same elements, 2 and 4.
- The **union** of sets A and B, denoted by $A \cup B$, is the set of elements that are in either A or B or both A and B.
- An **intersection** is a set whose elements are each in both of two other sets. The intersection of sets A and B, denoted by $A \cap B$, is the set of elements that are in both A and B.
- For example, if $A = \{a, b, c\}$ and $B = \{m, a, t, h\}$, then:
  - $A \cup B = \{a, b, c, m, t, h\}$
  - $A \cap B = \{a\}$
- An experiment is any process or action that has observable results. Some examples are:
  - an action, such as drawing a card from a deck or tossing a coin
  - a survey in which participants are asked questions
  - a study, such as taking note of what model vehicles are on a street or how many manufactured items are defective
- The results of an experiment are called outcomes. The set of all possible outcomes is called the sample space.
- As an example, consider experiment #1: toss a coin. The sample space for experiment #1 is {H, T}, where H indicates heads and T indicates tails. Now consider experiment #2: toss a coin and then toss the coin again. The sample space for experiment #2 is {HH, HT, TH, TT}. Note that HT and TH are different outcomes because HT indicates tossing a head and then a tail, whereas TH indicates tossing a tail and then a head.
- An event is an outcome or set of outcomes, so an event is a subset of the sample space.
- Consider this experiment: roll a 6-sided die.
  - The sample space is {1, 2, 3, 4, 5, 6}.
  - The event "Roll a number greater than 2" is {3, 4, 5, 6}.
  - The event "Roll an odd number less than 3" is {1}.
- A **universal set** is a set of all elements that are being considered in a particular situation. In a probability experiment, the universal set is the sample space.

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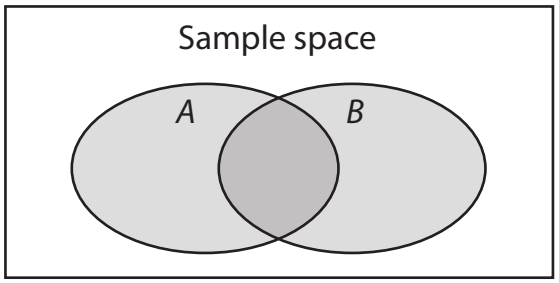
- The **complement** of set A, denoted by $\overline{A}$, is the set of elements that are in some universal set, but not in A.
- For a probability experiment, the complement of event A is the set of all outcomes in the sample space that are not in A. Consider this experiment again: roll a 6-sided die. The sample space is {1, 2, 3, 4, 5, 6}. If event A is described as "roll a number greater than 2," then $A = \{3, 4, 5, 6\}$ and $\overline{A} = \{1, 2\}$.
- When two or more objects are used in an experiment, those objects are distinct, even if they have identical appearances. For example, suppose an experiment consists of rolling two dice and you are considering the event "roll a 2 and a 3." There are two ways to roll a 2 and a 3; that is, the event consists of two outcomes. To understand why, assume that one die is red and the other is blue. You can roll a red 2 and a blue 3 or you can roll a blue 2 and a red 3. Or, if the dice are the same color, you can take a marker and label them "first die" and "second die." Then, after the dice are rolled you can distinguish the outcome "first die 2 and second die 3" from the outcome "first die 3 and second die 2."
- So, when an experiment consists of an action performed with two or more objects at the same time, you can think of the experiment as two or more separate actions with one object. Some examples:
  - Tossing two coins: You can think of this as tossing one coin two times. So the outcomes HT and TH are different. These outcomes comprise the event "toss one head and one tail."
  - Tossing three coins: You can think of this as tossing one coin three times. So the outcomes HHT, HTH, and THH are all different. These outcomes comprise the event "toss exactly two heads."
  - Rolling a pair of dice: You can think of this as rolling one die two times. So the outcome of rolling a 1 and then a 2 is different from the outcome of rolling a 2 and then a 1. These outcomes comprise the event "roll a sum of 3."
- Sometimes it is helpful to draw tables or diagrams to visualize outcomes and the relationships between events.
- A **Venn diagram** is a diagram that shows how two or more sets in a universal set are related. In this diagram, some members in event A also fit the criteria for members in event B, so the shapes overlap:



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Common Errors/Misconceptions

- confusing the meanings of *event* and *experiment*
- confusing *union* and *intersection* of sets
- neglecting order, thereby neglecting to identify different outcomes such as HT and TH

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Guided Practice 4.1.1

Example 1

The table that follows shows a group of students and the extra-curricular activities in which they participate.

Student	Chess club	Debate club	Band	German club
Asher			✔	
Ray				
Eva		✔		✔
Merida			✔	
Ysabel		✔	✔	
Ben	✔			
Nate	✔			✔

A student is chosen from the group at random. List the sample space and then describe each of the following events in terms of the student's activities.

{Eva, Nate} {Ysabel} {Ben, Nate, Eva} {Ray, Eva, Ben, Nate}

1. The sample space is the set of all possible outcomes.

   The sample space is {Asher, Ray, Eva, Merida, Ysabel, Ben, Nate}.

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2. Describe the event {Eva, Nate}.

   Eva is in the debate club and the German club.

   Nate is in the chess club and the German club.

   To write a description of {Eva, Nate}, consider using the word "or." The activities mentioned are: debate club, German club, and chess club. You might consider this description: "in the debate club, German club, or chess club." But that description would also include Ysabel, who is in the debate club, and Ben, who is in the chess club. So that description is not correct.

   Consider using the word "and." Identify the activity or activities Eva and Nate have in common. They are both in the German club. There are no other students in the German club, so the event can be described as "in the German club."

   A correct description is "the chosen student is in the German club."

3. Describe the event {Ysabel}.

   Ysabel is in the debate club and the band, and there are no other students in both of those activities.

   Therefore, a correct description is "the chosen student is in the debate club and the band."

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4. Describe the event {Ben, Nate, Eva}.

   Ben is in the chess club.

   Nate is in the chess club and the German club.

   Eva is in the debate club and the German club.

   There is no activity common to all three students, so the description will not use the word "and."

   Consider using the word "or." The activities mentioned are: chess club, German club, and debate club. You might consider this description: "in the chess club, German club, or debate club." But that description would also include Ysabel, who is in the debate club. So that description is not correct.

   A correct description is "the chosen student is in the chess club or German club."

5. Describe the event {Ray, Eva, Ben, Nate}.

   Ray is in no activity.

   Eva is in the debate club and the German club.

   Ben is in the chess club.

   Nate is in the chess club and the German club.

   Since Ray is in no activity, consider using the word "not." Identify any activity that none of the four students are in.

   A correct description is "the chosen student is not in the band."

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Example 2

Some students were asked what pets they have at home. The following table shows the results of the survey, with the students identified by numbers.

Student	Dog	Cat	Hamster	Bird	Fish
1					
2	✔	✔			
3					
4	✔				
5			✔		✔
6					✔
7	✔	✔		✔	
8					
9		✔		✔	
10	✔	✔			

A student is chosen from the group at random. Consider the following events.

D: The student has a dog.

C: The student has a cat.

H: The student has a hamster.

B: The student has a bird.

F: The student has a fish.

Describe each of the following events by listing outcomes.

C $\quad D \cap B$ $\quad H \cup F$ $\quad \overline{B}$ $\quad \overline{D \cap C}$ $\quad \overline{D \cup C}$

1. List the outcomes of C.

   Identify the students who have a cat. List those student numbers in set notation.

   $C = \{2, 7, 9, 10\}$

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2. List the outcomes of $D \cap B$.

   Identify the students who have a dog. List those student numbers in set notation.

   $D = \{2, 4, 7, 10\}$

   Identify the students who have a bird. List those student numbers in set notation.

   $B = \{7, 9\}$

   $D \cap B$ is the intersection of events D and B or the student(s) who have both a dog and a bird. Identify the outcome(s) common to both events.

   $D \cap B = \{7\}$

3. List the outcomes of $H \cup F$.

   Identify the students who have a hamster. List those student numbers in set notation.

   $H = \{5\}$

   Identify the students who have a fish. List those student numbers in set notation.

   $F = \{5, 6\}$

   $H \cup F$ is the union of events H and F or the students who have a hamster or a fish or who have both. Identify the outcomes that appear in either event or both events.

   $H \cup F = \{5, 6\}$

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4. List the outcomes of $\overline{B}$.

Identify the students who have a bird. List those student numbers in set notation.

$B = \{7, 9\}$

Sample space = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

$\overline{B}$ is the set of all outcomes that are in the sample space, but not in B. In other words, $\overline{B}$ is all the students who don't have a bird.

$\overline{B} = \{1, 2, 3, 4, 5, 6, 8, 10\}$

5. List the outcomes of $\overline{D \cap C}$.

To find the outcomes of $\overline{D \cap C}$, we must first determine the outcomes of $D \cap C$.

From step 2, we determined the outcomes in D to be as follows:

$D = \{2, 4, 7, 10\}$

From step 1, we determined the outcomes in C to be as follows:

$C = \{2, 7, 9, 10\}$

The intersection of D and C, then, is the outcome(s) that are in both D and C.

$D \cap C = \{2, 7, 10\}$

$\overline{D \cap C}$ is the set of all outcomes that are in the sample space, but not in $D \cap C$.

$\overline{D \cap C} = \{1, 3, 4, 5, 6, 8, 9\}$

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6. List the outcomes of $\overline{D \cup C}$.

   First, determine the outcomes of $D \cup C$.

   $D = \{2, 4, 7, 10\}$

   $C = \{2, 7, 9, 10\}$

   $D \cup C = \{2, 4, 7, 9, 10\}$

   $\overline{D \cup C}$ is the set of all outcomes that are in the sample space, but not in $D \cup C$.

   $\overline{D \cup C} = \{1, 3, 5, 6, 8\}$

Example 3

Hector has entered the following names in the contact list of his new cell phone: Alicia, Brisa, Steve, Don, and Ellis. He chooses one of the names at random to call. Consider the following events.

B: The name begins with a vowel.

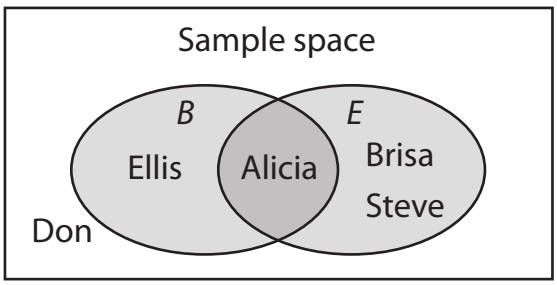
E: The name ends with a vowel.

Draw a Venn diagram to show the sample space and the events *B* and *E*. Then describe each of the following events by listing outcomes.

B E $B \cap E$ $B \cup E$ $\overline{B}$ $\overline{B \cup E}$

1. Draw a Venn diagram. Use a rectangle for the sample space. Use circles or elliptical shapes for the events *B* and *E*.

   Write the students' names in the appropriate sections to show what events they are in.



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2. List the outcomes of B.

B = {Ellis, Alicia}

3. List the outcomes of E.

E = {Alicia, Brisa, Steve}

4. List the outcomes of $B \cap E$.

B = {Ellis, Alicia}

E = {Alicia, Brisa, Steve}

$B \cap E$ is the intersection of events B and E. Identify the outcome(s) common to both events.

$B \cap E$ = {Alicia}

5. List the outcomes of $B \cup E$.

B = {Ellis, Alicia}

E = {Alicia, Brisa, Steve}

$B \cup E$ is the union of events B and E. Identify the outcomes that appear in either event or both events.

$B \cup E$ = {Ellis, Alicia, Brisa, Steve}

6. List the outcomes of $\overline{B}$.

B = {Ellis, Alicia}

Sample space = {Alicia, Brisa, Steve, Don, Ellis}

$\overline{B}$ is the set of all outcomes that are in the sample space, but not in B.

$\overline{B}$ = {Brisa, Steve, Don}

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7. List the outcomes of $\overline{B \cup E}$.

B = {Ellis, Alicia}

E = {Alicia, Brisa, Steve}

$B \cup E$ = {Ellis, Alicia, Brisa, Steve}

Sample space = {Alicia, Brisa, Steve, Don, Ellis}

$\overline{B \cup E}$ is the set of all outcomes that are in the sample space, but not in $B \cup E$.

$\overline{B \cup E}$ = {Don}

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Example 4

An experiment consists of tossing a coin three times. Consider the following events.

A: The first toss is heads.

B: The second toss is heads.

C: There are two consecutive heads.

D: There are two consecutive tails.

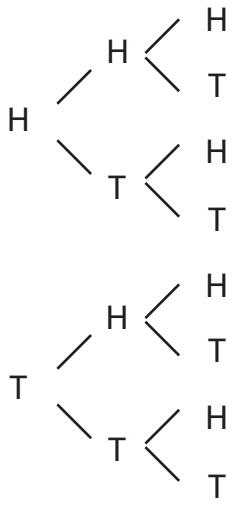
E: The first toss is heads and the second toss is heads.

F: There are neither two consecutive heads nor two consecutive tails.

List the sample space. Then express events *E* and *F* in terms of other events and list the outcomes of *E* and *F*.

1. The sample space is the set of all possible outcomes.

   The following tree diagram shows all possible outcomes, using H for heads and T for tails.



(*continued*)

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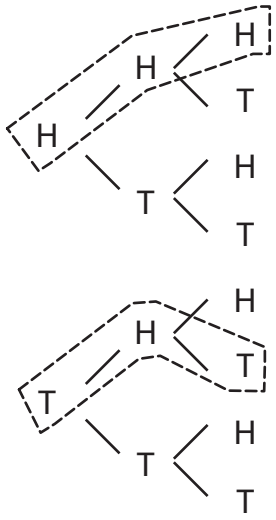
To identify all the possible outcomes, trace all the different paths from left to right. The following diagram shows two different paths. The first indicated path identifies the outcome HHH, which means heads on all three coin tosses. The second indicated path identifies the outcome THT, which means tails, then heads, then tails.



There are eight different paths, indicating the following sample space:

{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

2. To express event E in terms of other events, first note how the descriptions of A, B, and E are related.

   A: The first toss is heads.

   B: The second toss is heads.

   E: The first toss is heads and the second toss is heads.

   Using the word "and" in this way indicates that E is the intersection of A and B.

   $E = A \cap B$

   To list the outcomes of E, list all the outcomes that are in both A and B.

   $E = A \cap B = \{HHH, HHT\}$

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3. To express event F in terms of other events, first note how the descriptions of C, D, and F are related.

   C: There are two consecutive heads.

   D: There are two consecutive tails.

   F: There are neither two consecutive heads nor two consecutive tails.

   The union of C and D, denoted $C \cup D$, is the event that there are either two consecutive heads or two consecutive tails.

   $C \cup D$ = {HHH, HHT, HTT, THH, TTH, TTT}

   F is the event that there are *neither* two consecutive heads *nor* two consecutive tails. So F is the complement of $C \cup D$.

   $F = \overline{C \cup D}$

   To list the outcomes of F, list all the outcomes that are in the sample space, but not in $C \cup D$.

   Sample space = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

   $F = \overline{C \cup D}$ = {HTH, THT}

Example 5

An experiment consists of rolling a pair of dice. How many ways can you roll the dice so that the product of the two numbers rolled is less than their sum?

1. Begin by showing the sample space.

   This diagram of ordered pairs shows the sample space.

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

   Key: (2, 3) means 2 on the first die and 3 on the second die.

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2. Identify all the outcomes in the event "the product is less than the sum."

For (1, 1), the product is $1 \times 1 = 1$ and the sum is $1 + 1 = 2$, so the product is less than the sum.

For (1, 2), the product is $1 \times 2 = 2$ and the sum is $1 + 2 = 3$, so the product is less than the sum.

This table shows all the possible products and sums.

Outcome	Product	Sum	Product < sum	Outcome	Product	Sum	Product < sum
(1, 1	1	2	Yes	(4, 1)	4	5	Yes
(1, 2)	2	3	Yes	(4, 2)	8	6	No
(1, 3)	3	4	Yes	(4, 3)	12	7	No
(1, 4)	4	5	Yes	(4, 4)	16	8	No
(1, 5)	5	6	Yes	(4, 5)	20	9	No
(1, 6)	6	7	Yes	(4, 6)	24	10	No
(2, 1)	2	3	Yes	(5, 1)	5	6	Yes
(2, 2)	4	4	No	(5, 2)	10	7	No
(2, 3)	6	5	No	(5, 3)	15	8	No
(2, 4)	8	6	No	(5, 4)	20	9	No
(2, 5)	10	7	No	(5, 5)	25	10	No
(2, 6)	12	8	No	(5, 6)	30	11	No
(3, 1)	3	4	Yes	(6, 1)	6	7	Yes
(3, 2)	6	5	No	(6, 2)	12	8	No
(3, 3)	9	6	No	(6, 3)	18	9	No
(3, 4)	12	7	No	(6, 4)	24	10	No
(3, 5)	15	8	No	(6, 5)	30	11	No
(3, 6)	18	9	No	(6, 6)	36	12	No

By checking all the outcomes in the sample space, you can verify that the product is less than the sum for only these outcomes:

(1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6)
(2, 1) (3, 1) (4, 1) (5, 1) (6, 1)

3. Count the outcomes that meet the event criteria.

There are 11 ways to roll two dice so that the product is less than the sum.

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Problem-Based Task 4.1.1: Recognizing Equal Sets

SMP	
1 ✓	2
3 ✓	4
5	6
7	8 ✓

William is playing a game that involves rolling dice. On one of his turns he rolls a first die and then a second die. Consider the following events along with the sample space, *S*.

A: The first die is odd.

B: The sum is odd.

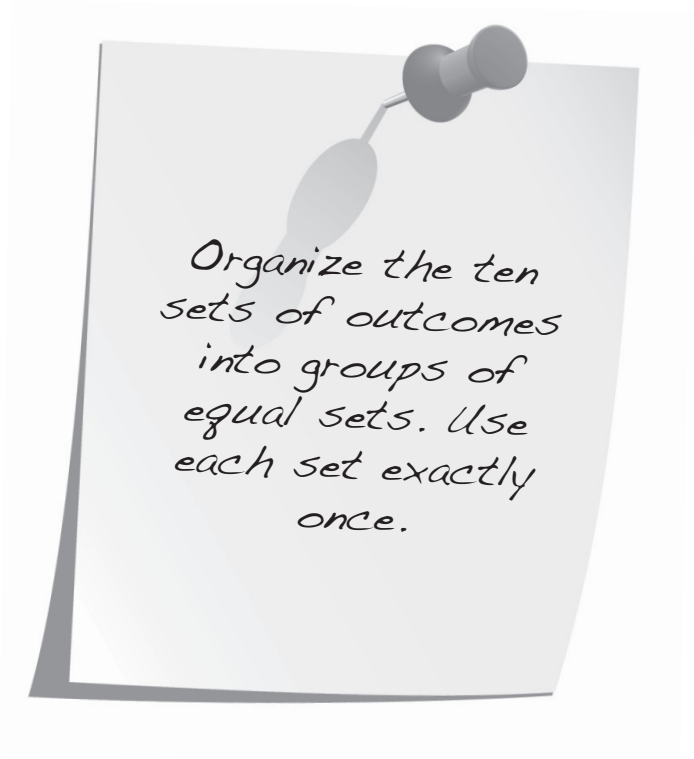
C: There is exactly one odd.

D: The sum is even.

E: The product is odd.

Organize the following ten sets of outcomes into groups of equal sets. Use each set exactly once.

$A, B, C, \overline{D}, E, A \cup E, A \cap E, B \cup D, \overline{B \cap D}, S$



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Problem-Based Task 4.1.1: Recognizing Equal Sets

Coaching

a. What is the sample space?

b. What are the outcomes in $\overline{D}$?

c. What are the outcomes in $A \cup E$?

d. What are the outcomes in $A \cap E$?

e. What are the outcomes in $B \cup D$?

f. What are the outcomes in $\overline{B \cap D}$?

g. Now list the ten sets you need to place into groups of equal sets, showing the outcomes in each.

h. Finally, organize the ten sets into groups of equal sets.

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Instruction

Problem-Based Task 4.1.1: Recognizing Equal Sets

Coaching Sample Responses

a. What is the sample space?

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

b. What are the outcomes in $\overline{D}$?

Note: For convenience, each outcome is written as a pair of single-digit integers. So, 11 represents (1, 1), 12 represents (1, 2), and so on.

$D = \{11, 13, 15, 22, 24, 26, 31, 33, 35, 42, 44, 46, 51, 53, 55, 62, 64, 66\}$

$\overline{D}$ is made up of all the elements in the sample space not found in D.

$\overline{D} = \{12, 14, 16, 21, 23, 25, 32, 34, 36, 41, 43, 45, 52, 54, 56, 61, 63, 65\}$

c. What are the outcomes in $A \cup E$?

$A = \{11, 12, 13, 14, 15, 16, 31, 32, 33, 34, 35, 36, 51, 52, 53, 54, 55, 56\}$

$E = \{11, 13, 15, 31, 33, 35, 51, 53, 55\}$

$A \cup E$ is made up of all the elements that are found in A or E or both.

$A \cup E = A = \{11, 12, 13, 14, 15, 16, 31, 32, 33, 34, 35, 36, 51, 52, 53, 54, 55, 56\}$

d. What are the outcomes in $A \cap E$?

$A = \{11, 12, 13, 14, 15, 16, 31, 32, 33, 34, 35, 36, 51, 52, 53, 54, 55, 56\}$

$E = \{11, 13, 15, 31, 33, 35, 51, 53, 55\}$

$A \cap E$ is made up of all the elements found in A and E.

$A \cap E = E = \{11, 13, 15, 31, 33, 35, 51, 53, 55\}$

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Instruction

e. What are the outcomes in $B \cup D$?

$B = \{12, 14, 16, 21, 23, 25, 32, 34, 36, 41, 43, 45, 52, 54, 56, 61, 63, 65\}$

$D = \{11, 13, 15, 22, 24, 26, 31, 33, 35, 42, 44, 46, 51, 53, 55, 62, 64, 66\}$

$B \cup D$ is made up of all the elements in B or D or both.

$B \cup D = S = \{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}$

f. What are the outcomes in $\overline{B \cap D}$?

$B = \{12, 14, 16, 21, 23, 25, 32, 34, 36, 41, 43, 45, 52, 54, 56, 61, 63, 65\}$

$D = \{11, 13, 15, 22, 24, 26, 31, 33, 35, 42, 44, 46, 51, 53, 55, 62, 64, 66\}$

$B \cap D = \varnothing$, the empty set.

$\overline{B \cap D}$ is the set of all outcomes that are in the sample space, but not in $B \cap D$.

There are no outcomes in $B \cap D$, so $\overline{B \cap D}$ is the entire sample space.

Therefore, $\overline{B \cap D} = S$.

g. Now list the ten sets you need to place into groups of equal sets, showing the outcomes in each.

$A = \{11, 12, 13, 14, 15, 16, 31, 32, 33, 34, 35, 36, 51, 52, 53, 54, 55, 56\}$

$B = \{12, 14, 16, 21, 23, 25, 32, 34, 36, 41, 43, 45, 52, 54, 56, 61, 63, 65\}$

$C = \{12, 14, 16, 21, 23, 25, 32, 34, 36, 41, 43, 45, 52, 54, 56, 61, 63, 65\}$

$\overline{D} = \{12, 14, 16, 21, 23, 25, 32, 34, 36, 41, 43, 45, 52, 54, 56, 61, 63, 65\}$

$E = \{11, 13, 15, 31, 33, 35, 51, 53, 55\}$

$A \cup E = A$

$A \cap E = E$

$B \cup D = S$

$\overline{B \cap D} = S$

$S = \{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}$

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h. Finally, organize the ten sets into groups of equal sets.

$B = C = \overline{D}$

$A \cup E = A$

$A \cap E = E$

$B \cup D = \overline{B \cap D} = S$

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4.1.1: Describing Events

The table below shows a group of students and some of the sports in which they participate. Use the table to complete problems 1 and 2.

Student	Soccer	Basketball	Softball	Volleyball
Omar		✔		
Julia			✔	
Mason				
Jacey	✔	✔		
Xin	✔			
Paul	✔			✔

1. A student is chosen from the group at random. Describe the event {Omar, Julia, Jacey}.

2. A student is chosen from the group at random. Describe the event {Jacey}.

Some students were asked what types of jobs they have had since beginning high school. The table below shows the results of the survey, with the students identified by their initials. Use the table and the information that follows to complete problems 3 and 4.

Student	Babysitting	Fast food	Retail	Office work	Lawn care
RJ	✔	✔		✔	
AS	✔		✔		
AL			✔	✔	
JC		✔			✔
LL	✔	✔	✔		
DB					
NP	✔				
DL					✔

A student is chosen from the group at random. Consider the following events.

B: The student has held a babysitting job.

F: The student has worked at a fast-food business.

R: The student has worked at a retail business.

O: The student has worked at an office.

L: The student has held a lawn-care job.

continued

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3. List the outcomes of $B \cap R$.

4. List the outcomes of $\overline{F \cup R}$.

Use the given information to complete problems 5 and 6.

Five students have the following locker combinations:

9–20–10 20–12–13 8–5–18 21–13–11 5–13–14

One of the combinations is chosen at random. Consider the following events.

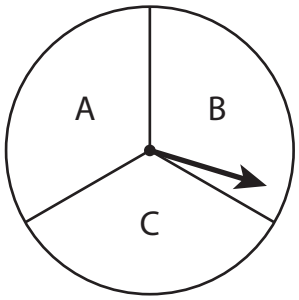
E: The chosen combination contains an even integer.

F: The chosen combination contains a factor of 36.

5. Draw a Venn diagram to show the sample space and the events *E* and *F*.

6. Describe the events $E \cap F$ and $\overline{E \cup F}$ by listing their outcomes.

An experiment consists of spinning the following spinner two times. Use the spinner diagram and the information that follows to complete problems 7 and 8.



Consider the following events.

P: two vowels

Q: no vowels

R: two consonants

S: exactly one vowel

T: exactly one consonant

continued

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7. Choosing from *P, Q, R,* and *T,* what event(s) are the same as *S*?

8. Choosing from *P, Q, R, S,* and *T,* what event(s) are the same as $\overline{P}$?

For problems 9 and 10, consider the experiment of rolling a pair of dice.

9. How many ways can you roll a pair of dice and get a sum of 7?

10. How many ways can you roll a pair of dice and get an even product?

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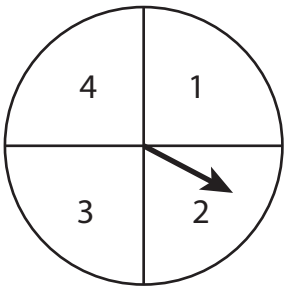
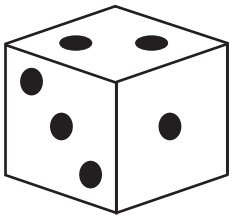
Lesson 4.1.2: The Addition Rule

Warm-Up 4.1.2

Jacquez is playing a board game. To find the number of spaces he gets to move, he spins the spinner and rolls a standard 6-sided die. Then he can choose either the sum or the product of the two results. He is hoping to move between 5 and 10 spaces (not including 5 or 10) on his next turn.





1. How many outcomes are in the sample space? Show those outcomes.

2. How many ways can Jacquez get what he is hoping for? Show those outcomes.

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Instruction

Lesson 4.1.2: The Addition Rule

Common Core State Standard

S–CP.7 Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model.★

Warm-Up 4.1.2 Debrief

1. How many outcomes are in the sample space? Show those outcomes.

   There are 24 outcomes:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)

   Key: (2, 3) means 2 on the spinner and 3 on the die.

2. How many ways can Jacquez get what he is hoping for? Show those outcomes.

   Of the 24 outcomes, there are 15 ways Jacquez can get what he is hoping for:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)

Connection to the Lesson

- Students will identify sample spaces and count the outcomes in sample spaces.
- "Getting a sum or the product between 5 and 10" describes an event; students will identify events and count the outcomes in the events.

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Instruction

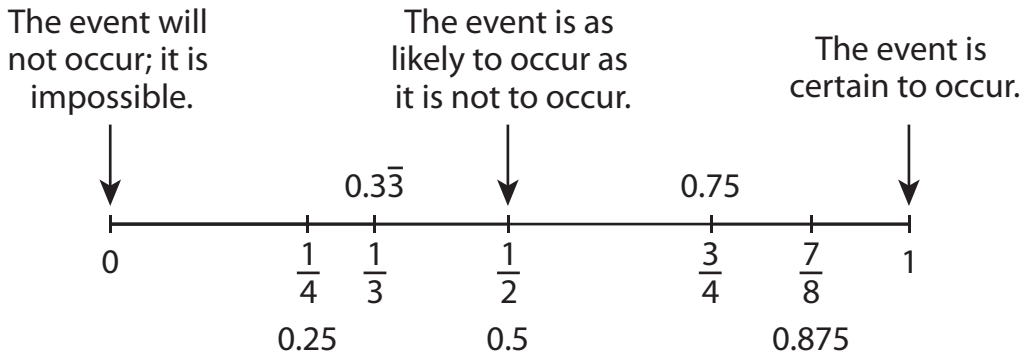
Prerequisite Skills

This lesson requires the use of the following skills:

- identifying sample spaces (7.SP.1)
- identifying events (7.SP.8c)
- describing events as subsets of sample spaces (S–CP.1)
- describing events as unions, intersections, or complements of other events (S–CP.1)
- simplifying fractions (4.NF.1)
- writing rational numbers in equivalent forms using fractions, decimals, and percents (7.EE.3)

Introduction

Remember that probability is a number from 0 to 1 inclusive or a percent from 0% to 100% inclusive that indicates how likely an event is to occur. A probability of 0 indicates that the event cannot occur. A probability of 1 indicates that the event is certain to occur. The following diagram shows some sample probabilities.



When all the outcomes of an experiment are equally likely, the **probability of an event *E***, denoted $P(E)$, is given by $P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}$.

For situations involving more than one event, the Addition Rule sometimes applies. You will learn about this rule.

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Instruction

Key Concepts

- Probability is the likelihood of an event.
- A **probability model** is a mathematical model for observable facts or occurrences that are assumed to be random. The facts or occurrences are called outcomes. Actions and processes that produce outcomes are called experiments.
- When all the outcomes of an experiment are assumed to be equally likely, then the probability model is a **uniform probability model**.
- In a uniform probability model, the probability of an event E, denoted $P(E)$, is given by the formula $P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}$.
- The frequency of an event is the number of times it occurs. The **relative frequency** of an event is the number of times it occurs divided by the number of times the experiment is performed.
- The probability of an event can be used to predict its relative frequency if the experiment is performed a large number of times. For example, the probability of getting a 3 by rolling a fair die is $\frac{1}{6}$. So if you roll a fair die 600 times, it is reasonable to predict that you will get a 3 about 100 times, or $\frac{1}{6}$ of the number of times you roll the die.
- According to the **Addition Rule,** if A and B are any two events, then the probability of A or B, denoted $P(A \text{ or } B)$, is given by the formula $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.
- Using set notation, the rule is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- The Addition Rule contains four probabilities: $P(A \text{ or } B)$, $P(A)$, $P(B)$, and $P(A \text{ and } B)$; if three of them are known, then the equation can be solved to find the fourth.
- If events A and B have no outcomes in common, then they are **mutually exclusive events**, also known as **disjoint events**. If A and B are mutually exclusive events, then they cannot both occur, so $P(A \text{ and } B) = 0$.
- This leads to the following special case of the Addition Rule for mutually exclusive events:

  $P(A \text{ or } B) = P(A) + P(B)$, or $P(A \cup B) = P(A) + P(B)$

Common Errors/Misconceptions

- confusing the meanings of "and" and "or" in probability models
- neglecting to apply the Addition Rule correctly for events that are not mutually exclusive, thereby neglecting to subtract $P(A \text{ and } B)$ when calculating $P(A \text{ or } B)$

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Instruction

Guided Practice 4.1.2

Example 1

Bobbi tosses a coin 3 times. What is the probability that she gets exactly 2 heads? Write your answer as a fraction, as a decimal, and as a percent.

1. Identify the sample space and count the outcomes.

   Sample space = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

   There are 8 outcomes in the sample space.

2. Identify the outcomes in the event and count the outcomes.

   There are 3 outcomes in the event "exactly 2 heads:" HHT, HTH, and THH.

3. Apply the formula for the probability of an event.

   Divide the number of outcomes for the event Bobbi is hoping for (3) by the total number of possible outcomes (8).

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}$$

$$P(\text{exactly 2 heads}) = \frac{3}{8} = 0.375 = 37.5\%$$

Example 2

Donte is playing a card game with a standard 52-card deck. He's hoping for a club or a face card on his first draw. What is the probability that he draws a club or a face card on his first draw?

1. Identify the sample space and count the outcomes.

   The sample space is the set of all cards in the deck, so there are 52 outcomes.

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Instruction

2. Identify the outcomes in the event and count the outcomes.

You can use a table to show the sample space. Then identify and count the cards that are either a club or a face card or both a club and a face card.

Suit	2	3	4	5	6	7	8	9	10	J	Q	K	A
Spade										✔	✔	✔	
Club	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔	✔
Diamond										✔	✔	✔	
Heart										✔	✔	✔	

The event "club or face card" has 22 outcomes.

3. Apply the formula for the probability of an event.

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}$$

$$P(\text{club or face card}) = \frac{22}{52} = \frac{11}{26} \approx 0.42$$

4. Apply the Addition Rule to verify your answer.

Let A be the event "club." There are 13 clubs, so A has 13 outcomes.

Let B be the event "face card." There are 12 face cards, so B has 12 outcomes.

The event "A and B" is the event "club and face card," which has 3 outcomes: jack of clubs, queen of clubs, and king of clubs.

Apply the Addition Rule.

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

$$P(\text{club or face card}) = P(\text{club}) + P(\text{face card}) - P(\text{club and face card})$$

$$P(\text{club or face card}) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52} = \frac{11}{26} \approx 0.42$$

The Addition Rule answer checks out with the probability found in step 3, so the probability of the event is approximately 0.42.

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Instruction

Example 3

Corrine is playing a board game. To find the number of spaces to move, she rolls a pair of dice. On her next roll she wants doubles or a sum of 10. What is the probability that she rolls doubles or a sum of 10 on her next roll? Interpret your answer in terms of a uniform probability model.

1. Identify the sample space and count the outcomes.

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

There are 36 outcomes in the sample space.

2. Apply the Addition Rule to find the probability that she rolls doubles or a sum of 10 on her next roll.

Let A be the event "doubles." Event A has 6 outcomes: (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), and (6, 6).

Let B be the event "sum of 10." Event B has 3 outcomes: (4, 6), (5, 5), and (6, 4).

The event "A and B" is the event "doubles and sum of 10," which has 1 outcome: (5, 5).

Apply the Addition Rule.

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$P(\text{doubles or sum of 10}) = P(\text{doubles}) + P(\text{sum of 10}) - P(\text{doubles and sum of 10})$

$$P(\text{doubles or sum of 10}) = \frac{6}{36} + \frac{3}{36} - \frac{1}{36} = \frac{8}{36} = \frac{2}{9} \approx 0.22 \approx 22\%$$

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Instruction

3. Interpret the answer in terms of a uniform probability model.

The probabilities used in the application of the Addition Rule, $\frac{6}{36}$, $\frac{3}{36}$, and $\frac{1}{36}$, are found by using the formula $P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}$.

This formula is a uniform probability model, which requires that all outcomes in the sample space be equally likely. It is reasonable to assume that the dice Corrine rolls are fair, so that all outcomes in the sample space are equally likely. Therefore, the answer is valid and can serve as a reasonable predictor. You can predict that the relative frequency of getting doubles or a sum of 10 will be about 22% for a large number of dice rolls. That is, you can predict that Corrine will get doubles or a sum of 10 about 22% of the time if she rolls the dice a large number of times.

Example 4

Students at Rolling Hills High School receive an achievement award for either performing community service or making the honor roll. The school has 500 students and 180 of them received the award. There were 125 students who performed community service and 75 students who made the honor roll. What is the probability that a randomly chosen student at Rolling Hills High School performed community service and made the honor roll?

1. Define the sample space and state its number of outcomes.

The sample space is the set of all students at the school; it has 500 outcomes.

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Instruction

2. Define events that are associated with the numbers in the problem and state their probabilities.

   Let A be the event "performed community service." Then $P(A)=\frac{125}{500}$.

   Let B be the event "made the honor roll." Then $P(B)=\frac{75}{500}$.

   The event "A or B" is the event "performed community service or made the honor roll," and can also be written $A\cup B$. $P(A\cup B)=\frac{180}{500}$ because 180 students received the award for either community service or making the honor roll.

3. Write the Addition Rule and solve it for $P(A \text{ and } B)$, which is the probability of the event "performed community service and made the honor roll." $P(A \text{ and } B)$ can also be written $P(A\cap B)$.

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$\frac{180}{500}=\frac{125}{500}+\frac{75}{500}-P(A\cap B)$ Substitute the known probabilities.

$\frac{180}{500}=\frac{200}{500}-P(A\cap B)$ Simplify.

$-\frac{20}{500}=-P(A\cap B)$ Subtract $\frac{200}{500}$ from both sides.

$\frac{20}{500}=P(A\cap B)$ Multiply both sides by –1.

The probability that a randomly chosen student at Rolling Hills High School has performed community service and is on the honor roll is $\frac{20}{500}=\frac{1}{25}$, or 4%.

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Problem-Based Task 4.1.2: Paying the Bonus?

SMP	
1	2
3 ✓	4 ✓
5	6
7 ✓	8

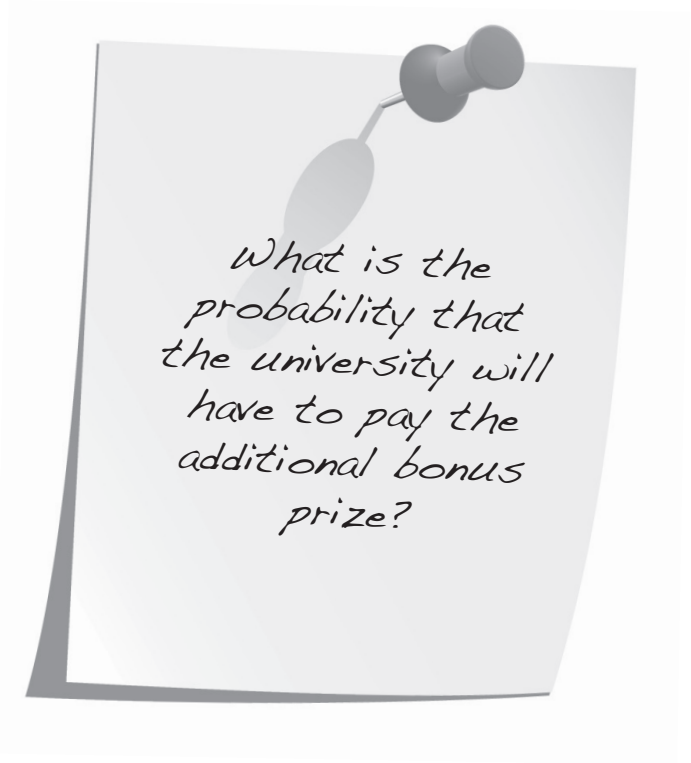
A university is offering a two-day conference for high school math teachers. Teachers can register for Monday or Tuesday or both days. The conference coordinator compiles the following data:

150 teachers registered for Monday.

95 teachers registered for Tuesday.

210 teachers registered in all.

One registered teacher selected at random will win a laptop computer as a door prize. If the selected teacher has registered for both days, he or she will win an additional bonus prize of $500. What is the probability that the university will have to pay the additional bonus prize?



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Problem-Based Task 4.1.2: Paying the Bonus?

Coaching

a. What is the sample space?

b. How many outcomes are in the sample space?

c. Let M represent the set of all teachers registered for Monday. What is $P(M)$, the probability that the selected teacher is registered for Monday?

d. Let T represent the set of all teachers registered for Tuesday. What is $P(T)$, the probability that the selected teacher is registered for Tuesday?

e. You need to find the probability that the university will have to pay the additional bonus prize of $500. The occurrence of what event will require the university to pay the additional bonus?

f. How can you restate the event "registered for both days," using the words "Monday" and "Tuesday"?

g. Using the letters P, M, and T, how can you represent the probability of the event "registered for Monday and Tuesday"?

h. There are 210 teachers registered in all. How can you restate the phrase "registered in all," using the words "Monday" and "Tuesday"?

i. Using the letters P, M, and T, how can you represent the probability of the event "registered for Monday or Tuesday"?

j. What is the probability that the selected teacher is registered for Monday or Tuesday? Explain.

k. What is the rule that relates the probabilities $P(M)$, $P(T)$, $P(M \text{ or } T)$, and $P(M \text{ and } T)$?

l. What is the probability that the university will have to pay the additional bonus prize? Show and explain your work.

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Instruction

Problem-Based Task 4.1.2: Paying the Bonus?

Coaching Sample Responses

a. What is the sample space?

The sample space is all the registered teachers.

b. How many outcomes are in the sample space?

210

c. Let M represent the set of all teachers registered for Monday. What is $P(M)$, the probability that the selected teacher is registered for Monday?

$$P(M)=\frac{150}{210}$$

d. Let T represent the set of all teachers registered for Tuesday. What is $P(T)$, the probability that the selected teacher is registered for Tuesday?

$$P(T)=\frac{95}{210}$$

e. You need to find the probability that the university will have to pay the additional bonus prize of \$500. The occurrence of what event will require the university to pay the additional bonus?

The university will have to pay the additional bonus if the selected teacher is registered for both days.

f. How can you restate the event "registered for both days," using the words "Monday" and "Tuesday"?

The event can be restated as "registered for Monday and Tuesday."

g. Using the letters P, M, and T, how can you represent the probability of the event "registered for Monday and Tuesday"?

The event can be represented as $P(M \text{ and } T)$.

h. There are 210 teachers registered in all. How can you restate the phrase "registered in all," using the words "Monday" and "Tuesday"?

The phrase can be restated as "registered for Monday or Tuesday."

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Instruction

i. Using the letters *P*, *M*, and *T*, how can you represent the probability of the event "registered for Monday or Tuesday"?

The event can be represented as $P(M \text{ or } T)$.

j. What is the probability that the selected teacher is registered for Monday or Tuesday? Explain.

$$P(M \text{ or } T) = \frac{210}{210} = 1$$

The teacher is selected from the sample space of the 210 registered teachers. All of those registered teachers are registered for Monday or Tuesday or both days, so it is certain that the selected teacher is registered for Monday or Tuesday.

k. What is the rule that relates the probabilities $P(M)$, $P(T)$, $P(M \text{ or } T)$, and $P(M \text{ and } T)$?

The Addition Rule relates the probabilities: $P(M \text{ or } T) = P(M) + P(T) - P(M \text{ and } T)$

l. What is the probability that the university will have to pay the additional bonus prize? Show and explain your work.

$P(M \text{ or } T) = P(M) + P(T) - P(M \text{ and } T)$

The addition rule says that you add the probabilities of each event and subtract out the probability of both events occurring. Without subtracting out the intersection of the events, the probabilities of those events are counted twice.

$$\frac{210}{210} = \frac{150}{210} + \frac{95}{210} - P(M \text{ and } T)$$

$$\frac{210}{210} = \frac{245}{210} - P(M \text{ and } T)$$

$$-\frac{35}{210} = -P(M \text{ and } T)$$

$$\frac{35}{210} = P(M \text{ and } T)$$

The probability that the university will have to pay the additional bonus prize is $\frac{35}{210} = \frac{1}{6} \approx 0.17$, or about 17%.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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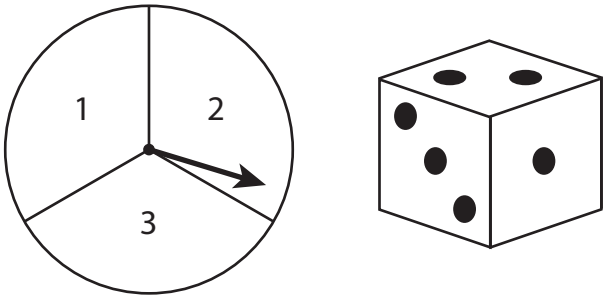
Practice 4.1.2: The Addition Rule

Use what you have learned about probability to solve the following problems.

1. Nasir tosses a coin 3 times. What is the probability that he gets at least 2 tails? Write your answer as a fraction, as a decimal, and as a percent.

Use the following diagram and information to complete problems 2 and 3.

Bianca is playing a board game. She spins the spinner and rolls a standard 6-sided die. Let s be the spinner result and let d be the die result.



2. What is the probability that $s + d > 3$ and $s \cdot d > 3$?

3. What is the probability that $s + d > 5$ or $s \cdot d < 10$?

4. Consuela is playing a card game with a standard 52-card deck. She wants a king or a diamond on her first draw. What is the probability that she gets a king or a diamond on her first draw?

Ladarius is playing a board game. To find the number of spaces to move, he rolls a pair of dice. Use this information to complete problems 5 and 6.

5. What is the probability that Ladarius rolls doubles or a sum of 7? Write your answer as a fraction, as a decimal, and as a percent.

6. What is the probability that Ladarius rolls a sum that is less than 4 or greater than 5?

continued

SWB p. 27; SRB Practice 4.1.2 p. U4-31

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Middletown High School has 240 students in the tenth grade. Use this information to complete problems 7 and 8.

7. The only tenth grade math courses are algebra and geometry. All the tenth grade students are taking at least one math course. There are 142 taking algebra and 120 taking geometry. What is the probability that a randomly chosen tenth grader is taking both algebra and geometry?

8. The only tenth grade music courses are vocal and instrumental. There are 89 tenth grade students taking at least one music course. There are 51 taking vocal and 58 taking instrumental. What is the probability that a randomly chosen tenth grader is taking both vocal and instrumental music?

Use the information to complete problems 9 and 10.

The table below lists the first name and date of birth for five students. The information is presented in letters (for the names) and digits (for the birth dates). The total characters for any given student is made up of both letters and digits in their data. The characters of the names are not case sensitive. For example, Trent's characters are t, r, e, n, t, 8, 1, 2, 9, and 7.

Trent	8–12–97
Ramiro	6–15–98
Mara	10–9–98
Pam	2–12–98
Fina	7–15–97

9. If a student is chosen at random from the group, what is the probability that the choice will contain a repeated letter or a repeated digit?

10. If a student is chosen at random from the group, what is the probability that the choice will contain more than 10 characters and a digit sum greater than 20?

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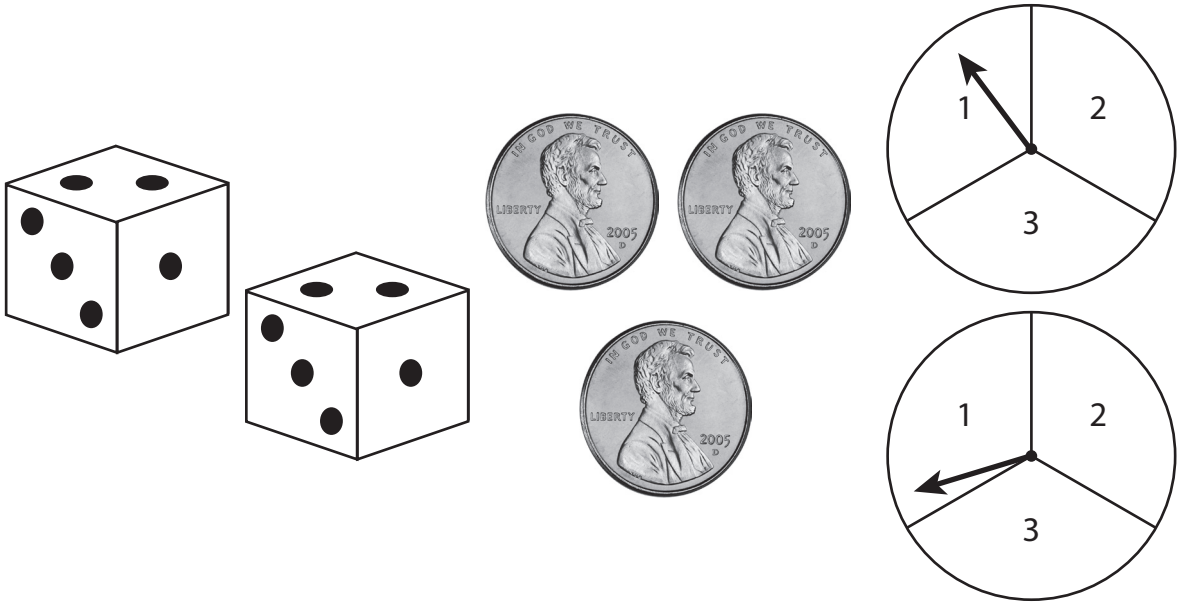
Lesson 4.1.3: Understanding Independent Events

Warm-Up 4.1.3

Tonya's class is studying probability by conducting probability experiments. Tonya can choose one of three experiments. If she gets an outcome with all matching results, she wins a prize. The experiments are rolling a pair of dice, tossing 3 coins, or spinning two spinners.



Which experiment should Tonya choose to have the best chance of winning the prize? Explain.

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Lesson 4.1.3: Understanding Independent Events

Common Core State Standard

S–CP.2 Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent.★

Warm-Up 4.1.3 Debrief

To determine the experiment with the best chance of winning the prize, determine the possible outcomes for each experiment and how many of those outcomes have matching results.

There are 36 possible outcomes for the dice; 6 of those outcomes have matching numbers for both dice.

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

$P(\text{all matching results}) = \frac{6}{36} = \frac{1}{6}$

There are 8 possible outcomes for the pennies, 2 of which match:

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

$P(\text{all matching results}) = \frac{2}{8} = \frac{1}{4}$

There are 9 possible outcomes for the spinners, 3 of which match:

(1, 1)	(1, 2)	(1, 3)
(2, 1)	(2, 2)	(2, 3)
(3, 1)	(3, 2)	(3, 3)

$P(\text{all matching results}) = \frac{3}{9} = \frac{1}{3}$

$\frac{1}{3}$ is greater than $\frac{1}{6}$ and $\frac{1}{4}$. Therefore, Tonya should choose the spinners because they provide the greatest probability of getting all matching results.

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Instruction

Connection to the Lesson

- The warm-up requires students to work with experiments, sample spaces, outcomes, and probabilities; students will work with all these concepts in the lesson.
- Students will learn another rule for probability.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- calculating probabilities and expressing them in equivalent forms using fractions, decimals, and percents (7.SP.8a)
- adding, subtracting, multiplying, and dividing rational numbers (7.EE.3)
- applying the Addition Rule, which includes probabilities of unions and intersections (S–CP.7)
- solving equations (8.EE.7b)

Introduction

In probability, events are either dependent or independent. Two events are **independent** if the occurrence or non-occurrence of one event has no effect on the probability of the other event. If two events are independent, then you can simply multiply their individual probabilities to find the probability that both events will occur. If events are **dependent**, then the outcome of one event affects the outcome of another event. So it is important to know whether or not two events are independent.

Two events A and B are independent if and only if $P(A \text{ and } B) = P(A) \bullet P(B)$. But sometimes you only know two of these three probabilities and you want to find the third. In such cases, you can't test for independence, so you might assess the situation or nature of the experiment and then make an assumption about whether or not the events are independent. Then, based on your assumption, you can find the third probability in the equation.

Key Concepts

- Two events A and B are independent if and only if they satisfy the following test:

  $P(A \text{ and } B) = P(A) \bullet P(B)$

- Using set notation, the test is $P(A \cap B) = P(A) \bullet P(B)$.
- Sometimes it is useful or necessary to make an assumption about whether or not events are independent, based on the situation or nature of the experiment. Then any conclusions or answers are based on the assumption.
- In a uniform probability model, all the outcomes of an experiment are assumed to be equally likely, and the probability of an event E, denoted $P(E)$, is given by

  $$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}.$$

SRB p. U4-34

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Instruction

- When this definition of probability is used and the relevant probabilities are known, then the definition of independent events can be used to determine, verify, or prove that two events are dependent or independent.
- The relative frequency of an event is the number of times it occurs divided by the number of times the experiment is performed (called trials) or the number of observations:

$$\text{relative frequency} = \frac{\text{number of times event occurs}}{\text{number of trials or observations}}$$

- Relative frequency can be used to estimate probability in some cases where a uniform probability model does not seem appropriate. Such cases include data collected by surveys or obtained by observations.
- When relative frequency is used to estimate the relevant probabilities, then the definition of independent events can be used to determine whether two events *seem to be* dependent or independent, *based on the data*.
- Probability and relative frequency are related as follows:
  - The probability of an event can be used to predict its relative frequency if the experiment is performed a large number of times. For example, the probability of getting a 3 by rolling a fair die is $\frac{1}{6}$. So if you roll a fair die 6,000 times, it is reasonable to predict that you will get a 3 about 1,000 times, or $\frac{1}{6}$ of the number of times you roll the die. But a prediction is not a guarantee. If you roll a die 6,000 times, the number of times you get a 3 might not be 1,000.
  - Relative frequency can be used to predict the probability of an event. In general, as the number of trials or observations increases, the prediction becomes stronger. For example, suppose you ask 40 people for their favorite ice cream flavor. If 8 say chocolate, then you can predict that the probability of a randomly selected person saying chocolate is $\frac{8}{40}$, or 20%. But 40 people make up a small sample, so this is not a very strong prediction. Now suppose you ask 4,000 people who are randomly selected using a good sampling method. If 740 say chocolate, then you can predict that the probability of a randomly selected person saying chocolate is $\frac{740}{4000}$, or 18.5%; this is a stronger prediction.

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Instruction

- Also remember the Addition Rule: If A and B are any two events, then the probability of A or B, denoted $P(A \text{ or } B)$, is given by $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.
- Using set notation, the rule is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Common Errors/Misconceptions

- thinking that the actual relative frequency of an event will equal the probability of the event; for example, thinking 100 tosses of a fair coin will yield 50 heads and 50 tails
- thinking that a probability based on actual relative frequency is a "true fact;" that is, not understanding that the probability is just a number that is only as good as the assumptions and statistical sampling methods it is based on
- thinking that a probability based on past actual relative frequency and obtained using reasonable assumptions and sound statistical sampling methods can be used to guarantee the actual future relative frequency of an event; that is, not realizing that even a probability obtained by the best possible means is only a predictor, never a guarantee

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Guided Practice 4.1.3

Example 1

Ms. Martinez is teaching her students about probability. She has a bag of 6 marbles; 2 are red and 4 are black. She asked her students to conduct the following experiments.

> Experiment A: Pick a random marble from the bag, then *replace it*, and then pick a random marble from the bag again.
>
> Experiment B: Pick a random marble from the bag, *do not replace it*, and then pick a random marble from the bag again.

For each experiment, determine whether getting red on the first pick and getting black on the second pick are independent events.

1. Assign variables as names of the events.

   Let *RF* be the event that a red marble is picked first.

   Let *BS* be the event that a black marble is picked second.

2. For experiment A, determine whether *RF* and *BS* are independent events.

   In experiment A, the marble is *replaced* in the bag after the first pick, so the conditions are identical on the first and second picks. That is, for both picks, the bag has 2 red marbles and 4 black marbles. Whether event *RF* does or does not occur has no effect on the probability that *BS* occurs.

   So for experiment A, *RF* and *BS* are independent events.

3. For experiment B, determine whether *RF* and *BS* are independent events.

   In experiment B, the marble is *not replaced* in the bag after the first pick, so the conditions are different on the first and second picks. There are 6 marbles in the bag for the first pick and 5 marbles in the bag for the second pick. If event *RF* occurs, then for the second pick the bag will have 1 red marble and 4 black marbles. If event *RF* does not occur, then for the second pick the bag will have 2 red marbles and 3 black marbles. So whether event *RF* does or does not occur does affect the probability of *BS*.

   Therefore, for experiment B, *RF* and *BS* are dependent events.

SWB p. 33; SRB p. U4-36

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Instruction

Example 2

Trevor tosses a coin 3 times. Consider the following events.

A: The first toss is heads.

B: The second toss is heads.

C: There are exactly 2 consecutive heads.

For each of the following pairs of events, determine if the events are independent.

A and *B* (This is $A \cap B$ in set notation.)

A and *C* (This is $A \cap C$ in set notation.)

B and *C* (This is $B \cap C$ in set notation.)

1. List the sample space.

   Sample space = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

2. Use the sample space to determine the relevant probabilities.

$P(A) = \frac{4}{8} = \frac{1}{2}$	There are 4 outcomes with heads first.
$P(B) = \frac{4}{8} = \frac{1}{2}$	There are 4 outcomes with heads second.
$P(C) = \frac{2}{8} = \frac{1}{4}$	There are 2 outcomes with exactly 2 consecutive heads.
$P(A \cap B) = \frac{2}{8} = \frac{1}{4}$	There are 2 outcomes with heads first and heads second.
$P(A \cap C) = \frac{1}{8}$	There is 1 outcome with heads first and exactly 2 consecutive heads.
$P(B \cap C) = \frac{2}{8} = \frac{1}{4}$	There are 2 outcomes with heads second and exactly 2 consecutive heads.

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Instruction

3. Use the definition of independence to determine if the events are independent in each specified pair.

$P(A \cap B) = P(A) \bullet P(B)$

$\frac{1}{4} = \frac{1}{2} \bullet \frac{1}{2}$ A and B are independent.

$P(A \cap C) = P(A) \bullet P(C)$

$\frac{1}{8} = \frac{1}{2} \bullet \frac{1}{4}$ A and C are independent.

$P(B \cap C) = P(B) \bullet P(C)$

$\frac{1}{4} \neq \frac{1}{2} \bullet \frac{1}{4}$ B and C are dependent.

Example 3

Landen owns a delicatessen. He collected data on sales of his most popular sandwiches for one week and recorded it in the following table.

Sandwiches Sold in One Week

Bread choice	Sandwich choice			
	Landen's club	Turkey melt	Roasted chicken	Veggie delight
Country white	44	25	25	8
Whole wheat	24	28	26	34
Sourdough	24	27	24	31

Each of the following statements describes a pair of events. For each statement, determine if the events seem to be independent based on the data in the table.

A random customer orders Landen's club sandwich on country white bread.

A random customer orders the roasted chicken sandwich on whole wheat bread.

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Instruction

1. Find the totals of all the categories.

Bread choice	Sandwich choice				Total
	Landen's club	Turkey melt	Roasted chicken	Veggie delight	
Country white	44	25	25	8	102
Whole wheat	24	28	26	34	112
Sourdough	24	27	24	31	106
Total	92	80	75	73	320

2. For the first statement, assign variables as names of the events.

   LC: A random customer orders Landen's club sandwich.

   CW: A random customer orders country white bread.

3. Use the data to determine the relevant probabilities.

$P(LC) = \frac{92}{320}$ There were 320 sandwiches sold, and 92 of them were Landen's club.

$P(CW) = \frac{102}{320}$ There were 320 sandwiches sold, and 102 of them were on country white bread.

$P(LC \cap CW) = \frac{44}{320}$ There were 320 sandwiches sold, and 44 of them were Landen's club on country white bread.

4. Use the definition of independence to determine if the events seem to be independent.

$P(LC \cap CW) = P(LC) \bullet P(CW)$

$\frac{44}{320} = \frac{92}{320} \bullet \frac{102}{320}$ Substitute probabilities and simplify.

$0.138 \neq 0.092$ LC and CW seem to be dependent, based on the data.

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Instruction

5. For the second statement, assign variables as names of the events.

   RC: A random customer orders a roasted chicken sandwich.

   WW: A random customer orders whole wheat bread.

6. Use the data to determine the relevant probabilities.

$P(RC) = \frac{75}{320}$ There were 320 sandwiches sold, and 75 of them were roasted chicken.

$P(WW) = \frac{112}{320}$ There were 320 sandwiches sold, and 112 of them were on whole wheat bread.

$P(RC \cap WW) = \frac{26}{320}$ There were 320 sandwiches sold, and 26 of them ere roasted chicken on whole wheat bread.

7. Use the definition of independence to determine if the events seem to be independent.

$P(RC \cap WW) = P(RC) \bullet P(WW)$

$\frac{26}{320} = \frac{75}{320} \bullet \frac{112}{320}$ Substitute probabilities and simplify.

$0.081 \approx 0.082$

RC and *WW* seem to be independent based on the data.

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Instruction

Example 4

The Town Cinema is next to Rhiannon's Bistro. A consultant surveyed 200 people who saw a particular movie at the Town Cinema and also ate at Rhiannon's Bistro. The survey indicated that 80% liked the movie and 60% liked both the movie and their meal at Rhiannon's Bistro. Assume that liking the movie and liking a meal at Rhiannon's Bistro are independent events. Based on the survey, what is the probability that a randomly chosen moviegoer will like a meal at Rhiannon's Bistro?

1. Define the relevant events and assign them variable names.

   Let M be the event that a randomly chosen person will like the movie.

   Let R be the event that a randomly chosen person will like a meal at Rhiannon's Bistro.

   Therefore, "M and R" is the event that a randomly chosen person will like both the movie and a meal at Rhiannon's Bistro.

2. Assign relevant probabilities based on the survey results.

   $P(M) = 80\%$ To assign probability based on data, use relative frequency.

   $P(M \text{ and } R) = 60\%$

3. Write the equation contained in the definition of independent events. Solve it for the desired probability.

   $P(M \text{ and } R) = P(M) \bullet P(R)$

   $0.60 = 0.80 \bullet P(R)$ Substitute probabilities using decimal equivalents of percents.

   $\frac{0.60}{0.80} = P(R)$ Divide both sides by 0.80.

   $0.75 = P(R)$ Simplify.

   Based on the survey, the probability that a randomly chosen person will like a meal at Rhiannon's Bistro is 0.75, or 75%.

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Instruction

Example 5

Joel is in a basketball game. He has just tied the score in the final seconds of the game. He was fouled while shooting, so he is awarded a free throw. Moreover, the referee calls a technical foul on the other team because of bad behavior, so Joel's team is awarded a second free throw attempt. The coach assigns a different player, Rico, to shoot the second free throw. Joel has made 32 of 50 free throws so far this season, and Rico has made 42 of 48. What is the probability that Joel or Rico will make a free throw to win the game?

1. Define the relevant events and assign them variable names.

   Let J be the event that Joel makes his free throw.

   Let R be the event that Rico makes his free throw.

   "J or R" is the event that either Joel or Rico makes a free throw.

   "J and R" is the event that both Joel and Rico make a free throw.

2. Assign relevant probabilities based on past performance over the season.

   $P(J) = \frac{32}{50}$ Joel has made 32 of 50 free throws so far this season.

   $P(R) = \frac{42}{48}$ Rico has made 42 of 48 free throws so far this season.

3. Decide if it is reasonable to assume that the events in the problem are independent.

   It is reasonable to assume that the probability of either player making his free throw is not affected by whether the other makes his free throw. Therefore, the events J and R are independent.

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Instruction

4. Use the Addition Rule and the definition of independent events. Solve the equation for the desired probability.

$P(J \text{ or } R) = P(J) + P(R) - P(J \text{ and } R)$	Write the Addition Rule.
$P(J \text{ or } R) = P(J) + P(R) - P(J) \bullet P(R)$	J and R are independent, so substitute $P(J) \bullet P(R)$ for $P(J \text{ and } R)$.
$P(J \text{ or } R) = \frac{32}{50} + \frac{42}{48} - \left(\frac{32}{50}\right)\left(\frac{42}{48}\right)$	Substitute probabilities.
$P(J \text{ or } R) = 0.64 + 0.875 - (0.64)(0.875)$	Substitute decimal equivalents.
$P(J \text{ or } R) = 0.64 + 0.875 - 0.56$	Simplify, then solve.
$P(J \text{ or } R) = 0.955$	

Based on past performance this season and assuming the events are independent, the probability that either Joel or Rico will make a free throw to win the game is 0.955, or 95.5%.

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Problem-Based Task 4.1.3: Tossing Coins

SMP	
1 ✓	2 ✓
3	4
5	6
7	8 ✓

Three events, A, B, and C, are independent if the following conditions are all satisfied:

$$P(A \cap B) = P(A) \bullet P(B)$$
$$P(A \cap C) = P(A) \bullet P(C)$$
$$P(B \cap C) = P(B) \bullet P(C)$$
$$P(A \cap B \cap C) = P(A) \bullet P(B) \bullet P(C)$$

Consider these coin-tossing experiments and associated events:

Experiment 1: Toss a coin 2 times.

- Event A: Get heads on the first toss.
- Event B: Get heads on the second toss.
- Event C: Get exactly one head.

Experiment 2: Toss a coin 3 times.

- Event A: Get heads on the first toss.
- Event B: Get heads on the second toss.
- Event C: Get heads on the third toss.

For each experiment, determine whether events A, B, and C are independent.

For each experiment, determine whether events A, B, and C are independent.

SWB p. 37

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Problem-Based Task 4.1.3: Tossing Coins

Coaching

a. For experiment 1, what is the sample space?

b. For experiment 1, what are the probabilities of the individual events A, B, and C?

c. For experiment 1, what are the probabilities of the events $A \cap B$, $A \cap C$, and $B \cap C$?

d. For experiment 1, what is the probability of the event $A \cap B \cap C$? Explain.

e. For experiment 1, are the events A, B, and C independent? Show your work.

f. For experiment 2, what is the sample space?

g. For experiment 2, what are the probabilities of the individual events A, B, and C?

h. For experiment 2, what are the probabilities of the events $A \cap B$, $A \cap C$, and $B \cap C$?

i. For experiment 2, what is the probability of the event $A \cap B \cap C$? Explain.

j. For experiment 2, are the events A, B, and C independent? Show your work.

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Instruction

Problem-Based Task 4.1.3: Tossing Coins

Coaching Sample Responses

a. For experiment 1, what is the sample space?

Sample space = {HH, HT, TH, TT}

b. For experiment 1, what are the probabilities of the individual events *A*, *B*, and *C*?

$P(A)=\frac{2}{4}=\frac{1}{2};\ P(B)=\frac{2}{4}=\frac{1}{2};\ P(C)=\frac{2}{4}=\frac{1}{2}$

c. For experiment 1, what are the probabilities of the events $A\cap B$, $A\cap C$, and $B\cap C$?

$P(A\cap B)=\frac{1}{4};\ P(A\cap C)=\frac{1}{4};\ P(B\cap C)=\frac{1}{4}$

d. For experiment 1, what is the probability of the event $A\cap B\cap C$? Explain.

$P(A\cap B\cap C)=0$ because there is no outcome with heads first, heads second, and exactly one head. The event $A\cap B\cap C$ is impossible.

e. For experiment 1, are the events *A*, *B*, and *C* independent? Show your work.

No, the events *A*, *B*, and *C* are not independent.

$P(A\cap B\cap C)\neq P(A)\bullet P(B)\bullet P(C)$

$0\neq\frac{1}{2}\bullet\frac{1}{2}\bullet\frac{1}{2}$

f. For experiment 2, what is the sample space?

Sample space = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

g. For experiment 2, what are the probabilities of the individual events *A*, *B*, and *C*?

$P(A)=\frac{4}{8}=\frac{1}{2};\ P(B)=\frac{4}{8}=\frac{1}{2};\ P(C)=\frac{4}{8}=\frac{1}{2}$

h. For experiment 2, what are the probabilities of the events $A\cap B$, $A\cap C$, and $B\cap C$?

$P(A\cap B)=\frac{2}{8}=\frac{1}{4};\ P(A\cap C)=\frac{2}{8}=\frac{1}{4};\ P(B\cap C)=\frac{2}{8}=\frac{1}{4}$

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i. For experiment 2, what is the probability of the event $A \cap B \cap C$? Explain.

$P(A \cap B \cap C) = \frac{1}{8}$ because there are 8 outcomes in the sample space and 1 of them has heads first, heads second, and heads third.

j. For experiment 2, are the events *A*, *B*, and *C* independent? Show your work.

Yes, the events *A*, *B*, and *C* are independent.

$P(A \cap B \cap C) = P(A) \bullet P(B) \bullet P(C)$

$\frac{1}{8} = \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2}$

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4.1.3: Understanding Independent Events

Use what you know about independent events to solve problem 1.

1. The table below shows four sets of values for $P(A)$, $P(B)$, and $P(A \cap B)$. Based on the definition of independence, determine if events A and B are independent in each case.

	$P(A)$	$P(B)$	$P(A \cap B)$	Are A and B independent? (yes/no)
a.	0.2	0.14	0.028	
b.	0.32	0.16	0.48	
c.	$\frac{1}{3}$	$\frac{3}{5}$	$\frac{4}{15}$	
d.	$\frac{7}{8}$	$\frac{2}{5}$	$\frac{7}{20}$	

Use the following information to complete problems 2 and 3.

> Paola is playing a word game in which she draws letter tiles from a bag without looking. The bag contains 7 tiles: 2 As, 3 Es, and 2 Rs. For each of problems 2 and 3, find the probability of getting an E first and getting an E second. In each problem, state whether the events are independent.

2. Paola takes a tile, then replaces it, and then takes a second tile.
3. Paola takes a tile, does not replace it, and then takes a second tile.

To complete problem 4, suppose that a married couple will have 3 children and suppose that having a boy or girl is equally likely each time. Consider the following events.

A: At least 2 consecutive children are of the same gender.

B: Exactly 2 consecutive children are of the same gender.

C: No 2 consecutive children are of the same gender.

4. For each pair of events, determine if the events are independent.

   A and B

   A and C

   B and C

continued

SWB p. 39; SRB Practice 4.1.3 p. U4-44

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Use the table and the given information to complete problems 5 and 6.

The Coolest Deal is a daily special sold at Ike's Ice Cream Parlor. One day, the Coolest Deal is a large cone with one topping. The following table shows the sales data for the Coolest Deal that day.

Coolest Deals Sold at Ike's

Topping choice	Ice cream choice			
	Vanilla	Chocolate	Cookie dough	Mint chip
Sprinkles	9	12	16	14
Hot fudge	11	4	16	15
Caramel	10	12	18	15

Using the data in the table, determine if the events stated in problems 5 and 6 seem to be independent. Show the work that supports your answer.

5. A random customer at Ike's orders caramel and cookie dough for the Coolest Deal.

6. A random customer at Ike's orders hot fudge and chocolate for the Coolest Deal.

Use the given information to solve problems 7–10.

7. For a statistics project, Tamara surveys a well-chosen sample that represents all the students at her school. She finds that 72% have at least one sibling (brother or sister) and 27% have at least one sibling and at least one pet in their home. Assume that having a sibling and having a pet are independent events. Based on the survey, what is the probability that a randomly chosen student at Tamara's school has at least one pet at home?

8. Emily and Nino are participating in an archery unit in their physical education class. Emily has hit inside the yellow region 16 times out of 40 shots. Nino has hit inside the yellow region 15 times out of 30 shots. They are now partners in a "take your best shot" tournament. In the tournament, each partner shoots once and the best shot counts. Assuming that Emily's result and Nino's result are independent of each other, what is the probability that Emily or Nino will hit inside the yellow region, based on past performance?

continued

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9. Mario's job evaluation has two components: punctuality (arriving on time) and task performance. To be rated satisfactory overall, he needs to be rated satisfactory for both components. His record shows the following: rated satisfactory overall 76% of work days and rated satisfactory in task performance 80% of work days. Assume that arriving on time and task performance are independent in Mario's case. What is the probability that he will arrive on time on his next work day, based on the data?

10. At Louie's Book Café, every sale is recorded as either "reading" or "café." Last month, 67.5% of people who visited Louie's spent money there, and 50% of the visitors made at least one café purchase. Assume that buying a reading item and buying a café item are independent events. What is the probability that the next visitor at Louie's will buy a reading item, based on last month's data? (*Hint*: Use the Addition Rule.)

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Progress Assessment

Circle the letter of the best answer.

1. The following table shows activities some students are involved in, with the students identified by their initials.

Student	Student council	Community service	Band	Drama	Sports team
AS	✔				✔
RV		✔		✔	
DB	✔		✔		
BB		✔	✔	✔	
NP	✔				✔
DG			✔		✔
DK	✔	✔		✔	

A student is chosen from the group at random. Consider the following events.

B: The student is in the band.

S: The student is on a sports team.

How many outcomes are in the event $\overline{B \cup S}$?

a. 1

b. 2

c. 5

d. 6

2. Raekwon, Sage, and Eleni work at a restaurant. On Monday, the boss assigns one of them at random to write the daily specials on the chalkboard in front of the restaurant. Then on Tuesday, the boss again assigns one of them at random to the same task. Which event has the most outcomes?

a. Raekwon is assigned at least once.

b. Raekwon is assigned exactly once.

c. Raekwon is not assigned.

d. The same person is assigned twice.

continued

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3. How many ways can you roll a pair of dice and get an odd product?

   a. 3

   b. 9

   c. 18

   d. 27

4. Ivan rolls a 6-sided die and then rolls it again. What is the probability that the first roll is odd or the sum of the rolls is odd?

   a. $\frac{1}{4}$

   b. $\frac{1}{2}$

   c. $\frac{3}{4}$

   d. 1

5. Hamilton High School has 200 eleventh-grade students. There are 165 eleventh graders in physical education, 60 eleventh graders in intramural sports, and 180 eleventh graders in physical education or intramural sports. What is the probability that a randomly chosen eleventh grader at Hamilton High School is in physical education and intramural sports?

   a. 0.075

   b. 0.225

   c. 0.375

   d. 0.525

6. Which event has the greatest probability if 3 coins are tossed?

   a. at least 2 tails

   b. no tail or exactly 1 tail

   c. exactly 1 tail or exactly 2 heads

   d. exactly 2 tails or exactly 2 heads

continued

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7. Byron has a bag that contains the following tiles.

Byron takes a tile from the bag without looking. Then he replaces it. Then he takes another tile from the bag without looking. Consider the following events.

EF: Byron takes E first.

ES: Byron takes E second.

EF and *ES*: Byron takes E first and E second.

Which option describes the events?

a. *EF* and *ES* are dependent, and the probability of *EF* and *ES* is 10%.

b. *EF* and *ES* are dependent, and the probability of *EF* and *ES* is 16%.

c. *EF* and *ES* are independent, and the probability of *EF* and *ES* is 10%.

d. *EF* and *ES* are independent, and the probability of *EF* and *ES* is 16%.

8. The expression for batting average in softball is $\frac{\text{number of hits}}{\text{number of times at bat}}$. Marcia's batting average is 0.420, and she is at bat. Ah-Reum's batting average is 0.300, and she will be at bat after Marcia. Assume that each player's performance is independent of the other's. What is the probability that Marcia or Ah-Reum will get a hit, based on their batting averages?

a. 0.360

b. 0.420

c. 0.594

d. 0.720

continued

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9. Ruben and Mack are electricians, working for a company called State Electrical Contractor. Mack has been on 60% of State's service calls. At least one of the two workers has been on 76% of State's service calls. Assume that Mack being on a service call and Ruben being on a service call are independent events. What is the probability that Ruben will be on the next service call, based on the data?

   a. 16%

   b. 24%

   c. 40%

   d. 56%

10. The table shows data on 200 students at a college.

Comparing Field of Study in College with High School Attended

High school attended	Field of study in college			
	Science, math, or engineering	Business or economics	Social sciences	Visual and performing arts
Washington	30	28	35	25
Lincoln	25	32	35	28
Jefferson	65	30	40	27

Each pair of events below describes a student chosen randomly at the college. Which pair of events seems to be dependent, based on the data in the table?

   a. The student is in science, math, or engineering and went to Jefferson High School.

   b. The student is in business or economics and went to Washington High School.

   c. The student is in social sciences and went to Washington High School.

   d. The student is in visual and performing arts and went to Jefferson High School.

continued

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Use the table and the given information to complete problem 11.

11. The table below shows the winning percentages of the four athletic teams at Bay High School that are active in league competition.

Winning Percentages at Bay High School

Team	Percent of games won
Boys' baseball	60%
Boys' lacrosse	50%
Girls' softball	80%
Girls' lacrosse	60%

All four teams are scheduled to play today. The school's athletic director wants to know the probability that at least one team will win today, based on the data. She assumes that each team's winning or losing is independent of all the other teams' results. Complete parts a–f to find the probability the director wants.

a. What is the probability that the boys' baseball team or boys' lacrosse team will win today?

b. What is the probability that a boys' team will win today?

c. What is the probability that the girls' softball team or girls' lacrosse team will win today?

d. What is the probability that a girls' team will win today?

e. What is the probability that a boys' team or a girls' team will win today?

f. What is the probability that at least one team will win today?

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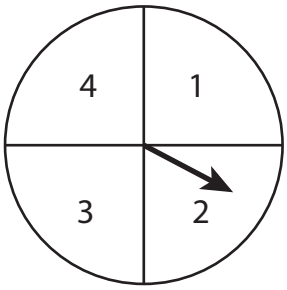
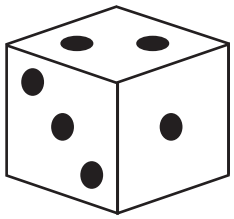
Pre-Assessment

Circle the letter of the best answer.

1. Solange spins a spinner with equal sections 1–4 and then rolls a standard 6-sided number cube. What is the probability that the sum of the spinner and number cube results is even if the product is even?





a. $\frac{1}{3}$

b. $\frac{1}{2}$

c. $\frac{3}{4}$

d. 1

2. Which of the following has the greatest probability if Caleb tosses a coin 3 times?

   a. He gets exactly 2 heads.

   b. He gets exactly 2 heads, given that the first toss is heads.

   c. The first toss is heads, given that he gets exactly 2 heads.

   d. The first toss is heads, given that he gets at least 2 heads.

continued

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3. The table below shows data about 108 pizzas sold in a pizzeria. Each pizza was sold with one topping.

Pizza shape	Pizza topping			
	Pepperoni	Mushroom	Onion	Chicken
Round	20	10	15	15
Square	16	8	18	6

Consider the following events.

R: A round pizza is sold. *P*: A pepperoni pizza is sold. *C*: A chicken pizza is sold.

S: A square pizza is sold. *O*: An onion pizza is sold.

Which pair of events is independent?

a. *S* and *C*
b. *R* and *O*
c. *S* and *O*
d. none of these

4. A sports club for girls has 80 members. Each member is currently involved in one activity, as shown in the table.

Grade	Activity		
	Volleyball	Basketball	Cheerleading
10	8	8	8
11	6	12	12
12	2	12	12

The following probabilities apply to a randomly chosen member of the club. Which probability is the greatest?

a. the probability that the girl is in the tenth grade
b. the probability that the girl is in the tenth grade, given that she is in volleyball
c. the probability that the girl is in the twelfth grade
d. the probability that the girl is in the twelfth grade, given that she is in basketball

5. Two cards are dealt from a standard 52-card deck, without replacement. What is the probability that both cards are spades?

a. 0.053
b. 0.059
c. 0.063
d. 0.064

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Lesson 2: Conditional Probability

Instruction

Common Core State Standards

SMP
1 ✓ 2 ✓
3 ✓ 4 ✓
5 6
7 ✓ 8

S–CP.3 Understand the conditional probability of *A* given *B* as *P*(*A* and *B*)/*P*(*B*), and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A,* and the conditional probability of *B* given A is the same as the probability of *B.*★

S–CP.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.*★

S–CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.*★

S–CP.6 Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A,* and interpret the answer in terms of the model.★

S–CP.8 (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interpret the answer in terms of the model.★

Essential Questions

1. What are the different ways to find a conditional probability?
2. How can you use a two-way frequency table to find probabilities and conditional probabilities?
3. How do you sometimes modify the test for independent events when using real-world data?
4. How is the Multiplication Rule applied to two independent events?
5. How is the Multiplication Rule applied to two dependent events?

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Instruction

WORDS TO KNOW

compound event	the combination of two or more simple events
compound probability	the probability of compound events
conditional probability of *B* given *A*	the probability that event B occurs, given that event A has already occurred. If A and B are two events from a sample space with $P(A) \neq 0$, then the conditional probability of B given A, denoted $P(B\|A)$, has two equivalent expressions: $P(B\|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\text{number of outcomes in } (A \text{ and } B)}{\text{number of outcomes in } A}$.
Multiplication Rule	the probability of two events, A and B, is $P(A \text{ and } B) = P(A) \bullet P(B\|A) = P(B) \bullet P(A\|B)$; for independent events A and B, the rule is $P(A \text{ and } B) = P(A) \bullet P(B)$.
simple event	an event that has only one outcome; sometimes called a single event
two-way frequency table	a frequency table that shows two categories of characteristics, one in rows and the other in columns. Each cell value is a frequency that shows how many times two different characteristics appear together, or how often characteristics are associated with a person, object, or type of item that is being studied.

Recommended Resources

- Interactivate. "Conditional Probability."

  http://www.walch.com/rr/00095

  This website presents conditional probability and compares it to simple probability in a question-and-answer format.

- Khan Academy. "Dependent Probability."

  http://www.walch.com/rr/00127

  This site provides engaging practice problems with available solutions and hints. Problems involve compound and conditional events.

- MathIsFun.com. "Conditional Probability."

  http://www.walch.com/rr/00096

  This site gives an overview of conditional probability and has practice questions.

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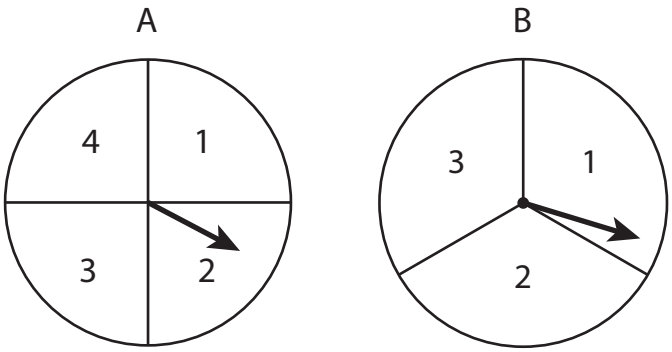
Lesson 4.2.1: Introducing Conditional Probability

Warm-Up 4.2.1

Howard is playing a carnival game. The object of the game is to predict the sum you will get by spinning spinner A and then spinner B. Howard predicts he will get a sum of 5.



1. List the sample space.

2. What is the probability that Howard gets a sum of 5? Explain.

3. Suppose Howard gets a 3 on spinner A. Now what is the probability that Howard gets a sum of 5? Explain.

4. How does getting a 3 on spinner A affect the probability that Howard gets a sum of 5? Explain.

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Instruction

Lesson 4.2.1: Introducing Conditional Probability

Common Core State Standards

S–CP.3 Understand the conditional probability of *A* given *B* as *P*(*A* and *B*)/*P*(*B*), and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A,* and the conditional probability of *B* given A is the same as the probability of *B.*★

S–CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.*★

S–CP.6 Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A,* and interpret the answer in terms of the model.★

Warm-Up 4.2.1 Debrief

1. List the sample space.

(1, 1)	(1, 2)	(1, 3)
(2, 1)	(2, 2)	(2, 3)
(3, 1)	(3, 2)	(3, 3)
(4, 1)	(4, 2)	(4, 3)

2. What is the probability that Howard gets a sum of 5? Explain.

   The sample space has a total of 12 outcomes and 3 of them result in a sum of 5.

(1, 1)	(1, 2)	(1, 3)
(2, 1)	(2, 2)	(2, 3)
(3, 1)	(3, 2)	(3, 3)
(4, 1)	(4, 2)	(4, 3)

   Divide the number of outcomes with the desired result (getting a sum of 5) by the total number of outcomes, 12.

$$\frac{3}{12} = \frac{1}{4}$$

   The probability of Howard getting a sum of 5 is $\frac{1}{4}$.

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Instruction

3. Suppose Howard gets a 3 on spinner A. Now what is the probability that Howard gets a sum of 5? Explain.

   Looking at the sample space, there are 3 outcomes that show getting a 3 for spinner A, and 1 of them results in a sum of 5.

   Divide the number of outcomes with the desired result (getting a sum of 5) by the new number of outcomes, 3.

   $\frac{1}{3}$

   The probability of Howard getting a sum of 5 given that he gets a 3 on spinner A is $\frac{1}{3}$.

4. How does getting a 3 on spinner A affect the probability that Howard gets a sum of 5? Explain.

   Compare the two probabilities, $\frac{1}{4}$ and $\frac{1}{3}$.

   Getting a 3 on spinner A increases the probability that Howard gets a sum of 5 because $\frac{1}{3} > \frac{1}{4}$.

Connection to the Lesson

- The warm-up requires students to distinguish between a simple probability (question 2) and a conditional probability (question 3).
- Students use their understanding of probability to calculate conditional probabilities.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- identifying sample spaces (7.SP.1)
- identifying the intersection of two sets (7.SP.3)
- simplifying complex fractions (A–APR.6)
- calculating simple probability (7.SP.7a)

Introduction

Let's say you and your friends draw straws to see who has to do some unpleasant activity, like cleaning out the class pet's cage. If everyone who draws before you keeps their straw, does that affect your odds of cleaning up after Fluffy the Hamster?

If you are drawing straws without replacing them, your friends' outcomes do have an effect on yours—if there are several long straws and one unfortunate short one, your odds of drawing the short straw increase with every long straw drawn.

In this lesson, we will look at conditional probability—that is, the probability that an event will occur based on the condition that another event has occurred.

Key Concepts

- The **conditional probability of *B* given *A*** is the probability that event *B* occurs, given that event *A* has already occurred.
- If *A* and *B* are two events from a sample space with $P(A) \neq 0$, then the conditional probability of *B* given *A*, denoted $P(B|A)$, has two equivalent expressions:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\text{number of outcomes in } (A \text{ and } B)}{\text{number of outcomes in } A}$$

- Here is an explanation of these equivalent expressions:

$$P(B|A) = \frac{\text{number of outcomes in } (A \text{ and } B)}{\text{number of outcomes in } A}$$

This uses subsets of the sample space.

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

This uses the calculated probabilities $P(A \text{ and } B)$ and $P(A)$.

- The second formula can be rewritten as $P(A \text{ and } B) = P(A) \bullet P(B|A)$.

SRB p. U4-50

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Instruction

- Using set notation, conditional probability is written like this:

$$P(B|A)=\frac{P(A\cap B)}{P(A)}$$

- The "conditional probability of B given A" only has meaning if event A has occurred. That is why the formula for $P(B|A)$ has the requirement that $P(A) \neq 0$.
- The conditional probability formula can be solved to obtain a formula for $P(A \text{ and } B)$:

$P(B\|A)=\frac{P(A \text{ and } B)}{P(A)}$	Write the conditional probability formula.
$P(A)\bullet P(B\|A)=\frac{P(A \text{ and } B)}{P(A)}\bullet P(A)$	Multiply both sides by $P(A)$.
$P(A)\bullet P(B\|A)=P(A \text{ and } B)$	Simplify.
$P(A \text{ and } B)=P(A)\bullet P(B\|A)$	Reverse the left and right sides.

- Remember that independent events are two events such that the probability of both events occurring is equal to the product of the individual probabilities. Two events A and B are independent if and only if $P(A \text{ and } B) = P(A) \bullet P(B)$. Using set notation, $P(A\cap B)=P(A)\bullet P(B)$. The occurrence or non-occurrence of one event has no effect on the probability of the other event.
- If A and B are independent, then the formula for $P(A \text{ and } B)$ is the equation used in the definition of independent events, as shown:

$P(A \text{ and } B)=P(A)\bullet P(B\|A)$	formula for $P(A \text{ and } B)$
$P(A \text{ and } B) = P(A) \bullet P(B)$	formula for $P(A \text{ and } B)$ if A and B are independent

- Using the conditional probability formula in different situations requires using different variables, depending on how the events are named. Here are some examples.

$P(B\|A)=\frac{P(A \text{ and } B)}{P(A)}$	Use this equation to find the probability of B given A.
$P(A\|B)=\frac{P(A \text{ and } B)}{P(B)}$	Use this equation to find the probability of A given B.

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- Another method that can be used to calculate conditional probabilities is dividing the number of outcomes in the intersection of A and B by the number of outcomes in a certain event:

$$P(A|B) = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } B}$$

- The following statements are equivalent. In other words, if any one of them is true, then the others are all true.
  - Events A and B are independent.
  - The occurrence of A has no effect on the probability of B; that is, $P(B|A) = P(B)$.
  - The occurrence of B has no effect on the probability of A; that is, $P(A|B) = P(A)$.
  - $P(A \text{ and } B) = P(A) \bullet P(B)$
- *Note*: For real-world data, these modified tests for independence are sometimes used:
  - Events A and B are independent if the occurrence of A has no *significant* effect on the probability of B; that is, $P(B|A) \approx P(B)$.
  - Events A and B are independent if the occurrence of B has no *significant* effect on the probability of A; that is, $P(A|B) \approx P(A)$.
- When using these modified tests, good judgment must be used when deciding whether the probabilities are close enough to conclude that the events are independent.

Common Errors/Misconceptions

- thinking that $P(A|B)$ represents the probability of A occurring and then B occurring
- confusing union and intersection of sets
- incorrectly finding the probabilities of A or B or the intersection of A and B
- applying the formula for probability incorrectly

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Instruction

Guided Practice 4.2.1

Example 1

Alexis rolls a pair of number cubes. What is the probability that both numbers are odd if their sum is 6? Interpret your answer in terms of a uniform probability model.

1. Assign variable names to the events and state what you need to find, using conditional probability.

   Let A be the event "Both numbers are odd."

   Let B be the event "The sum of the numbers is 6."

   You need to find the probability of A given B.

   That is, you need to find $P(A|B)$.

2. Show the sample space.

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

   Key: (2, 3) means 2 on the first cube and 3 on the second cube.

SWB p. 47; SRB p. U4-53

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Instruction

3. Identify the outcomes in the events.

The outcomes for each sample space are shaded.

***A*: Both numbers are odd.**	***B*: The sum of the numbers is 6.**
(1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6)	(1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6)
(2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (2, 6)	(2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (2, 6)
(3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (3, 6)	(3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (3, 6)
(4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6)	(4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6)
(5, 1) (5, 2) (5, 3) (5, 4) (5, 5) (5, 6)	(5, 1) (5, 2) (5, 3) (5, 4) (5, 5) (5, 6)
(6, 1) (6, 2) (6, 3) (6, 4) (6, 5) (6, 6)	(6, 1) (6, 2) (6, 3) (6, 4) (6, 5) (6, 6)

4. Identify the outcomes in the events $A \cap B$ and B.

Use the conditional probability formula: $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

$A \cap B$ = the outcomes that are in A and also in B.

$A \cap B$ = {(1, 5), (3, 3), (5, 1)}

B = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)}

5. Find $P(A \cap B)$ and $P(B)$.

$A \cap B$ has 3 outcomes; the sample space has 36 outcomes.

$$P(A \cap B) = \frac{\text{number of outcomes in } A \text{ and } B}{\text{number of outcomes in sample space}} = \frac{3}{36}$$

B has 5 outcomes; the sample space has 36 outcomes.

$$P(B) = \frac{\text{number of outcomes in } B}{\text{number of outcomes in sample space}} = \frac{5}{36}$$

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6. Find $P(A|B)$.

$P(A|B)=\dfrac{P(A\cap B)}{P(B)}$ Write the conditional probability formula.

$P(A|B)=\dfrac{\frac{3}{36}}{\frac{5}{36}}$ Substitute the probabilities found in step 5.

$P(A|B)=\dfrac{3}{36}\bullet\dfrac{36}{5}$ To divide by a fraction, multiply by its reciprocal.

$P(A|B)=\dfrac{3}{\cancel{36}_{1}}\bullet\dfrac{\cancel{36}^{1}}{5}$ Simplify.

$P(A|B)=\dfrac{3}{5}$

7. Verify your answer.

Use this alternate conditional probability formula:

$$P(A|B)=\frac{\text{number of outcomes in } A\cap B}{\text{number of outcomes in } B}.$$

$A\cap B$ has 3 outcomes: {(1, 5), (3, 3), (5, 1)}.

B has 5 outcomes: {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)}.

$$P(A|B)=\frac{\text{number of outcomes in } A\cap B}{\text{number of outcomes in } B}$$

$P(A|B)=\dfrac{3}{5}$

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8. Interpret your answer in terms of a uniform probability model.

The probabilities used in solving the problem are found by using the ratios $\frac{\text{number of outcomes in an event}}{\text{number of outcomes in the sample space}}$ and $\frac{\text{number of outcomes in an event}}{\text{number of outcomes in a subset of the sample space}}$.

These ratios are uniform probability models if all outcomes in the sample space are equally likely. It is reasonable to assume that Alexis rolls fair number cubes, so all outcomes in the sample space are equally likely. Therefore, the answer is valid and can serve as a reasonable predictor. You can predict the following: If you roll a pair of number cubes a large number of times and consider all the outcomes that have a sum of 6, then about $\frac{3}{5}$ of those outcomes will have both odd numbers.

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Example 2

Hamid rolls a number cube 3 times. Consider the following events.

A: The first roll is an odd number.

B: There are exactly 2 consecutive odd numbers.

Determine if *A* and *B* are independent events.

1. Investigate the sample space.

   The number cube is rolled 3 times, so the outcomes are ordered triples.

(1, 1, 1)	(1, 1, 2)	(1, 1, 3)	(1, 1, 4)	(1, 1, 5)	(1, 1, 6)
(1, 2, 1)	(1, 2, 2)	(1, 2, 3)	(1, 2, 4)	(1, 2, 5)	(1, 2, 6)
(1, 3, 1)	(1, 3, 2)	(1, 3, 3)	(1, 3, 4)	(1, 3, 5)	(1, 3, 6)
(1, 4, 1)	(1, 4, 2)	(1, 4, 3)	(1, 4, 4)	(1, 4, 5)	(1, 4, 6)
(1, 5, 1)	(1, 5, 2)	(1, 5, 3)	(1, 5, 4)	(1, 5, 5)	(1, 5, 6)
(1, 6, 1)	(1, 6, 2)	(1, 6, 3)	(1, 6, 4)	(1, 6, 5)	(1, 6, 6)
(2, 1, 1)	(2, 1, 2)	(2, 1, 3)	(2, 1, 4)	(2, 1, 5)	(2, 1, 6)
(2, 2, 1)	(2, 2, 2)	(2, 2, 3)	(2, 2, 4)	(2, 2, 5)	(2, 2, 6)
(2, 3, 1)	(2, 3, 2)	etc.			

   The ordered triples listed above comprise only a small part of the sample space, which contains $6 \cdot 6 \cdot 6 = 216$ outcomes.

   See step 2 for a more efficient method of determining the sample space.

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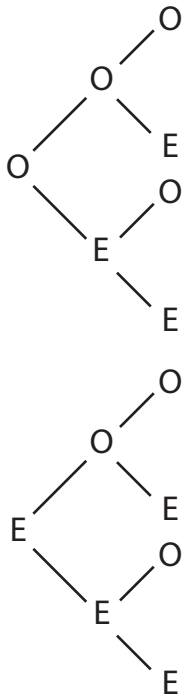
2. Create a tree diagram for the outcomes.

   For the events under consideration, you need only know whether each number cube roll is odd or even. The possible results of a number cube roll are 1, 2, 3, 4, 5, and 6. Half of the numbers on the number cube are odd and half of them are even, so odd and even are equally likely. Therefore, you can consider a sample space comprised of the outcomes "odd" and "even."

   The following tree diagram shows all possible outcomes, using O for odd and E for even.



   To identify all the possible outcomes of 3 number cube rolls, trace all the different paths from left to right. For example, the diagram that follows shows two different paths.

(*continued*)

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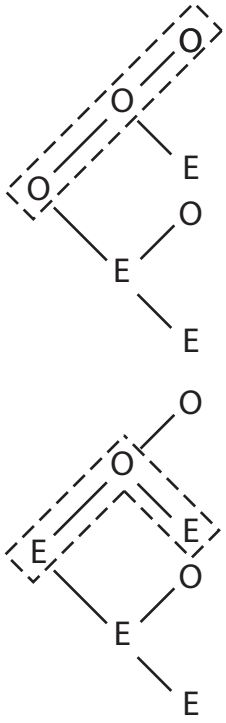

There are eight different paths that can be found from this new sample space.

Sample space = {OOO, OOE, OEO, OEE, EOO, EOE, EEO, EEE}

3. Events A and B are independent if $P(A|B)=P(A)$ and $P(B|A)=P(B)$. Find the probabilities of the relevant events.

A: The first roll is odd.

{OOO, OOE, OEO, OEE, EOO, EOE, EEO, EEE}

$$P(A)=\frac{4}{8}=\frac{1}{2}$$

B: There are exactly 2 consecutive odds.

{OOO, OOE, OEO, OEE, EOO, EOE, EEO, EEE}

$$P(B)=\frac{2}{8}=\frac{1}{4}$$

(*continued*)

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$A|B$: The first roll is odd, given that there are exactly 2 consecutive odds.

{OOE, EOO}

$$P(A|B)=\frac{1}{2}$$

$B|A$: There are exactly 2 consecutive odds, given that the first roll is odd.

{OOO, OOE, OEO, OEE}

$$P(B|A)=\frac{1}{4}$$

4. Determine if A and B are independent.

A and B are independent if $P(A|B)=P(A)$ and $P(B|A)=P(B)$.

First, determine if $P(A|B)=P(A)$.

$P(A|B)=P(A)$ Set $P(A|B)$ equal to $P(A)$.

$\frac{1}{2}=\frac{1}{2}$ Substitute probabilities for $P(A|B)$ and $P(A)$.

This verifies that $P(A|B)=P(A)$.

Now, determine if $P(B|A)=P(B)$.

$P(B|A)=P(B)$ Set $P(B|A)$ equal to $P(B)$.

$\frac{1}{4}=\frac{1}{4}$ Substitute probabilities for $P(B|A)$ and $P(B)$.

This verifies that $P(B|A)=P(B)$.

A and B are independent.

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5. Check your results.

The definition of independent events contains a different test for independence: A and B are independent if $P(A \cap B) = P(A) \bullet P(B)$. Use this other test to verify your answer.

A: The first roll is odd.

{OOO, OOE, OEO, OEE, EOO, EOE, EEO, EEE}

$$P(A) = \frac{4}{8} = \frac{1}{2}$$

B: There are exactly 2 consecutive odds.

{OOO, OOE, OEO, OEE, EOO, EOE, EEO, EEE}

$$P(B) = \frac{2}{8} = \frac{1}{4}$$

$A \cap B$: The first roll is odd and there are exactly 2 consecutive odds.

{OOO, OOE, OEO, OEE, EOO, EOE, EEO, EEE}

$$P(A \cap B) = \frac{1}{8}$$

Verify.

$P(A \cap B) = P(A) \bullet P(B)$	Set $P(A \cap B)$ equal to $P(A) \bullet P(B)$.
$\frac{1}{8} = \frac{1}{2} \bullet \frac{1}{4}$	Substitute probabilities for $P(A \cap B)$, $P(A)$, and $P(B)$.
$\frac{1}{8} = \frac{1}{8}$	Simplify.

A and B are independent.

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Instruction

Example 3

A vacation resort offers bicycles and personal watercrafts for rent. The resort's manager made the following notes about rentals:

- 200 customers rented items in all—100 rented bicycles and 100 rented personal watercrafts.
- Of the personal watercraft customers, 75 customers were young (30 years old or younger) and 25 customers were older (31 years old or older).
- 125 of the 200 customers were age 30 or younger. 50 of these customers rented bicycles, and 75 of them rented personal watercrafts.

Consider the following events that apply to a random customer.

Y: The customer is young (30 years old or younger).

W: The customer rents a personal watercraft.

Are Y and W independent? Compare $P(Y|W)$ and $P(W|Y)$ and interpret the results.

1. Determine if Y and W are independent.

   First, determine the probabilities of each event.

$P(Y)=\frac{125}{200}=0.625$	Of 200 customers, 125 were young.
$P(W)=\frac{100}{200}=0.5$	Of 200 customers, 100 rented a personal watercraft.
$P(Y\|W)=\frac{75}{100}=0.75$	Of 100 personal watercraft customers, 75 were young.
$P(W\|Y)=\frac{75}{125}=0.6$	Of 125 young customers, 75 rented a personal watercraft.

   Y and W are dependent because $P(Y|W)\neq P(Y)$ and $P(W|Y)\neq P(W)$.

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Instruction

2. Compare $P(Y|W)$ and $P(W|Y)$.

   $P(Y|W) = 0.75$

   $P(W|Y) = 0.6$

   $0.75 > 0.6$; therefore, $P(Y|W) > P(W|Y)$.

3. Interpret the results.

   $P(Y|W)$ represents the probability that a customer is young given that the customer rents a personal watercraft.

   $P(W|Y)$ represents the probability that a customer rents a personal watercraft given that the customer is young.

   The dependence of the events Y and W means that a customer's age affects the probability that the customer rents a personal watercraft; in this case, being young increases that probability because $P(W|Y) > P(W)$. The dependence of the events Y and W also means that a customer renting a personal watercraft affects the probability that the customer is young; in this case, renting a personal watercraft increases that probability because $P(Y|W) > P(Y)$.

   $P(Y|W) > P(W|Y)$ means that it is more likely that a customer is young given that he or she rents a personal watercraft than it is that a customer rents a personal watercraft given that he or she is young.

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Example 4

Last year at All Technical High School, 20% of the students received an academic award. That same year, 16% of the students at the school received an academic award and a service award. What is the probability that a student who receives an academic award also receives a service award?

1. Restate the question using a conditional probability.

   The original question asked, "What is the probability that a student who receives an academic award also receives a service award?"

   Rephrased in terms of a conditional probability, the question can be posed, "What is the probability that a student receives a service award given that he receives an academic award?"

2. Assign variable names to the relevant events and state what you need to find.

   Let S be the event "A student receives a service award."

   Let A be the event "A student receives an academic award."

   Then $S \cap A$ is the event "A student receives a service award and an academic award."

   Find $P(S|A)$.

3. Write the probabilities you know, based on the data given in the problem.

   $P(A) = 0.20$

   $P(S \cap A) = 0.16$

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4. Apply the conditional probability formula.

$P(S|A)=\frac{P(S\cap A)}{P(A)}$ Write the conditional probability formula for S and A.

$P(S|A)=\frac{0.16}{0.20}$ Substitute the probabilities determined in step 3.

$P(S|A)=0.8$ Simplify.

There is an 80% probability that a student who receives an academic award also receives a service award.

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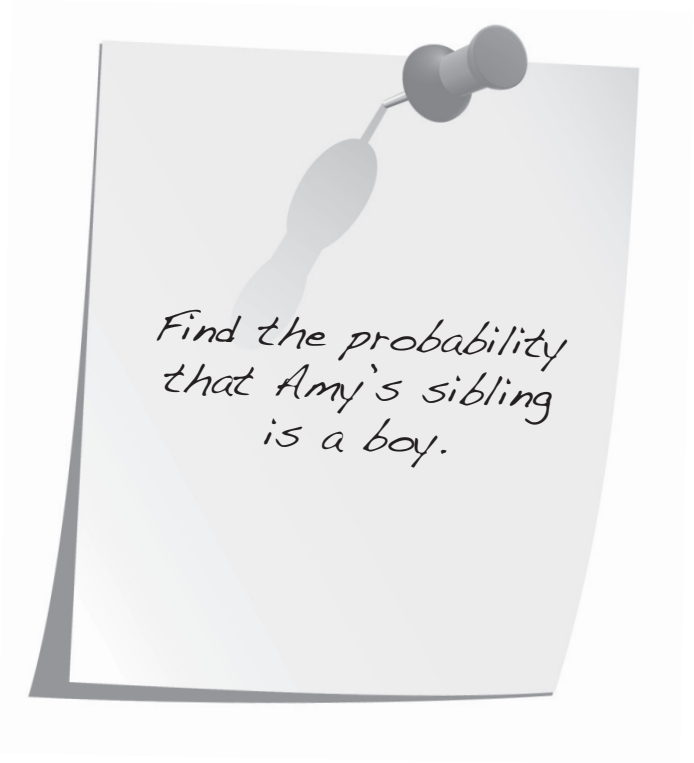
Problem-Based Task 4.2.1: Does She Have a Brother or a Sister?

SMP	
1	2
3 ✓	4 ✓
5	6
7 ✓	8

Amy has 1 sibling. Find the probability that Amy's sibling is a boy if given the following parameters:

Parameter 1: Nothing is known about the age of Amy or her sibling.

Parameter 2: It is known that Amy is older than her sibling.



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Problem-Based Task 4.2.1: Does She Have a Brother or a Sister?

Coaching

a. Restate the problem using conditional probabilities. Assign these variable names to the relevant events: Let *B* be the event "Amy's sibling is a boy." Let *N* be the event "Nothing is known about the age of Amy or her sibling." Let *AO* be the event "Amy is older than her sibling."

b. The problem is about gender. Amy is 1 of 2 children who were born. What is the sample space that shows all possible gender outcomes for the birth of 2 children? Use "b" for boy and "g" for girl.

c. Amy is a girl. What is the reduced sample space that applies to this situation?

d. Assume that the order of letters in each outcome corresponds to birth order. For example, "bg" means that a boy is born and then a girl is born. The first parameter has the condition that nothing is known about the age of Amy and her sibling. Does the sample space {bg, gb, gg} apply to Parameter 1?

e. Find the probability of Parameter 1, $P(B|N)$. Explain your reasoning.

f. Parameter 2 has the condition that Amy is older than her sibling. What is the further-reduced sample space that applies to Parameter 2? Explain your reasoning.

g. Find the probability of Parameter 2, $P(B|AO)$. Explain your reasoning.

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Instruction

Problem-Based Task 4.2.1: Does She Have a Brother or a Sister?

Coaching Sample Responses

a. Restate the problem using conditional probabilities. Assign these variable names to the relevant events: Let *B* be the event "Amy's sibling is a boy." Let *N* be the event "Nothing is known about the age of Amy or her sibling." Let *AO* be the event "Amy is older than her sibling."

Parameter 1 can be restated as "Find $P(B|N)$."

Parameter 2 can be restated as "Find $P(B|AO)$."

b. The problem is about gender. Amy is 1 of 2 children who were born. What is the sample space that shows all possible gender outcomes for the birth of 2 children? Use "b" for boy and "g" for girl.

The sample space is {bb, bg, gb, gg}.

c. Amy is a girl. What is the reduced sample space that applies to this situation?

Since there can't be two boys, the reduced sample space is {bg, gb, gg}.

d. Assume that the order of letters in each outcome corresponds to birth order. For example, "bg" means that a boy is born and then a girl is born. Parameter 1 has the condition that nothing is known about the age of Amy and her sibling. Does the sample space {bg, gb, gg} apply to Parameter 1?

Yes. The sample space consists of the following: a boy being born and then a girl, a girl being born and then a boy, or a girl being born and then another girl.

e. Find the probability of Parameter 1, $P(B|N)$. Explain your reasoning.
$P(B|N)=\frac{2}{3}$ because there are 3 outcomes and 2 of them involve a boy.

f. Parameter 2 has the condition that Amy is older than her sibling. What is the further-reduced sample space that applies to Parameter 2? Explain your reasoning.

{*gb*, *gg*}. If Amy is older, then she must have been born first. Therefore, *bg* is not a possible outcome for Parameter 2.

g. Find the probability of Parameter 2, $P(B|AO)$. Explain your reasoning.
$P(B|AO)=\frac{1}{2}$ because there are 2 outcomes and 1 of them involves a boy.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4.2.1: Introducing Conditional Probability

Timothy rolls a pair of number cubes. Use this information to complete problems 1 and 2.

1. What is the probability that both numbers are odd if their sum is 7? What is the probability that the sum of the numbers is 7 if both numbers are odd?

2. What is the probability that the sum is even if the product is even? What is the probability that the product is even if the sum is even?

For problems 3 and 4, list the sample space for the given events using a tree diagram and explain what your symbols represent. Use conditional probability to determine if events *A* and *B* are independent; show your work.

3. Hiro rolls a number cube 3 times.

   Event *A*: The second roll is odd.

   Event *B*: Exactly 2 consecutive rolls are odd.

4. The Millers have 3 children.

   Event *A*: The second oldest is a girl.

   Event *B*: At least 2 are girls.

Use the following information to complete problems 5 and 6.

A surveyor gathered data from 161 students regarding their interests and favorite entertainment activities. When asked to choose either "music and drama" or "athletics" as their greater overall interest, 80 students chose music and drama, while 81 students chose athletics. Students were then asked to choose among several entertainment activities, including watching movies and singing karaoke.

- 40 students chose movies; of those students, 19 had chosen music and drama, while 21 had chosen athletics.
- 40 students chose karaoke; of those students, 31 had chosen music and drama, while 9 had chosen athletics.

5. Consider the following events.

   MD: A student chooses music and drama.

   M: A student chooses movies.

   Are *MD* and *M* independent? Compare $P(MD|M)$ and $P(M|MD)$. Interpret what your answers mean.

continued

SWB p. 51; SRB Practice 4.2.1 p. U4-66

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6. Consider the following events.

   A: A student chooses athletics.

   K: A student chooses karaoke.

   Are A and K independent? Compare $P(A|K)$ and $P(K|A)$. Interpret what your answers mean.

Read each scenario and use the information to answer questions 7–10.

7. The buses at the Zoomy Express Bus Company depart as scheduled 80% of the time. The buses depart and arrive as scheduled 68% of the time. What is the probability that a Zoomy Express bus arrives as scheduled if it departs as scheduled?

8. A hotel restaurant has a 75% customer approval rating for its service. The restaurant has a 60% customer approval rating for its service and its food. What is the probability that a restaurant customer approves of the food if he or she approves of the service?

9. Benicio rolls a pair of number cubes. What is the probability that the sum is 6 if the numbers are different? What is the probability that the numbers are different if the sum is 6?

10. Naya tosses a coin 3 times. What is the probability that she gets all heads if the first toss is heads? What is the probability that the first toss is heads if she gets all heads?

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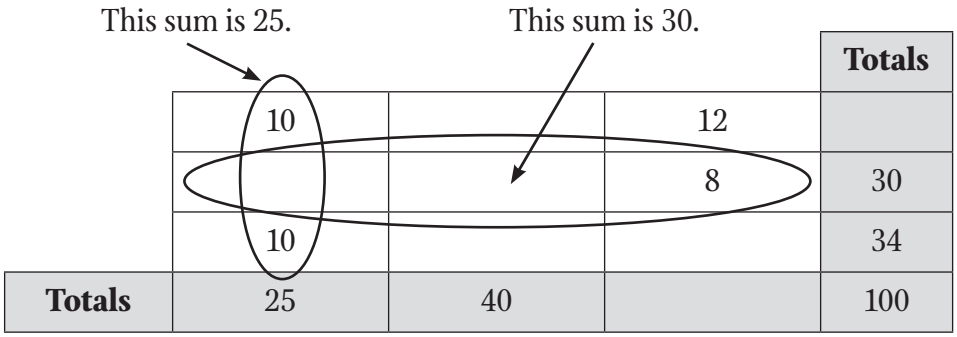
Lesson 4.2.2: Using Two-Way Frequency Tables

Warm-Up 4.2.2

Kira likes number puzzles and is tackling the puzzle below. The puzzle is incomplete. When it is complete, each total at the end of a row will be the sum of the numbers in that row, and each total at the end of a column will be the sum of the numbers in that column. Find all the missing numbers to solve the puzzle.



				Totals
	10		12	
			8	30
	10			34
Totals	25	40		100

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Lesson 4.2.2: Using Two-Way Frequency Tables

Common Core State Standards

S–CP.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.*★

S–CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.*★

S–CP.6 Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model.★

Warm-Up 4.2.2 Debrief

Completed puzzle:

				Totals
	10	**14**	12	**36**
	5	**17**	8	30
	10	**9**	**15**	34
Totals	25	40	**35**	100

Explanation:

For the first column, the sum given is 25 and the numbers provided add to 20. Therefore, the missing cell is 5.

The second row has a sum of 30. Using the information completed from the first column, we know that the first cell in the first row is 5. We are given 8 and the total is 30. Therefore, the missing cell is $30 – (8 + 5) = 17$.

The total row at the bottom sums to 100 and the first two cells are given as 25 and 40. Use $100 – (25 + 40) = 35$ to find the missing number.

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The third column we now know sums to 35, and 12 and 8 were given as two entries in that column. Use 35 – (12 + 8) = 15 to find the missing cell.

The third row now has two entries, 10 and 15. The row must sum to 34. Use 34 – (10 + 15) = 9 to find the missing cell.

The second column sums to 40 and we now have two entries in the column, 17 and 9. Use 40 – (17 + 9) = 14 to find the missing cell.

The sum is missing from the first row. All the cells have been filled in for the first row. Sum the cells to find the missing entry: 10 + 14 + 12 = 36.

Connection to the Lesson

- The rule for two-way frequency tables is the same as the rule for the diagram in the warm-up: Each total at the end of a row is the sum of the numbers in that row, and each total at the end of a column is the sum of the numbers in that column.
- Students will work with two-way frequency tables for calculating probabilities.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- using tally marks to record item counts (7.SP.7b)
- finding probabilities and conditional probabilities (S–CP.6)
- using probabilities and conditional probabilities to determine if events are independent (S–CP.3)

Introduction

Data can be presented in many different ways. In the previous lesson, you worked with data in conditional probabilities to interpret the relationships between various events. When you are presented with a great deal of data, two-way frequency tables can provide a convenient way to compare the frequency of data items and see the relationships among them.

Key Concepts

- Previously, you learned that there are two equivalent expressions for the conditional probability of B given A:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\text{number of outcomes in } (A \text{ and } B)}{\text{number of outcomes in } A}$$

- Conditional probability can be used to test for independence.
- Events A and B are independent events if $P(B|A) = P(B)$ or if $P(A|B) = P(A)$. (Note that if one is true, then the other is also true.)
- Remember that for real-world data, modified tests for independence are sometimes used:
  - Events A and B are independent if the occurrence of A has no *significant* effect on the probability of B; that is, $P(B|A) \approx P(B)$.
  - Events A and B are independent if the occurrence of B has no *significant* effect on the probability of A; that is, $P(A|B) \approx P(A)$.
- If A and B are two events from a sample space with $P(A) \neq 0$, then the conditional probability of B given A in set notation is $P(B|A) = \frac{P(A \cap B)}{P(A)}$.
- The conditional probability formula can be solved to obtain this formula for $P(A \text{ and } B)$:

$$P(A \text{ and } B) = P(A) \bullet P(B|A)$$

SRB p. U4-69

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Instruction

- A **two-way frequency table** is a frequency table that shows two categories of characteristics, one in rows and the other in columns. Each cell value is a frequency that shows how many times two different characteristics appear together, or how often characteristics are associated with a person, object, or type of item that is being studied.
- The following example shows a typical setup of a two-way frequency table.

Category 1 of characteristics	Category 2 of characteristics	
	Characteristic 1	**Characteristic 2**
Characteristic 1	*a*	*b*
Characteristic 2	*c*	*d*

- When probabilities and conditional probabilities are calculated from data in a two-way frequency table, then totals of characteristics are needed.

Category 1 of characteristics	Category 2 of characteristics		Total
	Characteristic 1	**Characteristic 2**	
Characteristic 1	*a*	*b*	
Characteristic 2	*c*	*d*	
Total			

- When a two-way frequency table is used to find probabilities and conditional probabilities, the characteristics represent events, and the frequencies are numbers of outcomes.

Common Errors/Misconceptions

- using a row total in a two-way frequency table when a column total should be used, and vice versa
- using a row or column total when the sample space total should be used

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Guided Practice 4.2.2

Example 1

The Student Council wants to host a school-wide activity. Council members survey 40 students, asking them to choose either a field trip, a dance, or a talent show. The following table shows the survey results, with the surveyed students numbered 1–40. Construct a two-way frequency table to summarize the data.

Student	Grade	Activity	Student	Grade	Activity
1	10	FT	21	10	D
2	12	D	22	10	FT
3	10	TS	23	12	D
4	10	FT	24	11	D
5	11	D	25	11	TS
6	12	D	26	12	D
7	10	TS	27	12	D
8	10	FT	28	10	D
9	10	FT	29	11	D
10	11	TS	30	11	D
11	12	D	31	12	FT
12	10	TS	32	10	TS
13	11	TS	33	12	D
14	10	FT	34	11	D
15	11	D	35	11	FT
16	10	FT	36	11	FT
17	12	D	37	11	TS
18	10	FT	38	12	TS
19	12	D	39	11	FT
20	11	TS	40	12	TS

Key: TS = Talent show, FT = Field trip, D = Dance

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Instruction

1. Set up a tally table.

   There are two characteristics associated with each student: that student's grade and that student's choice of activity. Set up a table that shows "Grade" and "Activity choice" as categories, and all the different characteristics in each category.

Grade	Activity choice		
	Talent show	Field trip	Dance
10			
11			
12			

2. Tally the data.

   For each student, draw a tally mark that corresponds to that student's grade and choice of activity in the appropriate cell of the data table. The tally marks for students 1–5 are shown in the incomplete tally table below.

Grade	Activity choice		
	Talent show	Field trip	Dance
10	I	II	
11			I
12			I

   The complete tally table below shows the tally marks for all the students.

Grade	Activity choice		
	Talent show	Field trip	Dance
10	IIII	𝍸 III	II
11	𝍸	III	𝍸 I
12	II	I	𝍸 IIII

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Instruction

3. Create a two-way frequency table.

   Count the tally marks in each cell of your tally table. Then, create another table (a two-way frequency table) to show your count results. These results are frequencies. The completed two-way frequency table is shown below.

Grade	Activity choice		
	Talent show	Field trip	Dance
10	4	8	2
11	5	3	6
12	2	1	9

   Add all the frequencies; verify that their sum is 40 (since 40 students were surveyed).

   $4 + 8 + 2 + 5 + 3 + 6 + 2 + 1 + 9 = 40$

Example 2

The completed two-way frequency table from Example 1 is shown. It shows the results of a survey designed to help the Student Council choose a school-wide activity.

Grade	Activity choice		
	Talent show	Field trip	Dance
10	4	8	2
11	5	3	6
12	2	1	9

Consider the following events that apply to a random student who participated in the survey.

TEN: The student is in the tenth grade.

TWELVE: The student is in the twelfth grade.

FT: The student prefers a field trip.

TS: The student prefers a talent show.

Compare $P(TEN|FT)$ and $P(FT|TEN)$. Are *TEN* and *FT* independent?

Compare $P(TWELVE|TS)$ and $P(TS|TWELVE)$. Are *TWELVE* and *TS* independent?

Interpret the results.

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Instruction

1. Find the totals of all the categories.

Grade	Activity choice			Total
	Talent show	Field trip	Dance	
10	4	8	2	14
11	5	3	6	14
12	2	1	9	12
Total	11	12	17	40

2. Compare $P(TEN|FT)$ and $P(FT|TEN)$.

$P(TEN|FT)=\frac{8}{12}\approx 0.667$ There were 12 votes for a field trip; 8 were by tenth graders.

$P(FT|TEN)=\frac{8}{14}\approx 0.571$ There were 14 votes by tenth graders; 8 were for a field trip.

$0.667 > 0.571$; therefore, $P(TEN|FT) > P(FT|TEN)$.

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Instruction

3. Determine if *TEN* and *FT* are independent.

Remember that events *A* and *B* are independent events if $P(B|A)=P(B)$ or if $P(A|B)=P(A)$.

Compare $P(TEN|FT)$ with *P*(*TEN*) and $P(FT|TEN)$ with *P*(*FT*).

$P(TEN|FT)=\frac{8}{12}\approx 0.667$ — There were 12 votes for a field trip; 8 were by tenth graders.

$P(TEN)=\frac{14}{40}=0.35$ — There were 40 votes in all; 14 were by tenth graders.

$0.667 \neq 0.35$; therefore, $P(TEN|FT) \neq P(TEN)$.

$P(FT|TEN)=\frac{8}{14}\approx 0.571$ — There were 14 votes by tenth graders; 8 were for a field trip.

$P(FT)=\frac{12}{40}=0.3$ — There were 40 votes in all; 12 were for a field trip.

$0.571 \neq 0.3$; therefore, $P(FT|TEN) \neq P(FT)$.

Based on the data, *TEN* and *FT* seem to be dependent because $P(TEN|FT) \neq P(TEN)$ and $P(FT|TEN) \neq P(FT)$.

4. Interpret the results for $P(TEN|FT)$ and $P(FT|TEN)$.

$P(TEN|FT)$ is the probability that a student is in the tenth grade given that he prefers a field trip.

$P(FT|TEN)$ is the probability that a student prefers a field trip given that he is in the tenth grade.

The fact that *TEN* and *FT* are dependent means that being in the tenth grade affects the probability that a student prefers a field trip, and preferring a field trip affects the probability that a student is in the tenth grade. In this case, being in the tenth grade increases the probability that a student prefers a field trip because $P(FT|TEN) > P(FT)$. Also, preferring a field trip increases the probability that a student is in the tenth grade because $P(TEN|FT) > P(TEN)$.

$P(TEN|FT) > P(FT|TEN)$ means that it is more likely that a student is in the tenth grade given that he prefers a field trip than it is that a student prefers a field trip given that he is in the tenth grade.

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Instruction

5. Compare $P(TWELVE|TS)$ and $P(TS|TWELVE)$.

$P(TWELVE|TS)=\frac{2}{11}\approx 0.182$ There were 11 votes for a talent show; 2 were by twelfth graders.

$P(TS|TWELVE)=\frac{2}{12}\approx 0.167$ There were 12 votes by twelfth graders; 2 were for a talent show.

$0.182 > 0.167$; therefore, $P(TWELVE|TS) > P(TS|TWELVE)$, but they are close in value. The values are approximately 18% and 17%.

6. Determine if *TWELVE* and *TS* are independent.

Events *A* and *B* are independent events if $P(B|A)=P(B)$ or if $P(A|B)=P(A)$.

Compare $P(TWELVE|TS)$ with $P(TWELVE)$ and $P(TS|TWELVE)$ with $P(TS)$.

$P(TWELVE|TS)=\frac{2}{11}\approx 0.182$ There were 11 votes for a talent show; 2 were by twelfth graders.

$P(TWELVE)=\frac{12}{40}=0.3$ There were 40 votes in all; 12 were by twelfth graders.

$0.182 \neq 0.3$; therefore, $P(TWELVE|TS) \neq P(TWELVE)$.

$P(TS|TWELVE)=\frac{2}{12}\approx 0.167$ There were 12 votes by twelfth graders; 2 were for a talent show.

$P(TS)=\frac{11}{40}=0.275$ There were 40 votes in all; 11 were for a talent show.

$0.167 \neq 0.275$; therefore, $P(TS|TWELVE) \neq P(TS)$.

Based on the data, *TWELVE* and *TS* are dependent because $P(TWELVE|TS) \neq P(TWELVE)$ and $P(TS|TWELVE) \neq P(TS)$.

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Instruction

7. Interpret the results for $P(TWELVE|TS)$ and $P(TS|TWELVE)$.

$P(TWELVE|TS)$ is the probability that a student is in the twelfth grade given that the student prefers a talent show.

$P(TS|TWELVE)$ is the probability that a student prefers a talent show given that the student is in the twelfth grade.

The fact that *TWELVE* and *TS* are dependent means that being in the twelfth grade affects the probability that a student prefers a talent show, and preferring a talent show affects the probability that a student is in the twelfth grade. In this case, being in the twelfth grade decreases the probability that a student prefers a talent show because $P(TS|TWELVE) < P(TS)$. And preferring a talent show decreases the probability that a student is in the twelfth grade because $P(TWELVE|TS) < P(TWELVE)$.

$P(TWELVE|TS) > P(TS|TWELVE)$, but they are close in value, differing by only about 1%. So it is almost equally likely that a student is in the twelfth grade given that the student prefers a talent show, as it is that a student prefers a talent show given that the student is in the twelfth grade.

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Instruction

Example 3

A cafeteria manager recorded the choices of 100 students who each chose one food item and one beverage. The table shows the data.

Beverage choice	Food choice	
	Burger	**Pizza**
Milk	34	26
Iced tea	24	16

Consider the following events that apply to a randomly chosen student who selects one food item and one beverage.

M: The student selects milk.

B: The student selects a burger.

P: The student selects pizza.

Compare $P(M|B)$ and $P(B|M)$. Are *M* and *B* independent?

Compare $P(M|P)$ and $P(P|M)$. Are *M* and *P* independent?

Interpret the results.

1. Find the totals of all the categories.

Beverage choice	Food choice		Total
	Burger	**Pizza**	
Milk	34	26	60
Iced tea	24	16	40
Total	58	42	100

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Instruction

2. Compare $P(M|B)$ and $P(B|M)$.

$P(M|B)=\frac{34}{58}\approx 0.586$ — 58 burgers were chosen; 34 were accompanied by milk.

$P(B|M)=\frac{34}{60}\approx 0.567$ — 60 servings of milk were chosen; 34 were accompanied by a burger.

$0.586 > 0.567$; therefore, $P(M|B) > P(B|M)$, but they are close in value. The values are approximately 59% and 57%.

3. Determine if M and B are independent.

Events A and B are independent if $P(B|A)=P(B)$ or if $P(A|B)=P(A)$.

Compare $P(M|B)$ with $P(M)$ and $P(B|M)$ with $P(B)$.

$P(M|B)=\frac{34}{58}\approx 0.586$ — 58 burgers were chosen; 34 were accompanied by milk.

$P(M)=\frac{60}{100}=0.6$ — There were 100 students, and 60 of them chose milk.

$0.586 \approx 0.6$; therefore, $P(M|B) \approx P(M)$.

$P(B|M)=\frac{34}{60}\approx 0.567$ — 60 servings of milk were chosen; 34 were accompanied by a burger.

$P(B)=\frac{58}{100}=0.58$ — There were 100 students, and 58 of them chose a burger.

$0.567 \approx 0.58$; therefore, $P(B|M) \approx P(B)$.

Based on the data, M and B seem to be independent because $P(M|B) \approx P(M)$ and $P(B|M) \approx P(B)$.

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Instruction

4. Interpret the results for $P(M|B)$ and $P(B|M)$.

   $P(M|B)$ represents the probability of a student choosing milk given that he chooses a burger.

   $P(B|M)$ represents the probability of a student choosing a burger given that he chooses milk.

   The conclusion that M and B are independent based on probabilities that are close in value means that choosing milk does not significantly affect the probability of choosing a burger, and choosing a burger does not significantly affect the probability of choosing milk.

   $P(M|B) > P(B|M)$, but they are close in value, differing by only about 1%. So it is almost equally likely that a student chooses milk given that the student chooses a burger, as it is that the student chooses a burger given that the student chooses milk. But because the choices are independent based on probabilities that are close in value, this can be restated more simply as follows: It is almost equally likely that a student chooses milk as that the student chooses a burger.

5. Compare $P(M|P)$ and $P(P|M)$.

   $P(M|P) = \frac{26}{42} \approx 0.619$ — 42 pizza servings were chosen; 26 were accompanied by milk.

   $P(P|M) = \frac{26}{60} \approx 0.433$ — 60 servings of milk were chosen; 26 were accompanied by pizza.

   $0.619 > 0.433$; therefore, $P(M|P) > P(P|M)$.

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Instruction

6. Determine if M and P are independent.

Events A and B are independent events if $P(B|A) = P(B)$ or if $P(A|B) = P(A)$.

Compare $P(M|P)$ with $P(M)$ and $P(P|M)$ with $P(P)$.

$P(M|P) = \frac{26}{42} \approx 0.619$ — 42 pizza servings were chosen; 26 were accompanied by milk.

$P(M) = \frac{60}{100} = 0.6$ — There were 100 students, and 60 of them chose milk.

$0.619 \approx 0.6$; therefore, $P(M|P) \approx P(M)$.

$P(P|M) = \frac{26}{60} \approx 0.433$ — 60 servings of milk were chosen; 26 were accompanied by pizza.

$P(P) = \frac{42}{100} = 0.42$ — There were 100 students, and 42 of them chose pizza.

$0.433 \approx 0.42$; therefore, $P(P|M) \approx P(P)$.

Based on the data, M and P seem to be independent because $P(M|P) \approx P(M)$ and $P(P|M) \approx P(P)$.

7. Interpret the results for $P(M|P)$ and $P(P|M)$.

$P(M|P)$ represents the probability that a student chooses milk given that the student chooses pizza.

$P(P|M)$ represents the probability that a student chooses pizza given that the student chooses milk.

The conclusion that M and P are independent based on probabilities that are close in value means that choosing milk does not significantly affect the probability of choosing pizza, and choosing pizza does not significantly affect the probability of choosing milk.

$P(M|P) > P(P|M)$ means it is more likely that a student chooses milk given that the student chooses pizza than it is that a student chooses pizza given that the student chooses milk. But since the choices are independent based on probabilities that are close in value, this can be restated as follows: It is more likely that a student chooses milk than that a student chooses pizza.

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Problem-Based Task 4.2.2: How Many in the Survey?

SMP
1 2 ✓
3 ✓ 4 ✓
5 6
7 8

Some high school students with part-time jobs were asked where they work. The table shows some of the data, with x and y representing missing values.

Grade	Place of employment		
	Taco Heaven	Burgers Your Way	Pizza Palace
10	1	5	y
11	x	1	5
12	11	3	4

Consider the following events and probabilities.

TEN: The student is in the tenth grade.

ELEVEN: The student is in the eleventh grade.

T: The student works at Taco Heaven.

P: The student works at Pizza Palace.

$$P(T|ELEVEN)=\frac{2}{5}$$

$$P(TEN|P)=\frac{2}{5}$$

Find x, y, and the total number of students surveyed.

Find x, y, and the total number of students surveyed.

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Problem-Based Task 4.2.2: How Many in the Survey?

Coaching

a. What is the formula for $P(T|ELEVEN)$ that uses subsets of the sample space?

b. What variable represents the number of outcomes in "*T* and *ELEVEN*"?

c. What expression represents the number of outcomes in "*ELEVEN*"?

d. What is the given value of $P(T|ELEVEN)$?

e. The answer to part a is an equation. Rewrite that equation, substituting the answers to parts b, c, and d.

f. The equation you wrote for part e is a proportion. Solve the proportion.

g. What is the formula for $P(TEN|P)$ that uses subsets of the sample space?

h. What variable represents the number of outcomes in "*TEN* and *P*"?

i. What expression represents the number of outcomes in "*P*"?

j. What is the given value of $P(TEN|P)$?

k. The answer to part g is an equation. Rewrite that equation, substituting the answers to parts h, i, and j.

l. The equation you wrote for part k is a proportion. Solve the proportion.

m. Now that the values of *x* and *y* have been calculated, use those values along with the given values in the table to find the total number of students surveyed.

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Instruction

Problem-Based Task 4.2.2: How Many in the Survey?

Coaching Sample Responses

a. What is the formula for $P(T|ELEVEN)$ that uses subsets of the sample space?

$$P(T|ELEVEN) = \frac{\text{number of outcomes in } (T \text{ and } ELEVEN)}{\text{number of outcomes in } (ELEVEN)}$$

b. What variable represents the number of outcomes in "*T* and *ELEVEN*"?

x

c. What expression represents the number of outcomes in "*ELEVEN*"?

$x + 1 + 5$, or $x + 6$

d. What is the given value of $P(T|ELEVEN)$?

$\frac{2}{5}$

e. The answer to part a is an equation. Rewrite that equation, substituting the answers to parts b, c, and d.

$$\frac{2}{5} = \frac{x}{x+6}$$

f. The equation you wrote for part e is a proportion. Solve the proportion.

$$\frac{2}{5} = \frac{x}{x+6}$$
$$2(x + 6) = 5x$$
$$2x + 12 = 5x$$
$$12 = 3x$$
$$4 = x$$

g. What is the formula for $P(TEN|P)$ that uses subsets of the sample space?

$$P(TEN|P) = \frac{\text{number of outcomes in } (TEN \text{ and } P)}{\text{number of outcomes in } (P)}$$

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Instruction

h. What variable represents the number of outcomes in "*TEN* and *P*"?

y

i. What expression represents the number of outcomes in "*P*"?

$y + 5 + 4$, or $y + 9$

j. What is the given value of $P(TEN|P)$?

$\frac{2}{5}$

k. The answer to part g is an equation. Rewrite that equation, substituting the answers to parts h, i, and j.

$$\frac{2}{5} = \frac{y}{y+9}$$

l. The equation you wrote for part k is a proportion. Solve the proportion.

$$\frac{2}{5} = \frac{y}{y+9}$$

$2(y + 9) = 5y$

$2y + 18 = 5y$

$18 = 3y$

$6 = y$

m. Now that the values of x and y have been calculated, use those values along with the given values in the table to find the total number of students surveyed.

Sum the values to find the total number of students surveyed.

$1 + 5 + y + x + 1 + 5 + 11 + 3 + 4$

$= 1 + 5 + 6 + 4 + 1 + 5 + 11 + 3 + 4$

$= 40$

40 students were surveyed.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4.2.2: Using Two-Way Frequency Tables

A survey was conducted of 20 baseball players. They were asked to choose infield, outfield, pitcher, or catcher as their favorite position. They were also asked whether they throw with their right arm or left arm. The table shows the survey results with the players numbered 1–20. Use the table to complete problems 1 and 2.

Player	Favorite position	Throwing arm	Player	Favorite position	Throwing arm
1	OF	R	11	P	R
2	P	R	12	OF	R
3	IF	R	13	IF	R
4	C	R	14	IF	R
5	IF	L	15	P	L
6	C	R	16	OF	L
7	IF	R	17	IF	R
8	OF	R	18	P	L
9	OF	R	19	IF	R
10	P	R	20	OF	R

Key: IF = Infield, OF = Outfield, P = Pitcher, C = Catcher, R = Right, L = Left

1. Set up and complete a tally table for the data.

2. Use your tally table to construct a two-way frequency table that summarizes the data.

continued

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Eastern High School's highest academic award category is Highest Honors. The next highest award is Academic Excellence. The table shows data about the awards by grade. Use the table and the events that follow to complete problems 3–6.

Grade	Award category	
	Highest Honors	**Academic Excellence**
10	36	44
11	32	43
12	30	40

TEN: A recognized student is in the tenth grade.

TWELVE: A recognized student is in the twelfth grade.

HH: A recognized student received the Highest Honors award.

AE: A recognized student received the Academic Excellence award.

3. Compare $P(TEN \mid HH)$ and $P(HH \mid TEN)$. Interpret what your answer means.

4. Are *TEN* and *HH* independent? Explain your reasoning and what your answer means.

5. Compare $P(TWELVE \mid AE)$ and $P(AE \mid TWELVE)$. Interpret what your answer means.

6. Are *TWELVE* and *AE* independent? Explain your reasoning and interpret what your answer means.

continued

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A cafeteria manager recorded the choices of 200 students who each chose one food item and one beverage. The table shows the data. Use the table and the events that follow to complete problems 7–10.

Beverage choice	Food choice			
	Pizza	Cold-cut sub	Chicken tenders	Salad
Juice	39	25	26	9
Milk	11	24	25	41

J: The student chooses juice.

M: The student chooses milk.

P: The student chooses pizza.

CC: The student chooses a cold-cut sub.

CT: The student chooses chicken tenders.

S: The student chooses salad.

7. Compare $P(J|P)$ and $P(P|J)$. Determine if *J* and *P* are independent. Show the numerical values of all the probabilities used in your answers.

8. Compare $P(J|CT)$ and $P(CT|J)$. Determine if *J* and *CT* are independent. Show the numerical values of all the probabilities used in your answers.

9. Compare $P(M|CC)$ and $P(CC|M)$. Determine if *M* and *CC* are independent. Show the numerical values of all the probabilities used in your answers.

10. Compare $P(M|S)$ and $P(S|M)$. Determine if *M* and *S* are independent. Show the numerical values of all the probabilities used in your answers.

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Lesson 4.2.3: The Multiplication Rule

Warm-Up 4.2.3

There are 14 students in Mr. Matthew's algebra class. The students' names and genders are listed in the table below. A single student is selected from the class at random.

Name	Gender
Allison	Female
Aiko	Female
Andrew	Male
Charlie	Male
Damara	Female
Dylan	Male
Ian	Male
Isabella	Female
Kara	Female
Lila	Female
Manny	Male
Oliver	Male
Octavia	Female
Sarah	Female

1. What is the probability that the student is male?

2. What is the probability that the student is female?

3. Is it more likely that the selected student is male or female?

4. What is the probability that the student's name begins with O?

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Instruction

Lesson 4.2.3: The Multiplication Rule

Common Core State Standard

S–CP.8 (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interpret the answer in terms of the model.★

Warm-Up 4.2.3 Debrief

1. What is the probability that the student is male?

   The probability of an event is: $\frac{\text{number of ways event can occur}}{\text{total number of events}}$. In this case, the probability being calculated is the probability of selecting 1 male.

   Let *M* stand for the event of selecting a male.

   The total number of events is the total number of students: 14. Selecting a male can occur by picking any of the male students, so to find the number of ways the event can occur, divide the number of male students, 6, by the total number of students, 14.

   $$P(M) = \frac{\text{number of male students}}{\text{total number of students}}$$

   $$P(M) = \frac{6}{14}$$

   The answer can be written as a fraction in simplest form, as a decimal, and as a percent.

   $$P(M) = \frac{6}{14} = \frac{3}{7} \approx 0.43 \approx 43\%$$

   The probability of selecting a male student is about 43%.

2. What is the probability that the student is female?

   Let *F* stand for the event of selecting a female.

   Divide the number of female students by the total number of students in order to find the probability of selecting a female student.

   $$P(F) = \frac{\text{number of female students}}{\text{total number of students}}$$

   $$P(F) = \frac{8}{14} = \frac{4}{7} \approx 0.57 \approx 57\%$$

   The probability of selecting a female student is about 57%.

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3. Is it more likely that the selected student is male or female?

   Compare the two probabilities, $P(M)$ and $P(F)$, to determine which gender is more likely to be selected.

   $P(M) = 0.43$

   $P(F) = 0.57$

   $0.43 < 0.57$

   The probability of selecting a female is higher, so it is more likely that a female will be selected than a male.

4. What is the probability that the student's name begins with O?

   Let O stand for the event of selecting a student whose name begins with O.

   Divide the number of students whose names begin with O by the total number of students.

   $$P(O) = \frac{\text{number of students whose names begin with O}}{\text{total number of students}}$$

   $$P(O) = \frac{2}{14} = \frac{1}{7} \approx 0.14 \approx 14\%$$

   The probability that the student's name begins with O is about 14%.

Connection to the Lesson

- Students will find the probability of two events, both dependent and independent, using the Multiplication Rule.
- Students will find the probability of simple events.
- Students will use probability to compare the likelihood of two events.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- determining the size of a sample space (7.SP.1)
- determining the number of ways an outcome can occur in a sample space (7.SP.1)
- finding the probability of simple events (7.SP.7a)
- comparing probabilities of different events (7.SP.7)
- identifying when two events are independent or dependent (S–CP.2★)

Introduction

The probabilities of simple events can be used to find the probability that two or more events occur. A **simple event** has only one outcome and is sometimes called a single event. Before the probability of both events occurring can be determined, it is necessary to decide if the events are dependent or if they are independent of each other.

Key Concepts

- If two events are independent, the outcome of one event does not affect the probability of the other event. For example, if a coin is flipped 2 times, the outcome of the first flip does not affect the outcome of the second flip.
- If two events are dependent, the outcome of one event affects the probability of the other event and conditional probability must be applied. For example, if 2 cards are drawn from a standard deck without replacement, the probability of choosing a card that is a club the second time is affected by what card is drawn the first time.
- The conditional probability of B given A is the probability that event B occurs, given that event A has already occurred. If A and B are two events from a sample space with $P(A) \neq 0$, then the conditional probability of B given A, denoted $P(B|A)$, has two equivalent expressions: $P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\text{number of outcomes in } (A \text{ and } B)}{\text{number of outcomes in } A}$.
- The conditional probability of event B given that A occurs is written as $P(B|A)$.

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Instruction

- If events A and B are independent, then $P(A|B) = P(A)$ and $P(B|A) = P(B)$, since the probability of event A will have no effect on event B and vice versa.
- **Compound probability** is the probability of a **compound event** (the combination of two or more simple events). In other words, it is the probability of two or more things happening at once, such as flipping a coin and rolling a die.
- The compound probability of two events A and B is written as $P(A \text{ and } B)$ or, in set notation, $P(A \cap B)$.
- If events A and B are *dependent*, then the probability that both events occur is $P(A \text{ and } B) = P(A) \bullet P(B|A) = P(B) \bullet P(A|B)$. This is called the **Multiplication Rule**.
- If events A and B are *independent*, then the probability that both events occur is the product of the probability of each event: $P(A \text{ and } B) = P(A) \bullet P(B)$. This is an extension of the Multiplication Rule.
- If A and B are independent, then $P(B|A) = P(B)$; therefore, by substitution, the rule can be rewritten as $P(A \text{ and } B) = P(A) \bullet P(B)$.
- Since these are independent events, the probability of event A will have no effect on event B, and vice versa, so we can multiply the probability of each event to find the probability of both events occurring.

Common Errors/Misconceptions

- incorrectly calculating probabilities of simple events, such as $P(A)$ and $P(B)$
- incorrectly classifying two events as dependent or independent
- forgetting to find the conditional probability of dependent events

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Instruction

Guided Practice 4.2.3

Example 1

Naveen bought some new clothes for the start of school. The clothes he purchased are in the table:

Item	Color
Shirt	Red
Shirt	Blue
Shirt	White
Shirt	Black
Pants	Blue
Pants	Gray
Pants	Tan

If Naveen randomly selects a shirt and a pair of pants, what is the probability that both the shirt and pants will be blue?

1. Use probability notation to write the problem to be solved.

   Name the events as follows:

   SB: The selected shirt is blue.

   PB: The selected pants are blue.

   A compound probability needs to be calculated because the probability of two events is being found: selecting a blue shirt and selecting blue pants.

   $P(SB \text{ and } PB)$

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Instruction

2. Determine if the two events are independent or dependent.

   Events are dependent if one event happens and the probability of the other event changes. Events are independent if neither event affects the probability of the other. If a blue shirt is selected, does the number of pants or the number of blue pants change?

   Before the shirt or pants are selected, there are 4 shirts and 1 of them is blue. There are 3 pairs of pants and 1 of them is blue. If a blue shirt is selected, there are still 3 total pairs of pants with 1 pair being blue.

   Also consider if the second event happens first—if a blue pair of pants is selected, does the number of shirts or the number of blue shirts change?

   If blue pants are selected, there are still 4 total shirts and 1 of those shirts is blue. Selecting blue pants does not change the total shirts or the number of blue shirts.

   Because selecting a blue shirt does not impact selecting blue pants, and selecting blue pants does not impact selecting a blue shirt, the two events are *independent*.

3. Find the probability of each event.

   The two events are $P(SB)$ and $P(PB)$.

   Divide the number of blue shirts by the total number of shirts to find $P(SB)$, the probability of selecting a blue shirt. Divide the number of blue pants by the total number of pants to find $P(PB)$, the probability of selecting blue pants.

   The probabilities can be left as fractions so that the compound probability is as exact as possible.

$$P(SB) = \frac{\text{number of blue shirts}}{\text{total number of shirts}} \qquad P(PB) = \frac{\text{number of blue pants}}{\text{total number of pants}}$$

$$P(SB) = \frac{1}{4} \qquad P(PB) = \frac{1}{3}$$

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4. Use the Multiplication Rule to find the probability of the compound event.

   The compound event is selecting both a blue shirt and blue pants: $P(SB \text{ and } PB)$.

   The Multiplication Rule is $P(A \text{ and } B) = P(A) \bullet P(B|A)$.

   Recall that if A and B are independent, then

   $P(A \text{ and } B) = P(A) \bullet P(B|A) = P(B) \bullet P(A|B)$, which can be rewritten as

   $P(A \text{ and } B) = P(A) \bullet P(B)$.

   Since the two events are independent, use $P(A \text{ and } B) = P(A) \bullet P(B)$.

   Substitute event SB in place of A and PB in place of B.

   $P(SB \text{ and } PB) = P(SB) \bullet P(PB)$

   Substitute known values for $P(SB)$ and $P(PB)$ and solve.

   $$P(SB \text{ and } PB) = P(SB) \bullet P(PB) = \frac{1}{4} \bullet \frac{1}{3} = \frac{1}{12} \approx 0.083$$

   The probability that both the shirt and pants Naveen selects will be blue is approximately 0.083 or about 8%.

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Instruction

Example 2

Mr. Harkins is holding a raffle to raise money for a class field trip. The number of tickets each student purchased is listed in the following table.

Student	Number of tickets
Dax	3
Mieko	1
Marcus	2
Taylor	1
Caroline	2
Laura	5
Christo	5
Isaiah	4

Mr. Harkins will pick 2 winning tickets by selecting 2 different tickets without replacement. What is the probability that both the first and second tickets selected are Christo's?

1. Use probability notation to write the problem to be solved.

   Name the events as follows:

   FC: The first ticket selected is Christo's.

   SC: The second ticket selected is Christo's.

   A compound probability needs to be calculated because the probability of two events is being found: selecting Christo's ticket first and selecting Christo's ticket second.

   $P(FC \text{ and } SC)$

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Instruction

2. Determine if the two events are independent or dependent.

   Events are dependent if one event happens and the probability of the other event changes. Events are independent if neither event affects the probability of the other. If 1 of Christo's tickets is selected, does the total number of tickets or the total number of Christo's tickets change?

   Determine the total number of tickets by adding the number of tickets purchased by each student.

   $3 + 1 + 2 + 1 + 2 + 5 + 5 + 4 = 23$

   Before any tickets are selected, there are a total of 23 tickets, 5 of which are Christo's. If 1 of Christo's tickets is selected, the total number of tickets is 1 less ($23 - 1 = 22$), and the number of Christo's tickets remaining is also reduced by 1: $5 - 1 = 4$.

   If the first selection is Christo's ticket, both the total number of tickets and the number of Christo's tickets changes. This will change the probability that the second selected ticket is also Christo's. Because the probability of selecting a second ticket of Christo's changes based on the selection of the first ticket, the two events are *dependent*.

3. Use the Multiplication Rule to determine which probability and conditional probabilities need to be calculated.

   The Multiplication Rule is $P(A \text{ and } B) = P(A) \bullet P(B|A)$. Substitute event FC in place of A and SC in place of B.

   $$P(FC \text{ and } SC) = P(FC) \bullet P(SC|FC)$$

   The probabilities that need to be calculated are:

   $P(FC)$

   $P(SC|FC)$

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Instruction

4. Find the probability and conditional probability.

Find $P(FC)$, the probability of the first selected ticket being Christo's. Divide the number of Christo's tickets by the total number of tickets.

$$P(FC) = \frac{\text{number of Christo's tickets}}{\text{total number of tickets}}$$

$$P(FC) = \frac{5}{23}$$

To find $P(SC|FC)$ (the probability of the second selected ticket being Christo's given that the first selected ticket was his), assume that 1 ticket has been selected from the total, and that it was Christo's. The tickets remaining with each student are:

Student	Number of tickets
Dax	3
Mieko	1
Marcus	2
Taylor	1
Caroline	2
Laura	5
Christo	5 – 1 = 4
Isaiah	4

The new total number of tickets is 22, and 4 of them are Christo's. If the first selected ticket is Christo's, then the probability that the second selected ticket is Christo's is:

$$P(SC|FC) = \frac{\text{number of Christo's tickets remaining}}{\text{total number of tickets remaining}}$$

$$P(SC|FC) = \frac{4}{22} = \frac{2}{11}$$

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Instruction

5. Use the Multiplication Rule to find the compound probability of the two events.

   Multiply the probability of the first event by the probability of the second event given that the first event occurs.

   Use the calculated probabilities to evaluate:

   $P(FC \text{ and } SC) = P(FC) \bullet P(SC|FC)$

   $P(FC \text{ and } SC) = P(FC) \bullet P(SC|FC) = \frac{5}{23} \bullet \frac{2}{11}$

   $P(FC \text{ and } SC) = P(FC) \bullet P(SC|FC) = \frac{10}{253} \approx 0.040$

   The probability that both the first and second selected tickets are Christo's is approximately 0.040 or about 4%.

Example 3

Cate created a card game. In Cate's game, a player draws 2 cards from a standard deck of cards, without replacing the first card before drawing the second. If the 2 cards are even numbers, the player wins. Face cards (ace, king, queen, and jack) are not considered even or odd. What is the probability that a player will win the game?

1. Use probability notation to write the problem to be solved.

   Name the events as follows:

   FE: The first card drawn is even.

   SE: The second card drawn is even.

   A compound probability needs to be calculated because the probability of two events is being found: the probability that a player will win based on both drawn cards being even.

   $P(FE \text{ and } SE)$

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Instruction

2. Determine if the two events are independent or dependent.

   Events are dependent if one event happens and the probability of the other event changes. Events are independent if neither event affects the probability of the other. If one of the even cards is selected, does the total number of cards or the total number of even cards change?

   The total number of cards is the total for the entire deck: 52. The card values that are even are 2, 4, 6, 8, and 10. Because there are 4 suits of cards, this means there are 4 cards in the deck for each even value. If 5 cards in each suit are even, and there are 4 suits, there are $4 \bullet 5 = 20$ even cards in the deck. If 1 of the even cards is selected, the total number of cards remaining is 1 less ($52 - 1 = 51$), and the number of even cards remaining is also reduced by 1: $20 - 1 = 19$.

   If the first selection is an even card, both the total number of cards and the number of even cards changes. This will change the probability that the second selected card is also even. Because the probability of selecting a second even card changes based on the selection of the first card, the two events are *dependent*.

3. Use the Multiplication Rule to determine which probability and conditional probabilities need to be calculated.

   The Multiplication Rule is $P(A \text{ and } B) = P(A) \bullet P(B|A)$. Substitute event *FE* in place of *A* and *SE* in place of *B*.

   $$P(FE \text{ and } SE) = P(FE) \bullet P(SE|FE)$$

   The probabilities that need to be calculated are:

   $P(FE)$

   $P(SE|FE)$

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Instruction

4. Find the probability and conditional probability.

   Find $P(FE)$, the probability of the first card drawn being even. Divide the number of even cards by the total number of cards.

   $$P(FE)=\frac{\text{number of even cards}}{\text{total number of cards}}$$

   $$P(FE)=\frac{20}{52}=\frac{5}{13}$$

   To find $P(SE|FE)$, the probability of the second drawn card being even given that the first drawn card is even, find the number of even cards remaining and the number of total cards remaining.

   There is 1 less even card (20 – 1 = 19), and 1 less total card (52 – 1 = 51).

   $$P(SE|FE)=\frac{\text{number of even cards remaining}}{\text{total number of cards remaining}}$$

   $$P(SE|FE)=\frac{19}{51}$$

5. Use the Multiplication Rule to find the compound probability of the two events.

   Multiply the probability of the first event by the probability of the second event given that the first event occurs.

   Use the calculated probabilities to evaluate:

   $$P(FE \text{ and } SE)=P(FE)\bullet P(SE|FE)$$

   $$P(FE \text{ and } SE)=P(FE)\bullet P(SE|FE)=\frac{5}{13}\bullet\frac{19}{51}$$

   $$P(FE \text{ and } SE)=P(FE)\bullet P(SE|FE)=\frac{95}{663}\approx 0.14$$

   The probability that a player will win the game is approximately 0.14 or about 14%.

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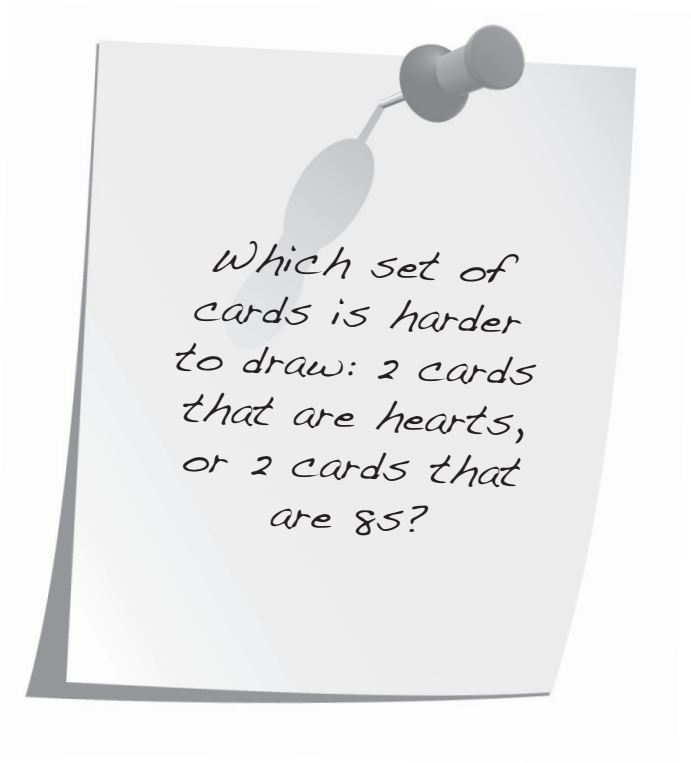
Problem-Based Task 4.2.3: Playing Cards

SMP	
1 ✓	2
3 ✓	4
5	6
7 ✓	8

Kayla is creating a card game. A player in the card game will draw 2 cards from a standard deck of cards, without replacing the first card drawn. Kayla wants sets of cards to be worth more points if they are harder to draw; that is, if the cards have a lower probability of being drawn from the deck. She is comparing drawing cards of a given suit and cards of a given value. Which set of cards is harder to draw: 2 cards that are hearts, or 2 cards that are 8s?



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Problem-Based Task 4.2.3: Playing Cards

Coaching

a. What are the suits in a standard deck of cards?

b. How many cards are there of each type of suit?

c. Are the events in the task independent or dependent? Explain your answer.

d. After drawing 1 card that is a heart, what is the probability that the second card is a heart?

e. What is the probability that 2 cards drawn from a deck are hearts?

f. What are the values of the cards in a standard deck?

g. How many of each value are in a deck?

h. After drawing 1 card that is an 8, what is the probability that the second card is an 8?

i. What is the probability that 2 cards drawn from a deck are both 8s?

j. Which set of cards is more difficult to draw: 2 cards that are hearts, or 2 cards that are 8s?

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Instruction

Problem-Based Task 4.2.3: Playing Cards

Coaching Sample Responses

a. What are the suits in a standard card deck?

There are 4 suits in a standard card deck: hearts, diamonds, spades, and clubs.

b. How many cards are there of each type of suit?

A deck of cards has 52 cards. There are an equal number of cards in each of the 4 suits, so there are $52 \div 4 = 13$ cards of each suit.

c. Are the events in the task independent or dependent? Explain your answer.

The events are dependent. The probability of choosing a card that is a heart the second time is affected by what card is drawn the first time. If a heart is drawn the first time, then there is now 1 less heart card in the total number of outcomes. The same is true for drawing an 8 on the second draw if an 8 is drawn the first time.

d. After drawing 1 card that is a heart, what is the probability that the second card is a heart?

After 1 card is drawn, there are $52 - 1 = 51$ cards remaining in the deck. There are 13 cards of each suit. After 1 heart is drawn, there are: $13 - 1 = 12$ hearts remaining. To find the probability that the second card is a heart given that the first card is a heart, divide the number of hearts remaining by the total cards remaining.

Let *FH* represent drawing a heart for the first card.

Let *SH* represent drawing a heart for the second card.

$$P(SH|FH) = \frac{\text{cards remaining that are hearts}}{\text{cards remaining in deck}}$$

$$P(SH|FH) = \frac{12}{51} = \frac{4}{17}$$

e. What is the probability that 2 cards drawn from a deck are both hearts?

There are two events: *FH*, the first card drawn is a heart, and *SH*, the second card drawn is a heart. The second event is dependent on the first. The Multiplication Rule can be used to find the probability that both events occur.

$$P(A \text{ and } B) = P(A) \bullet P(B|A)$$

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Substitute *FH* for *A* and *SH* for *B*.

$$P(FH \text{ and } SH) = P(FH) \bullet P\left(SH \middle| FH\right)$$

Find *P*(*FH*), the probability of the first drawn card being a heart.

$$P\left(FH\right) = \frac{\text{number of hearts}}{\text{total number of cards}}$$

$$P\left(FH\right) = \frac{13}{52} = \frac{1}{4}$$

From part d we determined $P\left(SH \middle| FH\right)$, the probability of the second drawn card being a heart given that the first drawn card is a heart.

$$P\left(SH \middle| FH\right) = \frac{4}{17}$$

Substitute the found probabilities into the equation for the Multiplication Rule and solve.

$$P\left(FH \text{ and } SH\right) = P(FH) \bullet P\left(SH \middle| FH\right) = \frac{1}{4} \bullet \frac{4}{17} = \frac{1}{17} \approx 0.059$$

f. What are the values of the cards in a standard deck?

The values are: 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king, and ace.

g. How many of each value are in a deck?

There are 13 values, and an equal number of each value in the deck.

There are 52 ÷ 13 = 4 cards of each value.

h. After drawing 1 card that is an 8, what is the probability that the second card is an 8?

After 1 card is drawn, there are 52 – 1 = 51 cards remaining in the deck. There are 4 cards of each value. After 1 card that is an 8 is drawn, there are 3 cards that are 8s remaining. To find the probability that the second card is an 8 given that the first card is an 8, divide the number of 8s remaining by the total cards remaining.

Let *FE* represent drawing an 8 for the first card.

Let *SE* represent drawing an 8 for the second card.

$$P\left(SE \middle| FE\right) = \frac{\text{cards remaining that are 8s}}{\text{cards remaining in deck}}$$

$$P\left(SE \middle| FE\right) = \frac{3}{51} = \frac{1}{17} \approx 0.059$$

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Instruction

i. What is the probability that 2 cards drawn from a deck are both 8s?

There are two events: *FE*, the first card drawn is an 8, and *SE*, the second card drawn is an 8. The second event is dependent on the first. The Multiplication Rule can be used to find the probability that both events occur.

$$P(A \text{ and } B) = P(A) \bullet P(B|A)$$

Substitute *FE* for *A* and *SE* for *B*.

$$P(FE \text{ and } SE) = P(FE) \bullet P(SE|FE)$$

Find *P*(*FE*), the probability of the first drawn card being an 8.

$$P(FE) = \frac{\text{number of 8s}}{\text{total number of cards}}$$

$$P(FE) = \frac{4}{52} = \frac{1}{13} \approx 0.077$$

From part h we determined $P(SE|FE)$, the probability of the second drawn card being an 8 given that the first drawn card is an 8.

$$P(SE|FE) = \frac{1}{17}$$

Substitute the found probabilities into the equation for the Multiplication Rule and solve.

$$P(FE \text{ and } SE) = P(FE) \bullet P(SE|FE) = \frac{1}{13} \bullet \frac{1}{17} = \frac{1}{221} \approx 0.0045$$

j. Which set of cards is more difficult to draw: 2 cards that are hearts, or 2 cards that are 8s?

The set that has the lower probability is less likely, and is more difficult to draw.

Compare the probability of *P*(*FH* and *SH*) with the probability of *P*(*FE* and *SE*).

$0.059 > 0.0045$

The probability of drawing 2 cards that are 8s is less than the probability of drawing 2 cards that are hearts, so drawing 2 cards that are 8s is more difficult.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4.2.3: The Multiplication Rule

Use the following information to answer questions 1–4. Round each answer to the nearest thousandth.

Harper keeps her textbooks in her locker. She has 1 math book, 2 French books, 2 history books, 1 biology book, and 3 English books. Harper grabs 2 textbooks at random.

1. What is the probability that both books are history books?

2. What is the probability that both books are English books?

3. What is the probability that 1 book is a math book, and the other is a biology book?

4. Is it more likely that Harper will pick 2 French books or 2 English books?

Use the following information to answer questions 5–8. Round each answer to the nearest thousandth.

Michael chooses 1 pair of socks and 1 pair of sneakers randomly each morning. The table below lists the colors and number of pairs of socks and sneakers he owns.

Item	Number of pairs by color				
	White	Gray	Black	Blue	Orange
Socks	6	2	1	2	0
Sneakers	1	0	2	0	1

5. What is the probability that Michael selects both white socks and white sneakers?

6. What is the probability that his socks are black and his sneakers are white?

7. What is the probability that his socks are gray and his sneakers are black?

8. Is it more likely that he chooses both black socks and black sneakers, or that he chooses both white socks and white sneakers?

continued

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Use the following information to answer questions 9 and 10. Round each answer to the nearest thousandth.

A factory produces tables for restaurants. The floor manager measures the first 15 tables to determine the cutting accuracy of a new machine that makes tabletops. The lengths of the first 15 tables are in the table below, in inches. The manager randomly chooses 2 tabletops to ship to a restaurant.

Table number	Tabletop length (in inches)
1	32
2	32
3	31
4	28
5	31
6	31
7	30
8	32
9	30
10	30
11	28
12	29
13	31
14	28
15	29

9. What is the probability that both of the randomly chosen tabletops are 30 inches long?

10. What is the probability that both tabletops are 31 through 32 inches long?

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Progress Assessment

Circle the letter of the best answer.

1. Paula rolls a pair of number cubes. What is the probability that she gets doubles (the same number on both cubes) if both numbers are odd?

   a. $\frac{1}{6}$

   b. $\frac{1}{4}$

   c. $\frac{1}{3}$

   d. $\frac{1}{2}$

2. The Leons have 3 children. Which pair of events is independent?

   a. The oldest child is a girl. Exactly 2 children are girls.

   b. The second oldest child is a girl. At least 2 children are girls.

   c. The oldest child is a boy. Exactly 2 consecutive children are boys.

   d. The second oldest child is a boy. Exactly 2 consecutive children are boys.

3. Waterfalls State Park has two popular hiking trails: Eagle Rock Trail and Falls Trail. On one particular day, 80 hiking groups used the trails: 40 groups used Eagle Rock Trail and 40 groups used Falls Trail. Of the 40 groups that used Eagle Rock Trail, 30 groups had children and 10 groups had no children. Consider the following events.

   E: A hiking group uses Eagle Rock Trail.

   C: A hiking group has children.

   Which statement is true about events E and C?

   a. Events E and C are dependent and $P(E|C) < P(C|E)$.

   b. Events E and C are dependent and $P(E|C) = P(C|E)$.

   c. Events E and C are independent and $P(E|C) < P(C|E)$.

   d. Events E and C are independent and $P(E|C) = P(C|E)$.

continued

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Assessment

4. Valley View School is a combined middle school and high school. The cafeteria serves optional breakfast to high school students and then to middle school students. One morning the cafeteria served breakfast to 40 high school students and then 60 middle school students. Of those 100 breakfasts, 60 were breakfast tacos. Of the 60 tacos, 24 were served to high school students and 36 were served to middle school students. Consider the following events.

   T: A breakfast taco is served.

   MS: A middle school student is served.

   Which statement is true about events *T* and *MS*?

   a. Events *T* and *MS* are dependent and $P(T|MS) = P(MS|T)$.

   b. Events *T* and *MS* are dependent and $P(T|MS) > P(MS|T)$.

   c. Events *T* and *MS* are independent and $P(T|MS) = P(MS|T)$.

   d. Events *T* and *MS* are independent and $P(T|MS) > P(MS|T)$.

5. A survey revealed that 40% of the students at Douglas High School are involved in an extracurricular activity. The survey also showed that 32% of the students at the school are involved in an extracurricular activity and also participated in the fall 5K fund-raiser. What is the probability that a student who is involved in an extracurricular activity also participated in the 5K fund-raiser?

   a. 8%

   b. 12.5%

   c. 25%

   d. 80%

6. The table below shows numbers of purchases in three stores and the corresponding gender of the customers for each number of purchases. What is the probability that a purchase is made by a male, given that the purchase is made in store B?

Gender	Store		
	Store A	Store B	Store C
Male	15	20	25
Female	20	15	15

   a. $\frac{7}{22}$

   b. $\frac{1}{3}$

   c. $\frac{6}{11}$

   d. $\frac{4}{7}$

continued

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7. The table below shows some data about a restaurant's pizza sales.

Crust	Pizza size		
	Small	Medium	Large
Thin	20	15	25
Thick	20	25	15

Consider the following events.

THIN: A thin-crust pizza is sold.

S: A small pizza is sold.

Which statement is true about events *THIN* and *S*?

a. Events *THIN* and *S* are dependent and $P(THIN|S) < P(S|THIN)$.

b. Events *THIN* and *S* are dependent and $P(THIN|S) > P(S|THIN)$.

c. Events *THIN* and *S* are independent and $P(THIN|S) < P(S|THIN)$.

d. Events *THIN* and *S* are independent and $P(THIN|S) > P(S|THIN)$.

8. A college admissions officer was curious about the major chosen by students at her college who had graduated from three high schools in a nearby city. She collected data on a sample of students from those high schools. The table shows the data.

High school	College major		
	Business	Pre-law	Computer science
Eastern	8	8	4
Western	6	6	8
Central	11	1	8

The following probabilities apply to a randomly chosen student in the sample. Which probability is the greatest?

a. the probability that a student went to Central

b. the probability that a student went to Western given that the student is a computer science major

c. the probability that a student went to Eastern given that the student is a business major

d. the probability that a student is a business major given that the student went to Western

continued

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Assessment

9. What is the probability that 2 cards selected from a standard deck of 52 cards without replacement are both odd numbers?

   a. 0.86
   b. 0.89
   c. 0.14
   d. 0.95

10. What is the probability that 2 cards selected from a standard deck of 52 cards without replacement are both multiples of 5?

   a. 0.00075
   b. 0.0045
   c. 0.011
   d. 0.021

Use what you have learned about probabilities to solve problem 11.

11. Dan and Lamar are brothers. Each Monday, Tuesday, and Wednesday they toss a coin to decide which one of them has to walk the family dog. Consider the following events.

   A: Dan walks the dog on Monday.

   B: Dan walks the dog on exactly 2 consecutive days.

   Complete the following.

   a. List the sample space.
   b. Find $P(B|A)$, first using probabilities and then using outcomes. Show your work.
   c. Determine if A and B are independent. Show your work and explain your method.
   d. Compare $P(B|A)$ and $P(A|B)$. Show your work and explain what your answer means.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Derek is painting his living room, dining room, and bedroom. He has narrowed down his paint choices to 8 colors. How many different ways can he paint the 3 rooms if he doesn't use any of the colors twice?

   a. 28
   b. 56
   c. 336
   d. 512

2. Yulia is decorating a cake. She wants to use 4 colors on the cake, and has 10 colors to choose from. How many different combinations of colors can she choose if she doesn't use the same color twice?

   a. 105
   b. 210
   c. 2,520
   d. 5,040

3. An event planner has found 6 types of entertainment for an upcoming event: a magician, a band, a deejay, a comedian, a sketch artist, and a storyteller. He is going to choose 2 different types of entertainment: 1 for the afternoon and 1 for the evening. What is the probability that the sketch artist is the afternoon entertainment?

   a. 0.17
   b. 0.33
   c. 0.67
   d. 0.84

4. Two cards are selected from a deck. The first card is replaced in the deck before the second card is selected. What is the probability that both cards are hearts?

   a. 0.0125
   b. 0.0625
   c. 0.20
   d. 0.25

5. Hiroshi is building his class schedule. He has time to take 5 classes in 1 day. He has narrowed down his choices to 8 classes, all of which are offered every class period: calculus, chemistry, anatomy, biology, physics, geography, philosophy, and psychology. If one schedule of all possible schedules is chosen at random, what is the probability that his first class of the day is calculus?

   a. 0.0125
   b. 0.025
   c. 0.125
   d. 0.25

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Lesson 3: Combinatorics

Instruction

SMP
1 ✓ 2 ✓
3 ✓ 4
5 6
7 ✓ 8

Common Core State Standard

S–CP.9 (+) Use permutations and combinations to compute probabilities of compound events and solve problems.★

Essential Questions

1. What is the difference between a permutation and a combination?
2. How are permutations and combinations related to a set of events or objects?
3. How can permutations and combinations be used to calculate probabilities?

WORDS TO KNOW

combination a subset of a group of objects taken from a larger group of objects; the order of the objects does not matter, and objects may be repeated. A combination of size r from a group of n objects can be represented using the notation ${}_nC_r$, where ${}_nC_r = \frac{n!}{(n-r)!r!}$.

factorial the product of an integer and all preceding positive integers, represented using a ! symbol; $n! = n \bullet (n-1) \bullet (n-2) \bullet \ldots \bullet 1$. For example, $5! = 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1$. By definition, $0! = 1$.

permutation a selection of objects where the order matters and is found either using n^r, if repetitions are allowed, or by using ${}_nP_r = \frac{n!}{(n-r)!}$, where n is the number of objects to select from and r is the number of objects being selected and ordered

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Instruction

Recommended Resources

- Khan Academy. "Permutations and combinations."

  http://www.walch.com/rr/00133

  This interactive quiz covers combinations and permutations. Videos and examples are available for support.

- MathIsFun.com. "Combinations and Permutations."

  http://www.walch.com/rr/00135

  This site offers an overview of both combinations and permutations. Practice questions and related activities are included to reinforce concepts.

- The Math Page. "Topics in Precalculus: Permutations and Combinations."

  http://www.walch.com/rr/00134

  This site offers definitions and examples for combinations and permutations, including sample problems that use the notation ${}_nP_r$.

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Lesson 4.3.1: Combinations and Permutations

Warm-Up 4.3.1

Mr. Isaacs needs to choose 2 students to hand out papers to the class. Four students volunteer. Two are boys, Siraj and Jonathan, and two are girls, Autumn and Quintana.

1. If Mr. Isaacs chooses Quintana as one of the students, what are all the different possible pairs of students?

2. If Autumn changes her mind and decides not to volunteer, what are all the different possible pairs of students who could be chosen?

3. If Mr. Isaacs wants to choose 1 boy and 1 girl, what are all the different possible pairs of students?

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Instruction

Lesson 4.3.1: Combinations and Permutations

Common Core State Standard

S–CP.9 (+) Use permutations and combinations to compute probabilities of compound events and solve problems.★

Warm-Up 4.3.1 Debrief

1. If Mr. Isaacs chooses Quintana as one of the students, what are all the different possible pairs of students?

   A table can help organize the different possible groups. Let the first student always be Quintana in each group. The second student in each row will be Siraj, Autumn, or Jonathan.

Student 1	Student 2
Quintana	Siraj
Quintana	Autumn
Quintana	Jonathan

2. If Autumn changes her mind and decides not to volunteer, what are all the different possible pairs of students who could be chosen?

   Use a table to make sure all the possible groups of students are counted. First, pair Quintana with each of the other volunteers remaining: Siraj and Jonathan. Next, pair Siraj with anyone he hasn't been paired with yet: Jonathan. Jonathan has already been paired with both Quintana and Siraj, so all the pairs have been listed.

Student 1	Student 2
Quintana	Siraj
Quintana	Jonathan
Siraj	Jonathan

3. If Mr. Isaacs wants to choose 1 boy and 1 girl, what are all the different possible pairs of students?

   In this table, list the girls in one column and the boys in the other. Make sure each girl is paired with each boy.

Student 1 (girl)	Student 2 (boy)
Quintana	Siraj
Quintana	Jonathan
Autumn	Siraj
Autumn	Jonathan

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Instruction

Connection to the Lesson

- In this lesson, students will find permutations and combinations of objects.
- Students will need to find combinations by finding the number of subsets of a group of objects.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- finding products and quotients (5.NBT.5, 5.NBT.6)
- simplifying fractions with common factors in the numerator and denominator (4.NF.1)

Introduction

Imagine an ice cream parlor in your town advertises that it has more than 10,000 possible combinations of flavors and toppings. Your friend is skeptical, and thinks the ice cream parlor isn't being truthful. You know how many different flavors of ice cream are served, and the number of possible toppings. How could you and your friend determine the real number of all the possible combinations? It would take quite a while to make a table or a list of all the combinations you could brainstorm, and you might not come up with all of the possibilities. A more efficient way of determining the answer is with permutations and combinations. Permutations and combinations are mathematical means of exploring the number of ways groups of objects can be made and arranged. Sometimes the order of the objects matters; other times, it is only which objects are chosen that matters.

Key Concepts

Permutations

- A **permutation** is an arrangement of some or all objects in a group where the order of the objects matters and the objects may or may not be repeated. For example, the permutations "a, b, c" and "a, c, b" count as two different permutations because the order of the letters is different.
- To find the permutations of a group of objects, list the different ways the objects (or a subset of the objects) can be arranged.
- To calculate the number of permutations, multiply the number of choices for each position. For example, if there are n choices for the first position, m for the second, and l for the third, the number of permutations is $n \bullet m \bullet l$.
- When determining the number of arrangements (permutations), keep in mind whether the same choice can be in more than one position, or if a choice can be repeated. Suppose there are n unique objects, and r of them are chosen. To find the total number of ways the n objects can be placed in a group of size r, if an object can be repeated, find the product: $n \bullet n \bullet n \bullet \ldots \bullet n = n^r$. If an object cannot be repeated, then the number of permutations of n objects in r positions is $n \bullet (n-1) \bullet (n-2) \bullet \ldots \bullet (n-r+1)$.

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Instruction

- A **factorial** is the product of an integer and all preceding positive integers, represented using a ! symbol; $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. By definition, $0! = 1$.
- Permutations, where repetition isn't allowed, can be represented using factorial notation. The number of permutations of n objects in r positions, without repeating any of the n objects, is ${}_nP_r = \frac{n!}{(n-r)!}$.

On a TI-83/84:

Step 1: Type the value of *n*.

Step 2: Press the [MATH] key.

Step 3: Arrow over to PRB to access the probability menu.

Step 4: Arrow down to 2: nPr. Press [ENTER].

Step 5: Input the value of *r* and press [ENTER]. Note this counts the number of permutations without repetition.

On a TI-Nspire:

Step 1: From the calculate window, press the [menu] key.

Step 2: Arrow down to 5: Probability. Press [enter].

Step 3: Arrow down to 2: Permutations. Press [enter].

Step 4: Where the screen says "nPr(", type [*n*][,][*r*] and press [enter]. Note this counts the number of permutations without repetition.

Combinations

- A **combination** is a subset of objects taken from a larger group of objects; the order of the objects does not matter, and objects may be repeated. A combination of n objects from a group of size r can be represented using ${}_nC_r$.
- When determining the number of combinations that can be made from a group of objects, the order of the objects doesn't matter. For example, the combinations "a, b, c" and "b, c, a" are the same, since they each contain the same three letters.
- To find the combinations of a group of objects, find the unique subsets from the original group of objects.

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Instruction

- To calculate the number of combinations, where repetition of objects isn't allowed, first find the number of permutations without repetition: $\frac{n!}{(n-r)!}$. Since the number of permutations counts combinations that are the same, but in a different order, the number of permutations needs to be divided by how many permutations there are of each combination. Since each combination has *r* objects, there are *r*! ways each combination can be arranged.

- The number of combinations of *n* objects in a group of size *r*, where objects cannot be repeated, is: $\frac{n!}{(n-r)!} \div r! = \frac{n!}{r!(n-r)!}$. Therefore, the formula for combinations is ${}_nC_r = \frac{n!}{(n-r)!r!}$.

On a TI-83/84:

Step 1: In a blank window, type the value of *n*.

Step 2: Press the [MATH] key.

Step 3: Arrow over to PRB to access the probability menu.

Step 4: Arrow down to 3: nCr. Press [ENTER].

Step 5: Input the value of *r* and press [ENTER]. Note this counts the number of combinations without repetition.

On a TI-Nspire:

Step 1: From the calculate window, press the [menu] key.

Step 2: Arrow down to 5: Probability. Press [enter].

Step 3: Arrow down to 3: Combinations. Press [enter].

Step 4: Where the screen says "nCr(", type [*n*][,][*r*] and press [enter]. Note this counts the number of combinations without repetition.

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Instruction

Common Errors/Misconceptions

- confusing permutations and combinations
- incorrectly calculating factorials
- forgetting to differentiate between permutations with repetition and permutations without repetition
- not dividing the number of permutations by the number of ways each combination can be arranged to find the number of combinations

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Instruction

Guided Practice 4.3.1

Example 1

Keenan is taking a multiple-choice quiz. Each question has 4 possible answers: a, b, c, or d. There are 6 questions. How many different ways can Keenan answer the quiz?

1. Determine whether permutations or combinations are being counted.

   When answering questions on a quiz, the order of the responses does matter. Answering "c, c, d, a, b, a" isn't the same as "a, b, a, d, c, c," even though the same 6 responses are chosen. Since the order matters, the number of permutations is being counted.

2. Determine if repetition is allowed.

   On this quiz, there are only 4 types of responses: a, b, c, or d. If a response is given for one question, it can still be given for any of the other questions. Repetition of the responses on different questions is allowed.

3. Find the number of permutations with repetition.

   There are 4 possible answers, so there are 4 objects to choose from for each question. There are 6 questions to answer, or positions.

   Therefore, the number of permutations is:

   $4 \bullet 4 \bullet 4 \bullet 4 \bullet 4 \bullet 4 = 4^6 = 4096$

   Alternately, the formula n^r can also be used. When choosing from n objects in r positions and allowing repetition, evaluate the expression n^r.

   There are 4 objects in 6 positions, or 4 answers for 6 questions.

   $n^r = 4^6 = 4096$ permutations

   Keenan can answer the quiz 4,096 different ways.

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Instruction

Example 2

Joely is answering a fill-in-the-blank section on an exam. There are 6 answer choices: a, b, c, d, e, and f, and 5 blanks to fill in. Each answer choice can only be used once, and one answer will not be chosen. How many different ways can Joely fill in the 5 blanks?

1. Determine whether permutations or combinations are being counted.

   When answering items on a test, selecting choices "a, b, and c" is not the same as selecting "b, c, and a," even though these two combinations contain the same choices. Since the order matters, the number of permutations is being counted.

2. Determine if repetition is allowed.

   It is stated that each answer choice can only be used to fill in 1 blank. For example, a student couldn't answer "a, b, a, b, a" to the 5 questions. Repetition of choices is not allowed.

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Instruction

3. Find the number of permutations without repetition.

There are 6 different choices for each of the 5 fill-in-the-blank questions. Let's organize the information in a table to identify how many different ways each fill-in-the-blank question can be answered.

Fill-in-the-blank question	Number of ways to answer	Explanation
1	6	None of the answers have been used yet, so any of the 6 choices can be used.
2	5	1 answer has been used already, so 6 – 1 = 5 choices remain.
3	4	2 answers have been used already, so 6 – 2 = 4 choices remain.
4	3	3 answers have been used already, so 6 – 3 = 3 choices remain.
5	2	4 answers have been used already, so 6 – 4 = 2 choices remain.

Find the product of the number of ways each question can be answered to find the total number of ways the questions can be answered:

$6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 = 720$

Alternately, the formula for permutations without repetition can also be used. When choosing n objects in r positions, and repetition isn't allowed, the number of permutations without repetition is ${}_nP_r = \frac{n!}{(n-r)!}$.

There are 6 choices for 5 answers, or 6 objects for 5 positions.

Therefore, the number of permutations is:

$${}_nP_r = \frac{n!}{(n-r)!} = \frac{6!}{(6-5)!} = \frac{6!}{1!} = \frac{6!}{1} = 6! = 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1 = 720$$

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Instruction

4. Confirm your answer using your graphing calculator.

   Use the steps appropriate to your calculator model.

   On a TI-83/84:

   Step 1: In a blank window, type [6], the value of n.

   Step 2: Press the [MATH] key.

   Step 3: Arrow over to PRB to access the probability menu.

   Step 4: Arrow down to 2: nPr.

   Step 5: Input the value of r, which is [5], and press [ENTER]. Note this counts the number of permutations without repetition.

   On a TI-Nspire:

   Step 1: From the calculate window, press the [menu] key.

   Step 2: Arrow down to 5: Probability. Press [enter].

   Step 3: Arrow down to 2: Permutations. Press [enter].

   Step 4: Where the screen says "nPr(", type [6][,][5], and press [enter]. Note this counts the number of permutations without repetition.

   Joely can answer the fill-in-the-blank questions 720 different ways.

Example 3

Mrs. Taylor is picking 4 students to organize a class field trip. There are 25 students in her class. How many different ways can she pick the 4 students?

1. Determine whether permutations or combinations are being counted.

   Mrs. Taylor is picking 4 students. If she picks students 1, 2, 3, and 4, and then makes a list with students 4, 3, 2, and 1, these two groups are the same, even though the order in which the students are listed is different. Arrangement, or order, of the students doesn't matter, so the number of combinations is being counted.

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2. Find the number of combinations.

Mrs. Taylor has 4 student spots to fill. Consider how many ways each spot can be filled.

Spot	Number of students who can be selected	Explanation
1	25	Any of the 25 students can be selected.
2	24	1 student has been selected, so 25 – 1 = 24 students remain.
3	23	2 students have been selected, so 25 – 2 = 23 students remain.
4	22	3 students have been selected, so 25 – 3 = 22 students remain.

Multiply the number of students who can be picked for each spot to determine the number of different possible permutations of selecting 4 students from a group of 25.

$25 \bullet 24 \bullet 23 \bullet 22 = 303{,}600$

But now groups that have the same students, but selected in a different order, have been counted more than once. Determine how many different ways the 4 spots can be arranged.

Consider the group "1 2 3 4," which consists of students 1, 2, 3, and 4. To find the number of ways this one group can be arranged, find the number of permutations of this combination of 4 students, n, in 4 spots, r, without repetition by substituting these values for n and r into the formula $\frac{n!}{(n-r)!}$.

$$\frac{n!}{(n-r)!} = \frac{4!}{(4-4)!} = \frac{4!}{0!} = \frac{4!}{1} = 4! = 4 \bullet 3 \bullet 2 \bullet 1 = 24$$

(*continued*)

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Instruction

There are 24 different ways each group of 4 students can be arranged. Divide the number of permutations, 303,600, by the number of ways each combination is arranged, 24, to find the total number of combinations.

$$\frac{303{,}600}{24} = 12{,}650$$

Alternately, you can use the formula for combinations without repetition, ${}_nC_r = \frac{n!}{(n-r)!r!}$. There are 25 objects to choose from, n, and the group size is 4, r:

$${}_nC_r = \frac{n!}{(n-r)!r!} = \frac{25!}{(25-4)!4!} = \frac{25!}{21!4!}$$

The fraction can be simplified before evaluating by reducing common factors in the numerator and denominator.

$${}_nC_r = \frac{25!}{21!4!}$$

$${}_nC_r = \frac{25 \bullet 24 \bullet 23 \bullet 22 \bullet \cancel{21!}}{\cancel{21!}\, 4!}$$

$${}_nC_r = \frac{25 \bullet 24 \bullet 23 \bullet 22}{4!}$$

$${}_nC_r = \frac{303{,}600}{24} = 12{,}650$$

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Instruction

3. Confirm your answer using your graphing calculator.

   Use the steps appropriate to your calculator model.

   On a TI-83/84:

   Step 1: In a blank window, type [25], the value of n.

   Step 2: Press the [MATH] key.

   Step 3: Arrow over to PRB to access the probability menu.

   Step 4: Arrow down to 3: nCr. Press [ENTER].

   Step 5: Input the value of r, which is [4], and press [ENTER]. Note this counts the number of combinations without repetition.

   On a TI-Nspire:

   Step 1: From the calculate window, press the [menu] key.

   Step 2: Arrow down to 5: Probability. Press [enter].

   Step 3: Arrow down to 3: Combinations. Press [enter].

   Step 4: Where the screen says "nCr(", type [25][,][4], and press [enter]. Note this counts the number of combinations without repetition.

   Mrs. Taylor can pick 12,650 different groups of students.

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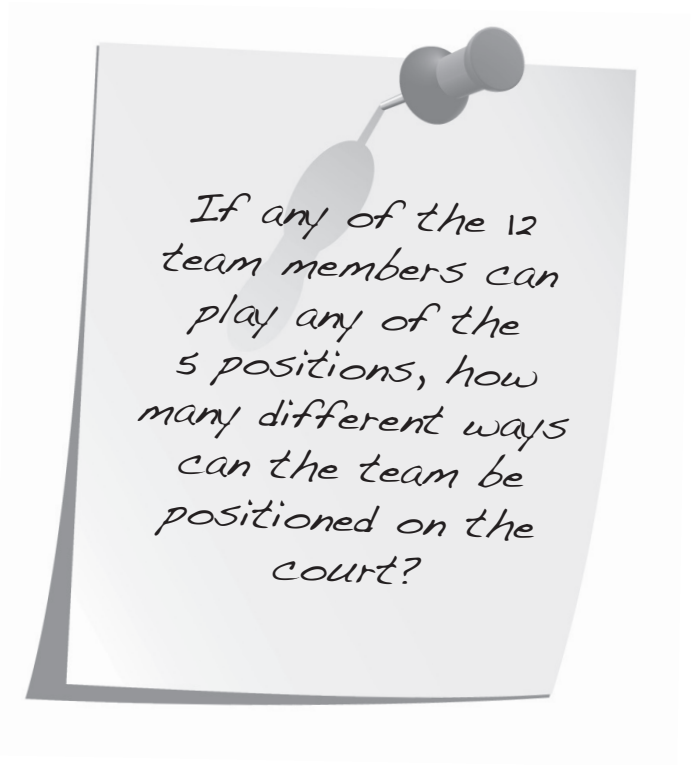
Problem-Based Task 4.3.1: On the Court

There are 5 different positions on a basketball team: point guard, shooting guard, power forward, center, and small forward. In professional basketball, only 12 different players per team are allowed to be ready to play in a game, with 5 allowed on the court at any time. If any of the 12 team members can play any of the 5 positions, how many different ways can the team be positioned on the court?

SMP	
1 ✓	2 ✓
3	4
5	6
7 ✓	8



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Problem-Based Task 4.3.1: On the Court

Coaching

a. Are combinations or permutations being counted?

b. Can there be repetition within the groups?

c. How many positions are there?

d. How many choices are there for each position?

e. How many different ways can the team be positioned?

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Problem-Based Task 4.3.1: On the Court

Coaching Sample Responses

a. Are combinations or permutations being counted?

There are 5 unique positions, so the arrangement of the people in the group matters. Since order matters, permutations are being counted.

b. Can there be repetition within the groups?

Each position is taken by 1 player. Since 1 player cannot be in more than 1 position, there can't be repetition.

c. How many positions are there?

There are 5 different positions.

d. How many choices are there for each position?

Position	Number of players who can be selected	Explanation
1	12	Any of the 12 players can be selected.
2	11	1 player has been selected, so 12 – 1 = 11 players remain.
3	10	2 players have been selected, so 12 – 2 = 10 players remain.
4	9	3 players have been selected, so 12 – 3 = 9 players remain.
5	8	4 players have been selected, so 12 – 4 = 8 players remain.

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e. How many different ways can the team be positioned?

Find the product of the number of players for each position.

12 • 11 • 10 • 9 • 8 = 95,040

A graphing calculator can also be used.

On a TI-83/84:

Step 1: In a blank window, type [12], the value of *n*.

Step 2: Press the [MATH] key.

Step 3: Arrow over to PRB to access the probability menu.

Step 4: Arrow down to 2: nPr. Press [ENTER].

Step 5: Input the value of *r*, which is [5], and press [ENTER]. Note this counts the number of permutations without repetition.

On a TI-Nspire:

Step 1: From the calculate window, press the [menu] key.

Step 2: Arrow down to 5: Probability.

Step 3: Arrow down to 2: Permutations.

Step 4: Where the screen says "nPr(", type [12][,][5], and press enter. Note this counts the number of permutations without repetition.

There are 95,040 ways the team can be positioned.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4.3.1: Combinations and Permutations

Use the following information to answer questions 1–4.

> A local soccer team has 18 members. No more than 11 members of the team can be on the field at one time. Any player can play any position. The positions are as follows: 3 forwards, 3 midfielders, 4 fullbacks, and 1 goalie.

1. If the 4 fullbacks are picked first, how many different ways can the coach pick the fullbacks?

2. The coach decides to pick the 3 forwards first. How many different ways can she pick the 3 forwards?

3. The coach instead decides to pick the forwards based on their field positions: left wing, center striker, and right wing. How many different ways can she pick the 3 forwards, if a player is assigned to a specific position?

4. After picking the 3 forwards, the coach picks the 3 midfielders. How many players does the coach have left to choose from? How many different ways can the coach pick the 3 midfielders?

Use the following information to answer questions 5–7.

> A small pizzeria offers 5 vegetarian toppings and 4 meat toppings from which customers can choose.

5. A customer wants to make a pizza with 3 different vegetarian toppings. How many different ways can the customer top the pizza?

continued

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6. The pizzeria offers a special on a large 2-topping pizza. How many different ways can a 2-topping pizza be ordered, if each topping is different?

7. If a customer chooses 4 different toppings, how many different pizzas can he create?

Use the following information to answer questions 8–10.

> Ashley is planning a day trip. She is deciding which places to visit, and in what order. Her possible destinations include a museum, an amusement park, a water park, a beach, a monument, an art gallery, and a golf course.

8. If Ashley visits 1 place in the morning, 1 in the afternoon, and 1 in the evening, how many different ways can she plan her trip?

9. Ashley decides to plan her trip without deciding when she will visit each place. How many different ways can she pick 3 places to visit?

10. Ashley thinks she can fit 4 places into her trip: 1 in the morning, 1 in the early afternoon, 1 in the late afternoon, and 1 in the evening. How many different ways can she plan her trip?

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Lesson 4.3.2: Probability with Combinatorics

Warm-Up 4.3.2

Felipe surveyed the students in his class to determine their ages. He recorded the results and the gender of each student surveyed in the table below.

Student	Age	Gender
1	14	Male
2	13	Male
3	14	Female
4	14	Male
5	12	Female
6	13	Female
7	14	Female
8	12	Male
9	13	Male
10	13	Female
11	13	Female
12	14	Female

1. If a student is randomly selected, what is the probability that the student will be female?

2. What is the probability that a randomly selected student is a 13-year-old female?

3. What is the probability that a randomly selected male is younger than 14?

4. If 2 students are randomly selected, what is the probability that both students are male? Use the events as follows:

   FM: The first student is male.

   SM: The second student is male.

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Instruction

Lesson 4.3.2: Probability with Combinatorics

Common Core State Standard

S–CP.9 (+) Use permutations and combinations to compute probabilities of compound events and solve problems.★

Warm-Up 4.3.2 Debrief

1. If a student is randomly selected, what is the probability that the student will be female?

   The probability of an event is $\frac{\text{number of ways event can occur}}{\text{total number of events}}$. In this case, the probability being found is the probability of selecting 1 female. This can be written as $P(\text{female})$.

   The total number of events is the total number of students: 12.

   Selecting a female can occur by picking any of the female students, so the number of ways this event can occur is the number of female students: 7.

   $$P(\text{female}) = \frac{\text{number of female students}}{\text{total number of students}} = \frac{7}{12}$$

   The answer can be written as a fraction in simplest form, or as a decimal.

   $$P(\text{female}) = \frac{7}{12} \approx 0.58$$

   The probability that a randomly selected student will be female is approximately 0.58.

2. What is the probability that a randomly selected student is a 13-year-old female?

   Divide the number of 13-year-old females by the total number of students. There are 3 females who are 13 years old.

   $$P(\text{13-year-old female}) = \frac{\text{number of 13-year-old females}}{\text{total number of students}}$$

   $$P(\text{13-year-old female}) = \frac{3}{12} = \frac{1}{4} = 0.25$$

   The probability that a randomly selected student is a 13-year-old female is 0.25.

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Instruction

3. What is the probability that a randomly selected male is younger than 14?

   In this case, a student is being selected from the group of male students. The total number of events is the total number of males, which is 5. The number of ways the event can occur is the number of 12- or 13-year-old males, which is 3. The probability can be written as *P*(12- or 13-year-old male given that the student is male), or $P(\text{12- or 13-year-old male} \mid \text{male})$.

   $$P(\text{12- or 13-year-old male} \mid \text{male}) = \frac{\text{number of 12- and 13-year-old males}}{\text{total number of males}}$$

   $$P(\text{12- or 13-year-old male} \mid \text{male}) = \frac{3}{5} = 0.6$$

   The probability that a randomly selected male is younger than 14 is 0.6.

4. If 2 students are randomly selected, what is the probability that both students are male? Use the events as follows:

   FM: The first student is male.

   SM: The second student is male.

   Since this is a compound event, the probability of these two events can be written as *P*(*FM* and *SM*). The first event is selecting a male. The second event is selecting a second male. The second event is dependent on the first event, because the number of unselected students changes and the number of males changes after the first event.

   The Multiplication Rule for compound events is $P(A \text{ and } B) = P(A) \bullet P(B \mid A)$.

   Substitute *FM*, "the first student is male," for *A* and substitute *SM*, "the second student is male," for *B*.

   $$P(FM \text{ and } SM) = P(FM) \bullet P(SM \mid FM)$$

   Find *P*(*FM*) and $P(SM \mid FM)$.

   P(*FM*) is the number of males divided by the number of students.

   $$P(FM) = \frac{\text{number of males}}{\text{number of students}} = \frac{5}{12}$$

   $P(SM \mid FM)$ is the number of males remaining after selecting a male divided by the number of students remaining after selecting 1 student.

   $$P(SM \mid FM) = \frac{\text{number of males remaining}}{\text{number of students remaining}}$$

   $$P(SM \mid FM) = \frac{5-1}{12-1} = \frac{4}{11}$$

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Instruction

Find the product of the determined probabilities to find the probability that 2 selected students are male.

$$P(FM \text{ and } SM) = P(FM) \bullet P(SM \mid FM) = \frac{5}{12} \bullet \frac{4}{11}$$

$$P(FM \text{ and } SM) = P(FM) \bullet P(SM \mid FM) = \frac{5}{33} \approx 0.15$$

The probability that 2 randomly selected students are male is approximately 0.15.

Connection to the Lesson

- In this lesson, students will use permutations and combinations to calculate probabilities of compound events.
- Students will find probabilities of single and conditional events.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- finding the probability of a single event (7.SP.7a)
- finding the probability of compound events, both independent and dependent (7.SP.8a)
- calculating the number of permutations and combinations (S–CP.9★)
- calculating factorials (S–CP.9★)
- simplifying fractions with factorials in the numerator and denominator (S–CP.9★)

Introduction

You have learned how to determine the number of outcomes in an event by using combinations and permutations. Now we will apply permutations and combinations to single and compound probabilities. Using the formulas for combinations and permutations allows you to determine the number of total outcomes and the number of desired outcomes without having to count.

Key Concepts

- Recall that the probability of an event is $\frac{\text{number of ways event can occur}}{\text{total number of events}}$.
- If a sample space or event includes arrangements or subsets of a group, the number of permutations or combinations can be used to find the size of the sample space or event.
- Remember the following:
  - A permutation is a selection of objects where the order matters and is found either using n^r, if repetitions are allowed, or by using ${}_nP_r = \frac{n!}{(n-r)!}$, where n is the number of objects to select from and r is the number of objects being selected and ordered.
  - A combination is a selection of objects where the order does *not* matter and is found using the formula ${}_nC_r = \frac{n!}{(n-r)!r!}$, where n is the number of objects to select from and r is the number of objects being selected.

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Instruction

Common Errors/Misconceptions

- incorrectly determining the number of ways outcomes of an event can occur
- incorrectly determining the size of a sample space
- misidentifying a compound event as independent when it is dependent, or vice versa
- incorrectly identifying a group as a permutation when it is a combination, or vice versa
- miscalculating permutations and combinations

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Instruction

Guided Practice 4.3.2

Example 1

There are 6 students presenting projects in a history class. The teacher is randomly determining the order in which the students will present. Each student only presents once. Brooke is one of the 6 students. What is the probability that Brooke will be the first student to present?

1. Determine the sample space.

   The probability to determine is the likelihood that Brooke presents first, or $P(BF)$. The sample space is the number of ways the students can be arranged. The number of ways the students can be arranged is the number of permutations of the 6 students. Since each student only presents once, the number of permutations without repetition needs to be calculated.

   The number of items to choose from, n, is 6.

   The number of items to arrange, r, is 6.

   The number of permutations of the 6 items can be found by substituting these values for n and r into the formula ${}_nP_r = \frac{n!}{(n-r)!}$.

   $${}_nP_r = \frac{n!}{(n-r)!} = \frac{6!}{(6-6)!} = \frac{6!}{0!} = \frac{6!}{1} = 6! = 720$$

2. Determine the number of ways the event can occur.

   The number of ways the event can occur is the number of permutations with Brooke presenting first. The remaining 5 students can be in any order. The number of ways the remaining 5 students can be arranged is the number of permutations with Brooke going first.

   $${}_nP_r = \frac{n!}{(n-r)!} = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5!}{1} = 5! = 120$$

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Instruction

3. Find the probability of the event using the determined sample space and number of ways the event can occur.

   The probability of an event is $\frac{\text{number of ways event can occur}}{\text{total number of events}}$.

   Use the determined information to find $P(BF)$, the probability of Brooke presenting first.

$$P(BF) = \frac{\text{number of ways Brooke can present first}}{\text{number of ways all students can be arranged}}$$

$$P(BF) = \frac{120}{720} = \frac{1}{6} \approx 0.17$$

   The probability that Brooke presents first is 0.17.

Example 2

Sean is playing the lottery. He randomly picks any 10 digits (0, 1, 2, 3, 4, 5, 6, 7, 8, or 9) in 5 places. If he gets the first 3 numbers correct, in the correct order, he wins a small prize. What is the probability that Sean gets the first 3 numbers correct and wins a prize?

1. Determine the sample space.

   The sample space is all of the ways Sean can pick his first 3 numbers. The order of the numbers matters, and the 10 digits can be repeated. Find the permutations of the 10 digits, $n = 10$, in the first 3 places, $r = 3$, using the formula n^r.

$$n^r = 10^3 = 1000$$

2. Determine the number of ways the event can occur.

   Let W represent the event of winning a prize. Sean wins a prize if he chooses 3 numbers in the same order as the 3 winning numbers. There is only one arrangement of the winning numbers, so there is only 1 way the event can occur.

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Instruction

3. Find the probability of the event using the determined sample space and number of ways the event can occur.

   The probability of an event is $\frac{\text{number of ways event can occur}}{\text{total number of events}}$.

   Use the determined information to find $P(W)$, the probability of Sean winning a prize.

   $$P(W) = \frac{\text{number of ways first 3 winning numbers are chosen}}{\text{number of ways Sean can pick his first 3 numbers}}$$

   $$P(W) = \frac{1}{1000} = 0.001$$

   There is a 0.001 probability that Sean gets the first 3 numbers correct and wins a prize.

Example 3

Max is selecting 2 cards from a standard deck. He selects 1 card, returns it, and selects a second card. What is the probability that he selects the same card twice?

1. Determine the size of the sample space.

   The sample space is the number of ways the 2 cards can be selected. There are 52 unique cards in a standard deck.

   There are 52 ways the first card can be selected, and 52 ways the second card can be selected.

   Therefore, the total number of ways the 2 cards can be selected is:

   $52 \cdot 52 = 2704$

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Instruction

2. Determine the number of ways the event can occur.

   Let T represent the event of selecting the same card twice. To find the number of ways this can happen, consider each selection.

   The first card can be any card, so there are 52 choices.

   The second card must be the same exact card as the first, so there is only 1 card that can be selected.

   The number of ways 2 identical cards can be selected is the product of the number of ways the first card can be picked (52) and the number of ways the second card can be picked (1).

   $52 \cdot 1 = 52$

3. Find the probability of the event using the determined sample space and the number of ways the event can occur.

   The probability of an event is $\frac{\text{number of ways event can occur}}{\text{total number of events}}$.

   Use the determined information to find $P(T)$, the probability of selecting the same card twice.

$$P(T) = \frac{\text{number of ways 2 identical cards can be selected}}{\text{number of ways 2 cards can be selected}}$$

$$P(T) = \frac{52}{2704} = \frac{1}{52} \approx 0.019$$

   The probability that Max selects the same card twice is approximately 0.019.

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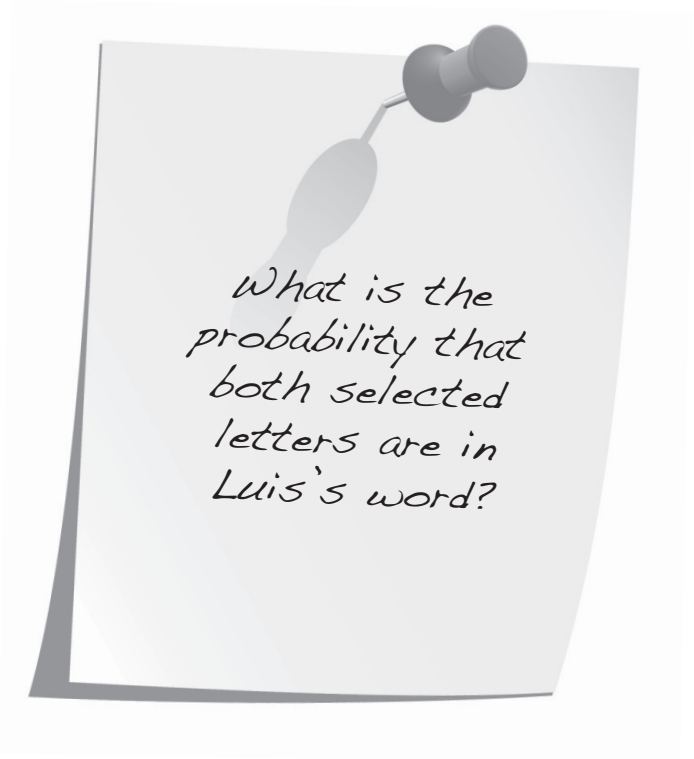
Problem-Based Task 4.3.2: The Apple Game

SMP	
1 ✓	2 ✓
3 ✓	4
5	6
7 ✓	8

Luis and Rose are playing a word game on the whiteboard. Luis has drawn an apple tree with apples on it. He has chosen a secret word and written blanks for each letter of the word. Rose must guess the letters of the word one at a time; if she's correct, Luis will fill in the letter she guessed. If she's incorrect, Luis will erase one of the apples on the tree. Rose keeps guessing letters until she knows what the word is, or until all the apples have been erased.

Luis's secret word is *young*. Rose decides to randomly guess a consonant first, then a vowel. What is the probability that both selected letters are in Luis's word? Let Y be a consonant.



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Problem-Based Task 4.3.2: The Apple Game

Coaching

a. What is the probability being determined?

b. What is the size of the sample space?

c. How many different ways can Rose correctly guess the 2 letters?

d. What is the probability that Rose correctly selects the consonant and the vowel?

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Instruction

Problem-Based Task 4.3.2: The Apple Game

Coaching Sample Responses

a. What is the probability being determined?

The probability being determined is whether Rose's randomly guessed consonant and vowel are in Luis's word. Let *CV* represent the event of correctly guessing a consonant and vowel: $P(CV)$.

b. What is the size of the sample space?

The sample space is all of the ways Rose can guess a consonant and a vowel.

If Y is considered a consonant, there are 21 consonants and 5 vowels in the alphabet.

Rose can guess any of the consonants in her first guess and any of the vowels in her second guess.

The number of combinations is the product of the number of consonants and the number of vowels.

$21 \cdot 5 = 105$

c. How many different ways can Rose correctly guess the 2 letters?

Luis's secret word is *young*, which has 3 consonants (Y, N, and G) and 2 vowels (O and U).

Any of the 3 consonants can be the first guess, and any of the 2 vowels can be the second guess.

The number of ways Rose can correctly guess 2 letters in the word is the product of the number of consonants in the word and the number of vowels in the word.

$3 \cdot 2 = 6$

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d. What is the probability that Rose correctly selects the consonant and the vowel?

The probability of an event is $\frac{\text{number of ways event can occur}}{\text{total number of events}}$.

Use the determined information to find $P(CV)$, the probability of correctly guessing a consonant and a vowel.

Divide the number of ways correct letters can be guessed by the total number of ways the letters can be guessed.

$$P(CV) = \frac{\text{number of ways Rose can guess correctly}}{\text{number of ways a consonant and a vowel can be guessed}}$$

$$P(CV) = \frac{6}{105} = \frac{2}{35} \approx 0.057$$

The probability that both selected letters are in Luis's word is 0.057.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4.3.2: Probability with Combinatorics

The table below lists items in Bryan's closet. Use the table to answer questions 1–4.

Item	Number of each color				
	Black	**Blue**	**White**	**Red**	**Tan**
Pants	1	1	1	1	1
Shirt	2	0	1	1	0

1. Bryan randomly selects 2 items from his closet. What is the probability that he selects 2 pairs of pants?

2. Bryan randomly selects 2 items from his closet. What is the probability that he selects a pair of pants then selects a shirt?

3. Bryan randomly selects 2 items. What is the probability that both selected items are black?

4. Bryan randomly selects a pair of pants, then randomly selects a shirt. What is the probability that both items are white?

continued

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Use the following information to answer questions 5–10.

Kennedy is picking members of a dodgeball team. She chooses 8 people from 12 who have offered to play, from the list below.

Pippa	Antonio	Serena
Harrison	Kai	Christina
Deion	Vida	Jalen
Richard	Akib	Andrea

5. What is the probability that the first person Kennedy picks is Deion?

6. What is the probability that she picks, in any order, the following 8 people: Pippa, Antonio, Harrison, Kai, Deion, Vida, Richard, and Akib?

7. What is the probability that Serena, Christina, and Jalen are among the 8 picked, in any order?

8. What is the probability that Kennedy picks Akib first and Andrea second, if the order of her picks matters?

9. What is the probability that Vida is picked, in any place?

10. What is the probability that the last 4 picks are Deion, Kai, Richard, and Antonio, in that order, if the order of her picks matters?

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Progress Assessment

Circle the letter of the best answer.

1. Larissa is planning a road trip. She has found 12 places that she wants to visit, but only has time to stop at 6 places. She is visiting the places in a specific order. How many different ways can she plan her trip?

   a. 72
   b. 924
   c. 5,544
   d. 665,280

2. A student council has 4 officers: president, vice president, treasurer, and secretary. There are 15 students running for the 4 positions. Each student can only hold 1 position. How many different ways can the positions be assigned?

   a. 1,365
   b. 5,460
   c. 32,760
   d. 50,625

3. Coach Lee is choosing a batting lineup for an upcoming baseball game. There are 14 players on the team and all players are listed on the batting lineup. What is the probability that Liam is first to bat?

   a. 0.0051
   b. 0.071
   c. 0.14
   d. 0.28

4. A bakery provides options for cupcake toppings: cookie crumbs, colored sprinkles, chocolate chips, peanuts, cherries, and gummy bears. A customer picks 2 toppings. What is the probability that both chosen toppings are chocolate chips?

   a. 0.028
   b. 0.056
   c. 0.20
   d. 0.25

5. A math teacher has saved all the problems that she has written for her end-of-year exam over the past 5 years. She has collected a total of 20 questions, 4 from each year. If she picks 4 of these questions for her new exam, without repeating a question, what is the probability that all 4 are from the same year?

   a. 0.0010
   b. 0.00021
   c. 0.00083
   d. 0.0041

continued

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6. A family is trying to decide which movies to watch tonight. There are 6 movies the family wants to watch. If 2 movies are randomly picked, in any order, how many different ways can tonight's movies be picked?

   a. 6
   b. 12
   c. 15
   d. 30

7. A track team has 7 people eligible to run the 100-meter hurdles. A team member can only run the race once, and 4 people are needed to run the race. How many different ways can the 4 people be selected, if order doesn't matter?

   a. 28
   b. 35
   c. 70
   d. 840

8. A cell phone company assigns the last 4 digits of a phone number randomly, in a specific order. The numbers can repeat, and are chosen from 10 digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. What is the probability that the 4 ordered numbers are all the same?

   a. 0.00001
   b. 0.0009
   c. 0.0001
   d. 0.001

9. There are 11 people on a swim team who can swim the 50-meter butterfly stroke, including the team's 4 best butterfly swimmers. The coach needs to choose 4 people to compete in a relay. What is the probability that the 4 chosen swimmers are the team's best swimmers, if the swimmers are chosen randomly?

   a. 0.0030
   b. 0.012
   c. 0.27
   d. 0.73

10. A jar contains 16 jelly beans, each a different flavor. Phil knows that there are 2 flavors in the jar that he doesn't like, but can't tell the jelly beans apart by looking in the jar. If he randomly selects 3 jelly beans, what is the probability that 2 of them are the flavors he doesn't like?

    a. 0.0018
    b. 0.013
    c. 0.018
    d. 0.025

continued

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Assessment

Use the given information to answer the questions that follow.

11. There are 9 students working on the student yearbook. The students are deciding who will work in each of 3 roles: main editor, photo editor, and photographer. A student cannot have more than 1 role.

    a. How many different ways can the 3 roles be randomly assigned?

    b. One of the students is chosen by the others to be the photographer, based on her experience. How many different ways can the remaining roles be randomly assigned?

    c. Chiara is one of the students working on the yearbook, and she isn't the photographer. What is the probability that she will be the photo editor?

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Pre-Assessment

Circle the letter of the best answer.

1. Which described number generator is random?
   a. Organize numbers from least to greatest. Choose the fourth number.
   b. Choose a value from 2 through 12 by rolling two dice and finding the sum.
   c. Select a card from a stack containing these cards: 2, 3, 4, 5, 3, 7, and 8.
   d. Put two of each of the values from 1 through 11 in a hat and select a value.

2. Giselle, Leila, and Mira are playing a game with two dice. If doubles are rolled, Giselle wins. If two even numbers are rolled that are not doubles, then Leila wins. If two odd numbers are rolled that are not doubles, then Mira wins. If an odd and an even number are rolled, then no one wins. Which player will most likely win the game?
   a. Giselle
   b. Leila
   c. Mira
   d. no one

3. A candy store sells three main types of candy: hard candy, chewy candy, and chocolate. The probability that a customer will purchase any type of candy is independent. The table below shows the average cost for a purchase of each type of candy and the given probabilities of each purchase, assuming the customer buys 1 type of candy. Use the table to answer the question that follows.

Candy type	*P*(customer will purchase)	Average purchase cost
Hard candy	0.21	$4.00
Chewy candy	0.35	$5.50
Chocolate	0.44	$8.00

   A customer enters the store. What is the expected value of his purchase, assuming he buys one type of candy?
   a. $2.10
   b. $5.83
   c. $6.29
   d. $17.50

continued

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4. A basketball team is down by 2 points. They need 2 points to tie the game and 3 or more points to win. A field goal is worth 2 points, and a 3-point shot is worth 3. There are two open team members on the court: Amir and Blake. Dwight is deciding to whom to pass the ball. He knows the following probabilities:

Player	*P*(catches the pass)	*P*(makes a field goal)	*P*(makes a 3-point shot)
Amir	0.77	0.57	0.25
Blake	0.85	0.55	0.17

Which actions will most likely earn the team at least 2 points?

a. Dwight passes to Amir, then Amir attempts a field goal.

b. Dwight passes to Amir, then Amir attempts a 3-point shot.

c. Dwight passes to Blake, then Blake attempts a field goal.

d. Dwight passes to Blake, then Blake attempts a 3-point shot.

5. Raoul is applying to different colleges. Based on his grades, he has determined approximate probabilities for being accepted into the schools. He has also estimated his probability of earning a yearly scholarship at each school, if accepted, in the amounts shown. Assume that each school has the same yearly cost.

College	*P*(getting accepted)	*P*(getting a scholarship if accepted)	Scholarship amount
A	0.85	0.95	$2,000
B	0.45	0.60	$7,000
C	0.20	0.05	$8,000
D	0.50	0.40	$6,000

At which school does Raoul have the highest expected value?

a. A

b. B

c. C

d. D

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Lesson 4: Making and Analyzing Decisions

Instruction

Common Core State Standards

SMP
1 ✓ 2
3 ✓ 4 ✓
5 6
7 ✓ 8 ✓

S–MD.6 (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).★

S–MD.7 (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).★

Essential Questions

1. What does it mean if a game is fair?
2. How can you determine if a random number generator is random?
3. How is expected value related to the probability of an event?

WORDS TO KNOW

expected value	an estimate of value that is determined by finding the product of a total value and a probability of a given event
random number generator	a tool to select a number without following a pattern, where the probability of any number in the set being generated is equal

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Instruction

Recommended Resources

- Google Sites. "Lottery Webquest."

  http://www.walch.com/rr/00136

  This site provides a multi-step investigation of fairness and expected value using a lottery drawing as an example. It provides an opportunity to conduct research and find real-time data to include in analysis, and includes supporting videos and an activity to design a lottery.

- Shodor. "Ideas that Lead to Probability."

  http://www.walch.com/rr/00137

  This website provides a lesson, an activity, and vocabulary support to assess the fairness of a game. Supporting and extension lessons are provided throughout the lesson description.

- Stat Trek: Teach yourself statistics. "Random Number Generator."

  http://www.walch.com/rr/00138

  This site allows the user to generate a list of random numbers based on the user's criteria.

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Lesson 4.4.1: Making Decisions

Warm-Up 4.4.1

Dakota is playing a board game with her friends. She rolls a 6-sided die, with sides numbered 1 through 6, to determine the number of spaces she will move.

1. To take the lead in the game, Dakota needs to roll a 6. What is the probability that she rolls a 6?

2. If Dakota rolls any odd number, she can send another player back to Start. What is the probability that she rolls an odd number?

3. If Dakota rolls a 2, she gets to roll again. What is the probability that Dakota rolls a 2, and then rolls an odd number?

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Instruction

Lesson 4.4.1: Making Decisions

Common Core State Standard

S–MD.6 (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).★

Warm-Up 4.4.1 Debrief

1. To take the lead in the game, Dakota needs to roll a 6. What is the probability that she rolls a 6?

   The probability of rolling a 6 is $P(6)=\frac{\text{number of ways a 6 can be rolled}}{\text{number of ways to roll the die}}$. The die has 6 different sides, each with a different number: 1, 2, 3, 4, 5, and 6. Only one of these sides is numbered 6.

   Therefore, there is only 1 way to roll a 6, and there are 6 ways the die can be rolled.

   $$P(6)=\frac{\text{number of ways a 6 can be rolled}}{\text{number of ways to roll the die}}=\frac{1}{6}\approx 0.17$$

2. If Dakota rolls any odd number, she can send another player back to Start. What is the probability that she rolls an odd number?

   The probability of rolling an odd number is $P(\text{odd})=\frac{\text{number of ways an odd number can be rolled}}{\text{number of ways to roll the die}}$.

   The 6 sides of the die are: 1, 2, 3, 4, 5, and 6. Sides 1, 3, and 5 are all odd, so there are 3 ways to roll an odd number.

   $$P(\text{odd})=\frac{\text{number of ways an odd number can be rolled}}{\text{number of ways to roll the die}}$$

   $$P(\text{odd})=\frac{3}{6}=\frac{1}{2}=0.5$$

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Instruction

3. If Dakota rolls a 2, she gets to roll again. What is the probability that Dakota rolls a 2, and then rolls an odd number?

   In this case, Dakota will roll the die twice. Because the result of the second roll will not be changed by the first roll, the two events are independent. The probability of both events happening is the product of the probability of each roll.

   $P(2 \text{ and odd}) = P(2) \bullet P(\text{odd})$

   The probability of rolling a 2 is $P(2) = \frac{\text{number of ways a 2 can be rolled}}{\text{number of ways to roll the die}}$. Only 1 side contains a 2, and there are 6 possible ways to roll the die.

   $$P(2) = \frac{\text{number of ways a 2 can be rolled}}{\text{number of ways to roll the die}} = \frac{1}{6}$$

   As seen in problem 2, the probability of rolling an odd number is as follows:

   $$P(\text{odd}) = \frac{\text{number of ways an odd number can be rolled}}{\text{number of ways to roll the die}}$$

   $$P(\text{odd}) = \frac{3}{6} = \frac{1}{2}$$

   Therefore, the probability of rolling a 2 and then an odd number is

   $$P(2 \text{ and odd}) = P(2) \bullet P(\text{odd}) = \frac{1}{6} \bullet \frac{1}{2} = \frac{1}{12} \approx 0.083.$$

Connection to the Lesson

- In this lesson, students will assess whether given situations are fair.
- Students will need to find probabilities of simple and compound events to assess fairness.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- finding simple probabilities (7.SP.7a)
- finding compound probabilities (7.SP.8a)

Introduction

Probabilities have a wide range of applications, including determining whether a situation is fair or not. A situation is fair if each outcome is equally likely. In this section, you will evaluate whether a given situation is fair based on the probabilities of the outcomes. You will also assess whether a random number generator will truly produce a random result based on probabilities.

Key Concepts

- The likelihood of events occurring can be given as a probability.
- Events that are equally likely have the same probability.
- If event A is more likely than event B, then the probability of event A is higher than the probability of event B.
- If a situation is fair, each outcome is equally likely.
- If a game is fair, each player has the same probability of winning.
- To maintain fairness, random numbers are often used in making selections.
- A **random number generator** is a tool to select a number without following a pattern such that each number in the set has an equal chance of being selected.
- A random number generator selects one number from a set of numbers. The set can be small or large, but there is a smallest and largest value.
- A random number generator does not select numbers using a repeated pattern, and the probability of selecting one number is not greater than the probability of selecting any other number.
- To determine whether a random number generator is selecting numbers randomly, take a look at how the numbers are being selected. If the numbers are being selected in a pattern or if it is more likely that certain numbers will be selected, then the generator is not random.

SRB p. U4-122

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Instruction

- The results from a random number generator do not always reveal whether the generator is random. For example, if the results of a random number generator are 5, 2, 2, 7, 9, it may appear that selecting a 2 is more likely than selecting other numbers. This may or may not be true. Always take a closer look at how the numbers are selected, instead of which numbers are selected.

Common Errors/Misconceptions

- incorrectly calculating probabilities
- using a list of randomly generated numbers to determine if the generator is random instead of examining the generation method

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Instruction

Guided Practice 4.4.1

Example 1

Jose and Cash are playing a game with 2 dice. Each die has 6 sides, numbered 1 through 6. Jose earns a point if either of the dice is even. Cash earns a point if both dice are odd. Is the game fair?

1. Find all the possible outcomes for the game.

   There are 2 dice. The dice can either be even or odd. The different ways the dice can be rolled are:

First die	Second die
Even	Even
Even	Odd
Odd	Even
Odd	Odd

   There are 4 possible outcomes.

2. Determine the probability of each outcome.

   Rolling one die does not influence the roll of the other, so the two rolls are independent. The probability of the two events occurring is the product of the probabilities of each event. The probability that any die is odd is $\frac{\text{number of ways to roll an odd number}}{\text{number of ways the die can be rolled}}$. There are 6 sides on each die, numbered 1, 2, 3, 4, 5, and 6. Of these, 3 are odd numbers: 1, 3, and 5. The probability that a roll is odd is as follows:

$$P(\text{odd}) = \frac{\text{number of ways to roll an odd number}}{\text{number of ways the die can be rolled}}$$

$$= \frac{3}{6} = \frac{1}{2}$$

(*continued*)

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Similarly, there are 3 even numbers on each die: 2, 4, and 6. The probability that a roll is even is as follows:

$$P(\text{even}) = \frac{\text{number of ways to roll an even number}}{\text{number of ways the die can be rolled}}$$

$$= \frac{3}{6} = \frac{1}{2}$$

Use these probabilities to find the probabilities of rolling each of the possible combinations.

First die	Second die	Probability
Even	Even	$\frac{1}{2} \bullet \frac{1}{2} = \frac{1}{4}$
Even	Odd	$\frac{1}{2} \bullet \frac{1}{2} = \frac{1}{4}$
Odd	Even	$\frac{1}{2} \bullet \frac{1}{2} = \frac{1}{4}$
Odd	Odd	$\frac{1}{2} \bullet \frac{1}{2} = \frac{1}{4}$

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3. Find the probability that each player earns a point.

The table can be used to help find the probabilities. Determine who will win each type of roll.

First die	Second die	Probability	Winner
Even	Even	$\frac{1}{4}$	Jose
Even	Odd	$\frac{1}{4}$	Jose
Odd	Even	$\frac{1}{4}$	Jose
Odd	Odd	$\frac{1}{4}$	Cash

The probability that Jose will win is the probability that any of the first three pairs are rolled. He can roll even even, even odd, or odd even. The probability that he will win is: P(even even or even odd or odd even). Since each pair is independent, his probability of winning is $P(\text{both even}) + P(\text{one even and one odd}) + P(\text{one odd and one even}) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} = 0.75$.

The probability that Cash will win is the probability that the last pair, odd odd, is rolled. His probability of winning is $P(\text{both odd}) = \frac{1}{4} = 0.25$.

4. Use the probabilities to determine if the game is fair.

The probability that Jose will earn a point is 0.75, and the probability that Cash will earn a point is 0.25. Because the probability that Jose will earn a point is greater than Cash's probability, it is more likely that Jose will earn a point. Because it is more likely Jose will earn a point, the game is not fair.

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Example 2

Morgan creates a random number generator to help her pick a winning combination of numbers. There are 2 winning numbers. The available numbers range from 1 through 15. To select the first number, she writes each number on a different piece of paper, puts the 15 pieces of paper in a hat, and picks one without looking. To select the second number, Morgan created the following table, so that her second number is based on the first number she draws from the hat.

First number	Second number
1	14
2	15
3	5
4	10
5	7
6	6
7	11
8	12
9	2
10	4
11	3
12	1
13	9
14	13
15	8

Is Morgan's selection of the 2 winning numbers random?

1. Determine if selecting the first number follows a pattern.

   Morgan selects the numbers from a hat, without looking. This does not follow a pattern.

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2. Determine if selecting any given value of the first number is more likely.

   Morgan has written each of the 15 different numbers on a different piece of paper. The probability that any of the 15 numbers are selected is $\frac{\text{number of times a number is in the hat}}{\text{number of numbers in the hat}} = \frac{1}{15}$. Each number has the same probability of being selected, so there are no values that are more likely.

3. Identify whether the generator is random for the first winning number.

   Because Morgan is selecting the first number without following a pattern and the probability of selecting each number is the same, the generator is randomly selecting the first winning number.

4. Determine if selecting the second number follows a pattern.

   The second number is selected based on the value of the first number, using the table below.

First number	Second number
1	14
2	15
3	5
4	10
5	7
6	6
7	11
8	12
9	2
10	4
11	3
12	1
13	9
14	13
15	8

(*continued*)

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For example, every time she picks 10 as the first number, the second winning number will be 4; every time she picks 13 as the first number, the second winning number will be 9, and so on. Matching each first number with only one second number is a pattern. The second number is not random; it is based on the result of the first number.

5. Determine if selecting any given value of the second number is more likely.

   The values in the table show that each number, 1 through 15, is only listed once as a winning number. Because the second number depends on the first, the probability that any of the second numbers will win is equal to the probability that the first number is selected. For example, the probability that 8 is the second number is the probability that 15 is the first number, since 15 and 8 are always chosen together as the first and second winning numbers. The probabilities of selecting each of the first numbers are equal, and each second number is different, so the probabilities of selecting each second number are also equal. None of the second numbers are more likely.

6. Identify whether the generator is random for the second winning number.

   The selection of the second winning number, even though each value is equally likely, follows a set pattern. Therefore, selecting the second winning number is not random.

7. Identify whether the generator is random for selecting a winning pair of numbers.

   Because selecting the second number is not random, Morgan's random number generator is not random for selecting a winning pair of numbers.

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Example 3

Mr. Won is taking his class on a field trip. He wants to find a random way to choose a student to be the class leader on the trip. There are 36 students in his class. He assigns each student a unique number from 1 through 36. He decides to use 2 dice to determine which student is selected. Each die has 6 sides, numbered 1 though 6. The product of the dice will be the number of the student. Is his selection fair?

1. Find all possible outcomes of Mr. Won's number generator.

   Mr. Won is using the product of two dice to determine a student's number. Find the possible products of the two 6-sided dice.

		First die					
		1	**2**	**3**	**4**	**5**	**6**
Second die	**1**	1	2	3	4	5	6
	2	2	4	6	8	10	12
	3	3	6	9	12	15	18
	4	4	8	12	16	20	24
	5	5	10	15	20	25	30
	6	6	12	18	24	30	36

   Organize the results in the table so that the products are from least to greatest. Each product represents a student's assigned number.

Die 1	Die 2	Product	Die 1	Die 2	Product
1	1	1	5	2	10
1	2	2	2	6	12
2	1	2	3	4	12
1	3	3	4	3	12
3	1	3	6	2	12
1	4	4	3	5	15
2	2	4	5	3	15
4	1	4	4	4	16
1	5	5	3	6	18
5	1	5	6	3	18
1	6	6	4	5	20
2	3	6	5	4	20
3	2	6	4	6	24
6	1	6	6	4	24
2	4	8	5	5	25
4	2	8	5	6	30
3	3	9	6	5	30
2	5	10	6	6	36

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2. Determine if selecting the student number follows a pattern.

   Selecting a number depends on the rolling of 2 dice. Rolling dice does not follow a particular pattern, so the number is not selected following a pattern.

3. Determine if any given student number from 1 through 36 is equally likely to be selected.

   The numbers 1 through 36 have all been assigned to different students. There are 36 different ways Mr. Won can roll the 2 dice. Each of these different pairs is equally likely. For example, calculate the probability of rolling 1 and a 1. The probability of rolling a 1 is $P(1) = \frac{\text{number of ways a 1 can be rolled}}{\text{number of ways to roll the die}} = \frac{1}{6}$. Each roll is independent, so the probability of both events is $P(1 \text{ and } 1) = P(1) \bullet P(1) = \frac{1}{6} \bullet \frac{1}{6} = \frac{1}{36}$.

   The probability of each of the roll pairs can be found in the same way.

   The probability of rolling each pair, or each product, is equally likely.

(*continued*)

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However, some of the numbers between 1 and 36 cannot be represented as a product of the 2 dice, and some numbers appear more than once. For example, the product 2 can occur two different ways, and the product 4 can occur three different ways, but there is no product equal to 35. The probability of rolling a product of 2 is P(rolling 1 then 2 or rolling 2 then 1). Each pair of rolls is independent. P(1 then 2 or 2 then 1) = P(1 then 2) + P(2 then 1). The probability of rolling each pair is $\frac{1}{36}$, so the probability of rolling 1 then 2 or rolling 2 then 1 is $P(1 \text{ then } 2) + P(2 \text{ then } 1) = \frac{1}{36} + \frac{1}{36} = \frac{2}{36} = \frac{1}{18}$. However, the probability of rolling a product of 35 is 0, because 35 is not the product of any rolls. Given that two numbers have different probabilities, there are at least two numbers that are not equally likely. Selecting a student number from 1 though 36 is not equally likely.

4. Identify whether the method fairly selects students.

   While the student numbers are selected without following a pattern, the selection of each student number is not equally likely. Therefore, the method does not fairly select students.

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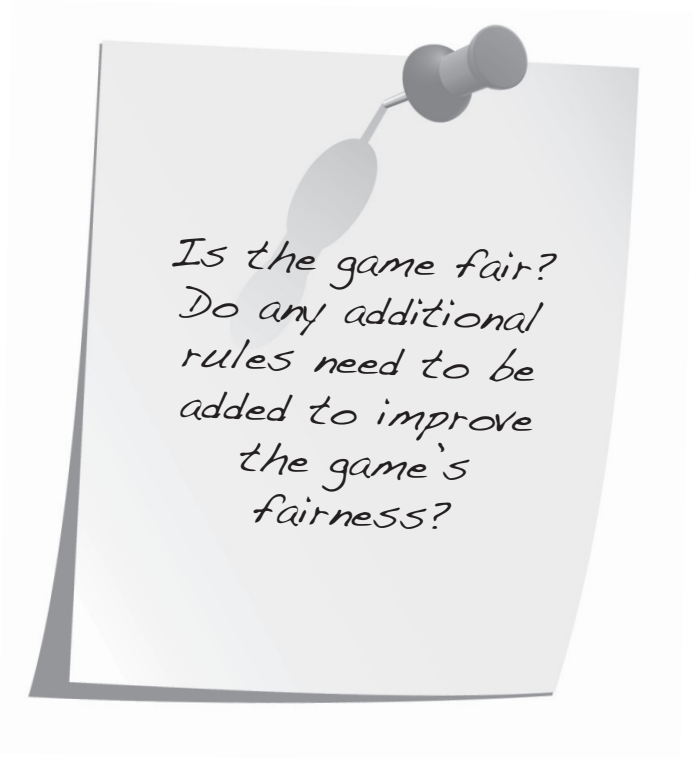
Problem-Based Task 4.4.1: Two of a Kind

SMP	
1 ✓	2
3 ✓	4
5	6
7	8 ✓

Shelby is creating a new card game. There are 10 cards in a stack. There are 5 different numbers, 1 through 5, and there are 2 of each number in the stack. The cards are shuffled. A player selects 2 cards from the stack. If the 2 cards have the same value, the player wins. If not, the cards are returned to the stack and reshuffled. Then, the next player selects 2 cards. The first player to select a pair wins. Is the game fair? Do any additional rules need to be added to improve the game's fairness?



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Problem-Based Task 4.4.1: Two of a Kind

Coaching

a. How do you win the game?

b. What is the probability that the first player wins?

c. If the first player doesn't win, what is the probability that the second player wins?

d. Does the order of the players impact a player's likelihood of winning?

e. How can you improve the game's fairness?

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Instruction

Problem-Based Task 4.4.1: Two of a Kind

Coaching Sample Responses

a. How do you win the game?

You win the game by selecting 2 cards with the same value. For example, if you select a 2 and then another 2, you win the game.

b. What is the probability that the first player wins?

To win, the second card must match the first. The first card can be any value. After selecting the first card, there are 10 – 1 = 9 cards remaining in the stack. Only 1 of these cards will have the same value as the first. The probability that this card is selected is the probability that the first player wins:

$$P(\text{first player wins}) = P(\text{second card has the same value as first})$$

$$P(\text{first player wins}) = \frac{\text{cards in deck with same value as first}}{\text{cards remaining in deck}}$$

$$P(\text{first player wins}) = \frac{1}{9}$$

c. If the first player doesn't win, what is the probability that the second player wins?

If the first player doesn't win, the selected cards are replaced in the stack and reshuffled. The probability that the second player wins is the probability that the second card selected has the same value as the first.

$$P(\text{second player wins}) = P(\text{second card has the same value as first})$$

$$P(\text{second player wins}) = \frac{\text{cards in deck with same value as first}}{\text{cards remaining in deck}}$$

$$P(\text{second player wins}) = \frac{1}{9}$$

d. Does the order of the players impact a player's likelihood of winning?

If the first player wins, then no other players get to play. Similarly, if the second player wins, the third player and any remaining players also don't get a chance to play. Each player has the same likelihood of winning, since the probability that a player will select a pair is the same. However, any players whose turn follows that of a winning player automatically lose.

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e. How can you improve the game's fairness?

The selection of the cards is a fair way to select a winner, since selecting a pair (and therefore, winning) has the same probability for each player. To improve the game's fairness, the order of the players should also be fair. The order should be randomly selected using a random number generator. Ways to randomly select the order include rolling a die, selecting numbers or names from a hat without looking, or selecting shuffled cards with different values.

The game is fair, based on the probability of selecting a winning hand, but could be improved by randomly selecting the order of the players.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4.4.1: Making Decisions

For problems 1–3, determine whether the selection in each scenario is random. Describe why or why not.

1. A music teacher is determining the order in which her students will perform in an upcoming concert. She decides to order the students alphabetically by the last letter of their first name.

2. Jordana makes a spinner with 10 equally sized spaces. She numbers each space 1 through 10. She creates a list of 5 numbers selected using the spinner.

3. Tenzin has 2 dice. One is labeled 1 through 6, and the other is labeled A through F. Each unique pair of a number and a letter is assigned a number from 1 through 36. Tenzin rolls the 2 dice to select a number from 1 through 36.

Read the following scenario and use the information in it to complete problems 4–7.

> Three friends are playing a game. A spinner has 3 equally sized spaces, numbered 1 through 3. If a player lands on a 1, the player's turn is over. If a player lands on a 2, she takes a second spin. If a player lands on a 2 on the second spin, the player's turn is over. If a player lands on a 3, she earns 1 point. Each player will play an equal number of turns.

4. What are the possible results of a player's turn? Use a table to help organize the spins and whether the player earns a point.

5. What is the probability of each possible type of turn?

6. What is the probability that a player earns a point?

7. Is the game fair? Explain.

continued

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Read the following scenario and use the information in it to complete problems 8–10.

> A car dealership is having a contest. The first 10 customers to enter the contest are each given 2 raffle tickets. The remaining 20 customers are each given 1 raffle ticket. There is 1 contest winner, selected by randomly choosing one of the raffle tickets.

8. Spencer is one of the first 10 customers to enter the contest. What is the probability that he will win the raffle?

9. Hope is one of the remaining 20 customers to enter the contest. What is the probability that she will win the raffle?

10. Is the game fair? Explain.

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Lesson 4.4.2: Analyzing Decisions

Warm-Up 4.4.2

A school sold 1,000 raffle tickets to raise money for extracurricular activities. 20 selected tickets will win third-place prizes, 10 selected tickets will win second-place prizes, and 1 selected ticket will win the first-place prize. Once a ticket is selected as a winning ticket, it is thrown away. First the third-place prizes are chosen, then the second-place prizes are chosen, and the first-place prize is chosen last. All tickets are selected randomly.

1. No tickets have been selected from the raffle. Preston bought 1 ticket. What is the probability he wins the first third-place prize?

2. All 20 third-place prizes have been awarded by selecting 20 raffle tickets. Sandeep has not won anything. He has purchased 2 raffle tickets. What is the probability that Sandeep wins the first second-place prize?

3. Marla bought 10 raffle tickets. The 20 third-place prizes have been awarded, and the 10 second-place prizes have been awarded. None of Marla's tickets have been selected. What is the probability that she wins the first-place prize?

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Lesson 4.4.2: Analyzing Decisions

Common Core State Standard

S–MD.7 (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).★

Warm-Up 4.4.2 Debrief

1. No tickets have been selected from the raffle. Preston bought 1 ticket. What is the probability he wins the first third-place prize?

   Only 1 ticket will be awarded the first third-place prize. It will be selected from all the tickets in the raffle, since it is the first ticket selected. Including Preston's ticket, there are 1,000 total tickets that can be selected. The probability that Preston's ticket is selected is:

   $$P(\text{first selected ticket is Preston's}) = \frac{\text{number of Preston's tickets}}{\text{number of tickets that can be selected}}$$

   $$P(\text{first selected ticket is Preston's}) = \frac{1}{1000} = 0.001$$

2. All 20 third-place prizes have been awarded by selecting 20 raffle tickets. Sandeep has not won anything. He has purchased 2 raffle tickets. What is the probability that Sandeep wins the first second-place prize?

   One ticket will be awarded the first second-place prize. 20 tickets have already been removed from the raffle, so the number of tickets remaining is 1000 – 20 = 980. Sandeep has 2 of the 980 tickets. Either of these can be selected as the first second-place prize.

   $$P(\text{first second-place prize is Sandeep's}) = \frac{\text{number of Sandeep's tickets}}{\text{number of tickets that can be selected}}$$

   $$P(\text{first second-place prize is Sandeep's}) = \frac{2}{980} \approx 0.002$$

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3. Marla bought 10 raffle tickets. The 20 third-place prizes have been awarded, and the 10 second-place prizes have been awarded. None of Marla's tickets have been selected. What is the probability that she wins the first-place prize?

   A total of 20 + 10 = 30 tickets have been removed from the raffle, since 30 prizes have been awarded. There are 1000 – 30 = 970 tickets remaining. Marla has all 10 of her tickets remaining in the lottery. Any of these can be selected for her to win the first-place prize.

$$P(\text{first-place prize is Marla's}) = \frac{\text{number of Marla's tickets}}{\text{number of tickets that can be selected}}$$

$$P(\text{first-place prize is Marla's}) = \frac{10}{970} \approx 0.010$$

Connection to the Lesson

- In this lesson, students will use probabilities to analyze situations and make decisions.
- Students will calculate simple and compound probabilities.
- Students will interpret written situations to calculate probabilities.

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Prerequisite Skills

This lesson requires the use of the following skills:

- calculating simple probabilities (7.SP.7a)
- calculating compound probabilities (7.SP.8a)
- determining the size of a sample space (7.SP.1)
- determining the size of an event in a sample space (7.SP.1)

Introduction

Probabilities can be used to analyze situations and to make decisions. Probabilities describe how likely it is that something will happen. These probabilities can be used to calculate estimates and determine which choice is more likely to get the desired outcome.

Key Concepts

- An **expected value** is an estimate of value that is determined by the product of a total value and a probability of a given event. For example, say the probability that a person in an office will drink coffee in the morning is 0.70. If there are 150 people in the office, the expected value of the number of people drinking coffee is found by multiplying the number of people in the office by the probability P of someone in the office drinking coffee in the morning:

  Number of people in the office • P(drink coffee in morning) = 150 • 0.70 = 105

- In other words, on any given day, it is expected that 105 people in this office will drink coffee.
- Probabilities can be compared to determine which event is more likely.
- An event with a higher probability is more likely.
- A probability can be based on an observation, and this observation can aid future decisions.

  For example, a shoe store sells running shoes. One type of running shoe is put on display in the window. In one month, 200 runners bought running shoes, and 80 of those runners purchased the shoe on display in the window. If this trend continues, the probability that the next runner buying shoes at the store will buy the shoes on display is $\frac{80}{200} = \frac{2}{5} = 0.4$.

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- When applying probabilities in decision-making, keep in mind that other components or factors may influence a decision. For example, if a store is analyzing the probability that a customer will purchase a shoe that is on display, other factors may influence why a customer will purchase that shoe, such as price, performance, style, etc. If any of these other factors change, the probability that the customer will buy the shoe on display may also change.

- The probability of an independent event is not influenced by previous results. For example, there are two sides of a coin, the head and the tail. The probability that heads is shown as the result of a coin toss is $P(\text{heads}) = \frac{1}{2} = 0.5$. Now, suppose Anna has tossed a coin 6 times, and each time the result has been tails. When Anna tosses the coin a seventh time, the probability that the result is heads is still $P(\text{heads}) = 0.5$, even though her previous results were tails.

Common Errors/Misconceptions

- incorrectly calculating probabilities when objects are removed from a sample space
- confusing calculations of "and" and "or" in compound events
- incorrectly determining an expected value

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Guided Practice 4.4.2

Example 1

A bike store offers a two-year warranty on new bikes. The warranty covers two parts, the bike frame and the gear shifter. If either of these parts breaks, the store will replace or repair it at no cost to the customer.

Repair type	Average cost
Bike frame	$225
Gear shifter	$110

Madeline buys a new bike. She does some research, and determines that the probability that her bike frame will break within two years is represented by the equation $P(\text{bike frame breaks}) = 0.15$. The probability that the gear shifter will break within two years is represented by the equation $P(\text{gear shifter breaks}) = 0.30$. The two-year warranty costs $75. Should Madeline buy the warranty?

1. Find the expected value of repairing each bike part.

   The expected value is the product of the repair costs and the probability of each repair.

   Set up an equation to determine the expected value of repairing the bike frame.

   Expected value = bike frame repair cost • P(bike frame breaks)

   Expected value = $225 • 0.15 = $33.75

   The expected value of repairing the bike frame is $33.75.

   Now set up an equation to determine the expected value of repairing the gear shifter.

   Expected value = gear shifter repair cost • P(gear shifter breaks)

   Expected value = $110 • 0.30 = $33.00

   The expected value of repairing the gear shifter is $33.00.

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2. Find the total expected value of the two repair costs.

   Sum the expected values of the two types of repairs.

   Expected bike frame costs + expected gear shifter costs = $33.75 + $33.00 = $66.75

3. Compare the expected repair costs to the cost of the warranty and determine whether or not Madeline should get the warranty.

   The warranty costs $75. The expected cost of the two repairs is $66.75. Given that the expected repair cost is less than the warranty, Madeline shouldn't buy the warranty.

Example 2

After a football team scores a touchdown, the team has two options. The team can try to kick an extra point through the goal posts from ten yards, or the team can try a two-point conversion. The two-point conversion requires the team to get the ball back into the end zone on one try from two yards away.

The Knights are down by one point. There is one minute left in the game. If the Knights kick the extra point, they will need to score again to win the game. If the Knights get the two-point conversion, they will need to keep the one-point lead until the game ends. Use the following probabilities to determine whether the Knights should attempt the kick or attempt the two-point conversion. Assume the probabilities of the events are independent. Which choice will give the Knights the higher probability of winning?

Activity	Success?	Probability
Kick a point	Yes	0.98
Kick a point	No	0.02
Two-point conversion	Yes	0.38
Two-point conversion	No	0.62
Keep a one-point lead with one minute left in the game	Yes	0.81
Take back the ball and score with one minute left in the game	Yes	0.16

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1. Describe the courses of events.

   Kicking the ball for the extra point or attempting the two-point conversion will lead to different game plans to try to win the game.

Action	Success?	Next action to win
Kick a point	Yes	Score
Kick a point	No	Score
Two-point conversion	Yes	Keep lead
Two-point conversion	No	Score

2. Find the probability of each path.

   Use the table with the different paths and the table of probabilities to find the probabilities of the paths.

Action	Success?	Probability	Next action to win	Probability
Kick a point	Yes	0.98	Score	0.16
Kick a point	No	0.02	Score	0.16
Two-point conversion	Yes	0.38	Keep lead	0.81
Two-point conversion	No	0.62	Score	0.16

   There are four paths that would lead to winning. The first two start with kicking an extra point, and the second two start with attempting a two-point conversion. Each path contains two independent events. Both events must occur. Find the probability of each path by finding the compound probability of the two events.

   P(successfully kick a point and score again) = P(successfully kick a point) • P(score again) = 0.98 • 0.16 = 0.1568

   P(unsuccessfully kick a point and score again) = P(unsuccessfully kick a point) • P(score again) = 0.02 • 0.16 = 0.0032

   P(successful two-point conversion and keep lead) = P(successful two-point conversion) • P(keep lead) = 0.38 • 0.81 = 0.3078

   P(unsuccessful two-point conversion and score again) = P(unsuccessful two-point conversion) • P(score again) = 0.62 • 0.16 = 0.0992

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3. Find the probability of winning after choosing to either kick or attempt the two-point conversion.

   If the team kicks, there are two ways they can win. Either the team scores an extra point by kicking and then scores again, or the attempt to kick for an extra point fails but the team scores afterward. The total probability of winning if the team attempts a kick is found by adding the probability of making the kick to the probability that the team misses the kick but scores again:

   P(team wins if kick attempted) =

   P(team wins if kick made or team wins if kick missed) =

   P(team wins if kick made) + P(team wins if kick missed) =

   0.1568 + 0.0032 = 0.16

   If the team attempts the two-point conversion, there are two ways they can win. Either the attempt is made and the team keeps the lead, or the attempt is missed and the team scores again. The total probability of winning if the team attempts the two-point conversion is found by adding the probability that the conversion is successful to the probability that the conversion is missed but the team scores again:

   P(team wins if two-point conversion attempted) =

   P(team wins if two-point conversion is made or team wins if two-point conversion is missed) =

   P(team wins if two-point conversion is made) + P(team wins if two-point conversion is missed) =

   0.3078 + 0.0992 = 0.407

4. Help the team decide whether to kick or attempt the two-point conversion.

   Summarize the results of your findings:

   P(team wins if kick attempted) = 0.16

   P(team wins if two-point conversion attempted) = 0.407

   The team has a higher probability of winning if the two-point conversion is attempted, so the team should try for the two points.

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Example 3

A restaurant has 5 tables that can be reserved. The restaurant manager knows that some people will make a reservation and then won't show up for dinner. The probability that a party will show up for dinner, if they have a reservation, is 0.8. The probability that a party won't show up is 0.2. The probability of each party showing up is independent. Even though the restaurant only has 5 tables to reserve, they usually accept at least 6 reservations because they know there may be some cancellations.

If 6 reservations are made, what is the probability that all parties will show up? Can the restaurant take 7 reservations without worrying about overbooking? The restaurant manager doesn't want a probability greater than 0.5 that more than 5 parties will show up.

1. Find the probability that all the parties will show up at the restaurant.

   The restaurant manager accepted 6 reservations. The probability that a party will show up is 0.8. It is given that the probability of each party showing up is independent. Find the compound probability that all 6 parties show up.

   $P(\text{all 6 parties show up}) =$

   $P(\text{party 1 shows}) \bullet P(\text{party 2 shows}) \bullet P(\text{party 3 shows}) \bullet P(\text{party 4 shows}) \bullet P(\text{party 5 shows}) \bullet P(\text{party 6 shows}) =$

   $0.8 \bullet 0.8 \bullet 0.8 \bullet 0.8 \bullet 0.8 \bullet 0.8 \approx 0.262$

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2. Find the probability that more than 5 parties show up if 7 reservations are made.

   If more than 5 parties show up, then at least 1 party won't be able to sit. Make a table showing the different ways that more than 5 parties can show up. In the table that follows, check marks indicate parties that show up.

Possible compound events	Which parties show up?						
	Party 1	Party 2	Party 3	Party 4	Party 5	Party 6	Party 7
All parties show up	✔	✔	✔	✔	✔	✔	✔
Parties 1–6 show up	✔	✔	✔	✔	✔	✔	✘
Parties 1–5 and 7 show up	✔	✔	✔	✔	✔	✘	✔
Parties 1–4 and 6–7 show up	✔	✔	✔	✔	✘	✔	✔
Parties 1–3 and 5–7 show up	✔	✔	✔	✘	✔	✔	✔
Parties 1–2 and 4–7 show up	✔	✔	✘	✔	✔	✔	✔
Parties 1 and 3–7 show up	✔	✘	✔	✔	✔	✔	✔
Parties 2–7 show up	✘	✔	✔	✔	✔	✔	✔

(*continued*)

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Instruction

Replace each check mark and X with the probability of each event. The probability of a party showing up is 0.8, and the probability of a party not showing up is 0.2.

Possible compound events	Individual probabilities						
	Party 1	Party 2	Party 3	Party 4	Party 5	Party 6	Party 7
All parties show up	0.8	0.8	0.8	0.8	0.8	0.8	0.8
Parties 1–6 show up	0.8	0.8	0.8	0.8	0.8	0.8	0.2
Parties 1–5 and 7 show up	0.8	0.8	0.8	0.8	0.8	0.2	0.8
Parties 1–4 and 6–7 show up	0.8	0.8	0.8	0.8	0.2	0.8	0.8
Parties 1–3 and 5–7 show up	0.8	0.8	0.8	0.2	0.8	0.8	0.8
Parties 1–2 and 4–7 show up	0.8	0.8	0.2	0.8	0.8	0.8	0.8
Parties 1 and 3–7 show up	0.8	0.2	0.8	0.8	0.8	0.8	0.8
Parties 2–7 show up	0.2	0.8	0.8	0.8	0.8	0.8	0.8

(*continued*)

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The product of the probability of all the events is the probability that all the events occur. Therefore, multiply across each row to determine the probability of each possible compound event's occurrence. Round to the nearest thousandth.

Possible compound events	Probability
All parties show up	0.210
Parties 1–6 show up	0.052
Parties 1–5 and 7 show up	0.052
Parties 1–4 and 6–7 show up	0.052
Parties 1–3 and 5–7 show up	0.052
Parties 1–2 and 4–7 show up	0.052
Parties 1 and 3–7 show up	0.052
Parties 2–7 show up	0.052

Since each scenario depicts a case where more than 5 parties show up, the sum of the probabilities of the different scenarios is the probability that more than 5 parties will show up.

$P(\text{more than 5 parties show up}) = 0.210 + 0.052 + 0.052 + 0.052 + 0.052 + 0.052 + 0.052 + 0.052 = 0.574$

3. Determine whether the restaurant can take 7 reservations.

   If 7 reservations are taken, the probability that more than 5 parties show up is about 0.574. The restaurant manager doesn't want greater than a 0.5 probability that more than 5 parties will show up, so the restaurant should not take 7 reservations.

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Problem-Based Task 4.4.2: Changing Channels

SMP
1 ✓ 2
3 ✓ 4 ✓
5 6
7 ✓ 8 ✓

A radio station conducted research to learn more about its listeners. The research shows that the song played just before a commercial break has an impact on whether or not listeners change the radio station. The station manager wants to keep as many listeners tuned during the commercials, to encourage advertisers to buy commercials.

The station manager is trying to determine which type of song the deejay should play just before a commercial break. The station plays pop, dance, and contemporary music. The probability that a listener will change the channel during any type of song is in the table below.

Song type	Probability of changing the channel	Probability of still listening
Pop	0.21	0.79
Dance	0.18	0.82
Contemporary	0.28	0.72

The probability that a listener will change the channel during the beginning of the commercial break depends on the song just prior to the commercial, as shown below.

Song type prior to commercial	Probability of changing the channel during the commercial break	Probability of still listening
Pop	0.55	0.45
Dance	0.60	0.40
Contemporary	0.32	0.68

Which type of song should the deejay play just prior to the commercial, to keep the most listeners during the commercial break? Assume that the probabilities take into consideration any new listeners.

Which type of song should the deejay play just prior to the commercial, to keep the most listeners during the commercial break?

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Problem-Based Task 4.4.2: Changing Channels

Coaching

a. When could a listener change the channel?

b. Create a table showing all the paths a listener could take during the song just prior to the commercial and during the commercial.

c. What is the probability that a listener will still be tuned to the radio station after each chosen path?

d. Which type of song should the deejay play prior to a commercial in order to keep as many listeners as possible during the commercial break?

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Problem-Based Task 4.4.2: Changing Channels

Coaching Sample Responses

a. When could a listener change the channel?

A listener could change the channel during a song or during a commercial break.

b. Create a table showing all the paths a listener could take during the song just prior to the commercial and during the commercial.

There are three different song types. The listener could stop listening during the song. If the listener is still tuned in after the song, the listener could stop listening during the commercial.

Song	Still listening during song?	Still listening during commercial?
Pop	Y	Y
Pop	Y	N
Pop	N	N
Dance	Y	Y
Dance	Y	N
Dance	N	N
Contemporary	Y	Y
Contemporary	Y	N
Contemporary	N	N

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c. What is the probability that a listener will still be tuned to the radio station after each chosen path?

For each path, write the probability at each point that a listener will either still be tuned in or not. Recall that the probability that a listener will change the channel during the commercial depends on the prior song choice. If a listener is not listening after the song, there is a probability of 0 that the listener will still be listening during the commercial.

Song	Still listening during song?	Probability	Still listening during commercial?	Probability
Pop	Y	0.79	Y	0.45
Pop	Y	0.79	N	0.55
Pop	N	0.21	N	0
Dance	Y	0.82	Y	0.40
Dance	Y	0.82	N	0.60
Dance	N	0.18	N	0
Contemporary	Y	0.72	Y	0.68
Contemporary	Y	0.72	N	0.32
Contemporary	N	0.28	N	0

There is only one path for each song type where the listener is still listening during the commercial. Find the product of the probability that the listener is still listening after the song and during the commercial.

P(still listening during the pop song and during the commercial) = 0.79 • 0.45 = 0.356

P(still listening during the dance song and during the commercial) = 0.82 • 0.40 = 0.328

P(still listening during the contemporary song and during the commercial) = 0.72 • 0.68 = 0.490

d. Which type of song should the deejay play prior to a commercial in order to keep as many listeners as possible during the commercial break?

The probability that a listener is still listening is greatest if the song played prior to the commercial is a contemporary song. Therefore, the deejay should play a contemporary song just prior to a commercial.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 4.4.2: Analyzing Decisions

Read the following scenario and use the given information to complete problems 1–4.

A clothing store owner uses the estimated number of customers at different times to estimate clothing sales and to determine the number of salespeople to schedule throughout the day. The estimated customer counts are in the table below.

Time span	Number of customers
9:00 A.M. to 11:00 A.M.	15
11:00 A.M. to 1:00 P.M.	25
1:00 P.M. to 3:00 P.M.	10
3:00 P.M. to 5:00 P.M.	20

The probability that a salesperson scheduled to work will come to work is 0.9. The probability that the salesperson will not come to work is 0.1. The probability that a customer will make a purchase is 0.4. The probability that a customer will not make a purchase is 0.6.

1. What is the estimated number of customers making a purchase between 11:00 A.M. and 1:00 P.M.?

2. The manager prefers that 5 people are working when the store opens at 9:00 A.M., so she schedules 5 people to work at 9:00 A.M. Can she be confident that 5 people will be working when the store opens? Use probability to describe your response.

3. The manager agrees that as long as 4 people are working at 9:00 A.M., the store will be ready to open. To be confident that at least 4 people will come to work, she would like a probability greater than 0.8 that at least 4 salespeople will be working the store at 9:00 A.M. What is the minimum number of salespeople that she should schedule to work, and why?

4. The manager finds that if the salesperson-to-customer ratio, $\frac{\text{number of salespeople}}{\text{number of customers}}$, is greater than 0.4, then the probability that each customer will make a purchase increases. The manager wants greater than a 0.8 probability that the salesperson-to-customer ratio is greater than 0.3 between 1:00 P.M. and 3:00 P.M., but wants to schedule the minimum number of salespeople. She schedules 5 people to work between 1:00 P.M. and 3:00 P.M. Will this meet her criteria? If not, recommend how many people she should schedule and explain why.

continued

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Read the following scenario and use the given information to complete problems 5–8.

Stores can use probabilities to estimate if online advertising is helping their businesses. Felix, the owner of a store called Hot Gadgets, is comparing the results of running Hot Gadgets advertisements on three different websites. Felix first determines the probability that a customer will click on the advertisement. He then determines the probability that a customer who clicked on the advertisement will make a purchase. Felix pays the websites for each click on a Hot Gadgets ad, regardless if the customer makes a purchase or not.

Website	Average number of different daily users	*P*(click on ad)	Pay per click	*P*(make a purchase after clicking on ad)
1	1,000	0.10	\$0.02	0.40
2	2,400	0.15	\$0.05	0.35
3	3,000	0.08	\$0.04	0.60

5. On which website do the highest number of customers click on Hot Gadgets' ad?

6. Explain to Felix which website generates the highest number of purchases. Note that *P*(make a purchase after clicking on ad) is the probability that a customer who is already on Hot Gadgets' site will make a purchase.

7. Which website costs Hot Gadgets the most each day?

8. Felix wants to reduce his advertising costs. Which website would you recommend is the best site for Hot Gadgets to advertise on? Keep in mind clicks, costs, and number of purchases.

continued

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Read the following scenario and use the given information to complete problems 9 and 10.

A car dealership offers a warranty on used cars purchased at the dealership. One five-year warranty completely covers three major components: air conditioning, power sunroof, and transmission. The warranty costs $1,000, and if any of the covered components require repair, the car owner pays nothing in repair costs. Karen researches the costs of these repairs for the car she would like to purchase, and finds the following:

Repair type	Average cost
Air conditioning	$1,200
Power sunroof	$600
Transmission	$3,000

She also finds information about the probability that each component will require a repair within five years, according to different consumer review websites. The probability that the air conditioning will need repairing is 0.64, the probability that the power sunroof will require repairing is 0.15, and the probability that the transmission will require repairing is 0.20.

9. Should Karen purchase the warranty? Explain.

10. The dealership offers a second type of warranty. This warranty is for two years and completely covers the same components, but costs $500. Karen finds the following probabilities for repair within two years: P(air conditioning repair) = 0.45, P(power sunroof repair) = 0.04, and P(transmission repair) = 0.10. Karen wants to purchase one of the two warranties. Which should she purchase? Explain.

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Progress Assessment

Circle the letter of the best answer.

1. Kevin draws a spinner with 6 equal spaces. He labels the spaces 1, 2, 3, 4, 2, and 6. He then spins the spinner to select a number. Which choice below correctly describes Kevin's number generator?

   a. The number generator is not random because the selection follows a pattern.

   b. The number generator is not random because the selection of each number is not equally likely.

   c. The number generator is not random because only one number is selected at a time.

   d. The number generator is random.

2. Yacoub and Zaina are playing a card game. There are 18 cards randomly shuffled in a stack, numbered 2 through 10. There are two cards of each value. Yacoub takes 1 card, replaces it, reshuffles the cards, then takes a second card. If his 2 cards have the same value, he earns a point. He then replaces his two cards. Zaina takes 1 card, keeps it, and selects a second card. If her 2 cards have the same value, she earns a point. Which statement is true about the game?

   a. It is more likely that Yacoub will earn a point than Zaina.

   b. It is more likely that Zaina will earn a point than Yacoub.

   c. It is equally likely that either Yacoub or Zaina earns a point.

   d. Only Yacoub can earn points, and Zaina cannot.

3. A clothing store sells three types of clothing: shirts, pants, and jackets. The probability that any customer will buy a shirt is 0.25, the probability that any customer will buy a pair of pants is 0.13, and the probability that any customer will buy a jacket is 0.09. The average price of a shirt is $30, the average price of a pair of pants is $65, and the average price of a jacket is $80. What is the expected value of a purchase made by any customer who walks in the store?

   a. $7.50

   b. $23.15

   c. $30.00

   d. $175.00

continued

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4. Jennah has an option to take four different routes to work. On each route, she can encounter traffic. If she does, the time to take each route changes.

Route	*P*(traffic)	Time with traffic, in minutes	*P*(no traffic)	Time with no traffic, in minutes
1	0.25	48	0.75	22
2	0.40	50	0.60	15
3	0.33	44	0.67	24
4	0.15	60	0.85	18

Which route should Jennah take if she hopes to spend the least amount of time commuting?

a. 1
b. 2
c. 3
d. 4

5. The manager at a car factory is ordering parts from a parts distributor. The manager is placing an order for one type of part. The manager knows that the probability that this part will be defective is 0.05. The manager wants to have a probability greater than 0.9 that at least 10 of the parts are not defective. She wants to place the minimum order required to meet this probability. How many parts should she order?

a. 10
b. 11
c. 12
d. 13

6. Which described number generator is random?

a. A set of numbers from 1 through 18 is listed from greatest to smallest. The second and fourth numbers are selected.
b. Three dice are rolled. A number from 1 through 18 is selected by finding the sum of the three dice.
c. Two spinners each have four spaces. The numbers 1 through 4 are on each spinner. Both spinners are spun, and the product is the selected number.
d. A stack of cards contains four each of the values 2 through 10. The cards are shuffled and a card is selected.

continued

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7. Rishi and Dixon are playing a dice game. Three dice are rolled. If any of the three dice are even, then Dixon earns a point. If none of the three dice are even, then Rishi earns a point. Which statement is true about the game?

   a. It is more likely that Rishi will earn a point than Dixon.

   b. It is more likely that Dixon will earn a point than Rishi.

   c. The game is fair.

   d. It is equally like that either Rishi or Dixon will earn a point.

8. Wesley is going to use an online sales website to sell a watch. He is deciding between four sites. The sites each tell him the selling price for his watch, and the probability that he will be able to sell the watch on that site.

Website	*P*(selling watch)	Selling price
A	0.48	$85
B	0.20	$120
C	0.63	$50
D	0.54	$72

   Which website has the highest expected value for Wesley's watch?

   a. A

   b. B

   c. C

   d. D

continued

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9. A medical treatment research trial analyzes how a treatment improves a patient's health. One group is given the new treatment, and another group is given a placebo, which has no medical effect on the body. The members in each group do not know if they are being given the new treatment or a placebo. A previous research trial on the treatment produced the following results.

Treatment groups	*P*(patient sees improvement)	*P*(patient sees no improvement)
New treatment	0.72	0.28
Placebo	0.15	0.85

Robert is being randomly assigned to one of the two treatment groups. What is the probability that he will see improvement during the trial?

a. 0.36
b. 0.425
c. 0.435
d. 0.565

10. A baker decides how many loaves of bread to make each morning. She loses money on any loaves of bread that aren't sold. The probability that a loaf will sell is based on how many loaves she bakes. She sells each loaf for $3, and loses $0.50 for each loaf not sold.

Number of loaves baked	*P*(loaf sells)	Sale price	*P*(loaf doesn't sell)	Money lost per unsold loaf
100	0.95	$3	0.05	$0.50
150	0.90	$3	0.10	$0.50
200	0.88	$3	0.12	$0.50
250	0.80	$3	0.20	$0.50

She wants to earn as much money as possible. The total money she earns is the expected value of loaves sold minus expected loss of loaves unsold. Among the choices in the table, how many loaves should she bake?

a. 100
b. 150
c. 200
d. 250

continued

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Assessment

Read the following scenario and use the given information to answer the questions.

11. A private pilot has capacity for 6 passengers in his airplane. He charges $300 a person for a local one-way flight. If the ticket is cancelled 24 hours before the scheduled plane departure, the pilot refunds the ticket price. He will try to resell the cancelled seat at a higher cost of $500. If the ticket isn't cancelled 24 hours before departure, the pilot keeps the $300, and doesn't resell the seat.

    P(no cancellation 24 hours prior) = 0.80

    P(cancellation 24 hours prior) = 0.20

    P(resell cancelled seat) = 0.60

    a. What is the probability that all 6 of the pilot's passengers will cancel 24 hours before departure?

    b. What is the expected value of reselling all 6 seats 24 hours prior to departure?

    c. What is the expected value of each airplane ticket, taking into consideration no cancellation, cancellation, and reselling the ticket?

    d. If the pilot overbooks a flight and sells 7 tickets at $300, what is the probability that none of the 7 people cancel 24 hours prior to the flight?

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Unit 4 Assessment

Circle the letter of the best answer.

1. Some students were asked what types of jobs they have held. The table below shows the results of the survey.

Student	Swim club	Lawn care	Fast food	Retail	Babysitting
Alana		✔			
Myron	✔			✔	
Djimon			✔		✔
Leon		✔	✔	✔	
Charo			✔		
Benaiah				✔	
Evita		✔		✔	
Terrell	✔				✔

A student is chosen from the group at random. Consider the following events.

L: The student has worked in lawn care.

R: The student has worked in retail.

How many outcomes are in the event $\overline{L \cap R}$?

a. 2
b. 3
c. 5
d. 6

2. How many ways can you get an even product by rolling two number cubes?

a. 3
b. 9
c. 18
d. 27

3. Abdu, Reilly, Chandra, and Delroy are on a Student Council committee. For the January meeting, they put their names in a hat and draw one name at random to decide who will take notes. They do the same thing for the February meeting. What is the probability that Abdu will be chosen at least once or Reilly will be chosen at least once?

a. $\frac{1}{2}$
b. $\frac{3}{4}$
c. $\frac{7}{8}$
d. $\frac{15}{16}$

continued

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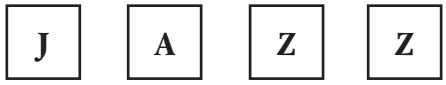
4. Meadow Ridge High School has 820 students. There are 88 students in vocal music, 142 students in instrumental music, and 190 students in vocal or instrumental music. Which option shows the approximate probability that a randomly chosen student at Meadow Ridge High School is in both vocal and instrumental music?

   a. 4.9%
   b. 6.6%
   c. 16.6%
   d. 29.8%

5. Mateo is playing a board game that involves rolling two number cubes. He wants to roll doubles or a sum of 6. Suppose Mateo rolls the pair of number cubes 100 times. Which option shows the best prediction of the number of times Mateo will get doubles or a sum of 6?

   a. 3
   b. 17
   c. 28
   d. 44

6. Jocelyn has a bag that contains the following tiles.

   Jocelyn takes a tile from the bag without looking. Then she puts the tile back in the bag. Next, she takes a tile from the bag again without looking. Consider the following events.

   ZF: She takes Z first.

   ZS: She takes Z second.

   Which statement is true?

   a. *ZF* and *ZS* are dependent, and $P(ZS|ZF)$ is $\frac{1}{3}$.
   b. *ZF* and *ZS* are dependent, and $P(ZS|ZF)$ is $\frac{1}{2}$.
   c. *ZF* and *ZS* are independent, and $P(ZS|ZF)$ is $\frac{1}{3}$.
   d. *ZF* and *ZS* are independent, and $P(ZS|ZF)$ is $\frac{1}{2}$.

continued

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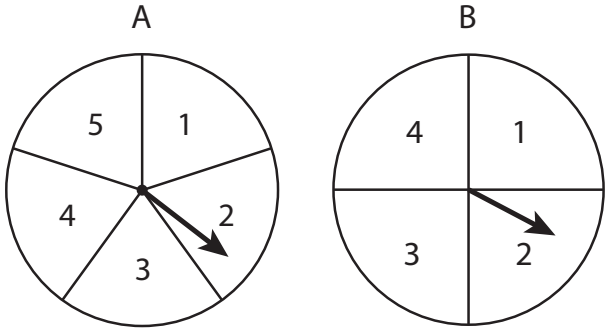
7. Lisa and Kia are in a basketball game. Prior to this game, Lisa has made 6 free throws out of 20 attempts, and Kia has made 10 free throws out of 16 attempts. Lisa is at the free-throw line, ready to shoot. Kia will attempt a free throw after Lisa attempts hers. Based on their records, which option shows the best approximation of the probability that at least one of them will make the free throw?

   a. 46%
   b. 63%
   c. 74%
   d. 93%

8. Remy spins spinner A and then spins spinner B. What is the probability that his first spin is odd if the product of his spins is even?



   a. 0
   b. $\frac{3}{7}$
   c. $\frac{1}{2}$
   d. $\frac{3}{5}$

9. Two cards are dealt from a standard 52-card deck, without replacement. What is the probability that both cards are clubs?

   a. 0.053
   b. 0.059
   c. 0.063
   d. 0.064

continued

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10. Jamil is a political science major. As part of an assignment, he collected data about how the citizens in three towns voted in an election for a county sheriff. The table below shows the data.

Town	Candidate		Total
	Blake	Valdez	
Greenville	645	890	1,535
Clinton	520	1,610	2,130
Springdale	950	1,242	2,192
Total	2,115	3,742	5,857

The following probabilities apply to a randomly chosen voter in the sample. Which probability is the greatest?

a. the probability that the voter lives in Springdale

b. the probability that the voter voted for Blake

c. the probability that the voter lives in Springdale, given that she or he voted for Blake

d. the probability that the voter voted for Blake, given that she or he lives in Springdale

11. A school librarian surveyed 300 students. He listed three novels and three movies, and asked the students to choose one from each category as their favorite. The table below shows the survey results.

Favorite novel	Favorite movie			Total
	Nine Dogs	***Henry Porter***	***Summerwater***	
The Hidden Gnome	25	70	25	120
Mosquito Nights	22	22	31	75
Tomato	33	28	44	105
Total	80	120	100	300

Consider the following events, which apply to a randomly chosen student in the survey sample.

HG: The student chose *The Hidden Gnome* as his or her favorite novel.

S: The student chose *Summerwater* as his or her favorite movie.

Which statement is true about events *HG* and *S*?

a. Events *HG* and *S* are dependent and $P(HG|S) < P(S|HG)$.

b. Events *HG* and *S* are dependent and $P(HG|S) > P(S|HG)$.

c. Events *HG* and *S* are independent and $P(HG|S) < P(S|HG)$.

d. Events *HG* and *S* are independent and $P(HG|S) > P(S|HG)$.

continued

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12. A bag has 10 jelly beans in it, each of a different flavor. Salima only likes 3 of the flavors. If she randomly selects 5 jelly beans, what is the probability that 3 of them are flavors she likes?

    a. 0.083
    b. 0.053
    c. 0.6
    d. 0.5

Use what you have learned in this unit to complete problems 13–15.

13. A group of children at a summer camp were asked to vote for a favorite daytime activity and a favorite evening activity. The table shows the data, with x and y representing missing values.

Evening activity	Daytime activity		
	Swimming	**Kayaking**	**Horseback riding**
Movie night	8	x	12
Skit performance	5	3	y
Costume party	8	7	14

Consider the following events and probabilities.

M: The camper voted for movie night.

SP: The camper voted for skit performance.

K: The camper voted for kayaking.

H: The camper voted for horseback riding.

$$P(M|K)=\frac{1}{3}$$

$$P(H|SP)=\frac{1}{2}$$

Find x, y, and the total number of campers surveyed. Show your work.

continued

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14. The events below apply to the experiment of rolling two number cubes.

D: The number cubes show doubles.

EP: The product is even.

Use these events to solve the following problems.

a. List the sample space.

b. Find $P(D|EP)$, first using probabilities and then using outcomes. Show your work.

c. Determine if *D* and *EP* are independent. Show your work and explain your method.

d. Compare $P(D|EP)$ and $P\left(EP|D\right)$. Show your work and explain what your answer means.

15. An electronics store offers an extended warranty with the purchase of all tablet computers. You are about to purchase a tablet, so you are thinking about buying the extended warranty. The warranty costs $50 for 3 years. The probability that the tablet will break in the first 3 years and be covered under the warranty is 0.17. The average cost of repairs is $120.

a. What is the expected cost of repairs based on the probability?

b. Should you purchase the warranty? Explain your reasoning.

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Answer Key

Lesson 1: Events

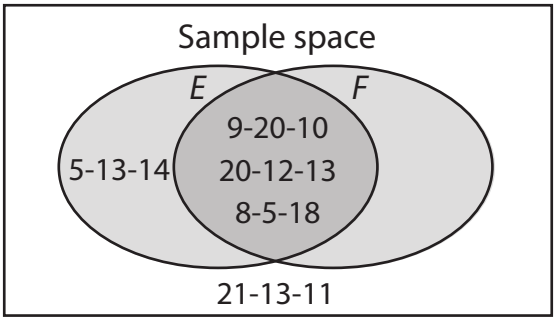
Pre-Assessment, p. U4-1

1. a
2. c
3. b
4. a
5. c

Warm-Up 4.1.1, p. U4-6

1. 16 ways. Explanation: You can use a table to show all the cards in the deck. Then identify and count the cards that are either a heart or an ace or both a heart and an ace.
2. 36 ways. Explanation: Count the cards that are neither hearts nor aces. Or subtract the first answer from 52: 52 – 16 = 36.

Practice 4.1.1: Describing Events, p. U4-30

1. The chosen student plays softball or basketball.
2. The chosen student plays soccer and basketball.
3. {AS, LL}
4. {DB, NP, DL}
5. 

6. $E \cap F$ = {12, 18}; $\overline{E \cup F}$ = {5, 11, 13, 21}
7. *T*
8. *Q* and *R*
9. 6
10. 27

Warm-Up 4.1.2, p. U4-33

1. There are 24 outcomes.
2. There are 15 ways.

Practice 4.1.2: The Addition Rule, p. U4-46

1. 1/2 = 0.5 = 50%
2. 13/18
3. 1 or 100%
4. 4/13
5. 1/3 ≈ 0.33 = 33%
6. 29/36
7. 11/120
8. 20/89
9. 1
10. 1/5

Warm-Up 4.1.3, p. U4-48

Tonya should choose the spinners because they provide the greatest probability of getting all matching results. Explanation: There are 36 possible outcomes for the dice; 6 of those outcomes have matching numbers for both dice:

$P(\text{all matching results}) = \frac{6}{36} = \frac{1}{6}$.

There are 8 possible outcomes for the dice, 2 of which match:

$P(\text{all matching results}) = \frac{2}{8} = \frac{1}{4}$.

There are 9 possible outcomes for the spinners, 3 of which match:

$P(\text{all matching results}) = \frac{3}{9} = \frac{1}{3}$.

$\frac{1}{3}$ is greater than $\frac{1}{6}$ and $\frac{1}{4}$.

Practice 4.1.3: Understanding Independent Events, p. U4-66

1. a. yes
   b. no
   c. no
   d. yes
2. 9/49, or about 18.4%; independent
3. 1/7, or about 14.3%; dependent
4. *A* and *B* are dependent; *A* and *C* are dependent; *B* and *C* are dependent.
5. Independent. Supporting work: *P*(caramel and cookie dough) = 18/152 ≈ 0.118; *P*(caramel) • *P*(cookie dough) = (55/152) • (50/152) ≈ 0.119. Therefore, *P*(caramel and cookie dough) ≈ *P*(caramel) • *P*(cookie dough).
6. Dependent. Supporting work: *P*(hot fudge and chocolate) = 4/152 ≈ 0.026; *P*(hot fudge) • *P*(chocolate) = (46/152) • (28/152) ≈ 0.056. Therefore, *P*(hot fudge and chocolate) ≠ *P*(hot fudge) • *P*(chocolate).
7. 0.375 or 37.5%
8. 0.7 or 70%
9. 0.95 or 95%
10. 0.35 or 35%

Progress Assessment, p. U4-69

1. b
2. a
3. b
4. c
5. b
6. d
7. d
8. c
9. c
10. a

11. a. 0.8 or 80%
    b. 0.8 or 80%
    c. 0.92 or 92%
    d. 0.92 or 92%
    e. 0.984 or 98.4%
    f. 0.984 or 98.4%

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Lesson 2: Conditional Probability

Pre-Assessment, p. U4-74

1. d
2. d
3. a
4. a
5. b

Warm-Up 4.2.1, p. U4-78

1. Sample space =
   {(1, 1) (1, 2) (1, 3)
   (2, 1) (2, 2) (2, 3)
   (3, 1) (3, 2) (3, 3)
   (4, 1) (4, 2) (4, 3)}
2. $\frac{1}{4}$; the sample space has 12 outcomes and 3 of them have a sum of 5; $\frac{3}{12}=\frac{1}{4}$.
3. $\frac{1}{3}$; there are 3 outcomes that show a 3 for spinner A, and 1 of them has a sum of 5.
4. Getting a 3 on spinner A increases the probability that Howard gets a sum of 5 because $\frac{1}{3}>\frac{1}{4}$.

Practice 4.2.1: Introducing Conditional Probability, p. U4-100

1. $P(\text{both numbers are odd} \mid \text{sum} = 7) = 0$; $P(\text{sum} = 7 \mid \text{both numbers are odd}) = 0$
2. $P(\text{sum is even} \mid \text{product is even}) = 9/27 = 1/3$; $P(\text{product is even} \mid \text{sum is even}) = 9/18 = 1/2$
3. Sample space = {OOO, OOE, OEO, OEE, EOO, EOE, EEO, EEE}, where O represents odd and E represents even; A and B are dependent. Explanation: $P(A \mid B) \neq P(A)$ or $P(B \mid A) \neq P(B)$ (either inequality suffices). Probabilities: $P(A) = 4/8 = 1/2$; $P(A \mid B) = 2/2 = 1$; $P(B) = 2/8 = 1/4$; $P(B \mid A) = 1/2$.
4. Sample space = {bbb, bbg, bgb, bgg, gbb, gbg, ggb, ggg}, where "b" represents boy and "g" represents girl; A and B are dependent. Explanation: $P(A \mid B) \neq P(A)$ or $P(B \mid A) \neq P(B)$ (either inequality suffices). Probabilities: $P(A) = 4/8 = 1/2$; $P(A \mid B) = 3/4$; $P(B) = 4/8 = 1/2$; $P(B \mid A) = 3/4$.
5. MD and M are independent because $P(M \mid MD) \approx P(M)$ and $P(MD \mid M) \approx P(MD)$; $P(M \mid MD) = 19/80 \approx 0.238$; $P(M) = 40/161 \approx 0.248$; $P(MD \mid M) = 19/40 \approx 0.475$; $P(MD) = 80/161 \approx 0.497$. Compare $P(MD \mid M)$ and $P(M \mid MD)$: $P(MD \mid M) > P(M \mid MD)$.
   Interpretation: $P(M \mid MD)$ represents the probability that a student chooses music and drama given that he chooses movies. $P(MD \mid M)$ represents the probability that a student chooses movies given that he chooses music and drama. The independence of the events M and MD means that choosing movies does not affect the probability of choosing music and drama, and vice versa. $P(MD \mid M) > P(M \mid MD)$ means that it is more likely that a student chooses music and drama given that he chooses movies than it is that a student chooses movies given that he chooses music and drama.
6. A and K are dependent because $P(A \mid K) \neq P(A)$ and $P(K \mid A) \neq P(K)$; $P(A \mid K) = 9/40 \approx 0.225$; $P(A) = 81/161 \approx 0.503$; $P(K \mid A) = 9/81 \approx 0.111$; $P(K) = 40/161 \approx 0.248$. Compare $P(K \mid A)$ and $P(A \mid K)$: $P(K \mid A) < P(A \mid K)$.
   Interpretation: $P(K \mid A)$ represents the probability that a student chooses karaoke given that he chooses athletics. $P(A \mid K)$ represents the probability that a student chooses athletics given that he chooses karaoke. The dependence of the events K and A means that choosing karaoke does affect the probability of choosing athletics, and vice versa. $P(K \mid A) < P(A \mid K)$ means that it is less likely that a student chooses karaoke given that he chooses athletics than it is that a student chooses athletics given that he chooses karaoke.
7. 0.85 or 85%
8. 0.80 or 80%
9. $P(\text{sum} = 6 \mid \text{numbers are different}) = 4/30 = 2/15$; $P(\text{numbers are different} \mid \text{sum} = 6) = 4/5$
10. $P(\text{all heads} \mid \text{first toss is heads}) = 1/4$; $P(\text{first toss is heads} \mid \text{all heads}) = 1$

Warm-Up 4.2.2, p. U4-102

				Totals
	10	**14**	12	**36**
	5	**17**	8	30
	10	**9**	**15**	34
Totals	25	40	**35**	100

Practice 4.2.2: Using Two-Way Frequency Tables, p. U4-122

1.

Favorite position	**Throwing arm**	
	Right	**Left**
IF	𝍸 I	I
OF	𝍸	I
P	III	II
C	II	

2.

Favorite position	**Throwing arm**	
	Right	**Left**
IF	6	1
OF	5	1
P	3	2
C	2	0

3. $P(TEN \mid HH) < P(HH \mid TEN)$; this means the probability that a recognized student is in the tenth grade given that she receives the Highest Honors award is less than the probability that a recognized student receives the Highest

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Honors award given that she is in the tenth grade.

4. $P(TEN) = 80/225 \approx 0.356$; $P(HH) = 98/225 \approx 0.436$; $P(TEN \mid HH) = 36/98 \approx 0.367$; $P(HH \mid TEN) = 36/80 = 0.45$. Based on the data, the events *TEN* and *HH* seem to be independent because $P(TEN \mid HH) \approx P(TEN)$, or because $P(HH \mid TEN) \approx P(HH)$.
5. $(TWELVE \mid AE) < P(AE \mid TWELVE)$; this means the probability that a recognized student is in the twelfth grade given that she receives the Academic Excellence award is less than the probability that a recognized student receives the Academic Excellence award given that she is in the twelfth grade.
6. $P(TWELVE) = 70/225 \approx 0.311$; $P(AE) = 127/225 \approx 0.564$; $P(TWELVE \mid AE) = 40/127 \approx 0.315$; $P(AE \mid TWELVE) = 40/70 \approx 0.571$. Based on the data, the events *TWELVE* and *AE* seem to be independent because $P(TWELVE \mid AE) \approx P(TWELVE)$, or because $P(AE \mid TWELVE) \approx P(AE)$.
7. $P(J \mid P) > P(P \mid J)$; *J* and *P* are dependent; $P(J) = 99/200 = 0.495$; $P(J \mid P) = 39/50 = 0.78$; $P(P) = 50/200 = 0.25$; $P(P \mid J) = 39/99 \approx 0.394$
8. $P(J \mid CT) > P(CT \mid J)$; *J* and *CT* are independent; $P(J) = 99/200 = 0.495$; $P(J \mid CT) = 26/51 = 0.510$; $P(CT) = 51/200 = 0.255$; $P(CT \mid J) = 26/99 \approx 0.263$
9. $P(M \mid CC) > P(CC \mid M)$; *M* and *CC* are independent; $P(M) = 101/200 = 0.505$; $P(M \mid CC) = 24/49 \approx 0.490$; $P(CC) = 49/200 = 0.245$; $P(CC \mid M) = 24/101 \approx 0.238$
10. $P(M \mid S) > P(S \mid M)$; *M* and *S* are dependent; $P(M) = 101/200 = 0.505$; $P(M \mid S) = 41/50 = 0.82$; $P(S) = 50/200 = 0.25$; $P(S \mid M) = 41/101 \approx 0.406$

Warm-Up 4.2.3, p. U4-125

1. approximately 0.43, or about 43%
2. approximately 0.57, or about 57%
3. It is more likely that the selected student is female.
4. approximately 0.14, or about 14%

Practice 4.2.3: The Multiplication Rule, p. U4-144

1. 0.028
2. 0.083
3. 0.028
4. The probability of selecting 2 English books is greater, so it is more likely that Harper will select 2 English books than that she will select 2 French books.
5. 0.136
6. 0.023
7. 0.091
8. The probability of selecting white socks and white sneakers is greater than the probability of selecting black socks and black sneakers, so it is more likely that both the socks and sneakers selected are white.
9. 0.029
10. 0.2

Progress Assessment, p. U4-146

1. c
2. c
3. b
4. c
5. d
6. d
7. d
8. b
9. c
10. d
11. a. Sample space = {DDD, DDL, DLD, DLL, LDD, LDL, LLD, LLL}, where D represents Dan and L represents Lamar

    b. $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{\frac{1}{8}}{\frac{1}{2}} = \dfrac{1}{8} \bullet \dfrac{2}{1} = \dfrac{2}{8} = \dfrac{1}{4}$;

    $P(B \mid A) = \dfrac{\text{number of outcomes in } (B \cap A)}{\text{number of outcomes in } A} = \dfrac{1}{4}$

    c. *A* and *B* are independent because $P(B) = P(B \mid A)$;

    $P(B) = \dfrac{\text{number of outcomes in } B}{\text{number of outcomes in sample space}} = \dfrac{2}{8} = \dfrac{1}{4}$

    and $P(B \mid A) = \dfrac{1}{4}$ as shown in part b.

    d. $P(A \mid B) = \dfrac{\text{number of outcomes in } (A \cap B)}{\text{number of outcomes in } B} = \dfrac{1}{2}$

    and $P(B \mid A) = \dfrac{1}{4}$ as shown in part b. Therefore, $P(A \mid B) > P(B \mid A)$, which means it is more likely that Dan walks the dog on Monday given that he walks the dog on exactly 2 consecutive days than it is that Dan walks the dog on exactly 2 consecutive days given that he walks the dog on Monday.

Lesson 3: Combinatorics

Pre-Assessment, p. U4-150

1. c
2. b
3. a
4. b
5. c

Warm-Up 4.3.1, p. U4-153

1. Quintana and Siraj; Quintana and Autumn; Quintana and Jonathan
2. Quintana and Siraj; Quintana and Jonathan; Siraj and Jonathan
3. Quintana and Siraj; Quintana and Jonathan; Autumn and Siraj; Autumn and Jonathan

Practice 4.3.1: Combinations and Permutations, p. U4-171

1. 3,060
2. 816
3. 4,896
4. 15 people to choose from; 455

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5. 10
6. 36
7. 126
8. 210
9. 35
10. 840

Warm-Up 4.3.2, p. U4-173

1. 0.58
2. 0.25
3. 0.6
4. 0.15

Practice 4.3.2: Probability with Combinatorics, p. U4-187

1. 0.28
2. 0.28
3. 0.083
4. 0.05
5. 0.083
6. 0.0020
7. 0.25
8. 0.0076
9. 0.67
10. 0.000084

Progress Assessment, p. U4-189

1. d
2. c
3. b
4. a
5. a
6. c
7. b
8. d
9. a
10. d
11. a. 504
    b. 56
    c. 0.125

Lesson 4: Making and Analyzing Decisions

Pre-Assessment, p. U4-192

1. d
2. d
3. c
4. c
5. b

Warm-Up 4.4.1, p. U4-196

1. $P(6) = \frac{1}{6} \approx 0.17$
2. $P(\text{odd}) = \frac{1}{2} = 0.5$
3. $P(2 \text{ and odd}) = \frac{1}{12} \approx 0.083$

Practice 4.4.1: Making Decisions, p. U4-214

1. The selection is not random because it follows a pattern.
2. The selection is random because each number has an equal probability of being chosen, and the selection does not follow a pattern.
3. The selection is random because each number has an equal probability of being chosen, and the selection does not follow a pattern.
4.

Spin 1	Spin 2	Result
1	no spin 2	no point
2	1	no point
2	2	no point
2	3	1 point
3	no spin 2	1 point

5.

Spin 1	Spin 2	Result	Probability
1	no spin 2	no point	$\frac{1}{3} \approx 0.33$
2	1	no point	$\frac{1}{9} \approx 0.11$
2	2	no point	$\frac{1}{9} \approx 0.11$
2	3	1 point	$\frac{1}{9} \approx 0.11$
3	no spin 2	1 point	$\frac{1}{3} \approx 0.33$

6. There are two paths in which a point is earned:

   spinning a 2 and then a 3, or spinning a 3.

   $P(\text{point earned}) = P(2,3) + P(3) = \frac{1}{9} + \frac{1}{3} = \frac{4}{9} \approx 0.44$
7. The probability of earning a point is not dependent on which player is taking a turn. Each player has an equal probability of earning a point, so the game is fair.
8. 0.05
9. 0.025
10. The probability of winning depends on when the customer entered the contest. The probability of winning is not the same for every customer, so the game is not fair.

Warm-Up 4.4.2, p. U4-216

1. $P(\text{first selected ticket is Preston's}) = \frac{1}{1{,}000} = 0.001$
2. $P(\text{first second-place prize is Sandeep's}) = \frac{2}{980} \approx 0.0020$
3. $P(\text{first-place prize is Marla's}) = \frac{10}{970} \approx 0.010$

Practice 4.4.2: Analyzing Decisions, p. U4-233

1. Approximately 10 people will make a purchase between 11:00 A.M. and 1:00 P.M.
2. Answers may vary. Sample response: P(all 5 scheduled come to work) = 0.59. There is a greater than 0.5 probability that all 5 will come to work, so it is more likely that all 5 will show up than less than all 5. However, if she wants to be confident that 5 people will be at work at 9:00 A.M., she should schedule an additional employee.

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3. Answers may vary. Sample response: P(if 5 scheduled, 4 or 5 come to work) = 0.92. The manager should schedule 5 people to work at 9:00 A.M. This gives greater than a 0.8 probability that at least 4 people will come to work.
4. Answers may vary. Sample response: A minimum of 4 people need to be working in order to have a salesperson-to-customer ratio that is greater than 0.3. However, 5 people need to be scheduled to work. If 5 people are scheduled, then there is a probability of 0.92 that at least 4 will come to work. The probability of 0.92 comes from the sum of the probabilities of the 6 different ways that at least 4 people come to work: $(0.9)^5 + [(0.9^4 \bullet (0.1)] + [(0.9^4 \bullet (0.1)] + [(0.9^4 \bullet (0.1)] + [(0.9^4 \bullet (0.1)] + [(0.9^4 \bullet (0.1)] = 0.92$. Five people need to be scheduled between 1:00 P.M. and 3:00 P.M.
5. Website 2, with 360 clicks
6. Website 3 generates the greatest number of purchases, with 144.
7. Website 2, at $18 a day
8. Answers may vary. Sample response: Given that website 3 has almost half the daily cost of website 2, at $9.60 a day, and it leads to more than 3 times the purchases of website 1, website 3 is the best advertising site for Hot Gadgets.
9. Yes, she should purchase the warranty. The expected cost of repairs is $1,458, and the warranty is only $1,000.
10. The expected cost of repairs under the two-year plan is $864, and this plan costs $500, so again it is a good idea to buy the warranty. Sample answer: She would be better off choosing the five-year warranty because the cost per year is less and the chance that the car breaks down increases with each passing year.

Progress Assessment, p. U4-236

1. b
2. a
3. b
4. d
5. c
6. d
7. b
8. a
9. c
10. d
11. a. 0.000064
    b. $139.97
    c. $300.00
    d. 0.210

Unit Assessment
p. U4-241

1. d
2. d
3. b
4. a
5. c
6. d
7. c
8. b
9. b
10. c
11. b
12. a
13. $x = 5$, $y = 8$, total = 70 campers. Student work should reflect the following:

$\frac{1}{3} = \frac{x}{x+10}$ $\qquad$ $\frac{1}{2} = \frac{y}{y+8}$

$x + 10 = 3x$ $\qquad$ $y + 8 = 2y$

$10 = 2x$ $\qquad$ $8 = y$

$5 = x$

total = 8 + 5 + 12 + 5 + 3 + 8 + 8 + 7 + 14 = 70

14. a. {(1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6)
(2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (2, 6)
(3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (3, 6)
(4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6)
(5, 1) (5, 2) (5, 3) (5, 4) (5, 5) (5, 6)
(6, 1) (6, 2) (6, 3) (6, 4) (6, 5) (6, 6)}

b. $P(D|EP) = \frac{P(D \cap EP)}{P(EP)} = \frac{\frac{3}{36}}{\frac{27}{36}} = \frac{3}{36} \bullet \frac{36}{27} = \frac{3}{27} = \frac{1}{9}$;

$P(D|EP) = \frac{\text{number of outcomes in } (D \cap EP)}{\text{number of outcomes in } EP} = \frac{3}{27} = \frac{1}{9}$

c. D and EP are dependent because $P(D) \neq P(D|EP)$;

$P(D) = \frac{\text{number of outcomes in } D}{\text{number of outcomes in sample space}} = \frac{6}{36} = \frac{1}{6}$

and $P(D|EP) = \frac{1}{9}$.

d. $P(EP|D) = \frac{\text{number of outcomes in } (D \cap EP)}{\text{number of outcomes in } D} = \frac{3}{6} = \frac{1}{2}$

and $P(D|EP) = \frac{1}{9}$. $P(EP|D) > P(D|EP)$, which means it is more likely that the number cubes show an even product given that they show doubles than it is that they show doubles given that they show an even product.

15. a. The expected cost of repairs is $20.40.
    b. Answers may vary. Sample answer: You should not purchase the warranty; based on the probability, it is unlikely that the tablet will break and the expected value is less than the cost of the warranty.

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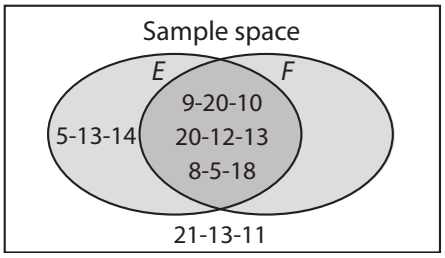
Student Book Answer Key

Lesson 1: Events

Practice 4.1.1: Describing Events, p. U4-21

1. The chosen house has a fireplace and oil heat.
2. The chosen house has a garage or a fireplace.
3. {Opal, Colin, Hugo, Carla, Rick}
4. {Monica, Opal, Josh, Elias, Carla, Sam}
5. 

6. $\overline{V}$ = {KXL 2677, MRB 5516, BCF 4435}; $V \cap S$ = {JEV 8591, LSU 7582}
7. *C*
8. *B*
9. 10
10. 24

Practice 4.1.2: The Addition Rule, p. U4-31

1. 3/4 = 0.75 = 75%
2. 2/3
3. 5/24
4. 8/13
5. 1/6 ≈ 0.17 = 17%
6. 1/18
7. 0
8. 7/206
9. 0
10. 4/5

Practice 4.1.3: Understanding Independent Events, p. U4-44

1. a. no
   b. yes
   c. no
   d. yes
2. 2/25, or 8%; independent
3. 1/10, or 10%; dependent
4. *A* and *B* are dependent; *A* and *C* are dependent; *B* and *C* are independent.
5. Dependent. Supporting work: *P*(Jones and less than high school) = 2/114 ≈ 0.017; *P*(Jones) • *P*(less than high school) = (35/114) • (12/114) ≈ 0.032. Therefore, *P*(Jones and less than high school) ≠ *P*(Jones) • *P*(less than high school).
6. Independent. Supporting work: *P*(Walker and advanced college) = 7/114 ≈ 0.061; *P*(Walker) • *P*(advanced college) = (44/114) • (20/114) ≈ 0.068. Therefore, *P*(Walker and advanced college) ≈ *P*(Walker) • *P*(advanced college).
7. 0.4 or 40%
8. 0.86 or 86%
9. 0.75 or 75%
10. 0.4 or 40%

Lesson 2: Conditional Probability

Practice 4.2.1: Introducing Conditional Probability, p. U4-66

1. $P(\text{both numbers are even} \mid \text{sum} = 8) = 3/5$; $P(\text{sum} = 8 \mid \text{both numbers are even}) = 3/9 = 1/3$
2. $P(\text{sum is odd} \mid \text{product is odd}) = 0$; $P(\text{product is odd} \mid \text{sum is odd}) = 0$
3. Sample space = {GGG, GGL, GLG, GLL, LGG, LGL, LLG, LLL}, where G represents greater than 3 on the first roll and L represents less than 4 on the third roll; *A* and *B* are independent. Explanation: $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$; Probabilities: $P(A) = 4/8 = 1/2$; $P(A \mid B) = 2/4 = 1/2$; $P(B) = 4/8 = 1/2$; $P(B \mid A) = 2/4 = 1/2$
4. Sample space = {bbb, bbg, bgb, bgg, gbb, gbg, ggb, ggg}, where "b" represents boy and "g" represents girl; *A* and *B* are dependent. Explanation: $P(A \mid B) \neq P(A)$ or $P(B \mid A) \neq P(B)$ (either inequality suffices). Probabilities: $P(A) = 4/8 = 1/2$; $P(A \mid B) = 2/3$; $P(B) = 3/8$; $P(B \mid A) = 2/3$.
5. *MD* and *C* are independent because $P(C \mid MD) \approx P(C)$ and $P(MD \mid C) \approx P(MD)$; $P(C \mid MD) = 20/80 = 0.25$; $P(C) = 41/161 \approx 0.255$; $P(MD \mid C) = 20/41 \approx 0.488$; $P(MD) = 80/161 \approx 0.497$. Compare $P(MD \mid C)$ and $P(C \mid MD)$: $P(MD \mid C) > P(C \mid MD)$.
   Interpretation: $P(C \mid MD)$ represents the probability that a student chooses a concert given that she chooses music and drama. $P(MD \mid C)$ represents the probability that a student chooses music and drama given that she chooses a concert. The independence of the events *C* and *MD* means that choosing a concert does not affect the probability of choosing music and drama, and vice versa.
   $P(MD \mid C) > P(C \mid MD)$ means that it is more likely that a student chooses music and drama given that she chooses a concert than it is that a student chooses a concert given that she chooses music and drama.
6. *A* and *B* are dependent because $P(A \mid B) \neq P(A)$ and $P(B \mid A) \neq P(B)$; $P(A \mid B) = 30/40 = 0.75$; $P(A) = 81/161 \approx 0.503$; $P(B \mid A) = 30/81 \approx 0.370$; $P(B) = 40/161 \approx 0.248$. Compare $P(B \mid A)$ and $P(A \mid B)$: $P(B \mid A) < P(A \mid B)$.
   Interpretation: $P(B \mid A)$ represents the probability that a student chooses bowling given that he chooses athletics. $P(A \mid B)$ represents the probability that a student chooses athletics given that he chooses bowling. The dependence of the events *B* and *A* means that choosing bowling does affect the probability of choosing athletics, and vice versa.
   $P(B \mid A) < P(A \mid B)$ means that it is less likely that a student chooses bowling given that he chooses athletics than it is that a student chooses athletics given that he chooses bowling.
7. 0.4 or 40%

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8. 0.6 or 60%
9. $P(\text{sum} = 7 \mid \text{numbers are different}) = 6/30 = 1/5$; $P(\text{numbers are different} \mid \text{sum} = 7) = 6/6 = 1$
10. $P(\text{exactly 2 heads} \mid \text{second toss is heads}) = 2/4 = 1/2$; $P(\text{second toss is heads} \mid \text{exactly 2 heads}) = 2/3$

Practice 4.2.2: Using Two-Way Frequency Tables, p. U4-82

1.

Sport	Gender	
	Male	Female
Football	𝍸 II	II
Basketball	II	III
Baseball	I	I
Softball	I	III

2.

Sport	Gender	
	Male	Female
Football	7	2
Basketball	2	3
Baseball	1	1
Softball	1	3

3. $P(TEN \mid V) < P(V \mid TEN)$; this means the probability that a work-study student is in the tenth grade given that she has a volunteer position is less than the probability that a work-study student has a volunteer position given that she is in the tenth grade.
4. $P(TEN) = 24/180 \approx 0.133$; $P(V) = 73/180 \approx 0.406$; $P(TEN \mid V) = 6/73 \approx 0.082$; $P(V \mid TEN) = 6/24 = 0.25$. Based on the data, the events *TEN* and *V* seem to be dependent because $P(TEN \mid V) \neq P(TEN)$, or because $P(V \mid TEN) \neq P(V)$.
5. $P(TWELVE \mid P) < P(P \mid TWELVE)$; this means the probability that a work-study student is in the twelfth grade given that she has a paid job is less than the probability that a work-study student has a paid job given that she is in the twelfth grade.
6. $P(TWELVE) = 79/180 \approx 0.439$; $P(P) = 107/180 \approx 0.594$; $P(TWELVE \mid P) = 65/107 \approx 0.608$; $P(P \mid TWELVE) = 65/79 \approx 0.823$. Based on the data, the events *TWELVE* and *P* seem to be dependent because $P(TWELVE \mid P) \neq P(TWELVE)$, or because $P(P \mid TWELVE) \neq P(P)$.
7. $P(M \mid P) > P(P \mid M)$; *M* and *P* are dependent; $P(M) = 80/160 = 0.5$; $P(M \mid P) = 12/41 \approx 0.293$; $P(P) = 41/160 \approx 0.256$; $P(P \mid M) = 12/80 = 0.15$
8. $P(F \mid V) > P(V \mid F)$; *F* and *V* are dependent; $P(F) = 80/160 = 0.5$; $P(F \mid V) = 15/40 = 0.375$; $P(V) = 40/160 = 0.25$; $P(V \mid F) = 15/80 \approx 0.188$
9. $P(M \mid MU) > P(MU \mid M)$; *M* and *MU* are independent; $P(M) = 80/160 = 0.5$; $P(M \mid MU) = 20/41 \approx 0.488$; $P(MU) = 41/160 \approx 0.256$; $P(MU \mid M) = 20/80 = 0.25$
10. $P(F \mid MA) > P(MA \mid F)$; *F* and *MA* are independent; $P(F) = 80/160 = 0.5$; $P(F \mid MA) = 18/38 \approx 0.474$; $P(MA) = 38/160 \approx 0.238$; $P(MA \mid F) = 18/80 = 0.225$

Practice 4.2.3: The Multiplication Rule, p. U4-96

1. 0.133
2. 0.029
3. 0.024
4. The probability of selecting 1 canister of coconut flakes and 1 canister of butterscotch chips is greater than the probability of selecting 2 canisters of butterscotch chips, so it is more likely that 1 canister of coconut flakes and 1 canister of butterscotch chips are selected.
5. 0.358
6. 0.039
7. The probability of receiving 1 salty snack and 1 sweet snack is greater than the probability of receiving 2 pretzels, so it is more likely that 1 sweet snack and 1 salty snack are received.
8. 0.148
9. 0.017
10. 0.151

Lesson 3: Combinatorics

Practice 4.3.1: Combinations and Permutations, p. U4-111

1. 252
2. 5,040
3. 1,000,000
4. 1,680
5. 306
6. 1,330
7. 8 commercials are needed; 121,080,960
8. 6 commercials are needed; 2,162,160
9. 1,475,789,056
10. 3,003

Practice 4.3.2: Probability with Combinatorics, p. U4-118

1. 0.0028
2. 0.048
3. 0.024
4. 0.48
5. 0.098
6. 0.000054
7. 0.056
8. 0.00020
9. 0.33
10. 0.25

Lesson 4: Making and Analyzing Decisions

Practice 4.4.1: Making Decisions, p. U4-132

1. The selection is random because each number has an equal probability of being chosen, and the selection does not follow a pattern.
2. The selection is not random because the selection follows a pattern.

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3. The selection is random because each number has an equal probability of being chosen, and the selection does not follow a pattern.
4.

Selected card	Winner
1	Piero
2	Piero
3	Ebony
4	Simone
5	Thaddeus
6	Ebony
7	Piero
8	Simone
9	Ebony
10	Thaddeus

5. Each card is selected randomly, so the card selection is fair. The probability of choosing any one card is $\frac{1}{10} = 0.1$.
6. Simone: 0.2; Thaddeus: 0.2; Ebony: 0.3; Piero: 0.3
7. The game is not fair. Ebony and Piero have a higher probability of winning than Simone and Thaddeus.
8. $P(\text{win 1:00 P.M. raffle}) = \frac{1}{240} \approx 0.0042$
9. Each raffle is fair, because each customer who makes a purchase receives one ticket. The winner is chosen at each time frame randomly.
10. The contest is not fair. Based on the customer estimates, $P(\text{win 10:00 A.M. raffle}) = \frac{1}{150} \approx 0.0067$, $P(\text{win 1:00 P.M. raffle}) = \frac{1}{240} \approx 0.0042$, and $P(\text{win 3:00 P.M. raffle}) = \frac{1}{60} \approx 0.017$. The customers who play at different times have different probabilities of winning.

Practice 4.4.2: Analyzing Decisions, p. U4-144

1. Monday: 129; Tuesday: 113; Wednesday: 158; Thursday: 88; Friday: 172
2. Monday: 139; Tuesday: 123; Wednesday: 168; Thursday: 98; Friday: 182
3. If 5 people are scheduled, then there is a 0.98 probability that at least 4 people arrive to work.
4. Troy should apply to companies B and C. Explanations may vary. Sample response: These two companies have the highest probabilities that an application will be forwarded to a hiring manager: company B, with a probability of 0.439, and company C, with a probability of 0.504.
5. Human resources personnel spend $867 reviewing 100 applications that are not forwarded to a manager.
6. The total cost is the HR cost in dollars + the program cost in dollars: 1275 + 5.00 = 1280, so the total cost is $1,280.
7. Answers may vary. Sample response: The greatest number of applications, approximately 50, is sent to a manager at company C. Company C also has the second-lowest cost associated with reviewing applications, at $950. Company C appears to have a better review process, because less money is being spent but more qualified applications are being sent to managers. Company B pays $1,085 to review the 100 applications, and company D pays $830 to review the 100 applications.
8. No; the expected value of the repair is $22.50. For replacement, the cost is expected to be $31.50 for television 1. The warranty is only $40, and is a higher price than the expected repair or replacement costs.
9. No; the expected value of the repair is $18 and for replacement the expected value is $28. The warranty is $70, and is greater than the cost of expected repairs or replacements.
10. Television 1. Explanations may vary. Sample response: Add the purchase price to the expected value of replacement because this is the most she will be expected to spend based on the probability. Television 1 cost in $: 450 + 31.5 = 481.5; television 2 cost in $: 560 + 28 = 588. Television 1 has a lower total cost between now and 5 years from now.

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Instruction

Goal: To provide opportunities for students to develop concepts and skills related to the counting principle and simple and compound probabilities for both independent and dependent events

Common Core State Standards

S–CP.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").★

S–CP.2 Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent.★

S–CP.3 Understand the conditional probability of *A* given *B* as $P(A \text{ and } B)/P(B)$, and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A,* and the conditional probability of *B* given A is the same as the probability of *B.*★

Student Activities Overview and Answer Key

Station 1

Students will be given 3 pieces of red yarn, 3 pieces of blue yarn, and tape. Students will be given five index cards that have the following written on them: "chicken," "tuna," "white," "wheat," and "Italian." Students will use the index cards and yarn to model the counting principle of independent events.

Answers

1. No, because there are three types of bread available.
2. chicken, white bread; chicken, wheat bread; chicken, Italian bread
3. tuna, white bread; tuna, wheat bread; tuna, Italian bread
4. multiplication; (2 types of meat)(3 types of bread) = 6 possibilities
5. (4)(3)(8)(4) = 384 different meals
6. No; they are independent events.

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Instruction

Station 2

Students will be given index cards with the following subjects written on them: "math," "science," "English," "history," "physical education," and "computer lab." Students will arrange the index cards to create a class schedule. They will derive the counting principle for dependent events by analyzing how to create all possible class schedules.

Answers

1. math, science, English, history, physical education, computer lab; 6
2. Answers will vary.
3. 5; 5
4. Answers will vary.
5. 4; 4
6. Answers will vary.
7. 3; 3
8. Answers will vary.
9. 2; 2
10. Answers will vary.
11. 1; 1
12. Answers will vary.
13. Multiply (6)(5)(4)(3)(2)(1) = 720 possible schedules
14. Dependent, because you can only use each class once in the schedule.

Station 3

Students will be given a number cube and fair coin. Students will use the number cube to model simple probability. Then they will use the fair coin to model simple probability. They will explore mutually exclusive events and provide real-world examples.

Answers

1. Answers will vary.
2. equal
3. 6
4. 1, 2, 3, 4, 5, 6
5. $P(5) = 1/6$; $P(6) = 1/6$

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6. $P(>4) = P(5) + P(6) = 1/6 + 1/6 = 1/3$; P(even number) = 1/2
7. H, T; 2
8. 1/2
9. 1/2 + 1/2 = 1/4; they are independent events.
10. casino and carnival games

Station 4

Students will be given a bag of 4 marbles that are red, green, yellow, and blue. They will also be given a fair coin. Students will use the marbles to model independent and dependent events. Then they will use the marbles and fair coin to model compound probability and mutually exclusive events.

Answers

1. 4
2. Answers will vary.
3. P(color drawn) = 1/4
4. red and green
5. red and yellow
6. red and blue
7. green and yellow
8. green and blue
9. yellow and blue
10. Answers will vary.
11. P(pair of marble colors) = 1/6
12. P(green) = 1/4
13. head, tail; 1/2
14. P(tail) = 1/2
15. independent
16. {GT, GH, RT, RH, BT, BH, YT, YH}
17. 8; P(green and tail) = 1/8
18. P(green and tail) = P(green) • P(tail) = 1/4 • 1/2 = 1/8

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Instruction

Materials List/Setup

Station 1 3 pieces of red yarn; 3 pieces of blue yarn; tape; 5 index cards that have the following written on them:

"chicken," "tuna," "white," "wheat," "Italian"

Station 2 6 index cards with the following written on them:

"math," "science," "English," "history," "physical education," "computer lab"

Station 3 number cube; fair coin

Station 4 bag of 4 marbles that are red, green, yellow, and blue; fair coin

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. How do you use the fundamental counting principle to find the number of possibilities of independent events?
2. How do you use the fundamental counting principle to find the number of possibilities of dependent events?
3. How do you find the simple probability of independent events?
4. How do you find the compound probability of independent events?
5. How do casinos and carnivals rely on probability to make money?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. Multiply together the number of possibilities for each event.
2. Multiply (number of possibilities) times (number of possibilities – 1) and so on.
3. The simple probability of one event equals the number of true outcomes/number of equally likely outcomes. The simple probability of two independent events equals the probability of the first event plus the probability of the second event.
4. The compound probability of two independent events is the probability of the first event multiplied by the probability of the second event.
5. Casinos and carnivals set up games that have the odds in their favor.

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Instruction

Possible Misunderstandings/Mistakes

- Not understanding the difference between independent and dependent events
- Forgetting to subtract 1 from the number of possibilities each time you perform multiplication using the counting principle for dependent events
- Not realizing that the simple probability of two or more independent events is found by adding the individual probabilities
- Not realizing that the compound probability of two or more independent events is found by multiplying each individual probability by the others

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Station 1

You will be given 3 pieces of red yarn, 3 pieces of blue yarn, and tape. You will also be given 5 index cards that have the following written on them: "chicken," "tuna," "white," "wheat," and "Italian." Use these materials and the problem scenario below to answer the questions.

> Jon is at the sandwich shop ready to eat lunch, but can't decide what sandwich to order. He wants to get either chicken or tuna. He has narrowed down the type of bread to white, wheat, or Italian.

1. Is Jon choosing between 2 sandwiches? Why or why not?

2. As a group, determine how many different chicken sandwiches he can pick from by taping the 3 pieces of red yarn to the "chicken" index card and each end of the yarn to the "white," "wheat," or "Italian" cards.

   What sandwich options have you created?

3. Repeat this process with the "tuna" index card and the blue yarn.

   What sandwich options have you created?

4. Based on the number of different sandwiches you have created, what operation (+, –, •, or ÷) can you use to find the number of sandwiches Jon can choose from? Explain your answer.

continued

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Use your observations from problems 1–4 to help you answer the next question.

5. A local restaurant offers a four-course meal that includes soup, salad, an entrée, and a dessert for $19.95. You can choose from 4 soups, 3 salads, 8 entrées, and 4 desserts.

   How many different four-course meals can you create? Show your work.

6. Does your choice of entrée depend on your choice of salad? What does this tell you about what type of events these are?

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Station 2

You will be given 6 index cards with the following written on them:

"math," "science," "English," "history," "physical education," "computer lab"

Use these index cards and the problem scenario below to answer the questions that follow.

The computer at school is creating a class schedule for a student named Elena. There are 6 class periods in the day.

1. List the possible classes Elena could have first period. How many possible classes did you list?

2. Select one of the index cards as Elena's first-period class. Place the index card underneath this paper.

   What class did you choose?

3. Look at your index cards. How many index cards are left?

   This means there are how many choices for Elena's second-period class?

4. Select one of the remaining index cards as Elena's second-period class. Place the index card underneath this paper.

   What class did you choose?

5. Look at your index cards. How many index cards are left?

   This means there are how many choices for Elena's third-period class?

continued

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6. Select one of the remaining index cards as Elena's third-period class. Place the index card underneath this paper.

   What class did you choose?

7. Look at your index cards. How many index cards are left?

   This means there are how many choices for Elena's fourth-period class?

8. Select one of the remaining index cards as Elena's fourth-period class. Place the index card underneath this paper.

   What class did you choose?

9. Look at your index cards. How many index cards are left?

   This means there are how many choices for Elena's fifth-period class?

10. Select one of the remaining index cards as Elena's fifth-period class. Place the index card underneath this paper.

    What class did you choose?

11. Look at your index cards. How many index cards are left?

    This means there are how many choices for Elena's sixth-period class?

12. Select the last remaining index card as Elena's sixth-period class. Place the index card underneath this paper.

    What class did you choose?

continued

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13. You created one possible class schedule for Elena. How can you use your answers in problems 1, 3, 5, 7, 9, and 11 to find ALL the possible class schedules Elena could have?

14. Were the choices you selected for each class period dependent on your choices for the other class periods? Why or why not?

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Station 3

You will be given a number cube and a fair coin. You will use these tools to explore simple probability.

1. As a group, roll the number cube. What number did you roll?

2. Did the number you rolled have less, equal, or more of a chance of being rolled than the other numbers on the number cube? Explain your answer.

3. How many numbers are on the number cube?

4. List these numbers: _____

   In probability, these numbers are known as your sample space or possible outcomes.

5. What is the probability of rolling a 5 on your number cube? Explain your answer.

   What is the probability of rolling a 6 on your number cube? Explain your answer.

continued

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6. How can you use your answers in problem 5 to find the probability P of rolling a number > 4 on your number cube?

   What is the P(number > 4)?

   What is the P(even number)?

As a group, examine the coin.

7. What is the sample space of the coin?

   How many possible outcomes can you have when you flip the coin?

8. What is the probability of tossing the coin heads-up?

9. What is the probability of tossing the coin heads-up, then tossing it heads-up again? Explain your answer.

10. What are some examples of simple probability used in the real world?

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Station 4

You will be given a bag of 4 marbles that are red, green, yellow, and blue. You will also be given a fair coin. You will use these tools to explore compound probability. Work as a group to answer the questions.

1. Place the marbles in the bag. How many different colored marbles are in the bag?

2. Pick 1 marble from the bag without looking. What color marble did you choose?

3. How can you use your answers from problems 1 and 2 to find the probability of choosing a marble of the color you found in problem 2?

Place the marble back in the bag. What are the possible different pairs of marble colors in the bag? A possible pair has been given for problem 4 as a model.

4. red and green
5. _____
6. _____
7. _____
8. _____
9. _____

10. Pick 2 marbles from the bag without looking. What pair of marble colors did you choose?

continued

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11. How can you use your answers from problems 4–9 to find the probability of choosing the pair of marble colors you found in problem 10? (*Hint*: These are dependent events.)

Place the marbles back in the bag.

12. What is the probability of choosing a green marble?

13. Examine your fair coin. List the possible outcomes that you could get if you tossed the coin.

What is the probability of each outcome?

14. What is the probability of tossing the coin tails-up?

15. Are these two events independent or dependent? Explain your answer.

16. List the sample space for finding the probability of choosing a green marble AND then tossing a coin tails-up: P(green and tail).

17. How many outcomes did you find in problem 16?

What is P(green and tail)?

18. What is a faster way to find the P(green and tail) than listing the entire sample space?

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APPENDIX

Problem-Based Task Implementation Guides

Introduction

This addendum contains a collection of Implementation Guides (IGs) designed to support the use of the Problem-Based Tasks found throughout this unit.

Each Implementation Guide includes an overview and suggestions for introducing and facilitating the task. These contain specific recommendations for selected Standards for Mathematical Practice (SMPs) to highlight, guidance on addressing common errors and misconceptions associated with the task, and ideas for monitoring and coaching students as they work through the task. Finally, there is guidance for debriefing the task, with special attention to the SMPs and relevant Key Concepts from the instruction, as well as information on suitable extensions and alternate strategies.

Throughout the Implementation Guides, suggestions are provided in a question/prompt format to support your efforts to elicit student thinking and facilitate classroom discourse.

We developed these Implementation Guides for our Problem-Based Tasks at the request of math educators, and with advice and feedback from mathematics supervisors and teachers. Please let us know if they are helpful. Visit us at www.walch.com, follow us on Twitter (@WalchEd), or e-mail suggestions to customerservice@walch.com.

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Implementation Guide

Problem-Based Task 4.1.1: Recognizing Equal Sets

Common Core State Standard

S–CP.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").★

Task Overview

Focus

How can a variety of events resulting from an experiment be organized into equal sets? What are the possible outcomes of rolling a pair of dice, and what are the events that can be created as a result of the experiment? In this task, students will first determine the outcomes for several events involving rolling two dice, and then they will analyze the subsets of these events and ultimately create a list of equal sets of events.

This activity will provide practice with:

- identifying the sample space of an experiment
- determining the outcomes of an event
- applying correct symbol notation
- distinguishing between the union and intersection of sets
- classifying the results of the union, intersection, and complements of sets
- organizing sets of outcomes into equal sets

Introduction

This task should be used to explore or to apply the skill of describing events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not"). Students should already be familiar with drawing and interpreting tree diagrams and Venn diagrams.

Begin by reading the problem and clarifying the meaning of the terms *element*, *event*, *outcome*, and *sample space*.

element	an item or member in a set
event	an outcome, or a set of outcomes, of an experiment
outcome	results of an experiment
sample space	the set of all possible outcomes of an experiment

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to organize sets of outcomes of an experiment into groups. In order to accomplish this goal, they will first determine the sample space of the experiment, and then they will identify the outcomes of specific events. They will use their knowledge of characteristics of subsets of a sample space, such as union, intersection, complement, and empty set, and they will also recognize and apply the corresponding symbols for these terms. They will organize their results in order to identify groups of equal sets.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments about their choices for the elements they choose for the sample space of the experiment, as well as their choices for the outcomes of each event. They will also justify their conclusions about which sets are equal by explaining their decisions about the unions, intersections, and complements of specific sets. When presenting their arguments to others, they will apply their knowledge and terminology about the characteristics of the specific events.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  Students will first complete the task of identifying the sample space for the experiment of rolling two dice. To accomplish this, they will use repeated reasoning as they organize their sample space into rows and columns, which are completed by reasoning through the ascending sequence of all possible outcomes of number combinations on the dice. Then they will use the repeated process of identifying elements from the sample space which belong in the set for each desired event.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- not identifying all of the elements in the sample space

  Remind students to organize the sample space for the experiment into rows and columns so that they can see the pattern of outcomes. Have them look for any gaps in the rows or columns. Also remind them that they should have the same number of rows and columns.

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- confusing *union* and *intersection* of sets

  Have students write the definitions of "union" and "intersection" on their papers. Also remind them that $\cup$ is the symbol for "union", and $\cap$ is the symbol for intersection. Remind them to be precise when using the symbols for each term.

- not recognizing the symbol for the complement of a set; i.e. not understanding what $\overline{D}$ means

  Remind students that the complement of a set is the elements that are in some universal set, but not in that particular set. Discuss the definition relating to an example in the task, such as $\overline{D}$ is all of the elements in the sample space that are *not* in the set D.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them to think of examples which relate to a classroom of a sample space and subsets. Ask them, "If a math class has 30 students, what is the sample space for the experiment of choosing one student at random? What are some possible subsets of the sample space of the 30 students in the classroom?" (**Answer**: The sample space for the experiment of choosing one student at random from 30 students is 30 students. Some possible subsets could be the number of brown haired students out of the 30 students, the number of students who are 5'5" and taller out of the 30 students, the number of students with blue jeans on that day out of 30 students, etc.)
- Before students begin the task, ask them, "In order to ensure that the experiment of rolling two dice is a completely fair and unbiased experiment, what must be true about the dice being used?" (**Answer**: In order for the experiment to be a fair one, each die must be exactly the same, and each die must have the values 1, 2, 3, 4, 5, and 6 on the six sides, so that each outcome is equally likely.)
- Before students begin the task, discuss the reasons why having a fair experiment is important. Ask them, "Suppose one of the dice had the values 1, 2, 2, 2, 3, and 4 on it, and the other dice had values of 1, 2, 3, 4, 5, and 6 on it. Why would the experiment be considered biased?" (**Answer**: The experiment would be considered biased because the first die is not a "fair" die. In other words, each number does not have an equal chance of being rolled, since the value of 2 is represented on three sides of one of the dice, and all the other numbers only appear once.)
- As students begin working on the task and have listed the sample space, ask them, "How many elements are in the sample space? How do you think you could have figured this number out before you listed all of the outcomes?"(**Answer:** There are 36 elements in the sample space. This number comes from the fact that there are 6 possible outcomes for rolling the first dice, and there are 6 possible outcomes for rolling the second dice. The total number of elements is $6 \bullet 6 = 36$.)

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- If students are having difficulty distinguishing between the union and intersection of sets, encourage them to come up with creative ways to remember the symbols. Ask them, "What are some techniques you can think of to remember the difference between union and intersection?"(**Answer:** Answers will vary, but some examples may include remembering that the symbol for union is a letter "U", which stands for "Union". Also, the word "union" means "coming together", so students may think of all of the elements "coming together".)
- If students are having difficulty finding the complement of a set, discuss some examples outside of the task that may help them relate the concept to the task. Ask them, "For the experiment of choosing a random student out of a classroom of 30 students, what would be the complement to the set of "all brown haired students in the classroom?"(**Answer:** The complement would be the elements that are in the sample space but are NOT in the set of "all brown haired students in the classroom". So the complement would be the set "all students in the classroom without brown hair".)
- As students are concluding the task, ask them to consider specifically how the sample space would expand if three dice were rolled. Ask them, "If you were asked to list all elements in the sample space if three dice were rolled, how many elements would this sample space contain? How can you relate the sample space of 36 when two dice are rolled to the sample space when three dice are rolled?"(**Answer:** Since there are 36 possible outcomes when two dice are rolled, which is a result of 6 • 6, then the number six is multiplied *three* times to represent the three rolls: 6 • 6 • 6 = 216. There are 216 possible outcomes if three dice are rolled, so the sample space has 216 elements.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their thoughts about how to first determine the sample space of the experiment, and then how to identify the elements for each specific event given. Specifically discuss how they chose the elements to represent the union, intersections, and complements of the given sets. Also ask for volunteers to explain how they interpreted symbols for each characteristic of the sets and how they used the definitions of these characteristics to determine the results. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process about creating the sample space, as well as how they organized the elements of the sample space in a useful and practical manner. Ask them to explain through examples how they chose the elements for each requested set, and also how they organized the results in order to identify equal sets. Focus on the use of precise mathematical language and clarity, specifically when describing the union, intersection, and complements of sets.

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Implementation Guide

Connecting to Key Concepts

Make explicit connections to key concepts:

- The results of an experiment are called **outcomes**. The set of all possible outcomes is called the **sample space**.

  In this task, student will recognize that rolling two dice is an experiment. They will also identify the sample space by listing all possible outcomes of the experiment.

- A **set** is a list or collection of items.

  Students will identify a variety of sets of elements from the sample space found from the experiment of rolling two dice.

- **Equal sets** are sets with all the same elements.

  Students will organize ten sets of outcomes into equal groups by identifying sets which contains the same elements as other sets.

- The **union** of sets A and B, denoted by $A \cup B$, is the set of elements that are in either A or B or both A and B.

  Students will determine and list the union of specific subsets of the sample space.

- An **intersection** is a set whose elements are each in both of two other sets. The intersection of sets A and B, denoted by $A \cap B$, is the set of elements that are in both A and B.

  Students will determine and list the intersection of specific subsets of the sample space.

- The **complement** of set A, denoted by $\overline{A}$, is the set of elements that are in some universal set, but not in A.

  Students will determine and list the complement of specific subsets of the sample space.

- An **empty set** is a set that has no elements. An empty set is denoted by $\varnothing$.

  Students will classify the results of the intersection of two sets as an empty set when they determine that the intersection of the sets does not have any elements. They will denote the empty set by the symbol $\varnothing$.

Extending the Task

- To extend the task, ask students to work with a partner and create their own experiment involving a different scenario. Provide ideas such as spinning a colored spinner, flipping four coins, or even combining scenarios such as "rolling a die and then flipping a coin." Ask them to first identify the sample space for their experiment and then create a list of possible events relating to their experiment. Remind them to include unions, intersections, and complements as they determine their lists. Encourage them to use correct terminology as well as the correct symbol notation. Ask for volunteers to share their experiments and present their list of events and results.

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- Another option for extending the task is to ask students to work through the original task with the experiment of rolling two dice, but this time have them create their own list of events, different from the ones already listed. Encourage them to come up with as many different events as possible. Once they have created their new lists of events, ask them to exchange lists with a partner, so each student will work through a different list of events. Encourage students to discuss their results with each other. Ask for volunteers to discuss their processes and justifications for their results.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by recognizing that the goal of the task was to organize sets of outcomes of an experiment into groups. I accomplished this goal by first determining the sample space of the experiment, and then I identified the outcomes of specific events. I persevered in solving the problem by using my knowledge of characteristics of subsets of a sample space, such as union, intersection, complement, and empty set, and I applied the corresponding symbols for these terms. Finally, I organized my results so that I could identify groups of equal sets.)
- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining the procedures and steps used when finding the sample space of the experiments, and then choosing the outcomes of each event. I justified my conclusions about which sets were equal by explaining how I found the unions, intersections, and complements of the specific sets. I also used correct terminology and symbol notation when explaining my results.)
- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** I looked for regularity in repeated reasoning as I first listed the sample space of the experiment. I then used repeated reasoning to organize the sample space into rows and columns. I also used the repeated process of identifying elements from the sample space which belong in the set for each desired event.)

Alternate Strategies or Solutions

Students may choose to create a visual representation of the elements in each event. They may draw a Venn diagram for each event in order to visualize the unions, intersections, and complements. Encourage students to be careful when placing elements in the appropriate sections of the Venn diagram, as well as when they identify the overlapping and non-overlapping parts.

Technology

No technology is needed for this task.

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Implementation Guide

Problem-Based Task 4.1.3: Tossing Coins

Common Core State Standard

S–CP.2 Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent.★

Task Overview

Focus

How can it be determined if three events are independent events? How can the probabilities of individual events be applied in order to determine the probability of the intersection of multiple events? In this task, students will first determine the individual probabilities for events of the experiments of tossing two coins and then three coins, and they will apply these probabilities in order to determine if the events are independent in each experiment.

This activity will provide practice with:

- identifying the sample space of an experiment
- analyzing the sample space of an experiment
- determining probabilities of individual events of an experiment
- applying the definition of the intersection of sets
- classifying events as independent or not independent

Introduction

This task should be used to explore or to apply the skill of understanding that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and using this characterization to determine if they are independent. Students will extend this skill to using three events. Students should already be familiar with calculating probabilities and applying the Addition Rule, which includes probabilities of intersections.

Begin by reading the problem and clarifying the meaning of *independent events* and *sample space*:

independent events	events in which the occurrence or non-occurrence of one event has no effect on the probability of the other event
sample space	the set of all possible outcomes of an experiment

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine whether events of experiments are independent events or not. They will first identify the sample spaces of each experiment, and then they will calculate the probability of each event. They will also apply their knowledge of independent events in order to draw conclusions about the independence of the events, as well as determining if the required conditions for events to be considered independent are satisfied.

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they make sense of the goal of the task by first recalling prior knowledge of finding sample spaces and individual probabilities of events in an experiment. They will examine and analyze the sample spaces for each experiment and use their analysis to determine the probabilities. When classifying events as independent or not, they will use their reasoning and knowledge of probability to conclude whether or not the events are independent. They will reason quantitatively as they perform the calculations necessary to determine individual probabilities, and they will recognize and apply symbol notation for the intersection of events when they multiply individual probabilities to find the value for the intersections.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  Students will first complete the task of identifying the sample space for the first experiment of tossing two coins. To accomplish this, they will use repeated reasoning as they determine a pattern of possible outcomes when two coins are tossed. They will also use the repeated process of calculating probabilities for each event and using these results when determining the probability of the intersection of the events. They will then classify the events as independent or not independent. Next, they will repeat these processes for the experiment of tossing three coins.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- not identifying all of the elements in the sample spaces for each experiment

  Remind students to first determine the number of outcomes that are in each sample space. For example, remind them that there are two possible outcomes when one coin is flipped, so the number of outcomes when two coins are flipped is $2 \cdot 2 = 4$.

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- mistaking the symbol for "intersection" as the symbol for "union"

  Have students write the definitions of "union" and "intersection" on their papers. Also remind them that $\cup$ is the symbol for "union", and $\cap$ is the symbol for intersection.

- incorrectly concluding that the three events for the experiment of tossing three coins are not independent events

  Remind students of the definition of independent events. Discuss the meaning of the three events, such that the outcome of getting heads when a coin is tossed once does not have any effect on getting heads when the coin is tossed a second time, or a third time. Also review the probability calculations in the task which verify that the three events are independent.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "What are some examples of real-life events that you hear about that are discussed in terms of probability?"(**Answer**: Answers will vary. Some examples include the probability of winning the lottery, the probability of rain in an area on a given day, or the probability of a candidate being elected to an office.)
- Before students begin the task, ask them, "What are some other terms used to describe probability? What other formats can probability be expressed in other than as a fraction?"(**Answer**: Some other terms used to describe probability are "chances", "odds", and "likelihood". Probability can also be expressed as a decimal, or a percent, or in words, such as "there is a one in four chance...".)
- As students begin working on the task and are finding the sample spaces for each experiment, ask them, "How can you determine the number of outcomes for the sample space of each experiment?"(**Answer**: For Experiment 1, the number of outcomes in the sample space is determined by multiplying the number of outcomes for one trial by the number of trials. There are two outcomes for tossing a coin, and there are two trials in this experiment: $2 \cdot 2 = 4$, so there are four outcomes in the sample space. For Experiment 2, there are three trials, so $2 \cdot 2 \cdot 2 = 8$, so there our eight outcomes in the sample space.)
- As students begin working on the task, if they are having difficulty determining the sample spaces for either experiment, ask them, "What are some ways the sample space can be determined? How can you visually represent the possible outcomes?"(**Answer**: A tree diagram, or other type of diagram, can be drawn to show the possible outcomes when two, and then three, coins are tossed.)

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- If students are having difficulty with applying the term "intersection" of the events, ask them, "Recall that the intersection of all three events means that all three must be possible in the context of the scenario. For the first experiment, is it possible to get a heads on the first toss, get heads on the second toss, and also have an outcome of exactly one heads? How could you change the third event so that the intersection would be possible?"(**Answer:** No, all three events happening at the same time is not possible for the first experiment. If the third event was changed to, "Get exactly two heads.", then it would be possible for all three events to happen at the same time.)
- If students are having difficulty applying the concept of "independent events", discuss examples with them. Ask them, "Suppose I pick a number that is written on a slip of paper out of a box. I record the number, and then I put the number back in the box. Then, I shake the box to mix the numbers up, and then I pick another number out of the box. Is picking a number the second time affected by picking a number the first time? Why or why not?"(**Answer:** Since the first number was put back in the box, picking a second number was not affected by picking the first number. These are independent events.)
- Ask students, "Suppose I draw two cards out of a standard deck of cards. After I draw the first card, I do not put it back in the deck. How does this affect the probability of any event involving drawing a second card from the deck?"(**Answer:** The sample space was originally 52 cards. When the first card was not put back in the deck, the sample space was reduced to 51. This will change the probability of future events because the denominator will now be 51 instead of 52 when calculating probabilities.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their thoughts about how to first determine the sample spaces for each experiment, and then how to calculate the probabilities for each event. Specifically discuss their reasons for choosing the values they use for the probabilities. Also ask for volunteers to explain the definition of the intersection of events, mathematically and in words which explain the context of the scenario. Ask them how they applied the symbol notation when choosing the values for the probability rules. Finally, ask them to defend and explain their conclusion about how one the events of one experiment were independent, and how the events of the other experiment were not independent. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process about creating the sample spaces, how they chose the values to use for calculating the probabilities of the individual events, and also how they classified the events as independent or not independent for each experiment. Ask them to explain how their results make sense in the context of the scenario. Focus on the use of precise mathematical language and clarity, specifically when describing the probabilities of an event, and the concept of independent events.

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Implementation Guide

Connecting to Key Concepts

Make explicit connections to key concepts:

- Two events are independent if and only if the probability of the intersection of the sets is equal to the product of their individual probabilities: $P(A \text{ and } B) = P(A) \bullet P(B)$.

  In this task, students will calculate the individual probabilities of individual events, as well as the intersection of two events.

- The set notation for the test for independent events is: $P(A \quad B) = P(A) \bullet P(B)$.

  Students will recognize the symbolic notation for the intersection of events and apply the definition when determining the probability.

- In a uniform probability model, all the outcomes of an experiment are assumed to be equally likely, and the probability of an event E, denoted $P(E)$, is given by:

  $$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}.$$

  Students will determine individual probabilities of events by applying the model for probability. They will examine the sample space and determine the number of possible outcomes for the event, and they will create a fraction with the number of outcomes of the event as the numerator, and the number of outcomes in the sample space as the denominator.

Extending the Task

- To extend the task, ask students to repeat the task with the experiment of tossing four coins. Ask them to find the sample space, and then create their own events for the experiment; events A, B, C, and D. Ask them to first create four events and mathematically prove that all four events are independent events, and then ask them to create four events which are not independent events. Remind them to use correct terminology as well as the correct symbol notation. Ask for volunteers to share their experiments and explain their events and results.

- Another option for extending the task is to provide a new experiment of spinning a spinner, numbered 1, 2, 3, and 4, two times. Remind them that each section on the spinner is the same size, so each number has an equally like chance of occurring. Ask them to first list the sample space for the experiment, and then ask them to create two events, events A and B. Ask them to find the individual probabilities of the events, the probability of the union of the events, the probability of the intersection of the events, and explain whether the events are independent or not. Ask for volunteers to share their experiments and explain their processes and justifications for their results.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the task is to determine whether events of experiments are independent events or not. I persevered in solving the problem by first finding the samples spaces for each experiment. Next, I determined the probability of each event in each experiment. Then, I found the intersection of the specified sets, and I used this information to see if the conditions for independent events were satisfied.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first made sense of the goal of the task by recalling prior knowledge of finding sample spaces and individual probabilities of events in an experiment. I also used abstract reasoning as I examined the outcomes in the sample spaces and used these outcomes when determining the probabilities for the events. I analyzed the results of the intersection of the events and made conclusions about whether or not the events were independent.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning when I performed the calculations necessary to find the individual probabilities, and I also applied symbol notation for the intersection of events when I multiplied individual probabilities to find the value for the intersections.)
- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** I looked for regularity in repeated reasoning as I first identified the sample space of the experiment by determining all possible outcomes when two coins are tossed. Then, I used the repeated process of calculating the probabilities of the events. I used these results to classify the events as independent or not independent. I repeated these same processes for the experiment of tossing three coins.)

Alternate Strategies or Solution

Students may choose to create a visual representation of the sample space for each experiment. They may choose to draw a tree diagram, or they may choose to draw their own diagrams of the coins with representations of the heads side and the tails side. They may use these diagrams to mark the elements of the sample space as they work through determining the individual probabilities. Encourage students to verify their work with their diagrams with the calculations using the probability models.

Technology

No technology is needed for this task.

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Implementation Guide

Problem-Based Task 4.2.1: Does She Have a Brother or a Sister?

Common Core State Standards

S–CP.3 Understand the conditional probability of *A* given *B* as *P*(*A* and *B*)/*P*(*B*), and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*.★

S–CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.*★

S–CP.6 Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpret the answer in terms of the model.★

Task Overview

Focus

How can a probability model be used to determine the likelihood of an event occurring, given that another event has already occurred? How is the conditional probability of an event calculated? In this task, students will apply the conditional probability model to determine the probability that a person has a sibling that is a boy.

This activity will provide practice with:

- interpreting verbal information and restating it using probability terminology
- identifying the sample space of an experiment
- analyzing the sample space of an experiment as it applies to the scenario
- determining conditional probabilities
- applying the conditional probability model
- justifying results as they apply to the context of the scenario

Introduction

This task should be used to explore or to apply the skills of finding the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpreting the answer in terms of the model, and the skill of recognizing and explaining the concepts of conditional probability and independence in everyday language and everyday situations. Students should already be familiar with identifying sample spaces and calculating simple probabilities.

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Begin by reading the problem and clarifying the meaning of *conditional probability* and *parameter*.

conditional probability of *B* given *A*	the probability that event *B* occurs, given that event *A* has already occurred.
parameter	a descriptive and measurable characteristic of a population

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments about their choices for the elements they choose for the sample space of the experiment, as well as their choices for components they use in the conditional probability model. They will also justify their conclusions about the probability that Amy's sibling is a boy by explaining their sequential processes of altering the sample space, as a result of the given parameters. When presenting their arguments to others, they will apply their knowledge about conditional probability and justify their conclusions by presenting their results and interpreting them in the context of the scenario.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled by using a conditional probability model. They will interpret the given verbal information and restate it using conditional probability notation. They will apply their knowledge of probability in the context of the given parameters. They will draw conclusions about the probability that Amy's sibling is a boy by analyzing and revising the sample space, as well as connecting the numerical probability and the meaning in context of the given scenario.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure and procedures involved in translating given information and restating it in probability terminology. They will also apply the structure of identifying the appropriate sample spaces given the parameters of the events. They will also recognize the structure of the notation of conditional probability and apply it for each step of the task.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- thinking that $P(A|B)$ represents the probability of A occurring and then B occuring

  Remind students of the format of conditional probability, such that $P(A|B)$ represents the probability of A occurring given that B has already occurred. Similarly, $P(B|A)$ represents the probability of B occurring given that A has already occurred.

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- not identifying all of the elements in the sample space

  Remind students to first determine the number of outcomes that are possible for the given scenario. Discuss with them that there are two outcomes possible, boy and girl, and there are two "trials". Therefore, there are 2 • 2 = 4 possible outcomes for Amy having a sibling.

- incorrectly reducing the sample space given each parameter

  Remind students to carefully consider each new parameter and apply it to the original sample space of four outcomes. Discuss with them how the sample space relates to the context of the scenario for each step of the task.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, review the concept of independent events with them. Ask them, "What does it mean if two events are independent?"(**Answer**: Two events are independent if the probability of one event occurring has no effect on the probability of the other event occurring.)
- Before students begin the task, discuss conditional probability statements. Ask them, "What are some examples of statements you might hear that indicate conditional probability?" (**Answer**: Answers will vary. Some examples may include, "What is the probability that you will get randomly selected from a class of 30, given that two of your classmates have already been chosen?" Or, "What are your chances of drawing two consecutive kings from a deck of cards, if the first king is not replaced in the deck?")
- Before students begin the task, ask them to make predictions about the probability that Amy's sibling is a boy, given the parameters. Ask them to assume that the probability of having a boy or a girl is equal, at 50% each. Ask them, "What do you predict the probability is that Amy's sibling is a boy, without doing any calculations? Explain your prediction."(**Answer**: Some students may predict that the probability is $\frac{1}{2}$, based solely on the fact that a sibling can either be a boy or a girl. Some students may predict that the probability is $\frac{1}{4}$, based on the fact that they know there will be four outcomes in the sample space.)
- As students begin working on the task and restate the verbal parameters into probability notation, ask them, "What does it mean that nothing is known about the age of Amy's sibling, in the context of the problem regarding birth order?"(**Answer:** This means that Amy's sibling could be older than her, which means he/she was born before Amy, or the sibling could be younger, which means that Amy was born first.)

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- As students begin working on the task and are identifying the sample space ask them, "How can you determine the number of outcomes for the sample space that shows all possible gender outcomes for the birth of two children?"(**Answer**: The number of outcomes in the sample space is determined by multiplying the number of outcomes for one trial by the number of trials. There are two possible outcomes for the gender, boy or girl. There are two children total, which represents two births. So, there are four possible gender outcomes for the birth or two children ($2 \bullet 2 = 4$).)
- As students are working on the task, "Why is it necessary to eliminate one of the possible outcomes from the original sample space?"(**Answer:** Since Amy is a girl, the possible outcome in the sample space of "Boy, Boy" has to be eliminated, because it is not possible for there to be two boys, when one of the children is already known to be a girl.)
- As students are completing the task, ask them to consider how the probability would be affected if one of the parameters was changed. Ask them, "If the second parameter was changed to, "It is known that Amy is younger than her sibling.", would this change the desired final probability in the problem?"(**Answer:** No, the probability that Amy's sibling is a boy would still be $\frac{1}{2}$, because the only thing that is affected is removing the possible outcome of "*gb*", instead of the outcome of "*bg*", from the sample space, since Amy would have been born second, if the parameter was changed.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their thoughts about how to first determine the sample space for the scenario of the birth of two children. Then, discuss how they interpreted the given parameters and rewrote them using conditional probability notation. Ask them to discuss their reasons for each sequential step of reducing the sample space, and also how they drew their conclusions about which outcome was eliminated each time. Finally, ask them to defend and explain their conclusion that the probability that Amy's sibling is a brother is one-half. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and procedure for identifying the sample space. Ask them how they translated the verbal information into conditional probability notation, based on the given scenario. Ask them to explain how their results make sense in the context of the scenario. Focus on the use of precise mathematical language and clarity, specifically when describing the sample space, as well as describing the events using conditional probability terminology and notation.

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Implementation Guide

Connecting to Key Concepts

Make explicit connections to key concepts:

- The **conditional probability of *B* given *A*** is the probability that event *B* occurs, given that event *A* has already occurred.

  In this task, students will be given the scenario where they will determine the probability that an event occurs after another event has already occurred. They will write these events using conditional probability notation.

- Using set notation, conditional probability is written like this: $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$.

  Students will examine the sample space for the individual events and determine the values that will be substituted into the conditional probability model.

- The "conditional probability of *B* given *A*" only has meaning if event *A* has occurred.

  Students will analyze the given events and recognize that the events have already occurred, so that they will be able to apply the conditional probability model to find the probability of the event, *B*.

Extending the Task

- To extend the task, ask students to repeat the task with the experiment of finding the probability that Amy is older than her sibling. Ask students to first make predictions about the resulting probability. Then, ask them to set up events similar to the ones in the original task. Ask them to carefully explain their steps for identifying the sample space, and sequentially reducing it in order to determine the probability that Amy is older than her sibling. Ask for volunteers to share and explain their results their results.

- Another option for extending the task is to ask students to work with a partner to create a list of real-world scenarios involving conditional probability. Ask them to consider examples from experiments they are familiar with (such as rolling dice, tossing coins, spinning a colored and/or numbered spinner, drawing cards, etc.) Also ask them to consider examples involving scenarios similar to the one in the original task. Ask them to create a possible experiment, with given parameters and events, for three of the examples on their lists. Ask for volunteers to share their experiments and explain their sample spaces and events. Discuss the validity of each experiment shared and brainstorm ideas about how the sample space and/or events could be revised if necessary.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining my choices for the elements I choose for the sample space of the experiment, given the parameters in the problem. I applied my knowledge of conditional probability and followed a sequential process in order to reach a conclusion about the probability that Amy's sibling is a boy. I also used correct terminology and symbol notation when explaining my results.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this scenario by recognizing that this scenario can be modeled by using a conditional probability model. I interpreted the given verbal information and restated it using conditional probability notation. I applied my knowledge of probability in the context of the given parameters. I made a conclusion about the probability that Amy's sibling is a boy by analyzing and revising the sample space, as well as connecting the numerical probability and the meaning in the context of the scenario.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** When solving this problem, I made use of structure by applying the structure and procedures involved in translating given information and restating it in probability terminology. I also applied the structure of identifying the appropriate sample spaces given the parameters of the events, and I recognized the structure of the notation of conditional probability and applied it for each step of the task.)

Alternate Strategies or Solutions

Students may choose to create a visual representation of the sample space the experiment. They may choose to draw a tree diagram, or they may choose to draw their own diagrams of the children with representations of the two genders. They may use these diagrams to mark the elements of the sample space as they work through determining the conditional probabilities. Encourage students to verify their work with their diagrams with the calculations using the probability models.

Technology

No technology is needed for this task.

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Implementation Guide

Problem-Based Task 4.2.3: Playing Cards

Common Core State Standard

S–CP.8 (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interpret the answer in terms of the model.★

Task Overview

Focus

How can a uniform probability model be used to compare probabilities? How is the Multiplication Rule applied in a probability model to determine probabilities of dependent events? In this task, students will apply the Multiplication Rule to calculate individual probabilities for drawing cards in two separate events in order to compare them and decide which event is more likely.

This activity will provide practice with:

- interpreting verbal information and restating it using probability terminology
- identifying the sample space of an experiment
- analyzing the sample space of an experiment as it applies to the scenario
- classifying events as dependent or independent
- applying the Multiplication Rule of probability
- calculating individual probabilities
- comparing probabilities in decimal format

Introduction

This task should be used to explore or to apply the skill of applying the general Multiplication Rule in a uniform probability model and interpreting the answer in terms of the model. Students should already be familiar with finding the probability of simple events and identifying when two events are independent or dependent.

Begin by reading the problem and clarifying the meaning of *dependent events*, *independent events*, and the *Multiplication Rule*.

dependent events	the outcome of one event affects the probability of the outcome of another event
independent events	the outcome of one event does not affect the probability of the outcome of another
Multiplication Rule	for two dependent events: $P(A \text{ and } B) = P(A) \bullet P(B|A)$

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine which set of cards is harder to draw, two eights, or two hearts. They will interpret this verbal information and recognize that the set of cards which has the lowest probability will be the harder to draw. They will apply their knowledge about independent and dependent events, and they will apply the Multiplication Rule and calculate the probabilities of each event. They will use these results to compare the probabilities and draw conclusions about which set of cards is harder to draw.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments about their choices for the use of the Multiplication Rule by explaining why the events are independent or dependent. They will use proper probability terminology and notation as they justify their probability calculations. When presenting their arguments to others, they will apply their knowledge about conditional probability and the Multiplication Rule, and they will justify their conclusions by presenting their results and interpreting them in the context of the scenario.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure and procedures involved in translating given information and restating it in conditional probability terminology. They will also apply the structure of the Multiplication Rule, as they justify each value used in the formula by explaining its meaning in the context of the scenario. They will use the same structure of the formula when calculating the probability for each event.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly calculating probabilities of simple events, such as $P(A)$ and $P(B)$

  Remind students to carefully consider and list the sample space when calculating probabilities of events. Remind them to consider if the sample space has changed given the occurrence of another event.

- incorrectly classifying two events as dependent or independent

  Have students write the definitions of dependent and independent events on their papers. Discuss some examples of each type of event, especially using the phrasing "with or without replacement".

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- incorrectly applying the Multiplication Rule

  Have students write the Multiplication Rule, in terms of probability notation, on their papers. Remind them to find the probability of each individual event before substituting those values into the formula.

- misinterpreting the event with the larger probability value as the event that is harder to achieve

  Once students have the probability values of the two events in decimal format, ask them to convert these values into percents. Remind them that the event with the lower percent value is less likely to happen, so in the context of the scenario, the event with the smaller percent is the event that is harder to achieve.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them to make predictions on which event they think is harder – drawing two eights from a deck of card, or drawing two hearts. Ask them, "Which set of cards do you think is harder to draw from a deck of cards – two eights, or two hearts? Explain your prediction." (**Answer**: Some students may realize that there are more hearts in a deck of cards than there are eights, and they may conclude that it will be harder to draw two eights. Some students may mistakenly think that it will be harder to draw two hearts because there are more of them than eights in a deck of cards. Assure students that their probability calculations will verify their predictions, one way or the other.)
- Before students begin the task, review the difference between independent and dependent events and commonly used terminology they might see. Ask them, "If a problem has the term, "with replacement", or "without replacement", does that mean that the events in the problem are independent or dependent events? Explain." (**Answer**: If a problem uses the term "with replacement", then after the first event occurs, the object, or item, that was chosen is placed back into the sample space. Therefore, the probability of the second event occurring has the same sample space as the first event, so the first event did not affect the second event. These are independent events. However, if the object or item is not placed back into the sample space, then the sample space is reduced, so the probability of the second event is affected. These are dependent events.)
- As students begin working on the task, ask them, "What are some examples of sets of two cards that would have equal probability of being drawn?" (**Answer**: Answers will vary. Some examples include: any two numerical cards, such as "two 4s and two 10s"; any two face cards, such as "two kings and two jacks"; any two of a suit, such as "two clubs and two diamonds", etc.)

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- If students are having difficulty understanding why the number of outcomes in the sample space when the second card is drawn is 51, ask them, "Since you are drawing two cards, what is the original sample space before drawing any cards? What happens to the sample space when one card is drawn and not put back into the deck?" (**Answer:** The original sample space is 52 cards. When the first card is drawn and not replaced, the sample space has now been reduced by one, so the new sample space is 51 cards.)
- As students are working through the calculations with the Multiplication Rule, ask them, "Why might it be helpful to state the result of the multiplication of the fractions in simplest terms?" (**Answer:** It might be helpful to reduce the fractions if possible if a calculator is not provided. The reduced fractions might help when comparing values.)
- If students are having difficulty comparing the decimal representations of the probabilities, ask, "What other formats can the probabilities be represented with, other than decimals? Which of these formats would be more practical for these probabilities? Explain your choice." (**Answer:** The probabilities can also be represented as percents or fractions. The probability that two eights are drawn is represented as 0.0045, which can be written as 0.45%, and as $\frac{45}{10{,}000}$ or $\frac{9}{2{,}000}$. The probability that two hearts are drawn is 0.059, which can be written as 5.9%, and as $\frac{59}{1{,}000}$. It is more practical to use the percent representations for these probabilities, because the fractional representations are not easily recognizable fractions.)
- Ask students, "If this same task were performed with the probability of three eights being drawn compared to the probability of three hearts being drawn, how would the sample space be affected, and which part of the fraction in the formula would change as a result of the sample space?" (**Answer:** At the beginning of the card game, the sample space is 52 cards. After one card is drawn, the sample space is 51 cards. If a third card were drawn, the sample space would then be 50 cards. The denominator in the fraction would change as a result of the sample space.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask for volunteers to discuss their initial predictions about which set of cards would be harder to draw. Specifically, ask them to explain their reasons, based on what they know about the sample space of a deck of cards. Then, discuss how they interpreted the given verbal information and translated it into conditional probability notation. Ask them to discuss their reasons for each value that they used when applying the Multiplication Rule to find the probability of the event of drawing two hearts, and then the event of drawing two eights. Finally, ask them to defend and explain their conclusion that it is harder to draw two eights than it is to draw two hearts. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and procedure for each step in the task. Ask them how they translated the verbal information into conditional probability notation, based on the given scenario. Ask them to explain how they applied the Multiplication Rule, and ask them to justify their results based on their calculations. Focus on the use of precise mathematical language and clarity, specifically when classifying the events as dependent or independent, as well as describing the probability of the events using conditional probability terminology and notation.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Before the probability of both events occurring can be determined, it is necessary to decide if the events are dependent or if they are independent of each other.

  In this task, students will first determine whether or not the events are dependent or independent, and then they will use the appropriate format of the Multiplication rule to find the probability of both events.

- If two events are dependent, the outcome of one event affects the probability of the other event and conditional probability must be applied.

  Students will conclude that once a card is drawn and not replaced, then the outcome of the second card that is drawn is affected by the first card being drawn. Therefore, the events are dependent.

- If events A and B are *dependent*, then the probability that both events occur is $P(A \text{ and } B) = P(A) \bullet P(B|A) = P(B) \bullet P(A|B)$. This is called the Multiplication Rule.

  Since the events in the task are dependent, students will apply the Multiplication Rule and determine the probability of each event in order to compare them so they can draw conclusions about which one is less likely to occur.

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Extending the Task

- To extend the task, ask students to create a new card game, but this time a player will draw three cards from the deck, without replacing the cards each time. Ask them to compare the probability for the event of drawing three clubs to the probability for the event of drawing three kings. Ask them to calculate the probabilities and state the comparison in percent format. Ask them to carefully explain their steps for identifying the sample space for each event, as well as the steps for determining each probability. Ask for volunteers to share and explain their results their results.
- Another option for extending the task is to ask students to work with a partner to create a new type of game involving choosing colored marbles out of a bag, or choosing names out of a hat, etc. Ask them to create two dependent events and compare the probabilities to determine which one is more likely to occur. Ask for volunteers to share their experiments and explain their processes for determining the probabilities and results. Discuss the validity of each experiment shared and brainstorm ideas about how parts of the experiment could be revised, if necessary.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the task is to determine which set of cards is harder to draw, two eights, or two hearts. I interpreted the verbal information given in the problem and recognized that the set of cards which has the lowest probability will be the harder to draw. I persevered in solving the problem by classifying the events as dependent, and then applying the Multiplication Rule in order to calculate the probabilities of each event. Then I used these results to compare the probabilities, and I drew conclusions about which set of cards is harder to draw.)
- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining my choices for classifying the events in the task as dependent, and then using the Multiplication Rule to calculate the probabilities of each event. I also explained how I translated the given verbal information into conditional probability terminology. In order to justify my conclusion about which set of cards was harder to draw, I presented my results and explained the values of the probabilities of each event, thus showing which one was greater in value.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** When solving this problem, I made use of structure by applying the structure and procedures involved in translating given information and restating it in conditional probability terminology. I also applied the structure of the Multiplication Rule and justified each value used in the formula by explaining its meaning in the context of the scenario. I used this same structure of the formula when calculating the probability for each event.)

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Alternate Strategies or Solutions

Students may choose to create a visual representation of the sample space for each event. They may choose to draw a tree diagram, or a Venn diagram, in order to better understand the possible outcomes for the events. They may use these diagrams to mark the elements of the sample space as they work through the steps in the task, specifically choosing the values to use in the formula for the Multiplication Rule. Encourage students to verify their work with their diagrams with the calculations using the Multiplication Rule probability model.

Technology

Scientific calculators may be used for converting probability represented in fractional format to decimal format.

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Implementation Guide

Problem-Based Task 4.3.1: On the Court

Common Core State Standard

S–CP.9 (+) Use permutations and combinations to compute probabilities of compound events and solve problems.★

Task Overview

Focus

How can permutations or combinations be used to solve problems involving the number of possible outcomes of a real-world scenario? What is the difference between a permutation and a combination? In this task, students will use permutations to determine how many different ways a group of members of a basketball team can be positioned on the court.

This activity will provide practice with:

- analyzing verbal information to make sense of the context of a problem
- differentiating between permutations and combinations
- identifying the number of positions in an arrangement
- calculating a product
- interpreting a product in the context of the scenario

Introduction

This task should be used to explore or to apply the skill of using permutations and combinations to compute probabilities of compound events and solve problems. Students should already be familiar with finding products and simplifying fractions with common factors in the numerator and denominator.

Begin by reading the problem and clarifying the meaning of *combination*, *factorial*, and *permutation*.

combination a subset of a group of objects taken from a larger group of objects; the order of the objects does not matter, and objects may be repeated. A combination of size r from a group of n objects can be represented using the notation ${}_nC_r$, where ${}_nC_r = \frac{n!}{(n-r)!r!}$.

factorial the product of an integer and all preceding positive integers, represented using a ! symbol; $n! = n \bullet (n-1) \bullet (n-2) \bullet \cdots \bullet 1$. For example, $5! = 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1$. By definition, $0! = 1$.

permutation a selection of objects where the order matters and is found either using n^r, if repetitions are allowed, or by using ${}_nP_r = \frac{n!}{(n-r)!}$, where n is the number of objects to select from and r is the number of objects being selected and ordered

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine how many different ways that members of a basketball can be positioned on the court. They will first determine that the problem requires the use of permutations instead of combinations, and then they will apply their knowledge about permutations to identify how many players can possibly be arranged in each of the given positions. When performing their calculations, they will also explain whether or not repetitions are allowed in the context of the scenario.

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they make sense of the goal of the task by applying their knowledge about the difference between permutations and combinations, as well as their knowledge about when repetitions are allowed or not. They will reason through the process of determining how many players, out of 12, can be placed in the 5 given positions on the court. They will reason quantitatively as they perform the calculations necessary to determine the number of ways the team members can be positioned on the court.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure and procedures involved in calculating a permutation. They will organize the positions for each player and sequentially determine how many players can possibly be chosen for each position. They will observe and apply a pattern in order to perform the correct calculations.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- confusing permutations and combinations

  Have students write the definitions of permutations and combinations on their papers. Emphasize the terminology of "order matters" when permutations are involved, and that the order of the objects does not matter with combinations. Discuss several examples of each type.

- forgetting to differentiate between permutations with repetition and permutations without repetition

  Remind students to consider the context of the scenario. Discuss with them how a person cannot be in two positions at one time, so in other words, "people cannot repeat". Discuss examples of scenarios with and without repetition.

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- incorrectly representing the number of positions when performing calculations

  Remind students that there are five positions on the basketball team, so there must be five blanks represented when choosing the number of players that can be placed in each position.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them to think of examples involving people where order does NOT matter when choosing a group. Ask them, "In this task, the arrangement of the people in the group matters because the positions are unique, so order matters. What are some examples of scenarios involving selecting a group of people where order does not matter?" (**Answer**: Answers will vary. Some examples include choosing 4 students to help the teacher pass out papers to the class, choosing six lottery numbers, etc.)
- As students are beginning the task, ask them, "Why are permutations being counted in this scenario instead of combinations?" (**Answer:** Since there are 5 unique positions on the basketball team, the arrangement of the players in the group matters. Therefore, the order matters, and permutations are used when the order matters in a problem.)
- If students are having difficulty understanding why repetitions are not allowed in this task, discuss an example of a scenario where repetitions are allowed. Ask them, "If you are taking a multiple choice test, with options a, b, c, and d, and you wanted to know how many different possible combinations you could get for a 5-problem test, would repetitions be allowed? Explain." (**Answer**: Since this is a multiple-choice test, all four options (a, b, c, and d) can be used for each problem. So, for problem 1, there are four possible outcomes. For problems 2, 3, 4, and 5, there are also four possible outcomes. There would be a total of 4 • 4 • 4 • 4 • 4 = 1,024 possible combinations of answers for a 5-problem test.)
- To ensure understanding of why this task does not allow repetitions, ask students, "In the context of the scenario of this task, why are repetitions not allowed?" (**Answer:** Once a player has been chosen for a position, that player cannot be chosen again, since he cannot play two positions at the same time. Therefore, repeating players is not allowed.)
- As students are working through the task, ask them, "Why would representing this scenario by drawing a tree diagram not be a very practical method?" (**Answer**: A tree diagram would not be very practical because the numbers are so large, and the diagram would take up too much room because of all the different outcomes for each branch.)
- As students are working through the task, ask them, "What is another method that can be used to solve this problem, other than the one presented in this task?" (**Answer:** The permutation formula, ${}_nP_r = \frac{n!}{(n-r)!}$, could be used, with the values of 12 for n, and 5 for r.)

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- As students are completing the task, ask them about the use of graphing calculators when working through problems involving larger values. Ask them, "If a problem asked you to find how many different ways a group of 11 players can be chosen for positions on a football team of 55 players (assuming every player could play any position), why would it be difficult to work through the problem without the use of a calculator? What would the calculations look like?" (**Answer:** If a calculator was not available for this problem, the calculations would be: 55 • 54 • 53 • 52 • 51 • 50 • 49 • 48 • 47 • 46 • 45. This would be a difficult calculation to perform without the use of some type of calculator.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their reasoning for their choice of using permutations instead of combinations in this task. Specifically, ask them to explain how the context of the scenario affected the order of the arrangement, as well as why or why not repetitions were allowed. Then, discuss how they determined the number of players that could possibly be placed in each of the five positions. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and procedure for each step in the task. If some students chose to use the permutation formula, ${}_nP_r = \frac{n!}{(n-r)!}$, ask them about any difficulty they encountered when working through the formula. Also discuss if they think the method presented in the task is more efficient than using the formula to get the final result. Focus on the use of precise mathematical language and clarity, specifically when discussing permutations versus combinations, and the use of terms such as "order matters" and "repetitions".

Connecting to Key Concepts

Make explicit connections to key concepts:

- A **permutation** is an arrangement of some or all objects in a group where the order of the objects matters and the objects may or may not be repeated. For example, the permutations "a, b, c" and "a, c, b" count as two different permutations because the order of the letters is different.

  In this task, students will distinguish between the use of permutations and combinations. They will conclude that this scenario involves permutations, because the order of the objects matters.

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- To find the permutations of a group of objects, list the different ways the objects (or a subset of the objects) can be arranged.

  Students will create a chart which lists, and explains, the numbers of players that can be in each position.

- To calculate the number of permutations, multiply the number of choices for each position. For example, if there are n choices for the first position, m for the second, and l for the third, the number of permutations is $n \bullet m \bullet l$.

  Students will multiply the number of players that can be chosen for the first position (12), by the number of players that can be chosen by the second position (11), and so on. When all five positions are filled, they will calculate the product of $12 \bullet 11 \bullet 10 \bullet 9 \bullet 8$, for a result of 95,040.

- When determining the number of arrangements (permutations), keep in mind whether the same choice can be in more than one position, or if a choice can be repeated. Suppose there are n unique objects, and r of them are chosen. To find the total number of ways the n objects can be placed in a group of size r, if an object can be repeated, find the product: $n \bullet n \bullet n \bullet \cdots \bullet n = nr$. If an object cannot be repeated, then the number of permutations of n objects in r positions is $n \bullet (n-1) \bullet (n-2) \bullet \cdots \bullet (n-r+1)$. Since the number of objects cannot be repeated students will use the formula $n \bullet (n-1) \bullet (n-2) \bullet \cdots \bullet (n-r+1)$, using the values of 12 for n, and 5 for r.

Extending the Task

- To extend the task, ask students to create a new scenario that can be counted using combinations instead of permutations (rolling dice is a possible answer). Ask them to set up their scenario similar to the one in the original task. Remind them to show all calculations, as well as written explanations about why the scenario uses combinations, and why the numbers were chosen for each part of the calculations. Have students exchange their scenario with a partner, and ask each person to review and provide feedback on the other person's scenario. Ask for volunteers to share their scenarios and explain their results.

- Another option for extending the task is to ask students to work together with a partner and discuss how the original task could be altered to allow for repetitions. Have students determine which parameters would need to change, and how the context of the scenario would need to change. Ask for volunteers to share their results for altering the task. As a group, discuss ideas and brainstorm various scenarios which would allow repetitions.

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Implementation Guide

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the task is to determine the number of ways that 5 out of 12 players can be positioned on a basketball court. I realized that the scenario involved permutations instead of combinations, because the order matters. I also realized that players cannot be in two positions at the same time, so repetitions are not allowed. I persevered in solving the problem by deciding how many players could go in each consecutive position, based on the fact that there was one less player each time.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first made sense of the goal of the task by applying my knowledge about the difference between permutations and combinations, as well as my knowledge about when repetitions are allowed or not. I reasoned through the process of determining how many players, out of 12, could be placed in the 5 given positions on the court.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning when I performed the calculations necessary to determine the number of ways the team members could be positioned on the court.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** When solving this problem, I made use of structure by applying the structure and procedures involved in calculating a permutation. I organized the positions for each player and sequentially determined how many players could possibly be chosen for each position. I noticed a pattern when examining each consecutive position, and I applied this pattern in order to perform the correct calculations.)

Alternate Strategies or Solutions

- Students may choose to use the permutation formula, $_nP_r = \frac{n!}{(n-r)!}$, in order to solve this problem. They may recognize that the variable n represents the total number of objects to select from, which is 12 in this task, and r represents the number of positions, which is 5 in this task. Once they substitute the values in the formula for n and r, they will apply their knowledge of factorials, and then perform the calculations. Remind students to be careful when substituting in values and reducing factors in the numerator and denominator in the formula. Encourage students to check their results with the results found when the goal of the task is achieved with the method used in the task. Some students may use a calculator which has a factorial key, in order to more efficiently find the value of 12! and 7!.

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- Some students may choose to draw pictorial representations of 12 players and assign names to them. Then they may create a visual representation of the five positions in order to better understand how many players can be placed in each position. Remind them that once a player is in a position, he cannot be placed in another position.

Technology

Students may use scientific calculators in order to find the product of their calculations in the task, or they may use graphing calculators, which have the factorial key.

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Problem-Based Task 4.4.1: Two of a Kind

Common Core State Standard

S–MD.6 (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).★

Task Overview

Focus

How can probabilities be used to determine if a card game is fair? What aspects of a game can be changed in order to ensure fairness? In this task, students will determine the probabilities of events that correspond to rules of a game, and they will draw conclusions about the fairness of the game based on these probabilities.

This activity will provide practice with:

- interpreting the rules of a game based on verbal information
- identifying a sample space
- calculating probabilities of events
- comparing and examining probabilities of events
- analyzing a scenario and drawing conclusions based on results
- making decisions about the fairness of a game

Introduction

This task should be used to explore or to apply the skill of using probabilities to make fair decisions. Students should already be familiar with finding simple and compound probabilities.

Begin by reading the problem and clarifying the meaning of the terms *fair*, *likelihood*, and *random number generator*.

fair	each outcome in a situation is equally likely
likelihood	the chances of an outcome occurring
random number generator	a tool to select a number without following a pattern, where the probability of any number in the set being generated is equal

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine whether or not the card game Shelby created is a fair game. They will first examine the given parameters and rules of the game, then they will calculate the probabilities of each player winning the game, based on the rules. They will also analyze the probabilities to draw conclusions about the fairness of the game, and they will make decisions about creating additional rules in order to improve the fairness of the game.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments about the choices they make in regards to the fairness of the card game that Shelby created. They will justify their choices by explaining their probability calculations for each player. They will defend their results and draw conclusions about whether each player has an equally likely chance of winning the game. Finally, they will justify their decisions about whether or not the rules of the game need to be improved, based on their analysis of the current rules.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  Students will first recognize that in order to make a determination about the fairness of the current rules of the game, they will need to calculate probabilities to conclude if each player has an equally likely chance of winning the game. They will use repeated reasoning as they follow the same procedure when calculating each player's probability. They will also use repeated reasoning as they summarize their findings and extend the procedure to determining the probability of sequential players winning the game.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly calculating probabilities

  Remind students to first identify the sample space of the card game. Also remind them to consider the sample space once one card is drawn and removed from the stack of cards.

- incorrectly assessing the impact of the order of the players in regards to their chances of winning the game

  Have students write the object of the game down on their papers. Remind them to consider how a student can win if they do not get the chance to go first. Discuss the advantages a player has if they are chosen to go first.

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- incorrectly concluding that the rules of the game should not be improved

  Remind students to carefully examine the chances for each player to win. Remind them that if the first player wins, then no other player has a chance of winning. Discuss with them how it is necessary to have a fair way of choosing the order of the players so that each has an equally likely chance of winning.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them to make predictions on the fairness of the game before they begin working through the steps and the probability calculations. Ask them, "After reading the rules and set-up of the game, do you think the game is fair? Do you think any of the rules will need to be changed? Explain your thoughts." (**Answer:** Answers will vary. Some students may predict that the game is fair because each player follows the same procedure each time. They may overlook the fact that the person who gets to go first has an advantage. Some students may observe that there is no current system for selecting the order of the players, and therefore some rules will need to be made such that the player who goes first is randomly selected.)
- Before students begin the task, ask them, "What does it mean that a game is fair? What are some circumstances that might cause a game to be considered unfair?" (**Answer**: A game is fair if each player has an equally likely chance of winning. If a player cheats and does not follow the rules, that could cause the game to be unfair. Also, if a player has more experience playing the game, that could also cause the game to be unfair.)
- Before students begin the task, ask them, "What are some fair ways to choose which player goes first in a game?" (**Answer**: Some possible answers include: rolling a fair die to see who rolls the highest number, choosing names out of a hat, and using a random number generator.)
- To ensure that students understand how the game is won, ask them, "What are the possible card combinations that a player could draw that would cause them to win the game?" (**Answer:** To win the game, a player has to draw both cards with the same value. Therefore, the possible winning combinations are: 1 and 1, 2 and 2, 3 and 3, 4 and 4, and 5 and 5.)
- As students are working through the task, ask them to explain how they calculated the probability of each player drawing two identical cards. Ask them, "How did you decide on the values for the numerator and the denominator in the probability calculation for each player?" (**Answer**: The original sample space has 10 cards, or 10 possible outcomes. After one card is drawn, then there is one less card in the deck, so the denominator is 9. The numerator is represented by the number of cards still left in the deck with the same value of the first card chosen. There is only one match for each card, so 1 is in the numerator. The probability of each student winning is $\frac{1}{9}$.)

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- Ask students, "Why is the rule that defines the selection of the cards considered a fair way to select a winner?(**Answer:** Selecting a pair of cards is fair since each value of the cards (1, 2, 3, 4, 5) in the game is represented exactly twice. Therefore, the probability of selecting a pair of equal value cards is the same for each player.)
- If students are having difficulty understanding the concept of random number generation, ask them, "Why would rolling a die to determine which player goes first be a method of random number generation?" (**Answer:** Each number on a die has the same probability of being rolled, which is $\frac{1}{6}$. Since each number is equally likely to be rolled, this is an example of random number generation.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their predictions about the fairness of the game. Discuss their reasons for their thoughts based on the rules of the game. Ask them to discuss their procedures for determining the probabilities that the first player, and then the second player wins. Discuss the concept of "equally likely", and discuss how the rules of the games do or do not lend to the card game being a fair game. Ask them to explain their assessment of their results of the probability, and ask them to explain how they decided on methods of improving the rules of the game to ensure fairness in how players are selected. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and procedure for calculating the probabilities of each player winning, as well as their assessment of the results of the calculations. Ask them how they applied their knowledge about fairness and probability to determine how the rules of the game could be improved. Focus on the use of precise mathematical language and clarity, specifically when describing the probability calculations, as well as when discussing a random number generator.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Events that are equally likely have the same probability.

  In this task, students will conclude that the way to determine if each player has an equally likely chance of winning the game, then each player's probability will be the same.

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- If a game is fair, each player has the same probability of winning.

  Students will calculate probabilities in order to draw conclusions about the fairness of the card game. They will analyze the probabilities to determine if each outcome, or each player's chance of winning the game, is equally likely.

- To maintain fairness, random numbers are often used in making selections.

  Students will assess the rules of the game and conclude that a random number generator is a method that can be used to ensure fairness of the order of the players in the game.

Extending the Task

- To extend the task, choose several groups of 5 students and have students create cards that are numbered 1, 1, 2, 2, 3, 3, 4, 4, 5, 5. Ask them to play Shelby's card game by first determining a method of randomly selecting the order of the 5 players. Ask them to play several rounds of the game, recording their results each time. After several rounds, ask each group of students to start over and choose a different method of randomly generating the order of the players. After another round, ask for groups to volunteer and discuss their experiences with the game. Discuss the concept of expected value and experimental value, and ask students to brainstorm ways in which they can make the game fairer, so that players have better chances of winning.
- Another option for extending the task is to ask students to work with a partner to create their own game, different from the one in the original task. Ask them to consider possibilities of games involving rolling dice, or spinning a colored and/or numbered spinner. Ask them to write down precise rules, as well as explanations of why the rules are fair, or what parameters are necessary to ensure a fair game. Ask for volunteers to explain their games and rules. Discuss the validity of each game and brainstorm ideas about how the rules of the game could be revised if necessary.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by recognizing that the goal of the task is to determine whether are not the card game Shelby created is a fair game. I persevered in solving the problem by first examining the given parameters and rules of the game, and then I calculated the probabilities of each player winning the game, based on the rules. I analyzed the results of the probabilities and drew conclusions about what additional rules needed to be created in order to improve the fairness of the game.)

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- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining my choices about the fairness of the card game that Shelby created. I justified these choices by explaining their probability calculations for each player. I also explained how I drew conclusions about whether each player has an equally likely chance of winning the game. Then I applied the results of my analysis to determine that the rules of the game do need to be improved.)
- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** I looked for regularity in repeated reasoning as I first recognized that in order to make a determination about the fairness of the current rules of the game, I needed to calculate probabilities to conclude if each player has an equally likely chance of winning the game. I used repeated reasoning by following the same procedure when calculating each player's probability. I extended this reasoning when calculating the probability of each sequential player.)

Alternate Strategies or Solutions

Students may choose to create a visual representation of the game by drawing the cards, and creating fictional players. They may also choose to use probability models in order to calculate the probabilities. Encourage students to verify their work with their drawings and calculations with the ones presented in the task.

Technology

No technology is needed for this task.

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Common Core State Standards

Mathematics II

Integrated Pathway

Teacher Resource

Unit 5

WALCH

INTEGRATED MATH

This program was developed and reviewed by experienced math educators who have both academic and professional backgrounds in mathematics. This ensures: freedom from mathematical errors, grade level appropriateness, freedom from bias, and freedom from unnecessary language complexity.

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Standards Correlations

Each lesson in this Integrated Pathway: Mathematics II program was written specifically to address the Common Core State Standards. Each lesson lists the standards covered in all the sub-lessons, and each sub-lesson lists the standards addressed in that particular section. In this section, you'll find a comprehensive list mapping the sub-lessons to the CCSS.

Guide to Common Core State Standards Annotation

As you use this program, you will come across a symbol included with the Common Core standards for some of the lessons and activities. These symbols are explained below.

Symbol: ⋆

Denotes: Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (⋆).

From http://www.walch.com/CCSS/00003

Symbol: (+)

Denotes: College and Career Readiness Standards

Advanced mathematics standards that are required in higher-level courses such as advanced statistics may also be included in lower-level courses. These additional standards are denoted by (+). According to the Common Core State Standards Initiative, "the evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6–8."

From http://www.walch.com/CCSS/00004

Connections to Future Courses

This section provides a map between topics introduced in each unit of this course and subsequent courses where each topic is revisited and built upon.

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CCSS INTEGRATED PATHWAY: MATHEMATICS II STANDARDS CORRELATIONS

Unit 5: Similarity, Right Triangle Trigonometry, and Proof				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Line Segments**			
	5.1.1	Midpoints and Other Points on Line Segments	G–GPE.6	U5-3
Lesson 2	**Investigating Properties of Dilations**			
	5.2.1	Investigating Properties of Parallelism and the Center	G–SRT.1a	U5-34
	5.2.2	Investigating Scale Factors	G–SRT.1b	U5-56
Lesson 3	**Defining and Applying Similarity**			
	5.3.1	Defining Similarity	G–SRT.2	U5-80
	5.3.2	Applying Similarity Using the Angle-Angle (AA) Criterion	G–SRT.3	U5-103
Lesson 4	**Proving Similarity**			
	5.4.1	Proving Triangle Similarity Using Side-Angle-Side (SAS) and Side-Side-Side (SSS) Similarity	G–SRT.4	U5-130
	5.4.2	Working with Ratio Segments	G–SRT.4	U5-147
	5.4.3	Proving the Pythagorean Theorem Using Similarity	G–SRT.4	U5-167
	5.4.4	Solving Problems Using Similarity and Congruence	G–SRT.5	U5-187
Lesson 5	**Proving Theorems About Lines and Angles**			
	5.5.1	Proving the Vertical Angles Theorem	G–CO.9	U5-222
	5.5.2	Proving Theorems About Angles in Parallel Lines Cut by a Transversal	G–CO.9	U5-251
Lesson 6	**Proving Theorems About Triangles**			
	5.6.1	Proving the Interior Angle Sum Theorem	G–CO.10	U5-291
	5.6.2	Proving Theorems About Isosceles Triangles	G–CO.10	U5-313
	5.6.3	Proving the Midsegment of a Triangle	G–CO.10	U5-334
	5.6.4	Proving Centers of Triangles	G–CO.10	U5-363

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Lesson	Sub-lesson number	Title	Standard(s)	Pages
Lesson 7	**Proving Theorems About Parallelograms**			
	5.7.1	Proving Properties of Parallelograms	G–CO.11	U5-411
	5.7.2	Proving Properties of Special Quadrilaterals	G–CO.11	U5-438
Lesson 8	**Exploring Trigonometric Ratios**			
	5.8.1	Defining Trigonometric Ratios	G–SRT.6	U5-481
	5.8.2	Exploring Sine and Cosine As Complements	G–SRT.7	U5-508
Lesson 9	**Applying Trigonometric Ratios**			
	5.9.1	Calculating Sine, Cosine, and Tangent	G–SRT.8★	U5-534
	5.9.2	Calculating Cosecant, Secant, and Cotangent	G–SRT.8★	U5-557
	5.9.3	Problem Solving with the Pythagorean Theorem and Trigonometry	G–SRT.8★	U5-576
	5.9.4	Proving the Fundamental Pythagorean Identity	F–TF.8	U5-596

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TOPICS FROM FUTURE COURSES: CCSS INTEGRATED PATHWAY MATHEMATICS II

Math II	Topic introduced	Course/unit where addressed	How addressed
Unit 5	Similarity, special angle pairs formed by parallel lines cut by a transversal and other angle pairs, the properties of triangles and special triangles, and parallelograms	Math III Unit 4B	using planes to create two-dimensional figures and slice three-dimensional objects
	Trigonometric ratios and proofs of trigonometric identities	Math III Unit 3	proving and applying the laws of sines and cosines, studying periodic phenomena, and modeling using trigonometric functions
		Math IV	graphing the sine and cosine functions and representing and operating with vectors; extended to complex trigonometric identities

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Conceptual Activities

Use these interactive open education and/or Desmos resources to build conceptual understanding of mathematical ideas. (*Note*: Activity links will be monitored and repaired or replaced as necessary.)

- Desmos. "Lines, Transversals, and Angles."

  http://www.walch.com/ca/01033

  In this activity, students explore the relationship of angles formed by a transversal and a system of two lines. In particular, students consider what happens when the two lines are parallel versus when they are not.

- Desmos. "Polygraph: Angle Relationships."

  http://www.walch.com/ca/01034

  This activity is designed to spark vocabulary-rich conversations about angle relationships. Key vocabulary terms that may appear in student questions include *parallel*, *transversal*, *adjacent*, *opposite*, *alternate interior*, *corresponding*, *alternate exterior*, *vertical*, and *right*.

- Desmos. "Polygraph: Figure It Out."

  http://www.walch.com/ca/01035

  This activity is intended as an introduction to geometric notation and vocabulary. Depending on prior knowledge, students could use the following to distinguish figures: points, lines, rays, segments, parallel, perpendicular, angles, congruence, midpoints, bisectors, betweenness, collinearity, and more.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. What is the midpoint between (1, 1) and (4, 10)?

   a. (–1.5, 4.5)
   b. (–1.5, 5.5)
   c. (2.5, 4.5)
   d. (2.5, 5.5)

2. A line segment has endpoints at (6, 3) and (12, 11). A point on the segment has a distance that is $\frac{1}{4}$ of the length of the segment from endpoint (6, 3). What is the point?

   a. (10.5, 9)
   b. (7.5, 5)
   c. (4.5, 1)
   d. (13.5, 13)

3. What is the point that is $\frac{1}{3}$ the distance from the endpoint (5, 6) of the segment with endpoints (5, 6) and (–4, –6)?

   a. (2, 2)
   b. (–1, –2)
   c. (0.5, 0)
   d. (1, 3)

4. One endpoint of a segment is (2, 5). The midpoint of the segment is (–2, 6.5). What is the second endpoint of this segment?

   a. (6, 3.5)
   b. (–6, 8)
   c. (0.5, 5.75)
   d. (0, 5.75)

5. Tito and Shara are playing a game. Tito's game piece is at the point (–10, –3). Shara's game piece is at the point (2, 9). What are the coordinates of the point midway between the two game pieces?

   a. (6, 3)
   b. (–6, –6)
   c. (–4, 3)
   d. (3, –4)

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Lesson 1: Line Segments

Instruction

SMP	
1 ✓	2 ✓
3	4 ✓
5	6
7	8

Common Core State Standard

G–GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

Essential Questions

1. How can the midpoint of a line segment be found without measuring?
2. How are the midpoint formula and the distance formula related?

WORDS TO KNOW

distance formula formula that states the distance between (x_1, y_1) and (x_2, y_2) is equal to $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

line segment a part of a line that is noted by two endpoints, (x_1, y_1) and (x_2, y_2)

midpoint a point on a line segment that divides the segment into two equal parts

midpoint formula formula that states the midpoint of a segment created by connecting (x_1, y_1) and (x_2, y_2) is given by the formula $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Recommended Resources

- Math Open Reference. "Midpoint of a Line Segment (Coordinate Geometry)."

  http://www.walch.com/rr/00128

  Users can adjust a line segment by dragging the endpoints around a coordinate plane to see how the midpoint changes. The equations for each set of endpoints are displayed off to the side of the plane and change as the points are dragged.

- Purplemath. "The Midpoint Formula."

  http://www.walch.com/rr/00129

  This site explains why finding the midpoint can be useful and includes worked examples. A widget on the second page allows users to enter problems, view the steps to the solution, and then view a graph of the solution.

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Lesson 5.1.1: Midpoints and Other Points on Line Segments

Warm-Up 5.1.1

Two cars leave the same parking lot in Centerville at noon. One travels due north, and the other travels directly east. Suppose the northbound car is traveling at 60 mph and the eastbound car is traveling at 50 mph.

1. Make a table that shows the changes over time in the following: each car's distance from the parking lot and the distance between each car.

2. Suppose the northbound car is traveling at 40 mph, and after 2 hours the cars are 100 miles apart. How fast is the eastbound car going?

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Instruction

Lesson 5.1.1: Midpoints and Other Points on Line Segments

Common Core State Standard

G–GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

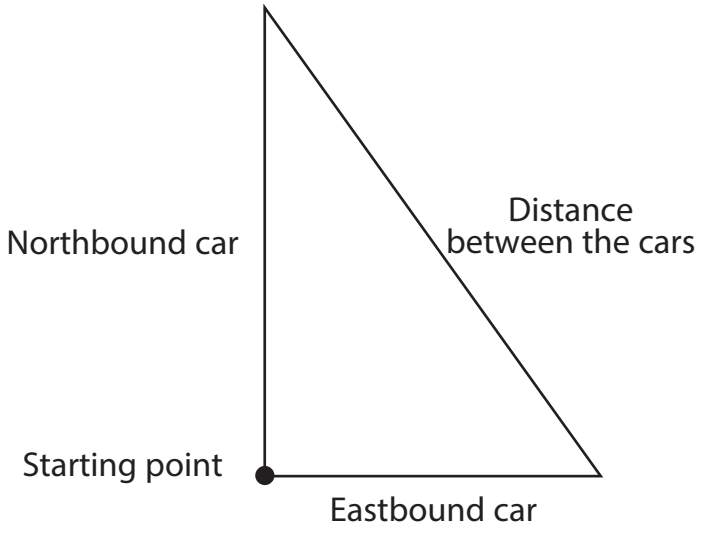
Warm-Up 5.1.1 Debrief

1. Make a table that shows the changes over time in the following: each car's distance from the parking lot and the distance between each car.

Time in hours	Northbound car's distance in miles	Eastbound car's distance in miles	Distance between cars in miles
1	60	50	78.1
2	120	100	156.2
3	180	150	234.3
4	240	200	312.4

Each car's distance from the starting point (the parking lot) is increasing by the amount of its rate of speed each hour.

It may be helpful to draw a diagram to explain this situation.



Notice a right triangle is created.

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Instruction

Use the Pythagorean Theorem to find the distance between the cars.

$a^2 + b^2 = c^2$

(Northbound car's distance)^2 + (Eastbound car's distance)^2 = (distance between the cars)^2

$(60)^2 + (50)^2 = (\text{distance between the cars})^2$

$3600 + 2500 = (\text{distance between the cars})^2$

$6100 = (\text{distance between the cars})^2$

$78.1 \approx$ distance between the cars

The distance between the cars is increasing by 78.1 miles each hour.

2. Suppose the northbound car is traveling at 40 mph, and after 2 hours the cars are 100 miles apart. How fast is the eastbound car going?

   The northbound car is traveling at 40 mph.

   After 2 hours, the northbound car has traveled 80 miles.

   Use the Pythagorean Theorem to determine how fast the eastbound car is going.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$80^2 + b^2 = 100^2$	Substitute the distance for the northbound car and the distance between the two cars.
$6400 + b^2 = 10{,}000$	Simplify.
$b^2 = 3600$	Subtract 6,400 from both sides.
$b = 60$	Take the square root of both sides.

   The eastbound car traveled 60 miles in 2 hours, or 30 miles in 1 hour. This is equal to 30 mph.

Connection to the Lesson

- Students will continue to develop their understanding of the Pythagorean Theorem and its relation to the distance formula.
- Students will find points on a line segment that partition the segment in a given ratio and use the distance formula to verify the accuracy of these points.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- plotting points on a coordinate plane (5.G.1)
- calculating distance in the coordinate plane (8.G.8)

Introduction

The use of the coordinate plane can be helpful with many real-world applications. Scenarios can be translated into equations, equations can be graphed, points can be found, and distances between points can be calculated; but what if you need to find a point on a line that is halfway between two points? Can this be easily done?

Key Concepts

- Lines continue infinitely in both directions. Their length cannot be measured.
- A **line segment** is a part of a line that is noted by two endpoints, (x_1, y_1) and (x_2, y_2).
- The length of a line segment can be found using the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)}$.
- The **midpoint** of a line segment is the point on the segment that divides it into two equal parts.
- Finding the midpoint of a line segment is like finding the average of the two endpoints.
- The **midpoint formula** is used to find the midpoint of a line segment. The formula is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- You can prove that the midpoint is halfway between the endpoints by calculating the distance from each endpoint to the midpoint.

Finding the Midpoint of a Line Segment in the Coordinate Plane

1. Determine the endpoints of the line segment, (x_1, y_1) and (x_2, y_2).
2. Substitute the values of (x_1, y_1) and (x_2, y_2) into the midpoint formula: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
3. Simplify.

SRB p. U5-2

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Instruction

- It is often helpful to plot the segment on a coordinate plane.
- Other points, such as a point that is one-fourth the distance from one endpoint of a segment, can be calculated in a similar way.

Finding the Point on a Line Segment with Any Given Ratio

1. Determine the endpoints of the line segment, (x_1, y_1) and (x_2, y_2).
2. Calculate the difference between the x-values: $|x_2 - x_1|$.
3. Multiply the difference by the given ratio, $\frac{m}{n}$.
4. Add the product to or subtract it from the x-value of the endpoint. This is the x-value of the point with the given ratio.
5. Calculate the difference between the y-values: $|y_2 - y_1|$.
6. Multiply the difference by the given ratio, $\frac{m}{n}$.
7. Add the product to or subtract it from the y-value of the endpoint. This is the y-value of the point with the given ratio.

Common Errors/Misconceptions

- misidentifying x_1, x_2 and y_1, y_2
- attempting to use the distance formula to locate the midpoint
- attempting to use the midpoint formula to find points that split the segment in a ratio other than ½
- using the wrong endpoint to locate the point given a ratio
- incorrectly adding or subtracting units from the identified endpoint

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Instruction

Guided Practice 5.1.1

Example 1

Calculate the midpoint of the line segment with endpoints (–2, 1) and (4, 10).

1. Determine the endpoints of the line segment.

   The endpoints of the segment are (–2, 1) and (4, 10).

2. Substitute the values of (x_1, y_1) and (x_2, y_2) into the midpoint formula.

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$ Midpoint formula

$$\left(\frac{(-2)+(4)}{2}, \frac{(1)+(10)}{2}\right)$$ Substitute (–2, 1) and (4, 10).

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Instruction

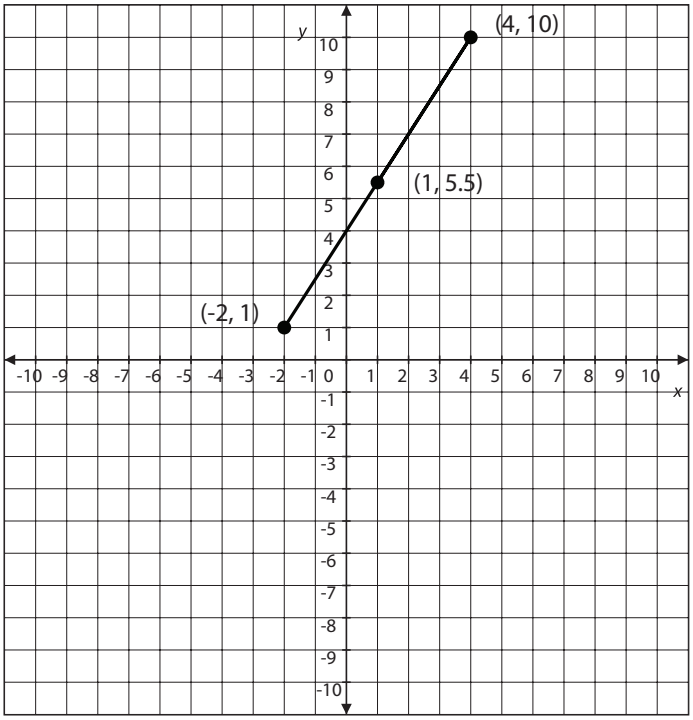
3. Simplify.

$$\left(\frac{(-2)+(4)}{2}, \frac{(1)+(10)}{2}\right)$$

$$\left(\frac{2}{2}, \frac{11}{2}\right)$$

(1, 5.5)

The midpoint of the segment with endpoints (–2, 1) and (4, 10) is (1, 5.5).



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Instruction

Example 2

Show mathematically that (1, 5.5) is the midpoint of the line segment with endpoints (–2, 1) and (4, 10).

1. Calculate the distance between the endpoint (–2, 1) and the midpoint (1, 5.5).

   Use the distance formula.

   $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance formula

   $= \sqrt{[(1) - (-2)]^2 + [(5.5) - (1)]^2}$ Substitute (–2, 1) and (1, 5.5).

   $= \sqrt{(3)^2 + (4.5)^2}$ Simplify as needed.

   $= \sqrt{9 + 20.25}$

   $= \sqrt{29.25}$

   The distance between (–2, 1) and (1, 5.5) is $\sqrt{29.25}$ units.

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Instruction

2. Calculate the distance between the endpoint (4, 10) and the midpoint (1, 5.5).

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance formula

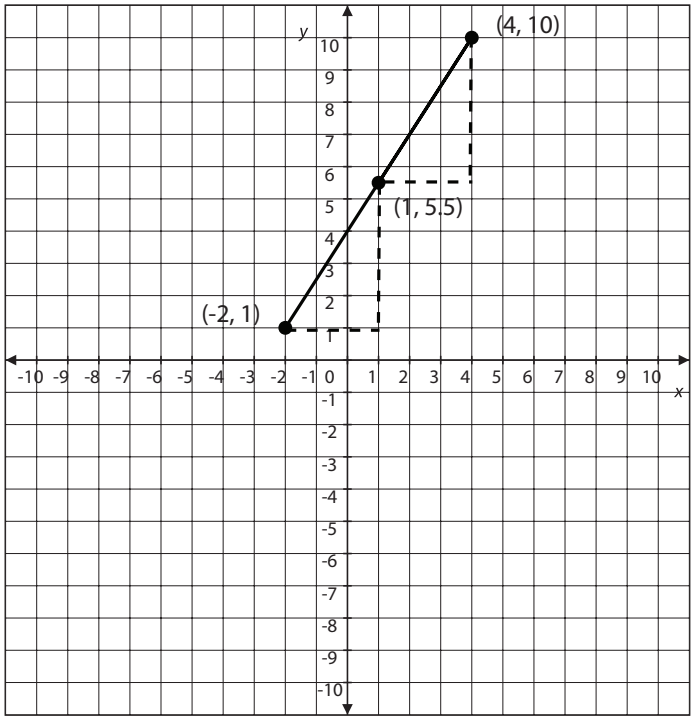
$= \sqrt{[(1) - (4)]^2 + [(5.5) - (10)]^2}$ Substitute (4, 10) and (1, 5.5).

$= \sqrt{(-3)^2 + (-4.5)^2}$ Simplify as needed.

$= \sqrt{9 + 20.25}$

$= \sqrt{29.25}$

The distance between (4, 10) and (1, 5.5) is $\sqrt{29.25}$ units.
The distance from each endpoint to the midpoint is the same, $\sqrt{29.25}$ units, proving that (1, 5.5) is the midpoint of the line segment.



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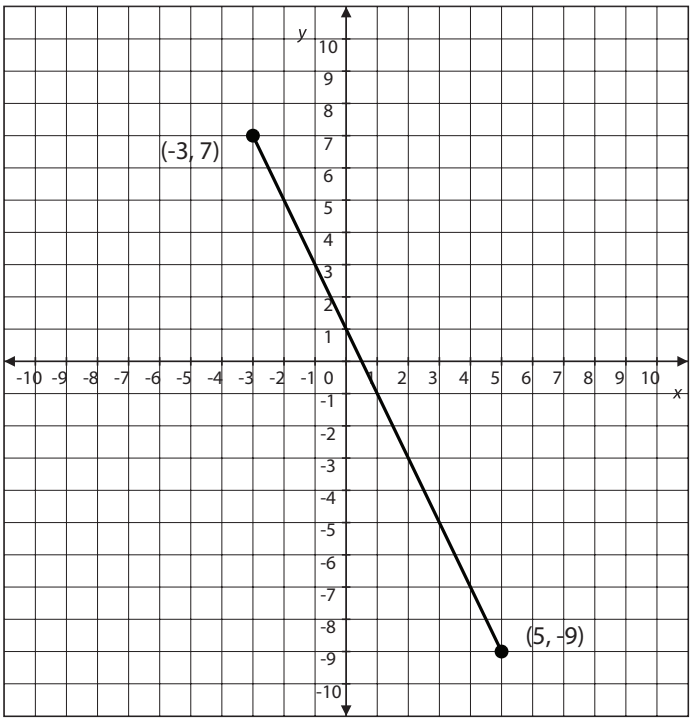
Instruction

Example 3

Determine the point that is $\frac{1}{4}$ the distance from the endpoint (–3, 7) of the segment with endpoints (–3, 7) and (5, –9).

1. Draw the segment (–3, 7) and (5, –9) on a coordinate plane.



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2. Calculate the difference between the *x*-values.

$\lvert x_2 - x_1 \rvert$	Distance between *x*-values
$= \lvert(-3) - (5)\rvert$	Substitute the *x*-values.
$= \lvert -8 \rvert$	Simplify.
$= 8$	

The difference between the *x*-values is 8 units.

3. Multiply the difference by the given ratio, $\frac{1}{4}$.

$$8 \bullet \frac{1}{4} = \frac{8}{4} = 2$$

4. The *x*-value is to the right of the original endpoint; therefore, add the product to the *x*-value of the endpoint. This is the *x*-value of the point with the given ratio.

$-3 + 2 = -1$

5. Calculate the difference between the *y*-values: $\lvert y_2 - y_1 \rvert$.

$\lvert y_2 - y_1 \rvert$	Distance between *y*-values
$= \lvert(7) - (-9)\rvert$	Substitute the *y*-values.
$= \lvert 16 \rvert$	Simplify.
$= 16$	

The difference between the *y*-values is 16 units.

6. Multiply the difference by the given ratio, $\frac{1}{4}$.

$$16 \bullet \frac{1}{4} = \frac{16}{4} = 4$$

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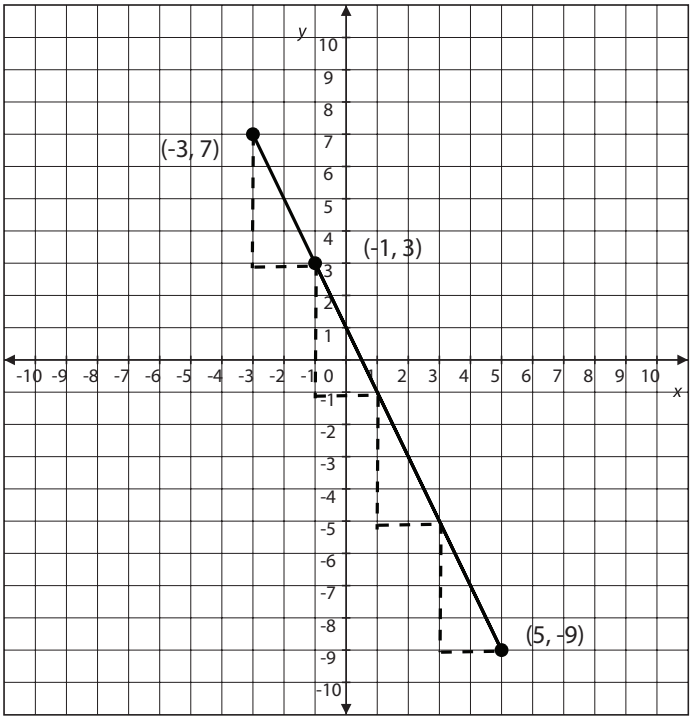
Instruction

7. The *y*-value is down from the original endpoint; therefore, subtract the product from the *y*-value of the endpoint. This is the *y*-value of the point with the given ratio.

   $7 - 4 = 3$

   The point that is $\frac{1}{4}$ the distance from the endpoint (–3, 7) of the segment with endpoints (–3, 7) and (5, –9) is (–1, 3).



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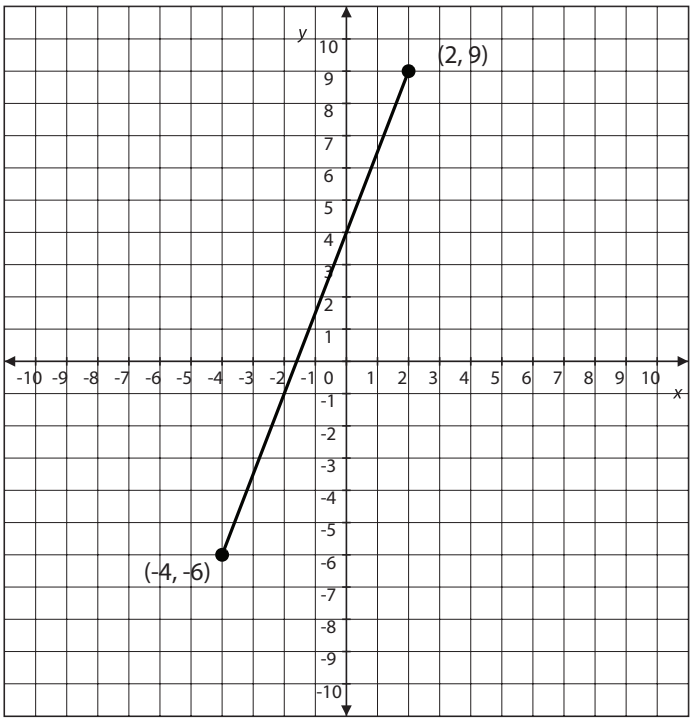
Instruction

Example 4

Determine the point that is $\frac{2}{3}$ the distance from the endpoint (2, 9) of the segment with endpoints (2, 9) and (–4, –6).

1. Draw the segment (2, 9) and (–4, –6) on a coordinate plane.



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Instruction

2. Calculate the difference between the x-values.

$\lvert x_2 - x_1 \rvert$	Distance between x-values
$\lvert (2) - (-4) \rvert$	Substitute the x-values.
$\lvert 6 \rvert$	Simplify.
6	

The difference between the x-values is 6 units.

3. Multiply the difference by the given ratio, $\frac{2}{3}$.

$$6 \bullet \frac{2}{3} = \frac{12}{3} = 4$$

4. Subtract the product from the x-value of the endpoint. This is the x-value of the point with the given ratio.

$2 - 4 = -2$

5. Calculate the difference between the y-values: $\lvert y_2 - y_1 \rvert$.

$\lvert y_2 - y_1 \rvert$	Distance between y-values
$\lvert (9) - (-6) \rvert$	Substitute the y-values.
$\lvert 15 \rvert$	Simplify.
15	

The difference between the y-values is 15 units.

6. Multiply the difference by the given ratio, $\frac{2}{3}$.

$$15 \bullet \frac{2}{3} = \frac{30}{3} = 10$$

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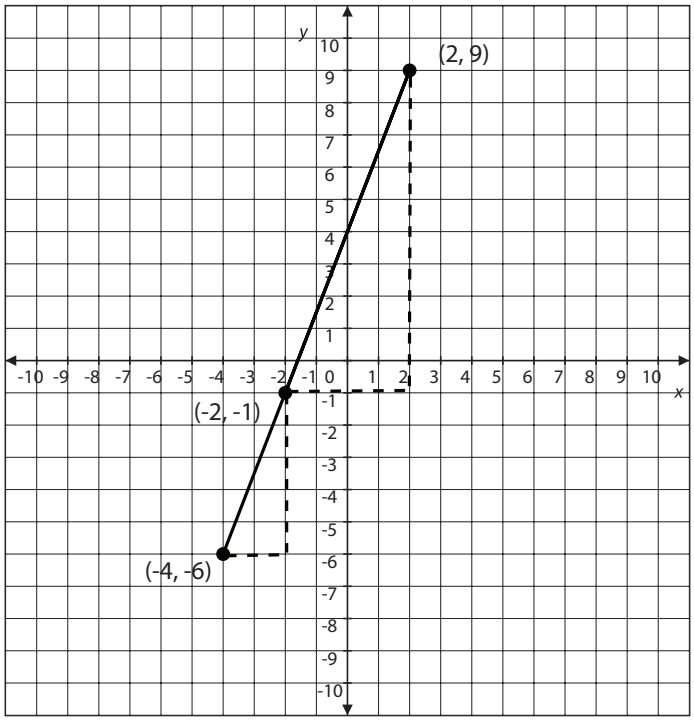
Instruction

7. Subtract the product from the *y*-value of the endpoint. This is the *y*-value of the point with the given ratio.

   $9 - 10 = -1$

   The point that is $\frac{2}{3}$ the distance from the endpoint (2, 9) of the segment with endpoints (2, 9) and (–4, –6) is (–2, –1).



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Instruction

Example 5

A line segment has one endpoint at (12, 0) and a midpoint of (10, –2). Locate the second endpoint.

1. Determine the endpoints of the line segment.

   One endpoint of the segment is (12, 0).

   The other endpoint is unknown.

2. Substitute the values of (x_1, y_1) into the midpoint formula and simplify.

   $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ Midpoint formula

   $\left(\frac{x+12}{2}, \frac{y+0}{2}\right)$ Substitute (12, 0).

3. Find the value of x.

   The midpoint (10, –2) is equal to $\left(\frac{x+12}{2}, \frac{y+0}{2}\right)$.

   $\left(\frac{x+12}{2}, \frac{y+0}{2}\right)=(10,-2)$

   Create an equation to find the value of x.

   $\frac{x+12}{2}=10$ Equation

   $x + 12 = 20$ Multiply both sides by 2.

   $x = 8$ Subtract 12 from both sides.

4. Find the value of y.

   Create an equation to find the value of y.

   $\frac{y+0}{2}=-2$ Equation

   $y + 0 = -4$ Multiply both sides by 2.

   $y = -4$ Simplify.

   The endpoint of the segment with one endpoint at (12, 0) and a midpoint at (10, –2) is (8, –4).

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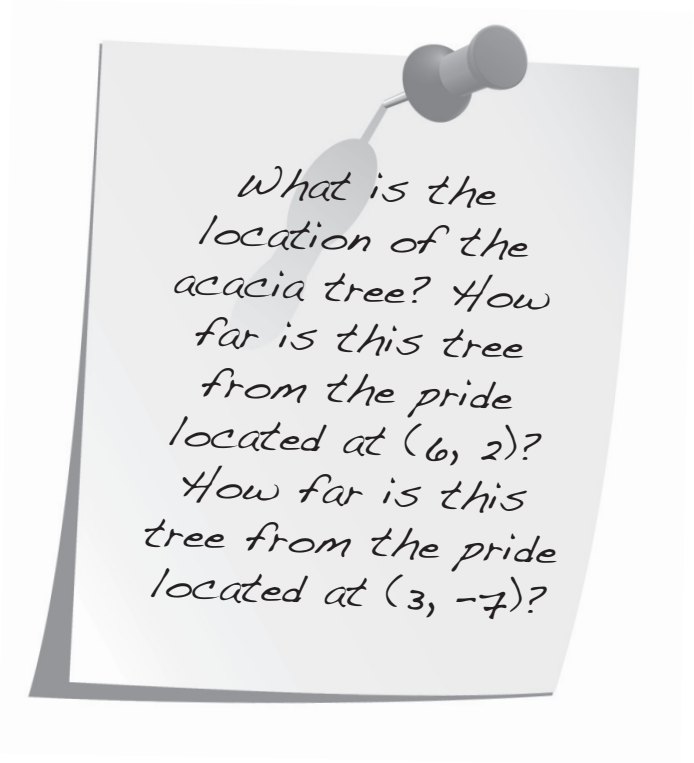
Problem-Based Task 5.1.1: Points of Shade

SMP	
1 ✓	2 ✓
3	4 ✓
5	6
7	8

Officials at Serengeti National Park created a map to represent an area of the park. Lion prides are located at the points (–4, –1), (1, 5), (6, 2), and (3, –7). Water holes are located at the points (1, –1) and (8, –4). Acacia trees provide shade for the prides. One tree is located halfway between each of the water holes. What is the location of the acacia tree? How far is this tree from the pride located at (6, 2)? How far is this tree from the pride located at (3, –7)? Describe the location of the acacia tree in terms of the prides located at (6, 2) and (3, –7). Explain your reasoning. Each unit represents 1,000 feet.



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Problem-Based Task 5.1.1: Points of Shade

Coaching

a. What are the coordinates of each of the water holes?

b. "Halfway" is another way of describing which mathematical term?

c. Which formula determines the halfway point between two coordinates?

d. What is the halfway point between the two water holes?

e. Which formula determines the distance between two points?

f. What is the distance between the acacia tree and the pride located at (6, 2)?

g. What is the distance between the acacia tree and the pride located at (3, –7)?

h. What do you notice about the location of the acacia tree in terms of the prides located at (6, 2) and (3, –7)?

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Instruction

Problem-Based Task 5.1.1: Points of Shade

Coaching Sample Responses

a. What are the coordinates of each of the water holes?

The coordinates of the water holes are (1, –1) and (8, –4).

b. "Halfway" is another way of describing which mathematical term?

Halfway describes the midpoint.

c. Which formula determines the halfway point between two coordinates?

The midpoint formula determines the halfway point between two coordinates. The formula is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

d. What is the halfway point between the two water holes?

Let (x_1, y_1) be (1, –1) and (x_2, y_2) be (8, –4).

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ Midpoint formula

$\left(\frac{(1)+(8)}{2}, \frac{(-1)+(-4)}{2}\right)$ Substitute (1, –1) and (8, –4).

$\left(\frac{9}{2}, \frac{-5}{2}\right)$ Simplify.

The midpoint between the two water holes is $\left(\frac{9}{2}, \frac{-5}{2}\right)$.

e. Which formula determines the distance between two points?

The distance formula determines the distance between two points. The formula is $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

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f. What is the distance between the acacia tree and the pride located at (6, 2)?

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ Distance formula

$=\sqrt{\left((6)-\left(\frac{9}{2}\right)\right)^2+\left((2)-\left(-\frac{5}{2}\right)\right)^2}$ Substitute $\left(\frac{9}{2},\frac{-5}{2}\right)$ and (6, 2).

$=\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{9}{2}\right)^2}$ Simplify as needed.

$=\sqrt{\frac{9}{4}+\frac{81}{4}}$

$=\sqrt{\frac{90}{4}}$

$=\frac{3\sqrt{10}}{2}$

The distance between the acacia tree and the pride located at (6, 2) is $\frac{3\sqrt{10}}{2}$ units, or approximately 4,743.4 feet.

g. What is the distance between the acacia tree and the pride located at (3, –7)?

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ Distance formula

$=\sqrt{\left((3)-\left(\frac{9}{2}\right)\right)^2+\left((-7)-\left(-\frac{5}{2}\right)\right)^2}$ Substitute $\left(\frac{9}{2},\frac{-5}{2}\right)$ and (3, –7).

$=\sqrt{\left(-\frac{3}{2}\right)^2+\left(-\frac{9}{2}\right)^2}$ Simplify as needed.

$=\sqrt{\frac{9}{4}+\frac{81}{4}}$

$=\sqrt{\frac{90}{4}}$

$\frac{3\sqrt{10}}{2}$

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Instruction

The distance between the acacia tree and the pride located at (3, –7) is $\frac{3\sqrt{10}}{2}$ units, or approximately 4,743.4 feet.

h. What do you notice about the location of the acacia tree in terms of the prides located at (6, 2) and (3, –7)?

The distance from the acacia tree to each of the prides is equal; therefore, the acacia tree is the midpoint between the prides.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 5.1.1: Midpoints and Other Points on Line Segments

For problems 1–3, the endpoints of a segment are given. Calculate the indicated point for each segment.

1. Determine the midpoint of the segment with endpoints (–10, –11) and (8, –17).

2. Determine the point that is $\frac{3}{4}$ the distance from the endpoint (–6, –16) of the segment with endpoints (34, 2) and (–6, –16).

3. Determine the point that is $\frac{1}{3}$ the distance from the endpoint (6, 24) of the segment with endpoints (–9, –18) and (6, 24).

For problems 4 and 5, find the coordinates of the second endpoint given one endpoint and the midpoint of the segment.

4. endpoint (–10, 3) and midpoint (3.5, 4)

5. endpoint (22, 0) and midpoint (7, 8)

continued

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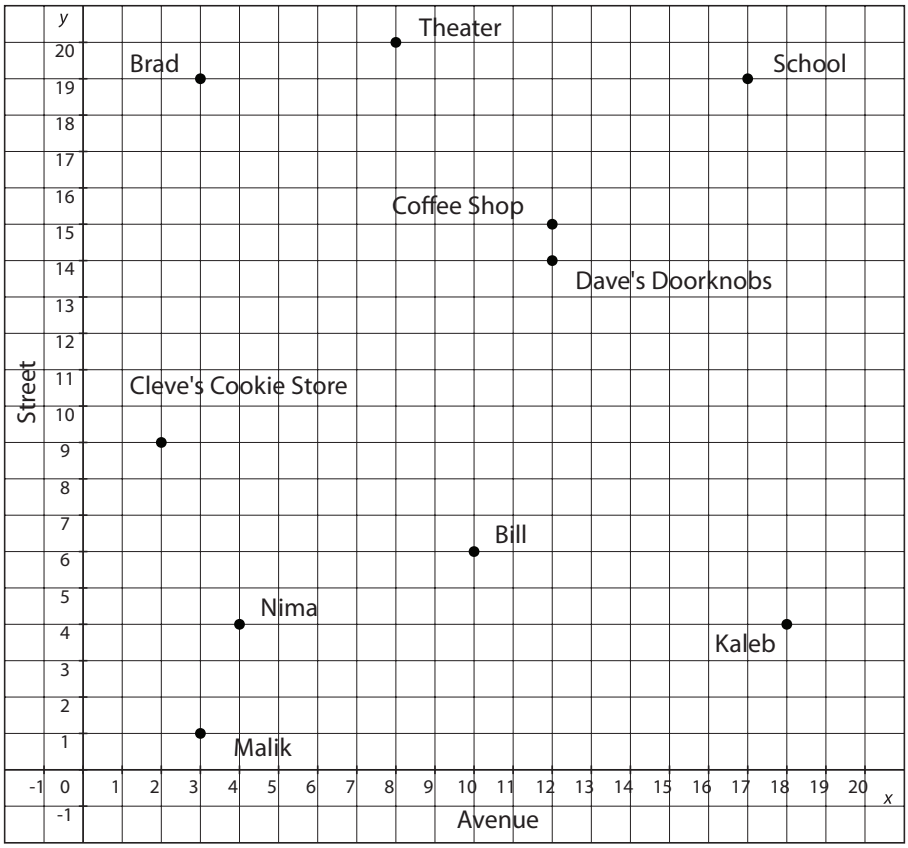
Use the map and the information given to solve problems 6–10.



6. Luis works at a theater on 8th Avenue and 20th Street. Kaleb lives at the corner of 18th Avenue and 4th Street. What is the intersection that is midway between them?

7. Nima lives at the corner of 4th Avenue and 4th Street. Bill lives at the corner of 10th Avenue and 6th Street. Their favorite bakery is located midway between them. What is the intersection that is at the location of the bakery?

8. Cleve's Cookie Store is located at the corner of 2nd Avenue and 9th Street. Dave's Doorknobs is located at the corner of 12th Avenue and 14th Street. Located $\frac{1}{5}$ of the distance from Cleve's Cookie Store is the post office. Where is the post office?

9. Malik and Brad both live on 3rd Avenue. Malik lives at the corner of 1st Street, and Brad lives at the corner of 19th Street. $\frac{2}{3}$ the distance from Malik's apartment to Brad's apartment is a market. Where is the market?

10. The main entrance to the high school is located at the corner of 17th Avenue and 19th Street. On his way from school to the bank, Luis stops at the coffee shop located at 12th Avenue and 15th Street. The coffee shop is the midpoint of this trip. What is the location of the bank?

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Progress Assessment

Circle the letter of the best answer.

1. What is the midpoint between (–2, 5) and (4, 8)?

   a. (1, 5)
   b. (1, 6.5)
   c. (3, 6.5)
   d. (3, 8)

2. One endpoint of a segment is (20, 20). The midpoint of the segment is (–2, 4). What is the second endpoint of this segment?

   a. (11, 12)
   b. (42, 36)
   c. (–24, –12)
   d. (–12, –6)

3. What is the point that is $\frac{2}{3}$ the distance from the endpoint (–3, 8) of the segment with endpoints (–3, 8) and (9, –7)?

   a. (17, –17)
   b. (1, 3)
   c. (–11, –18)
   d. (5, –2)

4. Tuan is standing at the point (–2, –8). Marc is standing at the point (6, 10). What are the coordinates of the point midway between Tuan and Marc?

   a. (2, 1)
   b. (2, 2)
   c. (4, 9)
   d. (4, 10)

5. Margaret lives at the corner of 2nd Avenue and 10th Street. Hugo lives at the corner of 6th Avenue and 18th Street. They decide to meet at a movie theater $\frac{1}{4}$ the distance from Hugo's apartment. Where do they meet?

   a. 5th Avenue and 16th Street
   b. 16th Avenue and 5th Street
   c. 3rd Avenue and 12th Street
   d. 12th Avenue and 3rd Street

continued

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6. What is the midpoint between (7, –6) and (–13, 2)?

   a. (3, 2)
   b. (–3, –2)
   c. (6, 4)
   d. (–10, –4)

7. One endpoint of a segment is (–9, –1). The midpoint of the segment is (–2, 3). What is the second endpoint of this segment?

   a. (3.5, 1)
   b. (–16, –4)
   c. (5, 7)
   d. (–5.5, 2)

8. What is the point that is $\frac{3}{4}$ the distance from the endpoint (6, –8) of the segment with endpoints (6, –8) and (–2, 10)?

   a. (4, –3.5)
   b. (2, 1)
   c. (–8, 23.5)
   d. (0, 5.5)

9. Tobie and Penny are playing a game. Tobie's game piece lands at the point (2, –3). The next turn, Penny's game piece lands at the point (0, 5). What are the coordinates of the point midway between the two game pieces?

   a. (1, 1)
   b. (1, –1)
   c. (1, –4)
   d. (–1, 4)

10. The grocery store is at the corner of 2nd Avenue and 9th Street. The library is at the corner of 8th Avenue and 21st Street. The school is located $\frac{2}{3}$ the distance from the grocery store. Where is the school located?

    a. 20th Avenue and 7th Street
    b. 4th Avenue and 8th Street
    c. 7th Avenue and 3rd Street
    d. 8th Avenue and 4th Street

Read the scenario and then solve the problem.

11. Johann's position is represented by the point (5, 18). Tia's position is represented by the point (–1, 6). What is the midway point between Johann and Tia? Use the distance formula to verify that you found the actual midpoint.

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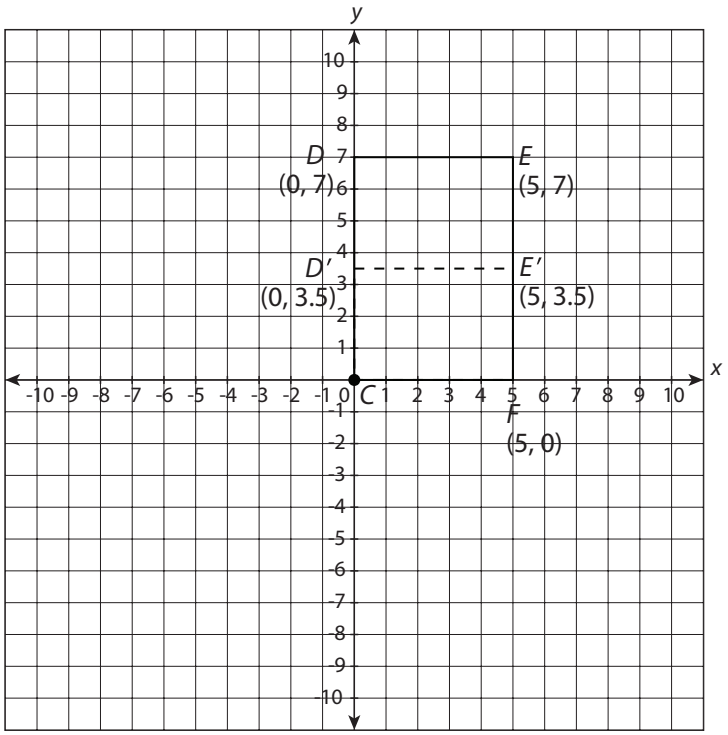
Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Does the graph below represent a dilation? Why or why not?



a. Yes, because the preimage sides are parallel with the corresponding image sides.

b. No, because the preimage sides are not parallel with the corresponding image sides.

c. Yes, because there is a single scale factor and a center of dilation.

d. No, because the scale factors of the image sides are not all consistent with the preimage sides.

continued

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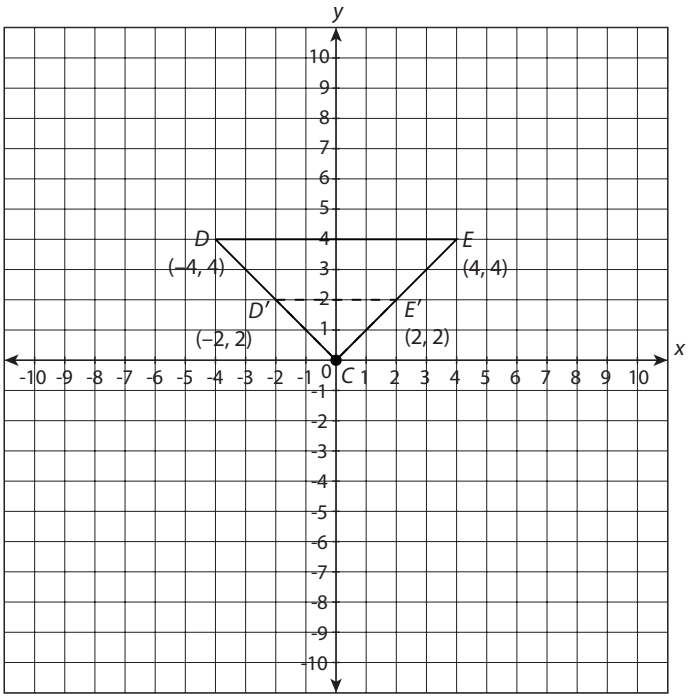
2. Determine the scale factor of the dilation below.



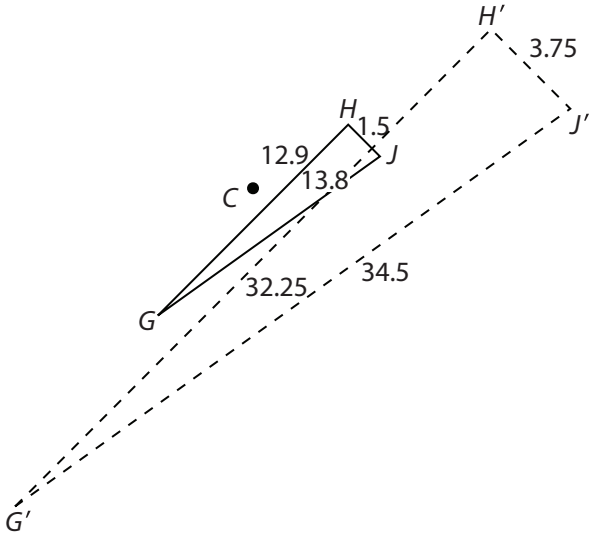
a. $k = 2$

b. $k = \frac{1}{4}$

c. $k = \frac{1}{2}$

d. $k = 1$

3. Determine the scale factor of the dilation below.



a. $k = 1.07$

b. $k = 2.5$

c. $k = 0.4$

d. $k = 9.2$

continued

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4. $\overline{AB}$ is 6.7 units long. If $\overline{AB}$ is dilated by a scale factor of $k = 3.2$, what is the length of $\overline{A'B'}$?

   a. 21.4 units

   b. 2.1 units

   c. 0.5 unit

   d. 1 unit

5. $\triangle FGH$ has vertices $F(3, -5)$, $G(8, -6)$, and $H(6, -7)$. If $\triangle FGH$ is dilated through the origin with a scale factor of $\frac{3}{4}$, what are the vertices of $\triangle F'G'H'$?

   a. $F'(-3, 5)$, $G'(-8, 6)$, and $H'(-6, 7)$

   b. $F'(2.25, -3.75)$, $G'(6, -4.5)$, and $H'(4.5, -5.25)$

   c. $F'\left(4, \frac{20}{3}\right)$, $G'\left(\frac{32}{3}, -8\right)$, and $H'\left(8, -\frac{28}{3}\right)$

   d. $F'(-5, 3)$, $G'(-6, -8)$, and $H'(-7, 6)$

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Lesson 2: Investigating Properties of Dilations

Instruction

Common Core State Standards

SMP	
1 ✓	2 ✓
3	4 ✓
5	6
7	8 ✓

G–SRT.1 Verify experimentally the properties of dilations given by a center and a scale factor:

a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.

b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

Essential Questions

1. How are the preimage and image similar in dilations?
2. How are the preimage and image different in dilations?
3. When are dilations used in the real world?

WORDS TO KNOW

center of dilation	a point through which a dilation takes place; all the points of a dilated figure are stretched or compressed through this point
collinear points	points that lie on the same line
compression	a transformation in which a figure becomes smaller; compressions may be horizontal (affecting only horizontal lengths), vertical (affecting only vertical lengths), or both
congruency transformation	a transformation in which a geometric figure moves but keeps the same size and shape; a dilation where the scale factor is equal to 1
corresponding sides	sides of two figures that lie in the same position relative to the figure
dilation	a transformation in which a figure is either enlarged or reduced by a scale factor in relation to a center point
enlargement	a dilation of a figure where the scale factor is greater than 1

SRB p. U5-17

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non-rigid motion	a transformation done to a figure that changes the figure's shape and/or size
reduction	a dilation where the scale factor is between 0 and 1
rigid motion	a transformation done to a figure that maintains the figure's shape and size
scale factor	a multiple of the lengths of the sides from one figure to the transformed figure. If the scale factor is larger than 1, then the figure is enlarged. If the scale factor is between 0 and 1, then the figure is reduced.
stretch	a transformation in which a figure becomes larger; stretches may be horizontal (affecting only horizontal lengths), vertical (affecting only vertical lengths), or both

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Instruction

Recommended Resources

- IXL Learning. "Transformations: Dilations: Find the Coordinates."

  http://www.walch.com/rr/00017

  This interactive website gives a series of problems and scores them immediately. If the user submits a wrong answer, a description and process for arriving at the correct answer are provided. These problems start with a graphed figure. Users are asked to input the coordinates of the dilated figure given a center and scale factor.

- IXL Learning. "Transformations: Dilations: Graph the Image."

  http://www.walch.com/rr/00018

  This interactive website gives a series of problems and scores them immediately. If the user submits a wrong answer, a description and process for arriving at the correct answer are provided. Users are asked to draw a dilation of the figure on the screen using a point that can be dragged, given a center and scale factor.

- IXL Learning. "Transformations: Dilations: Scale Factor and Classification."

  http://www.walch.com/rr/00019

  This interactive website gives a series of problems and scores them immediately. If the user submits a wrong answer, a description and process for arriving at the correct answer are provided. Users are required to choose whether the graphed figure is an enlargement or a reduction. Other problems ask users to enter the scale factor.

- MathIsFun.com. "Resizing."

  http://www.walch.com/rr/00020

  This website gives a brief explanation of the properties of dilations and how to perform them. The site also contains an interactive applet with which users can select a shape, a center point, and a scale factor. After users explore the applet, they may answer eight multiple-choice questions in order to check understanding.

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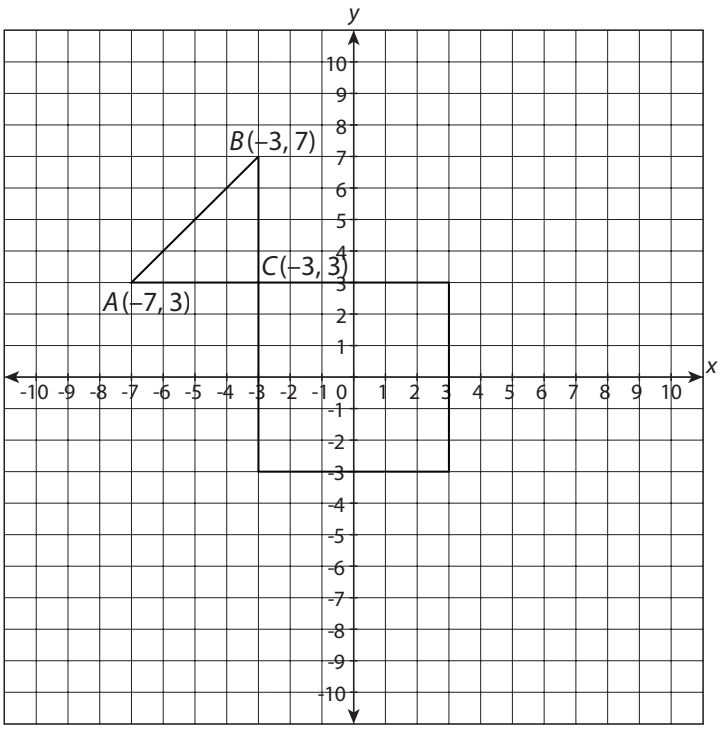
Lesson 5.2.1: Investigating Properties of Parallelism and the Center

Warm-Up 5.2.1

A college has a wide-open space in the center of the campus. Landscape architects have proposed gardens at the four corners of the open space. Each garden would be in the shape of a triangle. The first garden is pictured below.



1. Perform a transformation of the first garden to create a second garden at the opposite vertex of the open space. The opposite vertex lies in Quadrant IV. Write the coordinates of the new garden and state the transformation you used.

2. Is the transformation you performed a rigid motion? Explain.

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Instruction

Lesson 5.2.1: Investigating Properties of Parallelism and the Center

Common Core State Standard

G–SRT.1 Verify experimentally the properties of dilations given by a center and a scale factor:

a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.

Warm-Up 5.2.1 Debrief

1. Perform a transformation of the first garden to create a second garden at the opposite vertex of the open space. The opposite vertex lies in Quadrant IV. Write the coordinates of the new garden and state the transformation you used.

   There are several possibilities.

   One possibility is to reflect the garden over the line $y = x$.

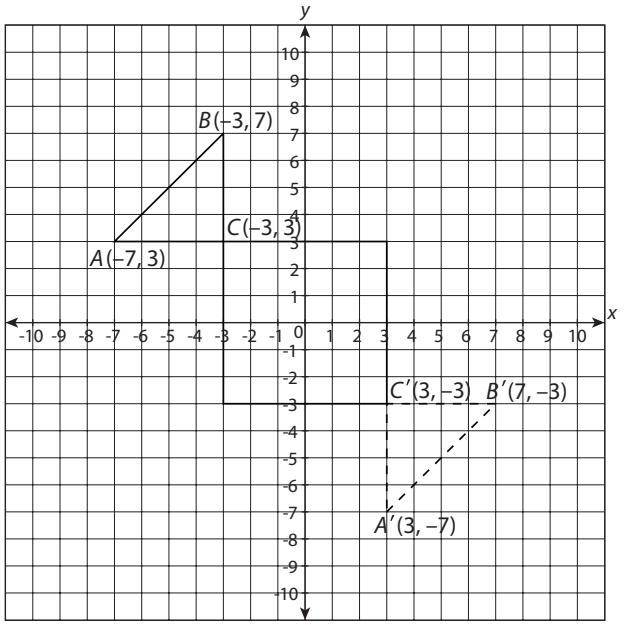
   $r_{y=x}(x, y) = (y, x)$

   $A' = r_{y=x}(A) = r_{y=x}(-7, 3) = (3, -7)$

   $B' = r_{y=x}(B) = r_{y=x}(-3, 7) = (7, -3)$

   $C' = r_{y=x}(C) = r_{y=x}(-3, 3) = (3, -3)$

   This results in the following graph:



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Instruction

A second possibility is to rotate the garden 180° counterclockwise about the origin.

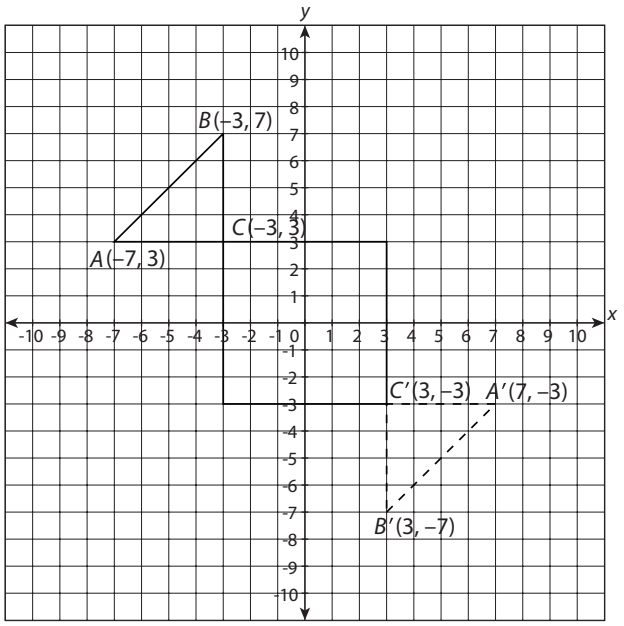
$R_{180}(x, y) = (-x, -y)$

$A' = R_{180}(A) = R_{180}(-7, 3) = (7, -3)$

$B' = R_{180}(B) = R_{180}(-3, 7) = (3, -7)$

$C' = R_{180}(C) = R_{180}(-3, 3) = (3, -3)$

This results in the following graph:



Observe that the shapes of the graphs for both possibilities are identical.

2. Is the transformation you performed a rigid motion? Explain.

   Reflections and rotations are rigid motions. They preserve angle and length measures. The result of either transformation is a congruent figure.

Connection to the Lesson

- Students will extend previous understandings of rigid motions to include non-rigid motions.
- Students will work with vertices on the coordinate plane.

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Prerequisite Skills

This lesson requires the use of the following skills:

- operating with fractions, including complex fractions (A–APR.6)
- operating with decimals (5.NBT.7)
- calculating slope (8.F.4)
- determining parallel lines (G–GPE.5)

Introduction

Think about resizing a window on your computer screen. You can stretch it vertically, horizontally, or at the corner so that it stretches both horizontally and vertically at the same time. These are non-rigid motions. **Non-rigid motions** are transformations done to a figure that change the figure's shape and/or size. These are in contrast to **rigid motions**, which are transformations to a figure that maintain the figure's shape and size.

Specifically, we are going to study non-rigid motions of dilations. **Dilations** are transformations in which a figure is either enlarged or reduced by a scale factor in relation to a center point.

Key Concepts

- Dilations require a center of dilation and a scale factor.
- The **center of dilation** is the point about which all points are stretched or compressed.
- The **scale factor** of a figure is a multiple of the lengths of the sides from one figure to the transformed figure.
- Side lengths are changed according to the scale factor, k.
- The scale factor can be found by finding the distances of the sides of the preimage in relation to the image.
- Use a ratio of corresponding sides to find the scale factor: $\frac{\text{length of image side}}{\text{length of preimage side}} = \text{scale factor}$

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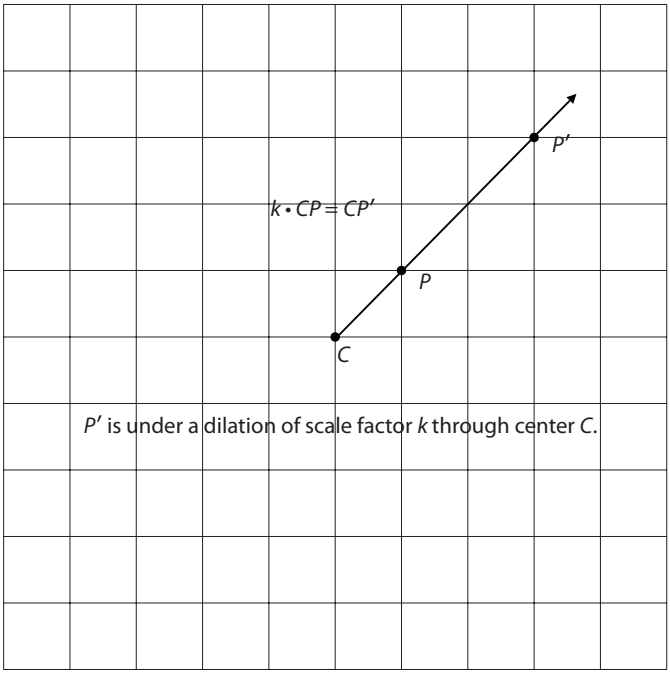
- The scale factor, k, takes a point P and moves it along a line in relation to the center so that $k \bullet CP = CP'$.



- If the scale factor is greater than 1, the figure is stretched or made larger and is called an **enlargement**. (A transformation in which a figure becomes larger is also called a **stretch**.)
- If the scale factor is between 0 and 1, the figure is compressed or made smaller and is called a **reduction**. (A transformation in which a figure becomes smaller is also called a **compression**.)
- If the scale factor is equal to 1, the preimage and image are congruent. This is called a **congruency transformation**.
- Angle measures are preserved in dilations.
- The orientation is also preserved.
- The sides of the preimage are parallel to the corresponding sides of the image.
- The **corresponding sides** are the sides of two figures that lie in the same position relative to the figures.
- In transformations, the corresponding sides are the preimage and image sides, so $\overline{AB}$ and $\overline{A'B'}$ are corresponding sides and so on.
- The notation of a dilation in the coordinate plane is given by $D_k(x, y) = (kx, ky)$. The scale factor is multiplied by each coordinate in the ordered pair.
- In this lesson, the center of dilation is usually the origin, (0, 0).

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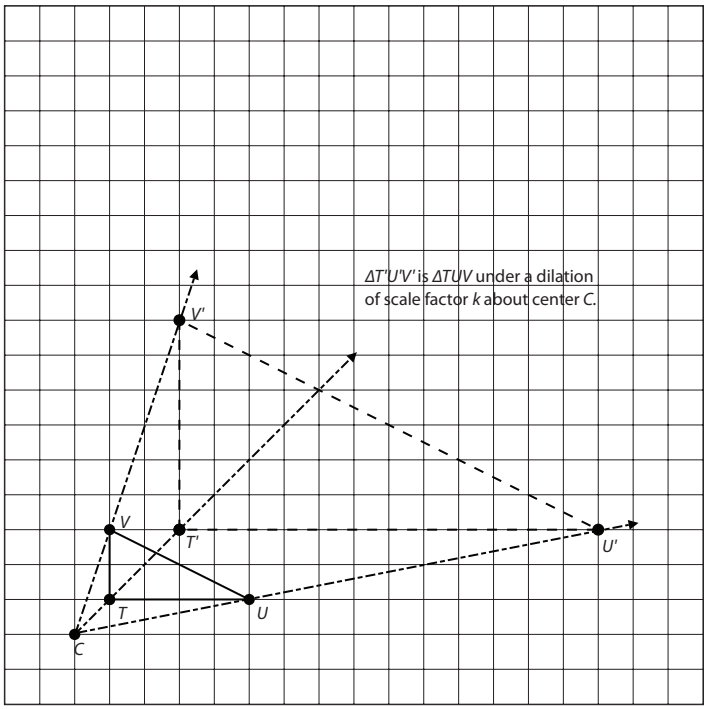
- If a segment of the figure being dilated passes through the center of dilation, then the image segment will lie on the same line as the preimage segment. All other segments of the image will be parallel to the corresponding preimage segments.
- The corresponding points in the preimage and image are **collinear points**, meaning they lie on the same line, with the center of dilation.



Properties of Dilations

1. Shape, orientation, and angles are preserved.
2. All sides change by a single scale factor, k.
3. Corresponding sides in the preimage and image are parallel.
4. The corresponding points of the figure are collinear with the center of dilation.

Common Errors/Misconceptions

- forgetting to check the ratio of all sides from the image to the preimage in determining if a dilation has occurred
- inconsistently setting up the ratio of the side lengths
- confusing enlargements with reductions and vice versa

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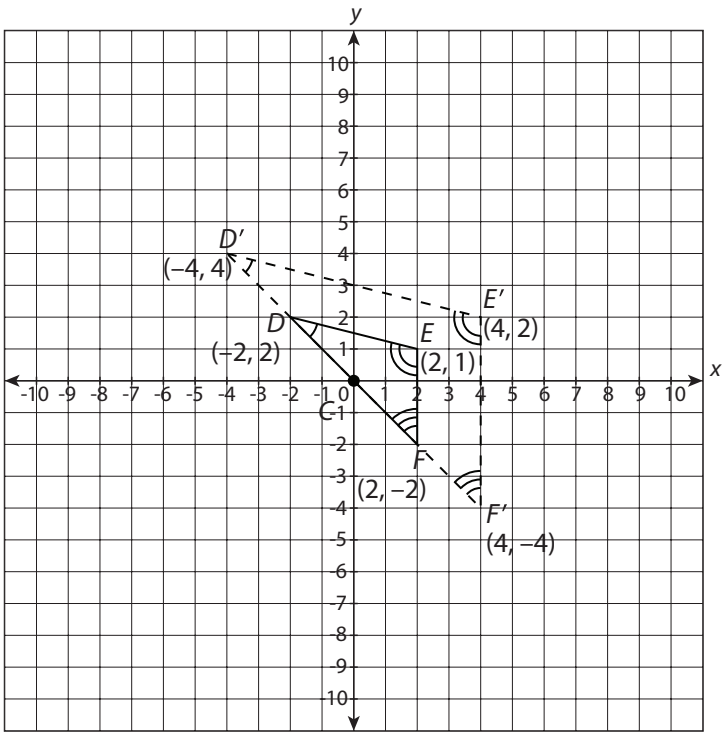
Guided Practice 5.2.1

Example 1

Is the following transformation a dilation? Justify your answer using the properties of dilations.



1. Verify that shape, orientation, and angles have been preserved from the preimage to the image.

   Both figures are triangles in the same orientation.

   $\angle D \cong \angle D'$

   $\angle E \cong \angle E'$

   $\angle F \cong \angle F'$

   The angle measures have been preserved.

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2. Verify that the corresponding sides are parallel.

$m_{\overline{DE}} = \frac{\Delta y}{\Delta x} = \frac{(2-1)}{(-2-2)} = \frac{1}{-4} = -\frac{1}{4}$ and $m_{\overline{D'E'}} = \frac{\Delta y}{\Delta x} = \frac{(4-2)}{(-4-4)} = \frac{2}{-8} = -\frac{1}{4}$;

therefore, $\overline{DE} \parallel \overline{D'E'}$.

By inspection, $\overline{EF} \parallel \overline{E'F'}$ because both lines are vertical; therefore, they have the same slope and are parallel.

The center of the dilation is on segment $\overline{DF}$, and we know that if a segment of the figure being dilated passes through the center of dilation, the image segment lies on the same line as the preimage, as can be seen in the diagram.

All corresponding sides are parallel.

3. Verify that the distances of the corresponding sides have changed by a common scale factor, k.

We could calculate the distances of each side, but that would take a lot of time. Instead, examine the coordinates and determine if the coordinates of the vertices have changed by a common scale factor.

The notation of a dilation in the coordinate plane is given by $D_k(x, y) = (kx, ky)$.

Divide the coordinates of each vertex to determine if there is a common scale factor.

$D(-2,2) \rightarrow D'(-4,4)$	$E(2,1) \rightarrow E'(4,2)$	$F(2,-2) \rightarrow F'(4,-4)$
$\frac{x_{D'}}{x_D} = \frac{-4}{-2} = 2; \frac{y_{D'}}{y_D} = \frac{4}{2} = 2$	$\frac{x_{E'}}{x_E} = \frac{4}{2} = 2; \frac{y_{E'}}{y_E} = \frac{2}{1} = 2$	$\frac{x_{F'}}{x_F} = \frac{4}{2} = 2; \frac{y_{F'}}{y_F} = \frac{-4}{-2} = 2$

Each vertex's preimage coordinate is multiplied by 2 to create the corresponding image vertex. Therefore, the common scale factor is $k = 2$.

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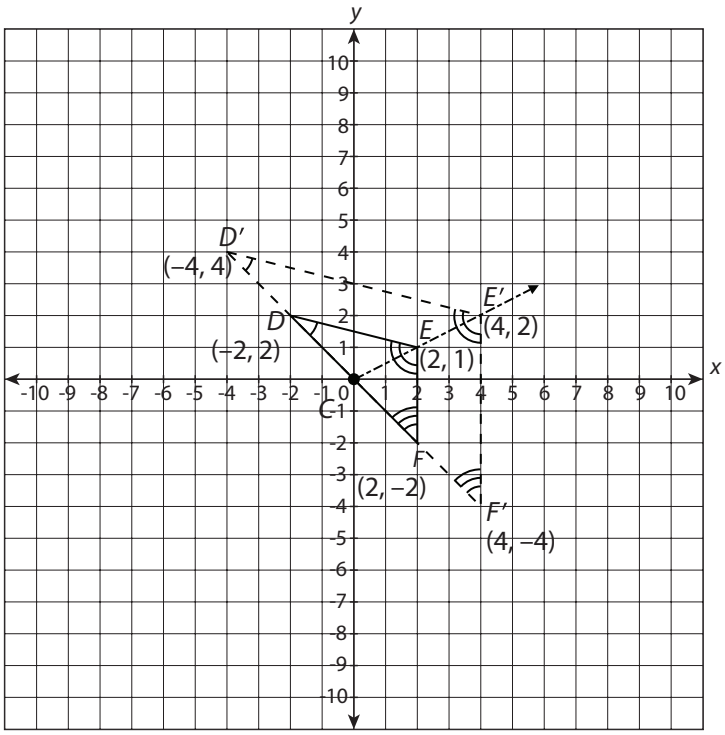
4. Verify that corresponding vertices are collinear with the center of dilation, *C*.



A straight line can be drawn connecting the center of the dilation, (0, 0), with the corresponding vertices. This means that the corresponding vertices are collinear with the center of dilation.

5. Draw conclusions.

The transformation is a dilation because the shape, orientation, and angle measures have been preserved. Additionally, the size has changed by a scale factor of 2. All corresponding sides are parallel, and the corresponding vertices are collinear with the center of dilation.

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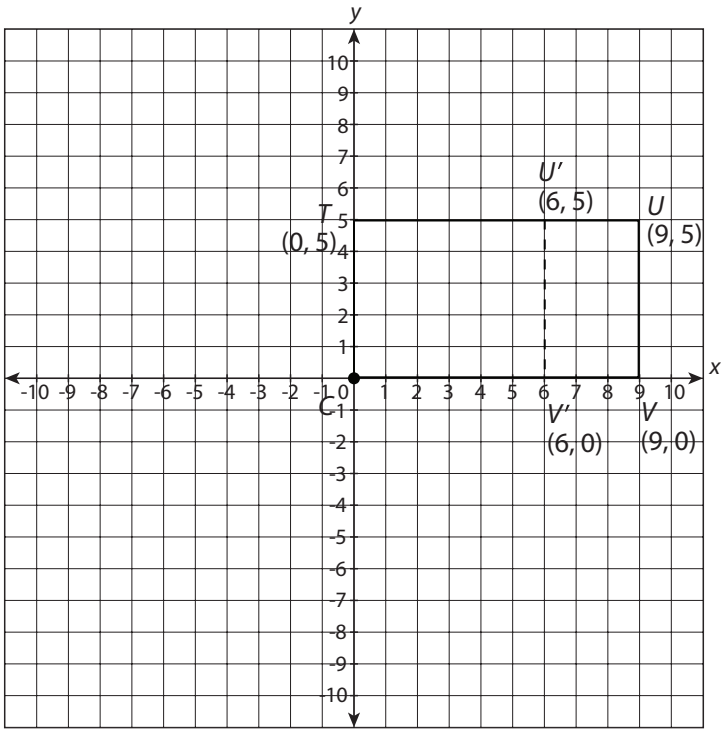
Instruction

Example 2

Is the following transformation a dilation? Justify your answer using the properties of dilations.



1. Verify that shape, orientation, and angles have been preserved from the preimage to the image.

   The preimage and image are both rectangles with the same orientation. The angle measures have been preserved since all angles are right angles.

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2. Verify that the corresponding sides are parallel.

   $\overline{TU'}$ is on the same line as $\overline{TU}$; therefore, $\overline{TU} \| \overline{TU'}$.

   $\overline{CV'}$ is on the same line as $\overline{CV}$; therefore, $\overline{CV} \| \overline{CV'}$.

   By inspection, $\overline{UV}$ and $\overline{U'V'}$ are vertical; therefore, $\overline{UV} \| \overline{U'V'}$.

   $\overline{TC}$ remains unchanged from the preimage to the image.

   All corresponding sides are parallel.

3. Verify that the distances of the corresponding sides have changed by a common scale factor, *k*.

   Since the segments of the figure are on a coordinate plane and are either horizontal or vertical, find the distance by counting.

In $\square TUVC$:	**In $\square TU'V'C$:**
$TU = VC = 9$	$TU' = V'C = 6$
$UV = CT = 5$	$U'V' = CT = 5$

   The formula for calculating the scale factor is:

   $$\text{scale factor} = \frac{\text{length of image side}}{\text{length of preimage side}}$$

   Start with the horizontal sides of the rectangle.

   $$\frac{TU'}{TU} = \frac{6}{9} = \frac{2}{3} \qquad \frac{V'C}{VC} = \frac{6}{9} = \frac{2}{3}$$

   Both corresponding horizontal sides have a scale factor of $\frac{2}{3}$.

   Next, calculate the scale factor of the vertical sides.

   $$\frac{U'V'}{UV} = \frac{5}{5} = 1 \qquad \frac{CT}{CT} = \frac{5}{5} = 1$$

   Both corresponding vertical sides have a scale factor of 1.

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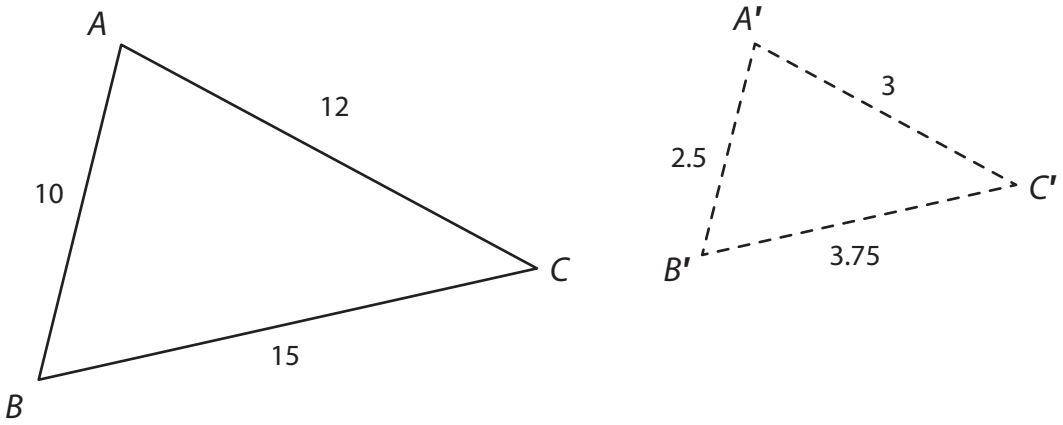
4. Draw conclusions.

   The vertical corresponding sides have a scale factor that is not consistent with the scale factor of $\frac{2}{3}$ for the horizontal sides. Since all corresponding sides do not have the same common scale factor, the transformation is NOT a dilation.

Example 3

The following transformation represents a dilation. What is the scale factor? Does this indicate enlargement, reduction, or congruence?



1. Determine the scale factor.

   Start with the ratio of one set of corresponding sides.

$$\text{scale factor} = \frac{\text{length of image side}}{\text{length of preimage side}}$$

$$\frac{A'B'}{AB} = \frac{2.5}{10} = \frac{1}{4}$$

The scale factor appears to be $\frac{1}{4}$.

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Instruction

2. Verify that the other sides maintain the same scale factor.

$\frac{B'C'}{BC}=\frac{3.75}{15}=\frac{1}{4}$ and $\frac{C'A'}{CA}=\frac{3}{12}=\frac{1}{4}$. Therefore,

$\frac{A'B'}{AB}=\frac{B'C'}{BC}=\frac{C'A'}{CA}=\frac{1}{4}$ and the scale factor, k, is $\frac{1}{4}$.

3. Determine the type of dilation that has occurred.

If $k > 1$, then the dilation is an enlargement.

If $0 < k < 1$, then the dilation is a reduction.

If $k = 1$, then the dilation is a congruency transformation.

Since $k=\frac{1}{4}$, k is between 0 and 1, or $0 < k < 1$.

The dilation is a reduction.

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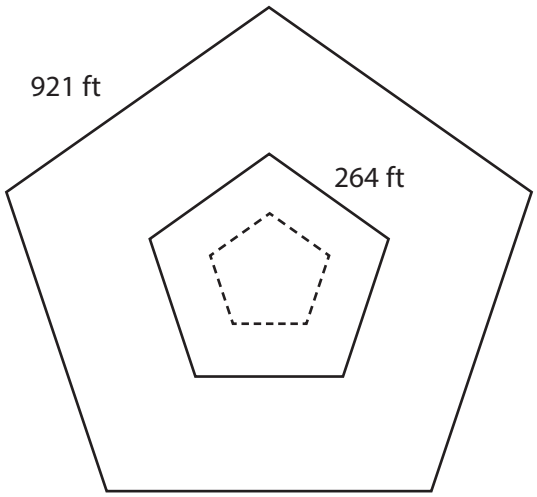
Problem-Based Task 5.2.1: Prettying Up the Pentagon

SMP	
1 ✓	2 ✓
3	4
5	6
7	8 ✓

The Pentagon, diagrammed below, is one of the world's largest office buildings. The outside walls are each 921 feet long and are a dilation of the inner walls through the center of the courtyard. The courtyard is the area inside the inner wall. Since the courtyard is surrounded by the inner walls, each side of the courtyard is the same length as the inner walls. The dashed lines represent a walkway that borders a garden. The walkway is a dilation of the inner wall of the office building.

A team of landscapers has been hired to update the courtyard. The landscapers need to know the perimeter of the walkway in order to install some temporary fencing while the courtyard is redone. What is the perimeter of the walkway if the dilation from the inner wall to the walkway has a scale factor of 0.25? What relationship does the scale factor have to the perimeters of the figures?



What relationship does the scale factor have to the perimeters of the figures?

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Problem-Based Task 5.2.1: Prettying Up the Pentagon

Coaching

a. What is the length of one side of the inner wall (the preimage)?

b. What is the scale factor used to dilate the inner wall to determine the walkway?

c. How do you calculate the length of each side of the walkway in the pentagonal figure in the courtyard?

d. What is the length of each side of the walkway?

e. How many sides are there in a pentagon?

f. What is the perimeter of the walkway?

g. What is the perimeter of the inner wall?

h. What is the scale factor of the perimeter of the inner wall to the perimeter of the walkway?

i. How do the scale factors of the lengths of each side of the inner wall and the walkway compare to the perimeters of the inner wall and the walkway?

j. What is the scale factor of the inner wall to the outer wall?

k. What is the perimeter of the outer wall?

l. What is the scale factor of the perimeter of the inner wall to the perimeter of the outer wall?

m. How do the scale factors of the individual side lengths of the inner and outer walls compare to the perimeters of the inner and outer walls?

n. What can you conclude about the scale factors of perimeters of dilated figures?

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Instruction

Problem-Based Task 5.2.1: Prettying Up the Pentagon

Coaching Sample Responses

a. What is the length of one side of the inner wall (the preimage)?

264 feet

b. What is the scale factor used to dilate the inner wall to determine the walkway?

0.25

c. How do you calculate the length of each side of the walkway in the pentagonal figure in the courtyard?

Multiply the preimage side by the scale factor: 264(0.25).

d. What is the length of each side of the walkway?

264(0.25) = 66

The length of each side of the walkway is 66 feet.

e. How many sides are there in a pentagon?

A pentagon has 5 sides.

f. What is the perimeter of the walkway?

5(66) = 330

The perimeter of the walkway is 330 feet.

g. What is the perimeter of the inner wall?

5(264) = 1320

The perimeter of the inner wall is 1,320 feet.

h. What is the scale factor of the perimeter of the inner wall to the perimeter of the walkway?

$$\frac{\text{perimeter of walkway}}{\text{perimeter of inner wall}} = \frac{330}{1320} = 0.25$$

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i. How do the scale factors of the lengths of each side of the inner wall and the walkway compare to the perimeters of the inner wall and the walkway?

The scale factor of each wall is the same as the scale factor of the perimeters.

j. What is the scale factor of the inner wall to the outer wall?

$$\frac{\text{length of inner wall}}{\text{length of outer wall}} = \frac{264}{921} \approx 0.29$$

k. What is the perimeter of the outer wall?

$5(921) = 4605$

The perimeter of the outer wall is 4,605 feet.

l. What is the scale factor of the perimeter of the inner wall to the perimeter of the outer wall?

$$\frac{\text{perimeter of inner wall}}{\text{perimeter of outer wall}} = \frac{1320}{4605} \approx 0.29$$

m. How do the scale factors of the individual side lengths of the inner and outer walls compare to the perimeters of the inner and outer walls?

The scale factors are the same.

n. What can you conclude about the scale factors of perimeters of dilated figures?

If a figure is dilated, the perimeter of the preimage to the image has the same scale factor as the dilation.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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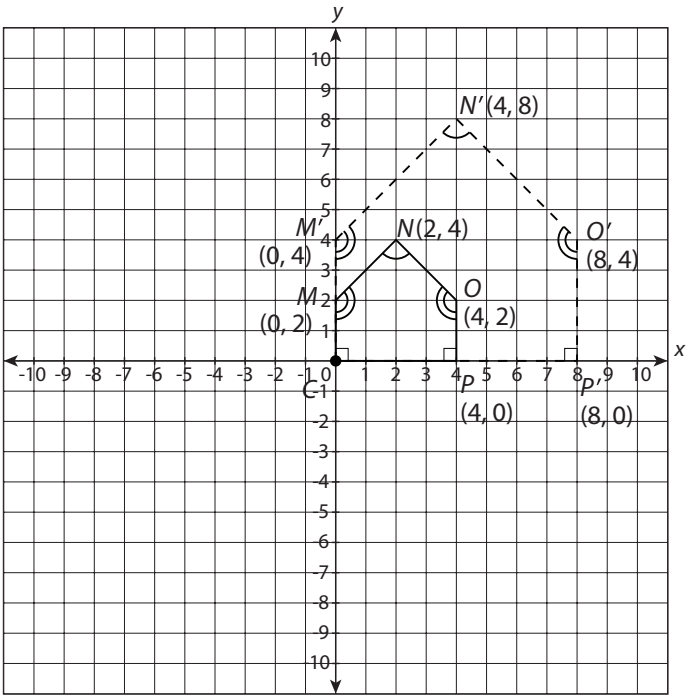
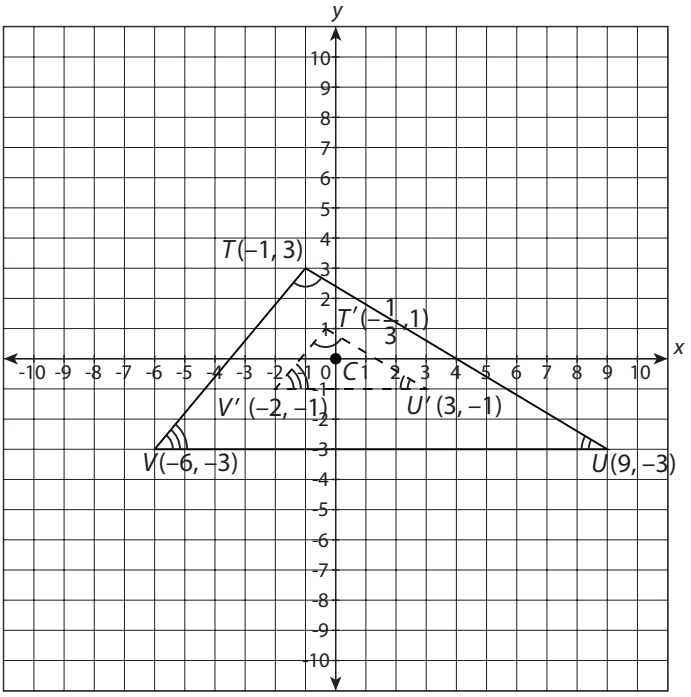
Practice 5.2.1: Investigating Properties of Parallelism and the Center

Determine whether each of the following transformations represents a dilation. Justify your answer using the properties of dilations.

1. Compare polygon *CMNOP* to polygon $CM'N'O'P'$.



2. Compare $\triangle TUV$ to $\triangle T'U'V'$.



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SWB p. 23; SRB Practice 5.2.1 p. U5-30

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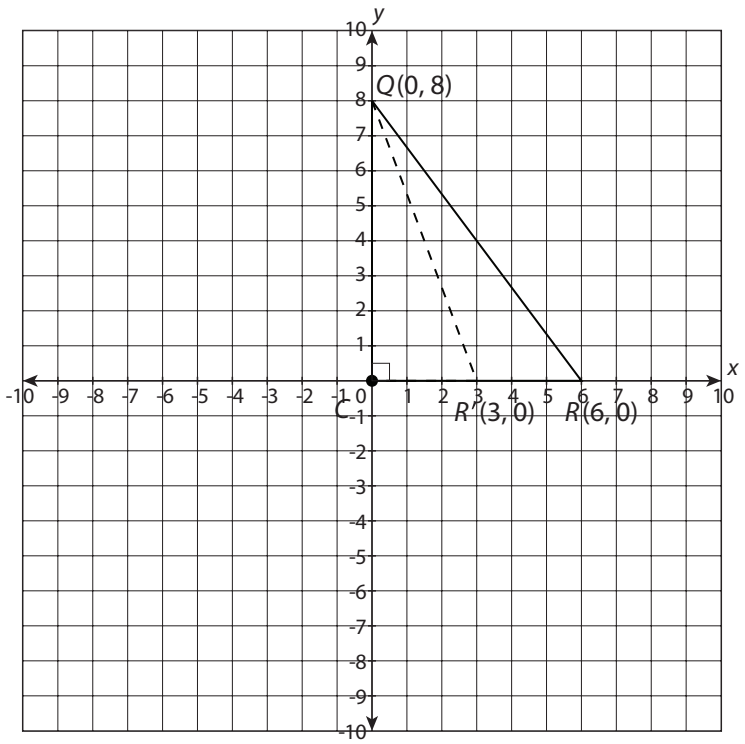
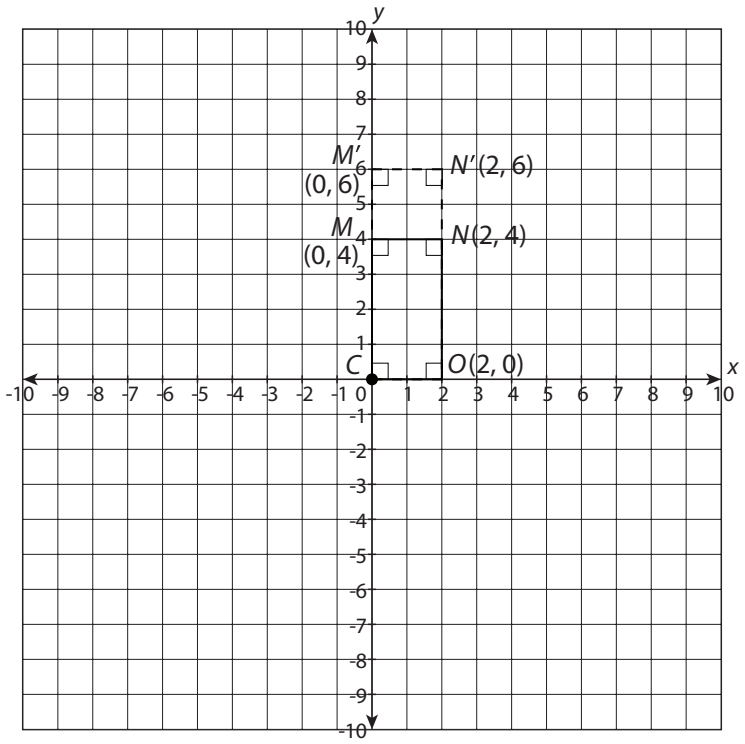
3. Compare $\triangle QRC$ to $\triangle QR'C$.



4. Compare $\square CMNO$ to $\square CM'N'O$.



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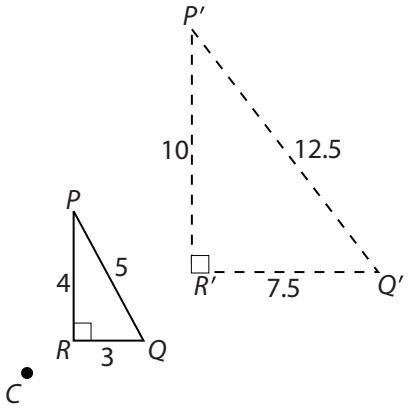
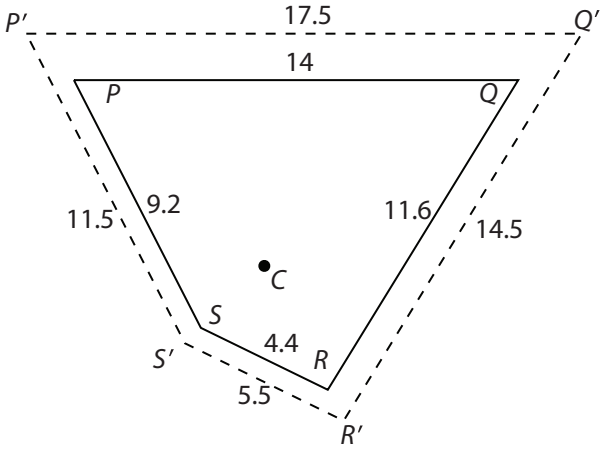
For problems 5 and 6, transformations represent dilations. Determine the scale factor and whether the dilation is an enlargement, a reduction, or a congruency transformation.

5.



6.



Use the given information in each problem that follows to answer the questions.

7. A right triangle has the following side lengths: $AB = 13$, $BC = 12$, and $CA = 5$. The triangle is dilated so that the image has side lengths $A'B' = \frac{26}{5}$, $B'C' = \frac{24}{5}$, and $C'A' = 2$. What is the scale factor? Does this represent an enlargement, a reduction, or a congruency transformation?

continued

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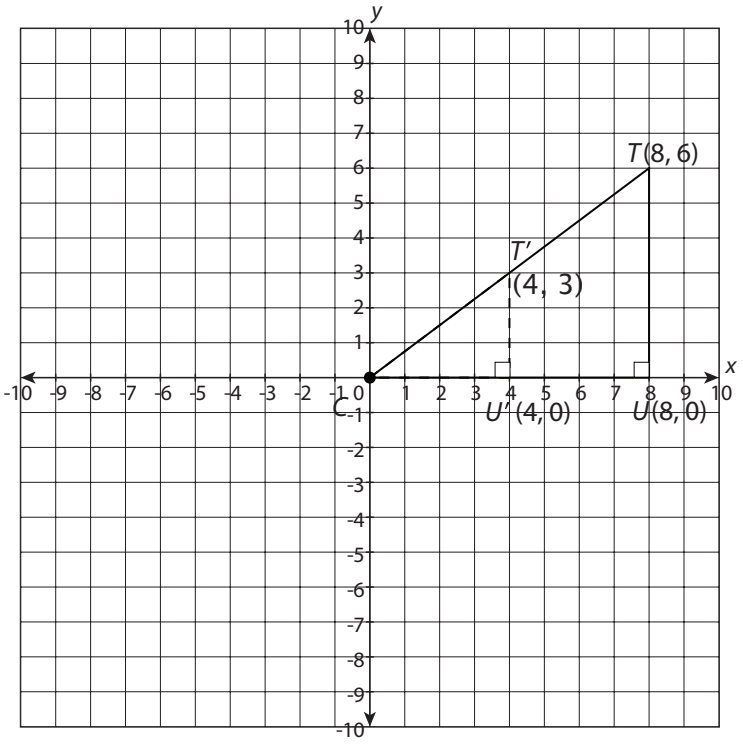
8. Derald is building a playhouse for his daughter. He wants the playhouse to look just like the family's home, so he's using the drawings from his house plans to create the plans for the playhouse. The diagram below shows part of a scale drawing of a roof truss used in the house. What is the scale factor of the roof truss from the house drawing to the playhouse drawing?



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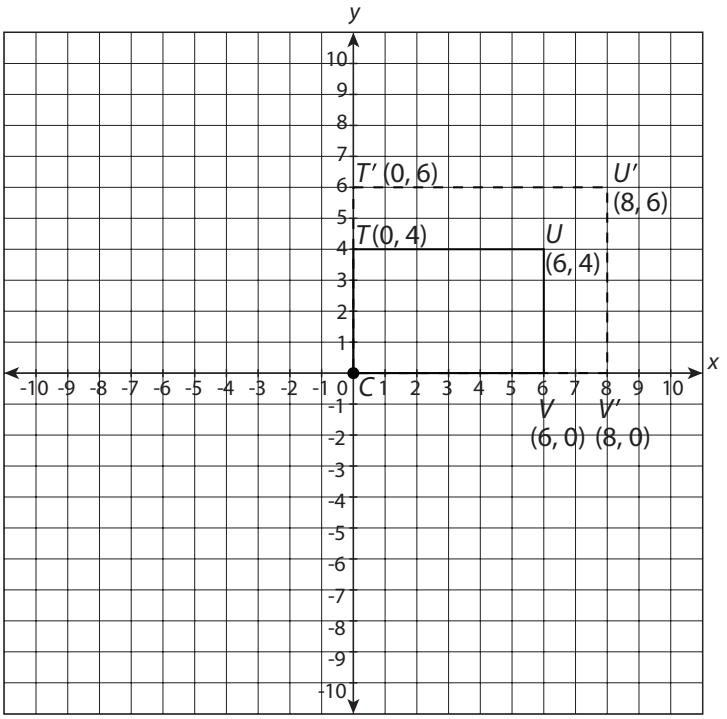
9. On Board, a luggage manufacturer, has had great success with a certain model of carry-on luggage. Feedback suggests that customers would prefer that the company sell different sizes of luggage with the same design as the carry-on. The graph below represents the top view of the original carry-on model and a proposed larger version of the same luggage. Does the new piece of luggage represent a dilation of the original piece of luggage? Why or why not?



10. A university wants to put in a courtyard for a new building. The courtyard is bounded by the coordinates P (–4, 0), Q (–2, –6), R (6, 2), and S (0, 4). The landscape architects created a dilation of the space through the center C (0, 0) to outline the garden. The garden is bounded by the points P' (–2.4, 0), Q' (–1.2, –3.6), R' (3.6, 1.2), and S' (0, 2.4). What is the scale factor? Does this represent an enlargement, a reduction, or a congruency transformation?

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Lesson 5.2.2: Investigating Scale Factors

Warm-Up 5.2.2

Hideki is babysitting his little sister and takes her to the park. While pushing her on the swing, he notices that if he pushes her so that she swings as high as the top of his head and then lets her go without pushing her, she only swings as high as his shoulders.

1. If Hideki is 6 feet tall and the height at his shoulders is 5.1 feet, what is the scale factor of the change in height of his sister's swing?

2. Rewrite this change as a percentage.

3. What is the reciprocal of this change in height?

4. Rewrite this reciprocal as a percentage.

5. What might it mean in the context of the problem if you applied the reciprocal of the change to the height when Hideki's sister is swinging as high as the top of his head?

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Instruction

Lesson 5.2.2: Investigating Scale Factors

Common Core State Standard

G–SRT.1 Verify experimentally the properties of dilations given by a center and a scale factor:

b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

Warm-Up 5.2.2 Debrief

1. If Hideki is 6 feet tall and the height at his shoulders is 5.1 feet, what is the scale factor of the change in height of his sister's swing?

$$\frac{\text{height after 1 swing}}{\text{initial height}} = \frac{5.1}{6} = 0.85$$

2. Rewrite this change as a percentage.

   0.85(100) = 85%

3. What is the reciprocal of this change in height?

   The reciprocal of $\frac{5.1}{6}$ is $\frac{6}{5.1}$.

4. Rewrite this reciprocal as a percentage.

$$\frac{6}{5.1} \approx 1.18$$

   1.18(100) = 118%

5. What might it mean in the context of the problem if you applied the reciprocal of the change to the height when Hideki's sister is swinging as high as the top of his head?

   Hideki might have pushed his sister again to cause her to swing higher.

Connection to the Lesson

- Students will need to be able to convert decimals to percents.
- Students can draw on the connection between scale factors and percent change in terms of when the dilation is an enlargement or a reduction. Students can use this connection to predict the effects of dilating a figure given a scale factor and center.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

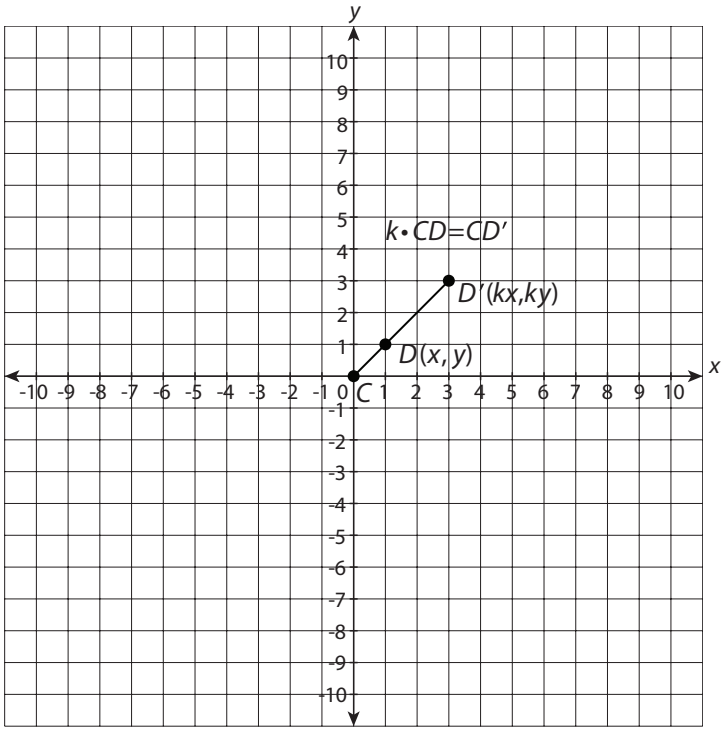
- operating with fractions, decimals, and percents (7.EE.2)
- converting among fractions, decimals, and percents (7.EE.2)

Introduction

A figure is dilated if the preimage can be mapped to the image using a scale factor through a center point, usually the origin. You have been determining if figures have been dilated, but how do you create a dilation? If the dilation is centered at the origin, use the scale factor and multiply each coordinate in the figure by that scale factor. If a distance is given, multiply the distance by the scale factor.

Key Concepts

- The notation for a dilation is as follows: $D_k(x, y) = (kx, ky)$.
- Multiply each coordinate of the figure by the scale factor when the center is at (0, 0).



- The lengths of each side in a figure also are multiplied by the scale factor.

SRB p. U5-35

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Instruction

- If you know the lengths of the sides in the preimage figure and you know the scale factor, you can calculate the lengths of the sides of the image by multiplying the preimage lengths by the scale factor.
- Remember that the dilation is an enlargement if $k > 1$, a reduction if $0 < k < 1$, and a congruency transformation if $k = 1$.

Common Errors/Misconceptions

- not applying the scale factor to both the x- and y-coordinates in the point
- improperly converting the decimal from a percentage
- missing the connection between the scale factor and the ratio of the image lengths to the preimage lengths

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Instruction

Guided Practice 5.2.2

Example 1

If $\overline{AB}$ has a length of 3 units and is dilated by a scale factor of 2.25, what is the length of $\overline{A'B'}$? Does this represent an enlargement or a reduction?

1. To determine the length of $\overline{A'B'}$, multiply the scale factor by the length of the segment.

   $AB = 3$; $k = 2.25$

   $A'B' = k \bullet AB$

   $A'B' = 2.25 \bullet 3 = 6.75$

   $\overline{A'B'}$ is 6.75 units long.

2. Determine the type of dilation.

   Since the scale factor is greater than 1, the dilation is an enlargement.

Example 2

A triangle has vertices G (2, –3), H (–6, 2), and J (0, 4). If the triangle is dilated by a scale factor of 0.5 through center C (0, 0), what are the image vertices? Draw the preimage and image on the coordinate plane.

1. Start with one vertex and multiply each coordinate by the scale factor, k.

   $D_k = (kx, ky)$

   $G' = D_{0.5}[G\ (2, -3)] = D_{0.5}(0.5 \bullet 2, 0.5 \bullet -3) = (1, -1.5)$

2. Repeat the process with another vertex. Multiply each coordinate of the vertex by the scale factor.

   $H' = D_{0.5}[H\ (-6, 2)] = D_{0.5}(0.5 \bullet -6, 0.5 \bullet 2) = (-3, 1)$

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Instruction

3. Repeat the process for the last vertex. Multiply each coordinate of the vertex by the scale factor.

$J' = D_{0.5}[J(0, 4)] = D_{0.5}(0.5 \bullet 0, 0.5 \bullet 4) = (0, 2)$

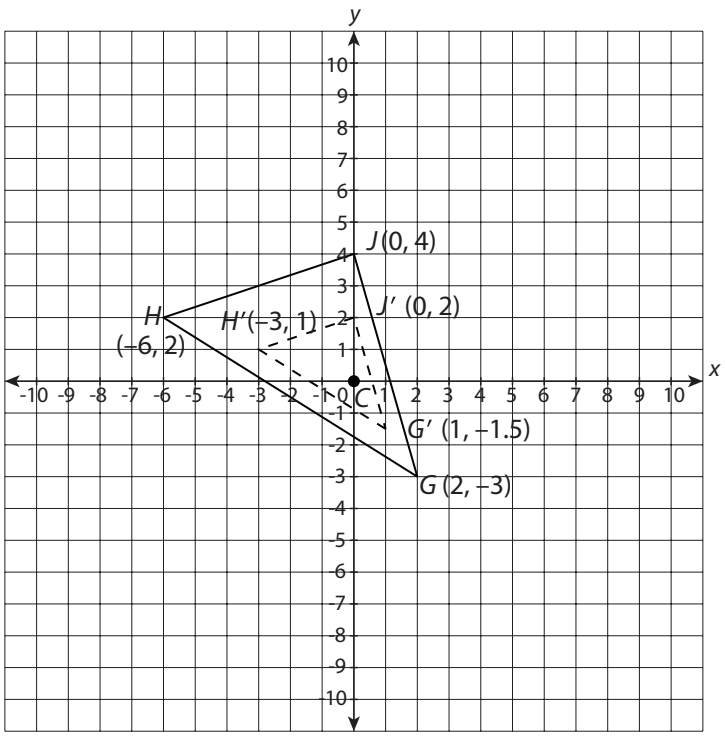
4. List the image vertices.

$G'(1, -1.5)$

$H'(-3, 1)$

$J'(0, 2)$

5. Draw the preimage and image on the coordinate plane.



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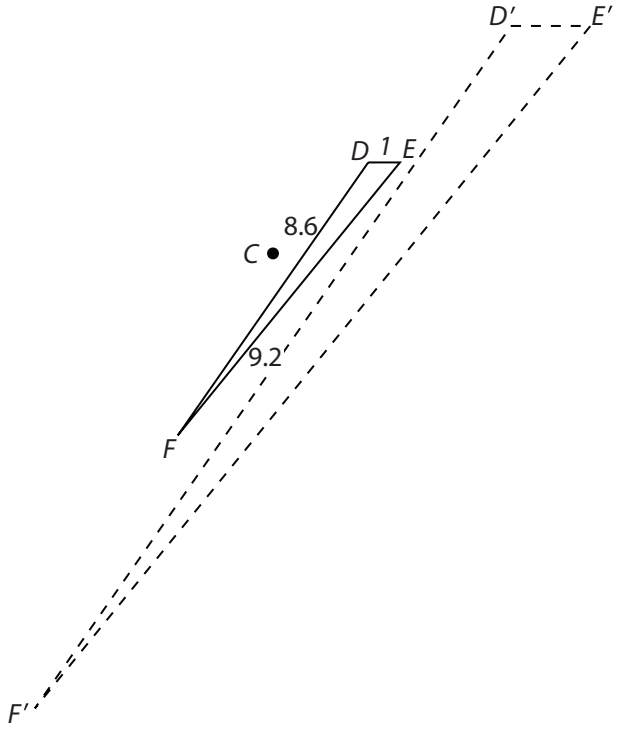
Instruction

Example 3

What are the side lengths of $\triangle D'E'F'$ with a scale factor of 2.5 given the following image and preimage, and the information that $DE = 1$, $EF = 9.2$, and $FD = 8.6$?



1. Choose a side to start with and multiply the scale factor (k) by that side length.

   $DE = 1$; $k = 2.5$

   $D'E' = k \bullet DE$

   $D'E' = 2.5 \bullet 1 = 2.5$

2. Choose a second side and multiply the scale factor by that side length.

   $EF = 9.2$; $k = 2.5$

   $E'F' = k \bullet EF$

   $E'F' = 2.5 \bullet 9.2 = 23$

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Instruction

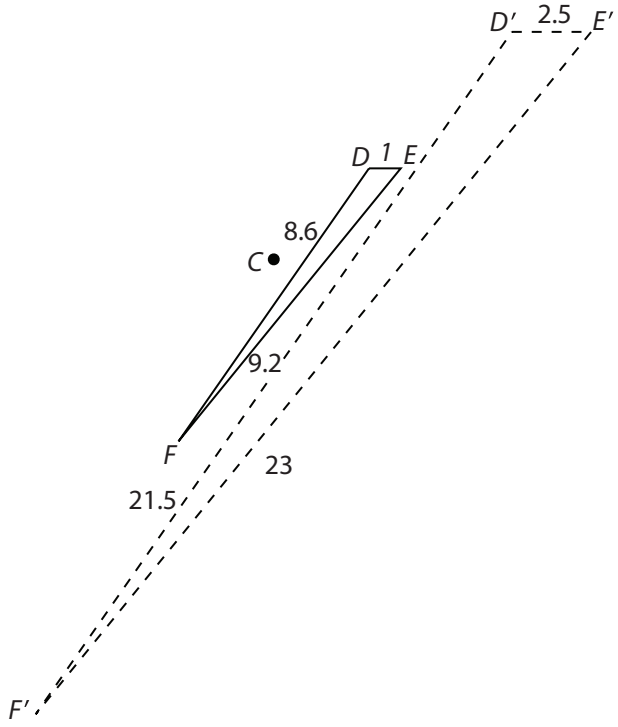
3. Choose the last side and multiply the scale factor by that side length.

   $FD = 8.6; k = 2.5$

   $F'D' = k \bullet FD$

   $F'D' = 2.5 \bullet 8.6 = 21.5$

4. Label the figure with the side lengths.



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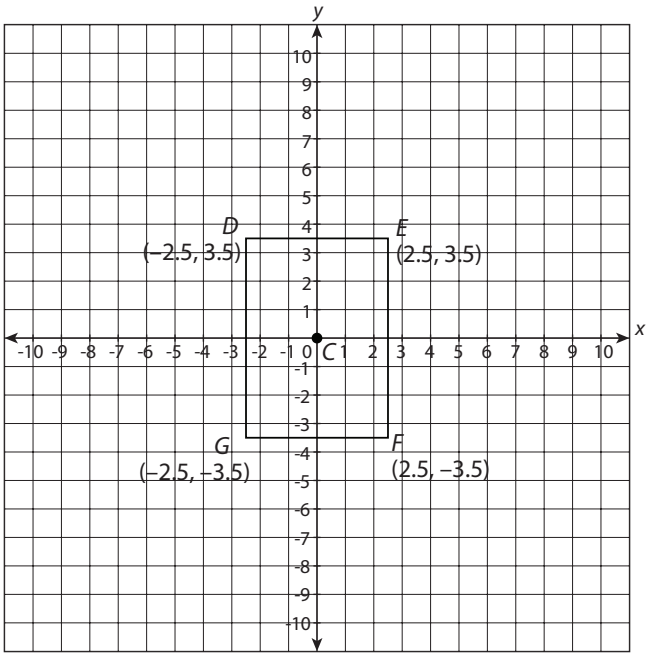
Problem-Based Task 5.2.2: The Bigger Picture

SMP	
1 ✓	2 ✓
3	4 ✓
5	6
7	8

A photographer wants to enlarge a 5×7 picture to an 8×10. However, she wants to preserve the image as it appears in the 5×7 without distorting the picture. Distortions happen when the width and height of the photo are not enlarged at the same scale. How can the photographer dilate a 5×7 picture to an 8×10 picture without distorting the picture? Describe a process for enlarging the picture so that the image is a dilation of the preimage. Give the coordinates for the image vertices. The preimage is pictured below with the center C (0, 0).



How can the photographer dilate a 5 × 7 picture to an 8 × 10 picture without distorting the picture?

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Problem-Based Task 5.2.2: The Bigger Picture

Coaching

a. What is the scale factor of the width from the preimage to the image?

b. What is the scale factor of the height from the preimage to the image?

c. How do these scale factors compare?

d. Which scale factor can you use consistently with the width and the height so that the picture will not be distorted?

e. How can you modify the preimage so that you can use the same scale factor and arrive at an 8×10 picture?

f. How can you determine numerically how to modify the picture?

g. What are the coordinates of the modified preimage?

h. What are the coordinates of the image after applying the scale factor?

i. Summarize your procedure and thinking process in dilating the original photograph.

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Instruction

Problem-Based Task 5.2.2: The Bigger Picture

Coaching Sample Responses

a. What is the scale factor of the width from the preimage to the image?

$$\frac{\text{width of image}}{\text{width of preimage}} = \frac{8}{5} = 1.6$$

b. What is the scale factor of the height from the preimage to the image?

$$\frac{\text{height of image}}{\text{height of preimage}} = \frac{10}{7} \approx 1.429$$

c. How do these scale factors compare?

The scale factor of the height is smaller than the scale factor for the width.

d. Which scale factor can you use consistently with the width and the height so that the picture will not be distorted?

Examine what happens when you apply the scale factor backward. In other words, find the reciprocal of each scale factor.

The scale factor for the width = 8/5. The reciprocal is 5/8 or 0.625.

The scale factor for the height = 10/7. The reciprocal is 7/10 or 0.7.

Use the smaller reciprocal scale factor since you can crop the original picture, but you cannot generate more of the original picture. This means when you go from the smaller preimage to the image, you will use the reciprocal of 5/8, which is 8/5, the scale factor of the width.

e. How can you modify the preimage so that you can use the same scale factor and arrive at an 8×10 picture?

Crop or trim the picture width so that the preimage width becomes smaller.

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f. How can you determine numerically how to modify the picture?

To determine by how much to trim the picture, set up a proportion with the unknown being the height of the original picture. Since we are using one scale factor for both dimensions, the second ratio is the ratio of the width of the image to the width of the preimage.

$$\frac{10}{x}=\frac{8}{5}$$
$$50=8x$$
$$x=6.25$$

The original height is 7 inches. The picture needs to be trimmed by 7 – 6.25 inches, or 0.75 inches, so that when it is enlarged, the proportions remain the same as the original picture. The image could be trimmed 0.75 inches at the bottom or the top or by a combination to avoid losing any of the important aspects of the picture.

g. What are the coordinates of the modified preimage?

Assuming equal trimming of the top and bottom of the picture, the amount to trim at each end is 0.75/2 = 0.375 inches. For negative coordinates, you must add 0.375 to show the trimming.

D: 3.5 – 0.375 = 3.125

E: 3.5 – 0.375 = 3.125

F: –3.5 + 0.375 = –3.125

G: –3.5 + 0.375 = –3.125

The coordinates of the modified preimage are as follows:

$D(-2.5, 3.5) \rightarrow D_m(-2.5, 3.125)$
$E(2.5, 3.5) \rightarrow E_m(2.5, 3.125)$
$F(2.5, -3.5) \rightarrow F_m(2.5, -3.125)$
$G(-2.5, -3.5) \rightarrow G_m(-2.5, -3.125)$

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h. What are the coordinates of the image after applying the scale factor?

Apply the scale factor of 8/5 or 1.6 to each coordinate in the modified preimage.

$D_m(-2.5, 3.125) \rightarrow D_m'(-4, 5)$

$E_m(2.5, 3.125) \rightarrow E_m'(4, 5)$

$F_m(2.5, -3.125) \rightarrow F_m'(4, -5)$

$G_m(-2.5, -3.125) \rightarrow G_m'(-4, -5)$

i. Summarize your procedure and thinking process in dilating the original photograph.

First, determine the scale factor to use by calculating the scale factor for each dimension of width and height. Use the scale factor that will allow you to trim the photograph rather than adding onto it. This means you use the scale factor with the smaller reciprocal because you will be going backward to determine how much to cut from the picture. The scale factor to use is the scale factor of the width, which is 8/5 or 1.6. The unknown is how long the image should be before dilation. You know that you want to end up with 10 inches in height. Apply the scale factor to 10 inches and the result is 6.25 inches. The picture height is 7 inches.

To modify the height of the picture, trim 0.75 inches from the picture. This can be done equally from the top and bottom (0.375 inches from the top and 0.375 inches from the bottom), from the top only, from the bottom only, or some combination of the top and bottom that is not equal. For simplicity, trim 0.375 inches from the top and the bottom. On the graph, this means moving each of the *y*-coordinates 0.375 units toward 0 parallel to the *y*-axis. The *x*-coordinates will remain the same because the width is not changing. Now, you apply the scale factor of 8/5 to each coordinate in all the vertices. The resulting image is a dilation of the trimmed picture.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 5.2.2: Investigating Scale Factors

Determine the lengths of the dilated segments given the preimage length and the scale factor.

1. $\overline{AB}$ is 2.25 units long and the segment is dilated by a scale factor of $k = 3.2$.

2. $\overline{GH}$ is 15.3 units long and is dilated by a scale factor of $k = \frac{2}{3}$.

3. $\overline{ST}$ is 20.5 units long and is dilated by a scale factor of $k = 0.6$.

4. $\overline{DE}$ is 30 units long and is dilated by a scale factor of $k = \frac{2}{3}$.

Determine the image vertices of each dilation given a center and scale factor.

5. $\triangle HJK$ has the following vertices: H (–7, –3), J (–5, –6), and K (–6, –8). What are the vertices under a dilation with a center at (0, 0) and a scale factor of 3?

6. $\triangle PQR$ has the following vertices: P (–6, 4), Q (5, 9), and R (–3, –6). What are the vertices under a dilation with a center at (0, 0) and a scale factor of $\frac{1}{2}$?

7. $\triangle MNO$ has the following vertices: M (–5, 8), N (7, –3), and O (–10, –4). What are the vertices under a dilation with a center at (0, 0) and a scale factor of 75%?

8. $\triangle ABD$ has the following vertices: A (6, 5), B (2, 2), and D (–3, 4). What are the vertices under a dilation with a center at (0, 0) and a scale factor of 140%?

9. $\triangle DEF$ has the following vertices: D (3, 2), E (6, 2), and F (–1.5, 4). What are the final vertices after 2 successive dilations with a center at (0, 0) and a scale factor of 2? What is the scale factor from $\triangle DEF$ to $\triangle D''E''F''$?

10. Miguel's family is renovating their kitchen. Miguel is comparing the floor plan for the old kitchen to the floor plan for the new kitchen. According to the floor plan for the new kitchen, the center island is going to be enlarged by a scale factor of 1.5. If in the old floor plan, the vertices of the original island are S (–3, 2), T (3, 2), U (3, –2), and V (–3, –2), what are the vertices of the new island? By what factor is the new island longer than the original? Assume each unit of the coordinate plane represents 1 foot.

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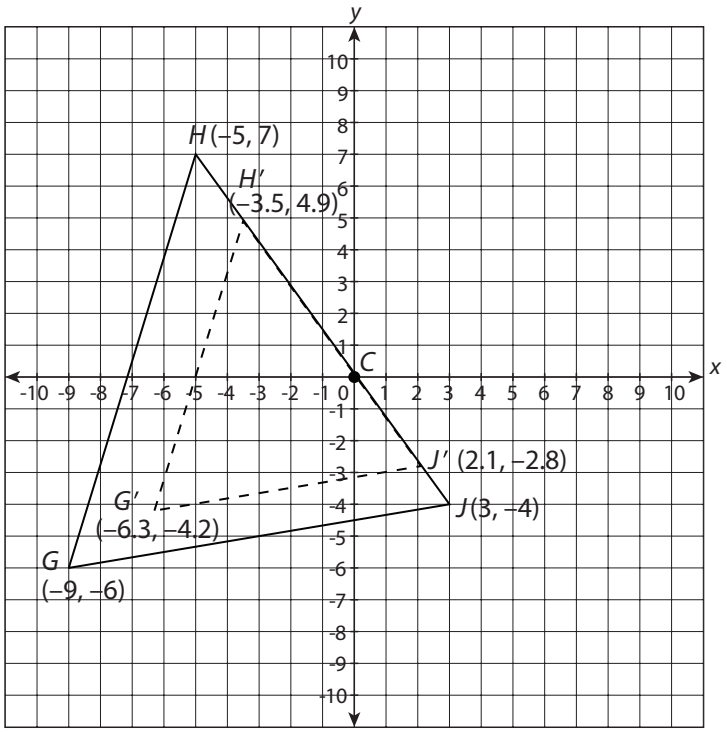
Progress Assessment

Circle the letter of the best answer.

1. Compare $\triangle GHJ$ to $\triangle G'H'J'$. Does the diagram below show a dilation? Why or why not?



a. Yes, the diagram represents a dilation because the preimage shares part of its side with the corresponding image side.

b. No, the diagram does not represent a dilation because the scale factors are inconsistent.

c. Yes, the diagram represents a dilation because the scale factors are consistent.

d. No, the diagram does not represent a dilation because the preimage shares part of its side with the corresponding image side.

continued

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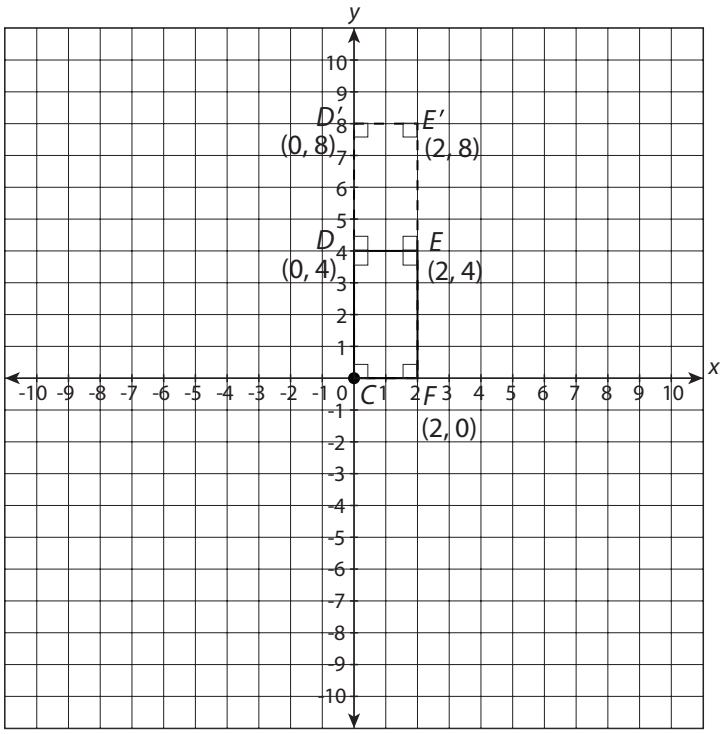
2. Compare $\square CDEF$ to $\square CD'E'F$. Does the diagram represent a dilation? Why or why not?



a. Yes, the diagram represents a dilation because all the angles are congruent within the figure and between the figures.

b. No, the diagram does not represent a dilation because the preimage points and corresponding image points are collinear with the center.

c. Yes, the diagram represents a dilation because the scale factors are consistent.

d. No, the diagram does not represent a dilation because the scale factors are inconsistent.

continued

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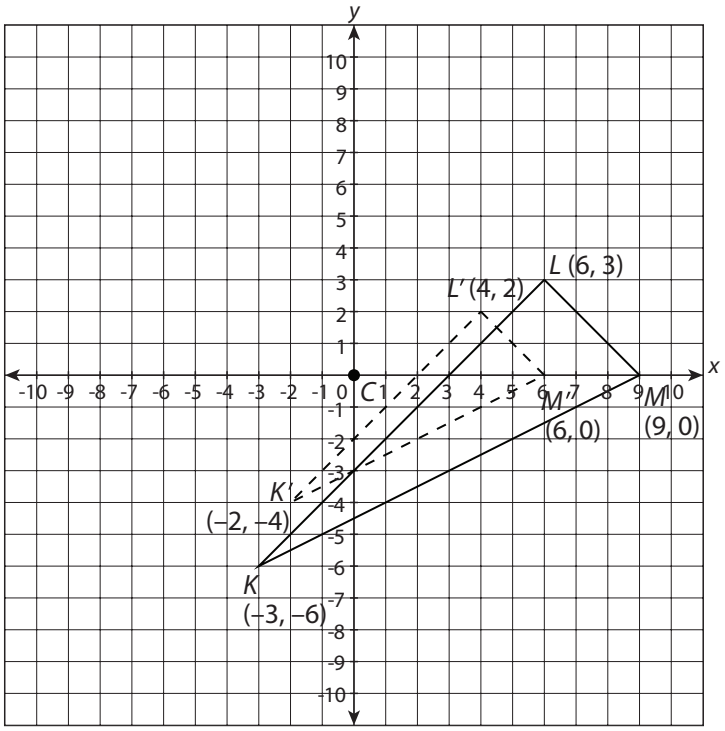
3. What is the scale factor of the dilation pictured below?



a. $k = \frac{2}{3}$

b. $k = \frac{3}{2}$

c. $k = \frac{1}{3}$

d. $k = 3$

4. If a dilation with a scale factor of $\frac{5}{4}$ takes place, what type of dilation is this?

   a. a reduction

   b. an enlargement

   c. a congruency transformation

   d. none of the above

continued

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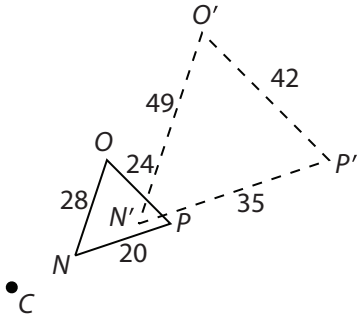
Assessment

5. What is the scale factor of the dilation pictured below?



a. $k = \frac{4}{7}$

b. $k = \frac{7}{4}$

c. $k = \frac{5}{7}$

d. $k = \frac{7}{5}$

6. If $\overline{AB}$ has a length of 16.4 units, what is the length of $\overline{A'B'}$ under a dilation with a scale factor of $k = 2.5$?

a. 8.2 units

b. 40 units

c. 41 units

d. 6.5 units

7. If $\overline{CD}$ has a length of 23.2 units, what is the length of $\overline{C'D'}$ under a dilation with a scale factor of 75%?

a. 1,740 units

b. 0.31 unit

c. 30.1 units

d. 17.4 units

8. If $\triangle QRS$ has vertices with the coordinates Q (–5, 6), R (2, 3), and S (–4, 0), what are the vertices of $\triangle Q'R'S'$ under a dilation with a scale factor of $\frac{5}{8}$ and the center at the origin, (0, 0)?

a. Q' (–3.125, 3.75), R' (1.25, 1.875), and S' (–2.5, 0)

b. Q' (–8, 9.6), R' (3.2, 4.8), and S' (–6.4, 0)

c. Q' (–3.125, 9.6), R' (1.25, 4.8), and S' (–2.5, 0)

d. Q' (–2.5, 3), R' (1, 1.5), and S' (–2, 0)

continued

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Assessment

9. If $\triangle TUV$ has vertices with the coordinates T (–3, –5), U (2, –1), and V (6, 2), what are the vertices of $\triangle T'U'V'$ under a dilation with a scale factor of 1.75 and the center at the origin, (0, 0)?

   a. T' (–5.25, –8.75), U' (3.5, –1.75), and V' (10.5, 3.5)

   b. T' (–1.71, –2.86), U' (1.14, –0.57), and V' (3.43, 1.14)

   c. T' (–4.5, –7.5), U' (3, –1.75), and V' (9, 3)

   d. T' (–2.25, –3.75), U' (1.5, –1.75), and V' (4.5, 1.5)

10. If $\triangle DEF$ has vertices with the coordinates D (–2, 4), E (8, –10), and F (–6, –2), what are the vertices of $\triangle D'E'F'$ under a dilation with a scale factor of 120% and the center at the origin, (0, 0)?

    a. D' (–24, –48), E' (96, –120), and F' (–72, –24)

    b. D' (–2.4, 4.8), E' (9.6, 12), and F' (–7.2, –2.4)

    c. D' (–0.24, –0.48), E' (0.96, –0.12), and F' (–0.72, –0.24)

    d. D' (–1.67, –3.33), E' (6.67, –8.33), and F' (–5, –1.67)

Use what you know about dilations to solve the following problem.

11. A publishing company makes cardboard posters of some of its book covers to use as advertising in bookstores. Each cardboard poster is at a scale factor of 180% compared to the original print book cover.

    a. Does this represent an enlargement, reduction, or congruency transformation? Explain.

    b. If the vertices of the original book's front cover are G (0, 0), H (0, 8), I (5, 8), and J (5, 0), what are the image vertices under the dilation with a center at the origin, (0, 0)?

    c. How does the perimeter of the preimage compare to the image? Assume each grid unit equals 1 inch.

    d. How does the area of the preimage compare to the image? Assume each grid unit equals 1 inch.

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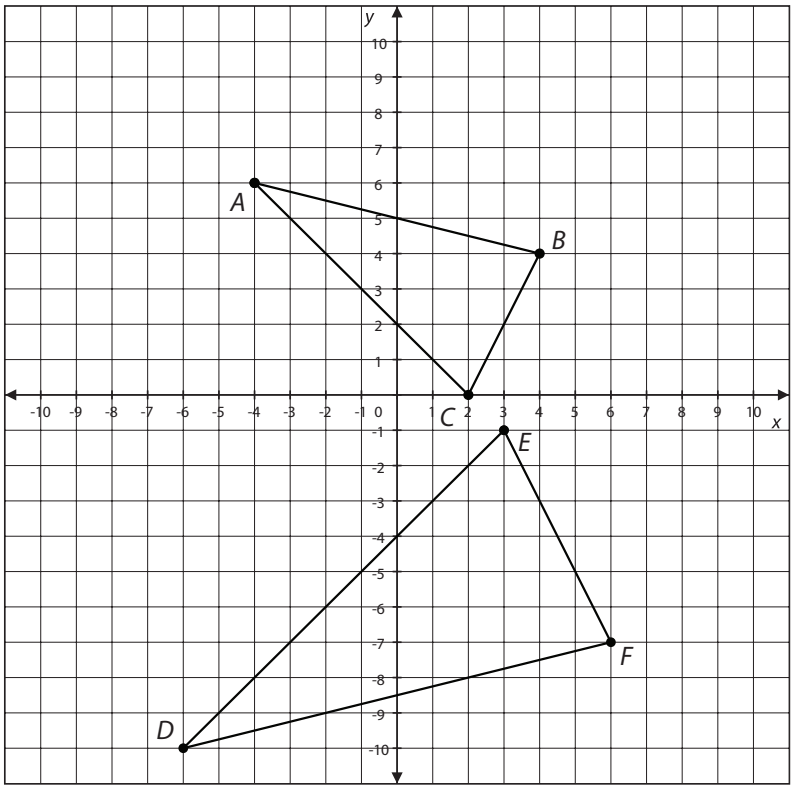
Pre-Assessment

Circle the letter of the best answer.

1. Are the two triangles similar? Why or why not?



a. They are not similar; $\triangle ABC$ has undergone a transformation that does not preserve congruency.

b. They are similar; $\triangle ABC$ has undergone a transformation that preserves congruency.

c. They are not similar; $\triangle ABC$ has undergone a transformation that does not preserve similarity.

d. They are similar; $\triangle ABC$ has undergone a transformation that preserves similarity.

continued

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2. $\triangle ABC$ has undergone a series of similarity transformations to produce $\triangle EFG$. Which of the following statements is NOT true?

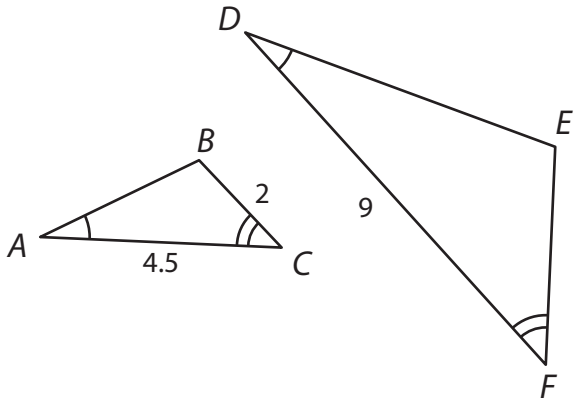
   a. The angles of $\triangle EFG$ are proportional to the angles of $\triangle ABC$.

   b. The angles of $\triangle EFG$ are the same as the angles of $\triangle ABC$.

   c. The sides of $\triangle EFG$ are proportional to the sides of $\triangle ABC$.

   d. $\triangle ABC$ is similar to $\triangle EFG$.

3. What is the length of $\overline{EF}$?



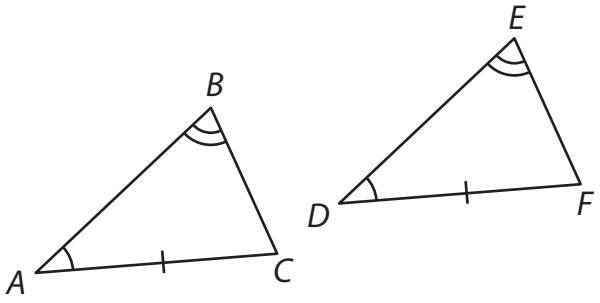
   a. 4 units

   b. 1 unit

   c. 9 units

   d. 2 units

4. Are the two triangles similar? Why or why not?



   a. Yes, they are similar because of the AA Similarity Statement.

   b. Yes, they are similar because of the ASA Congruence Statement.

   c. No, they are not similar because they are congruent.

   d. There is not enough information to determine similarity.

continued

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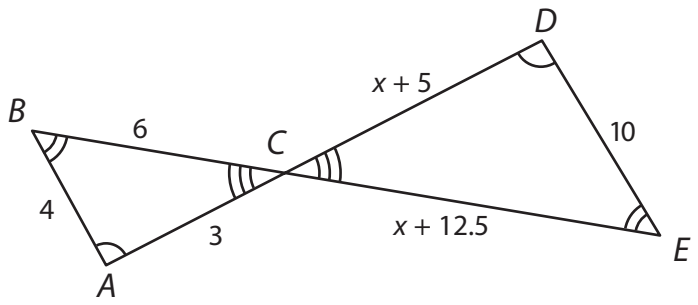
5. Identify the similar triangles. Find x and the missing side lengths.



a. $\triangle ABC \sim \triangle CDE$; $x = 2.5$; $DC = 7.5$; $EC = 15$

b. $\triangle ABC \sim \triangle DEC$; $x = 2.5$; $DC = 15$; $EC = 7.5$

c. $\triangle ABC \sim \triangle DEC$; $x = 2.5$; $DC = 7.5$; $EC = 15$

d. $\triangle ABC \sim \triangle DEC$; $x = 5.2$; $DC = 10.5$; $EC = 18$

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Lesson 3: Defining and Applying Similarity

Instruction

Common Core State Standards

SMP	
1 ✓	2 ✓
3	4
5 ✓	6
7 ✓	8

G–SRT.2 Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

G–SRT.3 Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

Essential Questions

1. What does it mean for two triangles to be similar?
2. How can you prove that two triangles are similar?
3. How can you use similar triangles to solve problems?

WORDS TO KNOW

Angle-Angle (AA) Similarity Statement	If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.
proportional	having a constant ratio to another quantity
ratio of similitude	a ratio of corresponding sides; also known as the scale factor
similar	two figures that are the same shape but not necessarily the same size; the symbol for representing similarity between figures is $\sim$
similarity transformation	a dilation; a transformation that results in the size of a figure changing, but not the shape

SRB p. U5-41

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Recommended Resources

- Analyzemath.com. "Similar Triangles Examples."

  http://www.walch.com/rr/00021

  This site provides examples and illustrations of similarity criteria.

- Math Open Reference. "Similar Triangles."

  http://www.walch.com/rr/00022

  This site includes a summary of similarity as well as links to tests of similarity.

- Math.com. "Similar Figures."

  http://www.walch.com/rr/00023

  This site provides a summary of similar figures and how to calculate lengths of sides of similar figures.

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Lesson 5.3.1: Defining Similarity

Warm-Up 5.3.1

A poster of the tallest buildings in the world hangs in a hallway. The scale on the poster is 0.5 inches = 45 feet.

1. The height of the tallest building in the world, Burj Khalifa in Dubai, is about 30.25 inches on the poster. What is the actual height of Burj Khalifa?

2. The height of the tallest building in the United States, Willis Tower (formerly known as Sears Tower), is about 16 inches on the poster. What is the actual height of Willis Tower?

3. How much taller is Burj Khalifa than Willis Tower?

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Instruction

Lesson 5.3.1: Defining Similarity

Common Core State Standard

G–SRT.2 Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

Warm-Up 5.3.1 Debrief

1. The height of the tallest building in the world, Burj Khalifa in Dubai, is about 30.25 inches on the poster. What is the actual height of Burj Khalifa?

   Set up a proportion. Let h represent the actual height of the building.

   $\frac{0.5}{45} = \frac{30.25}{h}$ Proportion

   $0.5h = 45(30.25)$ Simplify through cross-multiplication.

   $0.5h = 1361.25$ Solve for h.

   $h = 2722.5$

   The actual height of Burj Khalifa is approximately 2,722.5 feet.

2. The height of the tallest building in the United States, Willis Tower (formerly known as Sears Tower), is about 16 inches on the poster. What is the actual height of Willis Tower?

   Create a proportion to find the actual height of the building, h.

   $\frac{0.5}{45} = \frac{16}{h}$ Proportion

   $0.5h = 45(16)$ Simplify through cross-multiplication.

   $0.5h = 720$ Solve for h.

   $h = 1440$

   The actual height of Willis Tower is approximately 1,440 feet.

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3. How much taller is Burj Khalifa than Willis Tower?

   Find the difference of the calculated heights of the two buildings.

   2722.5 – 1440 = 1282.5

   Burj Khalifa is approximately 1,282.5 feet taller than Willis Tower.

Connection to the Lesson

- Students will use ratios and proportions to determine the ratio of similitude and calculate missing lengths of similar triangles.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- creating ratios (7.RP.2c)
- solving proportions (7.RP.3)
- identifying congruent triangles (G–CO.6)
- calculating the lengths of triangle sides using the distance formula (8.G.8)
- recognizing transformations performed as a combination of translations, reflections, rotations, and/or dilations (G–CO.2)

Introduction

Congruent triangles have corresponding angle measures that are the same and corresponding side lengths that are the same. If two triangles are congruent, they are also **similar**. Similar triangles have the same shape, but may be different in size. It is possible for two triangles to be similar but not congruent. Just like with determining congruency, it is possible to determine similarity based on the angle measures and lengths of the sides of the triangles.

Key Concepts

- To determine whether two triangles are similar, observe the angle measures and the side lengths of the triangles.
- When a triangle is transformed by a **similarity transformation** (a dilation, possibly combined with a translation, rotation, and/or reflection), the result is a triangle with a different position and size, but the same shape.
- If two triangles are similar, then their corresponding angles are congruent and the measures of their corresponding sides are **proportional**, or have a constant ratio.
- The ratio of corresponding sides is known as the **ratio of similitude**.
- The scale factor of the dilation is equal to the ratio of similitude.
- Similar triangles with a scale factor of 1 are congruent triangles.
- Like with congruent triangles, corresponding angles and sides can be determined by the order of the letters.
- If $\triangle ABC$ is similar to $\triangle DEF$, the vertices of the two triangles correspond in the same order as they are named.

SRB p. U5-43

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Instruction

- The symbol $\rightarrow$ shows that parts are corresponding.

  $\angle A \rightarrow \angle D$; they are congruent.

  $\angle B \rightarrow \angle E$; they are congruent.

  $\angle C \rightarrow \angle F$; they are congruent.

- The corresponding angles are used to name the corresponding sides.

  $\overline{AB} \rightarrow \overline{DE}$

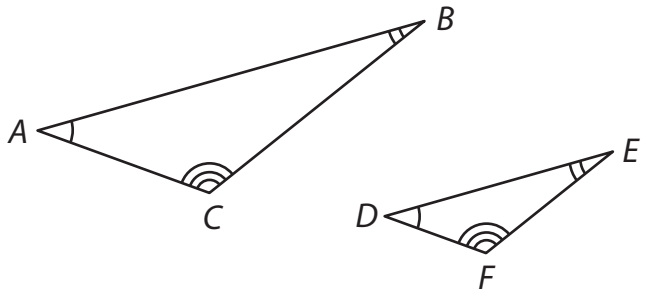
  $\overline{BC} \rightarrow \overline{EF}$

  $\overline{AC} \rightarrow \overline{DF}$

  $$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

- Observe the diagrams of $\triangle ABC$ and $\triangle DEF$.
- The symbol for similarity ($\sim$) is used to show that figures are similar.

  $\triangle ABC \sim \triangle DEF$



  $\angle A \cong \angle D$

  $\angle B \cong \angle E$

  $\angle C \cong \angle F$

  $$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Common Errors/Misconceptions

- incorrectly identifying corresponding parts of triangles
- assuming corresponding parts indicate congruent parts
- assuming alphabetical order indicates congruence
- changing the order of named triangles, causing parts to be incorrectly interpreted as congruent

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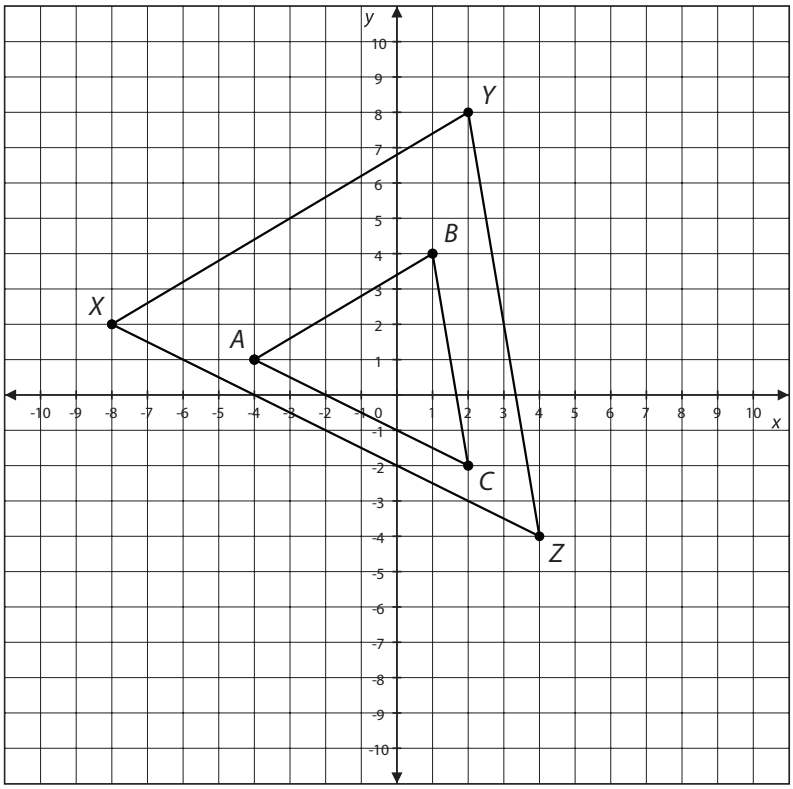
Guided Practice 5.3.1

Example 1

Use the definition of similarity in terms of similarity transformations to determine whether the two figures are similar. Explain your answer.



1. Examine the orientation of the triangles.

$\triangle ABC$		$\triangle XYZ$	
Side	**Orientation**	**Side**	**Orientation**
$\overline{AB}$	Top left side of triangle	$\overline{XY}$	Top left side of triangle
$\overline{AC}$	Bottom left side of triangle	$\overline{XZ}$	Bottom left side of triangle
$\overline{BC}$	Right side of triangle	$\overline{YZ}$	Right side of triangle

The orientation of the triangles has remained the same, indicating translation, dilation, stretch, or compression.

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2. Determine whether a dilation has taken place by calculating the scale factor.

First, identify the vertices of each triangle.

A (–4, 1), B (1, 4), and C (2, –2)

X (–8, 2), Y (2, 8), and Z (4, –4)

Then, find the length of each side of $\triangle ABC$ and $\triangle XYZ$ using the distance formula, $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

Calculate the distance of $\overline{AB}$.

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	Distance formula
$d=\sqrt{[(1)-(-4)]^2+[(4)-(1)]^2}$	Substitute (–4, 1) and (1, 4) for (x_1, y_1) and (x_2, y_2).
$d=\sqrt{(5)^2+(3)^2}$	Simplify.
$d=\sqrt{25+9}$	
$d=\sqrt{34}$	

The distance of $\overline{AB}$ is $\sqrt{34}$ units.

Calculate the distance of $\overline{BC}$.

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	Distance formula
$d=\sqrt{[(2)-(1)]^2+[(-2)-(4)]^2}$	Substitute (1, 4) and (2, –2) for (x_1, y_1) and (x_2, y_2).
$d=\sqrt{(1)^2+(-6)^2}$	Simplify.
$d=\sqrt{1+36}$	
$d=\sqrt{37}$	

The distance of $\overline{BC}$ is $\sqrt{37}$ units.

(*continued*)

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Instruction

Calculate the distance of $\overline{AC}$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance formula

$d = \sqrt{[(2) - (-4)]^2 + [(-2) - (1)]^2}$ Substitute (–4, 1) and (2, –2) for (x_1, y_1) and (x_2, y_2).

$d = \sqrt{(6)^2 + (-3)^2}$ Simplify.

$d = \sqrt{36 + 9}$

$d = \sqrt{45} = \sqrt{9 \bullet 5} = 3\sqrt{5}$

The distance of $\overline{AC}$ is $3\sqrt{5}$ units.

Calculate the distance of $\overline{XY}$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance formula

$d = \sqrt{[(2) - (-8)]^2 + [(8) - (2)]^2}$ Substitute (–8, 2) and (2, 8) for (x_1, y_1) and (x_2, y_2).

$d = \sqrt{(10)^2 + (6)^2}$ Simplify.

$d = \sqrt{100 + 36}$

$d = \sqrt{136} = \sqrt{4 \bullet 34} = 2\sqrt{34}$

The distance of $\overline{XY}$ is $2\sqrt{34}$ units.

(*continued*)

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Instruction

Calculate the distance of $\overline{YZ}$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance formula

$d = \sqrt{[(4) - (2)]^2 + [(-4) - (8)]^2}$ Substitute (2, 8) and (4, –4) for (x_1, y_1) and (x_2, y_2).

$d = \sqrt{(2)^2 + (-12)^2}$ Simplify.

$d = \sqrt{4 + 144}$

$d = \sqrt{148} = \sqrt{4 \bullet 37} = 2\sqrt{37}$

The distance of $\overline{YZ}$ is $2\sqrt{37}$ units.

Calculate the distance of $\overline{XZ}$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance formula

$d = \sqrt{[(4) - (-8)]^2 + [(-4) - (2)]^2}$ Substitute (–8, 2) and (4, –4) for (x_1, y_1) and (x_2, y_2).

$d = \sqrt{(12)^2 + (-6)^2}$ Simplify.

$d = \sqrt{144 + 36}$

$d = \sqrt{180} = \sqrt{36 \bullet 5} = 6\sqrt{5}$

The distance of $\overline{XZ}$ is $6\sqrt{5}$ units.

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Instruction

3. Calculate the scale factor of the changes in the side lengths.

   Divide the side lengths of $\triangle XYZ$ by the side lengths of $\triangle ABC$.

   $$\frac{XY}{AB}=\frac{2\sqrt{34}}{\sqrt{34}}=2$$

   $$\frac{YZ}{BC}=\frac{2\sqrt{37}}{\sqrt{37}}=2$$

   $$\frac{XZ}{AC}=\frac{6\sqrt{5}}{3\sqrt{5}}=2$$

   The scale factor is constant between each pair of corresponding sides.

4. Determine if another transformation has taken place.

   Multiply the coordinate of each vertex of the preimage by the scale factor, k.

   $D_k(x, y) = (kx, ky)$ Dilation by a scale factor of k

   $D_2(-4, 1) = [2(-4), 2(1)] = (-8, 2)$

   $D_2(1, 4) = [2(1), 2(4)] = (2, 8)$

   $D_2(2, -2) = [2(2), 2(-2)] = (4, -4)$

   You can map $\triangle ABC$ onto $\triangle XYZ$ by the dilation with a scale factor of 2.

5. State your conclusion.

   A dilation is a similarity transformation; therefore, $\triangle ABC$ and $\triangle XYZ$ are similar. The ratio of similitude is 2.

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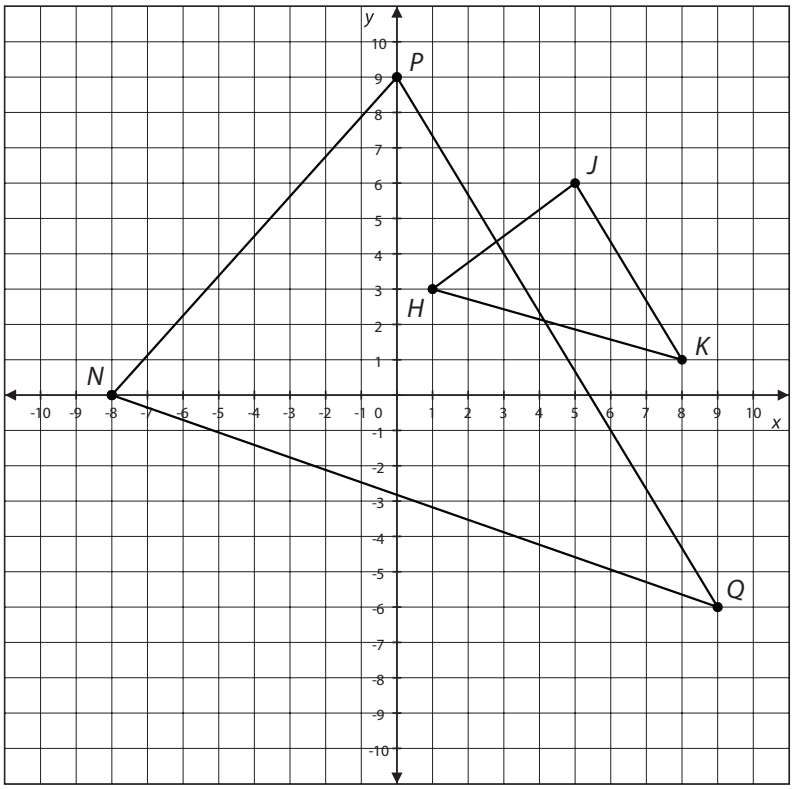
Instruction

Example 2

Use the definition of similarity in terms of similarity transformations to determine whether the two figures are similar. Explain your answer.



1. Examine the angle measures of the triangles.

   Use a protractor or construction methods to determine if corresponding angles are congruent.

   None of the angles of $\triangle HJK$ are congruent to the angles of $\triangle NPQ$.

2. Summarize your findings.

   Similarity transformations preserve angle measure.

   The angles of $\triangle HJK$ and $\triangle NPQ$ are not congruent.

   There are no sequences of transformations that will map $\triangle HJK$ onto $\triangle NPQ$.

   $\triangle HJK$ and $\triangle NPQ$ are not similar triangles.

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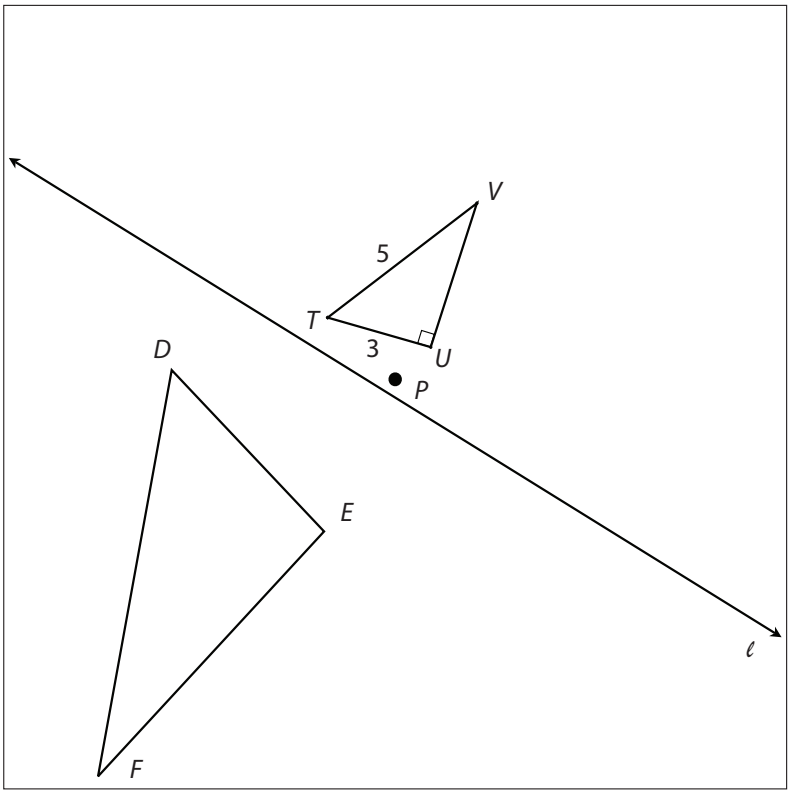
Instruction

Example 3

A dilation of $\triangle TUV$ centered at point P with a scale factor of 2 is then reflected over the line ℓ. Determine if $\triangle TUV$ is similar to $\triangle DEF$. If possible, find the unknown angle measures and lengths in $\triangle DEF$.



1. Determine if $\triangle TUV$ and $\triangle DEF$ are similar.

   The transformations performed on $\triangle TUV$ are dilation and reflection.

   The sequence of dilating and reflecting a figure is a similarity transformation; therefore, $\triangle TUV \sim \triangle DEF$.

2. Identify the pairs of congruent angles.

   Corresponding angles of similar triangles are congruent.

   $\angle T \cong \angle D$

   $\angle U \cong \angle E$

   $\angle V \cong \angle F$

   Since $\angle U$ is marked as a right angle, $\angle E$ must also be a right angle.

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Instruction

3. Identify the known lengths of $\triangle DEF$.

Corresponding sides of similar triangles are proportional.

The ratio of similitude is equal to the scale factor used in the dilation of the figure.

$$\frac{DE}{TU}=\frac{EF}{UV}=\frac{DF}{TV}=2$$

Since the length of $\overline{TU}$ is 3 units, the length of the corresponding side $\overline{DE}$ can be found using the ratio of similitude.

$$\frac{DE}{3}=2$$

$$DE = 6$$

$\overline{DE}$ is 6 units long.

Since the length of $\overline{TV}$ is 5 units, the length of the corresponding side $\overline{DF}$ can be found.

$$\frac{DF}{5}=2$$

$$DF = 10$$

$\overline{DF}$ is 10 units long.

Using the Pythagorean Theorem, we can determine that the length of $\overline{UV}$ is 4 units, so the length of the corresponding side $\overline{EF}$ can be found.

$$\frac{EF}{4}=2$$

$$EF = 8$$

$\overline{EF}$ is 8 units long.

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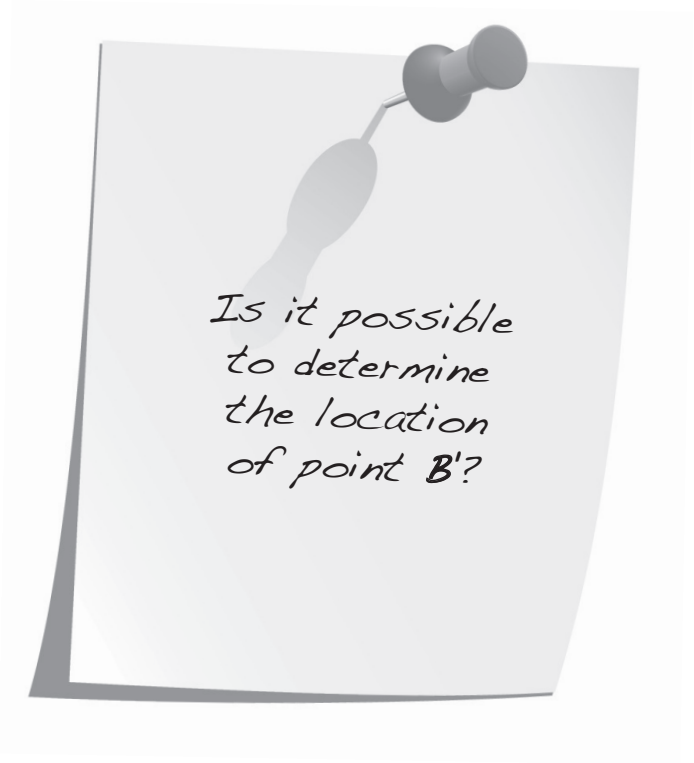
Problem-Based Task 5.3.1: Video Game Transformations

SMP	
1	2 ✓
3	4
5 ✓	6
7 ✓	8

The creators of a new video game are in the early design stages and are using the right triangle ABC on a coordinate plane to represent the movement of the character in the actual game setting. The coordinates of the points are A (–6, –9), B (–3, –9), and C (–3, –6). After a series of similarity transformations, the locations of the end points of the hypotenuse of the new image are A' (6, –4) and C' (4, –2). Is it possible to determine the location of point B'?



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Problem-Based Task 5.3.1: Video Game Transformations

Coaching

a. What is the length of hypotenuse $\overline{AC}$?

b. What is the length of $\overline{A'C'}$?

c. Is the length of $\overline{A'C'}$ shorter or longer than the length of $\overline{AC}$?

d. By what scale factor was the length of $\overline{AC}$ changed?

e. If a series of similarity transformations were carried out, what would you know about the lengths of the remaining sides of the image compared to the lengths of the preimage?

f. What is the length of $\overline{AB}$?

g. What is the length of $\overline{A'B'}$?

h. What is the length of $\overline{BC}$?

i. What is the length of $\overline{B'C'}$?

j. What are the possible locations of point B'?

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Instruction

Problem-Based Task 5.3.1: Video Game Transformations

Coaching Sample Responses

a. What is the length of hypotenuse $\overline{AC}$?

Use the distance formula to calculate the length of $\overline{AC}$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance formula

$d = \sqrt{[(-3) - (-6)]^2 + [(-6) - (-9)]^2}$ Substitute (–6, –9) and (–3, –6) for (x_1, y_1) and (x_2, y_2).

$d = \sqrt{(3)^2 + (3)^2}$ Simplify.

$d = \sqrt{18} = \sqrt{9 \bullet 2} = 3\sqrt{2}$

The length of $\overline{AC}$ is $3\sqrt{2}$ units.

b. What is the length of $\overline{A'C'}$?

Use the distance formula to calculate the length of $\overline{A'C'}$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance formula

$d = \sqrt{[(4) - (6)]^2 + [(-2) - (-4)]^2}$ Substitute (6, –4) and (4, –2) for (x_1, y_1) and (x_2, y_2).

$d = \sqrt{(-2)^2 + (2)^2}$ Simplify.

$d = \sqrt{8} = \sqrt{4 \bullet 2} = 2\sqrt{2}$

The length of $\overline{A'C'}$ is $2\sqrt{2}$ units.

c. Is the length of $\overline{A'C'}$ shorter or longer than the length of $\overline{AC}$?

The length of $\overline{AC}$ is $3\sqrt{2}$.

The length of $\overline{A'C'}$ is $2\sqrt{2}$ units.

The length of $\overline{A'C'}$ is shorter than the length of $\overline{AC}$.

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d. By what scale factor was the length of $\overline{AC}$ changed?

Divide the length of $\overline{A'C'}$ by the length of $\overline{AC}$.

$$\frac{A'C'}{AC}=\frac{2\sqrt{2}}{3\sqrt{2}}=\frac{2}{3}$$

$\overline{A'C'}$ is $\frac{2}{3}$ the length of $\overline{AC}$.

e. If a series of similarity transformations were carried out, what would you know about the lengths of the remaining sides of the image compared to the lengths of the preimage?

The lengths of the sides of the image are all proportional to the preimage.

Each side of $\triangle A'B'C'$ is $\frac{2}{3}$ the length of the sides of $\triangle ABC$.

f. What is the length of $\overline{AB}$?

Use the distance formula to calculate the length of $\overline{AB}$.

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ Distance formula

$d=\sqrt{[(-3)-(-6)]^2+[(-9)-(-9)]^2}$ Substitute (–6, –9) and (–3, –9) for (x_1, y_1) and (x_2, y_2).

$d=\sqrt{(3)^2+(0)^2}$ Simplify.

$d=\sqrt{9}=3$

The length of $\overline{AB}$ is 3 units.

g. What is the length of $\overline{A'B'}$?

Use the scale factor of $\frac{2}{3}$ to find the length of $\overline{A'B'}$.

$\frac{2}{3}\bullet 3=2$

The length of $\overline{A'B'}$ is 2 units.

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h. What is the length of $\overline{BC}$?

Use the distance formula to calculate the length of $\overline{BC}$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance formula

$d = \sqrt{[(-3) - (-3)]^2 + [(-6) - (-9)]^2}$ Substitute (–3, –9) and (–3, –6) for (x_1, y_1) and (x_2, y_2).

$d = \sqrt{(0)^2 + (3)^2}$ Simplify.

$d = \sqrt{9} = 3$

The length of $\overline{BC}$ is 3 units.

i. What is the length of $\overline{B'C'}$?

Use the scale factor of $\frac{2}{3}$ to find the length of $\overline{B'C'}$.

$\frac{2}{3} \bullet 3 = 2$

The length of $\overline{B'C'}$ is 2 units.

j. What are the possible locations of point B'?

There are two possible locations for point B'.

One location for point B' is 2 units to the right of C' and 2 units up from A'.

This location is at (4, –4).

The second location is 2 units down from C' and 2 units to the left of A'.

This location is at (6, –2).

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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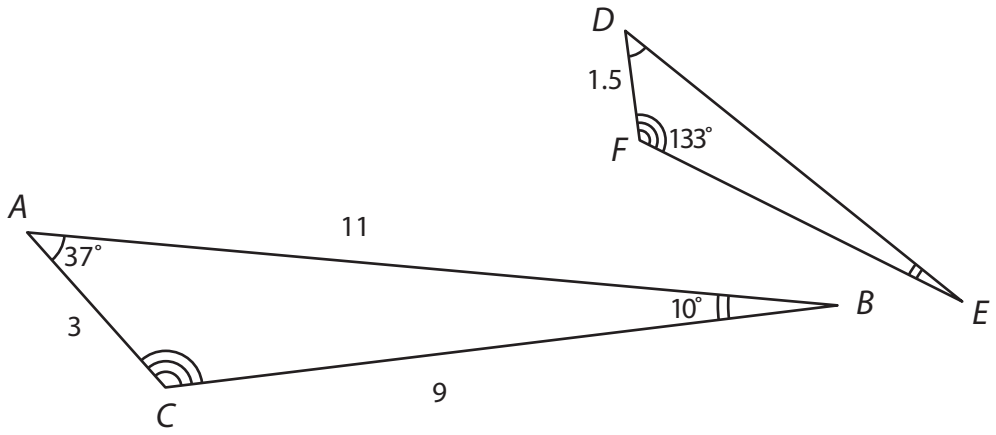
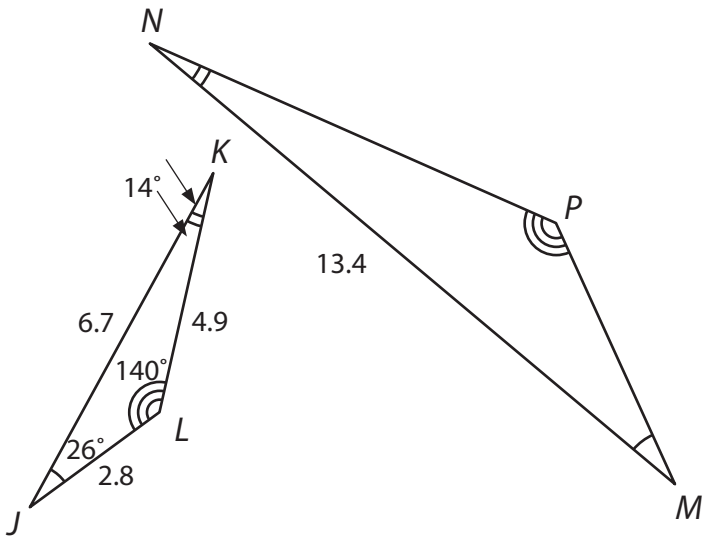
Practice 5.3.1: Defining Similarity

Find all the angle measures and side lengths for each triangle of the given similar pairs.

1. $\triangle ABC \sim \triangle DEF$



2. $\triangle JKL \sim \triangle MNP$



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SWB p. 49; SRB Practice 5.3.1 p. U5-53

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Determine if the two given triangles are similar. Use the definition of similarity in terms of similarity transformations to explain your answer.

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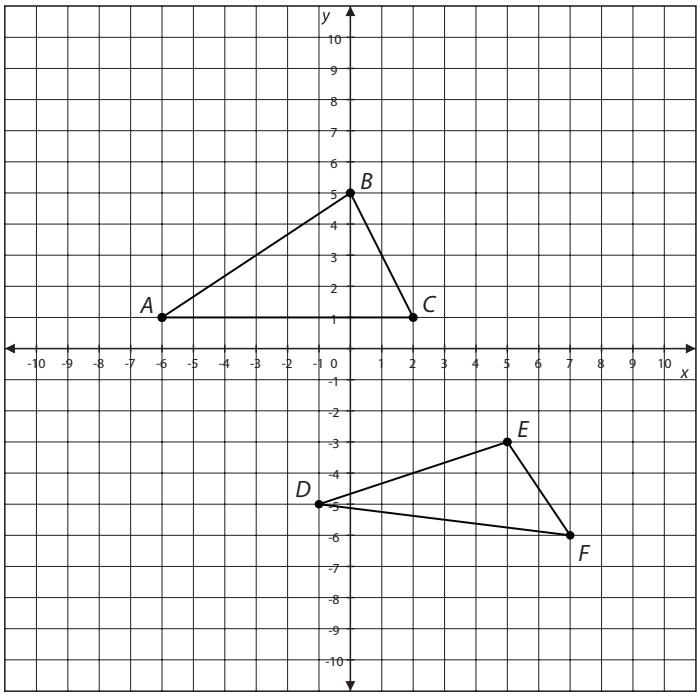

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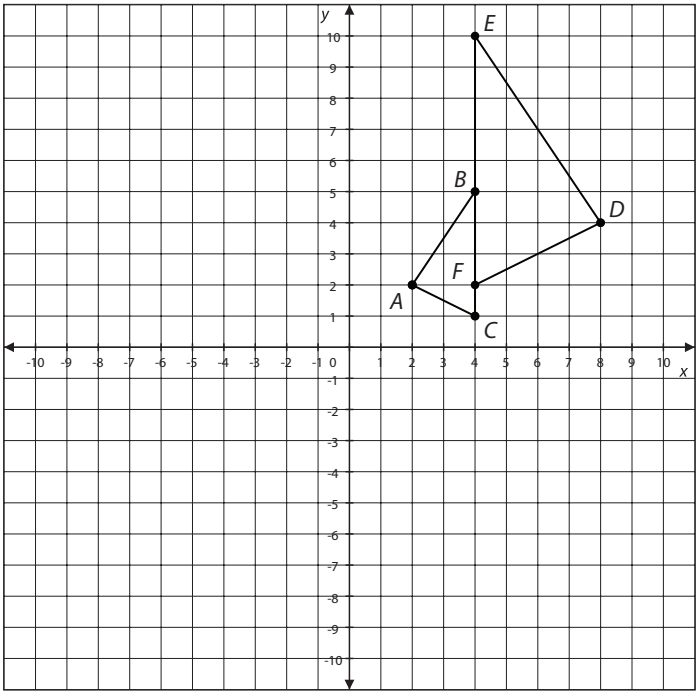

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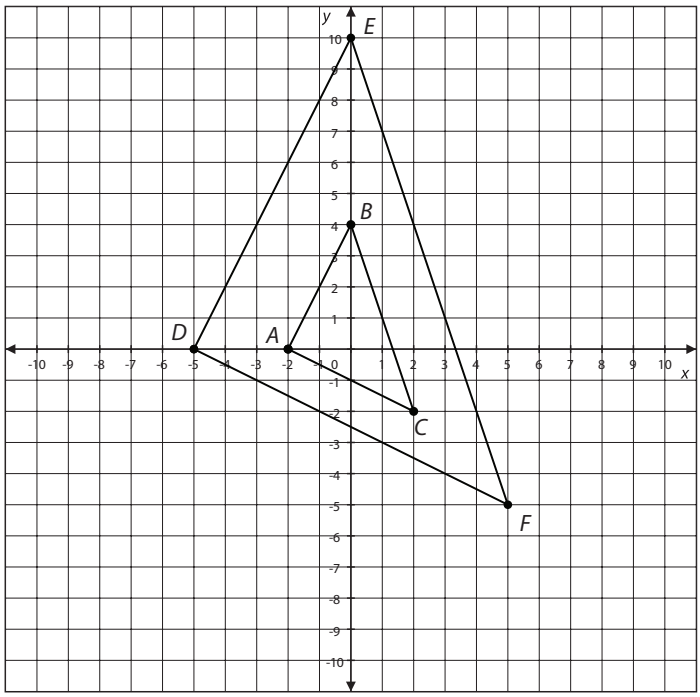

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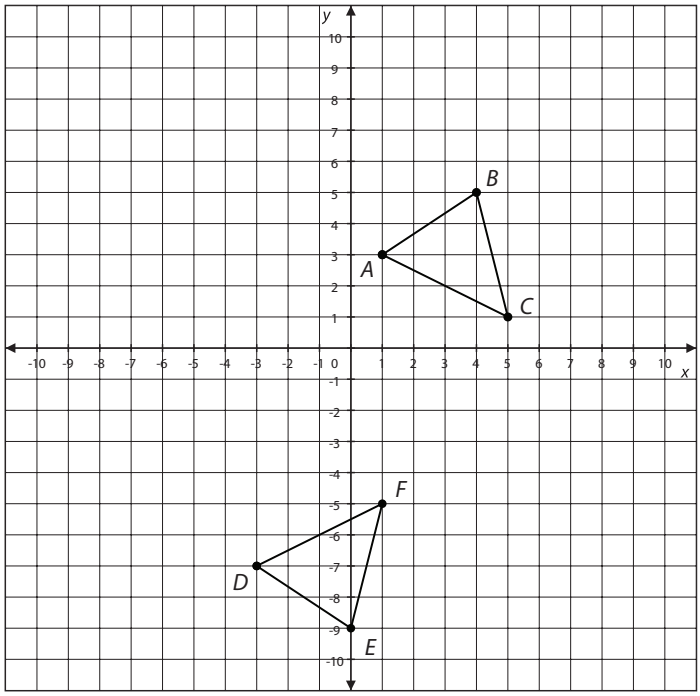

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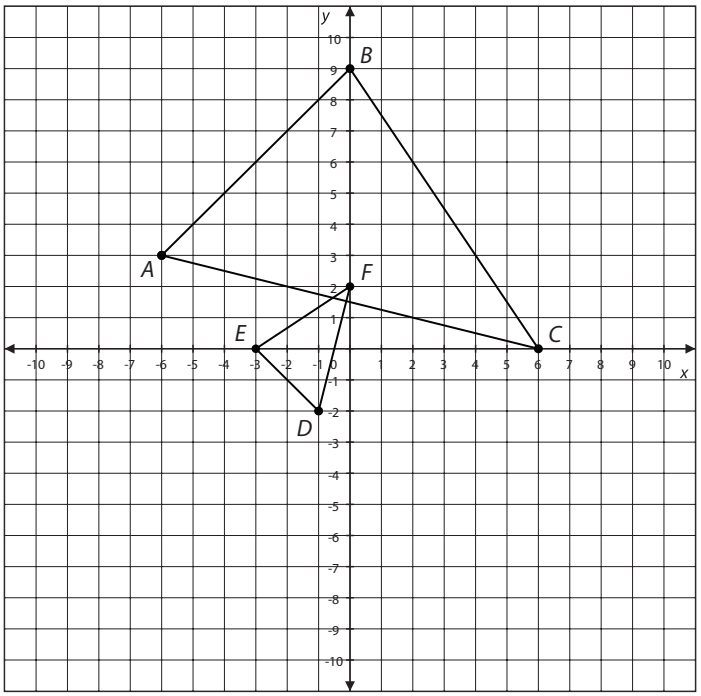

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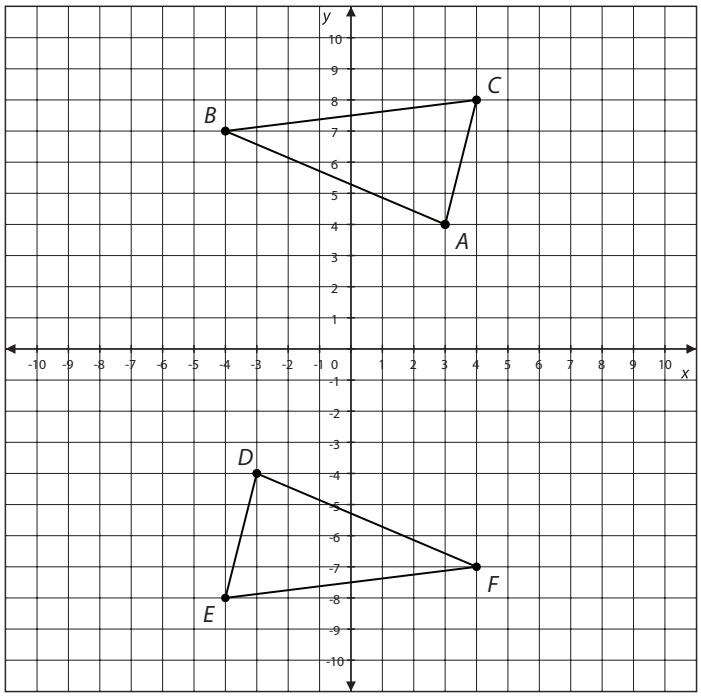

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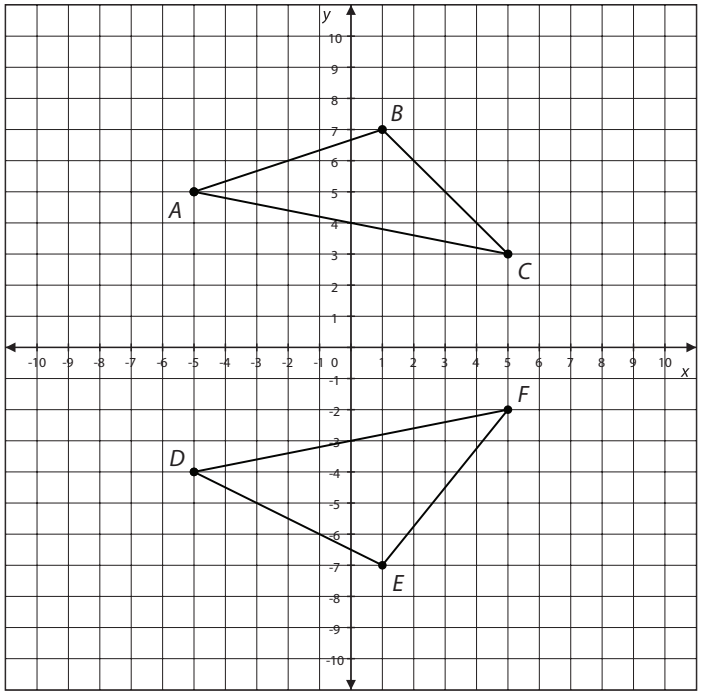

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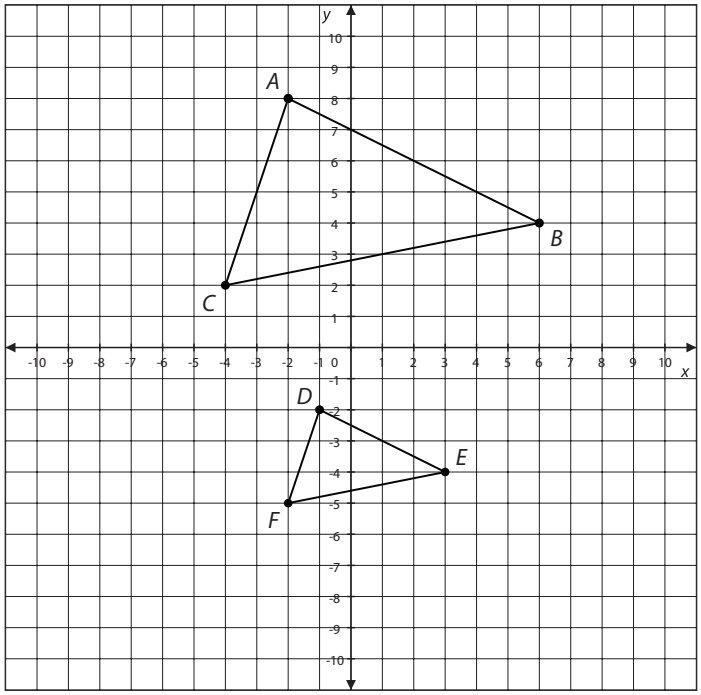

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Lesson 5.3.2: Applying Similarity Using the Angle-Angle (AA) Criterion

Warm-Up 5.3.2

The Statue of Liberty, a gift to the United States from France in 1886, stands 93 meters tall. A replica of the famous statue is 7.75 meters.

1. What is the scale factor comparing the height of the replica to the actual height of the Statue of Liberty?

2. The height of the base of the Statue of Liberty is approximately 46.5 meters. What is the height of the base of the Statue of Liberty replica?

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Instruction

Lesson 5.3.2: Applying Similarity Using the Angle-Angle (AA) Criterion

Common Core State Standard

G–SRT.3 Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

Warm-Up 5.3.2 Debrief

1. What is the scale factor comparing the height of the replica to the actual height of the Statue of Liberty?

   Create a ratio comparing the height of the replica to the actual height of the statue.

   $$\frac{\text{height of replica}}{\text{actual height}} = \frac{7.75 \text{ meters}}{93 \text{ meters}} = \frac{1}{12}$$

   The ratio comparing the two heights is $\frac{1}{12}$.

   The scale factor is $\frac{1}{12}$.

2. The height of the base of the Statue of Liberty is approximately 46.5 meters. What is the height of the base of the Statue of Liberty replica?

   Write a proportion to find the height of the base of the replica.

   $$\frac{\text{replica base height}}{\text{actual base height}} = \frac{1}{12}$$

   $$\frac{x}{46.5 \text{ meters}} = \frac{1}{12}$$

   $12x = 46.5$

   $x = 3.875$

   The height of the base of the Statue of Liberty replica is 3.875 meters.

Connection to the Lesson

- Students will continue to use ratios and proportions to determine the ratio of similitude and calculate missing lengths of similar triangles.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- understanding that the sum of the measures of the angles in a triangle is 180˚ (7.G.2)
- identifying both corresponding and congruent parts of triangles (G.CO.7)

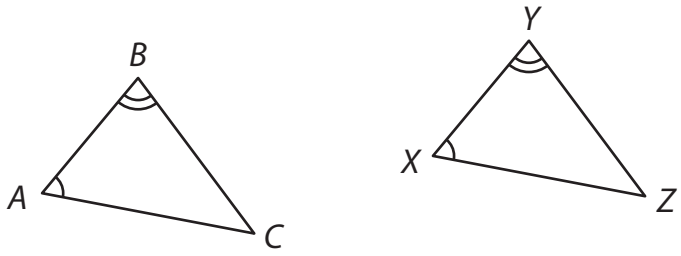
Introduction

When a series of similarity transformations are performed on a triangle, the result is a similar triangle. When triangles are similar, the corresponding angles are congruent and the corresponding sides are proportional. It is possible to determine if triangles are similar by measuring and comparing each angle and side, but this can take time. A set of similarity statements exists, similar to the congruence statements, that let us determine with less information whether triangles are similar. In this lesson, we will focus on the Angle-Angle Similarity Statement.

Key Concepts

- The **Angle-Angle (AA) Similarity Statement** is one statement that allows us to prove triangles are similar.
- The AA Similarity Statement allows that if two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

  $\triangle ABC \sim \triangle XYZ$



- Notice that it is not necessary to show that the third pair of angles is congruent because the sum of the angles must equal 180˚.
- Similar triangles have corresponding sides that are proportional.
- The Angle-Angle Similarity Statement can be used to solve various problems, including those that involve indirect measurement, such as using shadows to find the height of tall structures.

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Common Errors/Misconceptions

- misidentifying congruent parts because of the orientation of the triangles
- misreading similarity statements as congruent statements
- incorrectly creating proportions between corresponding sides

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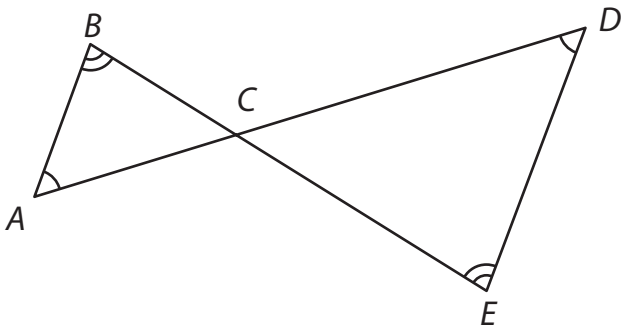
Guided Practice 5.3.2

Example 1

Explain why the triangles are similar and write a similarity statement.



1. Identify the given information.

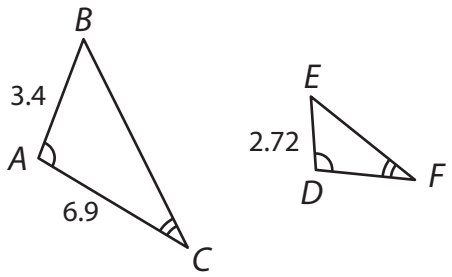
   According to the diagram, $\angle A \cong \angle D$ and $\angle B \cong \angle E$.

2. State your conclusion.

   $\triangle ABC \sim \triangle DEC$ by the Angle-Angle (AA) Similarity Statement.

Example 2

Explain why $\triangle ABC \sim \triangle DEF$, and then find the length of $\overline{DF}$.



1. Show that the triangles are similar.

   According to the diagram, $\angle A \cong \angle D$ and $\angle C \cong \angle F$.

   $\triangle ABC \sim \triangle DEF$ by the Angle-Angle (AA) Similarity Statement.

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2. Find the length of $\overline{DF}$.

Corresponding sides of similar triangles are proportional.

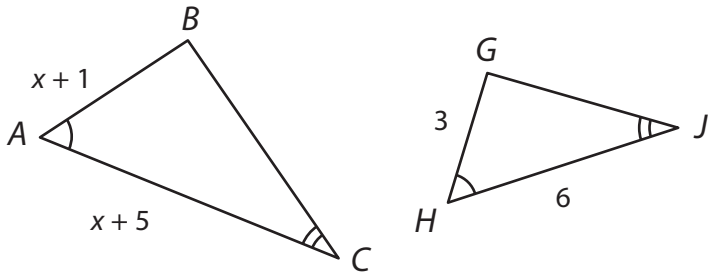
Create and solve a proportion to find the length of $\overline{DF}$.

$\frac{AB}{DE}=\frac{AC}{DF}$	Corresponding sides are proportional.
$\frac{3.4}{2.72}=\frac{6.9}{x}$	Substitute known values. Let x represent the length of $\overline{DF}$.
$(2.72)(6.9) = (3.4)(x)$	Solve for x.
$18.768 = 3.4x$	
$x = 5.52$	

The length of $\overline{DF}$ is 5.52 units.

Example 3

Identify the similar triangles. Find x and the measures of the indicated sides.



1. Show that the triangles are similar.

According to the diagram, $\angle A \cong \angle H$ and $\angle C \cong \angle J$.

$\triangle ABC \sim \triangle HGJ$ by the Angle-Angle (AA) Similarity Statement.

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2. Use the definition of similar triangles to find the value of x.

Corresponding sides of similar triangles are proportional.

Create and solve a proportion to find the value of x.

$\frac{AB}{HG} = \frac{AC}{HJ}$ Corresponding sides are proportional.

$\frac{x+1}{3} = \frac{x+5}{6}$ Substitute known values.

$(x + 1)(6) = (3)(x + 5)$ Solve for x.

$6x + 6 = 3x + 15$

$3x = 9$

$x = 3$

3. Find the unknown side lengths.

Use the value of x to find the unknown lengths of the triangles.

$AB = x + 1$

$= 3 + 1$

$= 4$

$AC = x + 5$

$= 3 + 5$

$= 8$

The length of $\overline{AB}$ is 4 units.

The length of $\overline{AC}$ is 8 units.

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Example 4

Suppose a person 5 feet 10 inches tall casts a shadow that is 3 feet 6 inches long. At the same time of day, a flagpole casts a shadow that is 12 feet long. To the nearest foot, how tall is the flagpole?

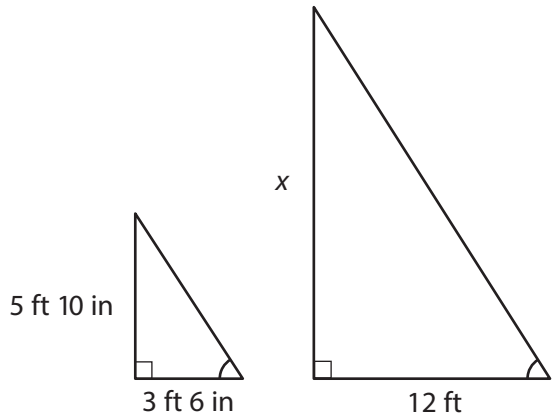
1. Identify the known information.

   The height of a person and the length of the shadow cast create a right angle.

   The height of the flagpole and the length of the shadow cast create a second right angle.

   You can use this information to create two triangles.

   Draw a picture to help understand the information.



2. Determine if the triangles are similar.

   Two pairs of angles are congruent.

   According to the Angle-Angle (AA) Similarity Statement, the triangles are similar.

   Corresponding sides of similar triangles are proportional.

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Instruction

3. Find the height of the flagpole.

Create and solve a proportion to find the height of the flagpole.

$\frac{5\frac{10}{12}}{x}=\frac{3\frac{6}{12}}{12}$ Corresponding sides are proportional. Let x represent the height of the flagpole.

$\frac{5\frac{5}{6}}{x}=\frac{3\frac{1}{2}}{12}$ Simplify.

$\left(5\frac{5}{6}\right)(12)=\left(3\frac{1}{2}\right)(x)$ Solve for x.

$70=\left(3\frac{1}{2}\right)(x)$

$x = 20$

The flagpole is 20 feet tall.

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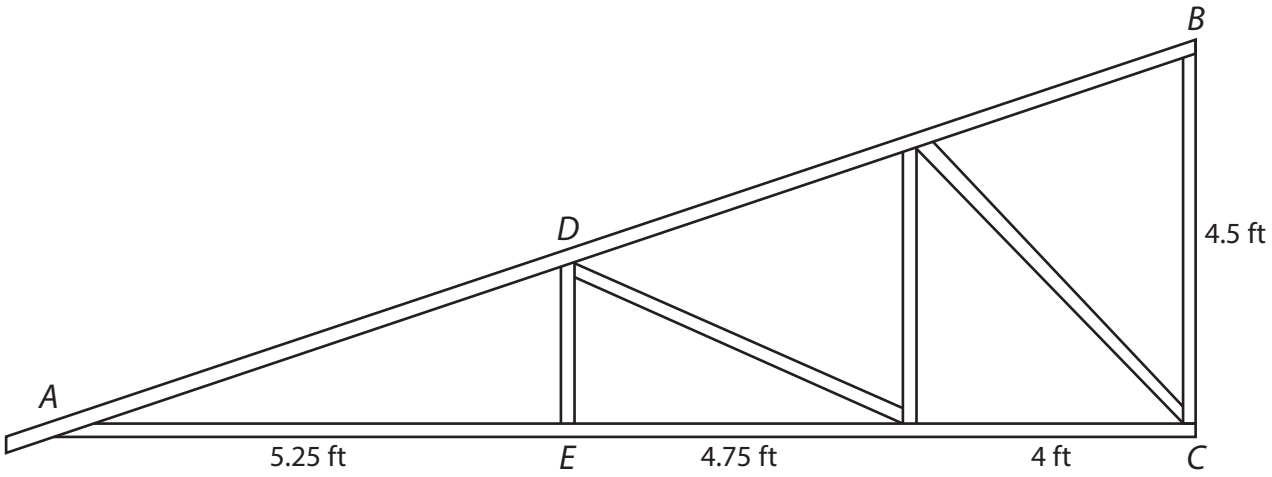
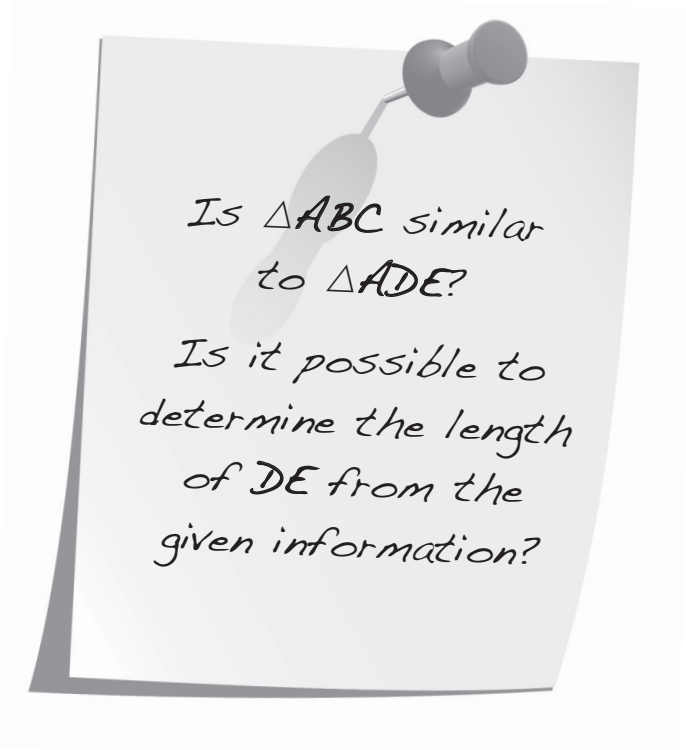
Problem-Based Task 5.3.2: True Trusses

SMP	
1 ✓	2
3	4
5	6
7 ✓	8

A mono truss is a type of building support structure that is in the shape of a right triangle. Contractors often use mono trusses when building roofs for small structures such as garages and sheds. The vertical pieces of this truss form 90° angles with the horizontal pieces in order to maximize the stability. Observe the diagram of a mono truss below. Is $\triangle ABC$ similar to $\triangle ADE$? Explain your reasoning. Is it possible to determine the length of $\overline{DE}$ from the given information? If so, calculate the length.





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Problem-Based Task 5.3.2: True Trusses

Coaching

a. Which angles of $\triangle ABC$ are congruent to angles of $\triangle ADE$? Name them, and explain why they are congruent.

b. Is there enough information to determine if $\triangle ABC$ is similar to $\triangle ADE$?

c. If two triangles are similar, what must be true about the lengths of their corresponding sides?

d. Identify the corresponding sides of $\triangle ABC$ and $\triangle ADE$.

e. What is the ratio between the length of $\overline{AC}$ and the length of $\overline{AE}$?

f. What is the length of $\overline{DE}$?

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Instruction

Problem-Based Task 5.3.2: True Trusses

Coaching Sample Responses

a. Which angles of $\triangle ABC$ are congruent to angles of $\triangle ADE$? Name them, and explain why they are congruent.

$\angle BCA$ of $\triangle ABC$ is congruent to $\angle DEA$ of $\triangle ADE$ because they are both right angles.

$\angle BAC$ of $\triangle ABC$ is congruent to $\angle DAE$ of $\triangle ADE$ because they are composed of the same angle, $\angle A$.

b. Is there enough information to determine if $\triangle ABC$ is similar to $\triangle ADE$?

Yes; according to the Angle-Angle (AA) Similarity Statement, if two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

c. If two triangles are similar, what must be true about the lengths of their corresponding sides?

Similar triangles have corresponding sides that are proportional.

d. Identify the corresponding sides of $\triangle ABC$ and $\triangle ADE$.

$\overline{AB} \rightarrow \overline{AD}$

$\overline{AC} \rightarrow \overline{AE}$

$\overline{BC} \rightarrow \overline{DE}$

e. What is the ratio between the length of $\overline{AC}$ and the length of $\overline{AE}$?

$$\frac{AC}{AE} = \frac{(5.25 + 4.75 + 4)}{5.25} = \frac{14}{5.25} = \frac{8}{3}$$

f. What is the length of $\overline{DE}$?

Create and solve a proportion using the ratio found in part e.

$$\frac{BC}{DE} = \frac{8}{3} = \frac{4.5}{x}$$

$x = 1.6875$

The length of $\overline{DE}$ is 1.6875 feet.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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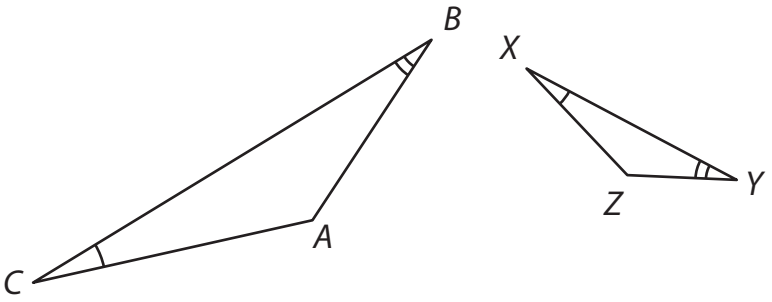
Practice 5.3.2: Applying Similarity Using the Angle-Angle (AA) Criterion

Decide whether each pair of triangles is similar. Explain your answer.

1.



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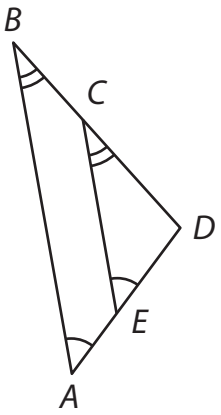

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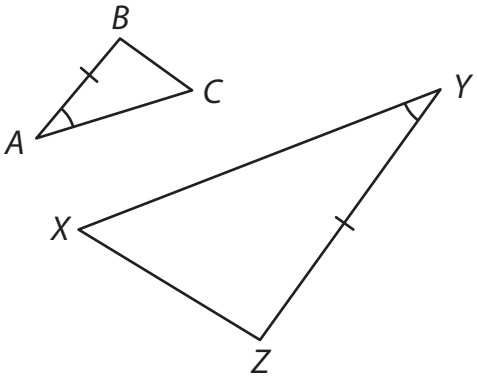

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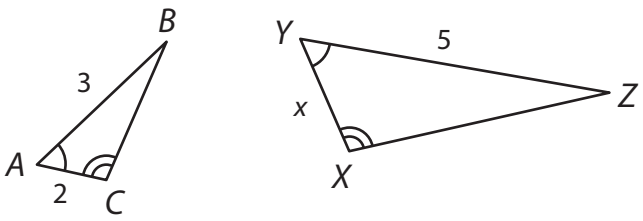
Identify the similar triangles. Find x and the measures of the indicated sides.

4.



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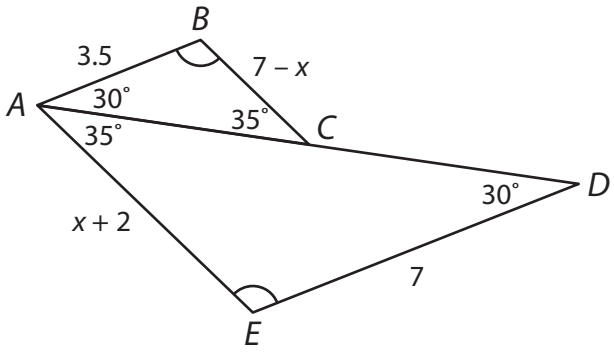

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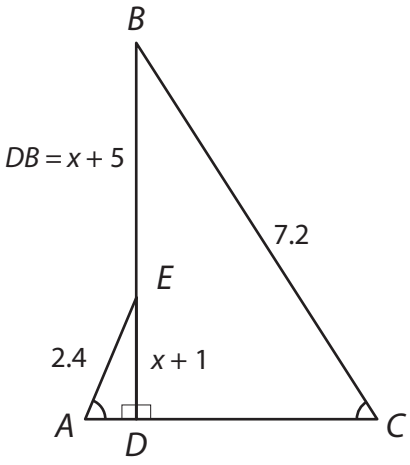

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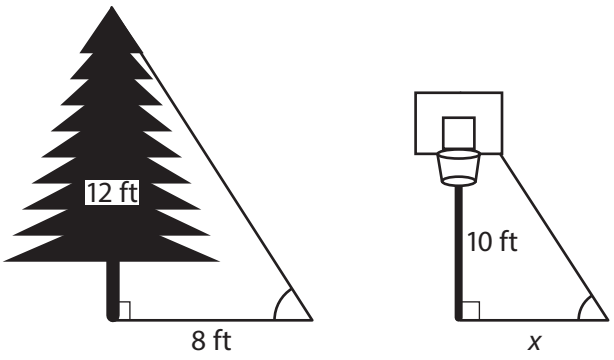
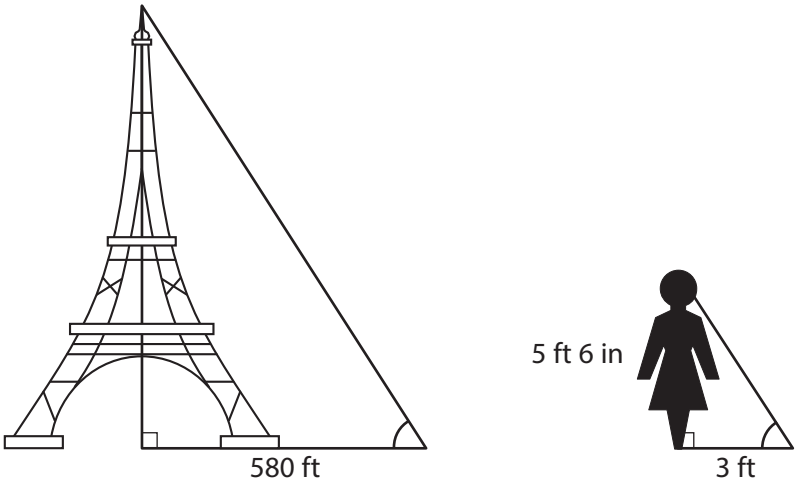
Use the definition of similarity to solve each problem.

7. At a certain time of day, a tree that is 12 feet tall casts a shadow that is 8 feet long. Find the length of the shadow that is created by a 10-foot-tall basketball hoop at the same time of day.



8. Sheila is standing near the Eiffel Tower in Paris, France. The shadow of the monument is 580 feet long, and Sheila's shadow is 3 feet long. If Sheila is 5 feet 6 inches tall, how tall is the monument?



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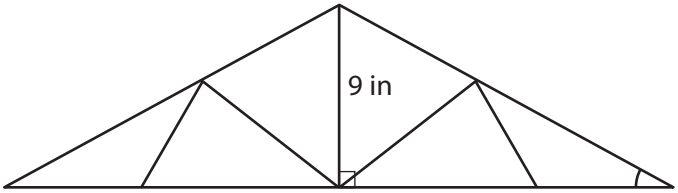
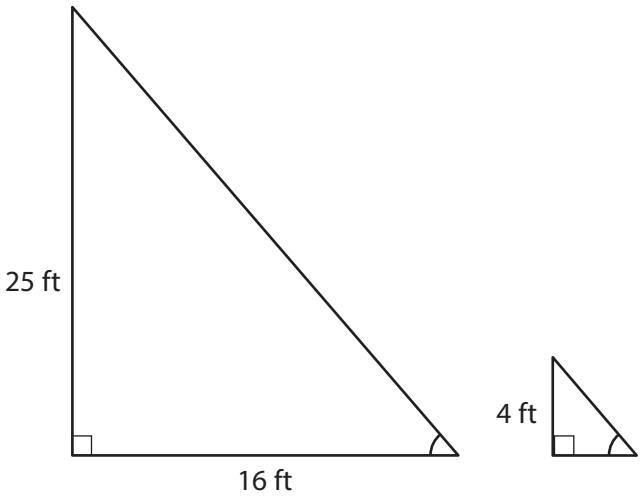
9. The support beams of truss bridges are triangles. James made a model of a truss bridge with a scale of 1 inch = 4 feet. If the height of the tallest triangle on the model is 9 inches, what is the height of the tallest triangle on the actual bridge?



10. A statue that is 25 feet tall casts a shadow that is 16 feet long. A cement post next to the statue is 4 feet tall. Find the length of the cement post's shadow.



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Progress Assessment

Circle the letter of the best answer.

1. Which graph does NOT illustrate the effect of a similarity transformation?

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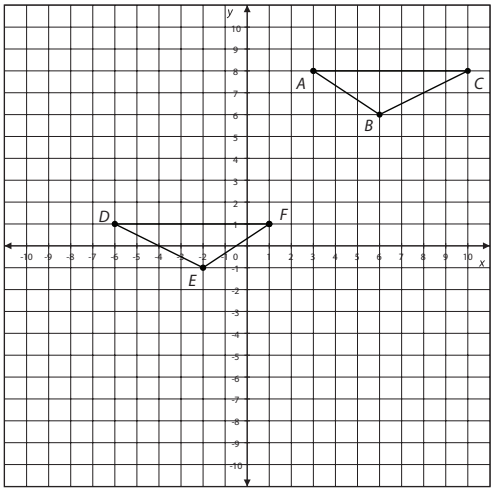
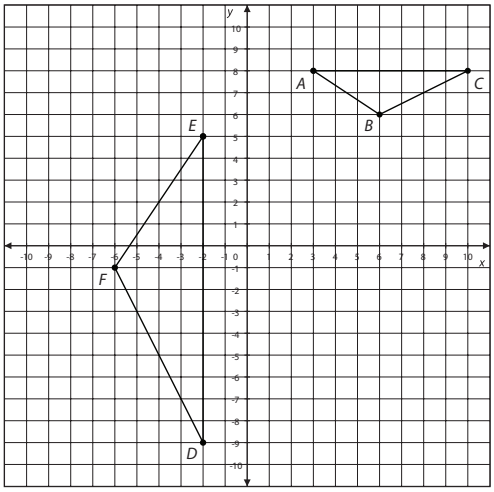


c.



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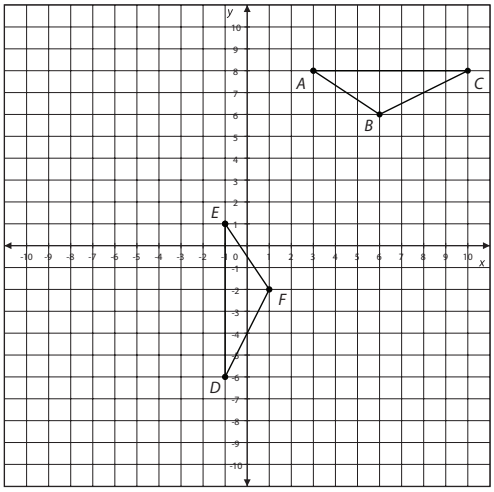

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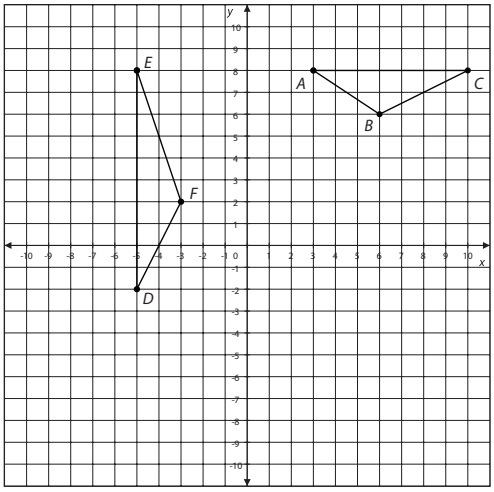

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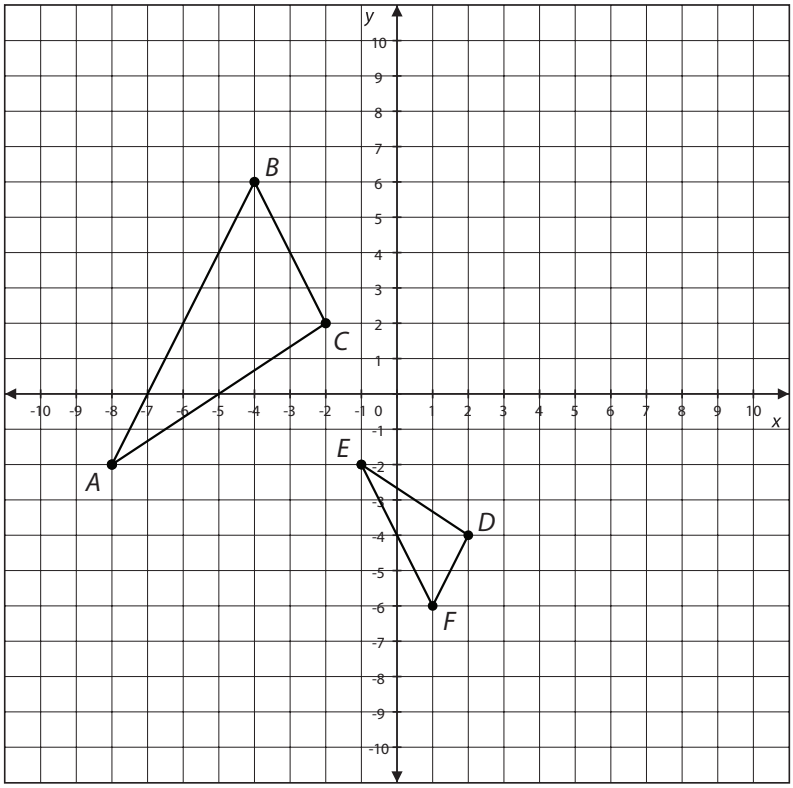
2. Are the two triangles similar? Why or why not?



a. They are not similar; $\triangle ABC$ has undergone a transformation that does not preserve congruency.

b. They are not similar; $\triangle ABC$ has undergone a transformation that does not preserve similarity.

c. They are similar; $\triangle ABC$ has undergone a transformation that preserves similarity.

d. They are similar; $\triangle ABC$ has undergone a transformation that preserves congruency.

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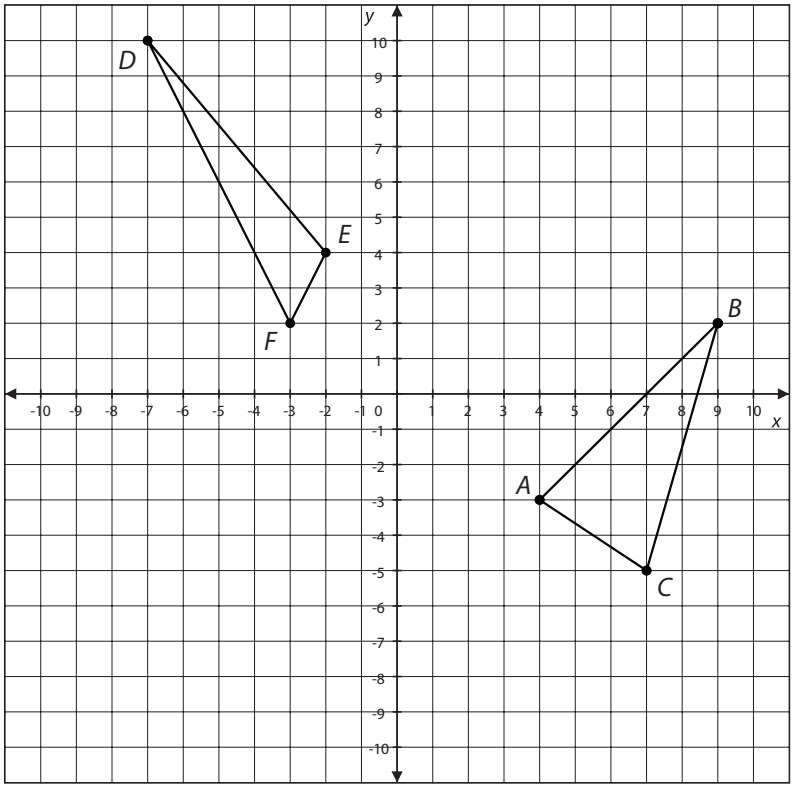
3. Are the two triangles similar? Why or why not?



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b. They are not similar; $\triangle ABC$ has undergone a transformation that does not preserve similarity.

c. They are similar; $\triangle ABC$ has undergone a transformation that preserves similarity.

d. They are similar; $\triangle ABC$ has undergone a transformation that preserves congruency.

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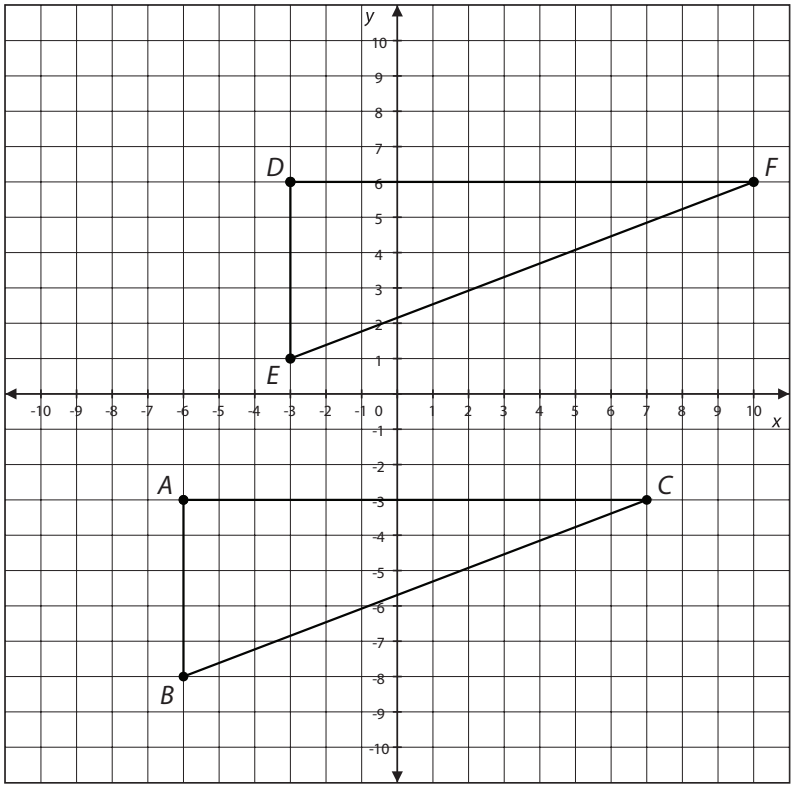
4. Are the two triangles similar? Why or why not?



a. They are not similar; $\triangle ABC$ has undergone a transformation that does not preserve congruency.

b. They are not similar; $\triangle ABC$ has undergone a transformation that does not preserve similarity.

c. They are similar; $\triangle ABC$ has undergone a transformation that preserves similarity.

d. They are not similar; $\triangle ABC$ has undergone a transformation that preserves congruency.

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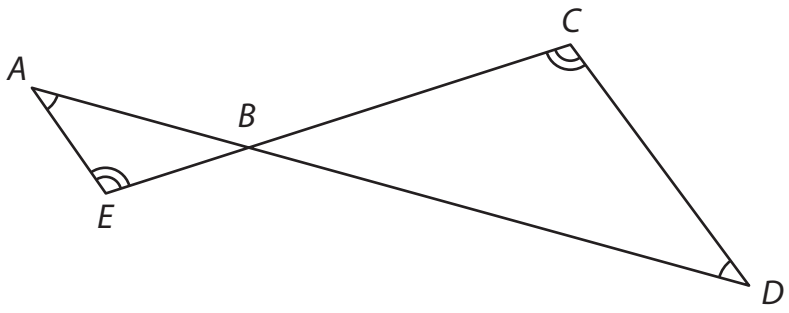
5. Are the two triangles similar? Why or why not?



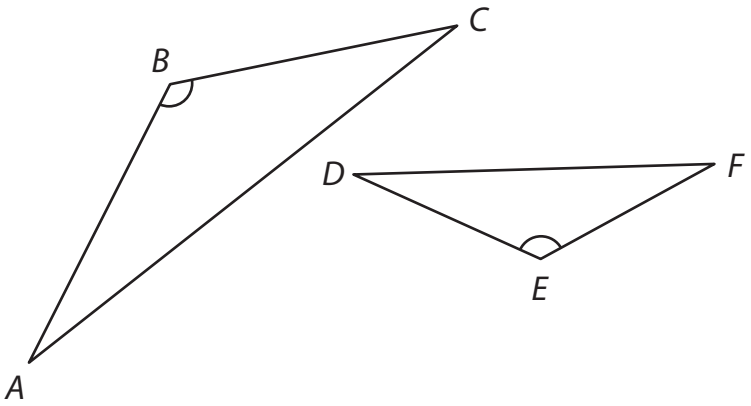
a. Yes, they are similar because of the AA Similarity Statement.

b. Yes, they are similar because of the ASA Congruence Statement.

c. No, they are not similar because congruence is not preserved.

d. It cannot be determined if the triangles are similar.

6. Are the two triangles similar? Why or why not?



a. Yes, they are similar because of the AA Similarity Statement.

b. Yes, they are similar because of the ASA Congruence Statement.

c. No, they are not similar because congruence is not preserved.

d. It cannot be determined if the triangles are similar.

continued

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Assessment

7. $\triangle ABC$ has undergone a series of similarity transformations to produce $\triangle EFG$. Which of the following statements is always true?

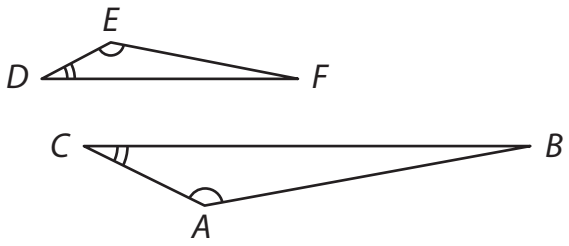
   a. The angles of $\triangle EFG$ are proportional to the angles of $\triangle ABC$.

   b. The angles of $\triangle EFG$ are the same as the angles of $\triangle ABC$.

   c. The sides of $\triangle EFG$ increase proportionally to the sides of $\triangle ABC$.

   d. The sides of $\triangle EFG$ decrease proportionally to the sides of $\triangle ABC$.

8. Identify the similar triangles.



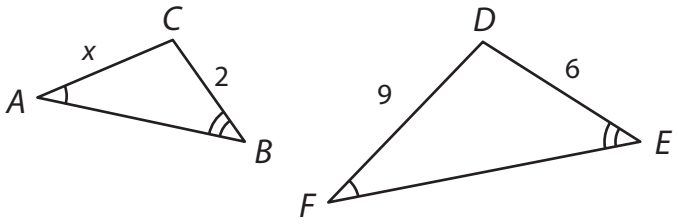
   a. $\triangle ABC \sim \triangle FED$

   b. $\triangle ABC \sim \triangle DEF$

   c. $\triangle ABC \sim \triangle EFD$

   d. There is not enough information to determine if the triangles are similar.

9. Identify the similar triangles. Find x.



   a. $\triangle ABC \sim \triangle FED$; $x = 3$

   b. $\triangle ABC \sim \triangle DEF$; $x = 3$

   c. $\triangle ABC \sim \triangle FED$; $x = \frac{1}{3}$

   d. $\triangle ABC \sim \triangle DEF$; $x = \frac{1}{3}$

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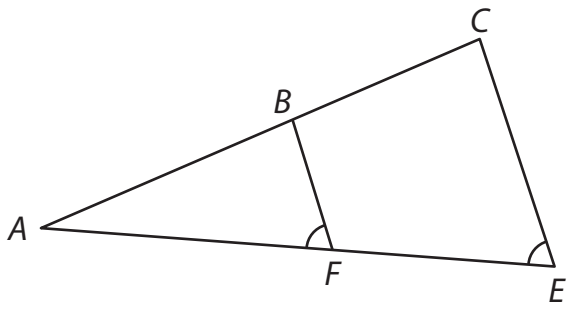
10. Identify the similar triangles.



a. $\triangle ACE \sim \triangle ABE$

b. $\triangle BFA \sim \triangle CEA$

c. $\triangle ACE \sim \triangle AFB$

d. $\triangle ACE \sim \triangle FBA$

Read the scenario below and answer the questions that follow.

11. Suppose you are standing near a building and you want to know its height. The building casts a 70-foot shadow. You cast a 4-foot shadow.

    a. If you are 5 feet 6 inches tall, how tall is the building?

    b. A nearby fence post is 3 feet tall. What is the approximate length of its shadow?

    c. Explain how the Angle-Angle (AA) Similarity Statement helps you solve this problem.

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Assessment

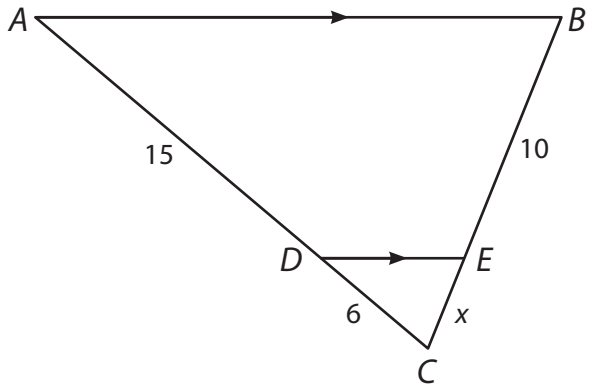
Pre-Assessment

Circle the letter of the best answer.

1. The length of a building's shadow is 107.2 feet. At the same time of day, a 1.8-foot-tall tree casts a shadow that is 2.4 feet long. What is the height of the building?

   a. 80.4 feet
   b. 142.9 feet
   c. 24.8 feet
   d. 463.1 feet

2. $\triangle ABC \sim \triangle DEC$. What is the length of $\overline{EC}$?



   a. 25 units
   b. 9 units
   c. 4 units
   d. There is not enough information to determine the length of $\overline{EC}$.

3. In $\triangle ABC$, the bisector of $\angle A$ divides $\overline{BC}$ into segments $\overline{BD}$ and $\overline{DC}$, with $BD = 24$ units and $DC = 30$ units. If $AC = 37.5$ units, what could be the length of $\overline{AB}$?

   a. 46.875 units
   b. 30 units
   c. 19.2 units
   d. There is not enough information to determine the length of $\overline{AB}$.

continued

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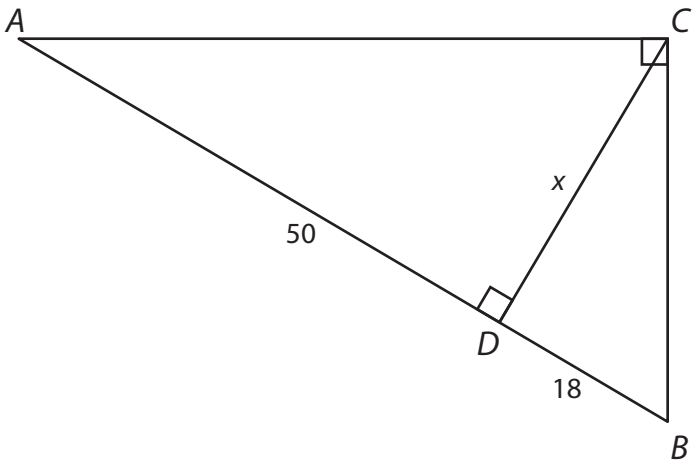
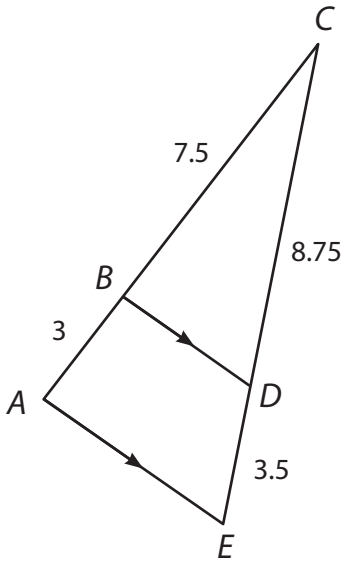
4. $\triangle ABC$ is a right triangle. Find the length of x, which is the altitude of $\triangle ABC$.



   a. 30 units
   b. 7.7 units
   c. 2.7 units
   d. There is not enough information to determine the length of x.

5. Which statement would justify that $\triangle ACE \sim \triangle BCD$?



   a. Angle-Angle (AA) Similarity Statement
   b. Side-Angle-Side (SAS) Similarity Statement
   c. Side-Side-Side (SSS) Similarity Statement
   d. It is not possible to determine if $\triangle ACE \sim \triangle BCD$.

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Lesson 4: Proving Similarity

Instruction

Common Core State Standards

SMP
1 ✓ 2
3 ✓ 4 ✓
5 6 ✓
7 ✓ 8

G–SRT.4 Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two sides proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.*

G–SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Essential Questions

1. How can you prove that two triangles are similar?
2. How does a line parallel to one side of a triangle divide the other two sides of the triangle?
3. How can you use triangle similarity to prove the Pythagorean Theorem?
4. How can you use similar triangles to solve problems?
5. How can you use similarity and congruence to solve problems?

WORDS TO KNOW

altitude	the perpendicular line segment from a vertex of a figure to its opposite side; height
angle bisector	a line, segment, or ray that divides an angle into two congruent angles
converse of the Pythagorean Theorem	If the sum of the squares of the measures of the two shorter sides of a triangle equals the square of the measure of the longest side, then the triangle is a right triangle.
flow proof	a graphical method of presenting the logical steps used to show an argument. In a flow proof, the logical statements are written in boxes and the reason for each statement is written below the box.
paragraph proof	statements written out in complete sentences in a logical order to show an argument
parallel lines	lines in a plane that do not share any points and never intersect; written as $\overleftrightarrow{AB} \parallel \overleftrightarrow{PQ}$
proof	a set of justified statements organized to form a convincing argument that a given statement is true

SRB p. U5-69

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Instruction

Reflexive Property of Congruent Segments	a segment is congruent to itself; $\overline{AB} \cong \overline{AB}$
Segment Addition Postulate	If B is a point on line segment $\overline{AC}$, then $AB + BC = AC$. Conversely, if $AB + BC = AC$, then B is on line segment $\overline{AC}$.
Side-Angle-Side (SAS) Similarity Statement	If the measures of two sides of a triangle are proportional to the measures of two corresponding sides of another triangle and the included angles are congruent, then the triangles are similar.
Side-Side-Side (SSS) Similarity Statement	If the measures of the corresponding sides of two triangles are proportional, then the triangles are similar.
Symmetric Property of Congruent Segments	If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$.
theorem	a statement that is shown to be true
Transitive Property of Congruent Segments	If $\overline{AB} \cong \overline{CD}$, and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.
two-column proof	numbered statements and corresponding reasons that show an argument in a logical order

Recommended Resources

- LearnZillion. "Prove the Pythagorean Theorem."

  http://www.walch.com/rr/00024

  This is a video that proves the Pythagorean Theorem through the use of similar triangles.

- Math Open Reference. "Similar Triangles."

  http://www.walch.com/rr/00025

  This site includes a summary of similarity and links to tests of similarity.

- Math Warehouse. "Triangle Angle Bisector Theorem."

  http://www.walch.com/rr/00026

  This site includes a summary of the Angle Bisector Theorem in addition to illustrated examples.

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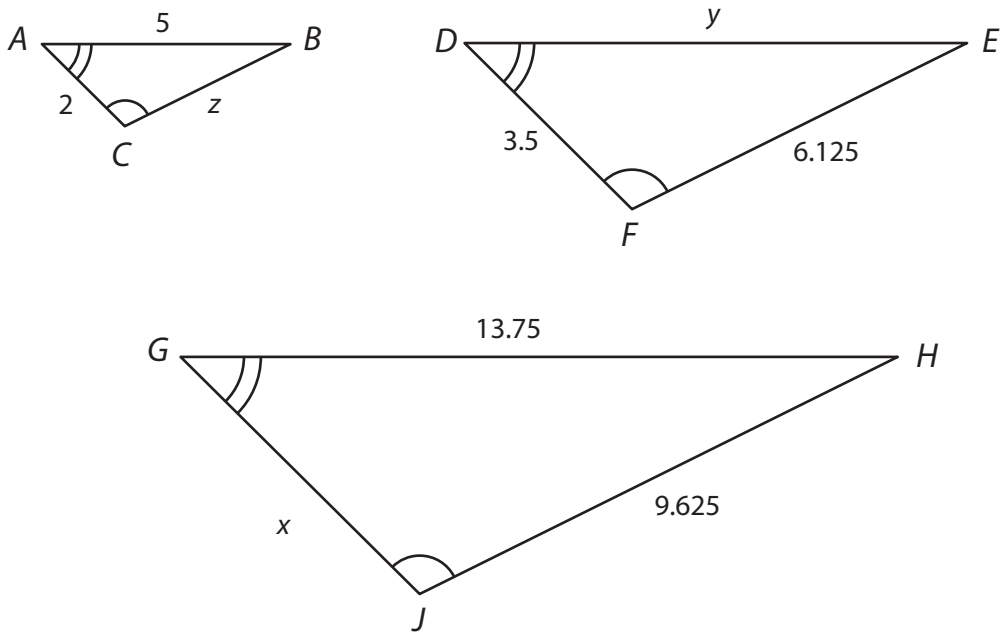
Lesson 5.4.1: Proving Triangle Similarity Using Side-Angle-Side (SAS) and Side-Side-Side (SSS) Similarity

Warm-Up 5.4.1

Dasha is making earrings to give her friend as a birthday gift. She's designed a pattern of 3 triangles stacked on top of one another. Dasha is missing some of the measurements for her project. The given units in the diagram are in centimeters.



1. Are the triangles similar? Explain your answer.

2. Is it possible to determine the missing lengths, x, y, and z? If so, find each value.

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Instruction

Lesson 5.4.1: Proving Triangle Similarity Using Side-Angle-Side (SAS) and Side-Side-Side (SSS) Similarity

Common Core State Standard

G–SRT.4 Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two sides proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.*

Warm-Up 5.4.1 Debrief

1. Are the triangles similar? Explain your answer.

   Two angles in each triangle are congruent, as noted by the arc marks.

   According to the Angle-Angle Similarity Statement, if two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

   $\triangle ABC \sim \triangle DEF \sim \triangle GHJ$

2. Is it possible to determine the missing lengths, x, y, and z? If so, find each value.

   If two triangles are similar, then the measures of their corresponding sides are proportional, or have a constant ratio.

   Create a proportion to find each of the unknown values.

   Find the value of x.

$\frac{AC}{GJ} = \frac{AB}{GH}$	Create a proportion.
$\frac{2}{x} = \frac{5}{13.75}$	Substitute.
$(2)(13.75) = (5)(x)$	Find the cross products.
$27.5 = 5x$	Simplify.
$x = 5.5$	Solve for x.

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Find the value of y.

$\frac{AC}{DF}=\frac{AB}{DE}$	Create a proportion.
$\frac{2}{3.5}=\frac{5}{y}$	Substitute.
$(2)(y) = (3.5)(5)$	Find the cross products.
$2y = 17.5$	Simplify.
$y = 8.75$	Solve for y.

Find the value of z.

$\frac{AC}{DF}=\frac{BC}{EF}$	Create a proportion.
$\frac{2}{3.5}=\frac{z}{6.125}$	Substitute.
$(2)(6.125) = (3.5)(z)$	Find the cross products.
$12.25 = 3.5z$	Simplify.
$z = 3.5$	Solve for z.

The missing values for x, y, and z are 5.5 cm, 8.75 cm, and 3.5 cm, respectively.

Connection to the Lesson

- Students will continue learning about similar triangles and similarity statements. Students will apply their understanding of the Angle-Angle Similarity Statement as they learn about the Side-Angle-Side and Side-Side-Side similarity statements.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- creating ratios (7.RP.2c)
- solving proportions (7.RP.3)
- identifying both corresponding and congruent parts of triangles (G–CO.7)

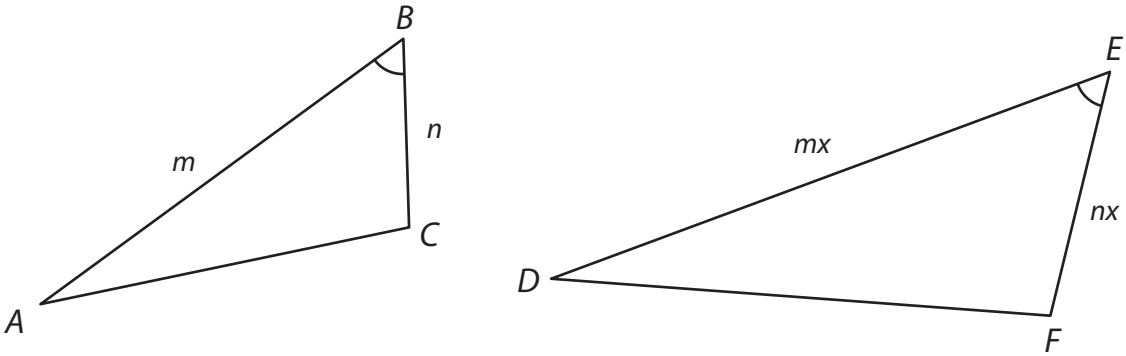
Introduction

There are several ways to show that two triangles are similar, just as there are several ways to show that two triangles are congruent. The Angle-Angle (AA) Similarity Statement is one of them. The Side-Angle-Side (SAS) and Side-Side-Side (SSS) similarity statements are two more ways to show that triangles are similar. In this lesson, we will prove that triangles are similar using the similarity statements.

Key Concepts

- The **Side-Angle-Side (SAS) Similarity Statement** asserts that if the measures of two sides of a triangle are proportional to the measures of two corresponding sides of another triangle and the included angles are congruent, then the triangles are similar.
- Similarity statements identify corresponding parts just like congruence statements do.

$\triangle ABC \sim \triangle DEF$



$\angle B \cong \angle E$

$DE = (x)AB$

$EF = (x)BC$

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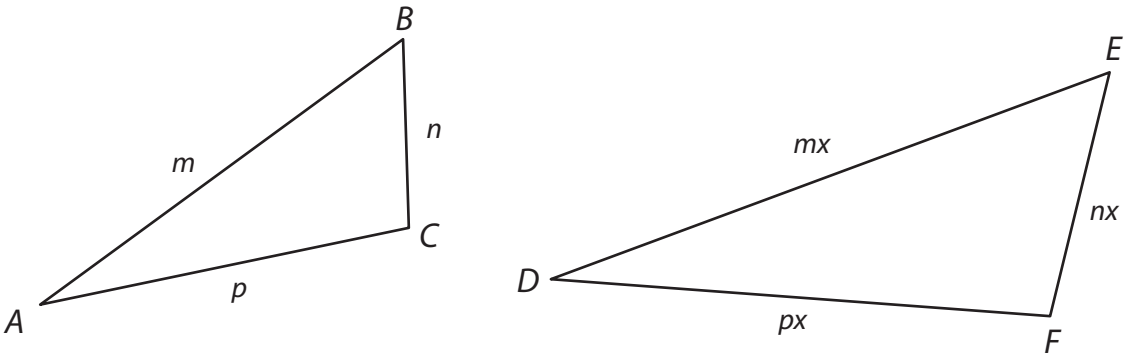
Instruction

- The **Side-Side-Side (SSS) Similarity Statement** asserts that if the measures of the corresponding sides of two triangles are proportional, then the triangles are similar.

  $\triangle ABC \sim \triangle DEF$



  $DE = (x)AB$

  $EF = (x)BC$

  $DF = (x)AC$

- It is important to note that while both similarity and congruence statements include an SSS and an SAS statement, the statements do not mean the same thing.
- Similar triangles have corresponding sides that are proportional, whereas congruent triangles have corresponding sides that are of the same length.
- Like with the Angle-Angle Similarity Statement, both the Side-Angle-Side and the Side-Side-Side similarity statements can be used to solve various problems.
- The ability to prove that triangles are similar is essential to solving many problems.
- A **proof** is a set of justified statements organized to form a convincing argument that a given statement is true.
- Definitions, algebraic properties, and previously proven statements can be used to prove a given statement.
- There are several types of proofs, such as paragraph proofs, two-column proofs, and flow diagrams.

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Instruction

- Every good proof includes the following:
  - a statement of what is to be proven
  - a list of the given information
  - a diagram including the given information
  - step-by-step statements that support your reasoning

Common Errors/Misconceptions

- misidentifying congruent parts because of the orientation of the triangles
- misreading similarity statements as congruency statements
- incorrectly creating proportions between corresponding sides

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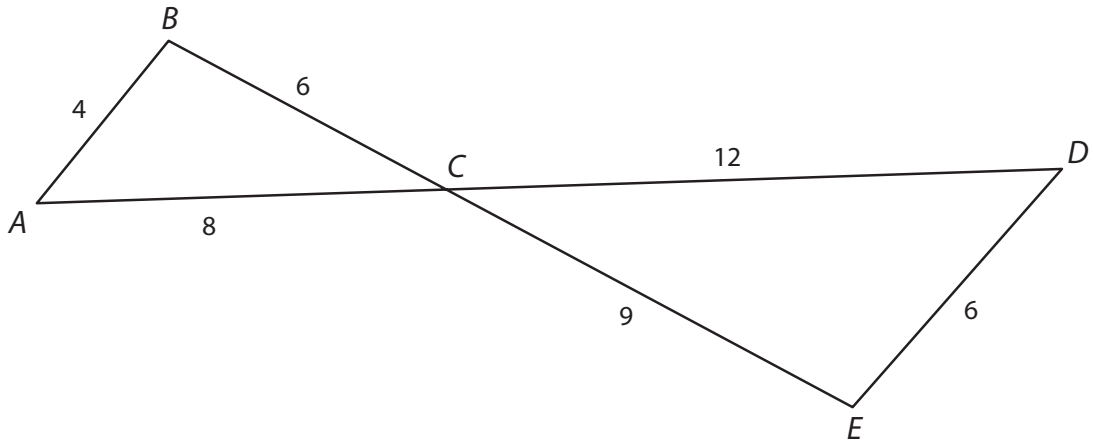
Instruction

Guided Practice 5.4.1

Example 1

Prove $\triangle ABC \sim \triangle DEC$.



1. Identify the given information.

   Lengths for each side of both triangles are given.

   $AB = 4$ $\quad DE = 6$

   $BC = 6$ $\quad EC = 9$

   $AC = 8$ $\quad DC = 12$

2. Compare the side lengths of both triangles.

   Pair the lengths of the sides of $\triangle ABC$ with the corresponding lengths of the sides of $\triangle DEC$ to determine if there is a common ratio.

   $\frac{AB}{DE} = \frac{4}{6} = \frac{2}{3}$ $\quad \frac{BC}{EC} = \frac{6}{9} = \frac{2}{3}$ $\quad \frac{AC}{DC} = \frac{8}{12} = \frac{2}{3}$

   Notice the common ratio, $\frac{2}{3}$; the side lengths are proportional.

3. State your conclusion.

   Similar triangles must have side lengths that are proportional.

   $\triangle ABC \sim \triangle DEC$ by the Side-Side-Side (SSS) Similarity Statement. ✓

SWB p. 71; SRB p. U5-75

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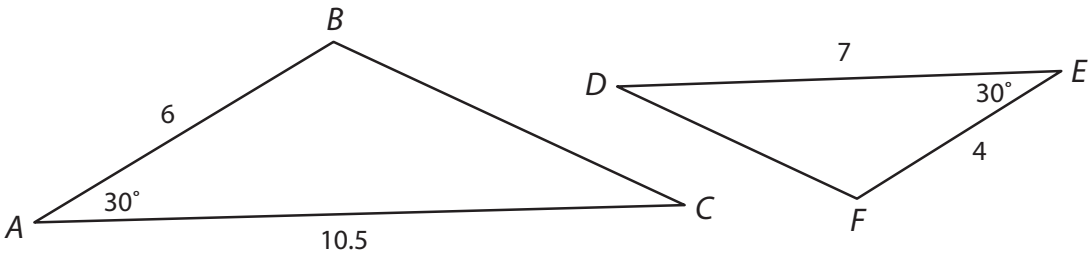
Instruction

Example 2

Determine whether the triangles are similar. Explain your reasoning.



1. Identify the given information.

   According to the diagram, $\angle A \cong \angle E$.

   Given the side lengths, both $\angle A$ and $\angle E$ are included angles.

2. Compare the given side lengths of both triangles.

   If the triangles are similar, then the corresponding sides are proportional.

   $$\frac{AB}{EF} = \frac{6}{4} = \frac{3}{2} \qquad \frac{AC}{ED} = \frac{10.5}{7} = \frac{3}{2}$$

   The side lengths are proportional.

3. State your conclusion.

   The measures of two sides of $\triangle ABC$ are proportional to the measures of two corresponding sides of $\triangle EFD$, and the included angles are congruent.

   $\triangle ABC \sim \triangle EFD$ by the Side-Angle-Side (SAS) Similarity Statement.

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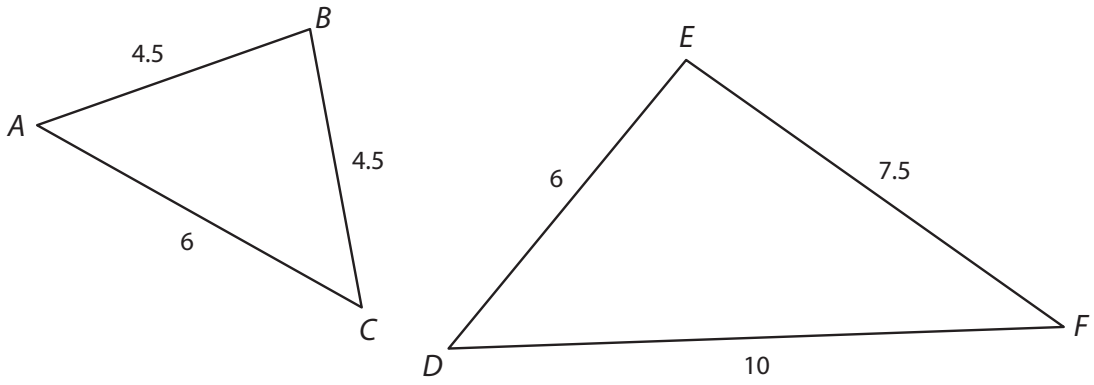
Instruction

Example 3

Determine whether the triangles are similar. Explain your reasoning.



1. Identify the given information.

   The measures of each side of both triangles are given.

2. Compare the side lengths of both triangles.

   Pair the lengths of the sides of $\triangle ABC$ with the corresponding lengths of the sides of $\triangle DEF$ to determine if there is a common ratio.

   $$\frac{AB}{DE}=\frac{4.5}{6}=\frac{3}{4} \qquad \frac{BC}{EF}=\frac{4.5}{7.5}=\frac{3}{5} \qquad \frac{AC}{DF}=\frac{6}{10}=\frac{3}{5}$$

   Notice there is not a common ratio; therefore, the side lengths are not proportional.

3. State your conclusion.

   Similar triangles must have side lengths that are proportional.

   $\triangle ABC$ is not similar to $\triangle DEF$.

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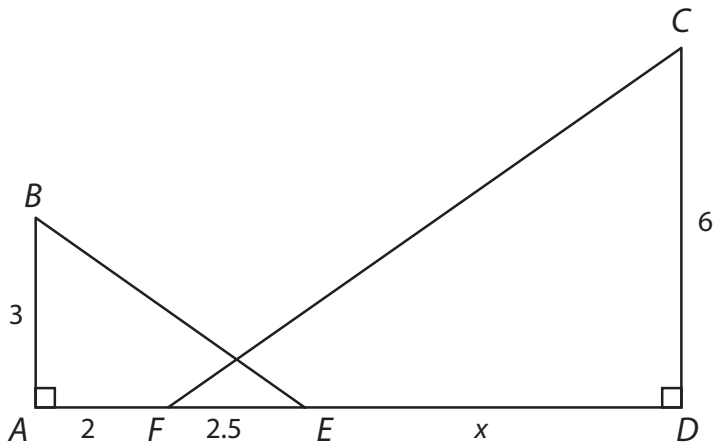
Instruction

Example 4

$\triangle ABE \sim \triangle DCF$, $\overline{AB}$ and $\overline{DC}$ are corresponding sides, and $\overline{AE}$ and $\overline{DF}$ are corresponding sides. Find the value of x.



1. Determine the scale factor of the triangle sides.

   $\frac{AB}{DC} = \frac{3}{6} = \frac{1}{2}$

   The scale factor is $\frac{1}{2}$.

2. Find the length of x.

   $\frac{AE}{DF} = \frac{2+2.5}{x+2.5} = \frac{4.5}{x+2.5} = \frac{1}{2}$

   Solve the proportion $\frac{4.5}{x+2.5} = \frac{1}{2}$ for x.

   $(4.5)(2) = (1)(x + 2.5)$ Find the cross products.

   $9 = x + 2.5$ Simplify.

   $x = 6.5$ Solve for x.

   The length of x is 6.5 units.

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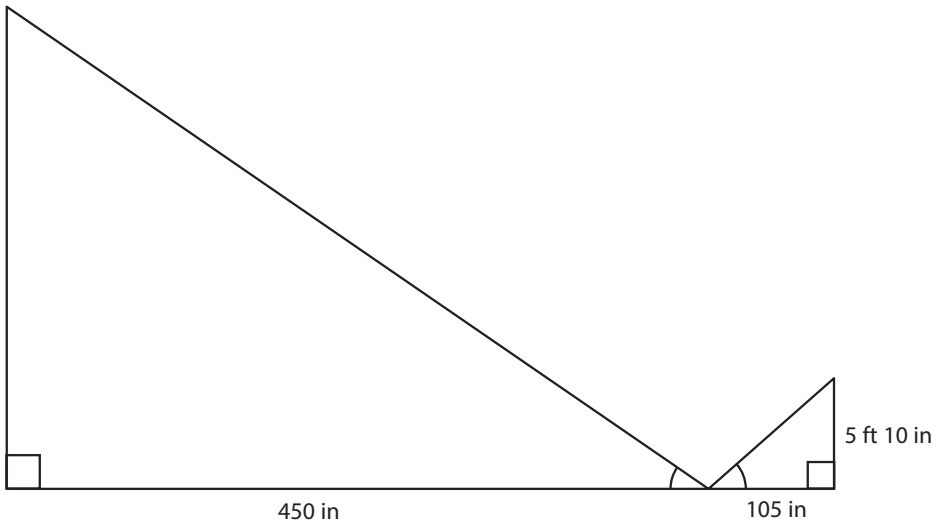
Problem-Based Task 5.4.1: Down, Down, Down

SMP	
1 ✓	2
3 ✓	4 ✓
5	6
7	8

Gutters are designed to channel rainwater away from roofs, preventing water damage and leaks. Downspouts lead down from the gutters to the ground. The amount of downspout needed depends on the height of the house. If the downspout for the gutters is too short, the water can cause flooding. Blair is using a mirror to measure the height of his house so he can determine the length of the downspout he should buy. He has placed the mirror on the ground and is standing far enough away from it that he can see the gutters of his house in the mirror. Blair, whose eyes are 5 feet 10 inches from the ground, measured his distance from the mirror as 105 inches. The distance from the mirror to the foundation of the house is 450 inches. Downspouts are sold in 5-foot sections. How many sections of downspout does Blair need to purchase?



How many sections of downspout does Blair need to purchase?

SWB p. 73

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Problem-Based Task 5.4.1: Down, Down, Down

Coaching

a. How far from the ground are Blair's eyes in inches?

b. Are the triangles in the diagram similar? Explain.

c. What are the characteristics of similar triangles?

d. Is it possible to determine the height of the house from the foundation to the gutters using the given information? Explain your answer.

e. What is the height of the house from the foundation to the gutters?

f. How many 5-foot sections of downspout are needed for Blair's house?

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Instruction

Problem-Based Task 5.4.1: Down, Down, Down

Coaching Sample Responses

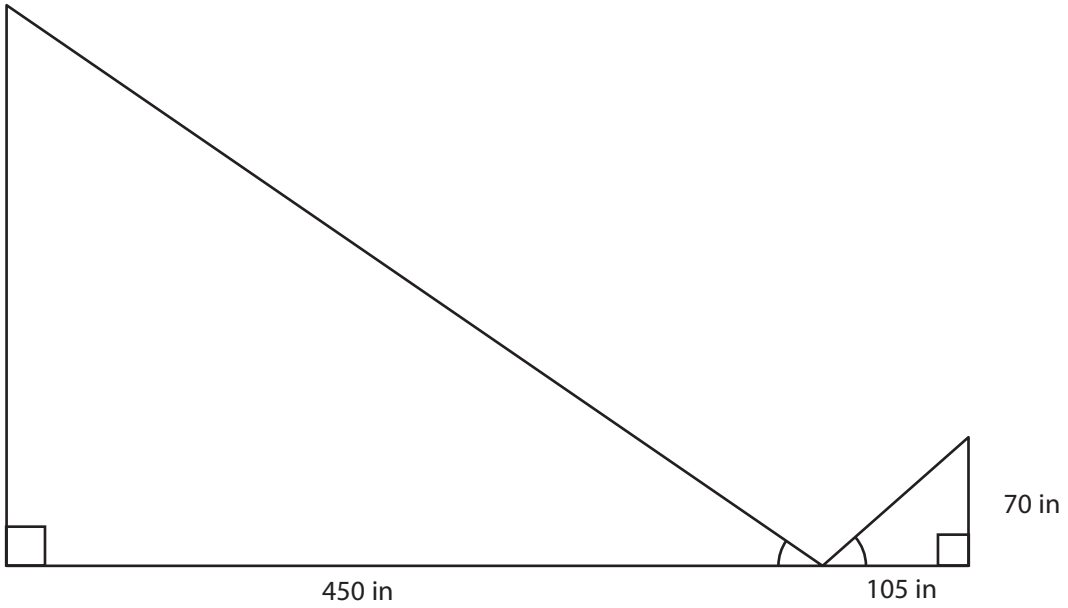
a. How far from the ground are Blair's eyes in inches?

There are 12 inches in a foot. Blair's eyes are 5 feet 10 inches from the ground.

To determine this height in inches, multiply the number of feet high his eyes are by 12, and then add the number of inches.

height from the ground = $5(12) + 10 = 60 + 10 = 70$

Blair's eyes are 70 inches from the ground.



b. Are the triangles in the diagram similar? Explain.

The angles of reflection from the mirror are congruent, and both triangles contain right angles, so the triangles in the diagram are similar because of the Angle-Angle (AA) Similarity Statement. This statement asserts that if two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

c. What are the characteristics of similar triangles?

The angles of similar triangles are congruent and the sides are proportional.

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Instruction

d. Is it possible to determine the height of the house from the foundation to the gutters using the given information? Explain your answer.

Yes, it is possible to determine the height of the house from the foundation to the gutters using the given information. You can do so by creating a proportion using the two similar triangles.

e. What is the height of the house from the foundation to the gutters?

Create a proportion to find the height of the house from the foundation to the gutters.

$$\frac{\text{Height of eyes from ground}}{\text{height from foundation to gutters}} = \frac{\text{distance from mirror to Blair}}{\text{distance from mirror to foundation}}$$

$\frac{70}{x} = \frac{105}{450}$ Substitute.

$(70)(450) = (105)(x)$ Find the cross products.

$31{,}500 = 105x$ Simplify.

$x = 300$ Solve for x.

The height of the house from the foundation to the gutters is 300 inches.

f. How many 5-foot sections of downspout are needed for Blair's house?

Determine the length of the needed downspout in feet.

Divide the number of inches by 12.

length of needed downspout in feet = $300 \div 12 = 25$

The length of the needed downspout in feet is 25 feet.

Each section of downspout is 5 feet.

Divide the number of feet needed by 5 to determine the number of sections needed.

number of sections needed = $25 \div 5 = 5$

Blair needs 5 sections of downspout.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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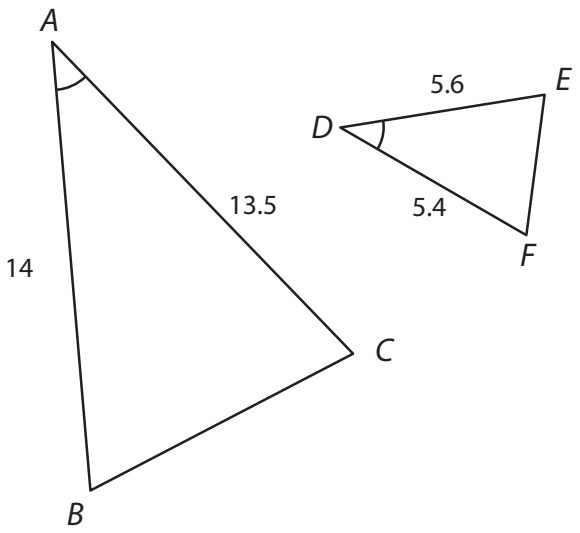
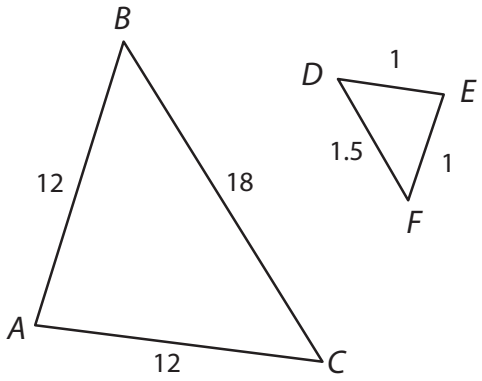
Practice 5.4.1: Proving Triangle Similarity Using Side-Angle-Side (SAS) and Side-Side-Side (SSS) Similarity

Prove that the triangles are similar.

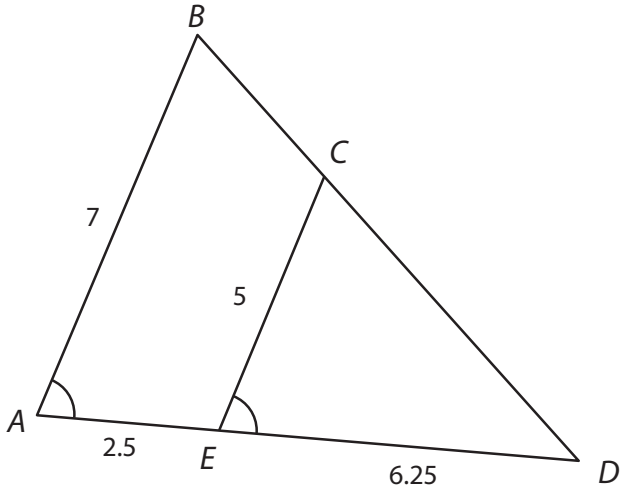
1.



2.



3.



continued

SWB p. 75; SRB Practice 5.4.1 p. U5-79

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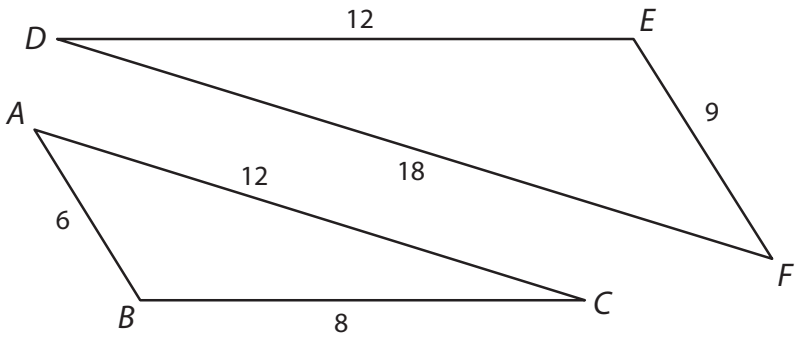
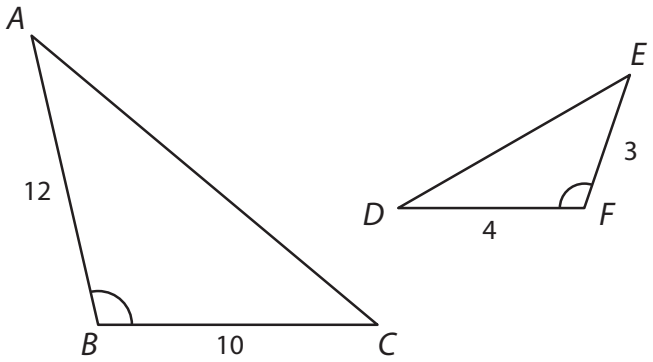
Determine whether the triangles are similar. If the triangles are similar, write a similarity statement.

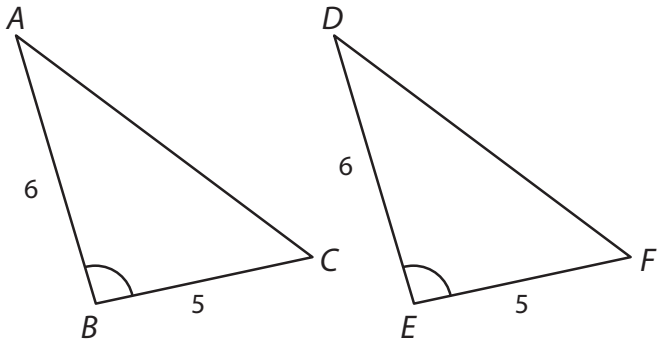
4.



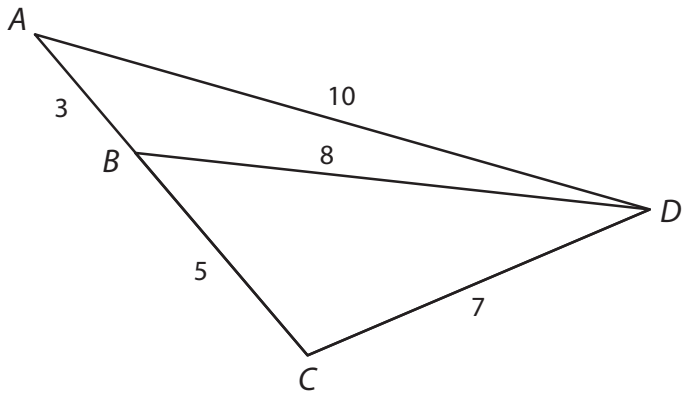
5.



6.



7. Determine whether $\triangle ACD$ and $\triangle BCD$ are similar. If so, write a similarity statement.



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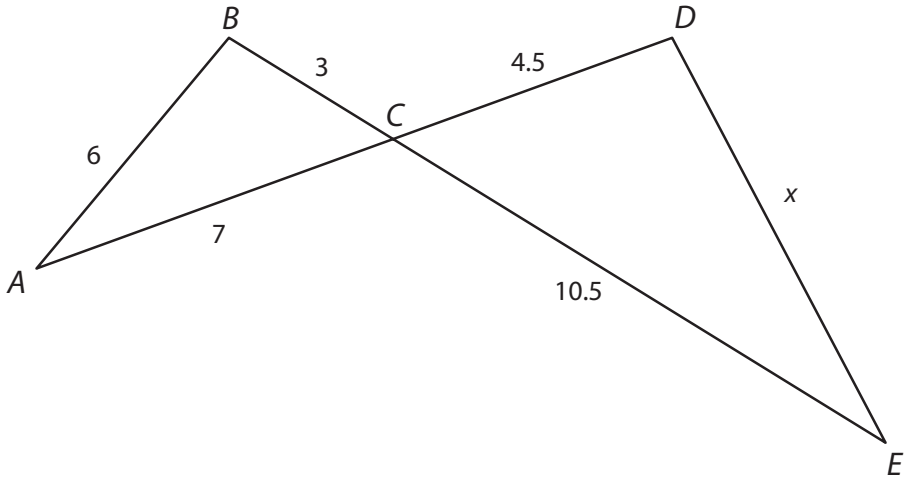
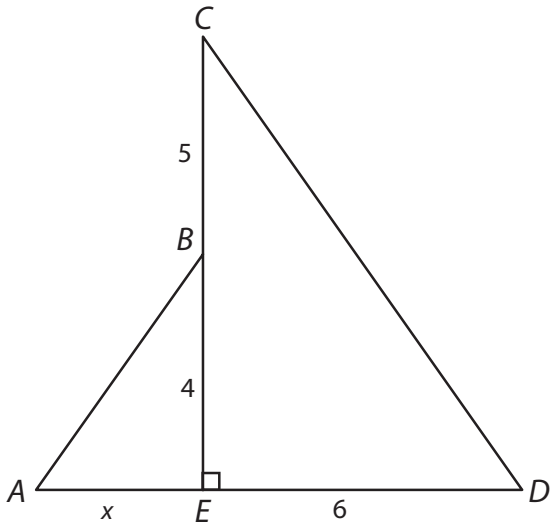
For problems 8–10, the two given triangles are similar. Find x.

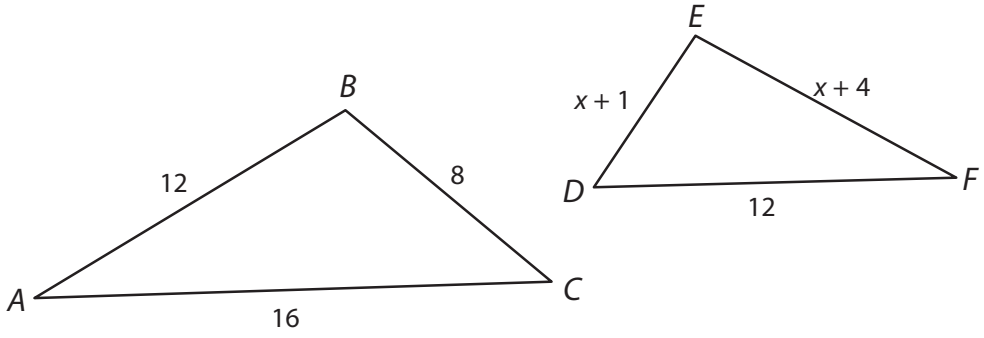
8.



9.



10.



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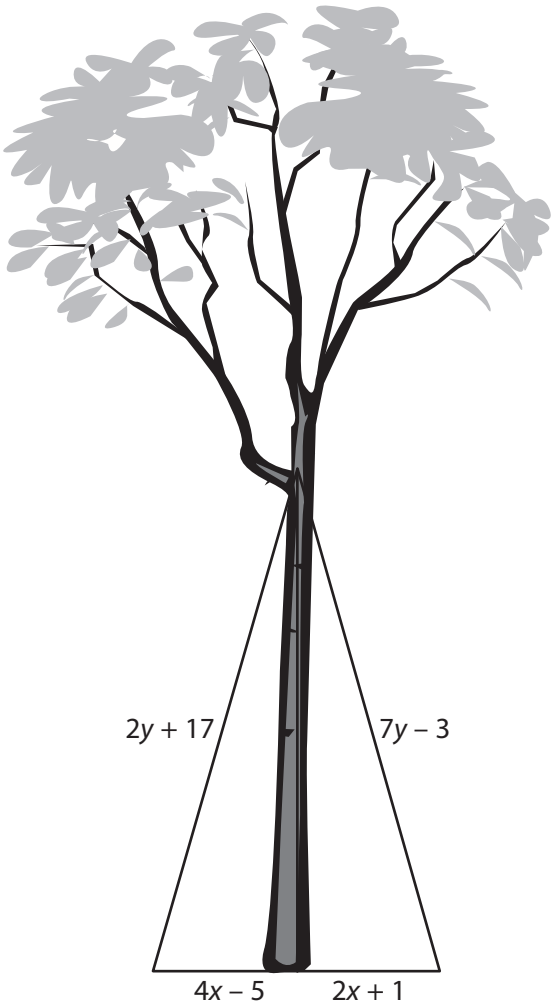
Lesson 5.4.2: Working with Ratio Segments

Warm-Up 5.4.2

Landscapers will often stake a sapling to strengthen the tree's root system. A typical method of staking a tree is to tie wires to both sides of the tree and then stake the wires to the ground. If done properly, the two stakes will be the same distance from the tree.



1. Assuming the distance from the tree trunk to each stake is equal, what is the value of x?

2. How far is each stake from the tree?

3. Assuming the distance from the tree trunk to each stake is equal, what is the value of y?

4. What is the length of the wire from each stake to the tie on the tree?

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Instruction

Lesson 5.4.2: Working with Ratio Segments

Common Core State Standard

G–SRT.4 Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two sides proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.*

Warm-Up 5.4.2 Debrief

1. Assuming the distance from the tree trunk to each stake is equal, what is the value of x?

   To find the value of x, set both expressions equal to each other.

$4x - 5 = 2x + 1$	Set the expressions equal to each other.
$2x - 5 = 1$	Subtract $2x$ from both sides of the equation.
$2x = 6$	Add 5 to both sides of the equation.
$x = 3$	Divide both sides by 3.

   The value of x is 3.

2. How far is each stake from the tree?

   To determine how far each stake is from the tree, substitute 3 for x in either expression.

$4x - 5$	Original expression
$4(3) - 5$	Substitute 3 for x.
$12 - 5 = 7$	Simplify.

$2x + 1$	Original expression
$2(3) + 1$	Substitute 3 for x.
$6 + 1 = 7$	Simplify.

   Each stake is 7 units from the tree trunk.

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Instruction

3. Assuming the distance from the tree trunk to each stake is equal, what is the value of y?

   To find the value of y, set both expressions equal to each other.

$2y + 17 = 7y - 3$	Set the expressions equal to each other.
$17 = 5y - 3$	Subtract $2y$ from both sides of the equation.
$20 = 5y$	Add 3 to both sides of the equation.
$y = 4$	Divide both sides by 5.

   The value of y is 4.

4. What is the length of the wire from each stake to the tie on the tree?

   To determine the length of the wire from each stake to the tie on the tree, substitute 4 for y in each expression.

$2y + 17$	Original expression
$2(4) + 17$	Substitute 4 for y.
$8 + 17 = 25$	Simplify.

$7y - 3$	Original expression
$7(4) - 3$	Substitute 4 for y.
$28 - 3 = 25$	Simplify.

   The length of the wire from each stake to the tie on the tree trunk is 25 units.

Connection to the Lesson

- Students will continue their work with triangles related to this example and expand on their understanding of similar triangles, specifically the relationship between angle bisectors and ratio segments.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- creating ratios (7.RP.2c)
- solving proportions (7.RP.3)
- identifying both corresponding and congruent parts of triangles (G–CO.7)
- understanding angle bisectors (G–CO.12)

Introduction

Archaeologists, among others, rely on the Angle-Angle (AA), Side-Angle-Side (SAS), and Side-Side-Side (SSS) similarity statements to determine actual distances and locations created by similar triangles. Many engineers, surveyors, and designers use these statements along with other properties of similar triangles in their daily work. Having the ability to determine if two triangles are similar allows us to solve many problems where it is necessary to find segment lengths of triangles.

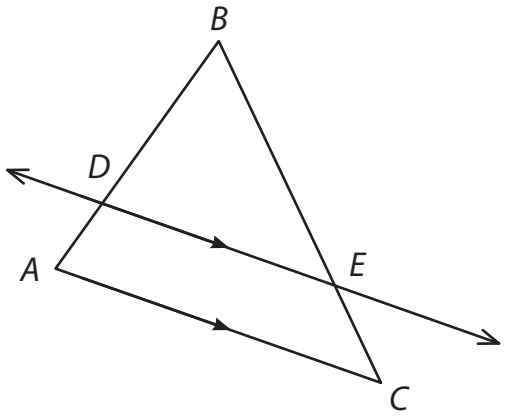
Key Concepts

- **Parallel lines** are lines in a plane that do not share any points and never intersect.
- If a line parallel to one side of a triangle intersects the other two sides of the triangle, then the parallel line divides these two sides proportionally.
- This is known as the Triangle Proportionality Theorem.

Theorem

Triangle Proportionality Theorem

If a line parallel to one side of a triangle intersects the other two sides of the triangle, then the parallel line divides these two sides proportionally.



- In the previous figure, $\overline{AC} \parallel \overline{DE}$; therefore, $\frac{AD}{DB} = \frac{CE}{EB}$.

SRB p. U5-83

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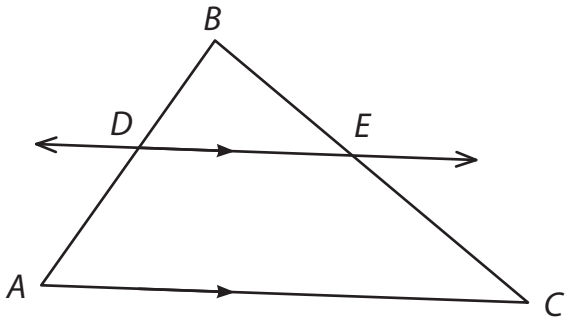
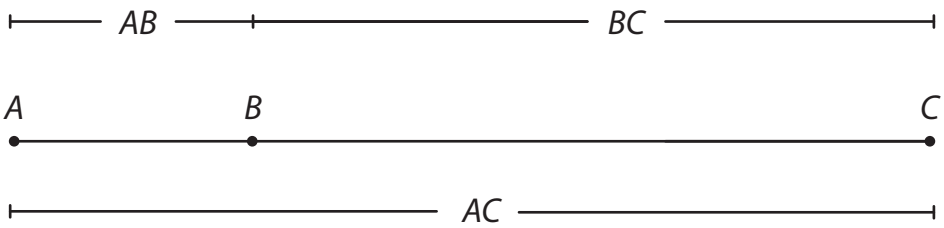
Instruction

- Notice the arrows in the middle of $\overline{DE}$ and $\overline{AC}$, which indicate that the segments are parallel.
- This theorem can be used to find the lengths of various sides or portions of sides of a triangle.
- In the previous figure, $\triangle ABC \sim \triangle DBE$ because of the Side-Angle-Side (SAS) Similarity Statement.
- It is also true that if a line divides two sides of a triangle proportionally, then the line is parallel to the third side.



- In the previous figure, $\frac{AD}{DB} = \frac{CE}{EB}$; therefore, $\overline{AC} \| \overline{DE}$.
- This is helpful when determining if two lines or segments are parallel.
- It is possible to determine the lengths of the sides of triangles because of the **Segment Addition Postulate**.
- This postulate states that if B lies on segment $\overline{AC}$ between A and C, then $AB + BC = AC$.
- It is also true that if $AB + BC = AC$, then B lies on segment $\overline{AC}$.



- Segment congruence is helpful when determining the lengths of sides of triangles.
- The **Reflexive Property of Congruent Segments** means that a segment is congruent to itself, so $\overline{AB} \cong \overline{AB}$.
- According to the **Symmetric Property of Congruent Segments**, if $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$.

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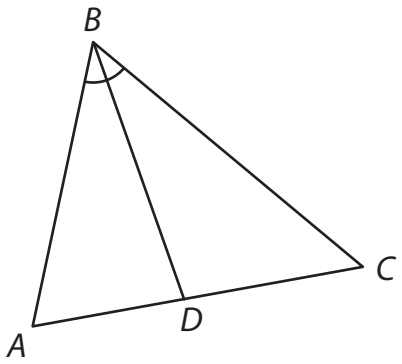
Instruction

- The **Transitive Property of Congruent Segments** allows that if $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.
- This information is helpful when determining segment lengths and proving statements.
- If one angle of a triangle is bisected, or cut in half, then the **angle bisector** of the triangle divides the opposite side of the triangle into two segments that are proportional to the other two sides of the triangle.
- This is known as the Triangle Angle Bisector Theorem.

Theorem

Triangle Angle Bisector Theorem

If one angle of a triangle is bisected, or cut in half, then the angle bisector of the triangle divides the opposite side of the triangle into two segments that are proportional to the other two sides of the triangle.



- In the previous figure, $\angle ABD \cong \angle DBC$; therefore, $\frac{AD}{DC} = \frac{BA}{BC}$.
- These theorems can be used to determine segment lengths as well as verify that lines or segments are parallel.

Common Errors/Misconceptions

- assuming a line parallel to one side of a triangle bisects the remaining sides rather than creating proportional sides
- interchanging similarity statements with congruence statements

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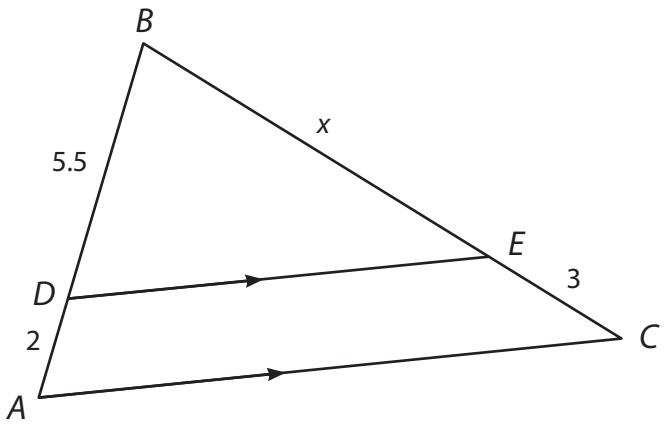
Instruction

Guided Practice 5.4.2

Example 1

Find the length of $\overline{BE}$.



1. Identify the given information.

   According to the diagram, $\overline{AC} \parallel \overline{DE}$, $DA = 2$, $BD = 5.5$, and $EC = 3$.

2. Find the length of $\overline{BE}$.

   Use the Triangle Proportionality Theorem to find the length of $\overline{BE}$.

$\frac{BD}{DA} = \frac{BE}{EC}$	Create a proportion.
$\frac{5.5}{2} = \frac{x}{3}$	Substitute the known lengths of each segment.
$(3)(5.5) = (2)(x)$	Find the cross products.
$16.5 = 2x$	Solve for x.
$x = 8.25$	

The length of $\overline{BE}$ is 8.25 units.

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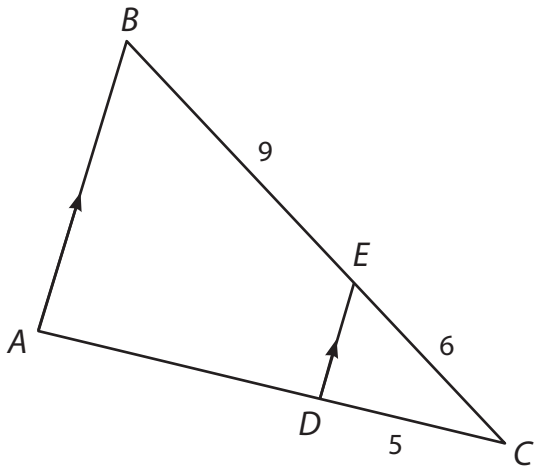
Instruction

Example 2

Using two different methods, find the length of $\overline{CA}$.



Method 1

1. Identify the given information.

   According to the diagram, $\overline{AB} \parallel \overline{DE}$, $EB = 9$, $CE = 6$, and $CD = 5$.

2. Find the length of $\overline{CA}$.

   Use the Triangle Proportionality Theorem to find the length of $\overline{CA}$.

   $\frac{CA}{CD} = \frac{CB}{CE}$ Create a proportion.

   $\frac{CA}{5} = \frac{9+6}{6}$ Substitute the known lengths of each segment.

   $(6)(CA) = (5)(9 + 6)$ Find the cross products.

   $(6)(CA) = (5)(15)$ Solve for CA.

   $(6)(CA) = 75$

   $CA = 12.5$

   The length of $\overline{CA}$ is 12.5 units.

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Method 2

1. Identify the given information.

   According to the diagram, $\overline{AB} \parallel \overline{DE}$, $EB = 9$, $CE = 6$, and $CD = 5$.

2. Find the length of $\overline{CA}$.

   Use the Triangle Proportionality Theorem to find the length of $\overline{CA}$.

   An alternative method for finding the length of $\overline{CA}$ is to first find the length of $\overline{DA}$.

   $\frac{CD}{DA} = \frac{CE}{EB}$ Create a proportion.

   $\frac{5}{x} = \frac{6}{9}$ Substitute the known lengths of each segment.

   $(5)(9) = (6)(x)$ Find the cross products.

   $45 = 6x$ Solve for x.

   $x = 7.5$

   The length of $\overline{CA}$ is equal to the sum of $\overline{CD}$ and $\overline{DA}$.

   $CD + DA = CA$

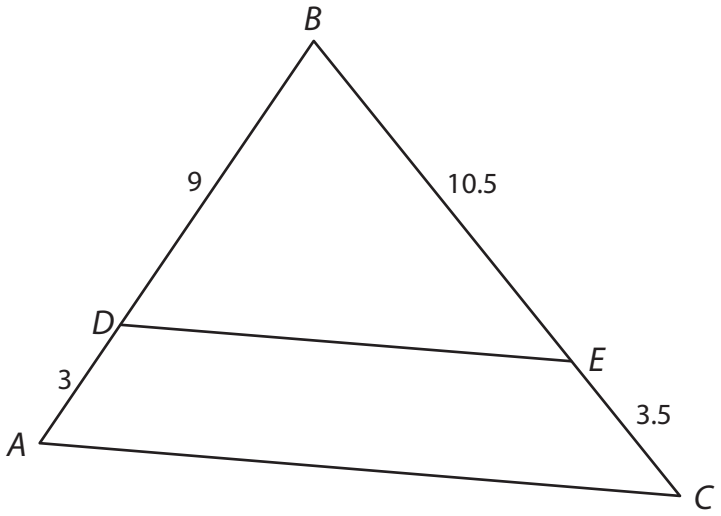
   $5 + 7.5 = 12.5$

   The length of $\overline{CA}$ is 12.5 units.

Instruction

Example 3

Prove that $\overline{DE} \| \overline{AC}$.



1. Determine if $\overline{DE}$ divides $\overline{BA}$ and $\overline{BC}$ proportionally.

$$\frac{BD}{DA} = \frac{9}{3} = 3$$

$$\frac{BE}{EC} = \frac{10.5}{3.5} = 3$$

2. State your conclusion.

$\frac{BD}{DA} = \frac{BE}{EC} = 3$; therefore, $\overline{DE} \| \overline{AC}$ because of the Triangle Proportionality Theorem.

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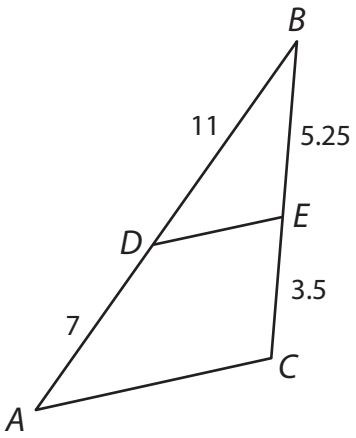
Instruction

Example 4

Is $\overline{DE} \| \overline{AC}$?



1. Determine if $\overline{DE}$ divides $\overline{BA}$ and $\overline{BC}$ proportionally.

$$\frac{BD}{DA} = \frac{11}{7}$$

$$\frac{BE}{EC} = \frac{5.25}{3.5} = \frac{3}{2}$$

2. State your conclusion.

$$\frac{11}{7} \neq \frac{3}{2}$$

$\frac{BD}{DA} \neq \frac{BE}{EC}$; therefore, $\overline{DE}$ is not parallel to $\overline{AC}$ because of the Triangle Proportionality Theorem.

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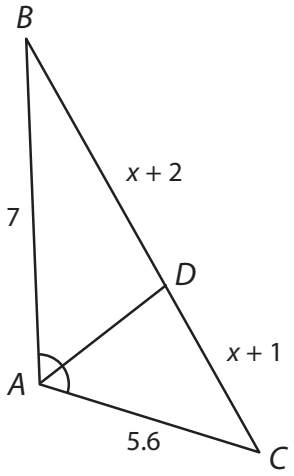
Instruction

Example 5

Find the lengths of $\overline{BD}$ and $\overline{DC}$.



1. Identify the given information.

   $\angle BAD \cong \angle DAC$, $AB = 7$, $AC = 5.6$, $BD = x + 2$, and $DC = x + 1$.

   $\frac{BD}{DC} = \frac{BA}{AC}$ because of the Triangle Angle Bisector Theorem.

2. Determine the lengths of $\overline{BD}$ and $\overline{DC}$.

$\frac{BD}{DC} = \frac{BA}{AC}$	Create a proportion.
$\frac{x+2}{x+1} = \frac{7}{5.6}$	Substitute the known lengths of each segment.
$(x + 2)(5.6) = (x + 1)(7)$	Find the cross products.
$5.6x + 11.2 = 7x + 7$	Solve for x.
$4.2 = 1.4x$	
$x = 3$	

   $BD = x + 2 = (3) + 2 = 5$

   $DC = x + 1 = (3) + 1 = 4$

3. State your conclusion.

   The length of $\overline{BD}$ is 5 units.

   The length of $\overline{DC}$ is 4 units.

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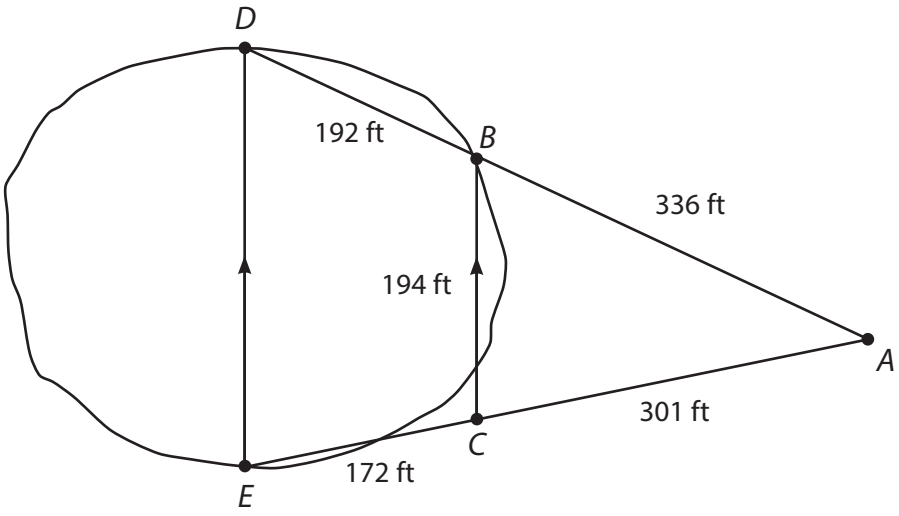
Problem-Based Task 5.4.2: Suddenly Sinking

SMP	
1 ✓	2
3 ✓	4 ✓
5	6
7	8

In August 2012, a sinkhole in Louisiana swallowed part of the Earth's surface. This sudden erosion of land forced many people to evacuate as engineers worked to determine the effects of the sinkhole. Officials began their assessment of the area by first determining the size of the sinkhole. Surveyors worked to locate points near the sinkhole that could be used to calculate its diameter. The initial measurements are shown below. As days passed, the diameter of the sinkhole increased by 15%. What was the diameter of the sinkhole after the increase?



What was the diameter of the sinkhole after the increase?

SWB p. 87

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Problem-Based Task 5.4.2: Suddenly Sinking

Coaching

a. Are the triangles in the diagram similar? Explain.

b. What are the characteristics of similar triangles?

c. Is it possible to determine the diameter of the initial sinkhole using the given information? Explain your answer.

d. Determine the length of $\overline{DE}$.

e. What was the diameter of the initial sinkhole?

f. How are percent increases calculated?

g. What is 15% written as a decimal?

h. What was the diameter after the sinkhole increased?

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Instruction

Problem-Based Task 5.4.2: Suddenly Sinking

Coaching Sample Responses

a. Are the triangles in the diagram similar? Explain.

The triangles in the diagram are similar. $\overline{BC}$ is parallel to $\overline{DE}$ of $\triangle ADE$.

According to the Triangle Proportionality Theorem, if a line parallel to one side of a triangle intersects the other two sides of the triangle, then the parallel line divides these two sides proportionally. And, according to the Side-Angle-Side Similarity Statement, if two sides of one triangle are proportional to two sides of another triangle, and their included angle is congruent, then the triangles are similar.

b. What are the characteristics of similar triangles?

The angles of similar triangles are congruent and the sides are proportional.

c. Is it possible to determine the diameter of the initial sinkhole using the given information? Explain your answer.

Yes, it is possible to determine the diameter of the initial sinkhole using the given information. You can do so by creating a proportion using the two similar triangles, $\triangle ABC$ and $\triangle ADE$.

d. Determine the length of $\overline{DE}$.

Create a proportion to find the length of $\overline{DE}$.

$\frac{AD}{DE}=\frac{AB}{BC}$	Create a proportion.
$\frac{336+192}{x}=\frac{336}{194}$	Substitute known values.
$\frac{528}{x}=\frac{336}{194}$	Simplify.
$(528)(194) = (336)(x)$	Find the cross products.
$102{,}432 = 336x$	Simplify.
$x \approx 305$ feet	

The length of $\overline{DE}$ is approximately 305 feet.

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e. What was the diameter of the initial sinkhole?

The diameter of the initial sinkhole was approximately 305 feet.

f. How are percent increases calculated?

One way to calculate a percent increase is to add the initial value to the product of the initial value and the decimal equivalent of the percent increase.

initial value + (percent increase)(initial value)

g. What is 15% written as a decimal?

15% can be written as 0.15.

h. What was the diameter after the sinkhole increased?

Add 15% of the initial diameter to the initial diameter.

$305 + (0.15)(305) = 350.75$

The diameter of the sinkhole after the increase was 350.75 feet.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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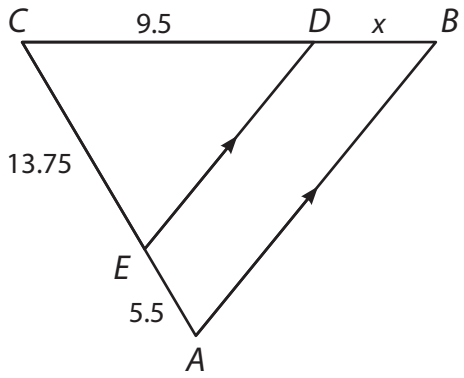
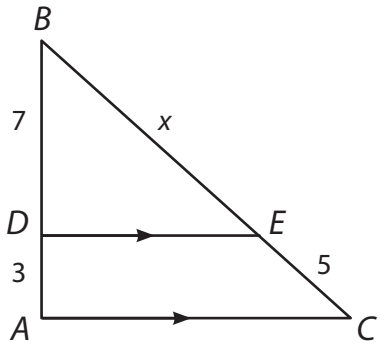
Practice 5.4.2: Working with Ratio Segments

Use the Triangle Proportionality Theorem and the Triangle Angle Bisector Theorem to find the unknown lengths of the given segments.

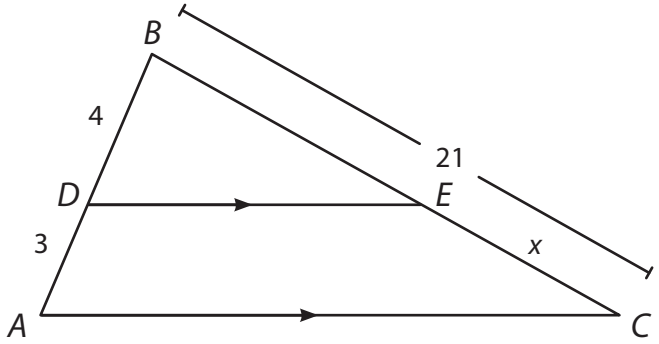
1. $\overline{BD}$



2. $\overline{BE}$



3. $\overline{EC}$



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SWB p. 89; SRB Practice 5.4.2 p. U5-92

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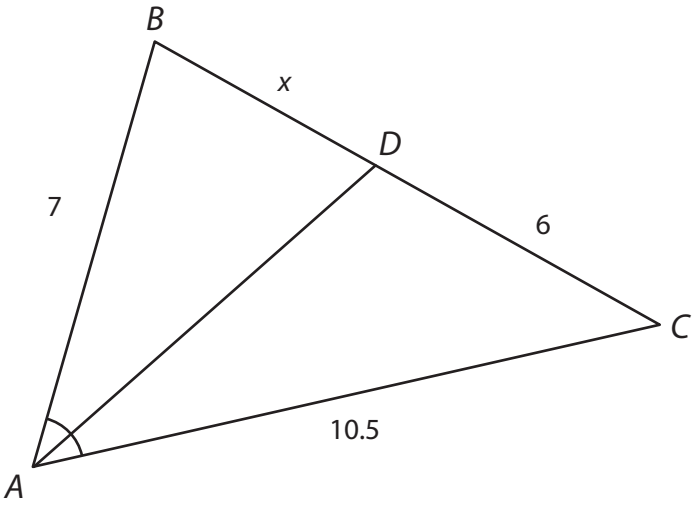
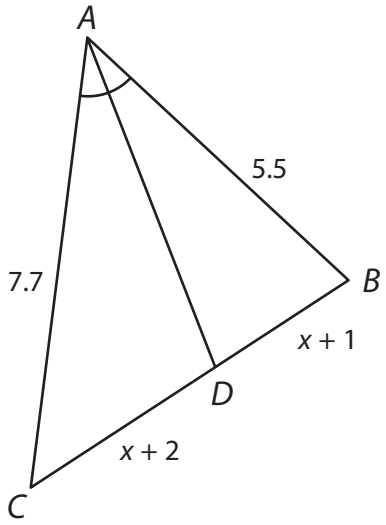
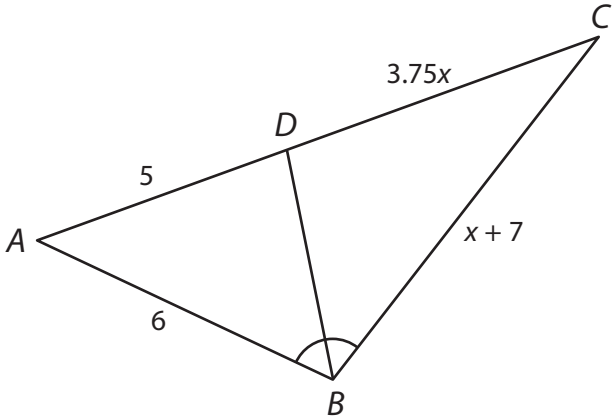
4. $\overline{BD}$



5. $\overline{CD}$; $\overline{BD}$



6. $\overline{CB}$; $\overline{CD}$



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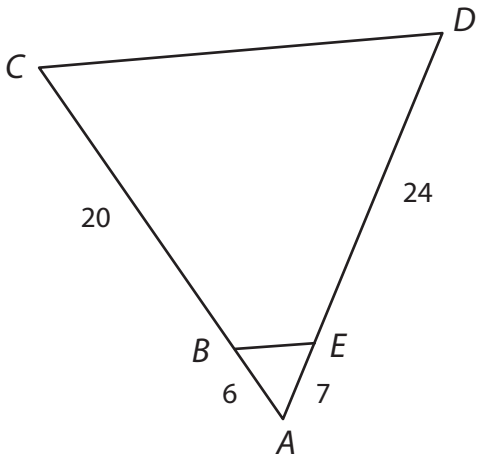
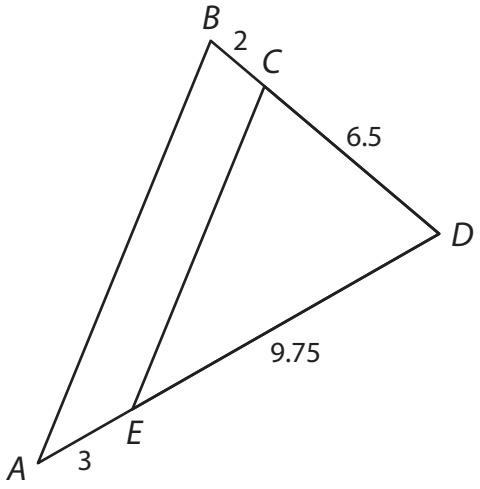
Use the Triangle Proportionality Theorem to determine if the given segments are parallel. Explain your reasoning.

7. Is $\overline{BE} \| \overline{CD}$?



8. Is $\overline{AB} \| \overline{EC}$?



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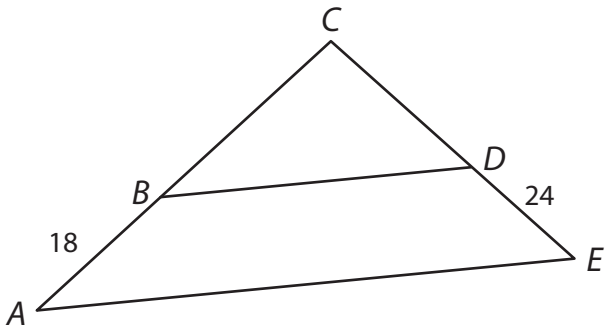
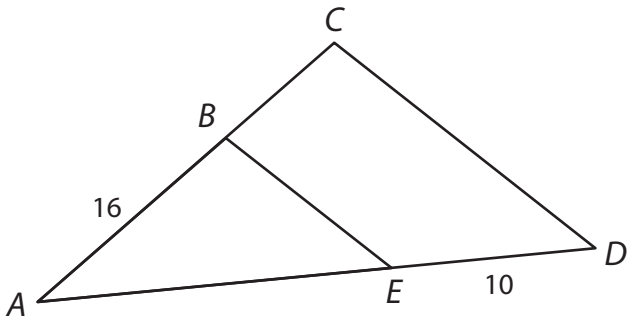
9. If $AC = 60$ units and $EC = 36$ units, is $\overline{AE} \parallel \overline{BD}$?



10. If $AC = 24$ units and $AD = 30$ units, is $\overline{BE} \parallel \overline{CD}$?



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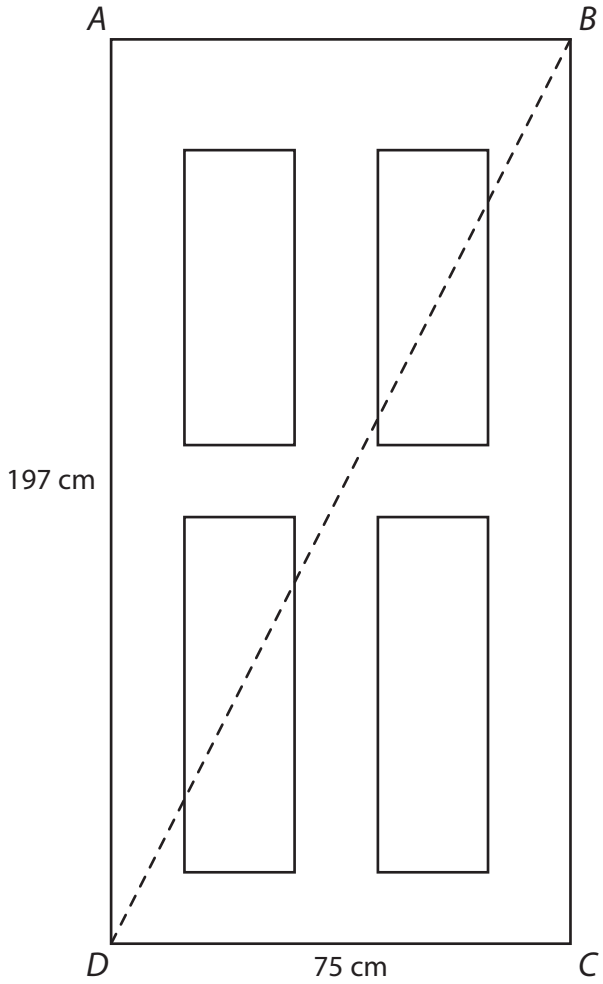
Lesson 5.4.3: Proving the Pythagorean Theorem Using Similarity

Warm-Up 5.4.3

Woodworkers must accurately cut and assemble each piece of wood to ensure that a project is "square." Every vertical piece should intersect every horizontal piece at a 90° angle. To determine if a project is square, woodworkers use the Pythagorean Theorem, which states that the sum of the squares of the two legs of a right triangle is equal to the square of the longest side. If the lengths of the two diagonals are equal, then the project is square. Use the diagram below of a door to solve the problems that follow.



1. A woodworker measured the length of one diagonal of the wooden door, $\overline{BD}$, to be 212 cm. The woodworker measured the length of $\overline{AD}$ to be 197 cm and the length of $\overline{DC}$ to be 75 cm. Calculate the length of $\overline{AC}$.
2. Is $\overline{BD}$ congruent to $\overline{AC}$?
3. Is the door "square"? Explain your answer.

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Instruction

Lesson 5.4.3: Proving the Pythagorean Theorem Using Similarity

Common Core State Standard

G–SRT.4 Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two sides proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.*

Warm-Up 5.4.3 Debrief

1. A woodworker measured the length of one diagonal of the wooden door, $\overline{BD}$, to be 212 cm. The woodworker measured the length of $\overline{AD}$ to be 197 cm and the length of $\overline{DC}$ to be 75 cm. Calculate the length of $\overline{AC}$.

   $\overline{AD}$ and $\overline{DC}$ are the two shorter sides of $\triangle ADC$.

   Use the Pythagorean Theorem to calculate the length of $\overline{AC}$.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$(197)^2 + (75)^2 = c^2$	Substitute values for a and b.
$38{,}809 + 5625 = c^2$	Simplify.
$44{,}434 = c^2$	Take the positive square root of both sides.

   $c = \sqrt{44{,}434} \approx 210.79$ cm

   The length of $\overline{AC}$ is approximately 210.79 cm.

2. Is $\overline{BD}$ congruent to $\overline{AC}$?

   The length of $\overline{BD}$ was measured to be 212 cm.

   The calculated length of $\overline{AC}$ is approximately 210.79 cm.

   Although the lengths are close, they are not congruent.

3. Is the door "square"? Explain your answer.

   To be square, the lengths of each diagonal must be congruent.

   The lengths are not congruent; therefore, the door, as assembled, is not square.

Connection to the Lesson

- Students have worked with the Pythagorean Theorem in the past and will deepen their understanding of this theorem while determining similar triangles.

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Prerequisite Skills

This lesson requires the use of the following skills:

- using the distance formula to find the lengths of sides of triangles (8.G.8)
- being familiar with the Pythagorean Theorem (8.G.7)
- working with and simplifying square roots (N–RN.2)
- identifying similar triangles (G–SRT.2)
- using similarity statements to find unknown lengths and measures of similar triangles (G–SRT.5)

Introduction

Geometry includes many definitions and statements. Once a statement has been shown to be true, it is called a **theorem**. Theorems, like definitions, can be used to show other statements are true. One of the most well known theorems of geometry is the Pythagorean Theorem, which relates the length of the hypotenuse of a right triangle to the lengths of its legs. The Pythagorean Theorem has many applications and can be very helpful when solving real-world problems. There are several ways to prove the Pythagorean Theorem; one way is by using similar triangles and similarity statements.

Key Concepts

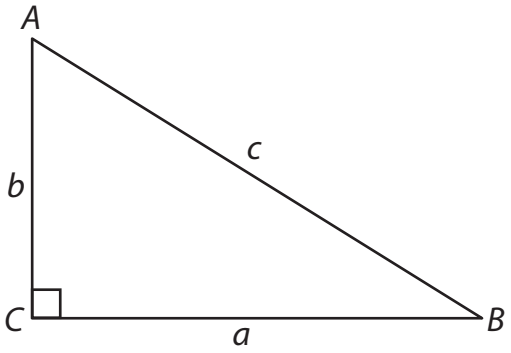
The Pythagorean Theorem

- The Pythagorean Theorem is often used to find the lengths of the sides of a right triangle, a triangle that includes one 90˚ angle.

Theorem

Pythagorean Theorem

The sum of the squares of the lengths of the legs (a and b) of a right triangle is equal to the square of the length of the hypotenuse (c).



$a^2 + b^2 = c^2$

SRB p. U5-96

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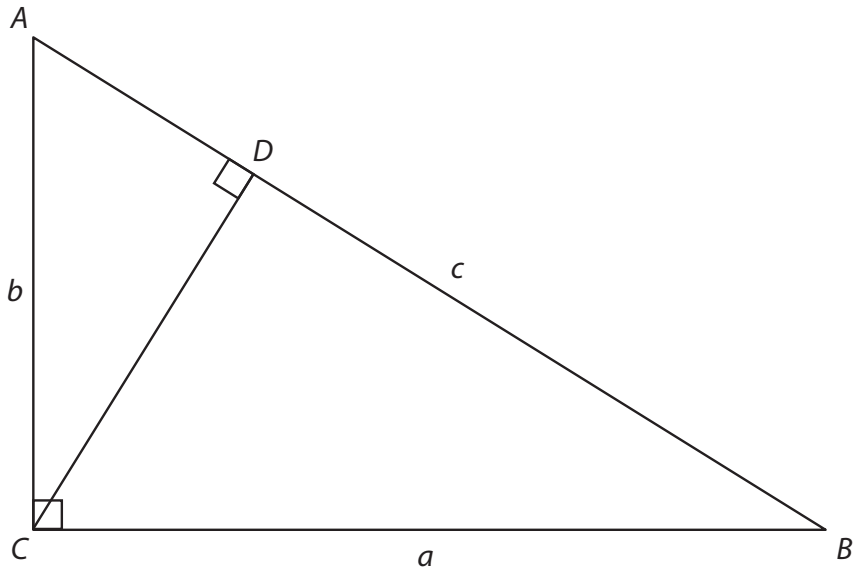
Instruction

- In the triangle diagram, angle C is 90°, as shown by the square.
- The longest side of the right triangle, c, is called the hypotenuse and is always located across from the right angle.
- The legs of the right triangle, a and b, are the two shorter sides.
- It is also true that if the sum of the squares of the measures of the two shorter sides of a triangle equals the square of the measure of the longest side, then the triangle is a right triangle.
- This is known as the **converse of the Pythagorean Theorem**.
- To prove the Pythagorean Theorem using similar triangles, you must first identify the similar triangles.
- In this example, there is only one triangle given.
- Begin by drawing the **altitude,** the segment from angle C that is perpendicular to the opposite side, c.



- Notice that by creating the altitude $\overline{CD}$, we have created two smaller right triangles, $\triangle ADC$ and $\triangle BDC$, within the larger given right triangle, $\triangle ACB$.
- $\angle ACB$ and $\angle ADC$ are 90° and are therefore congruent.
- $\angle A$ of $\triangle ADC$ is congruent to $\angle A$ of $\triangle ACB$ because of the Reflexive Property of Congruence.

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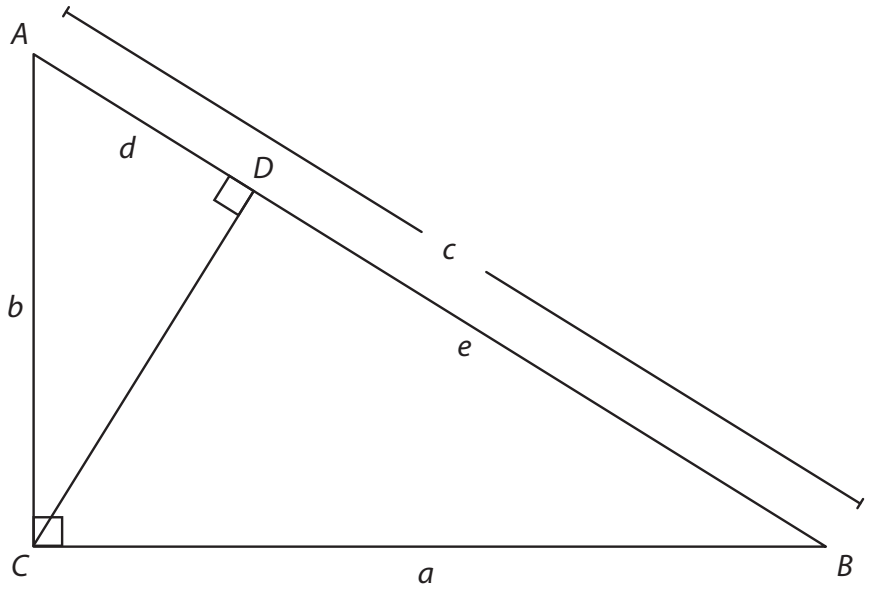
- According to the Angle-Angle (AA) Similarity Statement, if two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar; therefore, $\triangle ADC \sim \triangle ACB$.
- $\angle ACB$ and $\angle BDC$ are 90° and are therefore congruent.
- $\angle B$ of $\triangle BDC$ is congruent to $\angle B$ of $\triangle ACB$ because of the Reflexive Property of Congruence.
- Two angles in $\triangle BDC$ are congruent to two angles in $\triangle ACB$; therefore, $\triangle BDC \sim \triangle ACB$.
- Similarity is transitive. Since $\triangle ADC \sim \triangle ACB$ and $\triangle BDC \sim \triangle ACB$, then $\triangle ADC \sim \triangle BDC$.



- Corresponding sides of similar triangles are proportional; therefore, $\frac{c}{b}=\frac{b}{d}$ and $\frac{c}{a}=\frac{a}{e}$.
- Determining the cross products of each proportion leads to the Pythagorean Theorem.

$\frac{c}{b}=\frac{b}{d}$ $\qquad$ $\frac{c}{a}=\frac{a}{e}$

$cd = b^2$ $\qquad$ $ce = a^2$

$cd + ce = a^2 + b^2$	Add both equations.
$c(e + d) = a^2 + b^2$	Factor.
$c^2 = a^2 + b^2$	$(e + d)$ is equal to c because of segment addition.

- The converse of the Pythagorean Theorem can be useful when proving right triangles using similar triangles.

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Types of Proofs

- **Paragraph proofs** are statements written out in complete sentences in a logical order to show an argument.
- **Flow proofs** are a graphical method of presenting the logical steps used to show an argument.
- In a flow proof, the logical statements are written in boxes and the reason for each statement is written below the box.
- Another form of proof is a **two-column proof**.
- Two-column proofs include numbered statements and corresponding reasons that show an argument in a logical order.
- Two-column proofs appear in the Guided Practice examples that follow.

Common Errors/Misconceptions

- misidentifying the altitudes of triangles
- incorrectly simplifying expressions with square roots

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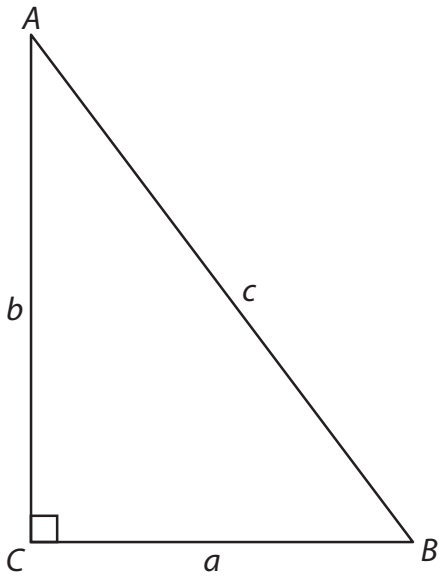
Guided Practice 5.4.3

Example 1

Write a two-column proof to prove the Pythagorean Theorem using similar triangles.

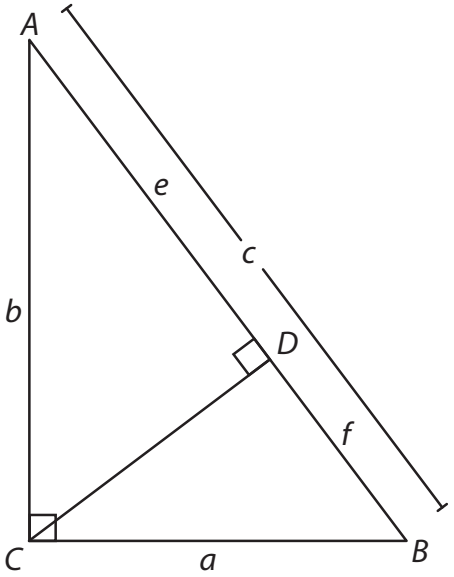


1. Draw in helpful information.

   Draw the altitude from $\angle C$ and label the point of intersection on $\overline{AB}$ as D.

   Label $\overline{AD}$ as e.

   Label $\overline{DB}$ as f.



SWB p. 97; SRB p. U5-100

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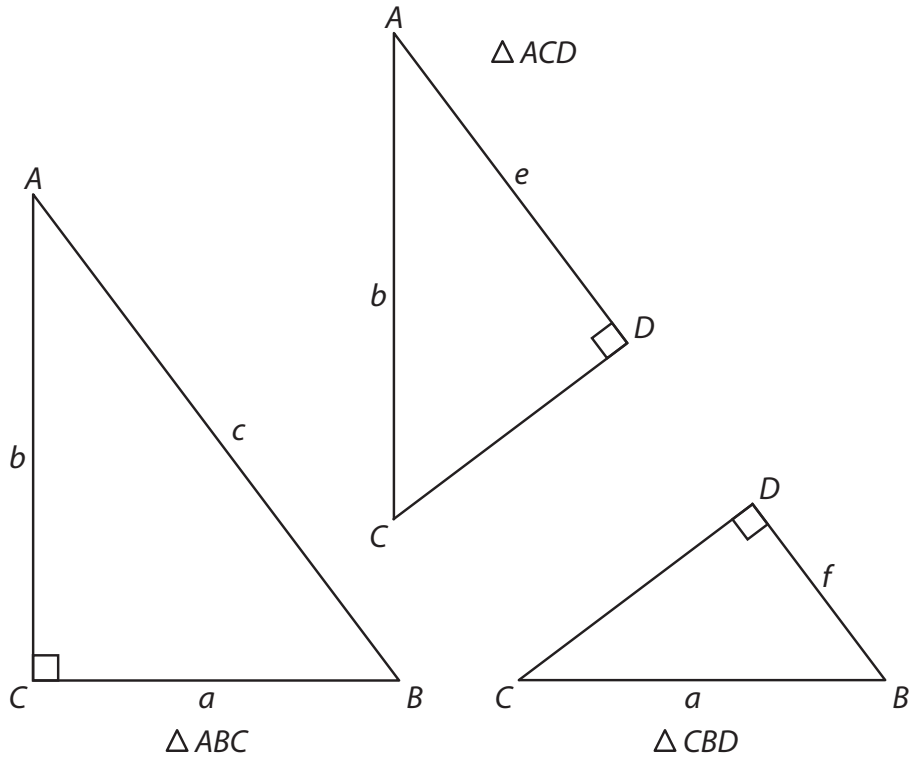
2. Identify the similar triangles.

   It is often helpful to redraw the triangles.



3. Create the two-column proof.

Statements	Reasons
1. $\triangle ABC$ with right $\angle C$	1. Given
2. $\triangle ABC \sim \triangle ACD$ $\triangle ABC \sim \triangle CBD$	2. If the altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original triangle and each other.
3. $\frac{c}{a}=\frac{a}{f}; \frac{c}{b}=\frac{b}{e}$	3. Definition of similar triangles; corresponding sides are proportional.
4. $cf = a^2; ce = b^2$	4. Multiplication Property of Equality
5. $cf + ce = a^2 + b^2$	5. Addition Property of Equality
6. $c(f + e) = a^2 + b^2$	6. Distributive Property of Equality
7. $e + f = c$	7. Segment Addition Postulate
8. $c(c) = a^2 + b^2$ or $c^2 = a^2 + b^2$	8. Substitution Property

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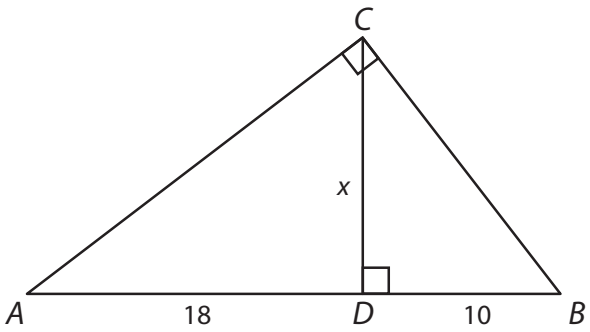
Example 2

Find the length of the altitude, x, of $\triangle ABC$.



1. Identify the similar triangles.

   $\triangle ABC$ is a right triangle.

   The altitude of $\triangle ABC$ is drawn from $\angle ACB$ to the opposite side, creating two smaller similar triangles.

   $\triangle ABC \sim \triangle ACD \sim \triangle CBD$

2. Use corresponding sides to write a proportion containing x.

   $$\frac{\text{shorter leg of } \triangle ACD}{\text{shorter leg of } \triangle CBD} = \frac{\text{longer leg of } \triangle ACD}{\text{longer leg of } \triangle CBD}$$

   $\frac{x}{10} = \frac{18}{x}$ Substitute values for each side.

   $(x)(x) = (10)(18)$ Find the cross products.

   $x^2 = 180$ Simplify.

   $x = 6\sqrt{5} \approx 13.4$ Take the positive square root of each side.

3. Summarize your findings.

   The length of the altitude, x, of $\triangle ABC$ is $6\sqrt{5}$ units, or approximately 13.4 units.

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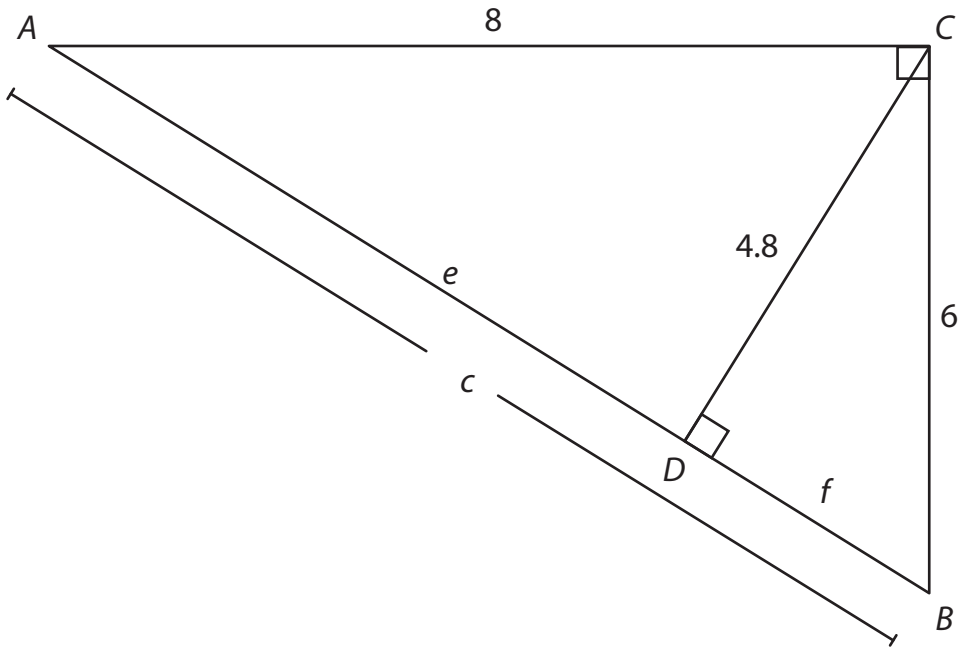
Example 3

Find the unknown values in the figure using similar triangles.



1. Identify the similar triangles.

   $\triangle ABC$ is a right triangle.

   The altitude of $\triangle ABC$ is drawn from $\angle ACB$ to the opposite side, creating two smaller similar triangles.

   $\triangle ABC \sim \triangle ACD \sim \triangle CBD$

2. Use corresponding sides to write a proportion containing c.

$$\frac{\text{hypotenuse of } \triangle ACD}{\text{hypotenuse of } \triangle ABC} = \frac{\text{shorter leg of } \triangle ACD}{\text{shorter leg of } \triangle ABC}$$

$\frac{8}{c} = \frac{4.8}{6}$	Substitute values for each side.
$(8)(6) = (c)(4.8)$	Find the cross products.
$48 = 4.8c$	Simplify.
$c = 10$	Solve for c.

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3. Use corresponding sides to write a proportion containing e.

$$\frac{\text{longer leg of } \triangle ACD}{\text{longer leg of } \triangle ABC} = \frac{\text{shorter leg of } \triangle ACD}{\text{shorter leg of } \triangle ABC}$$

$\frac{e}{8} = \frac{4.8}{6}$	Substitute values for each side.
$(e)(6) = (8)(4.8)$	Find the cross products.
$6e = 38.4$	Simplify.
$e = 6.4$	Solve for e.

4. Use corresponding sides to write a proportion containing f.

$$\frac{\text{longer leg of } \triangle CBD}{\text{longer leg of } \triangle ABC} = \frac{\text{shorter leg of } \triangle CBD}{\text{shorter leg of } \triangle ABC}$$

$\frac{4.8}{8} = \frac{f}{6}$	Substitute values for each side.
$(4.8)(6) = (8)(f)$	Find the cross products.
$28.8 = 8f$	Simplify.
$f = 3.6$	Solve for f.

5. Summarize your findings.

The length of c is 10 units.

The length of e is 6.4 units.

The length of f is 3.6 units.

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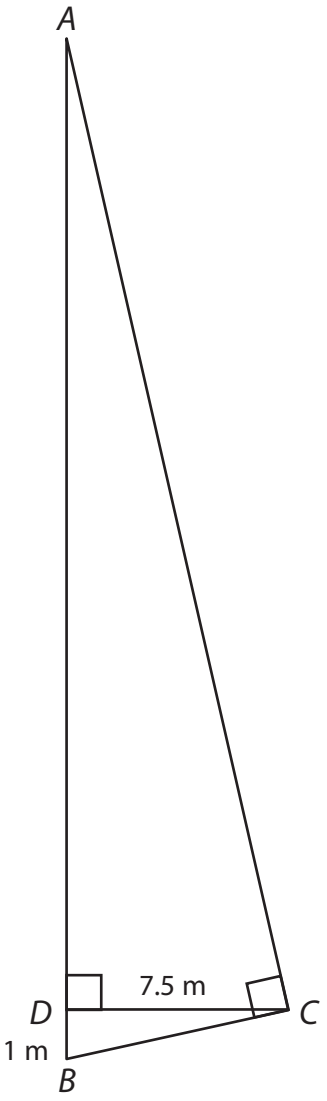
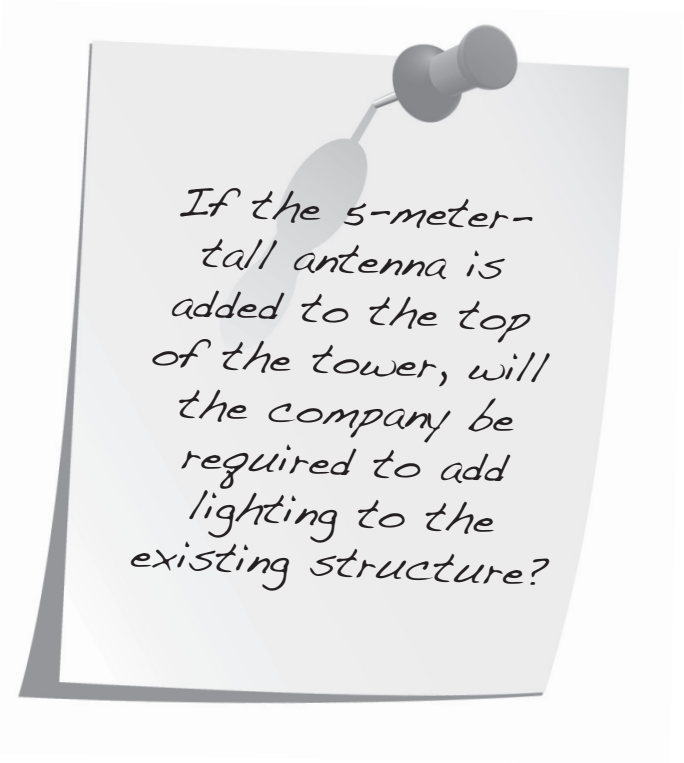
Problem-Based Task 5.4.3: Towering Heights

SMP	
1	2
3 ✓	4 ✓
5	6
7 ✓	8

Cell towers are tall structures that provide service to customers with cell phones, smartphones, tablets, and other wireless communication devices. As the use of wireless devices increases, the need for cell towers goes up. According to the Federal Aviation Administration (FAA) and the Federal Communications Commission (FCC), any structure taller than 61 meters must be lit up so aircraft pilots can see it.

A cell phone company is interested in placing a 5-meter-tall antenna on an existing tower in order to boost their cell signal without having to build a new tower. A surveyor standing 7.5 meters from the base of the tower calculates the height of the existing tower. The line of sight from the lens on the surveyor's tripod to the base of the tower is 1 meter above the ground. If the 5-meter-tall antenna is added to the top of the tower, will the company be required to add lighting to the structure? Explain your reasoning.





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Problem-Based Task 5.4.3: Towering Heights

Coaching

a. Are the triangles in the diagram similar? Explain.

b. Write a statement indicating similar triangles.

c. What are the characteristics of similar triangles?

d. Is it possible to determine the height of the tower using the given information? Explain your answer.

e. Determine the length of $\overline{AD}$.

f. What is the height of the tower?

g. How tall will the tower be once the antenna is added?

h. Will the height of the tower with the antenna exceed the maximum height stated by the FAA and FCC?

i. Will the company be required to add lighting to the tower once the antenna is added?

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Problem-Based Task 5.4.3: Towering Heights

Coaching Sample Responses

a. Are the triangles in the diagram similar? Explain.

Yes, the triangles are similar. The triangle created from the base of the tower to the top of the tower to the surveyor is a right triangle. The segment from the 90° angle to the tower is perpendicular to the tower and is the altitude of the triangle. By definition, the altitude of the right triangle creates two smaller, similar right triangles.

b. Write a statement indicating similar triangles.

$\triangle ABC \sim \triangle CDB \sim \triangle ADC$

c. What are the characteristics of similar triangles?

The angles of similar triangles are congruent and the sides are proportional.

d. Is it possible to determine the height of the tower using the given information? Explain your answer.

Yes, it is possible to determine the height of the tower using the given information by creating a proportion using the two smaller similar triangles, $\triangle CDB$ and $\triangle ADC$.

e. Determine the length of $\overline{AD}$.

Create a proportion to find the length of $\overline{AD}$.

$\frac{CD}{AD} = \frac{BD}{CD}$

$\frac{7.5}{x} = \frac{1}{7.5}$ Substitute known values.

$(7.5)(7.5) = (1)(x)$ Find the cross products.

$56.25 = x$ Simplify.

The length of $\overline{AD}$ is 56.25 meters.

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f. What is the height of the tower?

To find the height of the tower, add the length of $\overline{AD}$ to the length of $\overline{BD}$.

$AD + BD = 56.25 + 1 = 57.25$

The height of the tower is 57.25 meters.

g. How tall will the tower be once the antenna is added?

Add the height of the antenna to the height of the tower.

$5 + 57.25 = 62.25$

The tower will be 62.25 meters tall after the antenna is added.

h. Will the height of the tower with the antenna exceed the maximum height stated by the FAA and FCC?

Yes; 62.25 is greater than 61 meters, the maximum height stated by the FAA and FCC.

i. Will the company be required to add lighting to the tower once the antenna is added?

Yes; the tower with the antenna will exceed 61 meters, so lighting will have to be added.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 5.4.3: Proving the Pythagorean Theorem Using Similarity

For problems 1–9, find the unknown length(s) in each figure.

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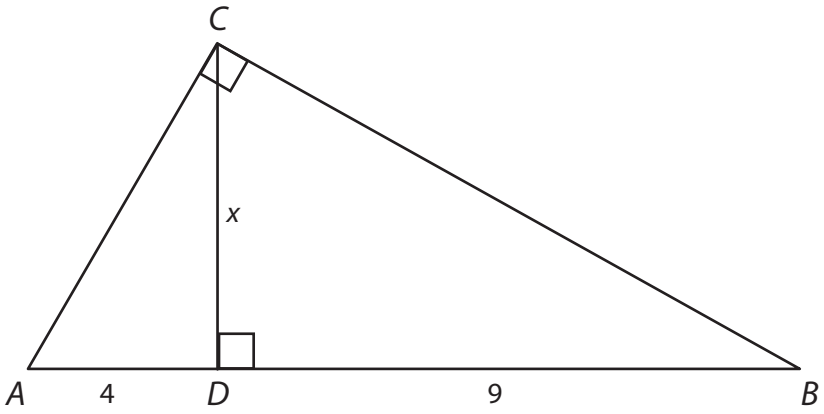

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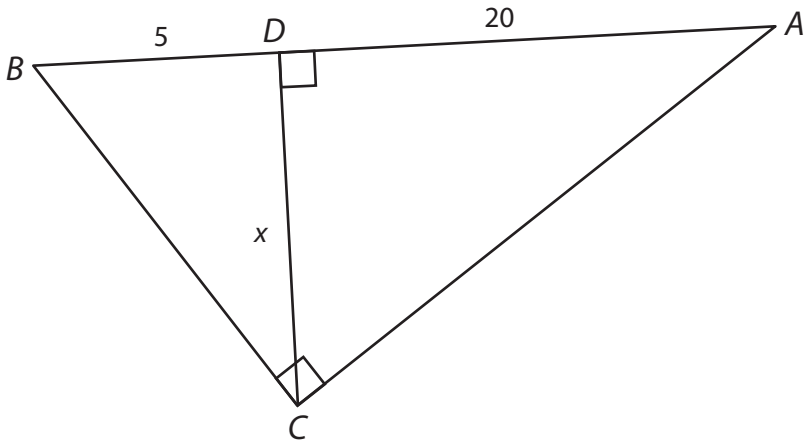

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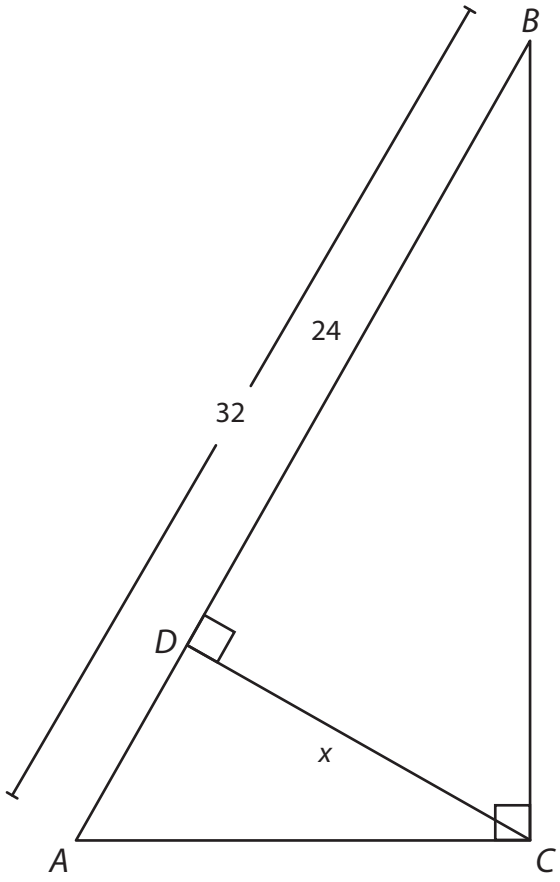

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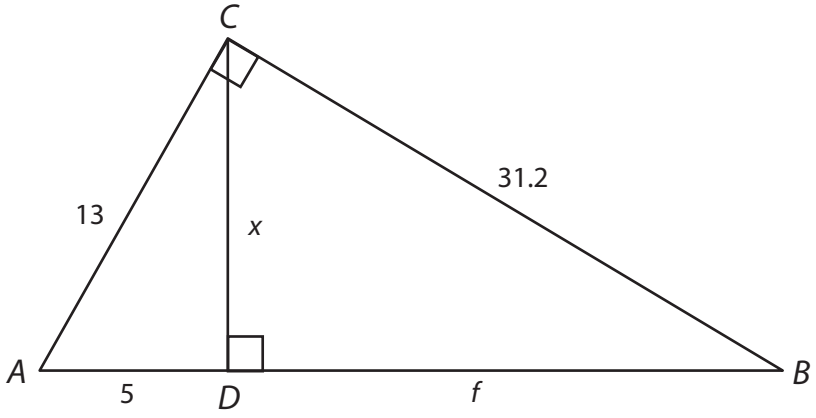

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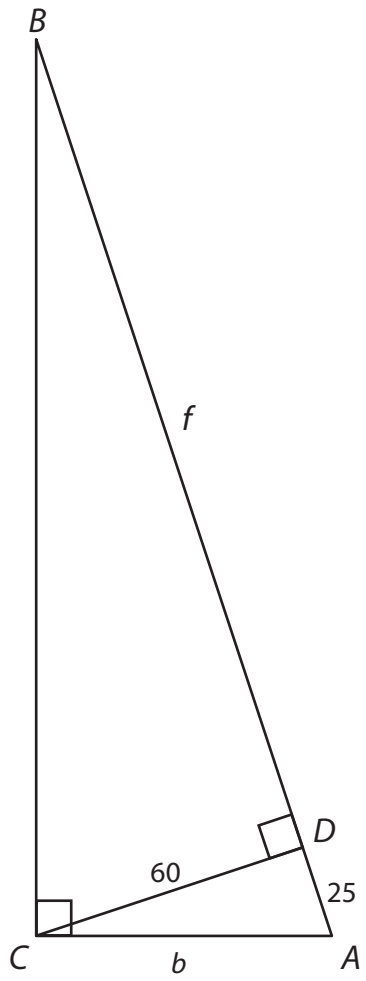

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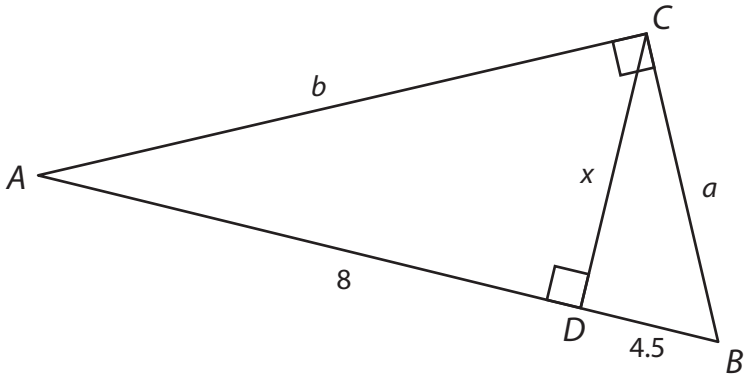

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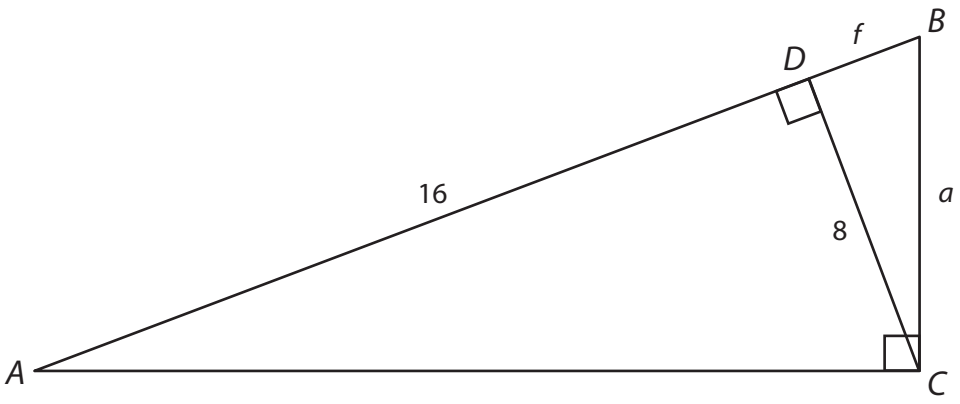

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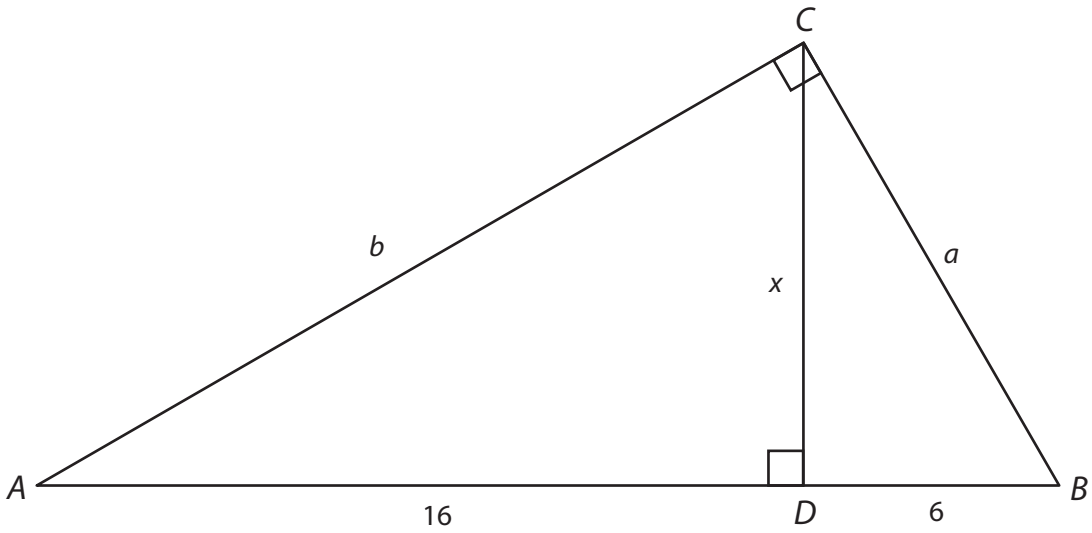

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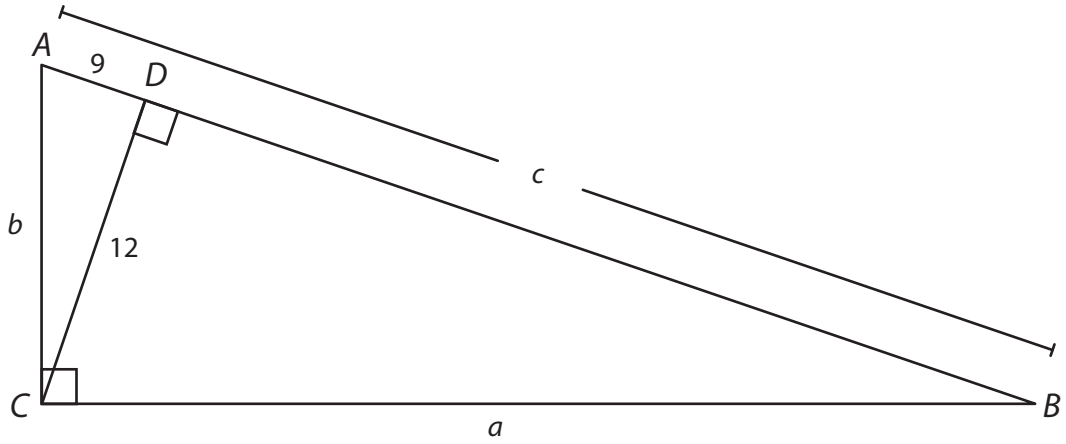
9.



Use the given information to solve problem 10.

10. Using similar triangles, write a two-column proof to prove the converse of the Pythagorean Theorem.

    Given: $\triangle ABC$, with $c^2 = a^2 + b^2$

    Prove: $\triangle ABC$ is a right triangle.

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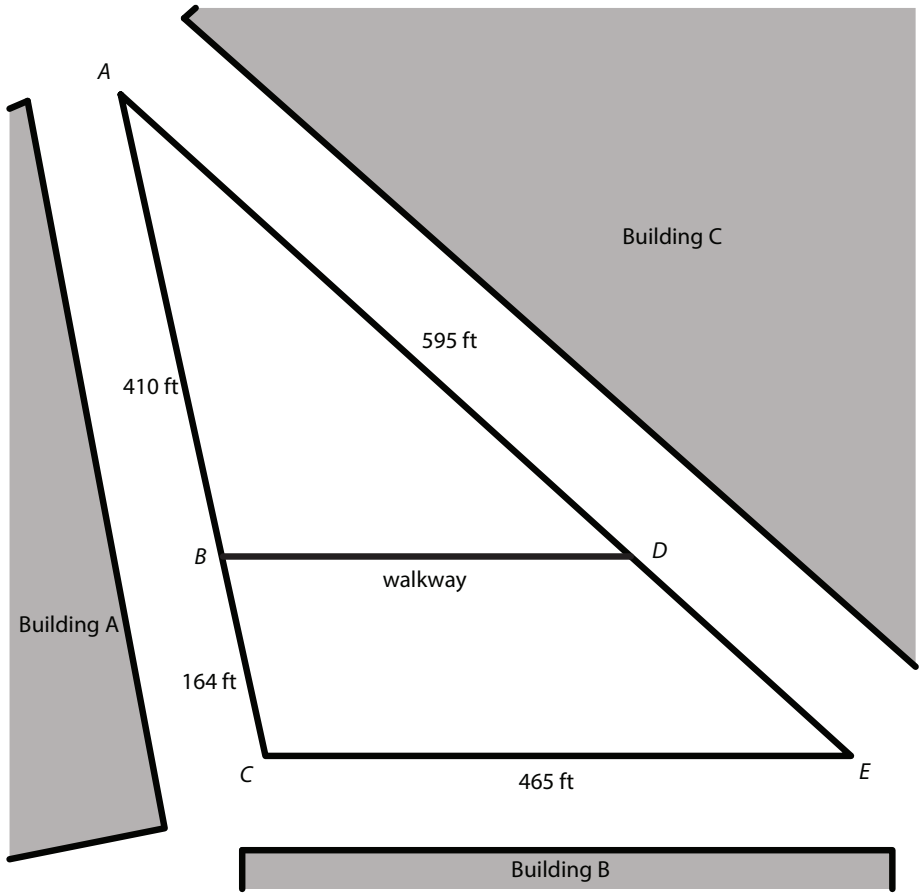
Lesson 5.4.4: Solving Problems Using Similarity and Congruence

Warm-Up 5.4.4

Three buildings border a triangular courtyard as shown in the diagram. A walkway runs parallel to the edge of the courtyard labeled $\overline{CE}$. Landscapers would like to install a picket fence along the outside of the courtyard. The fencing comes in 8-foot lengths.



1. Identify the similar triangles.

2. While preparing the sketch of the courtyard, landscapers forgot to measure the length of the courtyard represented by $\overline{DE}$. What is the length of $\overline{DE}$?

3. How many sections of fencing are needed?

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Instruction

Lesson 5.4.4: Solving Problems Using Similarity and Congruence

Common Core State Standard

G–SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Warm-Up 5.4.4 Debrief

1. Identify the similar triangles.

   The walkway, $\overline{BD}$, is parallel to the edge of the courtyard labeled $\overline{CE}$. If a line is parallel to one side of a triangle and intersects the other two sides in two different points, then it separates these sides into segments of proportional lengths. The parallel line creates two similar triangles.

   $\triangle ACE \sim \triangle ABD$

2. While preparing the sketch of the courtyard, landscapers forgot to measure the length of the courtyard represented by $\overline{DE}$. What is the length of $\overline{DE}$?

   Use the Triangle Proportionality Theorem to find the length of $\overline{DE}$.

   $\frac{AB}{BC} = \frac{AD}{DE}$ Create a proportion.

   $\frac{410}{164} = \frac{595}{DE}$ Substitute the known lengths of each segment.

   $(410)(DE) = (164)(595)$ Find the cross products.

   $(410)(DE) = 97{,}580$ Solve for DE.

   $DE = 238$

   The length of $\overline{DE}$ is 238 feet.

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3. How many sections of fencing are needed?

   Calculate the perimeter of the entire courtyard excluding the walkway.

$AB + BC + CE + AD + DE$	Equation for calculating perimeter
$410 + 164 + 465 + 595 + 238$	Substitute the lengths of each segment.
1872	Simplify.

   The perimeter of the courtyard is 1,872 feet.

   Determine the number of sections needed.

   The fencing comes in 8-foot lengths.

   Divide the total perimeter by 8.

   $1872 \div 8 = 234$

   234 sections of fencing are needed.

Connection to the Lesson

- Students will continue to work with similar triangles. Students will apply the properties of similar triangles to solve for unknown values related to real-world contexts.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- identifying similar triangles (G–SRT.2)
- using similarity statements to find unknown lengths and measures of similar triangles (G–SRT.5)
- using the distance formula to find lengths of sides of triangles (8.G.8)
- working with and simplifying square roots (N–RN.2)
- using the Pythagorean Theorem (8.G.7)

Introduction

Design, architecture, carpentry, surveillance, and many other fields rely on an understanding of the properties of similar triangles. Being able to determine if triangles are similar and understanding their properties can help you solve real-world problems.

Key Concepts

Similarity

- Similarity statements include Angle-Angle (AA), Side-Angle-Side (SAS), and Side-Side-Side (SSS).
- These statements allow us to prove triangles are similar.
- Similar triangles have corresponding sides that are proportional and corresponding angles that are congruent.
- It is important to note that while both similarity and congruence statements include an SSS and an SAS statement, the statements do not mean the same thing.
- Similar triangles have corresponding sides that are proportional, whereas congruent triangles have corresponding sides that are of the same length.

Triangle Theorems

- The Triangle Proportionality Theorem states that if a line parallel to one side of a triangle intersects the other two sides of the triangle, then the parallel line divides these two sides proportionally.
- This theorem can be used to find the lengths of various sides or portions of sides of a triangle.
- It is also true that if a line divides two sides of a triangle proportionally, then the line is parallel to the third side.

SRB p. U5-110

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Instruction

- The Triangle Angle Bisector Theorem states if one angle of a triangle is bisected, or cut in half, then the angle bisector of the triangle divides the opposite side of the triangle into two segments that are proportional to the other two sides of the triangle.
- The Pythagorean Theorem, written symbolically as $a^2 + b^2 = c^2$, is often used to find the lengths of the sides of a right triangle. A right triangle is a triangle that has one 90˚ angle.
- Drawing the altitude from the right angle to the opposite side creates two smaller right triangles that are similar to the original triangle.

Common Errors/Misconceptions

- misidentifying congruent angles because of the orientation of the triangles
- incorrectly creating proportions between corresponding sides
- assuming a line parallel to one side of a triangle bisects the remaining sides rather than creating proportional sides
- misidentifying the altitudes of triangles
- incorrectly simplifying expressions with square roots

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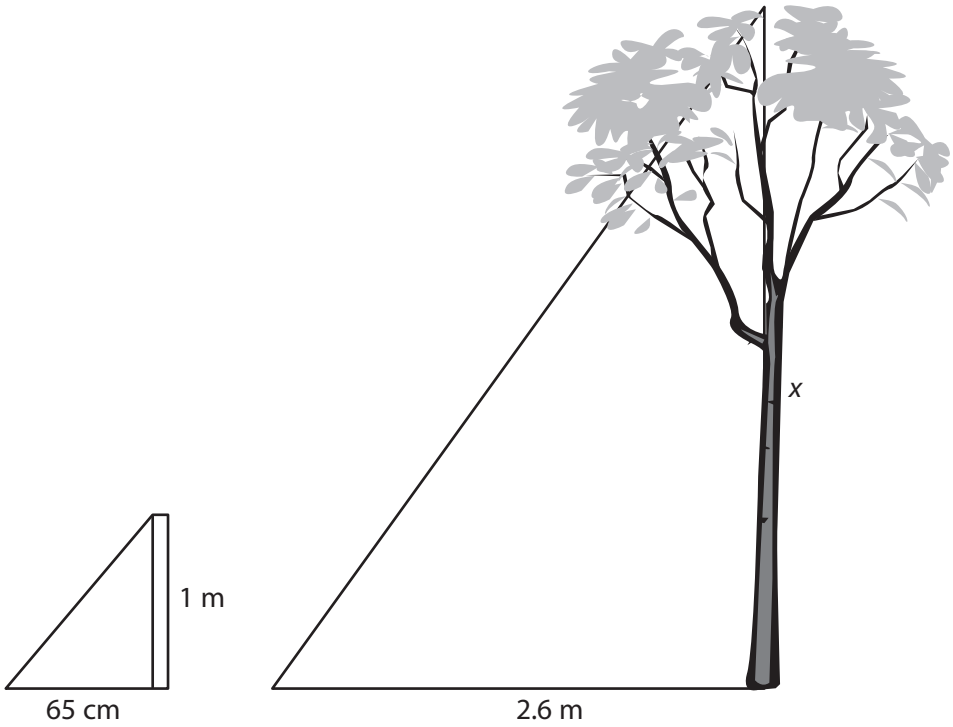
Instruction

Guided Practice 5.4.4

Example 1

A meterstick casts a shadow 65 centimeters long. At the same time, a tree casts a shadow 2.6 meters long. How tall is the tree?

1. Draw a picture to understand the information.



SWB p. 111; SRB p. U5-111

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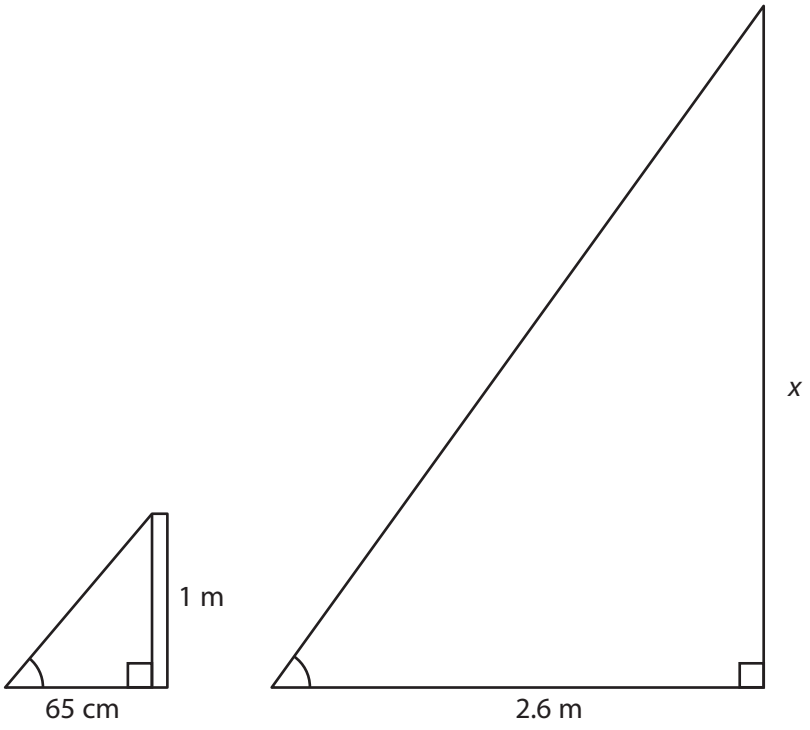
Instruction

2. Determine if the triangles are similar.

The rays of the sun that create the shadows are considered to be parallel, so the angles formed by the sun rays and the ground are congruent.

Right angles are formed between the ground and the meterstick as well as the ground and the tree.



Two angles of the triangles are congruent; therefore, by Angle-Angle Similarity, the triangles are similar.

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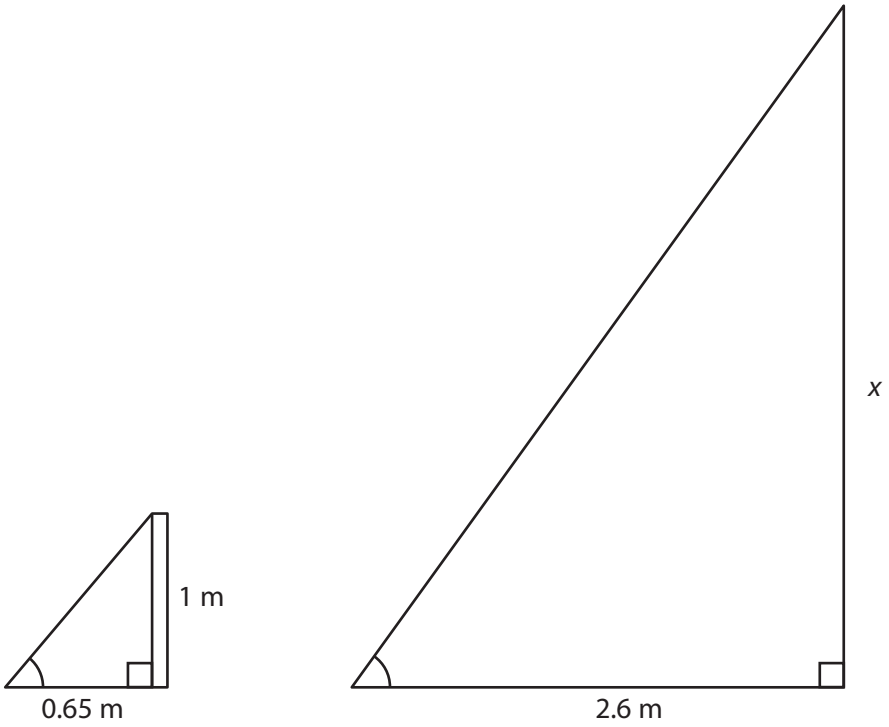
Instruction

3. Solve the problem.

Convert all measurements to the same units.



Similar triangles have proportional sides.

Create a proportion to find the height of the tree.

$\frac{\text{height of stick}}{\text{height of tree}} = \frac{\text{length of stick's shadow}}{\text{length of tree's shadow}}$ Create a proportion.

$\frac{1}{x} = \frac{0.65}{2.6}$ Substitute known values.

$(1)(2.6) = (0.65)(x)$ Find the cross products.

$2.6 = 0.65x$ Simplify.

$x = 4$ Solve for x.

The height of the tree is 4 meters.

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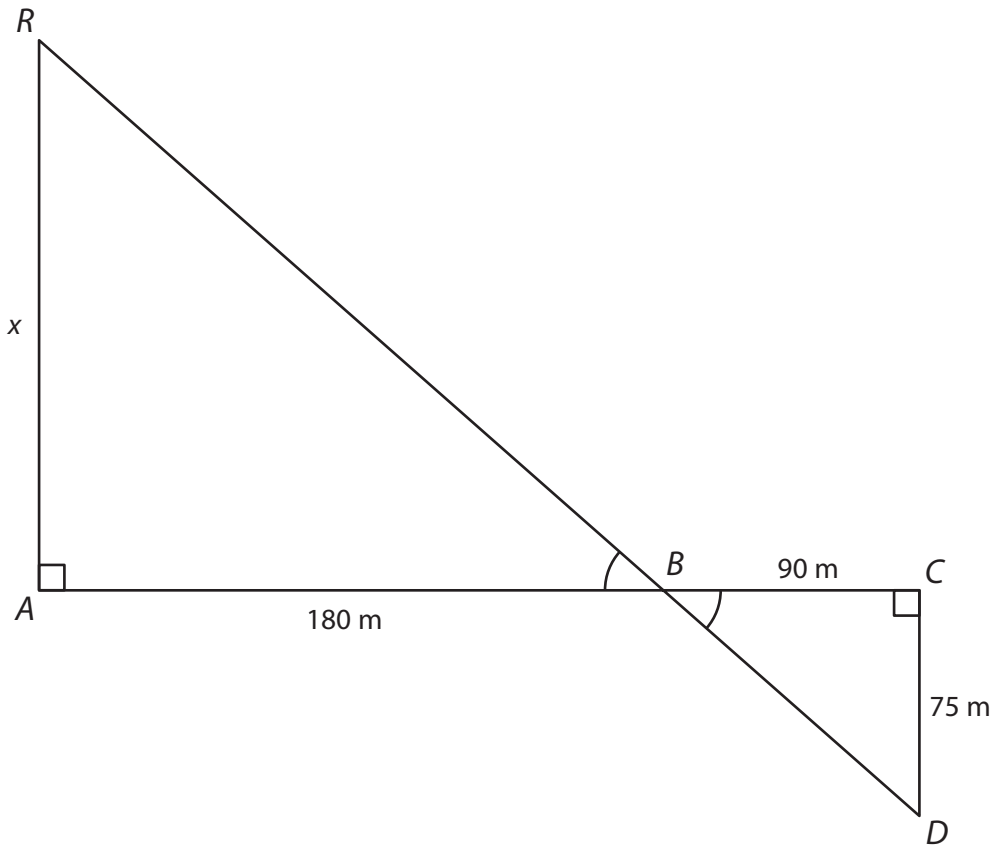
Instruction

Example 2

Finding the distance across a canyon can often be difficult. A drawing of similar triangles can be used to make this task easier. Use the diagram to determine $\overline{AR}$, the distance across the canyon.



1. Interpret the given information.

   A person standing at point A can sight a rock across the canyon at point R.

   Point C is selected so that $\overline{CA}$ is perpendicular to $\overline{AR}$, the distance across the canyon.

   Point D is selected so that $\overline{CD}$ is perpendicular to $\overline{CA}$ and can be easily measured.

   The point of intersection of $\overline{RD}$ and $\overline{CA}$, point B, can then be found.

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Instruction

2. Determine if the triangles are similar.

$\angle A$ and $\angle C$ both measure 90° and are congruent.

$\angle RBA \cong \angle DBC$

By the Angle-Angle Similarity Statement, $\triangle RBA \sim \triangle DBC$.

3. Solve the problem.

Similar triangles have proportional sides.

Create a proportion to find the distance across the canyon.

$\frac{AB}{BC} = \frac{AR}{CD}$	Create a proportion.
$\frac{180}{90} = \frac{x}{75}$	Substitute known values.
$(180)(75) = (90)(x)$	Find the cross products.
$13{,}500 = 90x$	Simplify.
$x = 150$	Solve for x.

The distance across the canyon is 150 meters.

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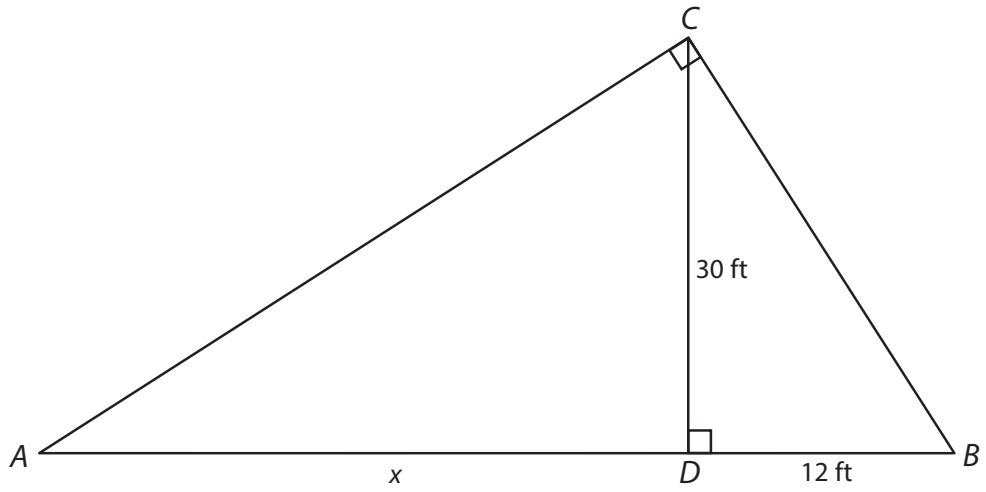
Instruction

Example 3

To find the distance across a pond, Rita climbs a 30-foot observation tower on the shore of the pond and locates points A and B so that $\overline{AC}$ is perpendicular to $\overline{CB}$. She then finds the measure of $\overline{DB}$ to be 12 feet. What is the measure of $\overline{AD}$, the distance across the pond?



1. Determine if the triangles are similar.

   $\triangle ABC$ is a right triangle with $\angle C$ the right angle.

   $\overline{CD}$ is the altitude of $\triangle ABC$, creating two similar triangles, $\triangle ACD$ and $\triangle CBD$.

   $\triangle ABC \sim \triangle ACD \sim \triangle CBD$

2. Solve the problem.

   Similar triangles have proportional sides.

   Create a proportion to find the distance across the pond.

$\frac{BD}{CD} = \frac{CD}{AD}$	Create a proportion.
$\frac{12}{30} = \frac{30}{x}$	Substitute known values.
$(12)(x) = (30)(30)$	Find the cross products.
$12x = 900$	Simplify.
$x = 75$	Solve for x.

   The distance across the pond is 75 feet.

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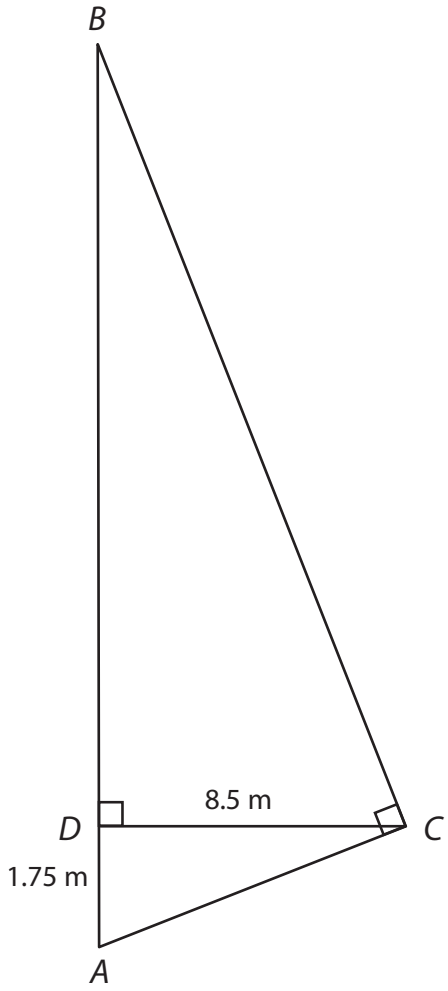
Instruction

Example 4

To estimate the height of a cliff, a surveyor positions herself so that her line of sight to the top of the cliff and her line of sight to the bottom of the cliff form a right angle. What is the height of the cliff to the nearest tenth of a meter?



1. Determine if the triangles are similar.

   $\triangle ABC$ is a right triangle with $\angle C$ the right angle.

   $\overline{CD}$ is the altitude of $\triangle ABC$, creating two similar triangles, $\triangle ACD$ and $\triangle CBD$.

   $\triangle ABC \sim \triangle ACD \sim \triangle CBD$

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Instruction

2. Solve the problem.

Similar triangles have proportional sides.

Create a proportion to find the height of the cliff.

$\frac{AD}{CD}=\frac{CD}{BD}$	Create a proportion.
$\frac{1.75}{8.5}=\frac{8.5}{x}$	Substitute known values.
$(1.75)(x) = (8.5)(8.5)$	Find the cross products.
$1.75x = 72.25$	Simplify.
$x \approx 41.29$	Solve for x.

The length of $\overline{BD}$ is approximately 41.29 meters; however, the measure of the cliff is represented by $\overline{AB}$.

Find the length of $\overline{AB}$.

$41.29 + 1.75 = 43.04$

The height of the cliff is approximately 43 meters.

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Problem-Based Task 5.4.4: Too Tall?

SMP	
1 ✓	2
3	4 ✓
5	6 ✓
7	8

Parks directors routinely assess the health of the trees in recreation areas. If trees are found to be diseased, they are often treated. If trees become too weak, they are removed before they become a danger to people and structures. Gorge Park is a rectangular park measuring 400 feet by 200 feet and is enclosed by a fence. A diseased tree needing removal stands in the center of the park. Tree removers must avoid having the tree fall on the fence. If necessary, the tree can be trimmed prior to being cut down. The 6-foot-tall parks director measured the length of the shadow cast by the tree to be 147 feet and the length of his own shadow to be 9 feet. Does the tree's trunk need to be trimmed prior to cutting it down to avoid damaging the fence?

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Problem-Based Task 5.4.4: Too Tall?

Coaching

a. Draw triangles to represent the height of the parks director, the diseased tree, and their shadows.

b. Are the triangles in your diagram similar? Explain.

c. What are the characteristics of similar triangles?

d. Is it possible to determine the height of the diseased tree using the given information? Explain your answer.

e. What is the height of the diseased tree?

f. Draw and label a diagram of Gorge Park and mark the location of the diseased tree.

g. Compare the height of the tree to the distance between the tree and each fence.

h. Does the tree need to be trimmed prior to cutting it down to avoid damaging the fence?

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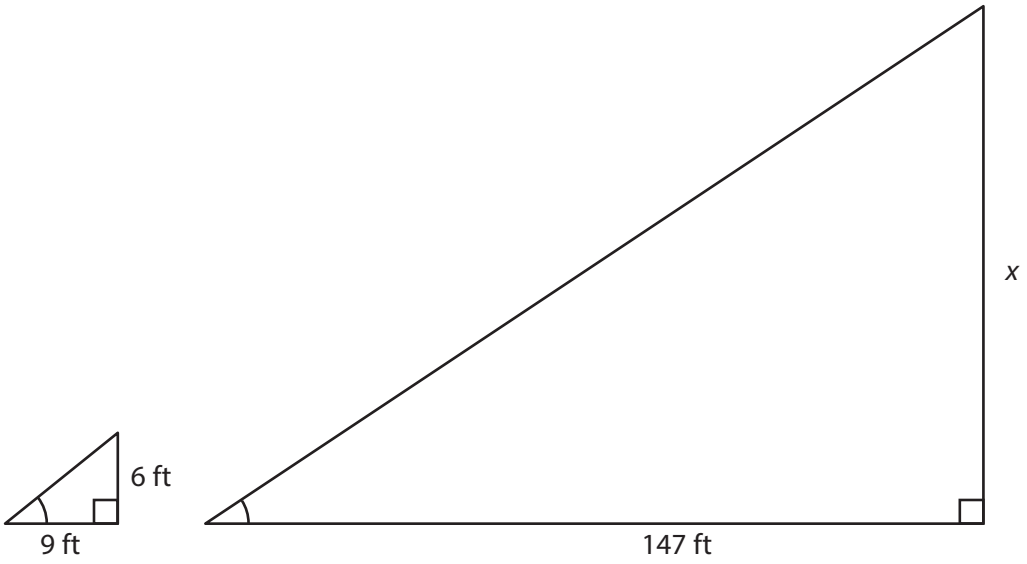
Instruction

Problem-Based Task 5.4.4: Too Tall?

Coaching Sample Responses

a. Draw triangles to represent the height of the parks director, the diseased tree, and their shadows.



b. Are the triangles in your diagram similar? Explain.

The triangles in the diagram are similar because of the Angle-Angle (AA) Similarity Statement, which states that if two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

c. What are the characteristics of similar triangles?

The angles of similar triangles are congruent and the sides are proportional.

d. Is it possible to determine the height of the diseased tree using the given information? Explain your answer.

Yes, it is possible to determine the height of the diseased tree using the given information by creating a proportion using the two similar triangles.

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Instruction

e. What is the height of the diseased tree?

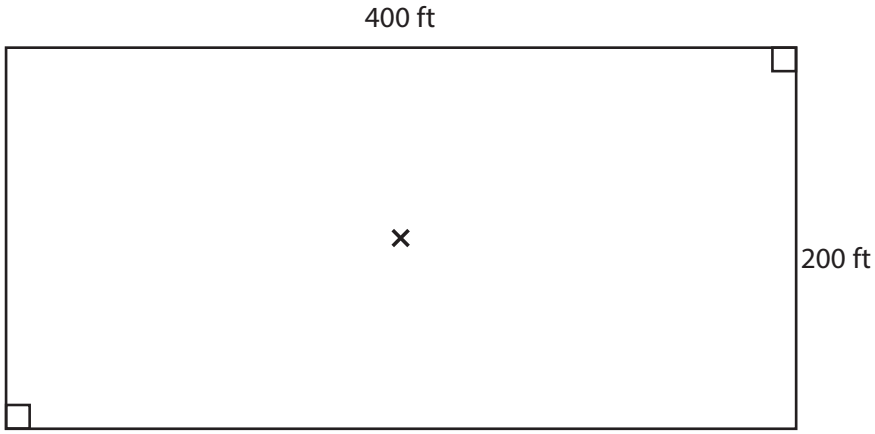
Create a proportion to find the height of the diseased tree.

$$\frac{\text{height of the director}}{\text{height of the tree}} = \frac{\text{length of the director's shadow}}{\text{length of the tree's shadow}}$$

$\frac{6}{x} = \frac{9}{147}$	Substitute known values.
$(6)(147) = (9)(x)$	Find the cross products.
$882 = 9x$	Simplify.
$x = 98$	Solve for x.

The height of the diseased tree is 98 feet.

f. Draw and label a diagram of Gorge Park and mark the location of the diseased tree.



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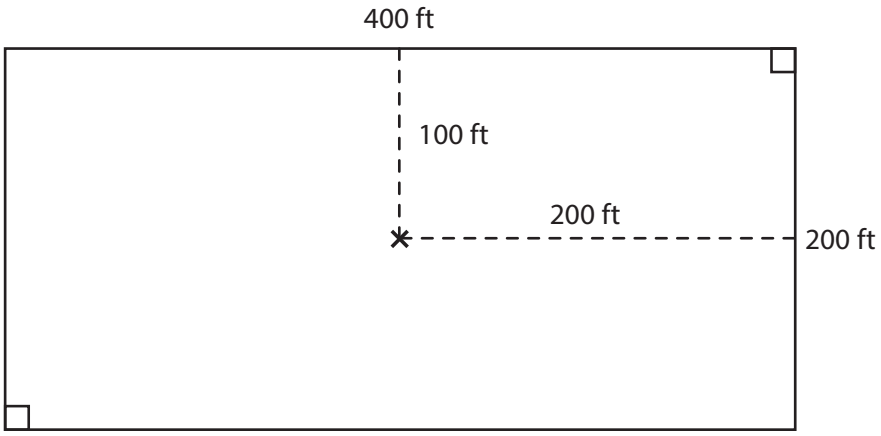
g. Compare the height of the tree to the distance between the tree and each fence.



The diseased tree is located in the center of the park. According to the diagram, there are 200 feet of park to the east and west of the tree and 100 feet of park north and south of the tree. The height of the tree is less than both of these distances to the fence.

h. Does the tree need to be trimmed prior to cutting it down to avoid damaging the fence?

The tree does not need to be trimmed. If cut properly, the tree will not fall on the fence.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 5.4.4: Solving Problems Using Similarity and Congruence

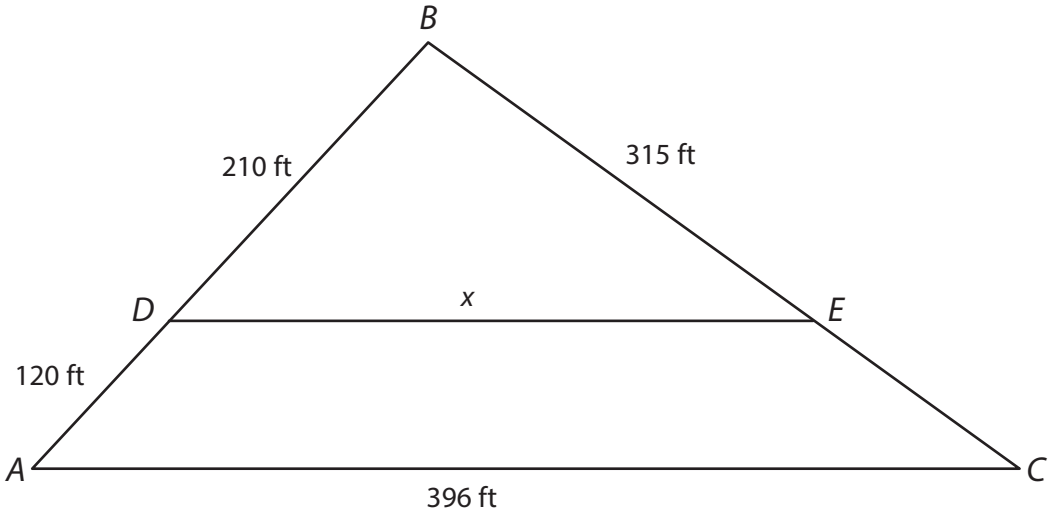
Use what you have learned about similar triangles to solve each problem.

1. A flat-roofed garage casts a shadow that is 9 meters long. At the same time, a 1.8-meter lamppost casts a shadow that is 2.7 meters long. What is the height of the garage?

2. A 12-foot statue casts a shadow that is 5 feet long. At the same time, a fence post casts a shadow that is 1.25 feet long. What is the height of the fence post?

For problems 3–10, use the information and the diagrams to solve each problem.

3. A piece of decorative trim is added to an asymmetrical roofline so that it is parallel to the bottom edge of the roof. What is the length of the decorative trim, $\overline{DE}$?



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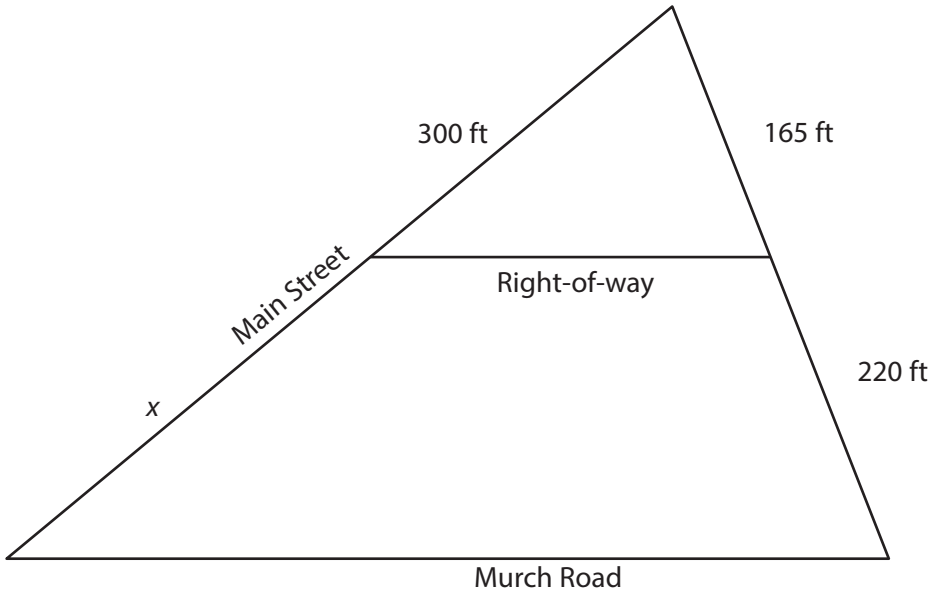
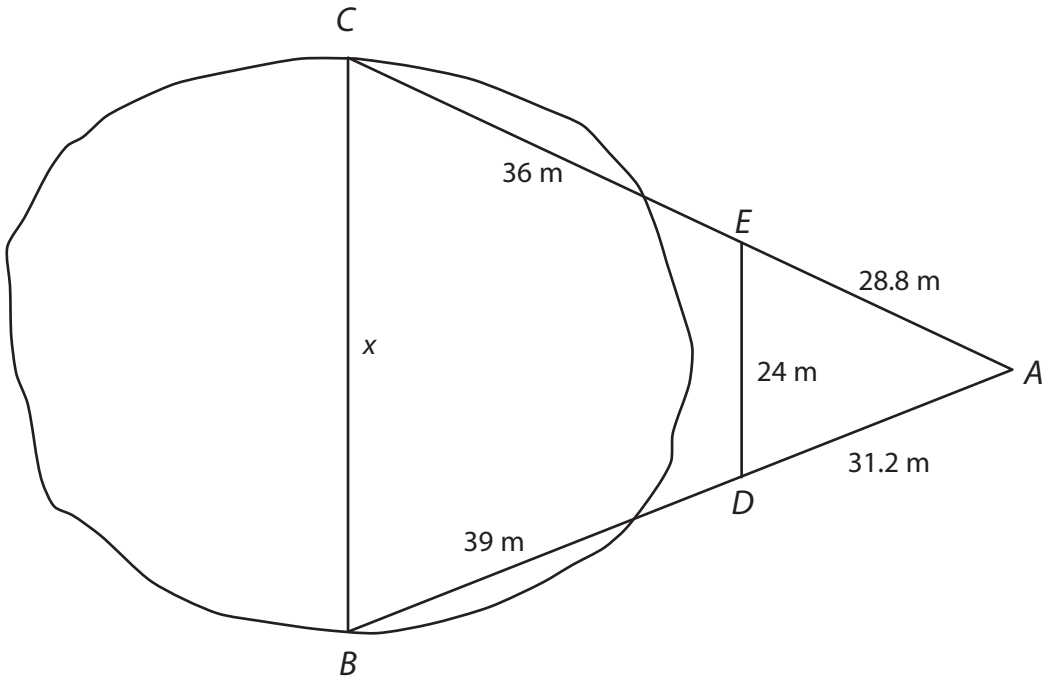
4. A right-of-way parallel to Murch Road is to be constructed on a triangular plot of land. What is the length of the plot of land along Main Street between Murch Road and the right-of-way?



5. To measure $\overline{BC}$, the distance across a lake, a surveyor stands at point *A* and locates points *B*, *C*, *D*, and *E*. What is the distance across the lake?



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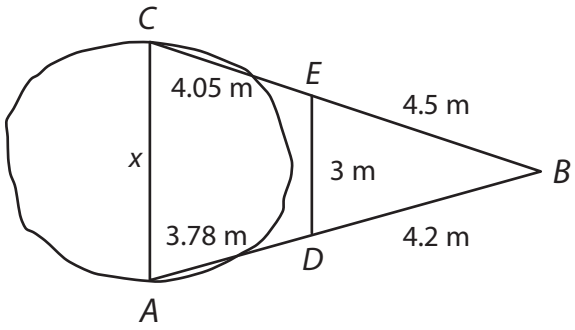
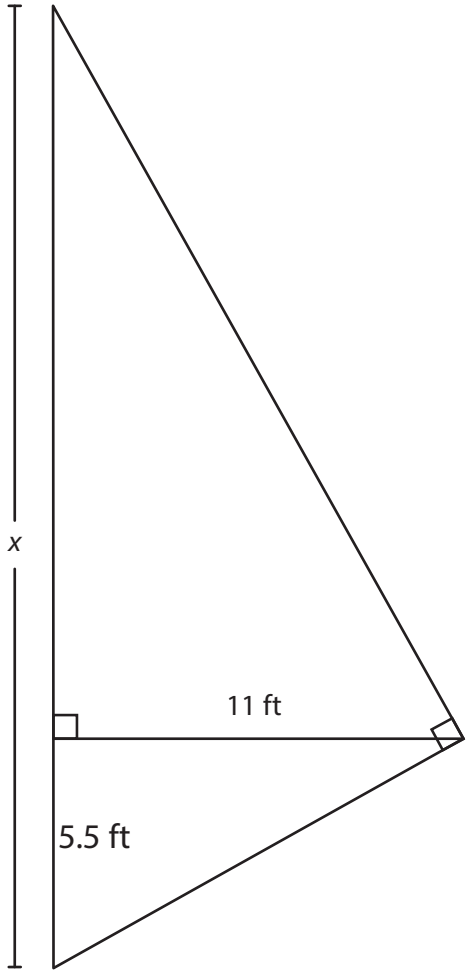
6. To measure $\overline{AC}$, the distance across a crater, an archeologist stands at point B and locates points A, C, D, and E. What is the distance across the crater?



7. To estimate the height of his school, a student positions himself so that his line of sight to the top of the school and his line of sight to the bottom form a right angle. What is the height of the school?



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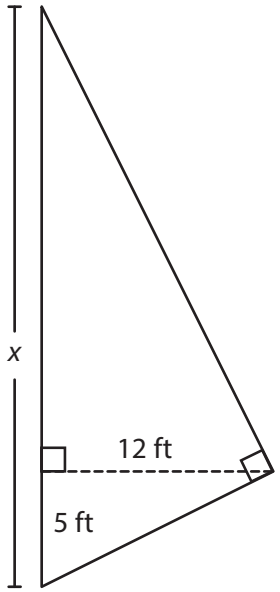
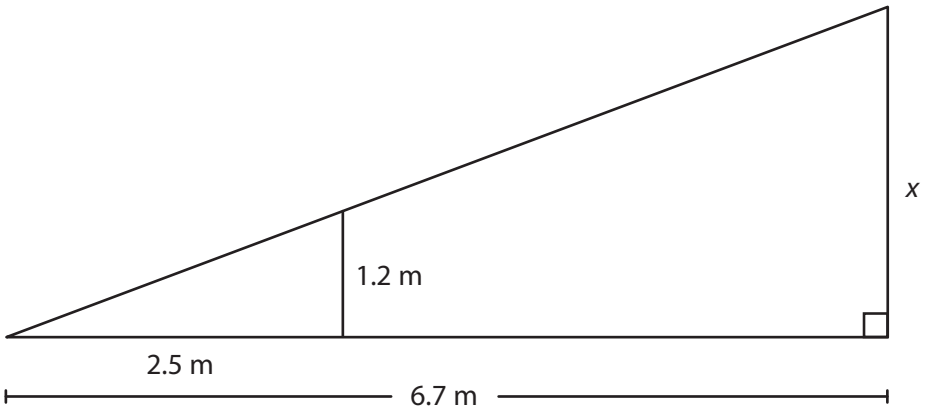
8. To estimate the height of a monument, Cala positions herself so that her line of sight to the top of the monument and her line of sight to the bottom form a right angle. What is the height of the monument?



9. The height of a ramp at a point 2.5 meters from its bottom edge is 1.2 meters. If the ramp runs for 6.7 meters along the ground, what is its height at its highest point, to the nearest tenth of a meter?



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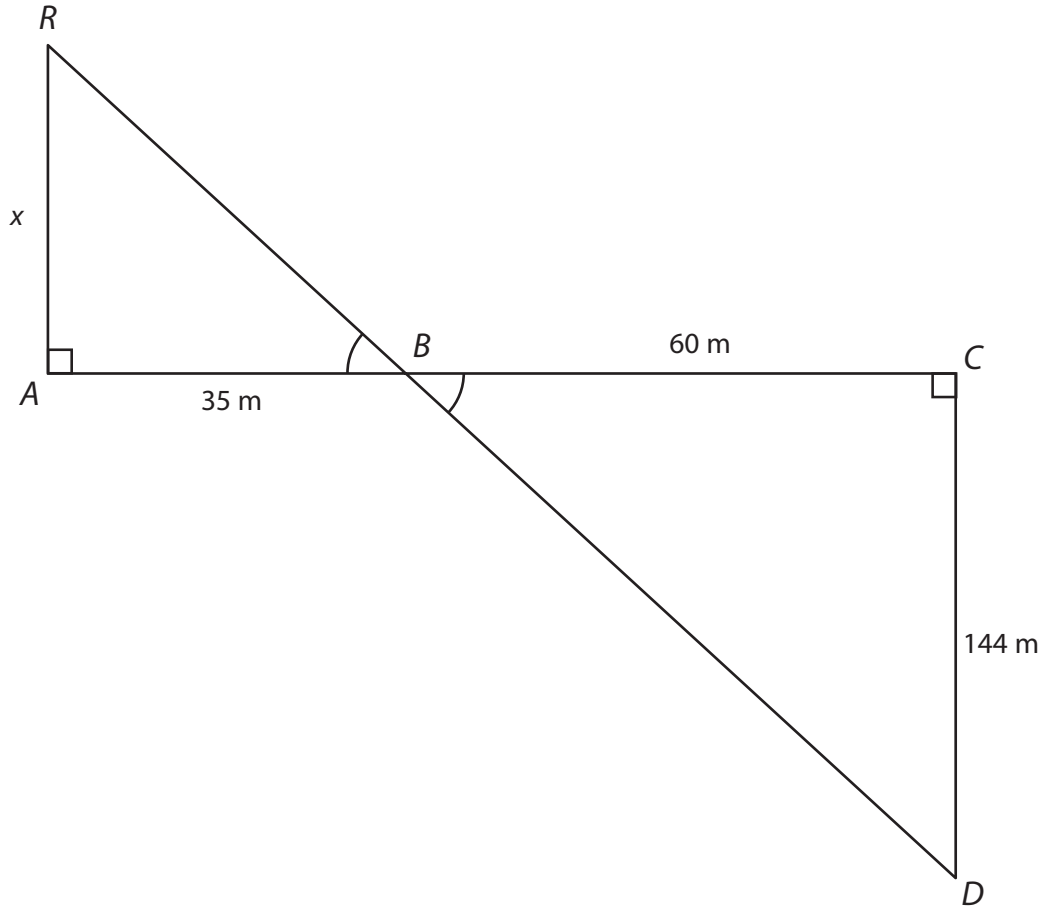
10. A geographer completed the following diagram to map a canyon's width. Determine $\overline{AR}$, the distance across the canyon.



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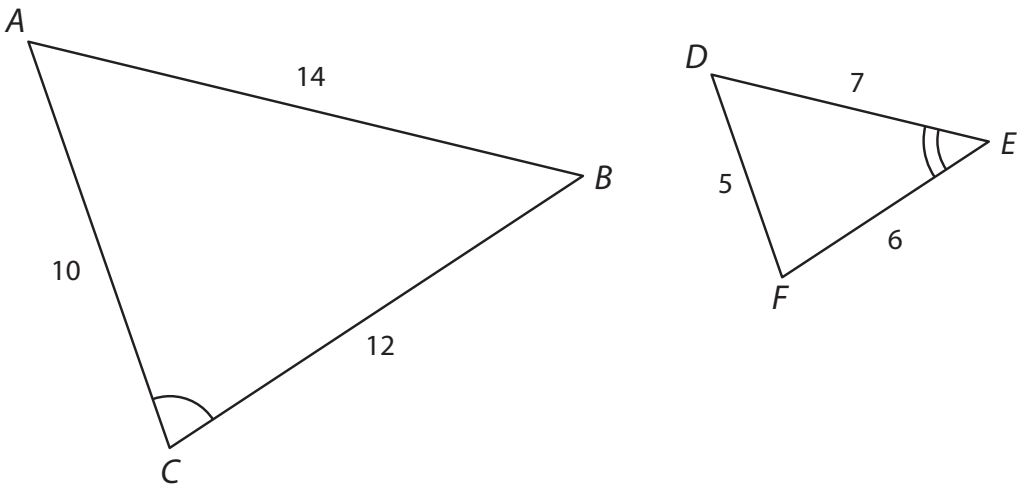
Progress Assessment

Circle the letter of the best answer.

1. Which statement would justify $\triangle ABC \sim \triangle DEF$?



a. Angle-Angle (AA) Similarity Statement

b. Side-Angle-Side (SAS) Similarity Statement

c. Side-Side-Side (SSS) Similarity Statement

d. It is not possible to determine if $\triangle ABC \sim \triangle DEF$.

continued

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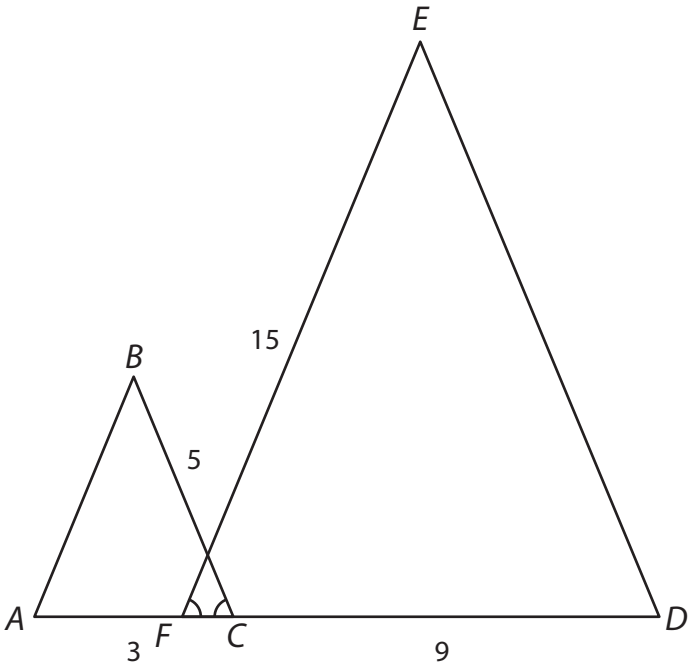
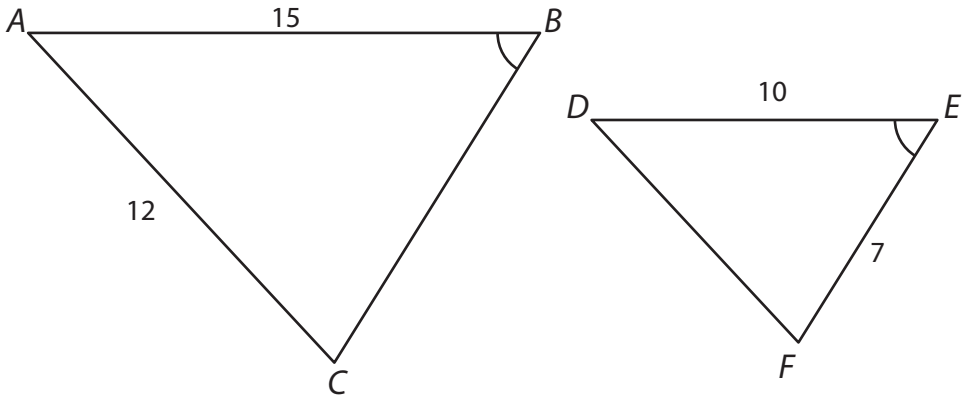
2. Which statement would justify $\triangle ABC \sim \triangle DEF$?



   a. Angle-Angle (AA) Similarity Statement
   b. Side-Angle-Side (SAS) Similarity Statement
   c. Side-Side-Side (SSS) Similarity Statement
   d. It is not possible to determine if $\triangle ABC \sim \triangle DEF$.

3. Which statement would justify $\triangle ABC \sim \triangle DEF$?



   a. Angle-Angle (AA) Similarity Statement
   b. Side-Angle-Side (SAS) Similarity Statement
   c. Side-Side-Side (SSS) Similarity Statement
   d. It is not possible to determine if $\triangle ABC \sim \triangle DEF$.

continued

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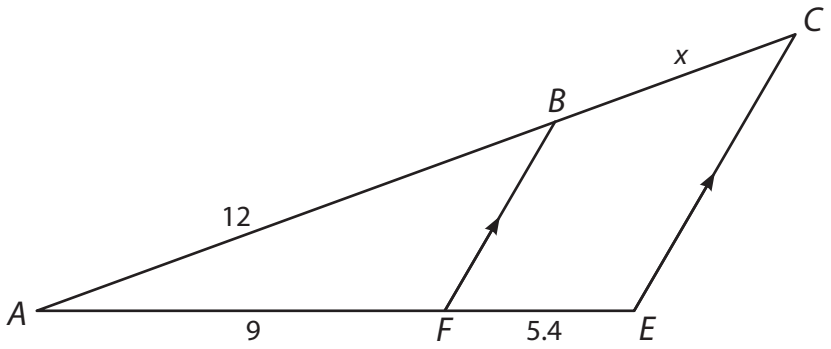
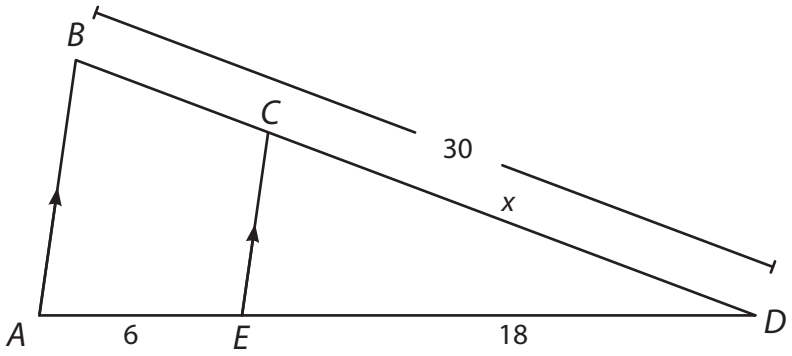
4. $\triangle ABF \sim \triangle ACE$. What is the length of $\overline{CB}$?



a. 7.2 units
b. 4.05 units
c. 20 units
d. There is not enough information to determine the length of $\overline{CB}$.

5. $\triangle ABD \sim \triangle ECD$. What is the length of $\overline{CD}$?



a. 90 units
b. 7.5 units
c. 22.5 units
d. There is not enough information to determine the length of $\overline{CD}$.

continued

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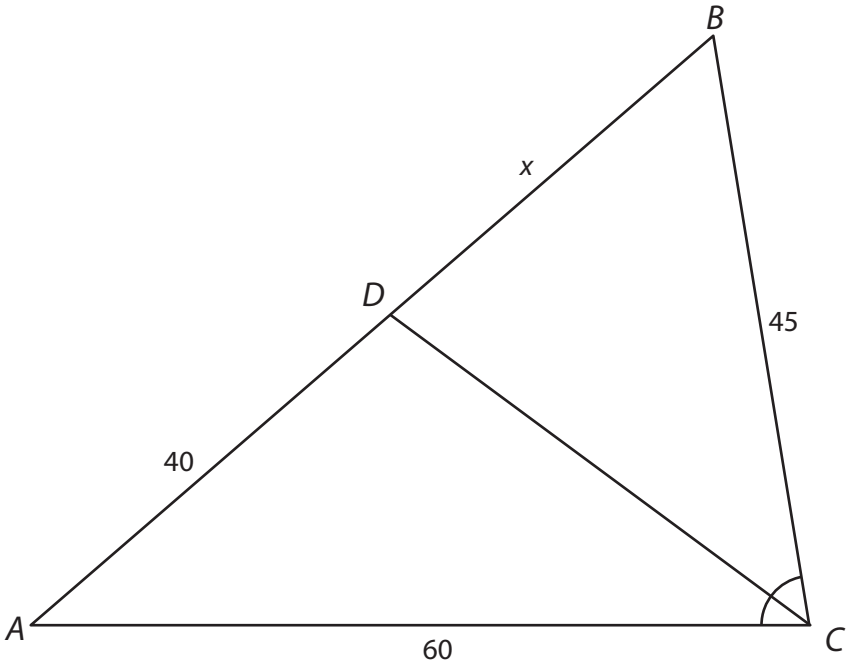
6. What is the length of $\overline{BD}$?



a. 53.3 units

b. 30 units

c. 67.5 units

d. There is not enough information to determine the length of $\overline{BD}$.

continued

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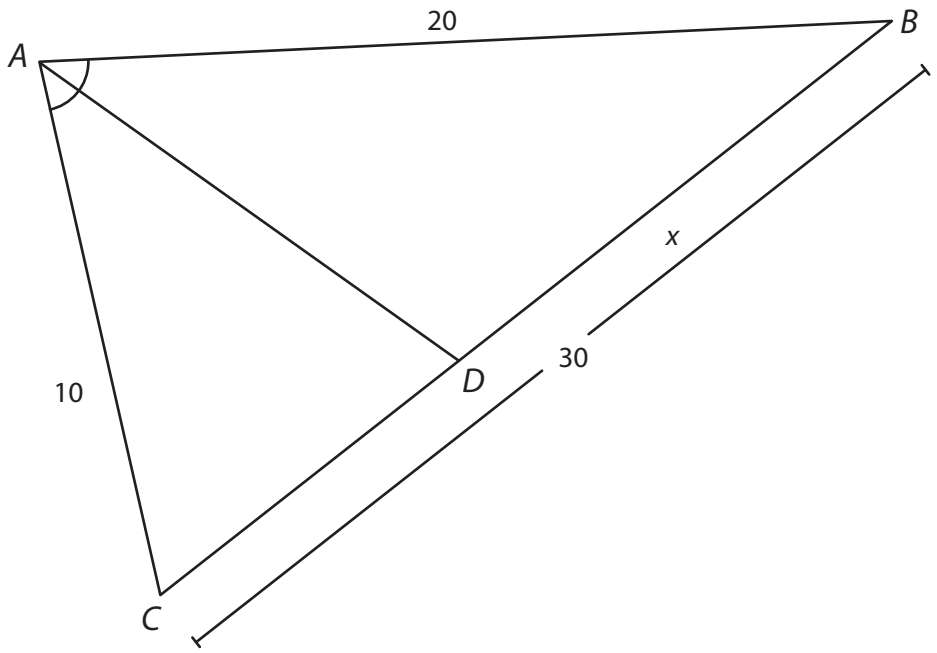
7. What is the length of $\overline{BD}$?



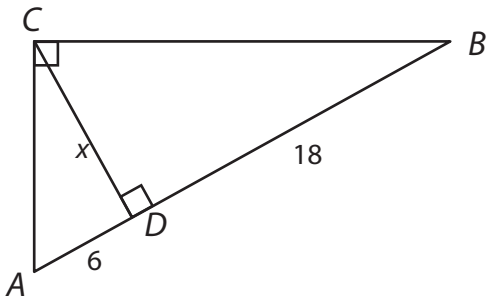
a. 5 units

b. 10 units

c. 20 units

d. There is not enough information to determine the length of $\overline{BD}$.

8. $\triangle ABC$ is a right triangle. Find the length of x, the altitude of $\triangle ABC$.



a. 54 units

b. $6\sqrt{3}$ units

c. $3\sqrt{6}$ units

d. There is not enough information to determine the length of x.

continued

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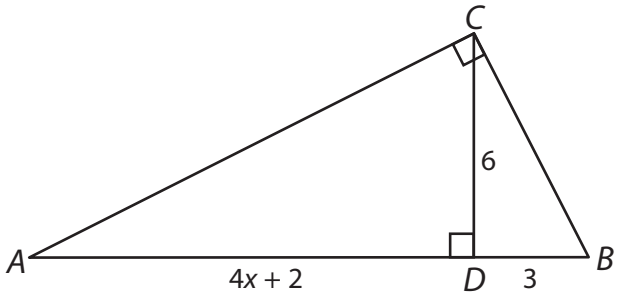
9. $\triangle ABC$ is a right triangle. Find the length of $\overline{AD}$.

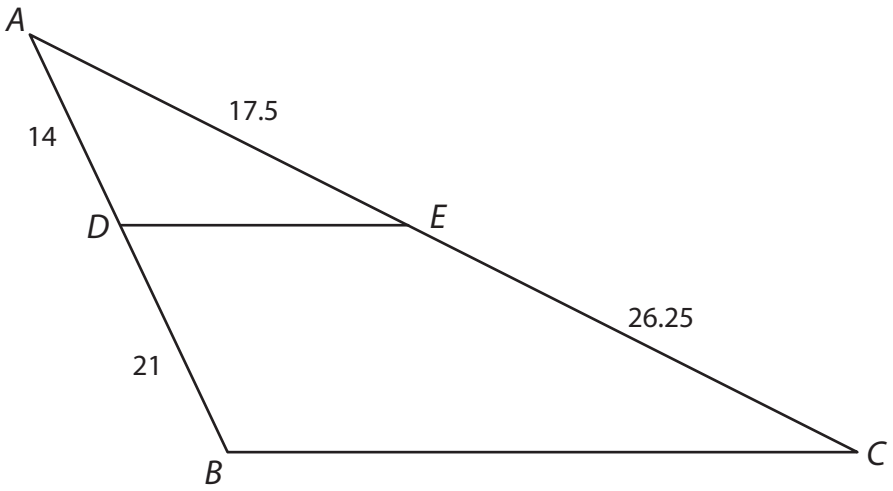


   a. 12 units
   b. 2.83 units
   c. 7 units
   d. There is not enough information to determine the length of $\overline{AD}$.

10. Which statement is NOT true about similar triangles?
   a. The angles of similar triangles are always congruent.
   b. The sides of similar triangles are proportional.
   c. One congruent angle measure indicates similarity.
   d. Similar triangles can have shared parts.

Use what you know about similar triangles to solve the following problem.

11. A classmate has presented the following diagram.



   a. Is $\triangle ABC \sim \triangle ADE$? Explain your answer.
   b. Is $\overline{DE} \| \overline{BC}$? Explain your answer.

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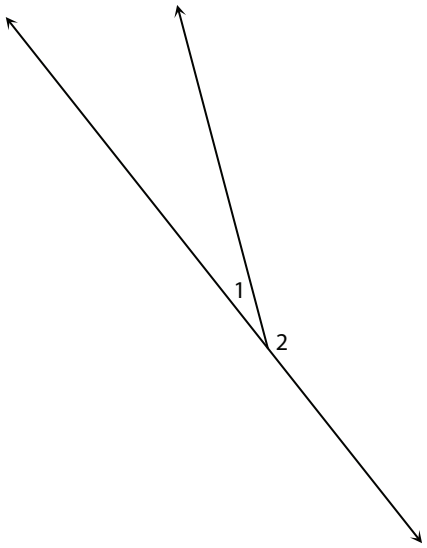
Pre-Assessment

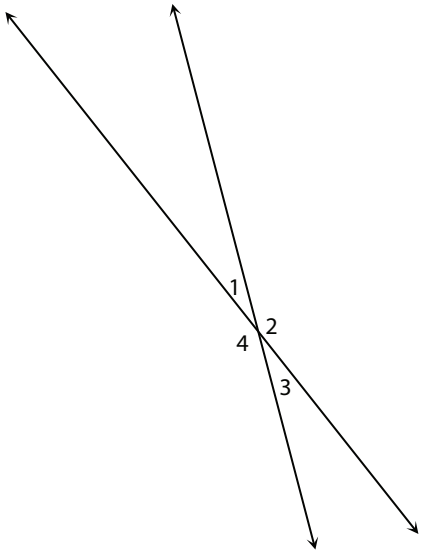
Circle the letter of the best answer.

1. In the diagram below, $\angle 1$ and $\angle 2$ are a linear pair. Find $m\angle 1$ if $m\angle 1 = 2x - 9$ and $m\angle 2 = 10x + 9$.



a. 17°
b. 159°
c. 163°
d. 21°

2. Find $m\angle 4$ in the diagram below if $m\angle 2 = 9(x + 6)$ and $m\angle 4 = 2(5x + 21)$.



a. 18°
b. 157°
c. 162°
d. 12°

continued

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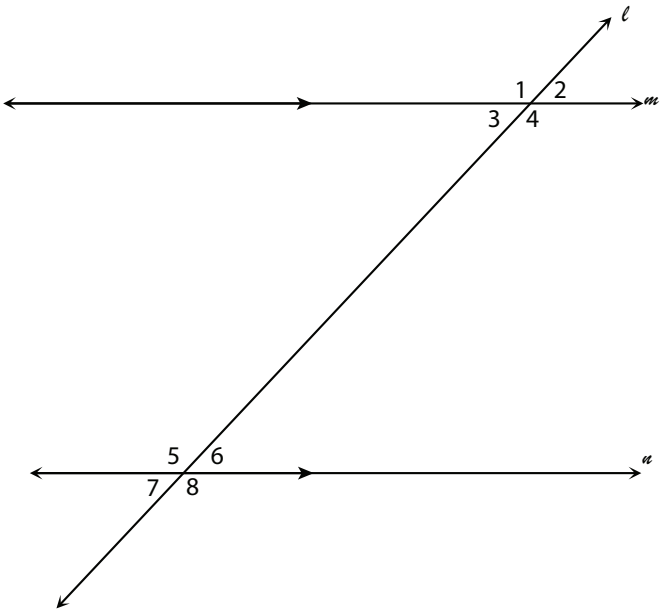
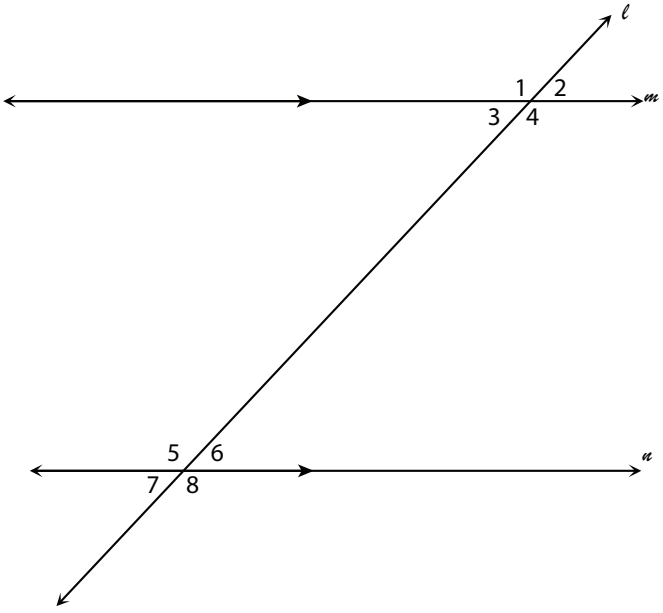
3. In the diagram below, ℓ is the transversal of the parallel lines m and n. Find $m\angle 3$ if $m\angle 4 = 5(x+11)$ and $m\angle 5 = 11x-17$.



a. 115°
b. 65°
c. 78°
d. 12°

4. In the diagram below, ℓ is the transversal of the parallel lines m and n. Find $m\angle 8$ if $m\angle 2 = 3x+12$ and $m\angle 8 = 4x$.



a. 24°
b. 6°
c. 156°
d. 46°

continued

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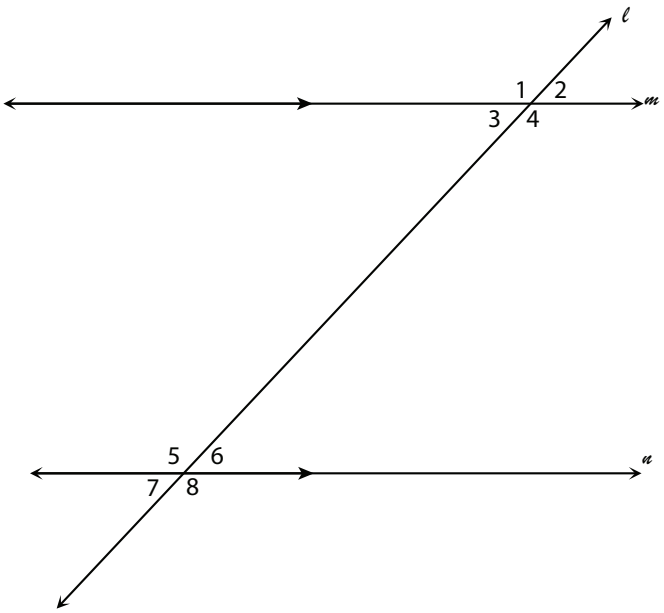
Assessment

5. In the diagram below, ℓ is the transversal of the parallel lines m and n. Find $m\angle 5$ if $m\angle 3 = 2(3x-4)$ and $m\angle 5 = 5(3x+4)$.



a. 8°

b. 172°

c. 40°

d. 140°

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Lesson 5: Proving Theorems About Lines and Angles

Instruction

Common Core State Standard

G–CO.9 Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*

SMP	
1 ✓	2 ✓
3 ✓	4
5	6
7 ✓	8

Essential Questions

1. How do angle relationships work together in two pairs of intersecting, opposite rays?
2. How do angle relationships work together in a set of parallel lines intersected by a transversal?
3. How are angle relationships important in the real world?
4. How do proofs apply to situations outside of mathematics?

WORDS TO KNOW

adjacent angles	angles that lie in the same plane and share a vertex and a common side. They have no common interior points.
alternate exterior angles	angles that are on opposite sides of the transversal and lie on the exterior of the two lines that the transversal intersects
alternate interior angles	angles that are on opposite sides of the transversal and lie within the interior of the two lines that the transversal intersects
complementary angles	two angles whose sum is 90°
corresponding angles	angles in the same relative position with respect to the transversal and the intersecting lines
equidistant	the same distance from a reference point
exterior angles	angles that lie outside a pair of parallel lines
interior angles	angles that lie between a pair of parallel lines
linear pair	a pair of adjacent angles whose non-shared sides form a straight angle

SRB p. U5-125

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Instruction

nonadjacent angles	angles that have no common vertex or common side, or have shared interior points
perpendicular bisector	a line that intersects a segment at its midpoint at a right angle
perpendicular lines	two lines that intersect at a right angle (90˚), written as $\overleftrightarrow{AB} \perp \overleftrightarrow{PQ}$. Line segments and rays can also be perpendicular.
plane	a flat, two-dimensional figure without depth that is determined by three non-collinear points and extends infinitely in all directions
postulate	a true statement that does not require a proof
proof	a set of justified statements organized to form a convincing argument that a given statement is true
right angle	an angle measuring 90˚
same-side exterior angles	angles that lie on the same side of the transversal and are outside the lines that the transversal intersects; sometimes called consecutive exterior angles
same-side interior angles	angles that lie on the same side of the transversal and are in between the lines that the transversal intersects; sometimes called consecutive interior angles
straight angle	an angle with rays in opposite directions; i.e., a straight line; a straight angle measures 180º.
supplementary angles	two angles whose sum is 180º
transversal	a line that intersects a system of two or more lines
vertical angles	nonadjacent angles formed by two pairs of opposite rays

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Recommended Resources

- Interactivate. "Angles."

  http://www.walch.com/rr/00027

  This website generates a set of parallel lines intersected by a transversal and prompts users to identify the angle relationships. The site provides immediate feedback.

- IXL Learning. "Geometry: Angles: Identify complementary, supplementary, vertical, adjacent, and congruent angles."

  http://www.walch.com/rr/00028

  This online quiz allows users to receive immediate feedback about their answers. If the answer is incorrect, the program explains how to solve the problem correctly. This site deals with complementary, supplementary, vertical, adjacent, and congruent angles.

- IXL Learning. "Geometry: Parallel and perpendicular lines: transversals of parallel lines: find angle measures."

  http://www.walch.com/rr/00029

  This online quiz deals with angle relationships in a set of parallel lines intersected by a transversal, and provides immediate feedback. For incorrect answers, the program explains how to solve the problem correctly.

- MathIsFun.com. "Vertically Opposite Angles."

  http://www.walch.com/rr/00030

  This website gives a brief explanation of vertical angles and provides a manipulative so users can investigate how angle measures change as the positions of the intersecting lines change. There is a short quiz at the end of the lesson, as well as links to other sites on angle relationships.

- Math-Play.com. "Angles Jeopardy Game."

  http://www.walch.com/rr/00031

  This online activity is provided in a game-show format that allows users to compete in teams or individually. The multiple-choice quiz questions pertain to angle relationships, and are worth points according to difficulty. Players receive immediate feedback.

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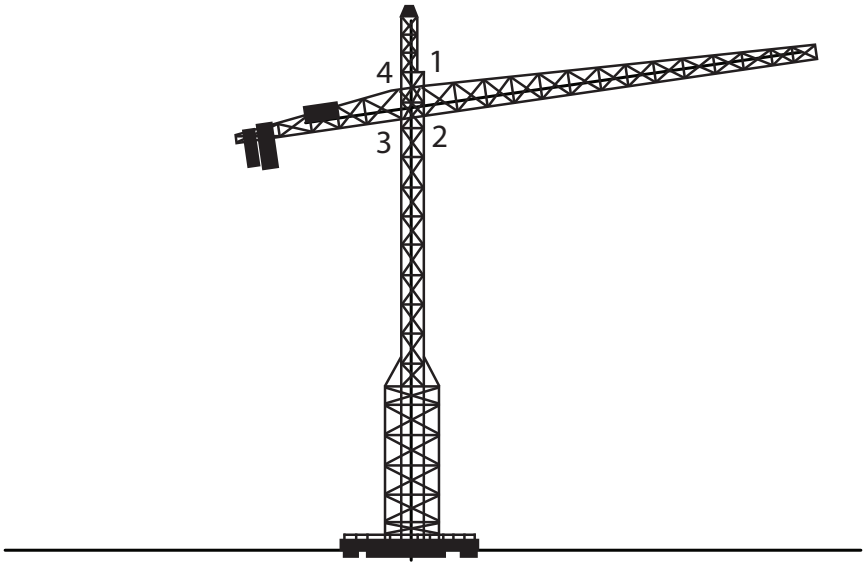
Lesson 5.5.1: Proving the Vertical Angles Theorem

Warm-Up 5.5.1

Metalbro is a construction company involved with building a new skyscraper in Dubai. The diagram below is a rough sketch of a crane that Metalbro workers are using to build the skyscraper. The vertical line represents the support tower and the other line represents the boom. For safety reasons, the boom cannot be more than 15° beyond the horizontal in either direction. A horizontal line forms a 90° angle with the support tower. A straight line forms a 180° angle.



1. What are the safety requirements for $m\angle 1$?
2. What are the safety requirements for $m\angle 2$?
3. What are the safety requirements for $m\angle 3$?
4. What are the safety requirements for $m\angle 4$?
5. Use your findings to fill in the table.

	Based on lower boundary of $\angle 1$	**Based on upper boundary of $\angle 1$**
$m\angle 1$		
$m\angle 2$		
$m\angle 3$		
$m\angle 4$		

6. What do you notice about these angles?

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Instruction

Lesson 5.5.1: Proving the Vertical Angles Theorem

Common Core State Standard

G–CO.9 Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*

Warm-Up 5.5.1 Debrief

1. What are the safety requirements for $m\angle 1$?

   The boom cannot be more than 15° beyond the horizontal in either direction. Since a horizontal line forms a 90° angle with a vertical line, add 15° to 90° to find the upper boundary of the boom angle, and then subtract 15° from 90° to find the lower boundary of the boom angle.

   Upper boundary: 90 + 15 = 105

   Lower boundary: 90 – 15 = 75

   The safety requirements for $m\angle 1$ are that the angle must be between 75° and 105°.

2. What are the safety requirements for $m\angle 2$?

   $\angle 1$ and $\angle 2$ form a straight line, so the sum of their angles is 180°.

   To find $m\angle 2$ when $m\angle 1 = 75$, set up an equation.

   $m\angle 1 + m\angle 2 = 180$

   $75 + m\angle 2 = 180$

   $m\angle 2 = 180 - 75$

   $m\angle 2 = 105$

   To find $m\angle 2$ when $m\angle 1 = 105$, set up an equation.

   $m\angle 1 + m\angle 2 = 180$

   $105 + m\angle 2 = 180$

   $m\angle 2 = 180 - 105$

   $m\angle 2 = 75$

   You could also see by inspection of the previous equation that 75 + 105 = 180.

   When $m\angle 1 = 75$, $m\angle 2 = 105$, and when $m\angle 1 = 105$, $m\angle 2 = 75$.

   The safety requirements for $m\angle 2$ are that the angle must be between 75° and 105°.

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3. What are the safety requirements for $m\angle 3$?

   $\angle 2$ and $\angle 3$ form a straight line, so the sum of their angles is 180°.

   To find $m\angle 3$ when $m\angle 2 = 105$, set up an equation or use inspection from the last problem.

   $m\angle 2 + m\angle 3 = 180$

   $105 + m\angle 3 = 180$

   $m\angle 3 = 180 - 105$

   $m\angle 3 = 75$

   To find $m\angle 3$ when $m\angle 2 = 75$, set up an equation or use inspection.

   $m\angle 2 + m\angle 3 = 180$

   $75 + m\angle 3 = 180$

   $m\angle 3 = 180 - 75$

   $m\angle 3 = 105$

   When $m\angle 2 = 105$, $m\angle 3 = 75$, and when $m\angle 2 = 75$, $m\angle 3 = 105$.

   The safety requirements for $m\angle 3$ are that the angle must be between 75° and 105°.

4. What are the safety requirements for $m\angle 4$?

   $\angle 3$ and $\angle 4$ form a straight line, so the sum of their angles is 180°.

   To find $m\angle 4$ when $m\angle 3 = 75$, set up an equation or use inspection.

   $m\angle 3 + m\angle 4 = 180$

   $75 + m\angle 4 = 180$

   $m\angle 4 = 180 - 75$

   $m\angle 4 = 105$

   To find $m\angle 4$ when $m\angle 3 = 105$, set up an equation or use inspection.

   $m\angle 3 + m\angle 4 = 180$

   $105 + m\angle 4 = 180$

   $m\angle 4 = 180 - 105$

   $m\angle 4 = 75$

   The safety requirements for $m\angle 4$ are that the angle must be between 75° and 105°.

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5. Use your findings to fill in the table.

	Based on lower boundary of $\angle 1$	**Based on upper boundary of $\angle 1$**
$m\angle 1$	75°	105°
$m\angle 2$	105°	75°
$m\angle 3$	75°	105°
$m\angle 4$	105°	75°

6. What do you notice about these angles?

   The angles that aren't next to each other but are opposite the point of intersection are congruent. $\angle 1$ and $\angle 3$ are congruent. $\angle 2$ and $\angle 4$ are congruent. If the angles are next to each other and form a line, they add up to 180°.

Connection to the Lesson

- Students will learn the names for these angle relationships.
- Students will use these angle relationships for solving problems and creating proofs.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

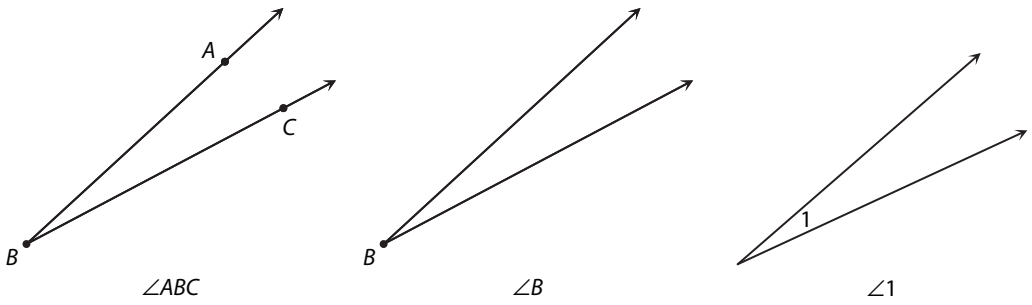
- identifying and labeling points, lines, and angles (G–CO.1)
- using the addition and subtraction properties of angles (G–CO.1)

Introduction

Think about crossing a pair of chopsticks and the angles that are created when they are opened at various positions. How many angles are formed? What are the relationships among those angles? This lesson explores angle relationships. We will be examining the relationships of angles that lie in the same plane. A **plane** is a two-dimensional figure, meaning it is a flat surface, and it extends infinitely in all directions. Planes are defined byrequire at least three non-collinear points. Planes are named using those points or a capital script letter. Since they are flat, planes have no depth.

Key Concepts

- Angles can be labeled with one point at the vertex, three points with the vertex point in the middle, or with numbers or variables written inside the angle. See the examples that follow.



- Be careful when using one vertex point to name the angle, as this can lead to confusion.
- If the vertex point serves as the vertex for more than one angle, three points or a number must be used to name the angle.

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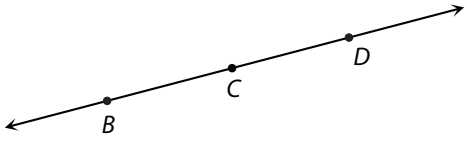
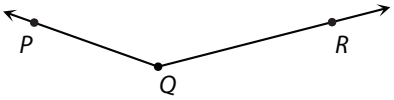
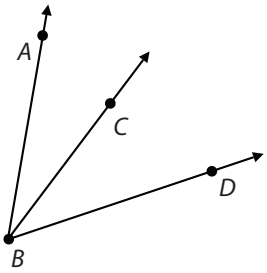
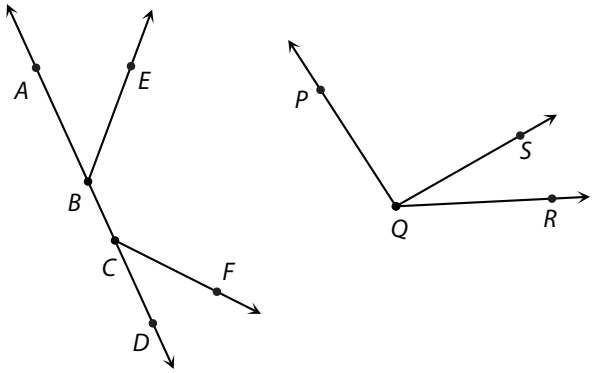
Instruction

- **Straight angles** are angles with rays in opposite directions—in other words, straight angles are straight lines. The measure of a straight angle is 180°.

Straight angle	Not a straight angle
	
$\angle BCD$ is a straight angle. Points *B*, *C*, and *D* lie on the same line.	$\angle PQR$ is not a straight angle. Points *P*, *Q*, and *R* do not lie on the same line.

- **Adjacent angles** are angles that lie in the same plane and share a vertex and a common side. They have no common interior points.
- **Nonadjacent angles** have no common vertex or common side, or have shared interior points.

Adjacent angles	Nonadjacent angles
	
$\angle ABC$ is adjacent to $\angle CBD$. They share vertex *B* and $\overrightarrow{BC}$. $\angle ABC$ and $\angle CBD$ have no common interior points.	$\angle ABE$ is not adjacent to $\angle FCD$. They do not have a common vertex. $\angle PQS$ is not adjacent to $\angle PQR$. They share common interior points within $\angle PQS$.

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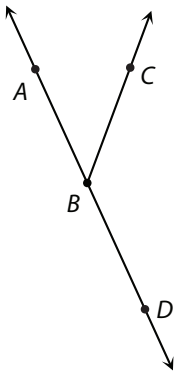
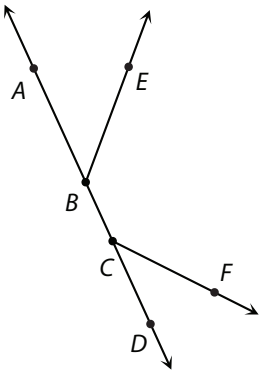
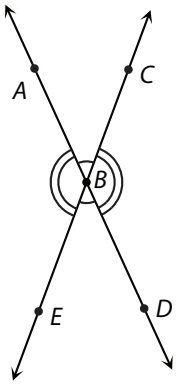
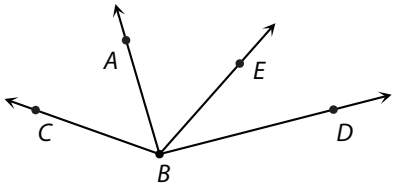
Instruction

- **Linear pairs** are pairs of adjacent angles whose non-shared sides form a straight angle.

Linear pair	Not a linear pair
	
$\angle ABC$ and $\angle CBD$ are a linear pair. They are adjacent angles with non-shared sides, creating a straight angle.	$\angle ABE$ and $\angle FCD$ are not a linear pair. They are not adjacent angles.

- **Vertical angles** are nonadjacent angles formed by two pairs of opposite rays.

Theorem
Vertical Angles Theorem Vertical angles are congruent.

Vertical angles	Not vertical angles
	
$\angle ABC$ and $\angle EBD$ are vertical angles. $\angle ABC \cong \angle EBD$ $\angle ABE$ and $\angle CBD$ are vertical angles. $\angle ABE \cong \angle CBD$	$\angle ABC$ and $\angle EBD$ are not vertical angles. $\overrightarrow{BC}$ and $\overrightarrow{BD}$ are not opposite rays. They do not form one straight line.

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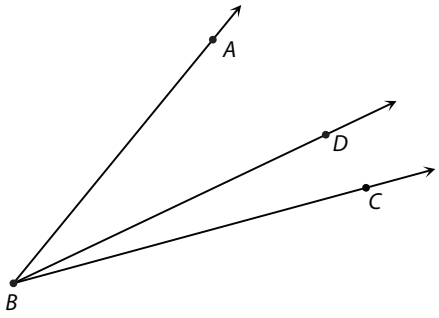
Instruction

Postulate

Angle Addition Postulate

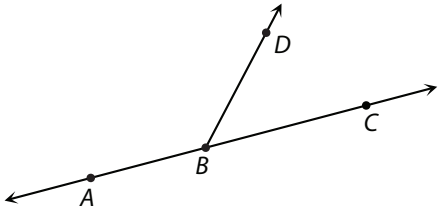
If D is in the interior of $\angle ABC$, then $m\angle ABD + m\angle DBC = m\angle ABC$.

If $m\angle ABD + m\angle DBC = m\angle ABC$, then D is in the interior of $\angle ABC$.



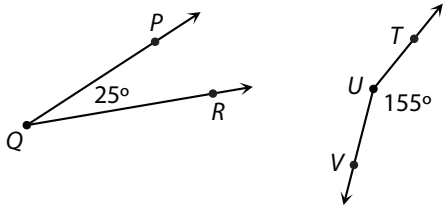
- Informally, the Angle Addition Postulate means that the measure of the larger angle is made up of the sum of the two smaller angles inside it. **Postulates** are true statements that don't need proofs.
- **Supplementary angles** are two angles whose sum is 180°.
- Supplementary angles can form a linear pair or be nonadjacent.
- In the following diagram, the angles form a linear pair.

$m\angle ABD + m\angle DBC = 180°$



- The next diagram shows a pair of supplementary angles that are nonadjacent.

$m\angle PQR + m\angle TUV = 180°$



Theorem

Supplement Theorem

If two angles form a linear pair, then they are supplementary.

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Instruction

- Angles have the same congruence properties that segments do.

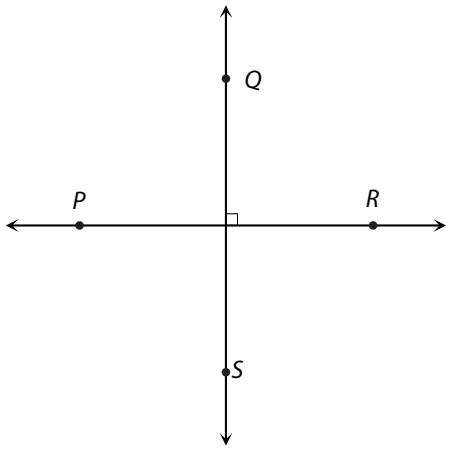
Theorem
Congruence of angles is reflexive, symmetric, and transitive. • Reflexive Property: $\angle 1 \cong \angle 1$ • Symmetric Property: If $\angle 1 \cong \angle 2$, then $\angle 2 \cong \angle 1$. • Transitive Property: If $\angle 1 \cong \angle 2$ and $\angle 2 \cong \angle 3$, then $\angle 1 \cong \angle 3$.

Theorem
Angles supplementary to the same angle or to congruent angles are congruent. If $m\angle 1 + m\angle 2 = 180$ and $m\angle 2 + m\angle 3 = 180$, then $\angle 1 \cong \angle 3$.

- **Perpendicular lines** are two lines that intersect at a right angle (90°). They form four adjacent and congruent **right angles**.

Theorem
If two congruent angles form a linear pair, then they are right angles. If two angles are congruent and supplementary, then each angle is a right angle.

- The symbol for indicating perpendicular lines in a diagram is a box at one of the right angles, as shown here.



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Instruction

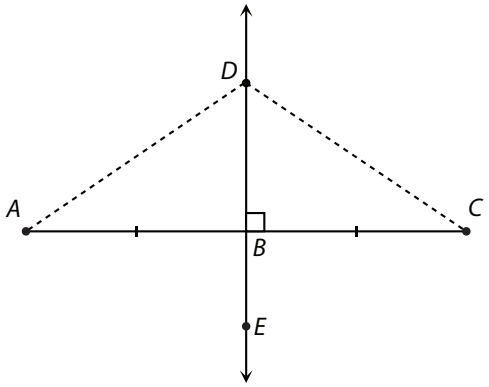
- The symbol for writing perpendicular lines is $\perp$, and is read as "is perpendicular to."
- In the previous diagram, $\overleftrightarrow{SQ} \perp \overleftrightarrow{PR}$.
- Rays and segments can also be perpendicular.
- In a pair of perpendicular lines, rays, or segments, only one right angle box is needed to indicate perpendicular lines.
- **Perpendicular bisectors** are lines that intersect a segment at its midpoint at a right angle; they are perpendicular to the segment.
- Any point along the perpendicular bisector is **equidistant,** or the same distance, from the endpoints of the segment that it bisects.

Theorem

Perpendicular Bisector Theorem

If a point lies on the perpendicular bisector of a segment, then that point is equidistant from the endpoints of the segment.

If a point is equidistant from the endpoints of a segment, then the point lies on the perpendicular bisector of the segment.



If $\overleftrightarrow{DE}$ is the perpendicular bisector of $\overline{AC}$, then $DA = DC$.

If $DA = DC$, then $\overleftrightarrow{DE}$ is the perpendicular bisector of $\overline{AC}$.

- **Complementary angles** are two angles whose sum is 90°.
- Complementary angles can form a right angle or be nonadjacent.

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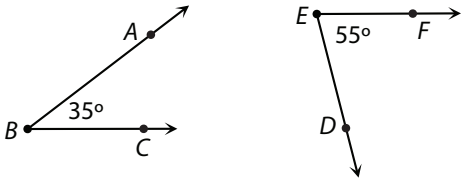
Instruction

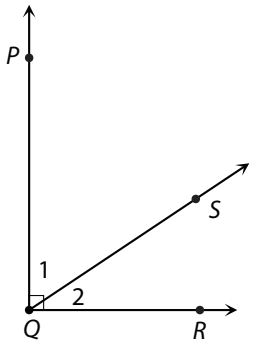
- The following diagram shows a pair of nonadjacent complementary angles.

  $m\angle B + m\angle E = 90^\circ$



- The next diagram shows a pair of adjacent complementary angles labeled with numbers.

  $m\angle 1 + m\angle 2 = 90^\circ$



Theorem
Complement Theorem If the non-shared sides of two adjacent angles form a right angle, then the angles are complementary. Angles complementary to the same angle or to congruent angles are congruent.

Common Errors/Misconceptions

- not recognizing the theorem that is being used or that needs to be used
- setting expressions equal to each other rather than using the Complement or Supplement Theorems
- mislabeling angles with a single letter when that letter is the vertex for adjacent angles
- not recognizing adjacent and nonadjacent angles

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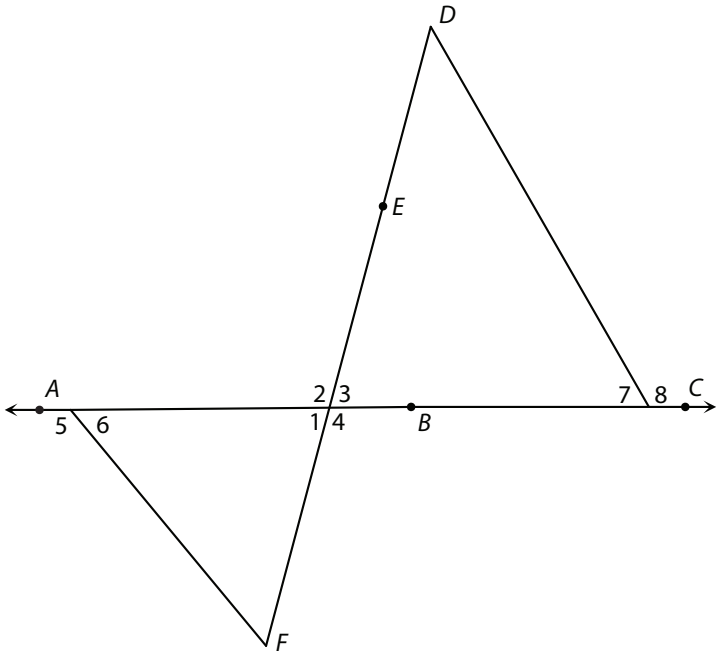
Instruction

Guided Practice 5.5.1

Example 1

Look at the following diagram. List pairs of supplementary angles, pairs of vertical angles, and a pair of opposite rays.



1. List pairs of supplementary angles.

   Supplementary angles have a sum of 180°.

   $\angle 5$ and $\angle 6$ are adjacent supplementary angles. They form a linear pair.

   $\angle 1$ and $\angle 4$ are adjacent supplementary angles. They form a linear pair.

   $\angle 2$ and $\angle 3$ are adjacent supplementary angles. They form a linear pair.

   $\angle 7$ and $\angle 8$ are adjacent supplementary angles. They form a linear pair.

   $\angle 1$ and $\angle 2$ are adjacent supplementary angles. They form a linear pair.

   $\angle 3$ and $\angle 4$ are adjacent supplementary angles. They form a linear pair.

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Instruction

2. List the vertical angles.

Vertical angles are nonadjacent angles that are formed by a pair of intersecting lines.

$\angle 1$ and $\angle 3$ are vertical angles. They are formed by the intersecting segments of $\overleftrightarrow{AC}$ and $\overleftrightarrow{DF}$.

$\angle 2$ and $\angle 4$ are vertical angles. They are formed by the intersecting segments of $\overleftrightarrow{AC}$ and $\overleftrightarrow{DF}$.

3. List a pair of opposite rays.

Opposite rays form a straight angle.

$\overrightarrow{BA}$ and $\overrightarrow{BC}$ are opposite rays.

Also, $\overrightarrow{EF}$ and $\overrightarrow{ED}$ are opposite rays. This can be misleading since what is pictured represents segments, but remember that segments are just parts of lines and the line extends in both directions infinitely. From the line, any number of rays can be named.

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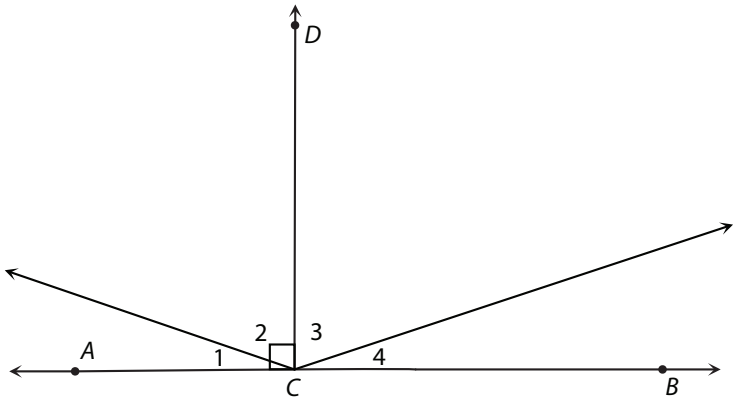
Instruction

Example 2

Prove the theorem that angles complementary to congruent angles are congruent using the given information.

In the following figure, prove that $\angle 1$ is congruent to $\angle 4$, given that $\overleftrightarrow{AC}$ is perpendicular to $\overrightarrow{CD}$ and $\angle 2$ is congruent to $\angle 3$.



1. Start by writing the given statements and the proof statement.

   Given: $\overleftrightarrow{AC} \perp \overrightarrow{CD}$; $\angle 2 \cong \angle 3$

   Prove: $\angle 1 \cong \angle 4$

2. Start writing what you know about perpendicular lines and complementary angles.

   Perpendicular lines form four adjacent and congruent right angles.

   Complementary angles have a sum of 90°, which is a right angle.

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Instruction

3. Determine where the right angles are located and use the Complement Theorem.

   $\angle ACD$ is a right angle because of the given information that $\overleftrightarrow{AC} \perp \overrightarrow{CD}$. $\angle ACD$ is made up of two adjacent complementary angles, $\angle 1$ and $\angle 2$. Therefore, $m\angle 1 + m\angle 2 = 90°$.

   $\angle BCD$ is a right angle because of the given information that $\overleftrightarrow{AC} \perp \overrightarrow{CD}$. $\angle BCD$ is made up of two adjacent complementary angles, $\angle 3$ and $\angle 4$. Therefore, $m\angle 3 + m\angle 4 = 90°$.

4. Use the definition of congruence.

   Since $\angle 2 \cong \angle 3$, $m\angle 2 = m\angle 3$. The definition of congruence states that if two angles are congruent, then the measures of their angles are equal.

5. Use substitution.

   Since $m\angle 1 + m\angle 2 = 90°$ and $m\angle 2 = m\angle 3$, $m\angle 1 + m\angle 3 = 90°$. Notice that $m\angle 3$ was substituted in for $m\angle 2$.

   Also, as stated in step 3, $m\angle 3 + m\angle 4 = 90°$. Since two expressions ($m\angle 1 + m\angle 3$ and $m\angle 3 + m\angle 4$) both equal 90°, set those two expressions equal to each other.

   $m\angle 1 + m\angle 3 = m\angle 3 + m\angle 4$

6. Use the Reflexive Property.

   $m\angle 3 = m\angle 3$

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Instruction

7. Use the Subtraction Property.

$m\angle 1+m\angle 3=m\angle 3+m\angle 4$	Set the expressions equal to each other.
$m\angle 1=m\angle 4$	Subtract $m\angle 3$ from both sides of the equation.

8. Use the definition of congruent angles.

$m\angle 1=m\angle 4$

$\angle 1\cong\angle 4$

9. Organize the information into a paragraph proof.

From the given information, $\overleftrightarrow{AC}$ is perpendicular to $\overrightarrow{CD}$. By the definition of perpendicular lines, these perpendicular lines create four right angles. Two of the right angles are $\angle ACD$ and $\angle BCD$. Each of the angles is made up of two smaller angles. By the Complement Theorem, $m\angle 1+m\angle 2=90°$ and $m\angle 3+m\angle 4=90°$. Since $\angle 2\cong\angle 3$, the measures of $\angle 2$ and $\angle 3$ are equal according to the definition of congruence. $m\angle 3$ can be substituted into the first complementary angle equation for $m\angle 2$ so that $m\angle 1+m\angle 3=90°$. Since two expressions are set equal to 90°, they are equal to each other; therefore, $m\angle 1+m\angle 3=m\angle 3+m\angle 4$. Congruence of angles is reflexive, meaning that $m\angle 3=m\angle 3$. This angle can be subtracted from both sides of the equation, leaving $m\angle 1=m\angle 4$. By the definition of congruent angles, $\angle 1\cong\angle 4$.

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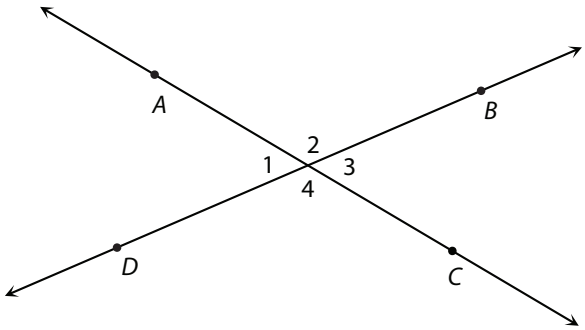
Instruction

Example 3

In the following diagram, $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are intersecting lines. If $m\angle 1 = 3x + 14$ and $m\angle 2 = 9x + 22$, find $m\angle 3$ and $m\angle 4$.



1. Use the Supplement Theorem.

   Since $\overleftrightarrow{BD}$ is a straight line, $m\angle 1 + m\angle 2 = 180°$.

2. Use substitution to find the value of x.

   Substitute the measures of $\angle 1$ and $\angle 2$ into the equation $m\angle 1 + m\angle 2 = 180°$.

   $m\angle 1 = 3x + 14$

   $m\angle 2 = 9x + 22$

$m\angle 1 + m\angle 2 = 180°$	Supplement Theorem
$(3x + 14) + (9x + 22) = 180$	Substitute $3x + 14$ and $9x + 22$ for $m\angle 1$ and $m\angle 2$.
$12x + 36 = 180$	Combine like terms.
$12x = 144$	Subtract 36 from both sides.
$x = 12$	Divide both sides by 12.

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3. Use substitution to find $m\angle 1$.

$m\angle 1 = 3x + 14$ and $x = 12$	Given
$m\angle 1 = 3(12) + 14$	Substitute 12 for x.
$m\angle 1 = 36 + 14$	Multiply.
$m\angle 1 = 50°$	Add.

4. Use substitution to find $m\angle 2$.

$m\angle 2 = 9x + 22$ and $x = 12$	Given
$m\angle 2 = 9(12) + 22$	Substitute 12 for x.
$m\angle 2 = 108 + 22$	Multiply.
$m\angle 2 = 130°$	Add.

5. Use the Vertical Angles Theorem to find $m\angle 3$ and $m\angle 4$.

$\angle 1$ and $\angle 3$ are vertical angles.

$\angle 1 \cong \angle 3$	Vertical Angles Theorem
$m\angle 1 = m\angle 3$	Definition of congruent angles
$50 = m\angle 3$	Substitute 50 for $m\angle 1$.

$\angle 2$ and $\angle 4$ are vertical angles.

$\angle 2 \cong \angle 4$	Vertical Angles Theorem
$m\angle 2 = m\angle 4$	Definition of congruent angles
$130 = m\angle 4$	Substitute 130 for $m\angle 2$.

$m\angle 3 = 50°$; $m\angle 4 = 130°$

The measure of $\angle 3$ is 50° and the measure of $\angle 4$ is 130°.

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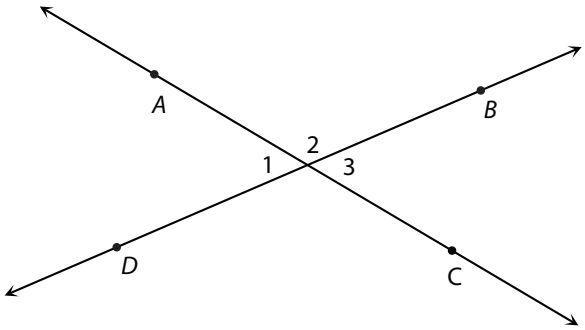
Instruction

Example 4

Prove that vertical angles are congruent given a pair of intersecting lines, $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$.

1. Draw a diagram and label three adjacent angles.



2. Start with the Supplement Theorem.

   Supplementary angles add up to 180°.

   $m\angle 1 + m\angle 2 = 180°$

   $m\angle 2 + m\angle 3 = 180°$

3. Use substitution.

   Both expressions are equal to 180°, so they are equal to each other. Rewrite the first equation, substituting $m\angle 2 + m\angle 3$ in for 180°.

   $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$

4. Use the Reflexive Property.

   $m\angle 2 = m\angle 2$

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Instruction

5. Use the Subtraction Property.

   Since $m\angle 2 = m\angle 2$, these measures can be subtracted from both sides of the equation $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$.

   This leaves $m\angle 1 = m\angle 3$.

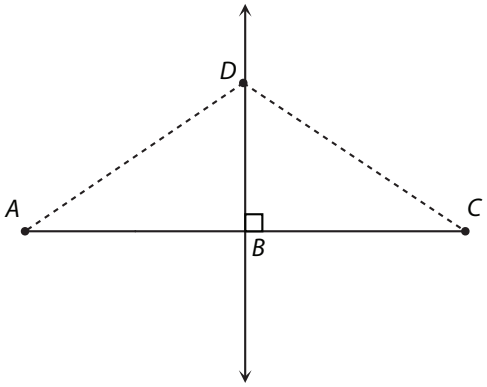
6. Use the definition of congruence.

   Since $m\angle 1 = m\angle 3$, by the definition of congruence, $\angle 1 \cong \angle 3$.

   $\angle 1$ and $\angle 3$ are vertical angles and they are congruent. This proof also shows that angles supplementary to the same angle are congruent.

Example 5

In the following diagram, $\overleftrightarrow{DB}$ is the perpendicular bisector of $\overline{AC}$. If $AD = 4x - 1$ and $DC = x + 11$, what are the values of AD and DC?



1. Use the Perpendicular Bisector Theorem to determine the lengths of AD and DC.

   If a point is on the perpendicular bisector of a segment, then that point is equidistant from the endpoints of the segment being bisected. That means $AD = DC$.

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Instruction

2. Use substitution to solve for x.

$AD = 4x - 1$ and $DC = x + 11$	Given equations
$AD = DC$	Perpendicular Bisector Theorem
$4x - 1 = x + 11$	Substitute $4x - 1$ for AD and $x + 11$ for DC.
$3x = 12$	Combine like terms.
$x = 4$	Divide both sides of the equation by 3.

3. Substitute the value of x into the given equations to determine the values of AD and DC.

$AD = 4x - 1$	$DC = x + 11$
$AD = 4(4) - 1$	$DC = (4) + 11$
$AD = 15$	$DC = 15$

AD and DC are each 15 units long.

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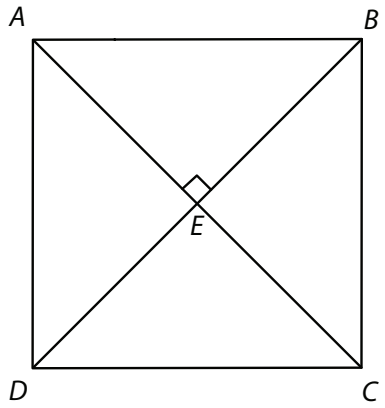
Problem-Based Task 5.5.1: Cutting Kitchen Tiles

SMP	
1 ✓	2 ✓
3 ✓	4
5	6
7	8

Maresol is retiling the backsplash over her kitchen stove, and has to cut square ceramic tiles into 4 congruent triangles to create the pattern she wants. If she doesn't cut each square into perfectly equal triangles, the tiles won't fit together properly. Before she cuts the first tile, she uses a pencil to draw two segments on the tile. The segments are perpendicular bisectors of each other. Use what you know about triangle congruency and perpendicular bisectors to prove that the 4 triangles are congruent.



Use what you know about triangle congruency and perpendicular bisectors to prove that the 4 triangles are congruent.

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Problem-Based Task 5.5.1: Cutting Kitchen Tiles

Coaching

a. What information are you given?

b. What is the definition of a perpendicular bisector?

c. Use the definition of a perpendicular bisector to determine which segments are congruent. Mark them in your diagram.

d. What does the Perpendicular Bisector Theorem state?

e. Write a statement of congruence using the Perpendicular Bisector Theorem.

f. Go to the next consecutive vertex and write a second statement of congruence using the Perpendicular Bisector Theorem.

g. Go to the next consecutive vertex and write a third statement of congruence using the Perpendicular Bisector Theorem.

h. Go to the next consecutive vertex and write a fourth statement of congruence using the Perpendicular Bisector Theorem.

i. On the diagram, mark the congruent sides that you developed in parts e–h.

j. What criteria do you need to prove triangles are congruent?

k. Are the four sections of the tile congruent?

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Instruction

Problem-Based Task 5.5.1: Cutting Kitchen Tiles

Coaching Sample Responses

a. What information are you given?

$\overline{AC}$ and $\overline{DB}$ are perpendicular bisectors of each other.

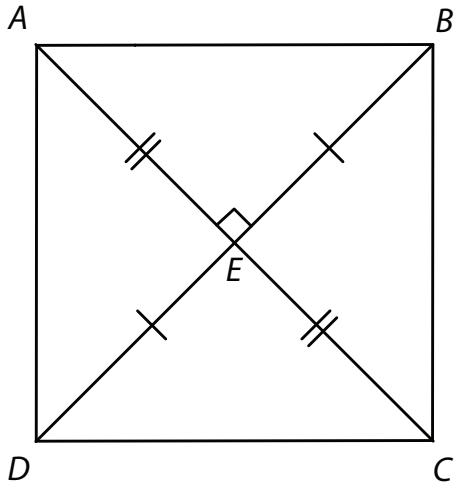
b. What is the definition of a perpendicular bisector?

A perpendicular bisector is a line that intersects a segment at a right angle at the midpoint of the segment.

c. Use the definition of a perpendicular bisector to determine which segments are congruent. Mark them in your diagram.

Since $\overline{AC}$ is the perpendicular bisector of $\overline{DB}$, then $\overline{DE} \cong \overline{EB}$.

Since $\overline{DB}$ is the perpendicular bisector of $\overline{AC}$, then $\overline{AE} \cong \overline{EC}$.



d. What does the Perpendicular Bisector Theorem state?

Any point on a perpendicular bisector is equidistant from the endpoints of the segment it bisects.

e. Write a statement of congruence using the Perpendicular Bisector Theorem.

Since point *A* is on the perpendicular bisector of $\overline{DB}$, $\overline{AD} \cong \overline{AB}$.

f. Go to the next consecutive vertex and write a second statement of congruence using the Perpendicular Bisector Theorem.

Since point *B* is on the perpendicular bisector of $\overline{AC}$, $\overline{BA} \cong \overline{BC}$.

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Instruction

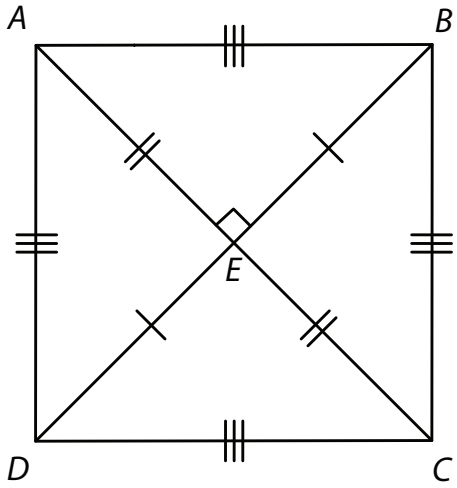
g. Go to the next consecutive vertex and write a third statement of congruence using the Perpendicular Bisector Theorem.

Since point *C* is on the perpendicular bisector of $\overline{DB}$, $\overline{CB} \cong \overline{CD}$.

h. Go to the next consecutive vertex and write a fourth statement of congruence using the Perpendicular Bisector Theorem.

Since point *D* is on the perpendicular bisector of $\overline{AC}$, $\overline{DC} \cong \overline{DA}$.

i. On the diagram, mark the congruent sides that you developed in parts e–h.



j. What criteria do you need to prove triangles are congruent?

Side-side-side (SSS), side-angle-side (SAS), or angle-side-angle (ASA).

k. Are the four sections of the tile congruent?

Yes, the four triangles are congruent. $\triangle ABE \cong \triangle CBE \cong \triangle CDE \cong \triangle ADE$ by side-side-side (SSS).

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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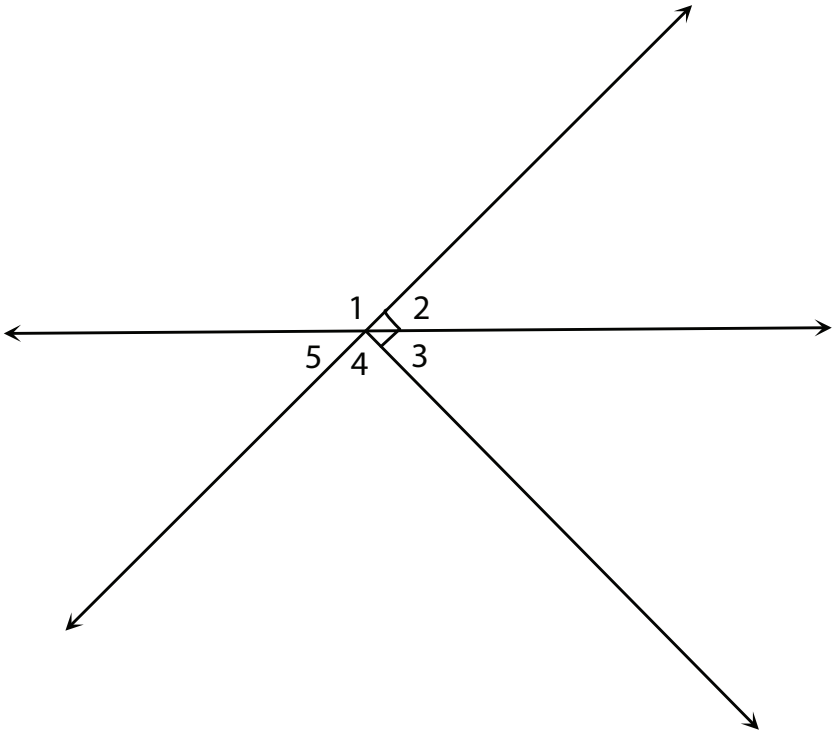
Practice 5.5.1: Proving the Vertical Angles Theorem

Use the following diagram to solve problems 1–4.



1. List two pairs of adjacent angles and two pairs of nonadjacent angles.

2. List two pairs of supplementary angles. Write a statement about those angles using the Supplement Theorem.

3. List a pair of vertical angles. Write a statement about those angles using the Vertical Angles Theorem.

4. List a pair of complementary angles. Write a statement about those angles using the Complement Theorem.

continued

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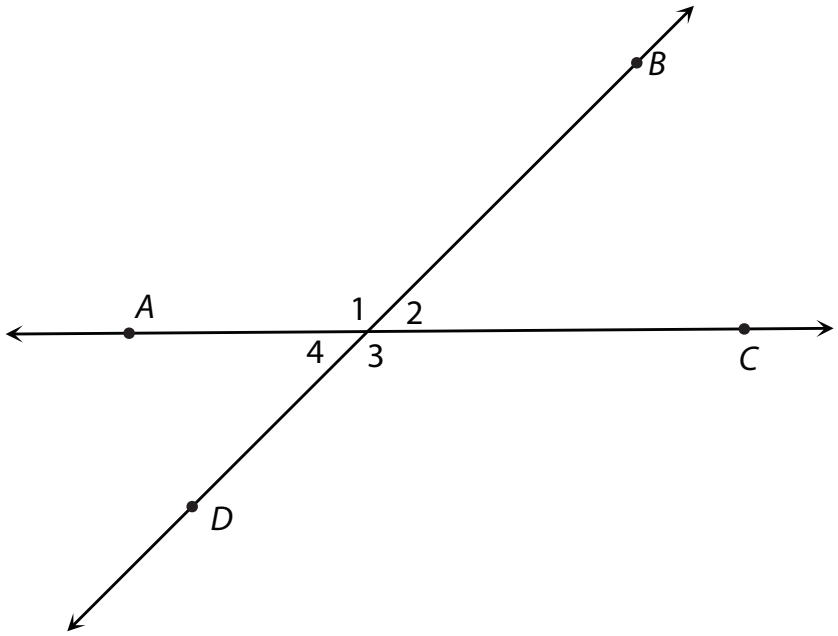
In the diagram that follows, $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect. Use this information to solve for the measures of the unknown angles in problems 5 and 6. Show and justify your work.



5. Find $m\angle 4$ if $m\angle 1 = 3x + 4$ and $m\angle 2 = 2x - 4$.

6. Find $m\angle 1$ if $m\angle 1 = 13x + 7$ and $m\angle 3 = 7x + 49$.

continued

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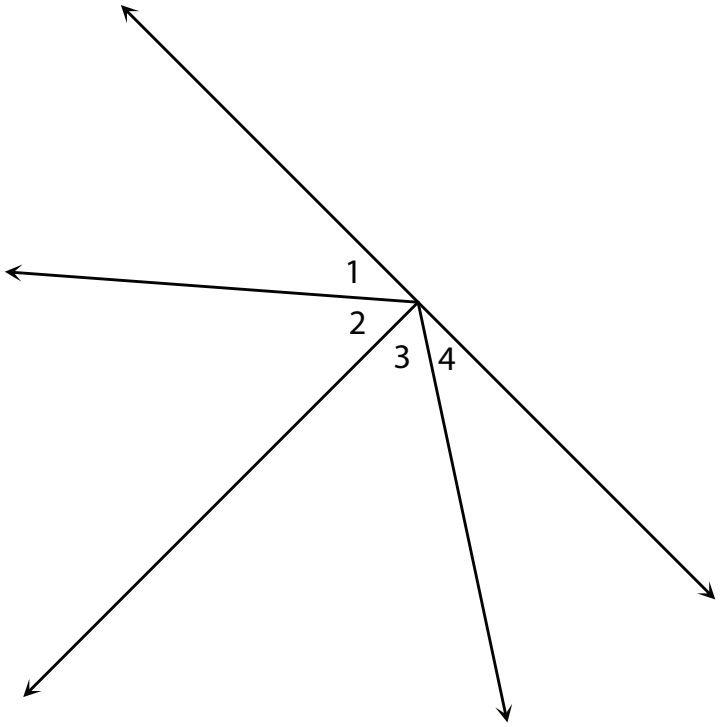
Use the diagram that follows to solve problems 7 and 8.



7. Find $m\angle 1$ given the following: $\angle 3$ and $\angle 4$ are complementary, $\angle 2 \cong \angle 3$, $m\angle 2 = 6x + 24$, and $m\angle 4 = 5x$.

8. Find $m\angle 1$ if $\angle 2$ and $\angle 3$ are complementary, $m\angle 1 = 4x - 23$, and $m\angle 4 = x + 38$.

continued

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Create your own diagram and use it to complete problem 9.

9. Prove that vertical angles are congruent.

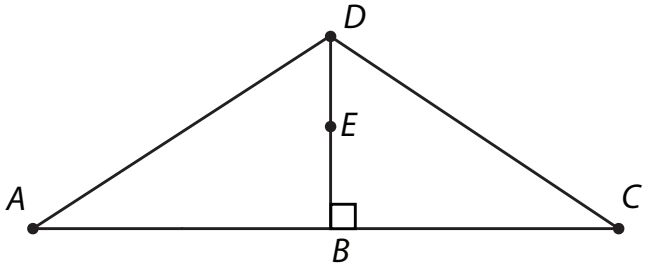
Use the diagram and the given information to complete problem 10.

10. Prove that a point on a perpendicular bisector is equidistant from the endpoints of the segment it bisects given that in $\triangle ACD$, $\overline{BD}$ is the perpendicular bisector of $\overline{AC}$ and point E is on $\overline{BD}$. Write your answer in a proof of your choosing.

Given: $\overline{DB}$ is the perpendicular bisector of $\overline{AC}$.

E is a point on $\overline{DB}$.

Prove: $EA = EC$



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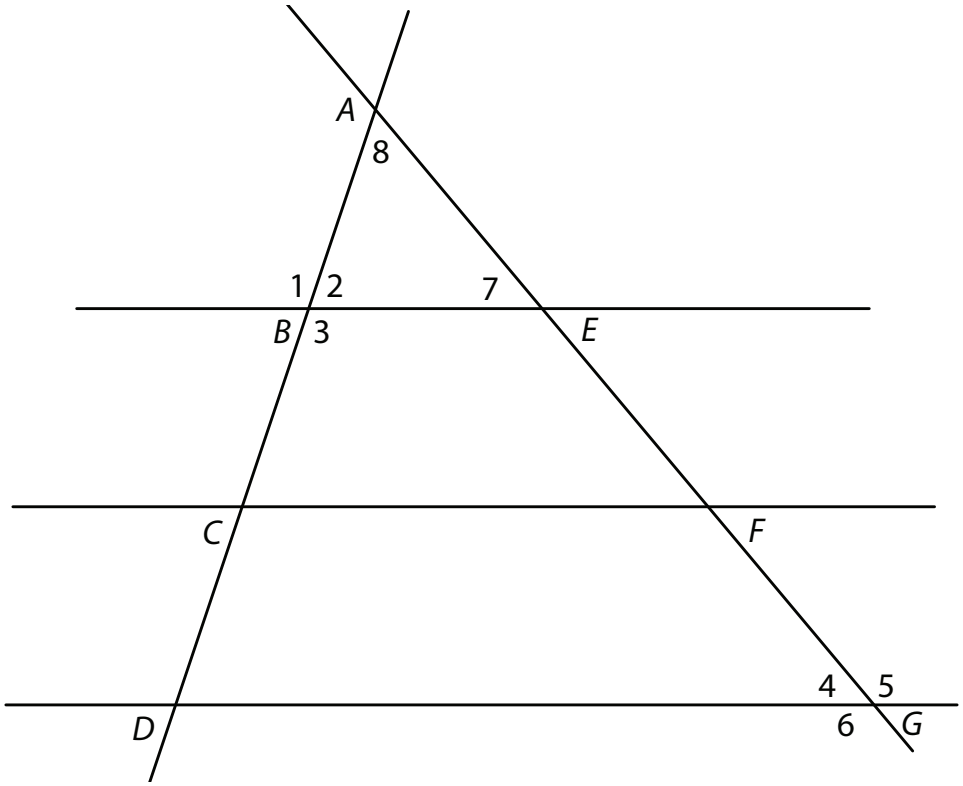
Lesson 5.5.2: Proving Theorems About Angles in Parallel Lines Cut by a Transversal

Warm-Up 5.5.2

City planners use geometry when building roads. Below is a portion of a city street map. In the diagram, $\triangle BAE \sim \triangle CAF \sim \triangle DAG$. Use what you know about similar triangles and angle relationships to answer the questions that follow.



1. If $m\angle 8 = 30^\circ$ and $m\angle 7 = 80^\circ$, find $m\angle 2$. Justify your reasoning.

2. Using the angle measures from problem 1, find the rest of the angle measures and state what angle relationship you used to find each angle measure. Use the following table to help organize the information.

Angle	Measure	Angle relationship used to determine measure
1		
2		
3		
4		
5		
6		
7		
8		

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Instruction

Lesson 5.5.2: Proving Theorems About Angles in Parallel Lines Cut by a Transversal

Common Core State Standard

G–CO.9 Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*

Warm-Up 5.5.2 Debrief

1. If $m\angle 8 = 30^\circ$ and $m\angle 7 = 80^\circ$, find $m\angle 2$. Justify your reasoning.

   The sum of the measures of the interior angles is 180°. Sum the measures of the two given angles and subtract that from 180°.

   $m\angle 2 + m\angle 7 + m\angle 8 = 180^\circ$

   $m\angle 2 + 80^\circ + 30^\circ = 180^\circ$

   $m\angle 2 = 180^\circ - 110^\circ$

   $m\angle 2 = 70^\circ$

   The measure of $\angle 2$ is 70°.

2. Using the angle measures from problem 1, find the rest of the angle measures and state what angle relationship you used to find the angle measure. Use the following table to help organize the information.

Angle	Measure	Angle relationship used to determine measure
1	110°	Linear pairs are supplementary. $\angle 1$ and $\angle 2$ are a linear pair.
2	70°	The sum of the interior angles of a triangle equals 180°.
3	110°	Vertical angles are congruent. $\angle 1$ and $\angle 3$ are vertical angles.
4	80°	Corresponding angles of similar triangles are congruent. $\angle 4$ and $\angle 7$ are corresponding angles in similar triangles.
5	100°	Linear pairs are supplementary. $\angle 4$ and $\angle 5$ are a linear pair.
6	100°	Vertical angles are congruent. $\angle 5$ and $\angle 6$ are vertical angles.
7	80°	Given
8	30°	Given

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Instruction

Connection to the Lesson

- Students will use corresponding angles of parallel lines intersected by a transversal. This warm-up gives students a visual of where corresponding angles lie that relates back to their work with similar triangles.
- The warm-up gives students practice with finding supplementary angles given the value of one angle. This will be extended to finding supplementary angles where the angles are given as expressions.
- Students will extend their knowledge of similar triangles where the Triangle Proportionality Theorem was used to determine angle relationships formed by parallel lines intersected by a transversal.

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Instruction

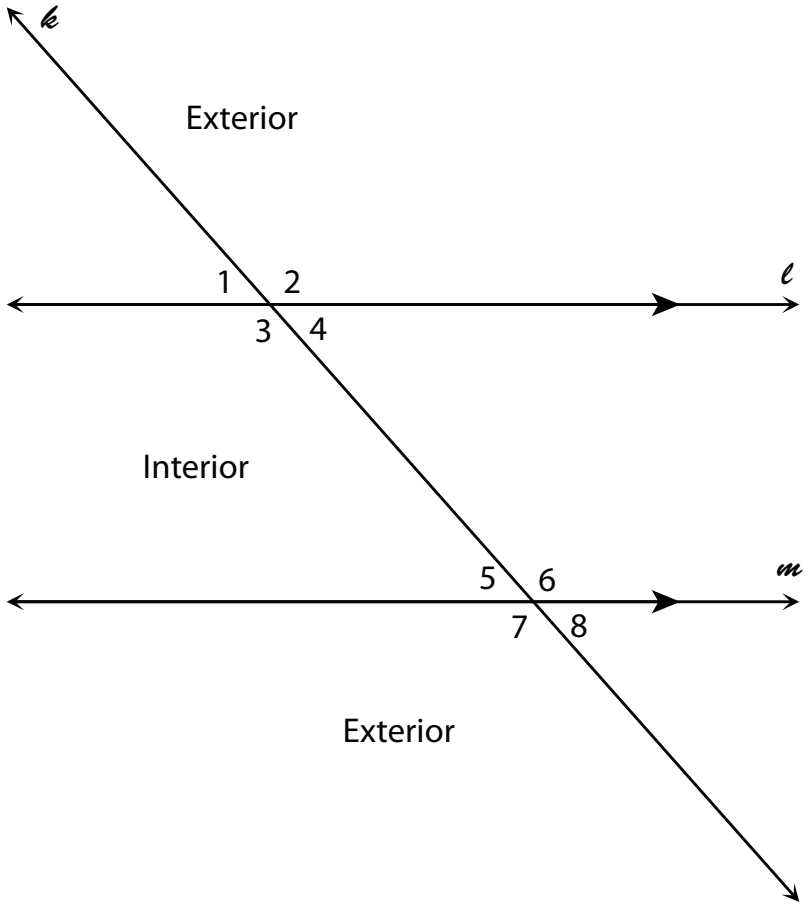
Prerequisite Skills

This lesson requires the use of the following skills:

- setting up and solving linear equations with a variable on both sides (8.EE.7b)
- applying the Supplement Theorem (G–CO.9)
- applying the Vertical Angles Theorem (G–CO.9)

Introduction

Think about all the angles formed by parallel lines intersected by a transversal. What are the relationships among those angles? In this lesson, we will prove those angle relationships. First, look at a diagram of a pair of parallel lines and notice the interior angles versus the exterior angles. The **interior angles** lie between the parallel lines and the **exterior angles** lie outside the pair of parallel lines. In the following diagram, line k is the transversal. A **transversal** is a line that intersects a system of two or more lines. Lines ℓ and m are parallel. The exterior angles are $\angle 1$, $\angle 2$, $\angle 7$, and $\angle 8$. The interior angles are $\angle 3$, $\angle 4$, $\angle 5$, and $\angle 6$.



SRB p. U5-149

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Instruction

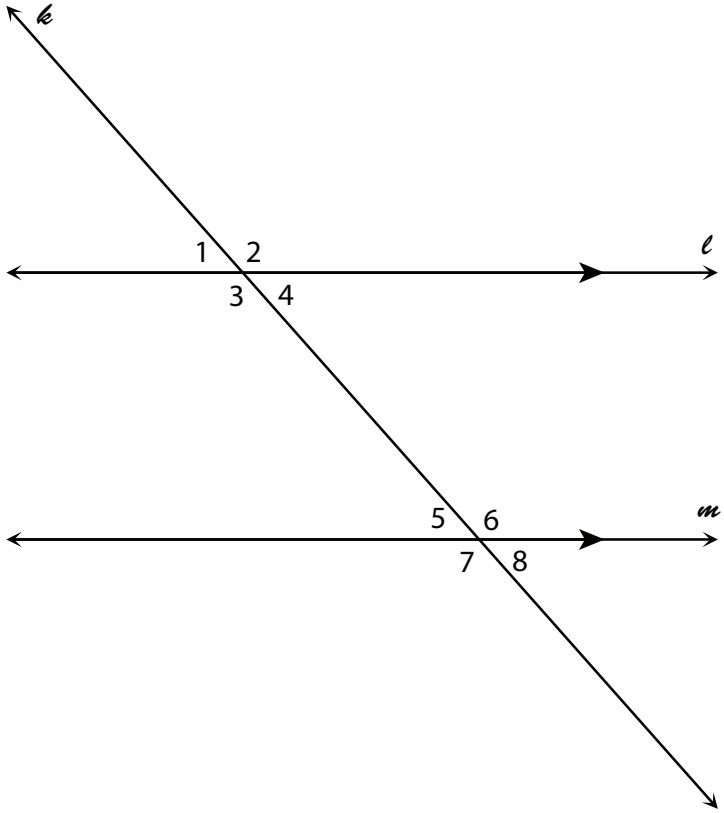
Key Concepts

- A straight line has a constant slope and parallel lines have the same slope.
- If a line intersects a set of parallel lines, then the angles in the same relative position have the same measures.
- Angles in the same relative position with respect to the transversal and the intersecting lines are **corresponding angles**.
- If the lines that the transversal intersects are parallel, then corresponding angles are congruent.

Postulate

Corresponding Angles Postulate

If two parallel lines are cut by a transversal, then corresponding angles are congruent.



Corresponding angles:

$\angle 1 \cong \angle 5,\ \angle 2 \cong \angle 6,\ \angle 3 \cong \angle 7,\ \angle 4 \cong \angle 8$

The converse is also true. If corresponding angles of lines that are intersected by a transversal are congruent, then the lines are parallel.

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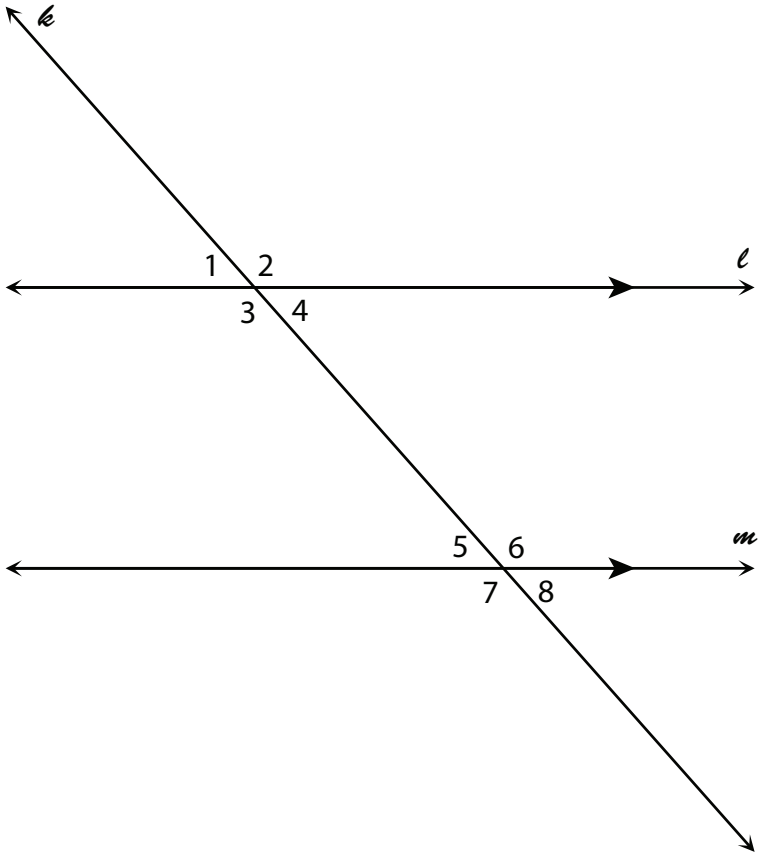
Instruction

- **Alternate interior angles** are angles that are on opposite sides of the transversal and lie on the interior of the two lines that the transversal intersects.
- If the two lines that the transversal intersects are parallel, then alternate interior angles are congruent.

Theorem

Alternate Interior Angles Theorem

If two parallel lines are intersected by a transversal, then alternate interior angles are congruent.



Alternate interior angles:

$\angle 3 \cong \angle 6,\ \angle 4 \cong \angle 5$

The converse is also true. If alternate interior angles of lines that are intersected by a transversal are congruent, then the lines are parallel.

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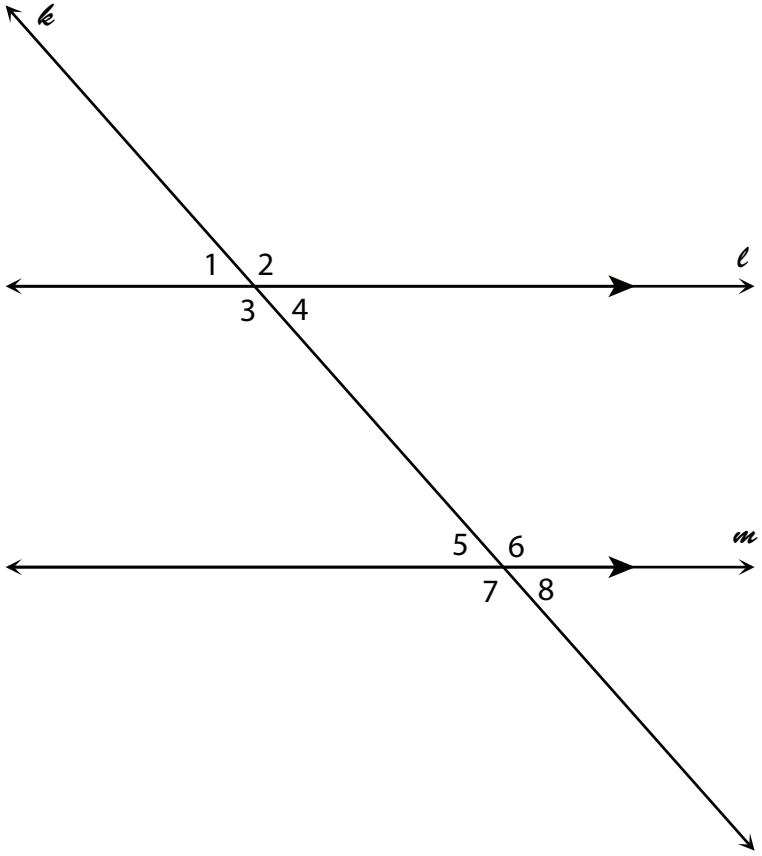
Instruction

- **Same-side interior angles** are angles that lie on the same side of the transversal and are in between the lines that the transversal intersects.
- If the lines that the transversal intersects are parallel, then same-side interior angles are supplementary.
- Same-side interior angles are sometimes called consecutive interior angles.

Theorem

Same-Side Interior Angles Theorem

If two parallel lines are intersected by a transversal, then same-side interior angles are supplementary.



Same-side interior angles:

$m\angle 3 + m\angle 5 = 180^\circ$

$m\angle 4 + m\angle 6 = 180^\circ$

The converse is also true. If same-side interior angles of lines that are intersected by a transversal are supplementary, then the lines are parallel.

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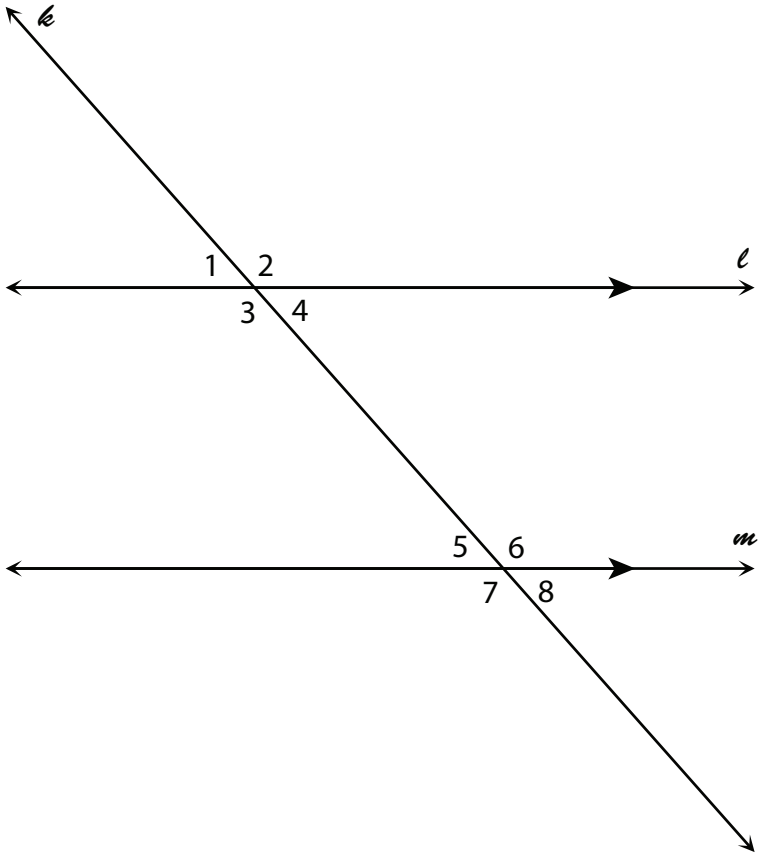
Instruction

- **Alternate exterior angles** are angles that are on opposite sides of the transversal and lie on the exterior (outside) of the two lines that the transversal intersects.
- If the two lines that the transversal intersects are parallel, then alternate exterior angles are congruent.

Theorem

Alternate Exterior Angles Theorem

If parallel lines are intersected by a transversal, then alternate exterior angles are congruent.



Alternate exterior angles:

$\angle 1 \cong \angle 8,\ \angle 2 \cong \angle 7$

The converse is also true. If alternate exterior angles of lines that are intersected by a transversal are congruent, then the lines are parallel.

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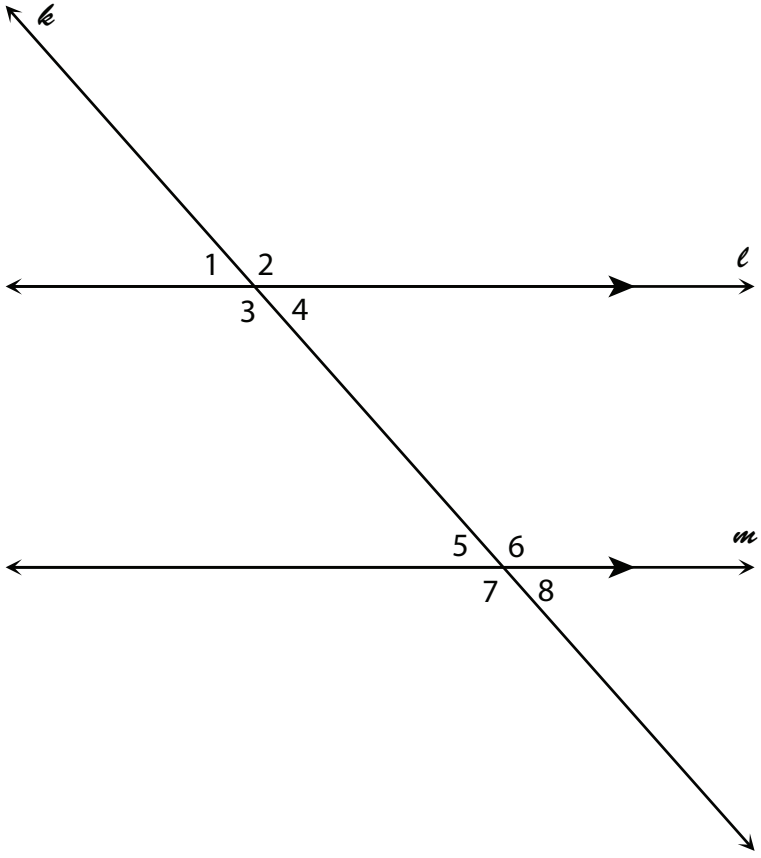
Instruction

- **Same-side exterior angles** are angles that lie on the same side of the transversal and are outside the lines that the transversal intersects.
- If the lines that the transversal intersects are parallel, then same-side exterior angles are supplementary.
- Same-side exterior angles are sometimes called consecutive exterior angles.

Theorem

Same-Side Exterior Angles Theorem

If two parallel lines are intersected by a transversal, then same-side exterior angles are supplementary.



Same-side exterior angles:

$m\angle 1 + m\angle 7 = 180^\circ$

$m\angle 2 + m\angle 8 = 180^\circ$

The converse is also true. If same-side exterior angles of lines that are intersected by a transversal are supplementary, then the lines are parallel.

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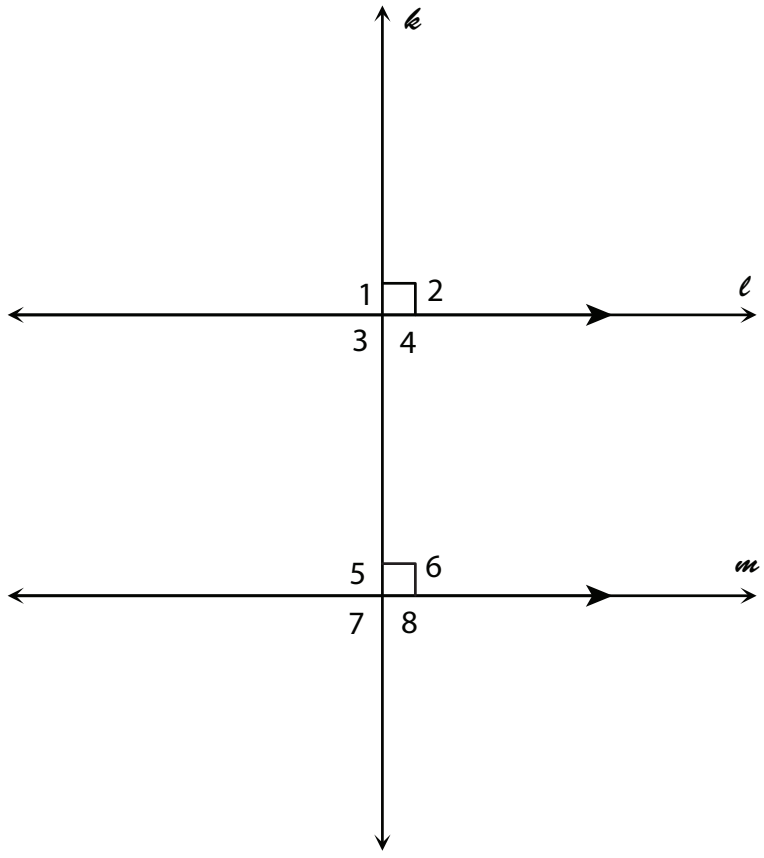
Instruction

- When the lines that the transversal intersects are parallel and perpendicular to the transversal, then all the interior and exterior angles are congruent right angles.

Theorem

Perpendicular Transversal Theorem

If a line is perpendicular to one line that is parallel to another, then the line is perpendicular to the second parallel line.



The converse is also true. If a line intersects two lines and is perpendicular to both lines, then the two lines are parallel.

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Instruction

Common Errors/Misconceptions

- setting expressions equal to each other instead of setting up expressions as a supplemental relationship and vice versa
- not being able to recognize the relative positions of the angles in a set of parallel lines intersected by a transversal
- misidentifying or not being able to identify the theorem or postulate to apply
- leaving out definitions or other steps in proofs
- assuming information not given in a diagram or problem statement that cannot be assumed
- assuming drawings are to scale

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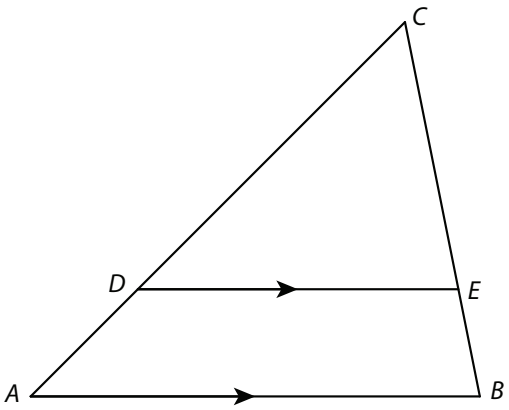
Instruction

Guided Practice 5.5.2

Example 1

Given $\overline{AB} \| \overline{DE}$, prove that $\triangle ABC \sim \triangle DEC$.

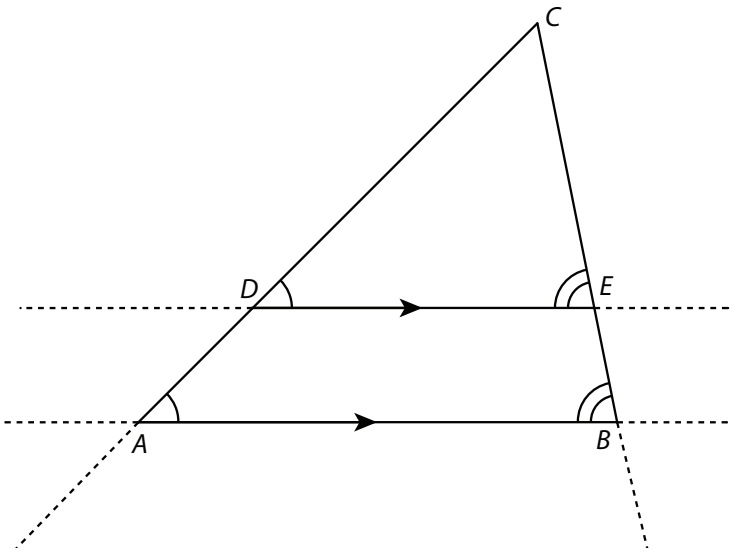


1. State the given information.

   $\overline{AB} \| \overline{DE}$

2. Extend the lines in the figure to show the transversals.

   Indicate the corresponding angles and mark the congruence of the corresponding angles with arcs.



$\angle CAB \cong \angle CDE$ and $\angle CBA \cong \angle CED$ because each pair is a set of corresponding angles.

SWB p. 141; SRB p. U5-156

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Instruction

3. Use the AA (angle-angle) criteria.

   When two pairs of corresponding angles of a triangle are congruent, the angles are similar.

   In this case, we actually know that all three pairs of corresponding angles are congruent because $\angle C \cong \angle C$ by the Reflexive Property.

4. Write the information in a two-column proof.

Statements	Reasons
1. $\overline{AB} \parallel \overline{DE}$	1. Given
2. $\angle CAB \cong \angle CDE$, $\angle CBA \cong \angle CED$	2. Corresponding Angles Postulate
3. $\triangle ABC \sim \triangle DEC$	3. AA Postulate

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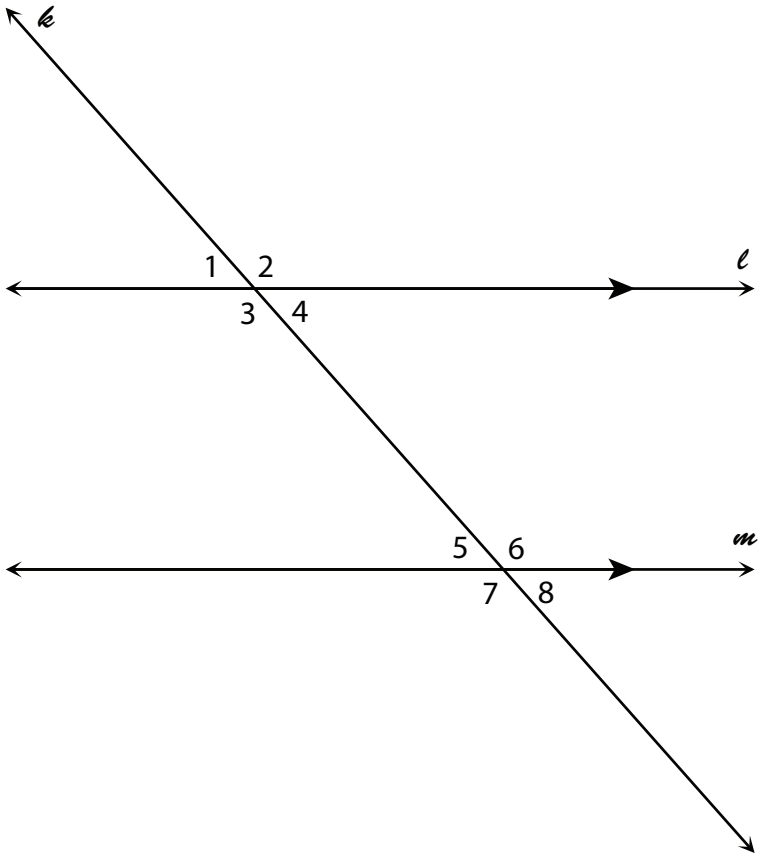
Instruction

Example 2

Given two parallel lines and a transversal, prove that alternate interior angles are congruent. In the following diagram, lines ℓ and m are parallel. Line k is the transversal.

Given: $\ell \parallel m$, and line k is a transversal.

Prove: $\angle 3 \cong \angle 6$



1. State the given information.

   $\ell \parallel m$, and line k is a transversal.

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Instruction

2. Use the Corresponding Angles Postulate.

   Corresponding angles are angles that lie in the same relative position with respect to the transversal and the lines the transversal intersects. If the lines that the transversal intersects are parallel, then corresponding angles are congruent. $\angle 3$ and $\angle 7$ are corresponding angles because they are both below the parallel lines and on the left side of the transversal.

   $\angle 3 \cong \angle 7$ because they are corresponding angles.

3. Use the Vertical Angles Theorem.

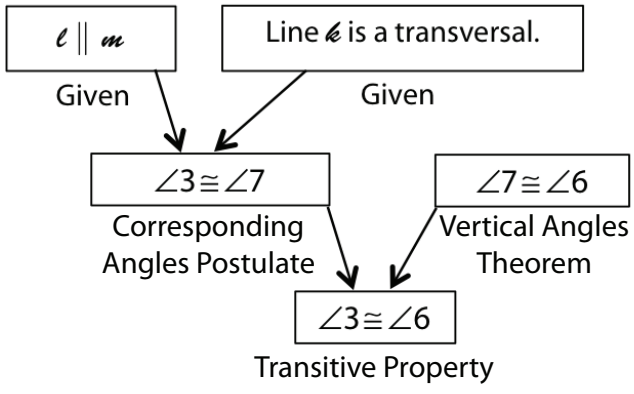
   Vertical angles are formed when a pair of lines intersect. Vertical angles are the nonadjacent angles formed by these intersecting lines. Vertical angles are congruent.

   $\angle 7 \cong \angle 6$ because they are vertical angles.

4. Use the Transitive Property.

   Since $\angle 3 \cong \angle 7$ and $\angle 7 \cong \angle 6$, $\angle 3 \cong \angle 6$.

5. Write the information in a flow proof.



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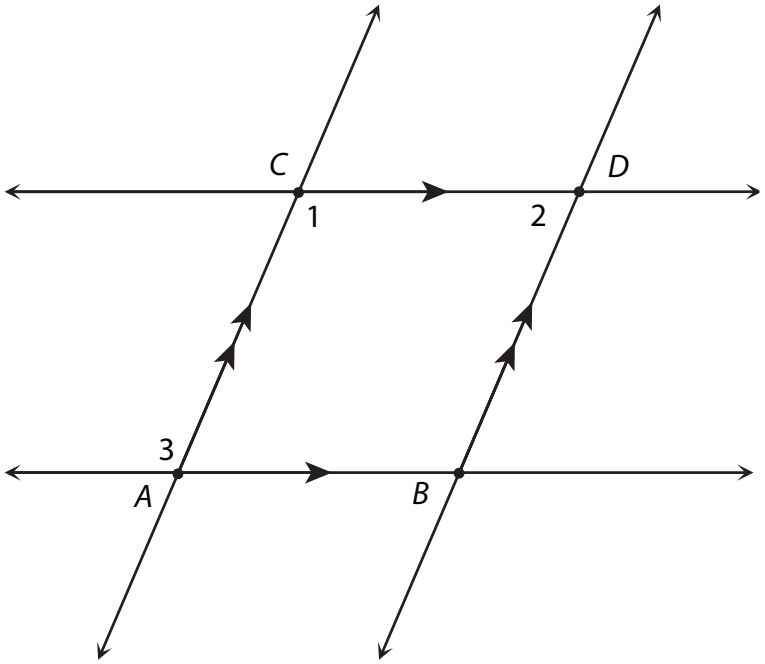
Instruction

Example 3

In the following diagram, $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$ and $\overleftrightarrow{AC} \| \overleftrightarrow{BD}$. If $m\angle 1 = 3(x+15)$, $m\angle 2 = 2x+55$, and $m\angle 3 = 4y+9$, find the measures of the unknown angles and the values of x and y.



1. Find the relationship between two angles that have the same variable.

   $\angle 1$ and $\angle 2$ are same-side interior angles and are both expressed in terms of x.

2. Use the Same-Side Interior Angles Theorem.

   Same-side interior angles are supplementary. Therefore, $m\angle 1 + m\angle 2 = 180^\circ$.

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Instruction

3. Use substitution and solve for x.

$m\angle 1 = 3(x+15)$ and $m\angle 2 = 2x+55$	Given
$m\angle 1 + m\angle 2 = 180^\circ$	Same-Side Interior Angles Theorem
$[3(x+15)] + (2x+55) = 180$	Substitute $3(x+15)$ for $m\angle 1$ and $2x+55$ for $m\angle 2$.
$(3x+45) + (2x+55) = 180$	Distribute.
$5x + 100 = 180$	Combine like terms.
$5x = 80$	Subtract 100 from both sides of the equation.
$x = 16$	Divide both sides by 5.

4. Find $m\angle 1$ and $m\angle 2$ using substitution.

$m\angle 1 = 3(x+15)\,;\, x = 16$
$m\angle 1 = 3[(16)+15]$
$m\angle 1 = 3(31)$
$m\angle 1 = 93^\circ$

$m\angle 2 = 2x+55\,;\, x = 16$
$m\angle 2 = 2(16)+55$
$m\angle 2 = 32+55$
$m\angle 2 = 87^\circ$

After finding $m\angle 1$, to find $m\angle 2$ you could alternately use the Same-Side Interior Angles Theorem, which says that same-side interior angles are supplementary.

$m\angle 1 + m\angle 2 = 180^\circ$
$(93) + m\angle 2 = 180$
$m\angle 2 = 180 - 93$
$m\angle 2 = 87^\circ$

5. Find the relationship between one of the known angles and the last unknown angle, $\angle 3$.

$\angle 1$ and $\angle 3$ lie on the opposite side of the transversal on the interior of the parallel lines. This means they are alternate interior angles.

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Instruction

6. Use the Alternate Interior Angles Theorem.

   The Alternate Interior Angles Theorem states that alternate interior angles are congruent if the transversal intersects a set of parallel lines. Therefore, $\angle 1 \cong \angle 3$.

7. Use the definition of congruence and substitution to find $m\angle 3$.

   $\angle 1 \cong \angle 3$, so $m\angle 1 = m\angle 3$.

   $m\angle 1 = 93^\circ$

   Using substitution, $93^\circ = m\angle 3$.

8. Use substitution to solve for y.

$m\angle 3 = 4y + 9$	Given
$93 = 4y + 9$	Substitute 93 for $m\angle 3$.
$84 = 4y$	Subtract 9 from both sides of the equation.
$y = 21$	Simplify.

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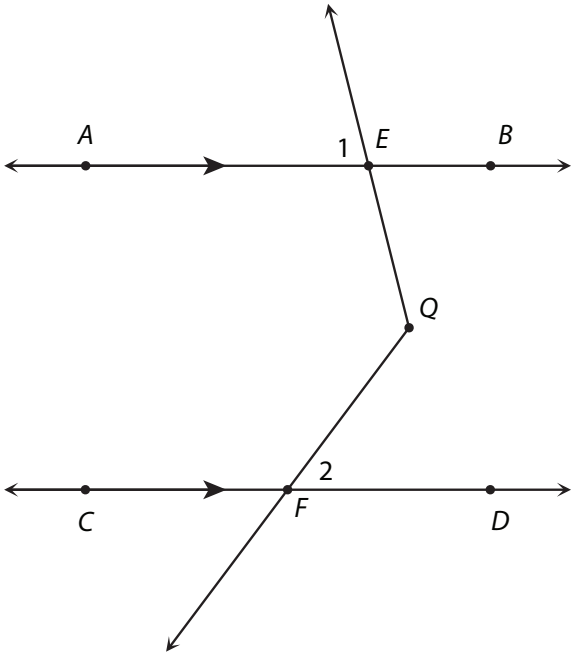
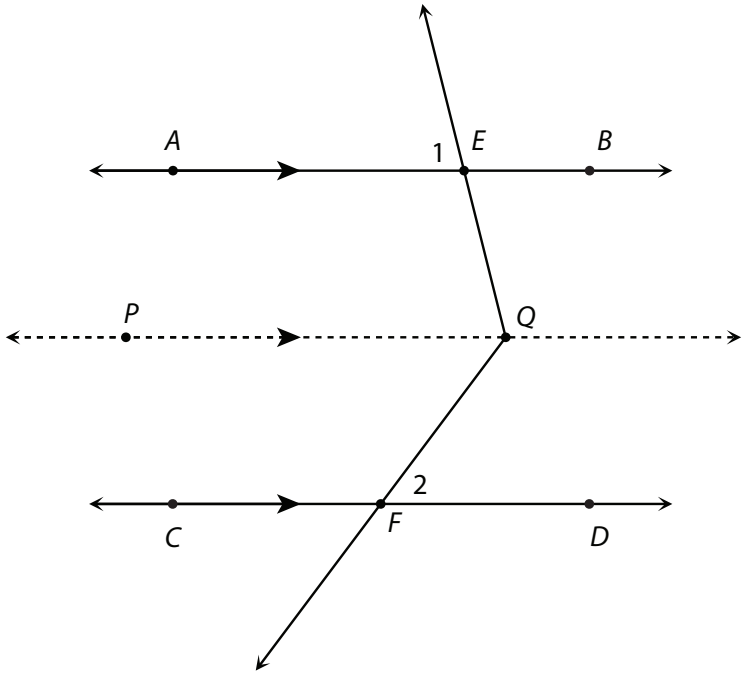
Instruction

Example 4

In the following diagram, $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$. If $m\angle 1 = 35°$ and $m\angle 2 = 65°$, find $m\angle EQF$.



1. Draw a third parallel line that passes through point Q.
   Label a second point on the line as P. $\overleftrightarrow{PQ} \| \overleftrightarrow{AB} \| \overleftrightarrow{CD}$.



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Instruction

2. Use $\overrightarrow{QE}$ as a transversal to $\overleftrightarrow{AB}$ and $\overleftrightarrow{PQ}$ and identify angle relationships.

   $\angle 1 \cong \angle BEQ$ because they are vertical angles.

   $\angle BEQ \cong \angle EQP$ because they are alternate interior angles.

   $\angle 1 \cong \angle EQP$ by the Transitive Property.

   It was given that $m\angle 1 = 35°$.

   By substitution, $m\angle EQP = 35°$.

3. Use $\overrightarrow{QF}$ as a transversal to $\overleftrightarrow{PQ}$ and $\overleftrightarrow{CD}$ and identify angle relationships.

   $\angle 2 \cong \angle FQP$ because they are alternate interior angles.

   It was given that $m\angle 2 = 65°$.

   By substitution, $m\angle FQP = 65°$.

4. Use angle addition.

   Notice that the angle measure we are looking for is made up of two smaller angle measures that we just found.

   $m\angle EQF = m\angle EQP + m\angle FQP$

   $m\angle EQF = 35° + 65°$

   $m\angle EQF = 100°$

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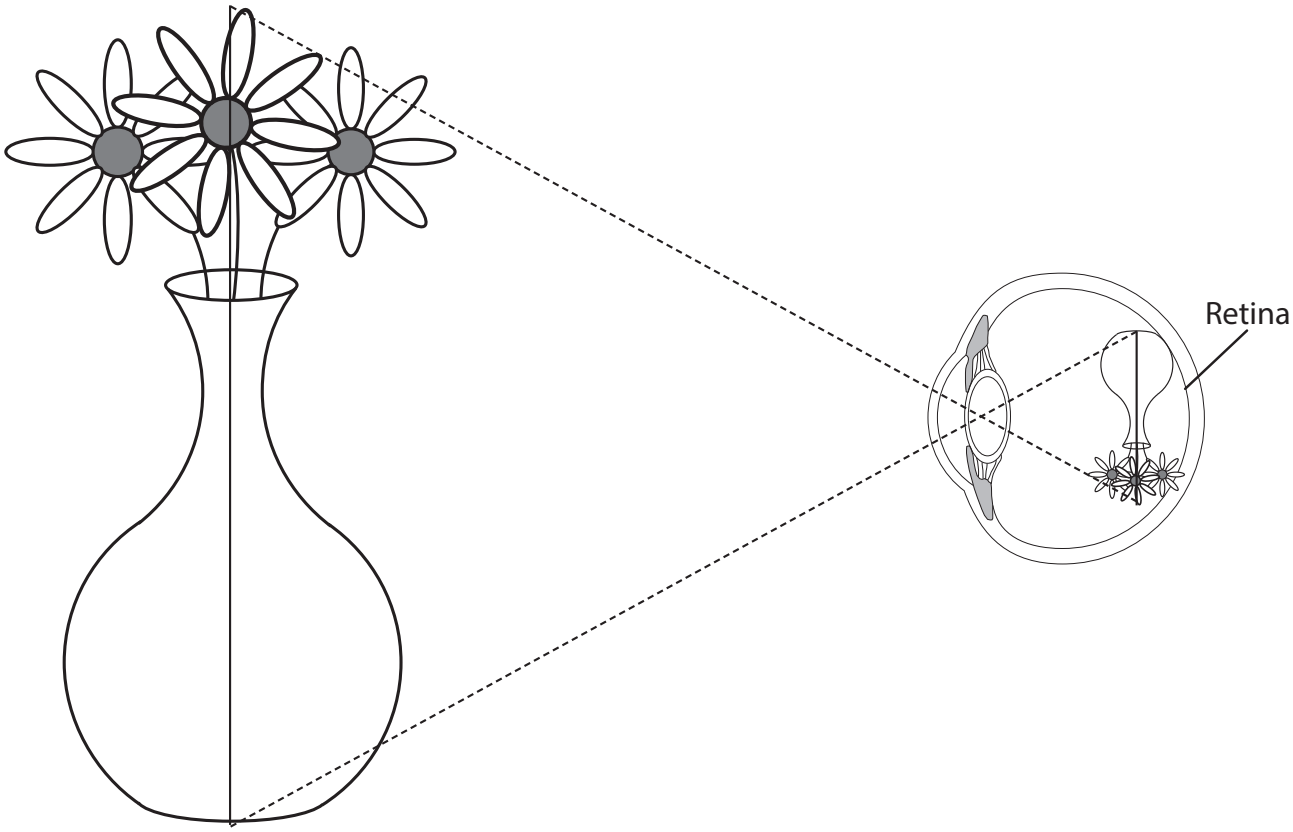
Problem-Based Task 5.5.2: Retinal Refraction

SMP	
1	2 ✓
3 ✓	4
5	6
7 ✓	8

When a person looks at an object, the light rays are refracted or distorted as they pass through the eye, and the image is transmitted upside down on the retina in the back of the eye. The object and its retinal image are similar. Prove that they are in proportion using similar triangles. A diagram is given below. Assume that both the person looking at the image and the image are vertical.



Prove that they are in proportion using similar triangles.

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Problem-Based Task 5.5.2: Retinal Refraction

Coaching

a. Label the angles on the diagram.

b. What information is given?

c. Which angles are congruent in parallel lines intersected by a transversal?

d. Which angles are congruent in the diagram and why?

e. What criteria are needed to prove that triangles are similar?

f. Which two triangles are similar?

g. What is the proof that the triangles are similar?

h. What is true about the parts of similar triangles?

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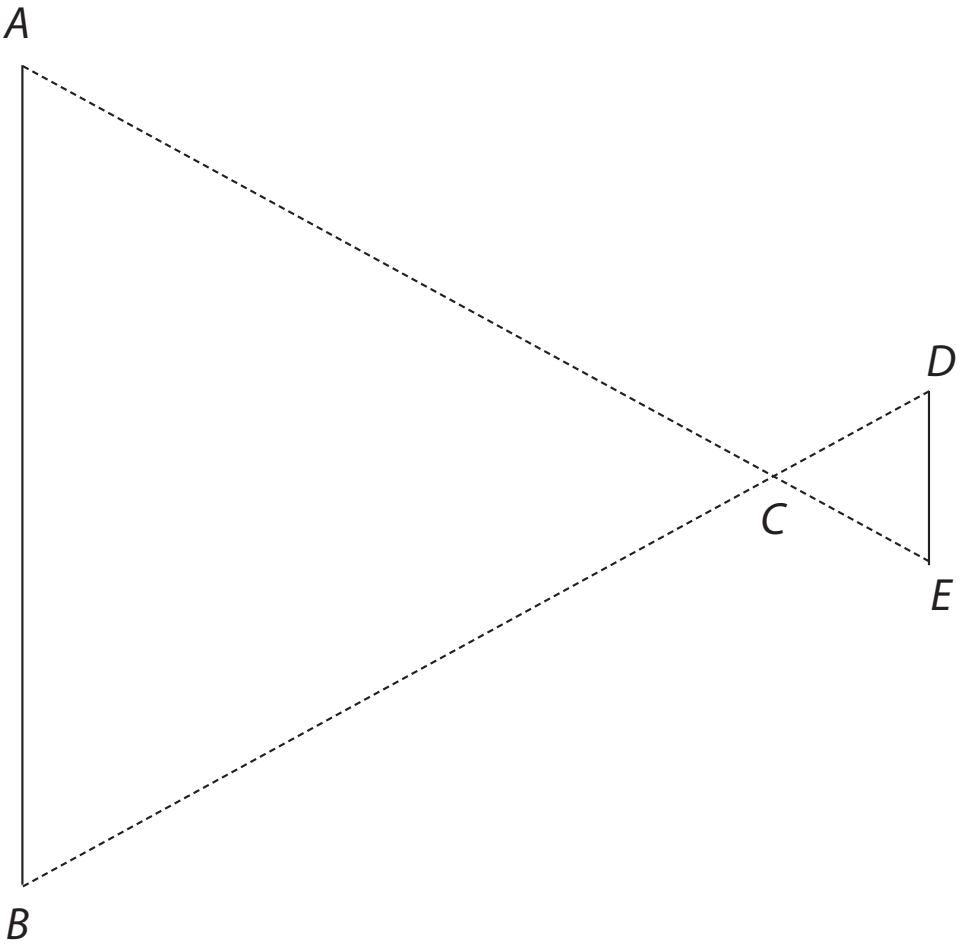
Problem-Based Task 5.5.2: Retinal Refraction

Coaching Sample Responses

a. Label the angles on the diagram.



b. What information is given?

$\overline{AB}$ and $\overline{DE}$ are both vertical. Therefore, $\overline{AB} \| \overline{DE}$.

c. Which angles are congruent in parallel lines intersected by a transversal?

Alternate interior angles are congruent and alternate exterior angles are congruent.

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d. Which angles are congruent in the diagram and why?

$\angle CED \cong \angle BAE$ by the Alternate Interior Angles Theorem.

$\angle CDE \cong \angle ABD$ by the Alternate Interior Angles Theorem.

$\angle ACB \cong \angle DCE$ by the Vertical Angles Theorem.

e. What criteria are needed to prove that triangles are similar?

The needed criteria are two angles (AA Similarity Statement), two sides and the included angle (SAS Similarity Statement), or all three sides (SSS Similarity Statement).

f. Which two triangles are similar?

$\triangle ABC \sim \triangle EDC$

g. What is the proof that the triangles are similar?

Statements	Reasons
1. $\overline{AB}$ and $\overline{DE}$ are both vertical.	1. Given
2. $\overline{AB} \parallel \overline{DE}$	2. Definition of parallel lines
3. $\angle CED \cong \angle BAE$	3. Alternate Interior Angles Theorem
4. $\angle ACB \cong \angle DCE$	4. Vertical Angles Theorem
5. $\triangle ABC \sim \triangle EDC$	5. Angle-Angle Similarity Postulate

h. What is true about the parts of similar triangles?

The corresponding parts of similar triangles are in proportion.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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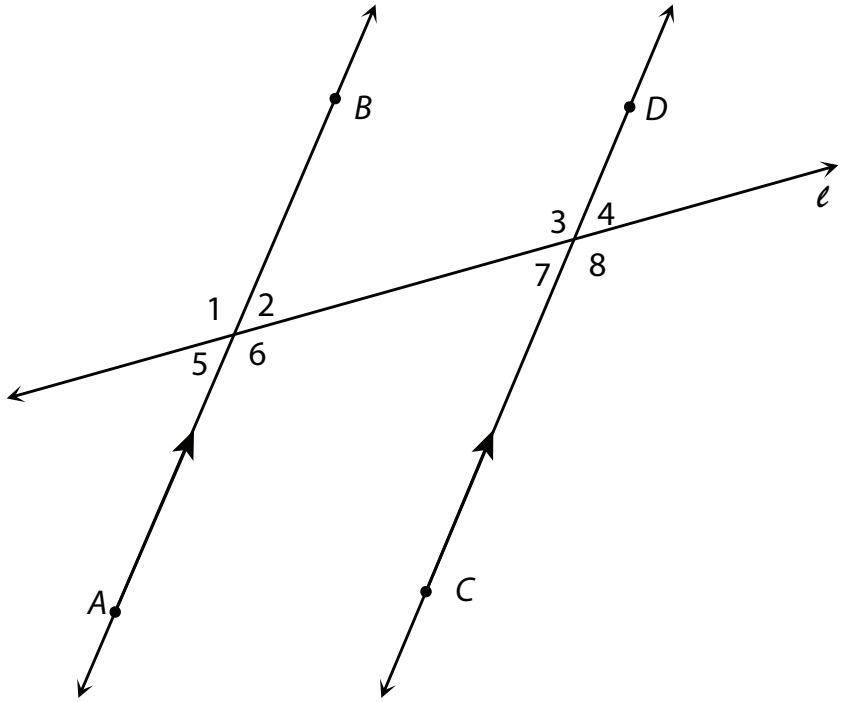
Practice 5.5.2: Proving Theorems About Angles in Parallel Lines Cut by a Transversal

Use the following diagram to solve problems 1–5, given that $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$ and line ℓ is the transversal. Justify your answers using angle relationships in parallel lines intersected by a transversal.



1. Find $m\angle 5$ if $m\angle 5 = 2(3x+13)$ and $m\angle 7 = 3x+50$.

2. Find $m\angle 2$ if $m\angle 2 = 4x+39$ and $m\angle 7 = 12x-17$.

3. Find $m\angle 6$ if $m\angle 6 = 7x+41$ and $m\angle 7 = 3x-1$.

4. Find $m\angle 4$ if $m\angle 4 = 2(5x-9)$ and $m\angle 5 = 3(x+8)$.

5. Find $m\angle 1$ if $m\angle 1 = 11x+35$ and $m\angle 4 = x+1$.

continued

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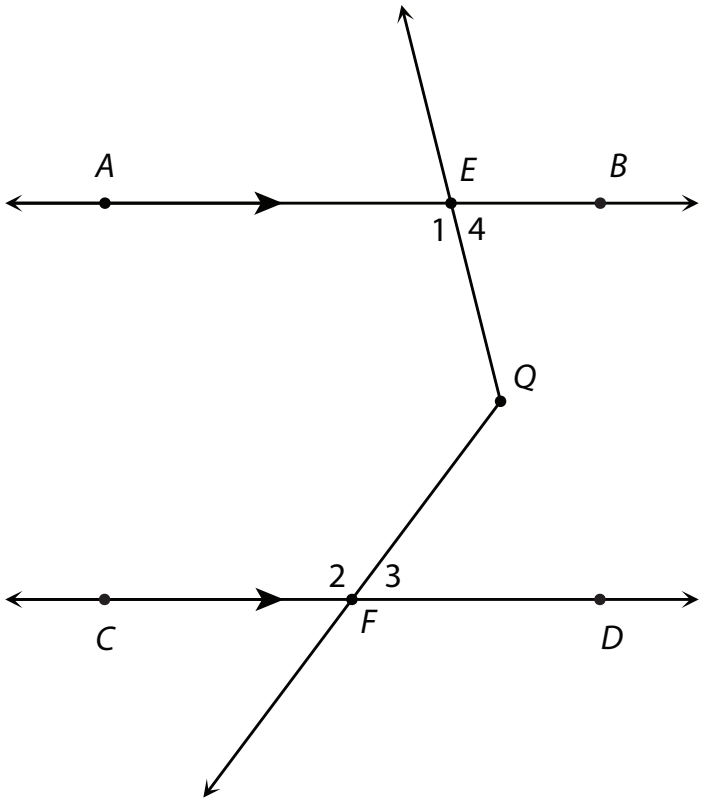
Use the following diagram to solve problems 6 and 7. Given: $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$.



6. Find $m\angle EQF$ if $m\angle 1 = 110°$ and $m\angle 2 = 135°$.

7. Find $m\angle EQF$ if $m\angle 1 = 117°$ and $m\angle 3 = 31°$.

continued

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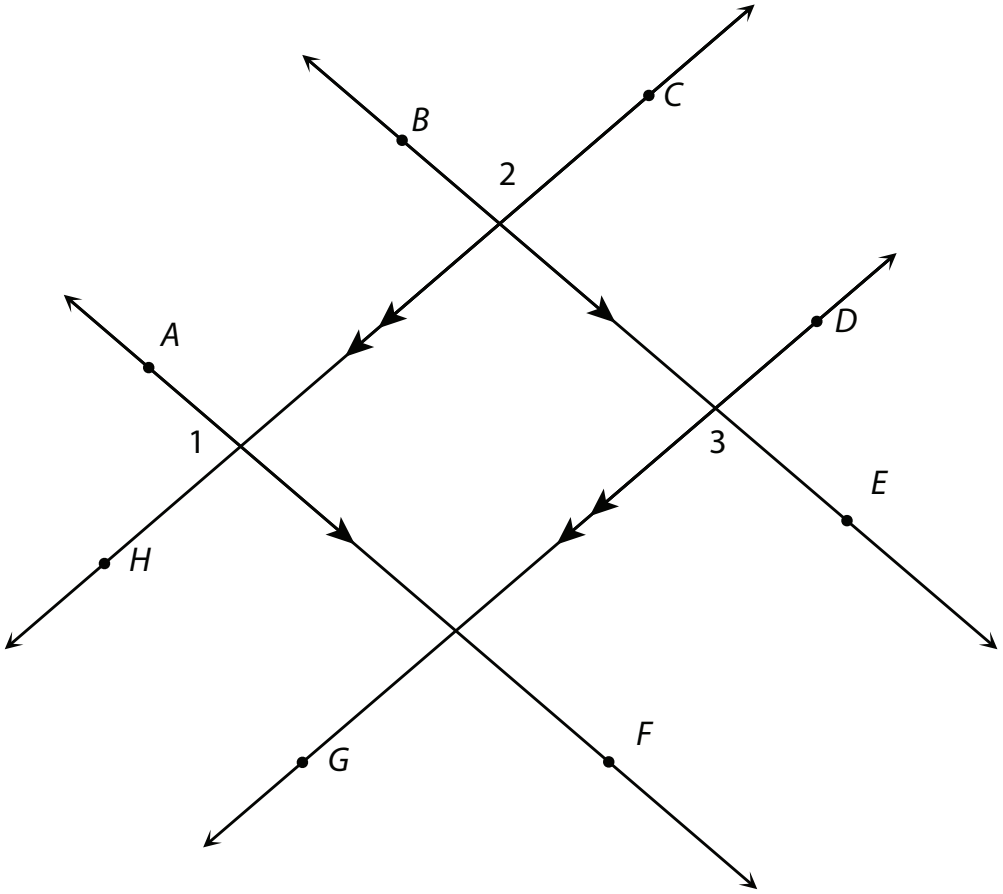
Use the following diagram to solve problem 8. Given: $\overleftrightarrow{AF} \| \overleftrightarrow{BE}$, $\overleftrightarrow{HC} \| \overleftrightarrow{GD}$, $m\angle 1 = 5x - 16$, $m\angle 2 = 6x - 13$, and $m\angle 3 = 10y - 9$.



8. Find the measures of the numbered angles and the values of x and y. Justify your reasoning.

Write proofs to complete problems 9 and 10. Use the following diagram as a reference.

9. Prove the Same-Side Interior Angles Theorem.

10. Prove that alternate exterior angles are congruent. Write your answer in a paragraph proof.

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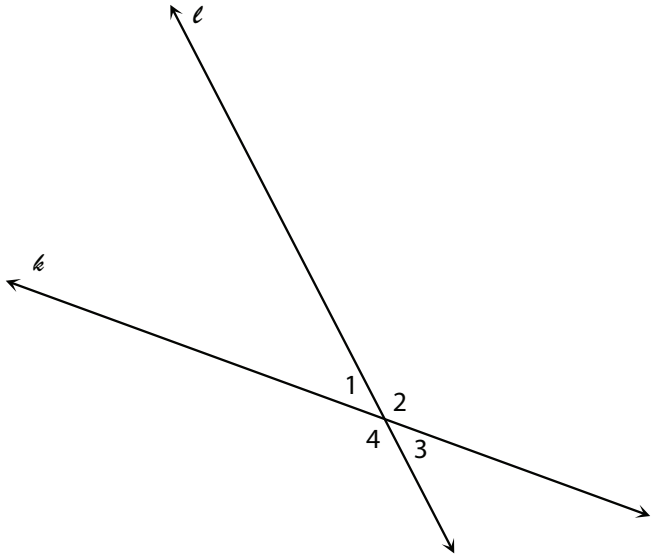
Progress Assessment

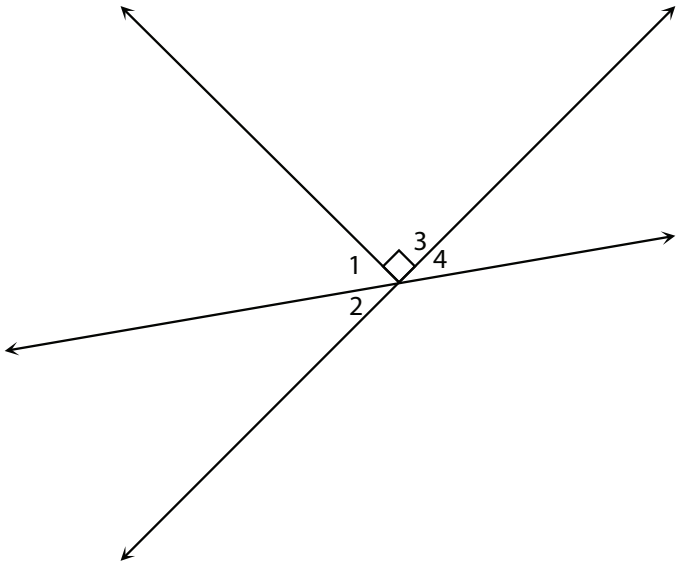
Circle the letter of the best answer.

1. If $m\angle 1 = 2x - 9$ and $m\angle 2 = 10x - 3$, find $m\angle 4$ given that lines k and ℓ intersect as shown below.



a. 157°
b. 164°
c. 16°
d. 23°

2. If $m\angle 1 = 4x$ and $m\angle 4 = 3x - 1$ in the diagram below, find $m\angle 2$ if $\angle 3$ is a right angle.



a. 142°
b. 52°
c. 38°
d. 27°

continued

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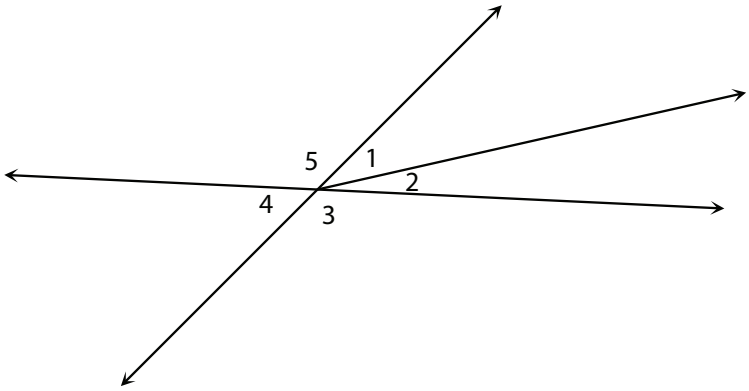
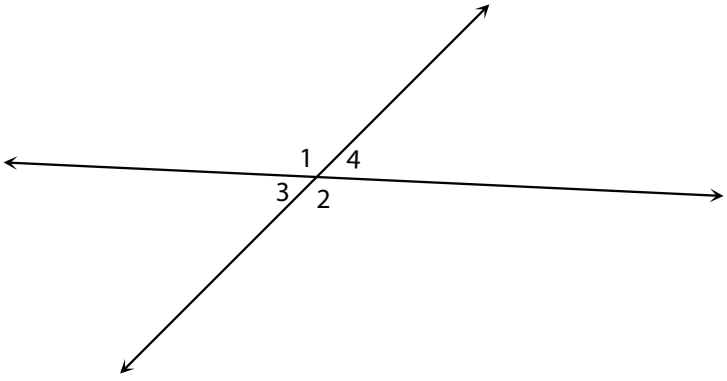
3. If $m\angle 1 = x + 7$, $m\angle 2 = 2(x + 2)$, and $m\angle 4 = 2(x + 13)$ in the diagram below, find $m\angle 4$.



a. 22°
b. 34°
c. 47°
d. 56°

4. In the proof of the Vertical Angles Theorem below, what are the reasons for the statements listed? Refer to the diagram given. In the diagram, $\angle 3$ and $\angle 1$ form a linear pair and $\angle 1$ and $\angle 4$ form a linear pair. Prove that $\angle 3 \cong \angle 4$.

Statements	Reasons
1. $\angle 3$ and $\angle 1$ form a linear pair. $\angle 1$ and $\angle 4$ form a linear pair.	1. ?
2. $\angle 3$ and $\angle 1$ are supplementary. $\angle 1$ and $\angle 4$ are supplementary.	2. ?
3. $\angle 3 \cong \angle 4$	3. ?



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a.

Statements	Reasons
1. $\angle 3$ and $\angle 1$ form a linear pair. $\angle 1$ and $\angle 4$ form a linear pair.	1. Inspection
2. $\angle 3$ and $\angle 1$ are supplementary. $\angle 1$ and $\angle 4$ are supplementary.	2. Supplement Theorem
3. $\angle 3 \cong \angle 4$	3. Transitive Property

b.

Statements	Reasons
1. $\angle 3$ and $\angle 1$ form a linear pair. $\angle 1$ and $\angle 4$ form a linear pair.	1. Given
2. $\angle 3$ and $\angle 1$ are supplementary. $\angle 1$ and $\angle 4$ are supplementary.	2. Supplement Theorem
3. $\angle 3 \cong \angle 4$	3. Angles supplementary to the same angle or to congruent angles are congruent.

c.

Statements	Reasons
1. $\angle 3$ and $\angle 1$ form a linear pair. $\angle 1$ and $\angle 4$ form a linear pair.	1. Given
2. $\angle 3$ and $\angle 1$ are supplementary. $\angle 1$ and $\angle 4$ are supplementary.	2. $m\angle 3 + m\angle 1 = 180$ $m\angle 1 + m\angle 4 = 180$
3. $\angle 3 \cong \angle 4$	3. Angles supplementary to the same angle or to congruent angles are congruent.

d.

Statements	Reasons
1. $\angle 3$ and $\angle 1$ form a linear pair. $\angle 1$ and $\angle 4$ form a linear pair.	1. Given
2. $\angle 3$ and $\angle 1$ are supplementary. $\angle 1$ and $\angle 4$ are supplementary.	2. Complement Theorem
3. $\angle 3 \cong \angle 4$	3. Angles supplementary to the same angle or to congruent angles are congruent.

continued

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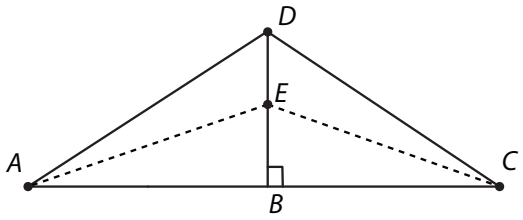
Assessment

5. Given the diagram and the information below, determine what information is missing from the proof.



Given: $\overline{DB}$ is the perpendicular bisector of $\overline{AC}$. E is a point on $\overline{DB}$.

Prove: $EA = EC$

Statements	Reasons
1. $\overline{DB}$ is the perpendicular bisector of $\overline{AC}$. E is a point on $\overline{DB}$.	1. Given
2. B is the midpoint of $\overline{AC}$.	2. Definition of perpendicular bisector
3. $\overline{AB} \cong \overline{BC}$	3. Definition of midpoint
4. $\angle DBA$ and $\angle DBC$ are right angles.	4. Definition of perpendicular lines
5. ?	5. All right angles are congruent.
6. $\overline{EB} \cong \overline{EB}$	6. ?
7. ?	7. SAS
8. $\overline{EA} \cong \overline{EC}$	8. CPCTC
9. $EA = EC$	9. Definition of congruence

a. $\angle DBA \cong \angle DBC$; Symmetric Property; $\triangle EBA \cong \triangle CBE$

b. $\angle DBA \cong \angle DBC$; Reflexive Property; $\triangle EBA \cong \triangle EBC$

c. $\angle EAC \cong \angle EAC$; Transitive Property; $\triangle EBA \cong \triangle EBC$

d. $\angle EAB \cong \angle ECB$; Symmetric Property; $\triangle EBA \cong \triangle CBE$

continued

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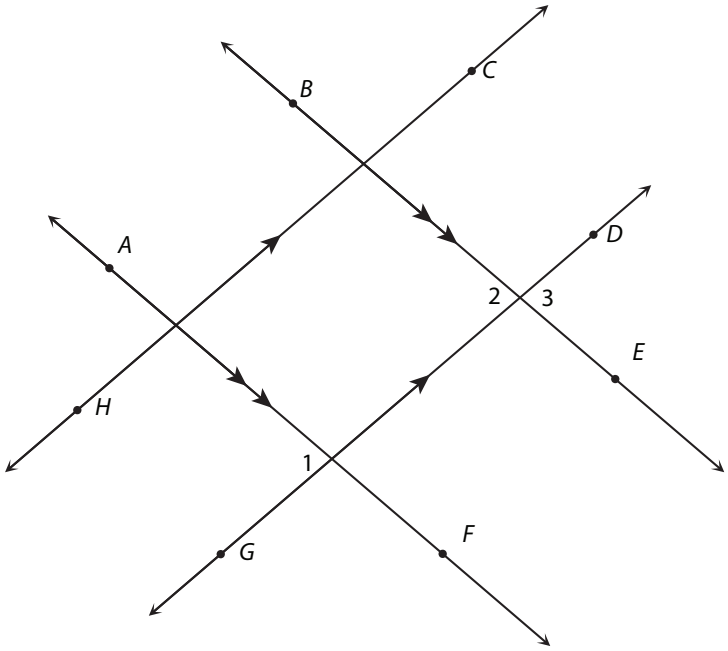
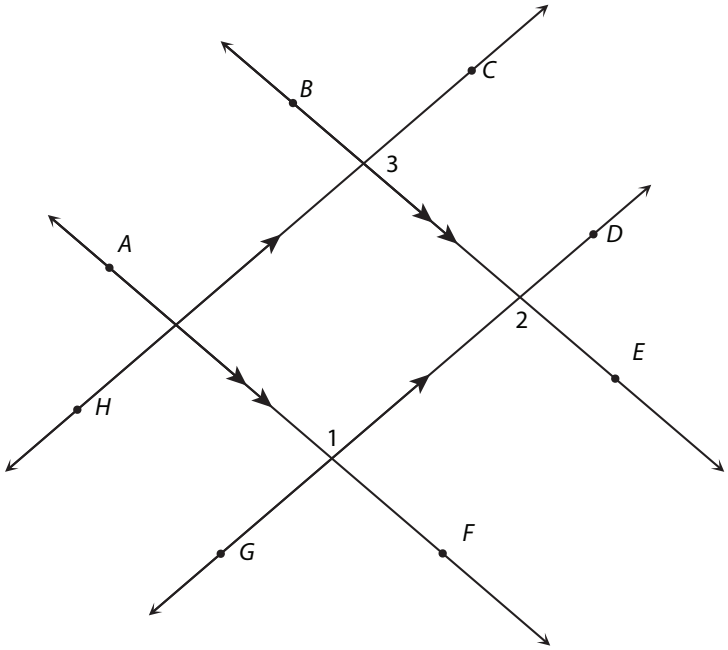
6. There are two sets of parallel lines in the diagram below. Find $m\angle 2$ if $m\angle 1 = 10x - 4$ and $m\angle 3 = 7x + 20$.



a. 14°
c. 76°
b. 90°
d. 104°

7. Find $m\angle 1$ if $m\angle 2 = 5x$ and $m\angle 3 = 6x - 7$.



a. 75°
c. 95°
b. 85°
d. 105°

continued

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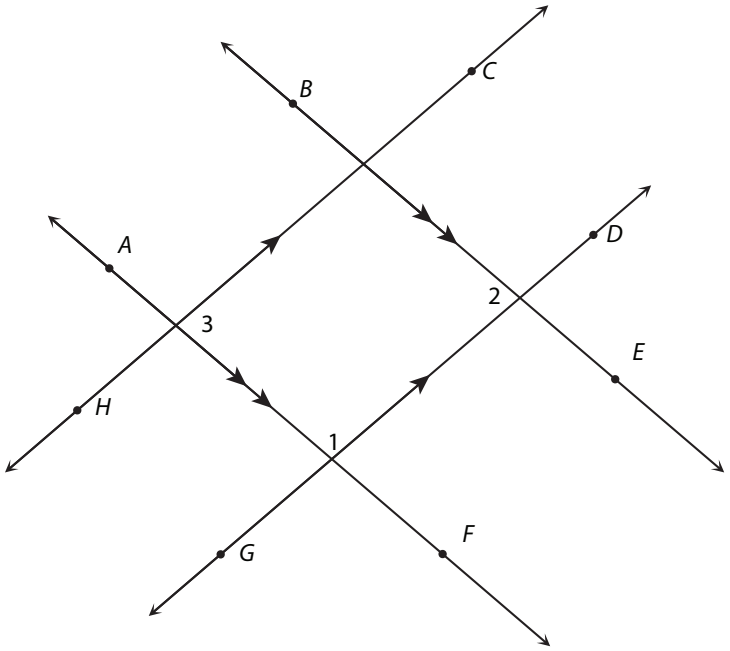
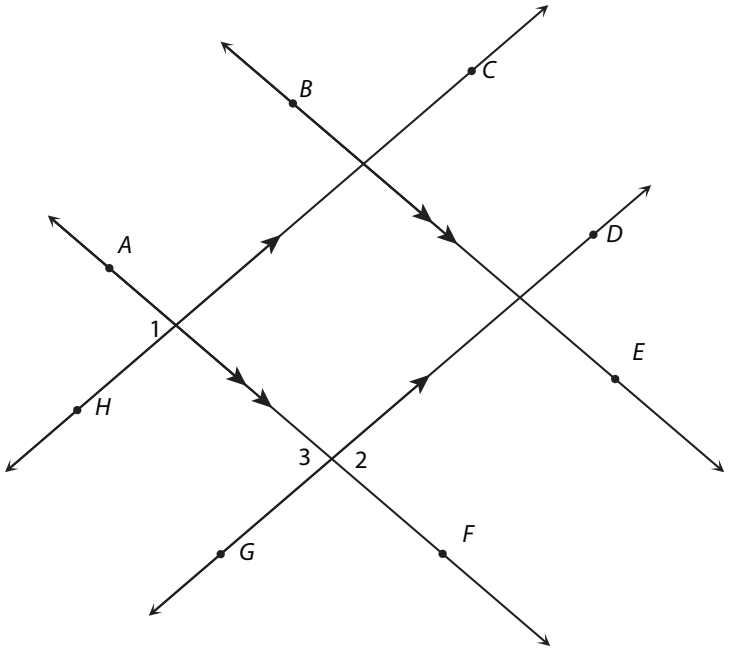
8. Find $m\angle 2$ if $m\angle 1 = 5x + 12$ and $m\angle 3 = 4x - 12$.



a. 112°
b. 20°
c. 180°
d. 68°

9. Find $m\angle 3$ if $m\angle 1 = 3x + 1$ and $m\angle 2 = 2(x + 7)$.



a. 40°
b. 50°
c. 90°
d. 140°

continued

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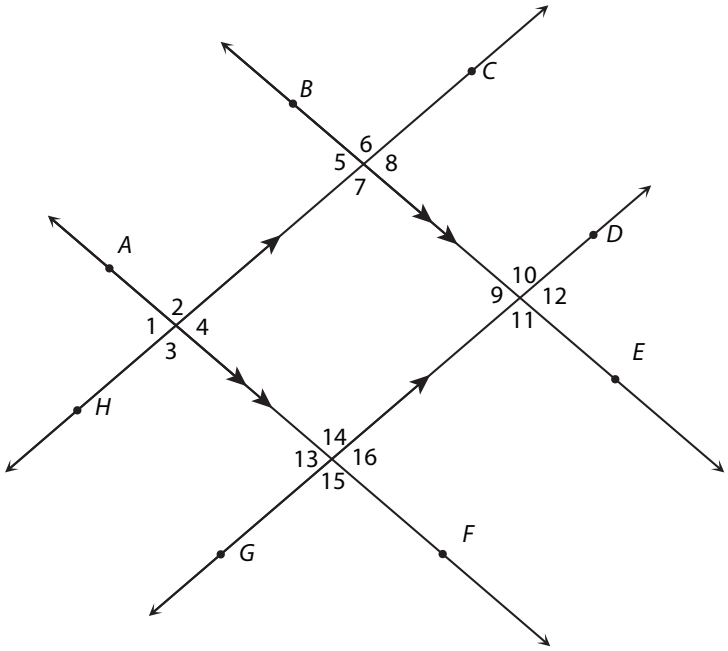
10. Given two sets of parallel lines in the diagram below, what is the relationship between $\angle 6$ and $\angle 15$?



a. $\angle 6$ and $\angle 15$ are congruent right angles because all right angles are congruent and two pairs of intersecting parallel lines are always perpendicular.

b. $\angle 6$ and $\angle 11$ are supplementary since alternate exterior angles are supplementary. Then, by corresponding angles, $\angle 11 \cong \angle 15$. Therefore, $\angle 6$ and $\angle 15$ are supplementary because angles that are supplementary to the same or congruent angles are congruent.

c. $\angle 6 \cong \angle 11$ since they are alternate exterior angles and alternate exterior angles are congruent. Then, by corresponding angles, $\angle 11 \cong \angle 15$. Therefore, $\angle 6 \cong \angle 15$ by the Transitive Property.

d. No relationship exists.

continued

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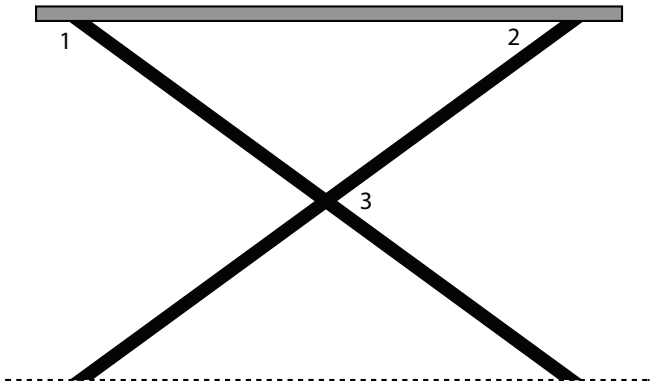
Assessment

Use the following diagram of a table to complete problem 11. The floor is represented by a dashed line.



11. What is $m\angle 3$ given that the tabletop is parallel with the floor, $m\angle 1 = 134$, and $m\angle 2 = 46$? Draw a diagram and justify your reasoning.

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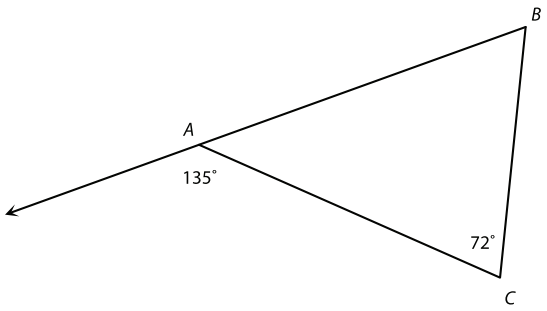
Pre-Assessment

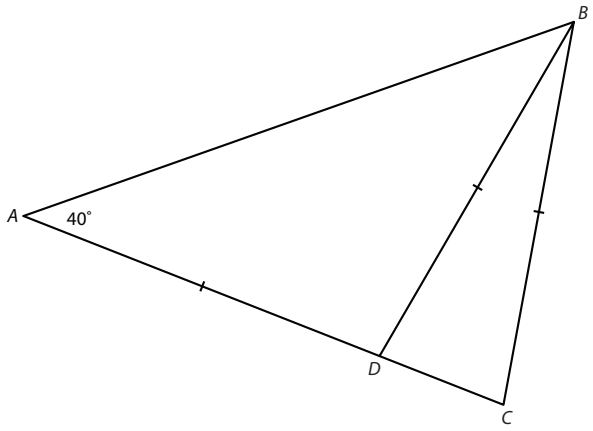
Circle the letter of the best answer.

1. What is the measure of $\angle B$?



a. 45°
b. 63°
c. 108°
d. 117°

2. What is the measure of $\angle C$?



a. 20°
b. 40°
c. 80°
d. 100°

continued

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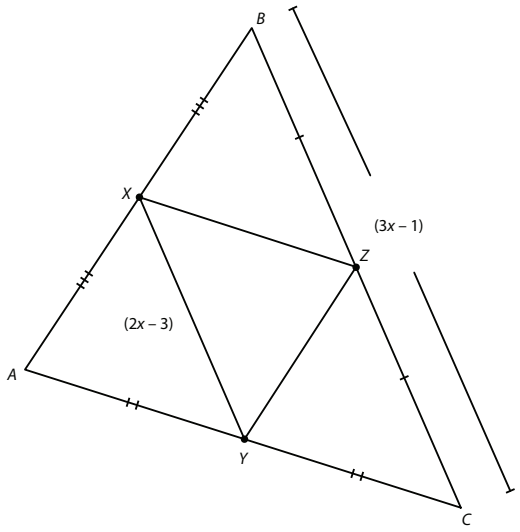
3. If $BC = 3x - 1$ and $XY = 2x - 3$, what is the length of XY?

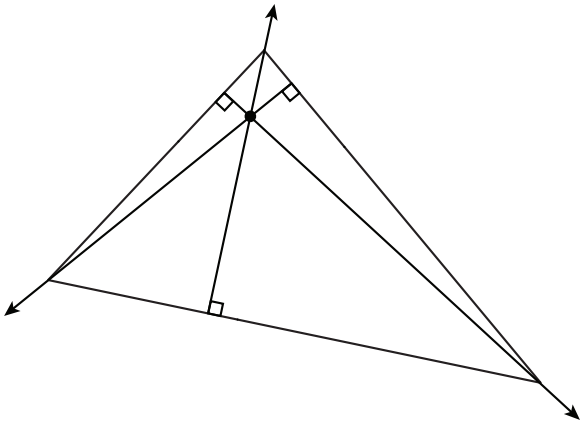


a. 5 units
b. 7 units
c. 14 units
d. 28 units

4. Which centers of a triangle must be inside the triangle?

a. centroid and incenter
b. circumcenter and incenter
c. centroid and orthocenter
d. orthocenter and incenter

5. Which center is in the triangle?



a. centroid
b. circumcenter
c. incenter
d. orthocenter

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Lesson 6: Proving Theorems About Triangles

Instruction

Common Core State Standard

G–CO.10 Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*

SMP	
1 ✓	2 ✓
3 ✓	4 ✓
5	6 ✓
7 ✓	8 ✓

Essential Questions

1. What is the relationship between an interior and exterior angle at the same vertex of a triangle?
2. What is the relationship between an exterior angle and the remote interior angles of a triangle?
3. How are properties of isosceles triangles used and applied?
4. What relationships exist between the midsegment and a triangle?
5. How can the point of concurrency be used to solve problems?
6. What are the similarities and differences among medians, altitudes, perpendicular bisectors, and angle bisectors of a triangle?
7. Is there only one center of a circle?

WORDS TO KNOW

acute triangle	a triangle in which all of the angles are acute (less than 90˚)
base	the side that is opposite the vertex angle of an isosceles triangle
base angle	an angle formed by the base and one of the legs of an isosceles triangle
centroid	the intersection of the medians of a triangle
circumcenter	the intersection of the perpendicular bisectors of a triangle
circumscribed circle	a circle that passes through all vertices of a polygon
concurrent lines	three or more lines that intersect at a single point
coordinate proof	a proof that involves calculations and makes reference to the coordinate plane

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Instruction

equiangular	having equal angles
equilateral triangle	a triangle with all three sides equal in length
exterior angle of a polygon	an angle formed by one side of a polygon and the extension of another side
incenter	the intersection of the angle bisectors of a triangle
inscribed circle	a circle that contains one point from each side of a polygon
interior angle of a polygon	an angle on the inside of a polygon that is formed by two consecutive sides of the polygon
isosceles triangle	a triangle with at least two congruent sides
legs	congruent sides of an isosceles triangle, or the two shorter sides of a right triangle
median of a triangle	the segment joining the vertex to the midpoint of the opposite side
midpoint	a point on a line segment that divides the segment into two equal parts
midpoint formula	a formula that states the midpoint of a segment created by connecting (x_1, y_1) and (x_2, y_2) is given by the formula $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
midsegment	a line segment joining the midpoints of two sides of a triangle
midsegment triangle	the triangle formed when all three of the midsegments of a triangle are connected
obtuse triangle	a triangle with one angle that is obtuse (greater than 90°)
orthocenter	the intersection of the altitudes of a triangle
point of concurrency	a single point of intersection of three or more lines
remote interior angles	interior angles that are not adjacent to an exterior angle
right triangle	a triangle with one angle that measures 90°
scalene triangle	a triangle with no congruent sides
vertex angle	angle formed by the legs of an isosceles triangle

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Recommended Resources

- IXL Learning. "Triangles: Midsegments of Triangles."

  http://www.walch.com/rr/00032

  This interactive website provides a series of problems related to midsegments of triangles and scores them immediately. If the user submits a wrong answer, a description and process for arriving at the correct answer are given.

- Math Open Reference. "Centroid of a Triangle."

  http://www.walch.com/rr/00033

  This website gives a brief explanation of the centroid of a triangle. An interactive applet allows users to change the size and shape of a triangle and observe the changes in the centroid. Also included are links to summaries of each of the triangle centers.

- Math Open Reference. "Isosceles Triangle."

  http://www.walch.com/rr/00034

  This website gives a brief explanation of the properties of isosceles triangles. Also included are links to finding the centers of triangles, as well as an interactive illustration demonstrating isosceles triangle properties.

- Math Warehouse. "Triangles."

  http://www.walch.com/rr/00035

  This website gives a brief explanation of the properties of triangles, including interior and exterior angles. The site also contains an interactive applet that allows users to change the measure of one angle of a triangle and observe the changes in the remaining angles.

- Mathwords.com. "Centers of a Triangle."

  http://www.walch.com/rr/00036

  This website contains a chart of the centers of a triangles and the lines used to find each center, as well as where the center is located on various triangles. Also included are links to interactive applets for each center.

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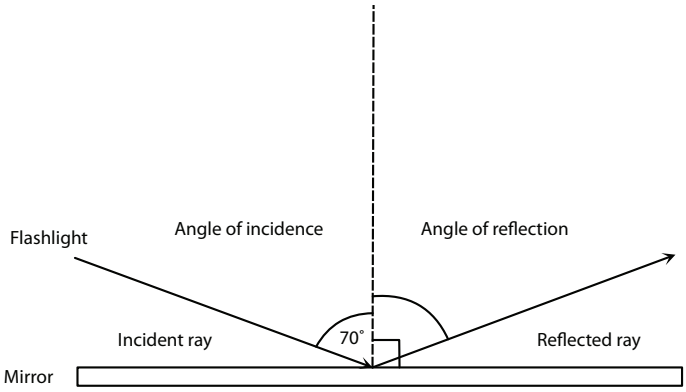
Lesson 5.6.1: Proving the Interior Angle Sum Theorem

Warm-Up 5.6.1

When a beam of light is reflected from a flat surface, the angle of incidence is congruent to the angle of reflection. The diagram shows a ray of light from a flashlight being reflected off a mirror. Use the diagram to answer the questions that follow.



1. What is the measure of the angle of reflection? Explain how you found your answer.

2. What is the measure of the angle created by the mirror and the flashlight beam? Explain how you found your answer.

3. Describe how to determine the measure of the angle created by the mirror and the reflected ray of light.

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Instruction

Lesson 5.6.1: Proving the Interior Angle Sum Theorem

Common Core State Standard

G–CO.10 Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*

Warm-Up 5.6.1 Debrief

1. What is the measure of the angle of reflection? Explain how you found your answer.

   The measure of the angle of reflection is congruent to the angle of incidence.

   The angle of incidence is 70˚, so the angle of reflection is also 70˚.

2. What is the measure of the angle created by the mirror and the flashlight beam? Explain how you found your answer.

   The angle created by the mirror and the flashlight beam is complementary to the angle of incidence because the sum of these angles is 90˚.

   The angle of incidence is 70˚.

   To find the measure of the angle created by the mirror and the flashlight beam, subtract 70˚ from 90˚.

   $90 - 70 = 20$

   The measure of this angle is 20˚.

3. Describe how to determine the measure of the angle created by the mirror and the reflected ray of light.

   There are several ways to determine the measure of the angle created by the mirror and the reflected ray.

   The angle created by the mirror and the reflected ray is congruent to the angle created by the mirror and the incident ray.

   This is true because we are told that the angle of incidence is congruent to the angle of reflection.

   Since the angle of incidence is 70˚, the angle of reflection is also 70˚.

   The angle created by the mirror and the flashlight is the complement of the angle of incidence.

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Instruction

The angle created by the mirror and the reflected ray is also the complement to the angle of reflection.

By subtracting the angle of reflection (70˚) from 90˚, we are able to determine that the measure of the angle created by the mirror and the reflected ray is 20˚.

Connection to the Lesson

- Students will continue to find unknown angle measures using previously learned angle relationships.
- Students' understanding of angle relationships will be expanded to include angles of triangles.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

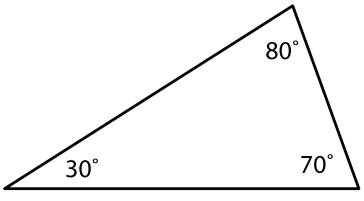
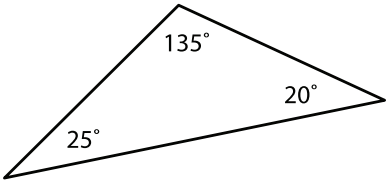
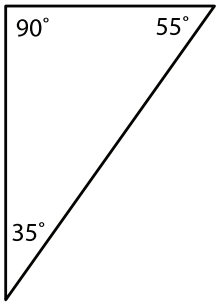
- identifying and using vertical angles, supplementary angles, and complementary angles to find unknown angle measures (G–CO.9)

Introduction

Think of all the different kinds of triangles you can create. What are the similarities among the triangles? What are the differences? Are there properties that hold true for all triangles and others that only hold true for certain types of triangles? This lesson will explore angle relationships of triangles. We will examine the relationships of interior angles of triangles as well as the exterior angles of triangles, and how these relationships can be used to find unknown angle measures.

Key Concepts

- Triangles can be classified by their angle measures or by their side lengths.
- Triangles classified by their angle measures can be acute, obtuse, or right triangles.
- All of the angles of an **acute triangle** are acute, or less than 90˚.
- One angle of an **obtuse triangle** is obtuse, or greater than 90˚.
- A **right triangle** has one angle that measures 90˚.

Acute triangle	Obtuse triangle	Right triangle
		
All angles are less than 90˚.	One angle is greater than 90˚.	One angle measures 90˚.

- Triangles classified by the number of congruent sides can be scalene, isosceles, or equilateral.
- A **scalene triangle** has no congruent sides.
- An **isosceles triangle** has at least two congruent sides.

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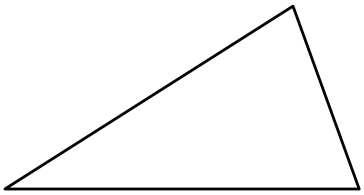
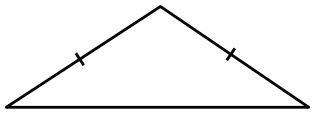
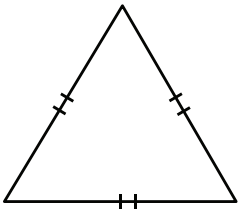
Instruction

- An **equilateral triangle** is a triangle with all three sides equal in length.

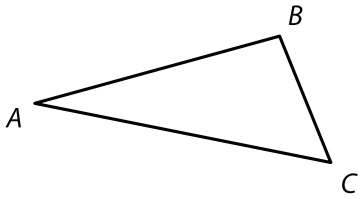
Scalene triangle	Isosceles triangle	Equilateral triangle
		
No congruent sides	At least two congruent sides	Three congruent sides

- It is possible to create many different triangles, but the sum of the angle measures of every triangle is 180°. This is known as the Triangle Sum Theorem.

Theorem

Triangle Sum Theorem

The sum of the angle measures of a triangle is 180°.



$m\angle A + m\angle B + m\angle C = 180^{\circ}$

- The Triangle Sum Theorem can be proven using the Parallel Postulate.
- The Parallel Postulate states that if given a line and a point not on the line, there exists only one line that is parallel to the given line.
- This postulate allows us to create a line parallel to one side of a triangle to prove angle relationships.

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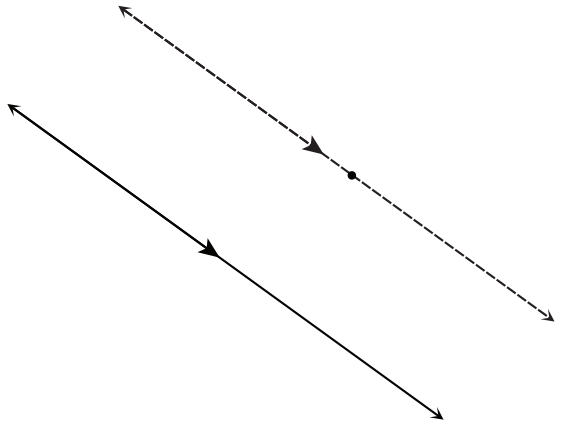
Instruction

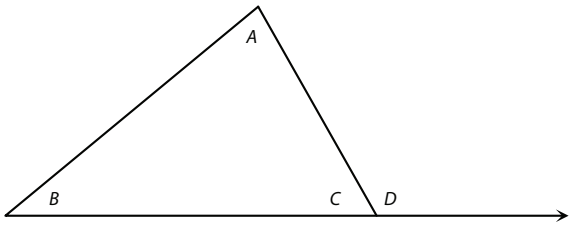
Postulate

Parallel Postulate

Given a line and a point not on it, there exists one and only one straight line that passes through that point and never intersects the first line.



- This postulate can be used to determine a missing angle measure by subtracting the known measures from 180°.
- Most often, triangles are described by what are known as the **interior angles** of triangles (the angles formed by two sides of the triangle). Interior angles can be thought of as the angles within the triangle.
- Triangles also have **exterior angles**, or angles formed by one side of the triangle and the extension of another side.
- The interior angles that are not adjacent to the exterior angle are called the **remote interior angles** of the exterior angle.
- The following illustration shows the differences among interior angles, exterior angles, and remote interior angles.



  - Interior angles: $\angle A$, $\angle B$, and $\angle C$
  - Exterior angle: $\angle D$
  - Remote interior angles of $\angle D$: $\angle A$ and $\angle B$

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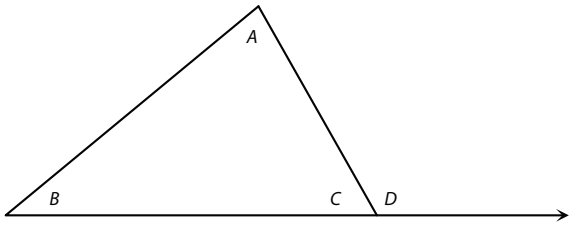
Instruction

- Notice that $\angle C$ and $\angle D$ are supplementary; that is, together they create a line and sum to 180°.
- The measure of an exterior angle is equal to the sum of the measure of its remote interior angles. This is known as the Exterior Angle Theorem.

Theorem

Exterior Angle Theorem

The measure of an exterior angle of a triangle is equal to the sum of the measures of its remote interior angles.



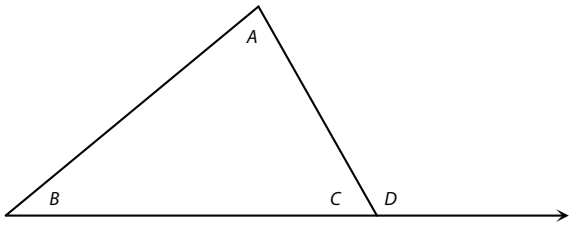
$m\angle D = m\angle A + m\angle B$

- This theorem can also be used to determine a missing angle measure of a triangle.
- The measure of an exterior angle will always be greater than either of the remote interior angles. This is known as the Exterior Angle Inequality Theorem.

Theorem

Exterior Angle Inequality Theorem

If an angle is an exterior angle of a triangle, then its measure is greater than the measure of either of its corresponding remote interior angles.



$m\angle D > m\angle A$

$m\angle D > m\angle B$

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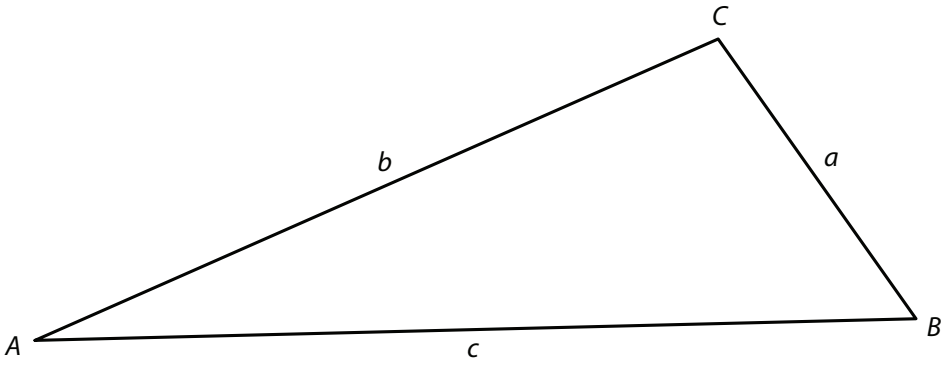
Instruction

- The following theorems are also helpful when finding the measures of missing angles and side lengths.

Theorem
If one side of a triangle is longer than another side, then the angle opposite the longer side has a greater measure than the angle opposite the shorter side.

Theorem
If one angle of a triangle has a greater measure than another angle, then the side opposite the greater angle is longer than the side opposite the lesser angle.



$m\angle A < m\angle B < m\angle C$

$a < b < c$

- The Triangle Sum Theorem and the Exterior Angle Theorem will be proven in this lesson.

Common Errors/Misconceptions

- incorrectly identifying the remote interior angles associated with an exterior angle
- incorrectly applying theorems to determine missing angle measures
- misidentifying or leaving out theorems, postulates, or definitions when writing proofs

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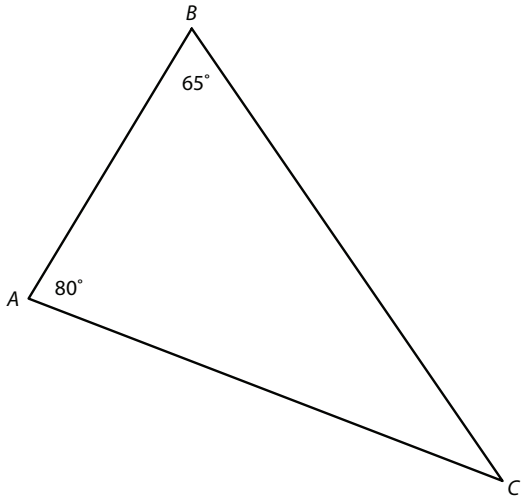
Instruction

Guided Practice 5.6.1

Example 1

Find the measure of $\angle C$.



1. Identify the known information.

   Two measures of the three interior angles are given in the problem.

   $m\angle A = 80°$

   $m\angle B = 65°$

   The measure of $\angle C$ is unknown.

2. Calculate the measure of $\angle C$.

   The sum of the measures of the interior angles of a triangle is 180°.

   Create an equation to solve for the unknown measure of $\angle C$.

$m\angle A + m\angle B + m\angle C = 180$	Triangle Sum Theorem
$(80) + (65) + m\angle C = 180$	Substitute values for $m\angle A$ and $m\angle B$.
$145 + m\angle C = 180$	Simplify.
$m\angle C = 35$	Solve for $m\angle C$.

3. State the answer in degrees.

   The measure of $\angle C$ is 35°.

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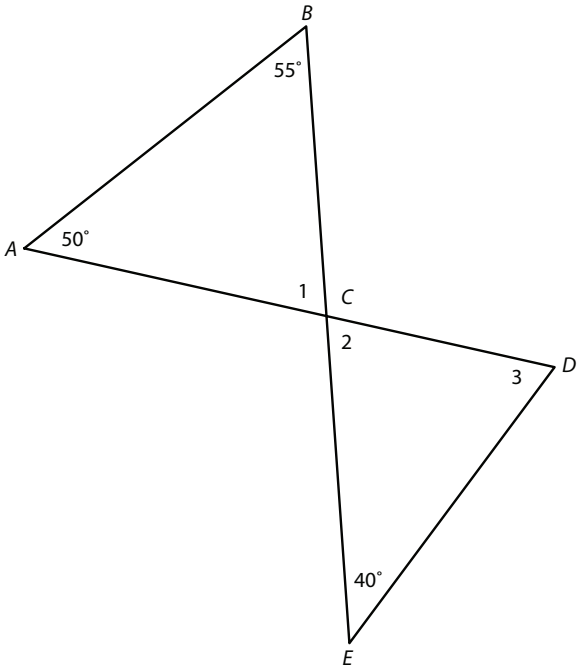
Instruction

Example 2

Find the missing angle measures.



1. Identify the known information.

   The figure contains two triangles, $\triangle ABC$ and $\triangle CDE$.

   The measures of two of the three interior angles of $\triangle ABC$ are given in the problem.

   $m\angle A = 50°$

   $m\angle B = 55°$

   The measure of $\angle BCA$ is unknown.

   The measure of one of the three interior angles of $\triangle CDE$ is given in the problem.

   $m\angle E = 40°$

   The measures of $\angle DCE$ and $\angle D$ are unknown.

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Instruction

2. Calculate the unknown measures.

The sum of the measures of the interior angles of a triangle is 180°.

Create an equation to solve for the unknown measure of $\angle BCA$.

$m\angle A + m\angle B + m\angle BCA = 180$	Triangle Sum Theorem
$(50) + (55) + m\angle BCA = 180$	Substitute values for $m\angle A$ and $m\angle B$.
$105 + m\angle BCA = 180$	Simplify.
$m\angle BCA = 75$	Solve for $m\angle BCA$.

$\angle BCA$ and $\angle DCE$ are vertical angles and are congruent.

$m\angle DCE = m\angle BCA = 75$

Create an equation to solve for the unknown measure of $\angle D$.

$m\angle DCE + m\angle D + m\angle E = 180$	Triangle Sum Theorem
$(75) + m\angle D + (40) = 180$	Substitute values for $m\angle DCE$ and $m\angle E$.
$115 + m\angle D = 180$	Simplify.
$m\angle D = 65$	Solve for $m\angle D$.

3. State the answer in degrees.

The measure of $\angle BCA$ is 75°.

The measure of $\angle DCE$ is 75°.

The measure of $\angle D$ is 65°.

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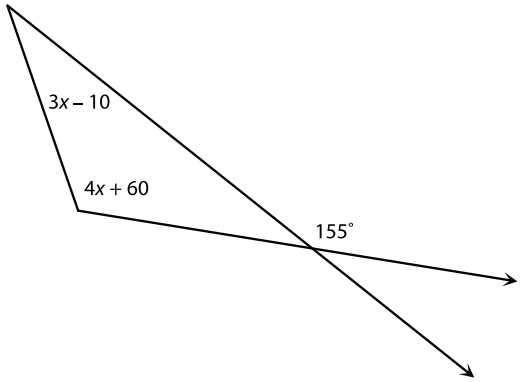
Instruction

Example 3

Find the missing angle measures.



1. Identify the known information.

   One exterior angle measures 155°.

   The measures of the remote interior angles are stated as expressions.

   The value of x is unknown.

2. Calculate the unknown measures.

   The measure of an exterior angle of a triangle is equal to the sum of the measures of its remote interior angles.

   Create an equation to solve for the unknown value of x.

$155 = (3x - 10) + (4x + 60)$	Exterior Angle Theorem
$155 = 7x + 50$	Simplify.
$105 = 7x$	Subtract 50 from both sides.
$x = 15$	Divide both sides by 7.

   The value of x is 15.

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Instruction

3. Determine the unknown measures using the value of x.

$3x - 10$	Unknown measure
$= 3(15) - 10$	Substitute 15 for x.
$= 35$	Simplify.

$4x + 60$	Unknown measure
$= 4(15) + 60$	Substitute 15 for x.
$= 120$	Simplify.

Check that the measures are correct.

$155 = (3x - 10) + (4x + 60)$	Original equation
$155 = (35) + (120)$	Substitute calculated values.
$155 = 155$	Simplify.

4. State the answer in degrees.

The measures of the remote interior angles are 35° and 120°.

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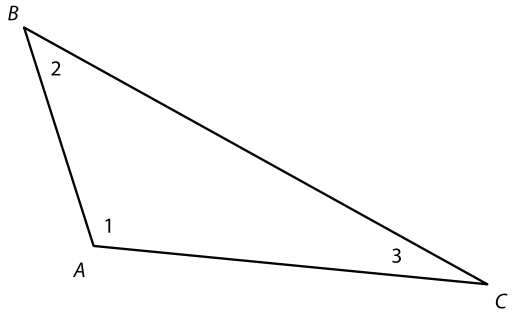
Instruction

Example 4

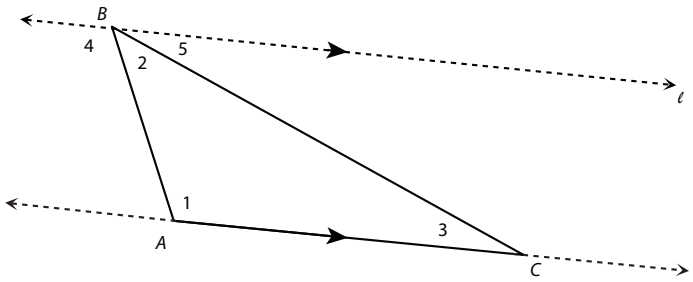
The Triangle Sum Theorem states that the sum of the angle measures of a triangle is 180°. Write a two-column proof of this theorem.



1. State the given information.

   Given: $\triangle ABC$

2. Draw a line, ℓ, through point B that is parallel to $\overline{AC}$ to aid in proving this theorem.



   $\ell \parallel \overline{AC}$

   $\overline{AB}$ and $\overline{BC}$ are both transversals (lines that intersect a set of lines).

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3. State known information about angles created by parallel lines cut by a transversal.

   Alternate interior angles are congruent.

   $\angle 1$ and $\angle 4$ are alternate interior angles and are congruent.

   $\angle 3$ and $\angle 5$ are alternate interior angles and are congruent.

4. Identify known information about congruent angles.

   Congruent angles have the same measure.

   $m\angle 1 = m\angle 4$

   $m\angle 3 = m\angle 5$

5. Write the information in a two-column proof.

Statements	Reasons
1. $\triangle ABC$	1. Given
2. Draw line ℓ through B parallel to $\overline{AC}$.	2. Parallel Postulate
3. $\angle 1 \cong \angle 4$	3. Alternate Interior Angles Theorem
4. $\angle 3 \cong \angle 5$	4. Alternate Interior Angles Theorem
5. $m\angle 1 = m\angle 4$	5. Definition of congruent angles
6. $m\angle 3 = m\angle 5$	6. Definition of congruent angles
7. $m\angle 4 + m\angle 2 + m\angle 5 = 180$	7. Angle Addition Postulate and the definition of a straight angle
8. $m\angle 1 + m\angle 2 + m\angle 3 = 180$	8. Substitution

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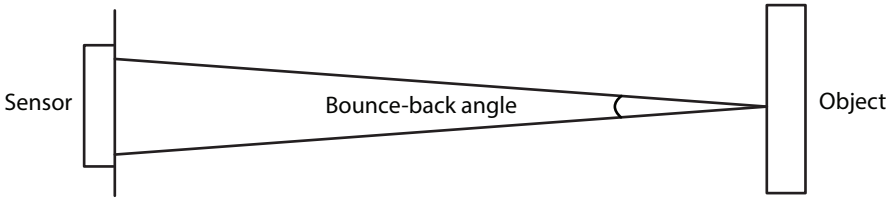
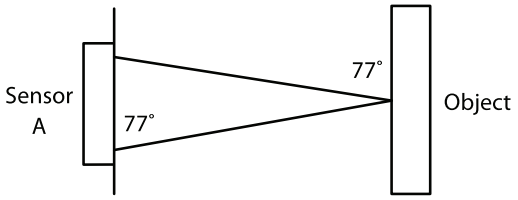
Problem-Based Task 5.6.1: Sensing Distance

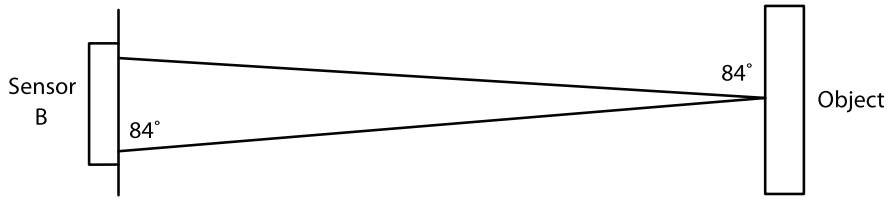
SMP	
1 ✓	2
3	4 ✓
5	6 ✓
7	8

Distance-measuring sensors frequently used in robotics send out a beam of infrared light that hits an object. The beam bounces off the object and returns to the sensor's detector, creating a triangle similar to the one shown.



The angles of the triangle vary depending on the sensor's distance from the object. The sensor uses the angles to determine how far away the object is. As the bounce-back angle increases, the calculated distance becomes more accurate. In each diagram that follows, the sensor is parallel to the object. Which of the sensors calculates a more accurate distance: Sensor A or Sensor B? Explain your reasoning.





Which of the sensors calculates a more accurate distance: Sensor A or Sensor B?

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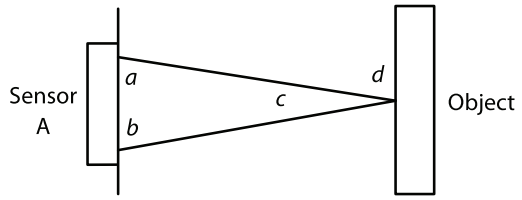
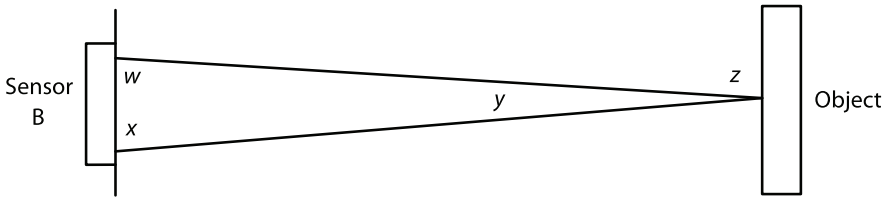
Problem-Based Task 5.6.1: Sensing Distance

Coaching

Use the diagram to answer the questions that follow.





a. What is the relationship between $\angle a$ and $\angle d$?

b. If the measure of $\angle d$ is 77°, what is the measure of $\angle a$?

c. What is the sum of the interior angles of a triangle?

d. What is the measure of $\angle c$?

e. What is the relationship between $\angle z$ and $\angle w$?

f. If the measure of $\angle z$ is 84°, what is the measure of $\angle w$?

g. What is the measure of $\angle y$?

h. Which sensor has a greater bounce-back angle?

i. Which sensor is more accurate?

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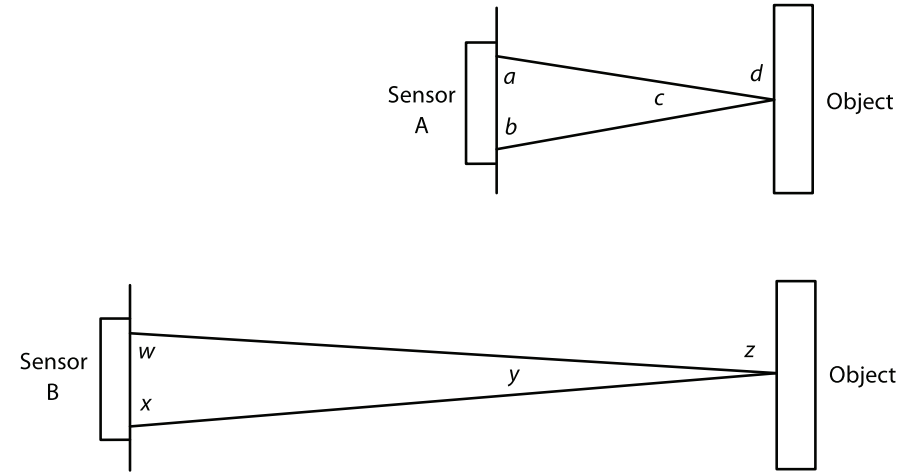
Instruction

Problem-Based Task 5.6.1: Sensing Distance

Coaching Sample Responses



a. What is the relationship between $\angle a$ and $\angle d$?

The sensor and the object are parallel; therefore, the segment from the sensor to the object acts as a transversal.

$\angle a$ and $\angle d$ are alternate interior angles and are congruent.

b. If the measure of $\angle d$ is 77°, what is the measure of $\angle a$?

The angles are congruent; therefore, $\angle a$ is also equal to 77°.

c. What is the sum of the interior angles of a triangle?

The sum of the interior angles of any triangle is 180°.

d. What is the measure of $\angle c$?

To determine the measure of $\angle c$, subtract the sum of $\angle a$ and $\angle b$ from 180.

$180 - (77 + 77) = 26$

The measure of $\angle c$ is 26°.

e. What is the relationship between $\angle z$ and $\angle w$?

$\angle z$ and $\angle w$ are alternate interior angles and are congruent.

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Instruction

f. If the measure of $\angle z$ is 84°, what is the measure of $\angle w$?

The angles are congruent; therefore $\angle w$ is also equal to 84°.

g. What is the measure of $\angle y$?

To determine the measure of $\angle y$, subtract the sum of $\angle w$ and $\angle x$ from 180.

Since the triangle is an isosceles triangle, $m\angle x = m\angle w = 84$.

$180 - (84 + 84) = 12$

The measure of $\angle y$ is 12°.

h. Which sensor has a greater bounce-back angle?

The bounce-back angle for Sensor A is 26°.

The bounce-back angle for Sensor B is 12°.

Sensor A has a greater bounce-back angle.

i. Which sensor is more accurate?

As the bounce-back angle increases, the calculated distance is more accurate.

Sensor A has a greater bounce-back angle; therefore, it is more accurate.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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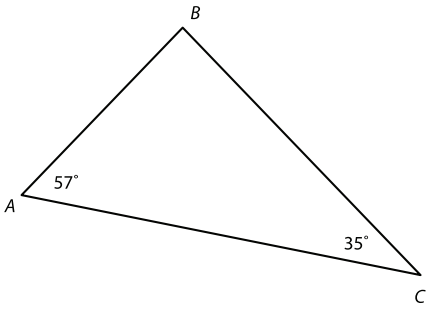
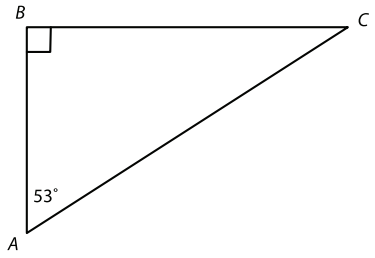
Practice 5.6.1: Proving the Interior Angle Sum Theorem

Use what you know about the sums of the interior and exterior angles of triangles to determine the measure of each identified angle.

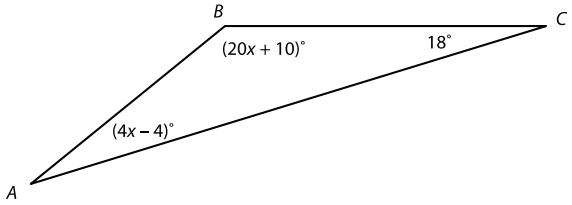
1. Find $m\angle B$.



2. Find $m\angle C$.



3. Find $m\angle A$ and $m\angle B$.



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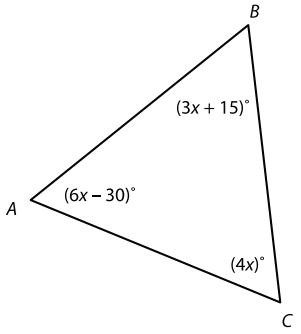
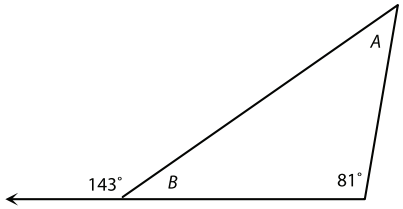
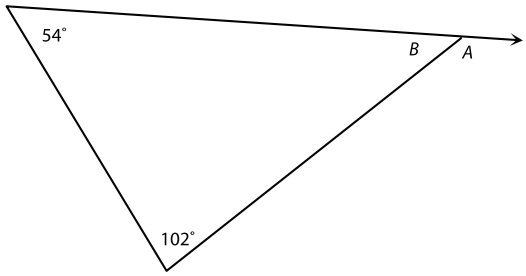
4. Find $m\angle A$, $m\angle B$, and $m\angle C$.



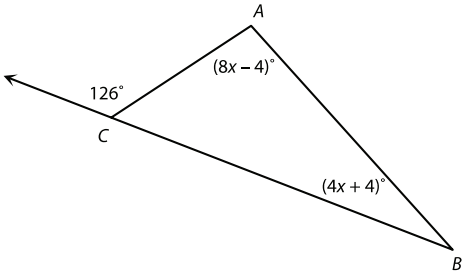
5. Find $m\angle A$ and $m\angle B$.



6. Find $m\angle A$ and $m\angle B$.



7. Find $m\angle CAB$ and $m\angle ABC$.



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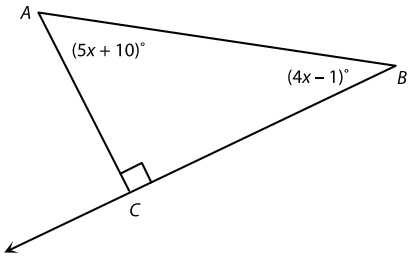
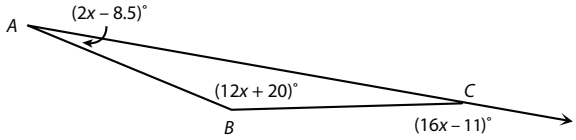
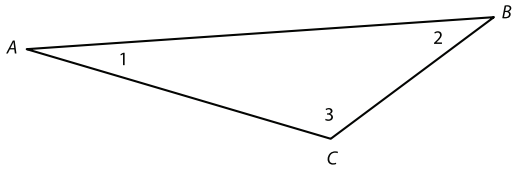
8. Find $m\angle CAB$ and $m\angle ABC$.



9. Find $m\angle CAB$ and $m\angle ABC$.



10. The Triangle Sum Theorem states that the sum of the angle measures of a triangle is 180°. Write a paragraph proof of this theorem, referring to the diagram that follows.



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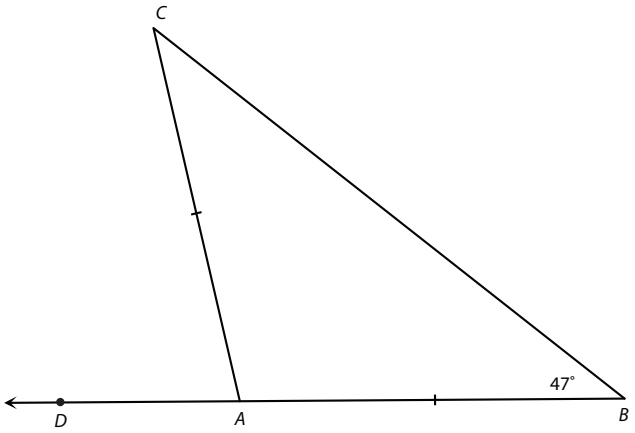
Lesson 5.6.2: Proving Theorems About Isosceles Triangles

Warm-Up 5.6.2

In the diagram, a captain is aboard a ship at point *B*, a lighthouse is located at point *C*, and the ship is sailing in the direction of $\overrightarrow{BD}$. There is a buoy floating in the water at point *A*. The captain measured $\angle CBD$ to be 47°. The ship is as far from the buoy as the buoy is from the lighthouse, so the measures of *AB* and *AC* are equal.



1. What kind of triangle is $\triangle ABC$? Explain your reasoning.

2. What is the measure of $\angle ACB$?

3. What is the measure of $\angle CAD$?

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Instruction

Lesson 5.6.2: Proving Theorems About Isosceles Triangles

Common Core State Standard

G–CO.10 Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*

Warm-Up 5.6.2 Debrief

1. What kind of triangle is $\triangle ABC$? Explain your reasoning.

   $\triangle ABC$ has two congruent sides, so by definition, $\triangle ABC$ is an isosceles triangle.

2. What is the measure of $\angle ACB$?

   By definition, an isosceles triangle has two congruent sides and two congruent angles. If AB is congruent to AC, then the measure of $\angle C$ is congruent to the measure of $\angle B$.

   $\angle B$ measures 47°; therefore, $\angle C$ also measures 47°.

3. What is the measure of $\angle CAD$?

   $\angle C$ and $\angle B$ are remote interior angles to $\angle CAD$, an exterior angle.

   The measure of an exterior angle is equal to the sum of the measure of its remote interior angles.

   The sum of the interior angles is found by adding $m\angle C$ and $m\angle B$.

   $47 + 47 = 94$

   The measure of $\angle CAD$ is 94°.

Connection to the Lesson

- Students will expand upon their understanding of isosceles triangles to find unknown side lengths and angle measures.
- Students will continue to use the Exterior Angle Theorem to find the unknown measures.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

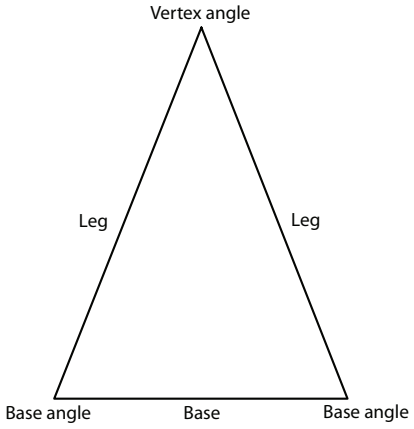
- classifying triangles (5.G.4)
- identifying and using vertical angles, supplementary angles, and complementary angles to find unknown angle measures (G–CO.9)
- applying the Triangle Sum Theorem and the Exterior Angle Theorem to find unknown measures of triangles (G–SRT.5)
- justifying congruence of triangles (G–SRT.5)

Introduction

Isosceles triangles can be seen throughout our daily lives in structures, supports, architectural details, and even bicycle frames. Isosceles triangles are a distinct classification of triangles with unique characteristics and parts that have specific names. In this lesson, we will explore the qualities of isosceles triangles.

Key Concepts

- Isosceles triangles have at least two congruent sides, called **legs**.
- The angle created by the intersection of the legs is called the **vertex angle**.
- The side that is opposite the vertex angle is the **base** of the isosceles triangle.
- Each of the remaining angles is referred to as a **base angle**. The intersection of a leg and the base of the isosceles triangle creates a base angle.



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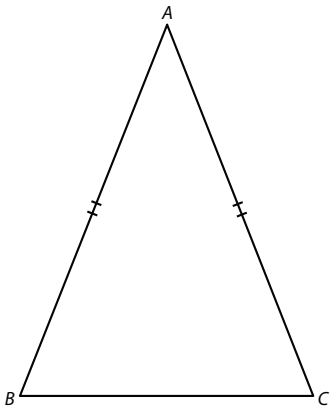
Instruction

- The following theorem is true of every isosceles triangle.

Theorem

Isosceles Triangle Theorem

If two sides of a triangle are congruent, then the angles opposite the congruent sides are congruent.



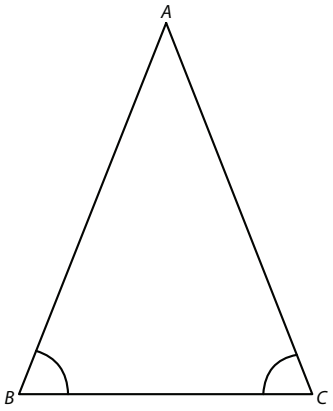
$m\angle B \cong m\angle C$

- If the Isosceles Triangle Theorem is reversed, then that statement is also true.
- This is known as the converse of the Isosceles Triangle Theorem.

Theorem

Converse of the Isosceles Triangle Theorem

If two angles of a triangle are congruent, then the sides opposite those angles are congruent.



$\overline{AB} \cong \overline{AC}$

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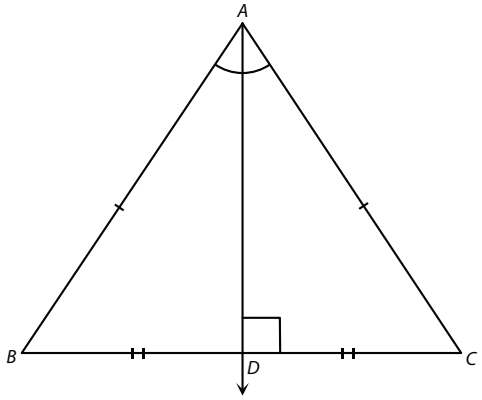
Instruction

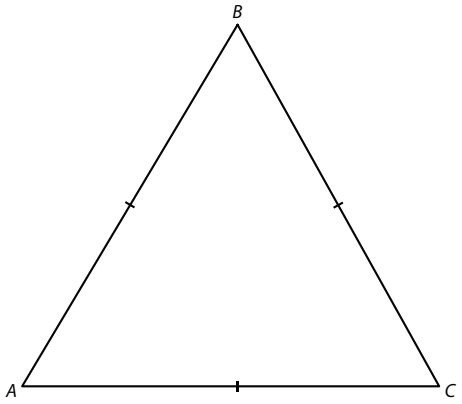
- If the vertex angle of an isosceles triangle is bisected, the bisector is perpendicular to the base, creating two right triangles.
- In the diagram that follows, D is the midpoint of $\overline{BC}$.



- Equilateral triangles are a special type of isosceles triangle, for which each side of the triangle is congruent.
- If all sides of a triangle are congruent, then all angles have the same measure.

Theorem

If a triangle is equilateral then it is **equiangular**, or has equal angles.



$\angle A \cong \angle B \cong \angle C$

- Each angle of an equilateral triangle measures 60° $(180 \div 3 = 60)$.

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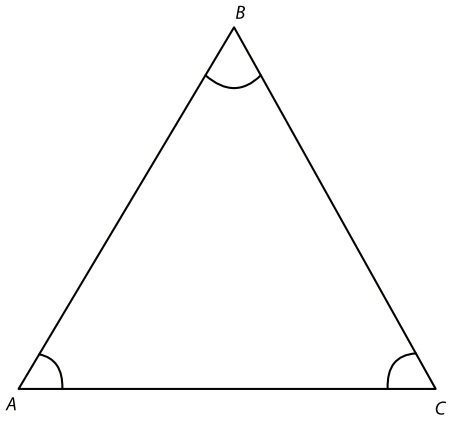
Instruction

- Conversely, if a triangle has equal angles, it is equilateral.

Theorem

If a triangle is equiangular, then it is equilateral.



$\overline{AB} \cong \overline{BC} \cong \overline{AC}$

- These theorems and properties can be used to solve many triangle problems.

Common Errors/Misconceptions

- incorrectly identifying parts of isosceles triangles
- not identifying equilateral triangles as having the same properties of isosceles triangles
- incorrectly setting up and solving equations to find unknown measures of triangles
- misidentifying or leaving out theorems, postulates, or definitions when writing proofs

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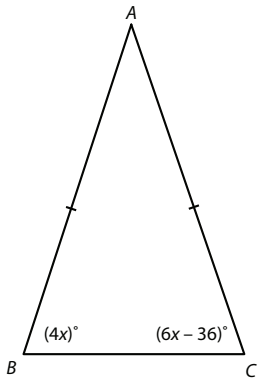
Guided Practice 5.6.2

Example 1

Find the measure of each angle of $\triangle ABC$.



1. Identify the congruent sides and angles.

   The legs of an isosceles triangle are congruent; therefore, $\overline{AB} \cong \overline{AC}$.

   The base of $\triangle ABC$ is $\overline{BC}$.

   $\angle B$ and $\angle C$ are base angles and are congruent.

2. Calculate the value of x.

   Congruent angles have the same measure.

   Create an equation.

$m\angle B = m\angle C$	The measures of base angles of isosceles triangles are equal.
$(4x) = (6x - 36)$	Substitute values for $m\angle B$ and $m\angle C$.
$-2x = -36$	Subtract $6x$ from both sides.
$x = 18$	Divide both sides by –2.

   The value of x is 18.

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Instruction

3. Calculate each angle measure.

$m\angle B = 4x$	Given
$m\angle B = 4(18)$	Substitute the known value of x into the expression for $m\angle B$.
$m\angle B = 72$	Solve for $m\angle B$.
$m\angle C = 6x - 36$	Given
$m\angle C = 6(18) - 36$	Substitute the known value of x into the expression for $m\angle C$.
$m\angle C = 72$	Solve for $m\angle C$.
$m\angle A + m\angle B + m\angle C = 180$	The sum of the angles of a triangle is 180°.
$m\angle A + (72) + (72) = 180$	Substitute the known values.
$m\angle A = 36$	Solve for $m\angle A$.

4. Summarize your findings.

$m\angle A = 36°$

$m\angle B = 72°$

$m\angle C = 72°$

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Instruction

Example 2

Determine whether $\triangle ABC$ with vertices A (–4, 5), B (–1, –4), and C (5, 2) is an isosceles triangle. If it is isosceles, name a pair of congruent angles.

1. Use the distance formula to calculate the length of each side.

Calculate the length of $\overline{AB}$.

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$AB=\sqrt{[(-1)-(-4)]^2+[(-4)-(5)]^2}$$ Substitute (–4, 5) and (–1, –4) for (x_1, y_1) and (x_2, y_2).

$$AB=\sqrt{(3)^2+(-9)^2}$$ Simplify.

$$AB=\sqrt{9+81}$$

$$AB=\sqrt{90}=3\sqrt{10}$$

Calculate the length of $\overline{BC}$.

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$BC=\sqrt{[(5)-(-1)]^2+[(2)-(-4)]^2}$$ Substitute (–1, –4) and (5, 2) for (x_1, y_1) and (x_2, y_2).

$$BC=\sqrt{(6)^2+(6)^2}$$ Simplify.

$$BC=\sqrt{36+36}$$

$$BC=\sqrt{72}=6\sqrt{2}$$

Calculate the length of $\overline{AC}$.

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$AC=\sqrt{[(5)-(-4)]^2+[(2)-(5)]^2}$$ Substitute (–4, 5) and (5, 2) for (x_1, y_1) and (x_2, y_2).

$$AC=\sqrt{(9)^2+(-3)^2}$$ Simplify.

$$AC=\sqrt{81+9}$$

$$AC=\sqrt{90}=3\sqrt{10}$$

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Instruction

2. Determine if the triangle is isosceles.

   A triangle with at least two congruent sides is an isosceles triangle.

   $\overline{AB} \cong \overline{AC}$, so $\triangle ABC$ is isosceles.

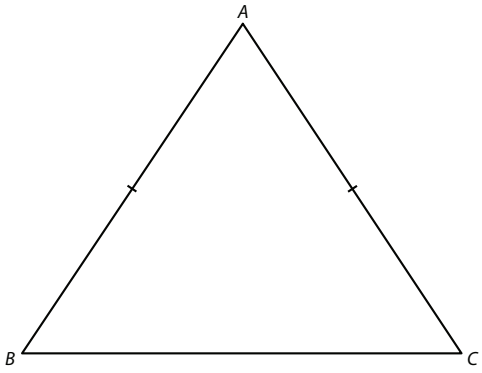
3. Identify congruent angles.

   If two sides of a triangle are congruent, then the angles opposite the sides are congruent.

   $\angle B \cong \angle C$

Example 3

Given $\overline{AB} \cong \overline{AC}$, prove that $\angle B \cong \angle C$.



1. State the given information.

   $\overline{AB} \cong \overline{AC}$

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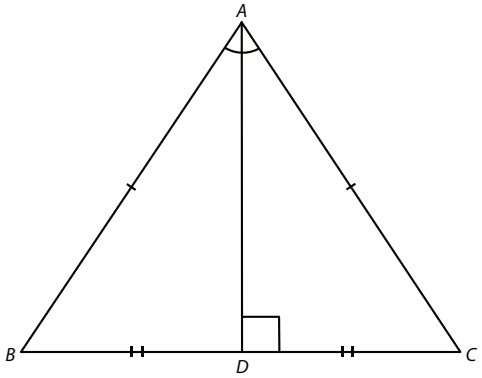
Instruction

2. Draw the angle bisector of $\angle A$ and extend it to $\overline{BC}$, creating the perpendicular bisector of $\overline{BC}$. Label the point of intersection D.

   Indicate congruent sides.



   $\angle B$ and $\angle C$ are congruent corresponding parts.

3. Write the information in a two-column proof.

Statements	Reasons
1. $\overline{AB} \cong \overline{AC}$	1. Given
2. Draw the angle bisector of $\angle A$ and extend it to $\overline{BC}$, creating a perpendicular bisector of $\overline{BC}$ and the midpoint of $\overline{BC}$.	2. There is exactly one line through two points.
3. $\overline{BD} \cong \overline{BC}$	3. Definition of midpoint
4. $\overline{AD} \cong \overline{AD}$	4. Reflexive Property
5. $\triangle ABD \cong \triangle ACD$	5. SSS Congruence Statement
6. $\angle B \cong \angle C$	6. Corresponding Parts of Congruent Triangles are Congruent

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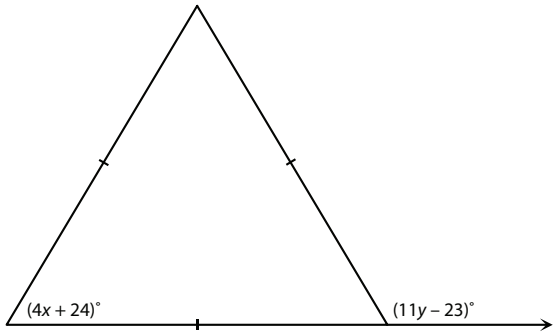
Example 4

Find the values of x and y.



1. Make observations about the figure.

   The triangle in the diagram has three congruent sides.

   A triangle with three congruent sides is equilateral.

   Equilateral triangles are also equiangular.

   The measure of each angle of an equilateral triangle is 60°.

   An exterior angle is also included in the diagram.

   The measure of an exterior angle is the supplement of the adjacent interior angle.

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2. Determine the value of x.

The measure of each angle of an equilateral triangle is 60°.

Create and solve an equation for x using this information.

$4x + 24 = 60$	Equation
$4x = 36$	Subtract 24 from both sides.
$x = 9$	Divide both sides by 4.

The value of x is 9.

3. Determine the value of y.

The exterior angle is the supplement to the interior angle.

The interior angle is 60° by the properties of equilateral triangles.

The sum of the measures of an exterior angle and interior angle pair is 180.

Create and solve an equation for y using this information.

$11y - 23 + 60 = 180$	Equation
$11y + 37 = 180$	Simplify.
$11y = 143$	Subtract 37 from both sides.
$y = 13$	Divide both sides by 11.

The value of y is 13.

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Instruction

Example 5

$\triangle ABC$ is equilateral. Prove that it is equiangular.

1. State the given information.

   $\triangle ABC$ is an equilateral triangle.

2. Plan the proof.

   Equilateral triangles are also isosceles triangles. Isosceles triangles have at least two congruent sides.

   $\overline{AB} \cong \overline{BC}$

   $\angle A$ and $\angle C$ are base angles in relation to $\overline{AB}$ and $\overline{BC}$.

   $\angle A \cong \angle C$ because of the Isosceles Triangle Theorem.

   $\overline{BC} \cong \overline{AC}$

   $\angle A$ and $\angle B$ are base angles in relation to $\overline{BC}$ and $\overline{AC}$.

   $\angle A \cong \angle B$ because of the Isosceles Triangle Theorem.

   By the Transitive Property, $\angle A \cong \angle B \cong \angle C$; so, $\triangle ABC$ is equiangular.

3. Write the information in a paragraph proof.

   Since $\triangle ABC$ is equilateral, $\overline{AB} \cong \overline{BC}$ and $\overline{BC} \cong \overline{AC}$. By the Isosceles Triangle Theorem, $\angle A \cong \angle C$ and $\angle A \cong \angle B$. By the Transitive Property, $\angle A \cong \angle B \cong \angle C$; therefore, $\triangle ABC$ is equiangular.

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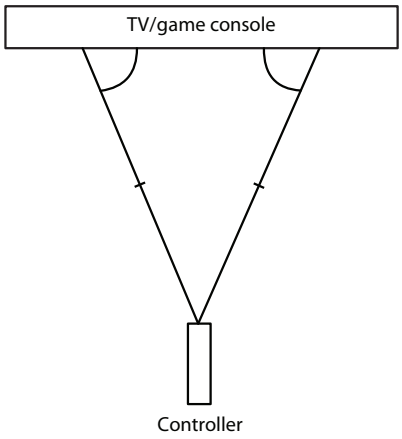
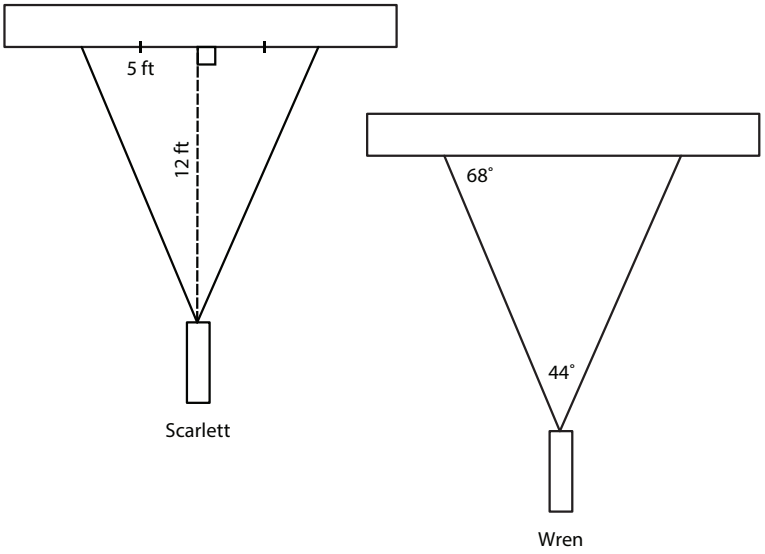
Problem-Based Task 5.6.2: Calibrating Consoles

SMP
1 ✓ 2
3 ✓ 4
5 6
7 ✓ 8

In order for a wireless game console controller to work correctly, users must first calibrate the system. The user stands with the controller in front of the system and allows the system to recognize the controller's location. Optimal performance is achieved when an isosceles triangle is created, as shown.



Scarlett and Wren have calibrated their game console controllers as shown in the following diagram. Did either of the girls calibrate her game console controller correctly? Explain your reasoning.



Did either of the girls calibrate her game console controller correctly?

SWB p. 171

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Problem-Based Task 5.6.2: Calibrating Consoles

Coaching

a. What is the definition of an isosceles triangle?

b. How can you determine the length of each side of the triangle Scarlett created?

c. What is the length of each side of the triangle Scarlett created?

d. Is the triangle Scarlett created an isosceles triangle?

e. How can you determine the measure of the third angle of the triangle Wren created?

f. What is the measure of the third angle of the triangle Wren created?

g. Is the triangle Wren created an isosceles triangle?

h. Did either girl calibrate her game console controller correctly? If so, who?

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Instruction

Problem-Based Task 5.6.2: Calibrating Consoles

Coaching Sample Responses

a. What is the definition of an isosceles triangle?

Isosceles triangles have at least two sides that are congruent and at least two base angles that are congruent.

b. How can you determine the length of each side of the triangle Scarlett created?

The triangle Scarlett created is made up of two smaller triangles.

The smaller triangles are congruent because of the Side-Angle-Side Congruence Statement.

Use the Pythagorean Theorem to calculate the lengths of the sides of the triangle Scarlett created.

c. What is the length of each side of the triangle Scarlett created?

$a^2 + b^2 = c^2$	Pythagorean Theorem
$5^2 + 12^2 = c^2$	Substitute 5 and 12 for the lengths of a and b.
$25 + 144 = c^2$	Simplify.
$169 = c^2$	
$c = \sqrt{169} = 13$	

Remember that the two smaller triangles are congruent.

Two sides of the triangle Scarlett created are each 13 feet long, and the third side is 10 feet long.

d. Is the triangle Scarlett created an isosceles triangle?

The triangle Scarlett created is an isosceles triangle because two of the sides are equal in length.

e. How can you determine the measure of the third angle of the triangle Wren created?

The sum of the interior angles of a triangle is 180°.

Subtract the given angles from 180 to determine the unknown angle measure.

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f. What is the measure of the third angle of the triangle Wren created?

$180 - (68 + 44) = 68$

The third angle measure of the triangle Wren created is 68°.

g. Is the triangle Wren created an isosceles triangle?

The triangle Wren created is an isosceles triangle because two of the angles are congruent.

h. Did either girl calibrate her game console controller correctly? If so, who?

Both girls calibrated their game console controllers correctly because the triangle created by each girl is an isosceles triangle.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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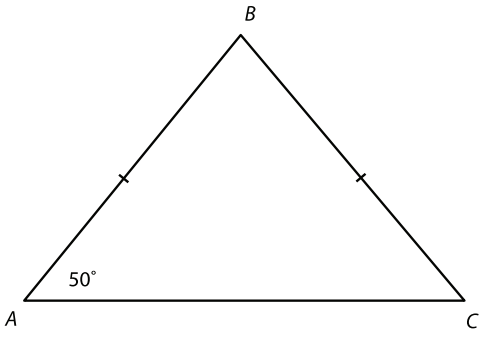
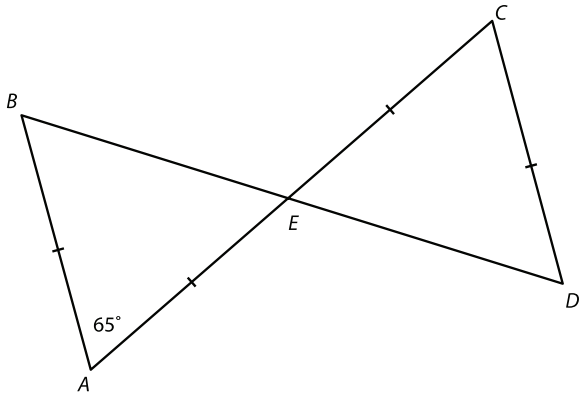
Practice 5.6.2: Proving Theorems About Isosceles Triangles

Use what you know about isosceles triangles to find each angle measure.

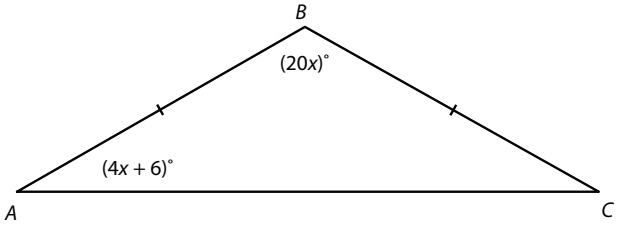
1. $m\angle B$ and $m\angle C$



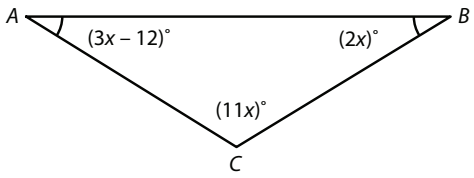
2. $m\angle B$, $m\angle C$, and $m\angle D$



3. $m\angle A$, $m\angle B$, and $m\angle C$



4. $m\angle A$, $m\angle B$, and $m\angle C$



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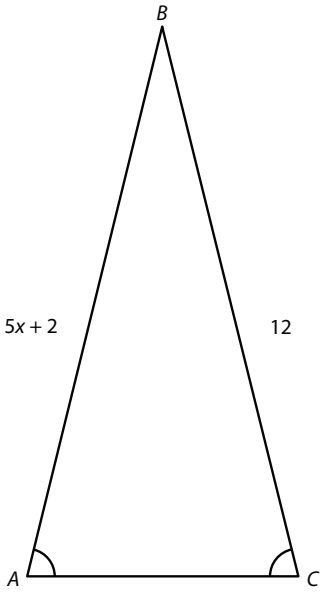
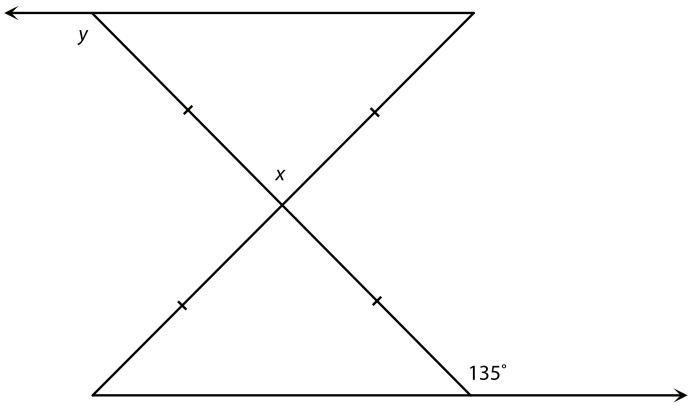
Find each value using the given information.

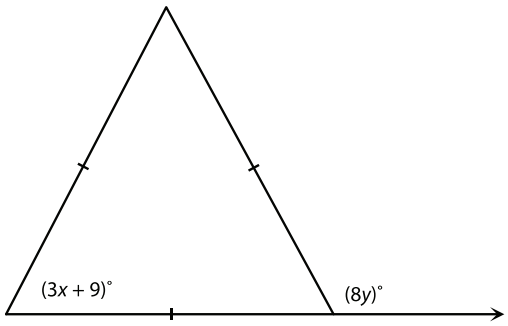
5. x



6. $m\angle x$ and $m\angle y$



7. x and y



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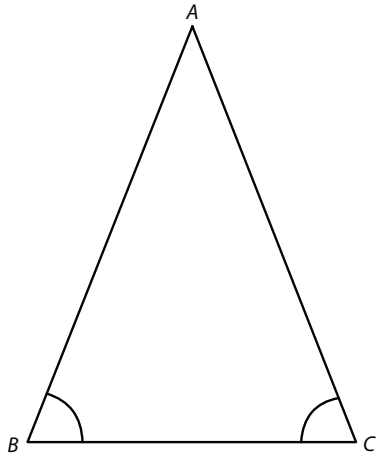
For problems 8 and 9, use the given vertices to determine whether $\triangle ABC$ is an isosceles triangle. If it is isosceles, name a pair of congruent angles.

8. A (0, 0), B (–8, 0), C (–4, –6)

9. A (1, 1), B (4, 4), C (6, –2)

Use your knowledge of triangle theorems to complete problem 10.

10. The converse of the Isosceles Triangle Theorem states that if two angles of a triangle are congruent, then the sides opposite those angles are congruent. Write a two-column proof of this statement.



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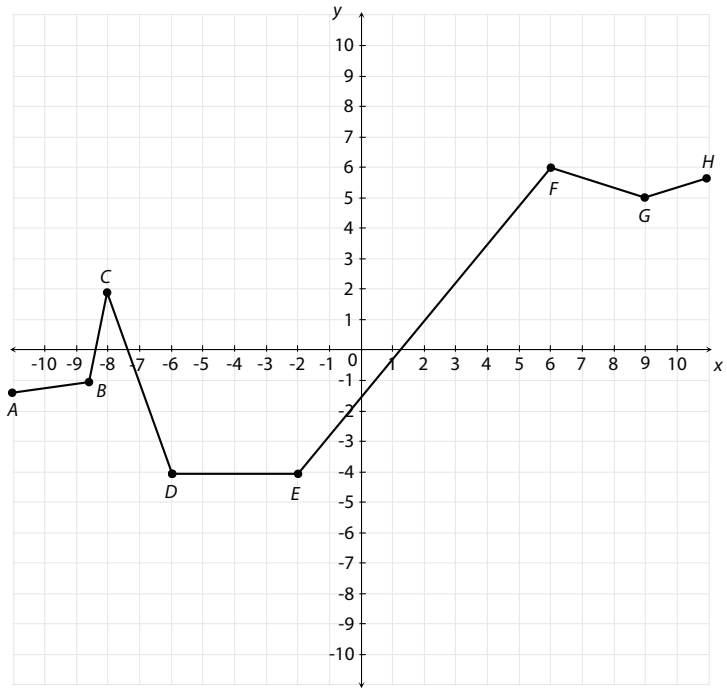
Lesson 5.6.3: Proving the Midsegment of a Triangle

Warm-Up 5.6.3

A portion of the Los Angeles marathon course is mapped on the coordinate plane shown. Each unit represents 600 feet. The locations of major landmarks along the course are labeled by points.



1. Officials need to verify the distance between points *C* and *D*. What is this distance?

2. A water station is planned midway between points *E* and *F*. What is the location of the water station?

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Lesson 5.6.3: Proving the Midsegment of a Triangle

Common Core State Standard

G–CO.10 Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*

Warm-Up 5.6.3 Debrief

1. Officials need to verify the distance between points *C* and *D*. What is this distance?

   Point *C* is located at (–8, 2) and point *D* is located at (–6, –4).

   To calculate the distance between points *C* and *D*, use the distance formula.

   $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance formula

   $d = \sqrt{[(-6) - (-8)]^2 + [(-4) - (2)]^2}$ Substitute (–8, 2) and (–6, –4) for (x_1, y_1) and (x_2, y_2).

   $d = \sqrt{(2)^2 + (-6)^2}$ Simplify.

   $d = \sqrt{4 + 36}$

   $d = \sqrt{40} \approx 6.3$ units

   Each unit on the map is 600 feet.

   To find the actual distance between points *C* and *D*, multiply the number of units by 600.

   6.3 • 600 = 3780

   The distance between points *C* and *D* is 3,780 feet.

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Instruction

2. A water station is planned midway between points *E* and *F*. What is the location of the water station?

   Point *E* is located at (–2, –4) and point *F* is located at (6, 6).

   To calculate the point that is midway between points *E* and *F*, use the midpoint formula.

   $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ Midpoint formula

   $\left(\frac{(-2)+(6)}{2}, \frac{(-4)+(6)}{2}\right)$ Substitute (–2, –4) and (6, 6) for (x_1, y_1) and (x_2, y_2).

   $\left(\frac{4}{2}, \frac{2}{2}\right)$ Simplify.

   (2, 1)

   The midpoint of $\overline{EF}$ is (2, 1).

   The water station is planned at the point (2, 1).

Connection to the Lesson

- Students will expand upon their understanding of distance and midpoints of segments and apply this knowledge to triangles.

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Prerequisite Skills

This lesson requires the use of the following skills:

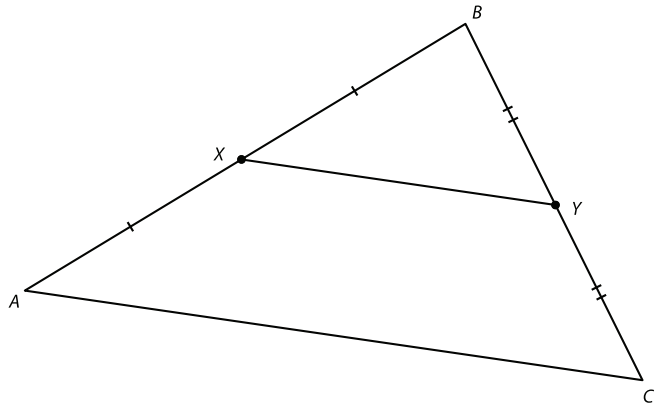
- calculating the midpoint of a segment (G–GPE.6)
- calculating slopes of lines (8.F.4)
- determining if lines are parallel based on slopes (G–GPE.5)

Introduction

Triangles are typically thought of as simplistic shapes constructed of three angles and three segments. As we continue to explore this shape, we discover there are many more properties and qualities than we may have first imagined. Each property and quality, such as the midsegment of a triangle, acts as a tool for solving problems.

Key Concepts

- The **midpoint** is the point on a line segment that divides the segment into two equal parts.
- The midpoint of a segment created by connecting (x_1, y_1) and (x_2, y_2) is given by the **midpoint formula**, $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- A **midsegment** of a triangle is a line segment that joins the midpoints of two sides of a triangle.



- In the previous diagram, the midpoint of $\overline{AB}$ is X.
- The midpoint of $\overline{BC}$ is Y.
- A midsegment of $\triangle ABC$ is $\overline{XY}$.

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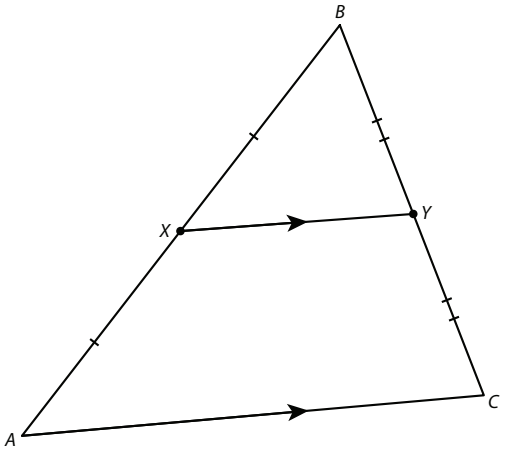
Instruction

- The midsegment of a triangle is parallel to the third side of the triangle and is half as long as the third side. This is known as the Triangle Midsegment Theorem.

Theorem

Triangle Midsegment Theorem

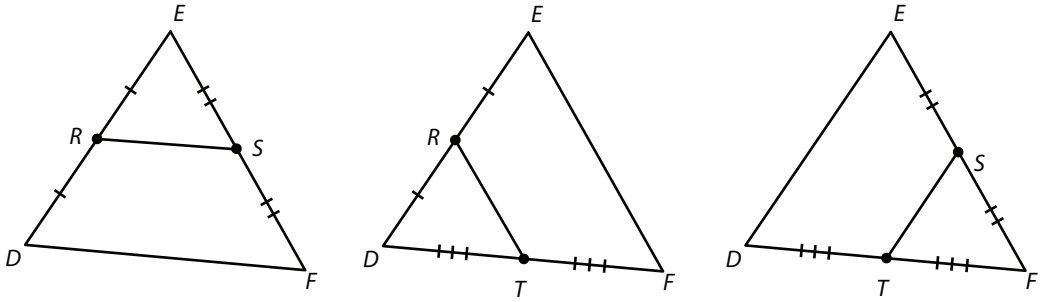
A midsegment of a triangle is parallel to the third side and is half as long.



$\overline{AC} \parallel \overline{XY}$

$XY = \frac{1}{2}AC$

- Every triangle has three midsegments.



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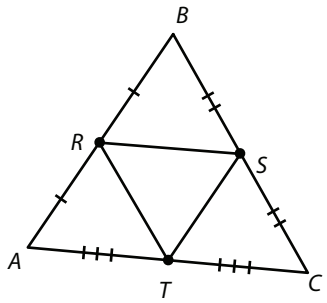
- Connecting all three of the midsegments of a triangle creates a **midsegment triangle**.



- In the previous diagram, $\triangle ABC \sim \triangle TSR$.
- Note that $\overline{AB}$ refers to the actual segment, and AB refers to the length of the segment.
- **Coordinate proofs**, which are proofs that involve calculations and make reference to the coordinate plane, are often used to prove many theorems.

Common Errors/Misconceptions

- assuming a segment that is parallel to the third side of a triangle is a midsegment
- incorrectly writing and solving equations to determine lengths
- incorrectly calculating slope
- incorrectly applying the Triangle Midsegment Theorem to solve problems
- misidentifying or leaving out theorems, postulates, or definitions when writing proofs

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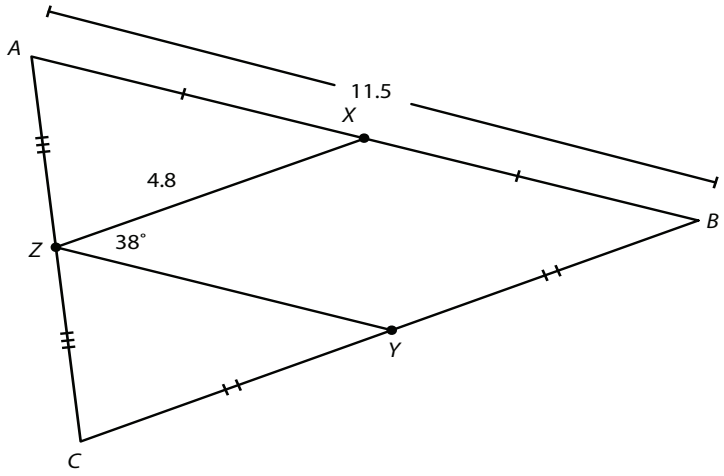
Guided Practice 5.6.3

Example 1

Find the lengths of $\overline{BC}$ and $\overline{YZ}$ and the measure of $\angle AXZ$.



1. Identify the known information.

   Tick marks indicate that X is the midpoint of $\overline{AB}$, Y is the midpoint of $\overline{BC}$, and Z is the midpoint of $\overline{AC}$.

   $\overline{XZ}$ and $\overline{YZ}$ are midsegments of $\triangle ABC$.

2. Calculate the length of $\overline{BC}$.

   $\overline{XZ}$ is the midsegment that is parallel to $\overline{BC}$.

   The length of $\overline{XZ}$ is $\frac{1}{2}$ the length of $\overline{BC}$.

   $XZ = \frac{1}{2}BC$ Triangle Midsegment Theorem

   $(4.8) = \frac{1}{2}BC$ Substitute 4.8 for XZ.

   $BC = 9.6$ Solve for BC.

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3. Calculate the length of $\overline{YZ}$.

$\overline{YZ}$ is the midsegment parallel to $\overline{AB}$.

The length of $\overline{YZ}$ is $\frac{1}{2}$ the length of $\overline{AB}$.

$YZ = \frac{1}{2}AB$	Triangle Midsegment Theorem
$YZ = \frac{1}{2}(11.5)$	Substitute 11.5 for AB.
$YZ = 5.75$	Solve for YZ.

4. Calculate the measure of $\angle AXZ$.

$\overline{YZ} \parallel \overline{AB}$	Triangle Midsegment Theorem
$m\angle AXZ = m\angle XZY$	Alternate Interior Angles Theorem
$m\angle AXZ = 38$	

5. State the answers.

$\overline{BC}$ is 9.6 units long. $\overline{YZ}$ is 5.75 units long. $m\angle AXZ$ is 38°.

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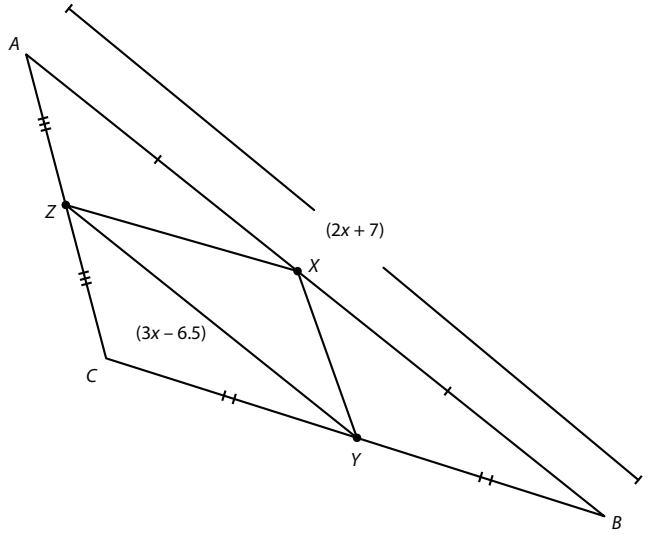
Example 2

If $AB = 2x + 7$ and $YZ = 3x - 6.5$, what is the length of $\overline{AB}$?



1. Identify the known information.

   Tick marks indicate that X is the midpoint of $\overline{AB}$, Y is the midpoint of $\overline{BC}$, and Z is the midpoint of $\overline{AC}$.

   $\overline{XY}$, $\overline{XZ}$, and $\overline{YZ}$ are the midsegments of $\triangle ABC$.

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Instruction

2. Calculate the length of $\overline{AB}$.

The length of $\overline{YZ}$ is $\frac{1}{2}$ the length of $\overline{AB}$.

$YZ = \frac{1}{2}AB$ Triangle Midsegment Theorem

$3x - 6.5 = \frac{1}{2}(2x + 7)$ Substitute values for YZ and AB.

$3x - 6.5 = x + 3.5$ Solve for x.

$2x - 6.5 = 3.5$

$2x = 10$

$x = 5$

Use the value of x to find the length of $\overline{AB}$.

$AB = 2x + 7$

$AB = 2(5) + 7$

$AB = 10 + 7$

$AB = 17$

The length of $\overline{AB}$ is 17 units.

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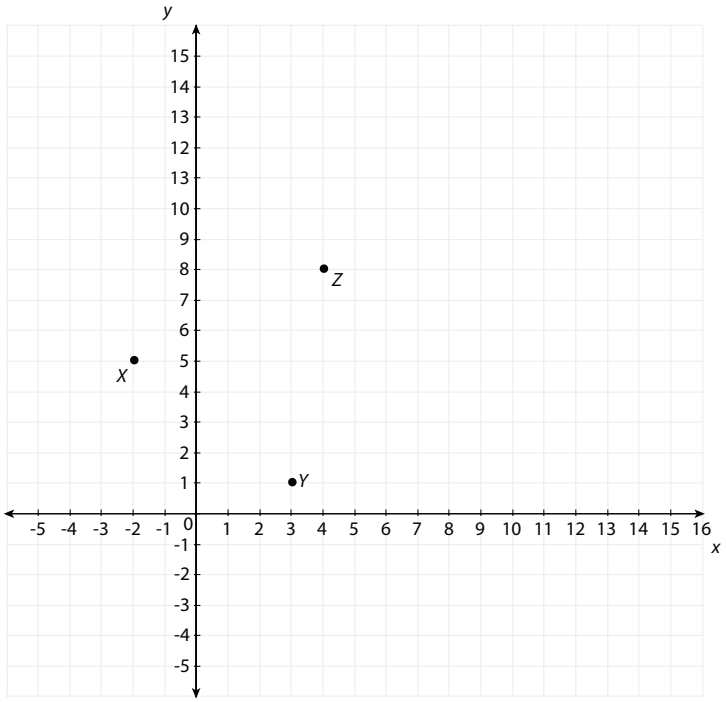
Example 3

The midpoints of a triangle are $X(-2, 5)$, $Y(3, 1)$, and $Z(4, 8)$. Find the coordinates of the vertices of the triangle.

1. Plot the midpoints on a coordinate plane.



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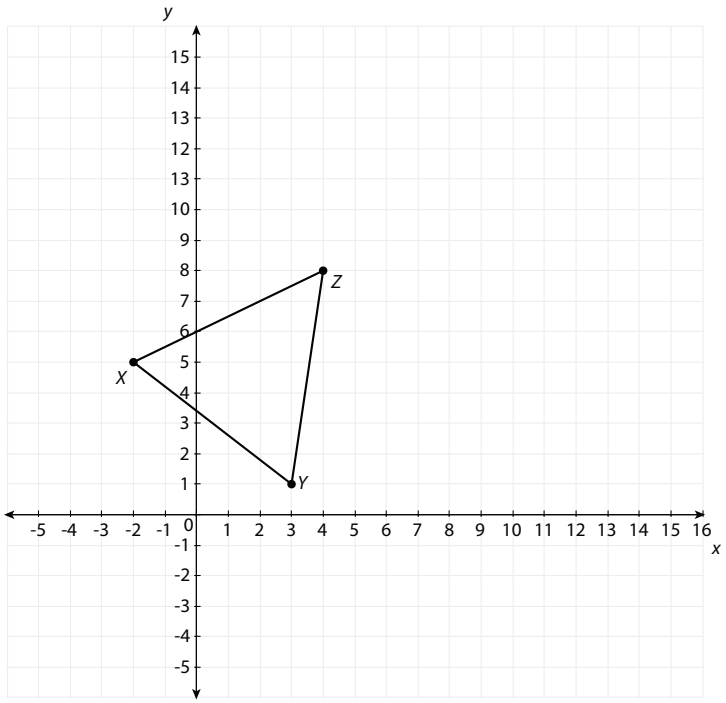
Instruction

2. Connect the midpoints to form the midsegments $\overline{XY}$, $\overline{YZ}$, and $\overline{XZ}$.



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3. Calculate the slope of each midsegment.

Calculate the slope of $\overline{XY}$.

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope formula

$m = \frac{(1)-(5)}{(3)-(-2)}$ Substitute (–2, 5) and (3, 1) for (x_1, y_1) and (x_2, y_2).

$m = \frac{-4}{5}$ Simplify.

The slope of $\overline{XY}$ is $-\frac{4}{5}$.

Calculate the slope of $\overline{YZ}$.

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope formula

$m = \frac{(8)-(1)}{(4)-(3)}$ Substitute (3, 1) and (4, 8) for (x_1, y_1) and (x_2, y_2).

$m = \frac{7}{1} = 7$ Simplify.

The slope of $\overline{YZ}$ is 7.

Calculate the slope of $\overline{XZ}$.

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope formula

$m = \frac{(8)-(5)}{(4)-(-2)}$ Substitute (–2, 5) and (4, 8) for (x_1, y_1) and (x_2, y_2).

$m = \frac{3}{6} = \frac{1}{2}$ Simplify.

The slope of $\overline{XZ}$ is $\frac{1}{2}$.

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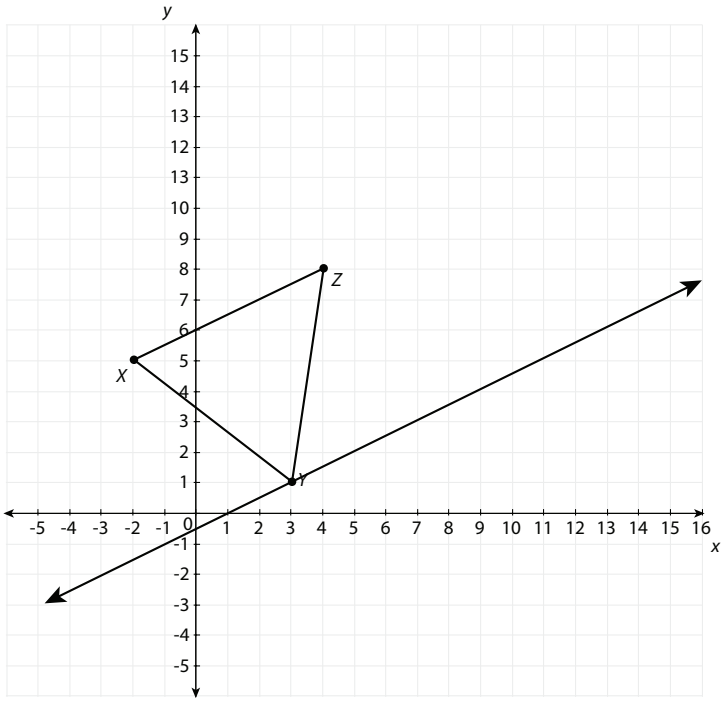
4. Draw the lines that contain the midpoints.

   The endpoints of each midsegment are the midpoints of the larger triangle.

   Each midsegment is also parallel to the opposite side.

   The slope of $\overline{XZ}$ is $\frac{1}{2}$.

   From point *Y*, draw a line that has a slope of $\frac{1}{2}$.



(*continued*)

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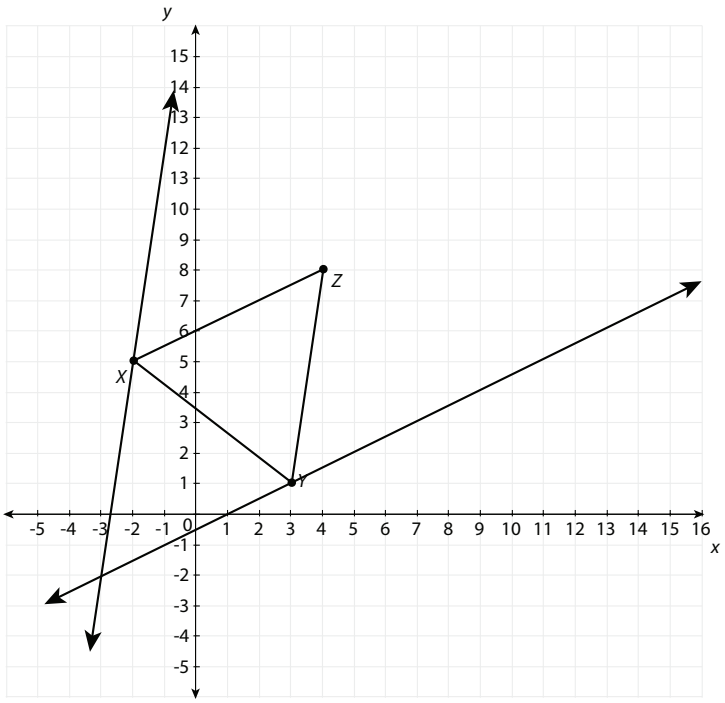
The slope of $\overline{YZ}$ is 7.

From point X, draw a line that has a slope of 7.



(*continued*)

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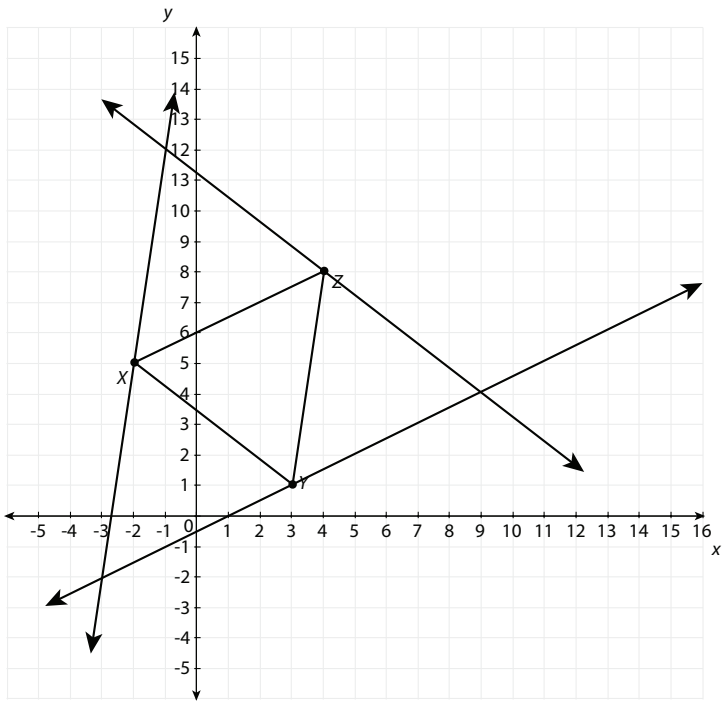
Instruction

The slope of $\overline{XY}$ is $-\frac{4}{5}$.

From point Z, draw a line that has a slope of $-\frac{4}{5}$.



The intersections of the lines form the vertices of the triangle.

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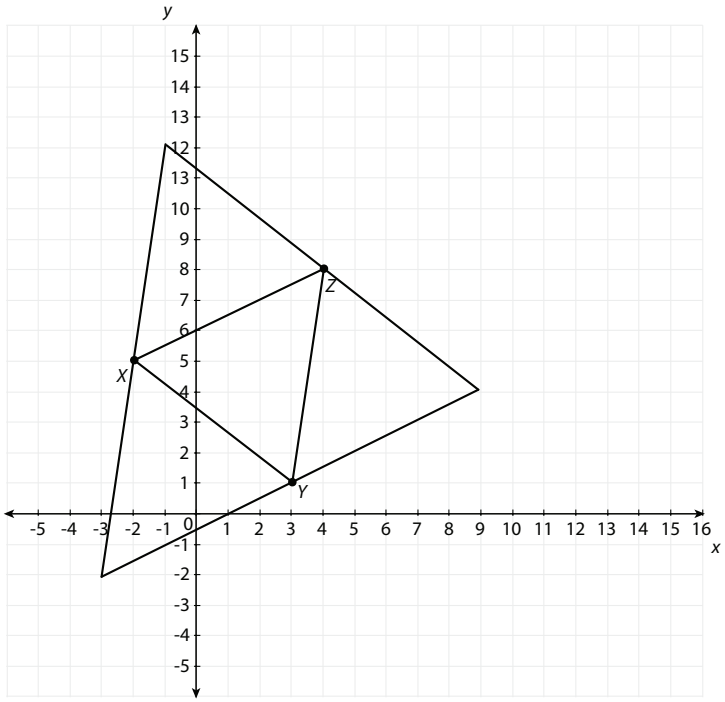
Instruction

5. Determine the vertices of the triangle.



The vertices of the triangle are (–3, –2), (9, 4), and (–1, 12).

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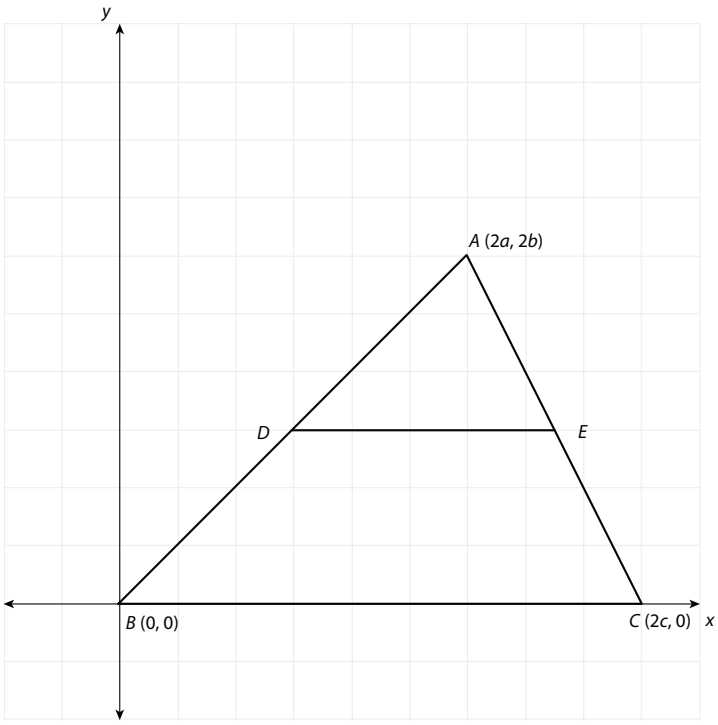
Instruction

Example 4

Write a coordinate proof of the Triangle Midsegment Theorem using the graph shown.



1. Identify the known information.

   According to the graph, $\triangle ABC$ has vertices located at $(2a, 2b)$, $(0, 0)$, and $(2c, 0)$.

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Instruction

2. Find the coordinates of D, which is the midpoint of $\overline{AB}$.

Use the midpoint formula to find the midpoint of $\overline{AB}$.

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ Midpoint formula

$\left(\frac{2a+0}{2}, \frac{2b+0}{2}\right)$ Substitute $(2a, 2b)$ and $(0, 0)$ for (x_1, y_1) and (x_2, y_2).

$\left(\frac{2a}{2}, \frac{2b}{2}\right)$ Simplify.

(a, b)

The midpoint of $\overline{AB}$ is (a, b).

3. Find the coordinates of E, which is the midpoint of $\overline{AC}$.

Use the midpoint formula to find the midpoint of $\overline{AC}$.

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ Midpoint formula

$\left(\frac{2a+2c}{2}, \frac{2b+0}{2}\right)$ Substitute $(2a, 2b)$ and $(2c, 0)$ for (x_1, y_1) and (x_2, y_2).

$\left(\frac{2a+2c}{2}, \frac{2b}{2}\right)$ Simplify.

$(a + c, b)$

The midpoint of $\overline{AC}$ is $(a + c, b)$.

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Instruction

4. Calculate the slope of the midsegment.

Use the slope formula to calculate the slope of $\overline{DE}$.

$m=\frac{y_2-y_1}{x_2-x_1}$	Slope formula
$m=\frac{(b)-(b)}{(a+c)-(a)}$	Substitute (a, b) and $(a+c, b)$ for (x_1, y_1) and (x_2, y_2).
$m=\frac{0}{c}=0$	Simplify.

The slope of $\overline{DE}$ is 0.

5. Calculate the slope of $\overline{BC}$.

Use the slope formula to calculate the slope of $\overline{BC}$.

$m=\frac{y_2-y_1}{x_2-x_1}$	Slope formula
$m=\frac{(0)-(0)}{(2c)-(0)}$	Substitute $(0, 0)$ and $(2c, 0)$ for (x_1, y_1) and (x_2, y_2).
$m=\frac{0}{2c}=0$	Simplify.

The slope of $\overline{BC}$ is 0.

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Instruction

6. Calculate the length of $\overline{DE}$.

Use the distance formula to calculate the length of the segment.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance formula

$d = \sqrt{[(a+c)-(a)]^2 + [(b)-(b)]^2}$ Substitute (a, b) and $(a + c, b)$ for (x_1, y_1) and (x_2, y_2).

$d = \sqrt{c^2 + 0}$ Simplify.

$d = \sqrt{c^2}$

$d = c$

The length of $\overline{DE}$ is c units.

7. Calculate the length of $\overline{BC}$.

Use the distance formula to calculate the length of the segment.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance formula

$d = \sqrt{[(2c)-(0)]^2 + [(0)-(0)]^2}$ Substitute $(0, 0)$ and $(2c, 0)$ for (x_1, y_1) and (x_2, y_2).

$d = \sqrt{(2c)^2 + 0}$ Simplify.

$d = \sqrt{4c^2}$

$d = 2c$

The distance of $\overline{BC}$ is $2c$ units.

8. State your conclusion.

The slopes of $\overline{DE}$ and $\overline{BC}$ are both 0; therefore, the segments are parallel.

The length of $\overline{DE}$ is c units and the length of $\overline{BC}$ is $2c$ units; therefore, $\overline{DE}$ is $\frac{1}{2}$ the length of $\overline{BC}$.

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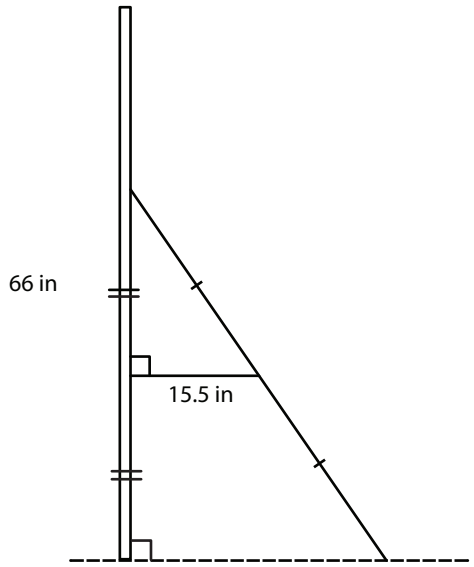
Problem-Based Task 5.6.3: Life-Size Support

SMP	
1 ✓	2
3	4 ✓
5	6 ✓
7	8

Alex is assembling a life-size cardboard cutout of his favorite musician. The package contains three pieces of cardboard: the cutout, which measures 66 inches; the horizontal support, which measures 15.5 inches; and the angled support, which attaches to the back of the cutout at a point that is $\frac{2}{3}$ the length of the cutout from the floor. The horizontal support acts as the midsegment connecting the cutout and the angled support. What is the length of the angled support rounded to the nearest inch?



What is the length of the angled support rounded to the nearest inch?

SWB p. 185

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Problem-Based Task 5.6.3: Life-Size Support

Coaching

a. If the cutout measures 66 inches, what is $\frac{2}{3}$ the length of the cutout?

b. The horizontal support acts as a midsegment connecting the cutout to the angled support. How does the length of the midsegment compare to the side it is parallel to?

c. If the horizontal support is 15.5 inches, what is the distance from the bottom of the angled support to the base of the cutout?

d. How can you determine the length of the angled support?

e. What is the length of the angled support rounded to the nearest inch?

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Instruction

Problem-Based Task 5.6.3: Life-Size Support

Coaching Sample Responses

a. If the cutout measures 66 inches, what is $\frac{2}{3}$ the length of the cutout?

To calculate $\frac{2}{3}$ the length of the cutout, multiply $\frac{2}{3}$ by 66.

$$66 \bullet \frac{2}{3} = 44$$

$\frac{2}{3}$ the length of the cutout is 44 inches.

b. The horizontal support acts as a midsegment connecting the cutout to the angled support. How does the length of the midsegment compare to the side it is parallel to?

The midsegment connects the midpoints of two sides of the triangle, is parallel to the third side, and is half as long as the third side.

c. If the horizontal support is 15.5 inches, what is the distance from the bottom of the angled support to the base of the cutout?

The horizontal support is a midsegment of the triangle; therefore, the support is half as long as the distance from the angled support to the base of the cutout.

Multiply 15.5 by 2 to find this distance.

$15.5 \bullet 2 = 31$

The distance from the angled support to the base of the cutout is 31 inches.

d. How can you determine the length of the angled support?

The cutout and the angled support create a right triangle, for which the angled support is the hypotenuse.

To find the length of the hypotenuse, use the Pythagorean Theorem.

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e. What is the length of the angled support rounded to the nearest inch?

$a^2 + b^2 = c^2$	Pythagorean Theorem
$31^2 + 44^2 = c^2$	Substitute 31 and 44 for the lengths of a and b.
$961 + 1936 = c^2$	Simplify.
$2897 = c^2$	

$c = \sqrt{2897} \approx 54$ inches

The length of the angled support rounded to the nearest inch is 54 inches.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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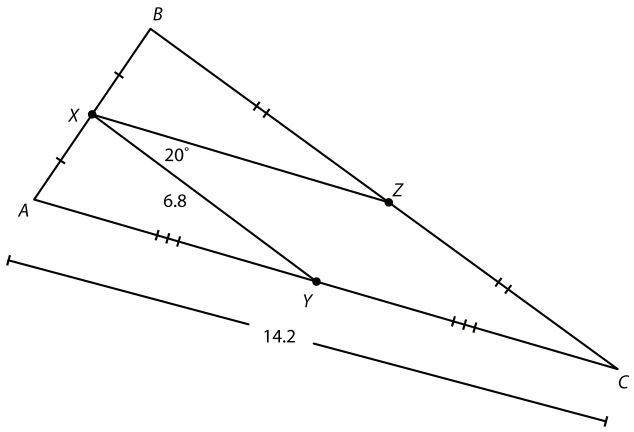
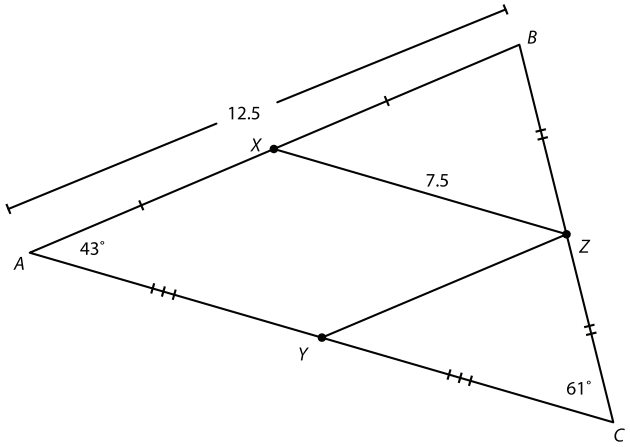
Practice 5.6.3: Proving the Midsegment of a Triangle

Use your knowledge of midsegments to solve each problem.

1. Find the lengths of $\overline{BC}$ and $\overline{XZ}$ and the measure of $\angle BZX$.



2. Find the lengths of $\overline{AC}$ and $\overline{YZ}$ and the measure of $\angle XZY$.



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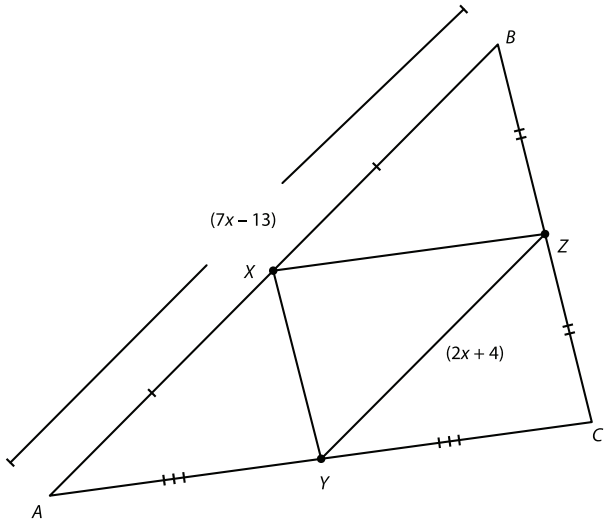
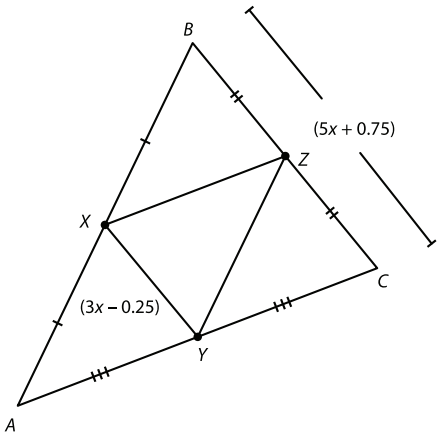
3. If $AB = 7x - 13$ and $YZ = 2x + 4$, what is the length of $\overline{YZ}$?



4. If $BC = 5x + 0.75$ and $XY = 3x - 0.25$, what is the length of $\overline{BC}$?



5. The midpoints of a triangle are X (–5, –4), Y (–3, 2), and Z (5, 2). Find the coordinates of the vertices of the triangle.

6. The vertices of a triangle are A (–5, –4), B (1, 10), and C (9, 0). Find the coordinates of the midpoints of the triangle.

continued

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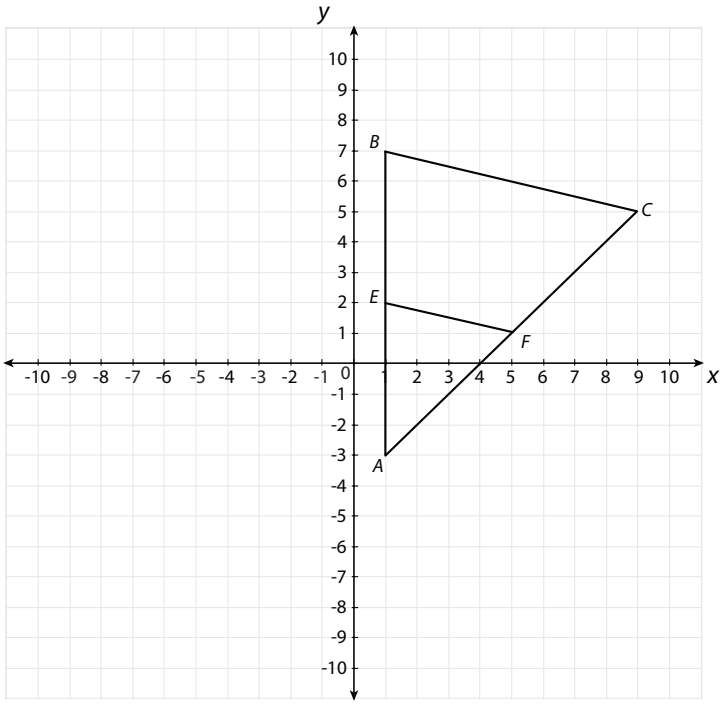
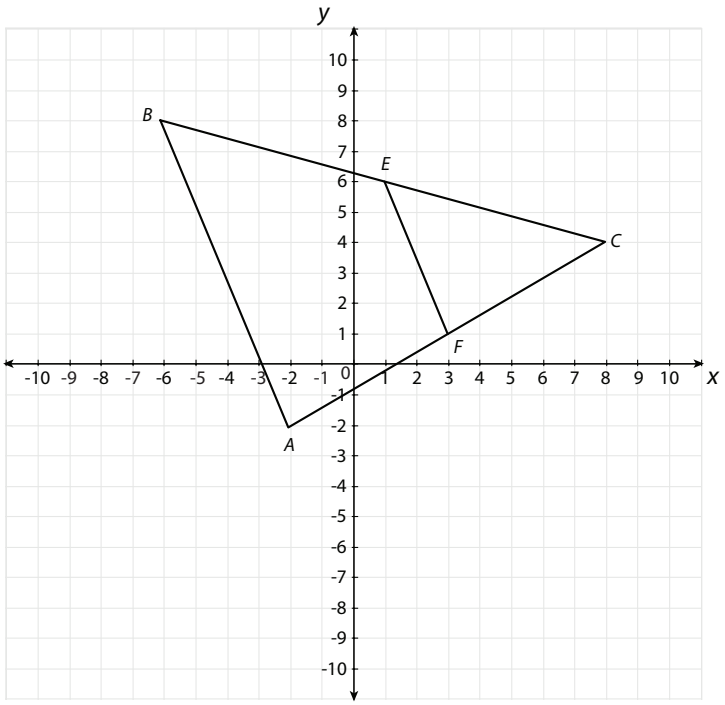
7. Use a coordinate proof to show that $\overline{EF} \parallel \overline{BC}$ and $EF = \frac{1}{2}BC$.



8. Use a coordinate proof to show that $\overline{EF} \parallel \overline{AB}$ and $EF = \frac{1}{2}AB$.



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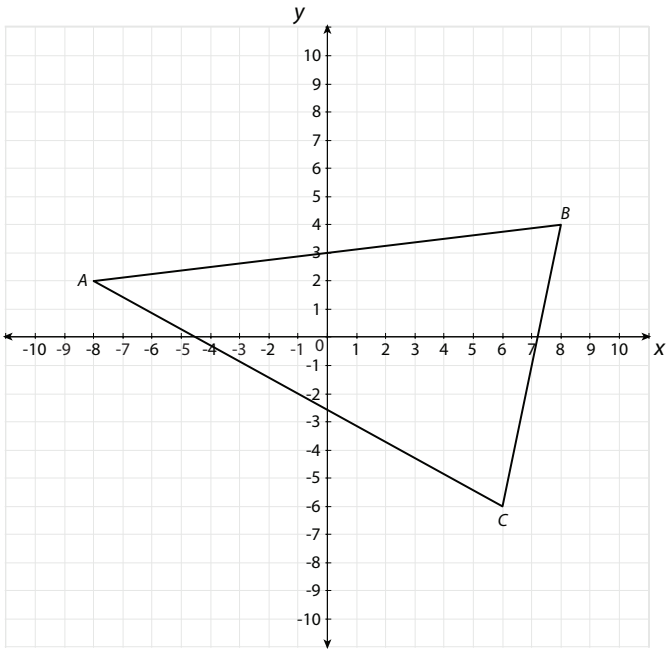
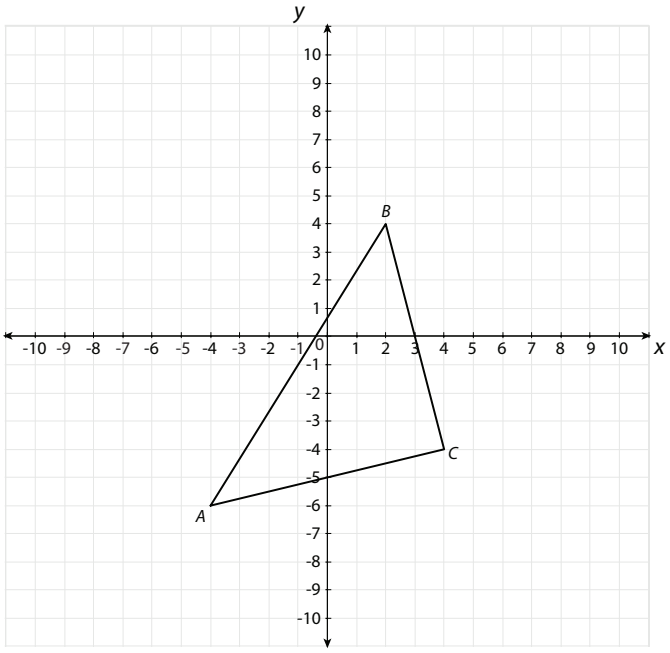
9. Determine the midpoints of $\overline{AC}$ and $\overline{BC}$. Label the points *E* and *F*. Show that $\overline{EF} \parallel \overline{AB}$ and $EF = \frac{1}{2}AB$.



10. Determine the midpoints of $\overline{AC}$ and $\overline{BC}$. Label the points *E* and *F*. Show that $\overline{EF} \parallel \overline{AB}$ and $EF = \frac{1}{2}AB$.



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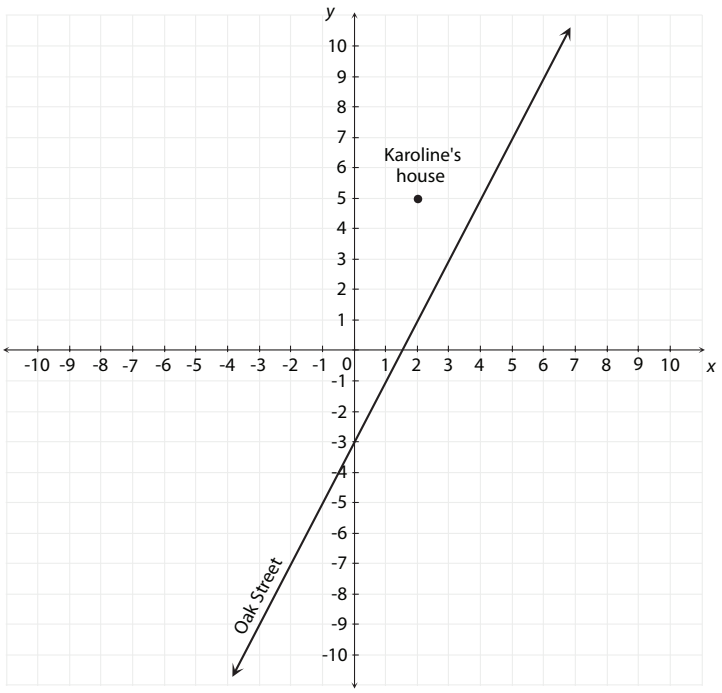
Lesson 5.6.4: Proving Centers of Triangles

Warm-Up 5.6.4

Each unit on the map represents 100 yards.



1. Karoline's house is located at the point (2, 5). Her driveway is perpendicular to Oak Street, which is represented by the equation $y = 2x - 3$. What is the equation of the line that represents Karoline's driveway?

2. What is the length of Karoline's driveway from her house to Oak Street?

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Lesson 5.6.4: Proving Centers of Triangles

Common Core State Standard

G–CO.10 Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*

Warm-Up 5.6.4 Debrief

1. Karoline's house is located at the point (2, 5). Her driveway is perpendicular to Oak Street, which is represented by the equation $y = 2x - 3$. What is the equation of the line that represents Karoline's driveway?

   The equation of the line that represents Oak Street is $y = 2x - 3$. Therefore, the slope of the line that represents Oak Street is 2.

   A line perpendicular to Oak Street has a slope that is the opposite reciprocal of 2.

   The slope of the line that is perpendicular is $-\frac{1}{2}$.

   Find the equation of the line perpendicular to Oak Street that passes through the point (2, 5).

   $y - y_1 = m(x - x_1)$ Point-slope form of a line

   $y - 5 = -\frac{1}{2}(x - 2)$ Substitute (2, 5) for (x_1, y_1) and $-\frac{1}{2}$ for m.

   $y - 5 = -\frac{1}{2}x + 1$ Simplify.

   $y = -\frac{1}{2}x + 6$

   The equation of the line that represents Karoline's driveway is $y = -\frac{1}{2}x + 6$.

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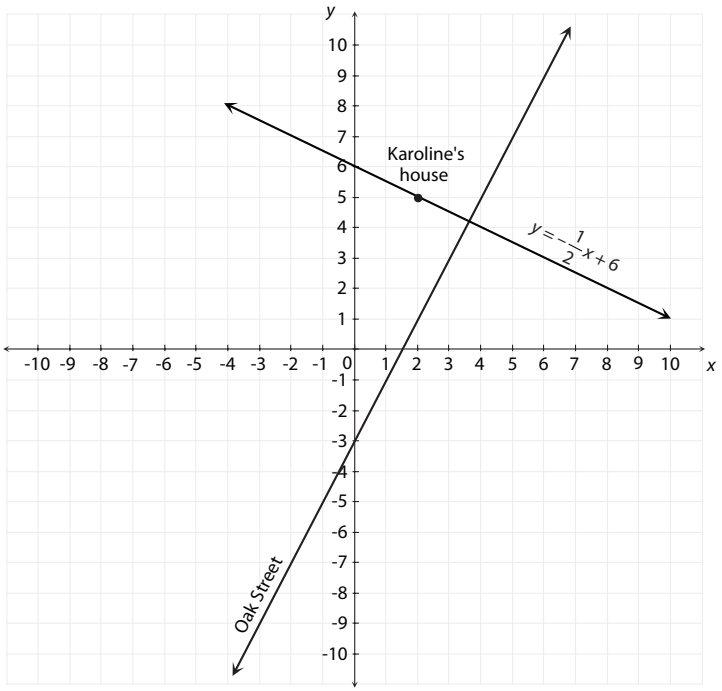

2. What is the length of Karoline's driveway from her house to Oak Street?

   Determine the point at which Karoline's driveway intersects with Oak Street.

   Set the equation representing Oak Street equal to the equation representing Karoline's driveway and solve for x.

$2x-3=-\frac{1}{2}x+6$

$\frac{5}{2}x-3=6$ Add $-\frac{1}{2}x$ to both sides of the equation.

$\frac{5}{2}x=9$ Add 3 to both sides of the equation.

$x=\frac{18}{5}=3.6$ Divide both sides by $\frac{5}{2}$.

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Substitute the value of x into either of the equations.

$y = 2x - 3$	Equation of Oak Street
$y = 2(3.6) - 3$	Substitute 3.6 for x.
$y = 7.2 - 3$	Simplify.
$y = 4.2$	

The point at which Karoline's driveway intersects with Oak Street is (3.6, 4.2).

Use the distance formula to calculate the distance from Karoline's house to the point of intersection with Oak Street.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance formula
$d = \sqrt{[(3.6) - (2)]^2 + [(4.2) - (5)]^2}$	Substitute (2, 5) and (3.6, 4.2) for (x_1, y_1) and (x_2, y_2).
$d = \sqrt{(1.6)^2 + (-0.8)^2}$	Simplify.
$d = \sqrt{2.56 + 0.64}$	
$d = \sqrt{3.2}$	

The distance from Karoline's house to the point of intersection with Oak Street is $\sqrt{3.2}$ units.

Each unit on the map represents 100 yards.

To find the actual length, multiply $\sqrt{3.2}$ by 100.

$\sqrt{3.2} \bullet 100 \approx 178.9$

The length of Karoline's driveway is approximately 178.9 yards.

Connection to the Lesson

- Students will use their understanding of slopes of perpendicular lines and distances of segments to determine centers of triangles.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- identifying and determining perpendicular bisectors and angle bisectors (G–CO.9)
- identifying and determining altitudes and medians of triangles (G–CO.10)
- calculating midpoints of segments (G–GPE.6)

Introduction

Think about all the properties of triangles we have learned so far and all the constructions we are able to perform. What properties exist when the perpendicular bisectors of triangles are constructed? Is there anything special about where the angle bisectors of a triangle intersect? We know triangles have three altitudes, but can determining each one serve any other purpose? How can the midpoints of each side of a triangle help find the center of gravity of a triangle? Each of these questions will be answered as we explore the centers of triangles.

Key Concepts

- Every triangle has four centers.
- Each center is determined by a different **point of concurrency**—the point at which three or more lines intersect.
- These centers are the circumcenter, the incenter, the orthocenter, and the centroid.

Circumcenters

- The perpendicular bisector is the line that is constructed through the midpoint of a segment at a right angle. In the case of a triangle, the perpendicular bisectors pass through the midpoints of each of the sides.
- The three perpendicular bisectors of a triangle are **concurrent lines**, or lines that intersect at one point.
- This point of concurrency is called the **circumcenter** of the triangle.
- The circumcenter of a triangle is equidistant, or the same distance, from the vertices of the triangle. This is known as the Circumcenter Theorem.

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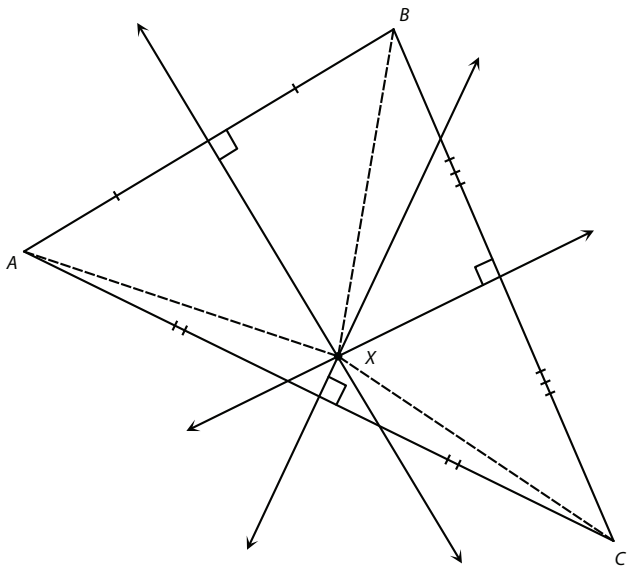
Instruction

Theorem

Circumcenter Theorem

The circumcenter of a triangle is equidistant from the vertices of a triangle.



The circumcenter of this triangle is at X.

$XA = XB = XC$

- The circumcenter can be inside the triangle, outside the triangle, or even on the triangle depending on the type of triangle.
- The circumcenter is inside acute triangles, outside obtuse triangles, and on the midpoint of the hypotenuse of right triangles.

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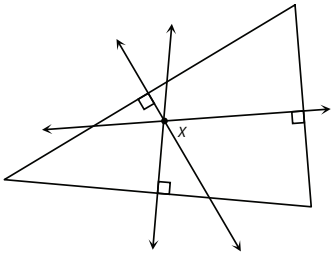
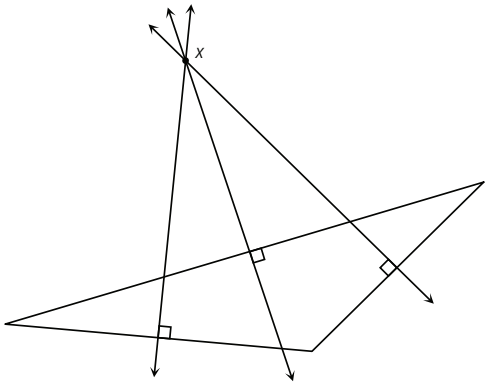
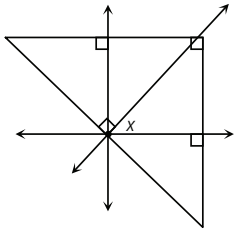
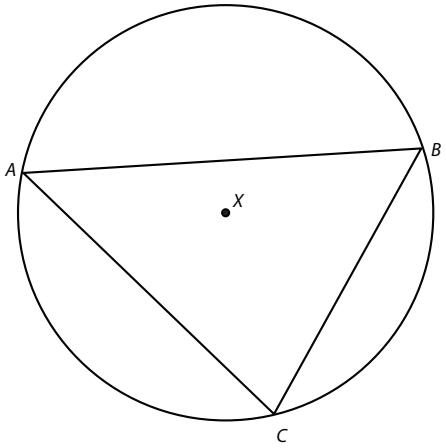
Instruction

- Look at the placement of the circumcenter, point *X*, in the following examples.

Acute triangle	Obtuse triangle	Right triangle
		
X is inside the triangle.	*X* is outside the triangle.	*X* is on the midpoint of the hypotenuse.

- The circumcenter of a triangle is also the center of the circle that passes through all of the vertices of the triangle. This circle is called the **circumscribed circle** of the triangle.



Incenters

- The angle bisectors of a triangle are rays that cut the measure of each vertex in half.
- The three angle bisectors of a triangle are also concurrent.
- This point of concurrency is called the **incenter** of the triangle.
- The incenter of a triangle is equidistant from the sides of the triangle. This is known as the Incenter Theorem.

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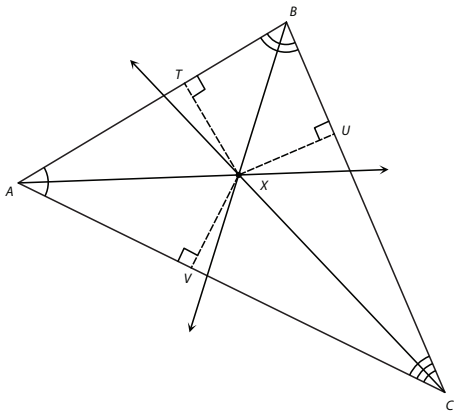
Instruction

Theorem

Incenter Theorem

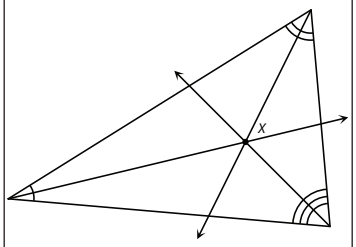
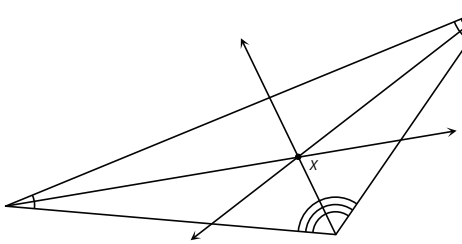
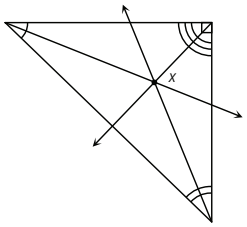
The incenter of a triangle is equidistant from the sides of a triangle.



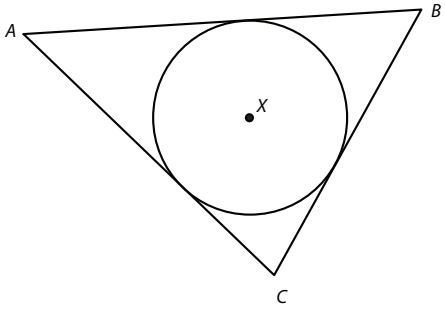
The incenter of this triangle is at point X.

$XT = XU = XV$

- The incenter is always inside the triangle.

Acute triangle	Obtuse triangle	Right triangle
		

- The incenter of a triangle is the center of the circle that contains one point from each side of a triangle. This is known as the circle that is **inscribed** in the triangle.



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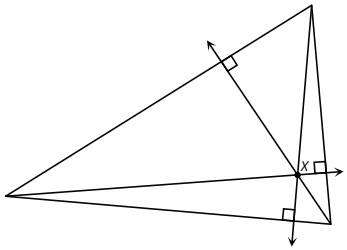
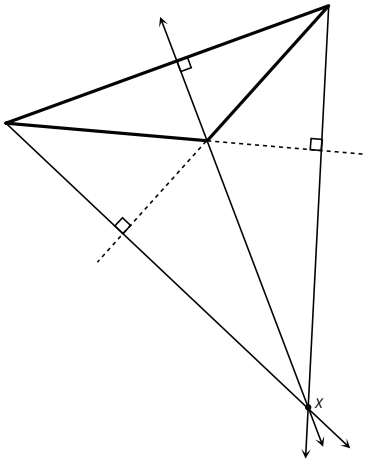
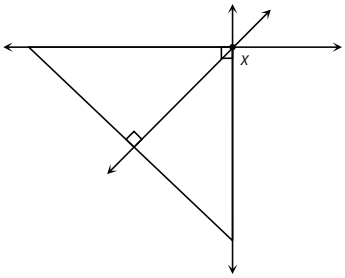
Instruction

Orthocenters

- The altitudes of a triangle are the perpendicular lines from each vertex of the triangle to its opposite side, also called the height of the triangle.
- The three altitudes of a triangle are also concurrent.
- This point of concurrency is called the **orthocenter** of the triangle.
- The orthocenter can be inside the triangle, outside the triangle, or even on the triangle depending on the type of triangle.
- The orthocenter is inside acute triangles, outside obtuse triangles, and at the vertex of the right angle of right triangles.
- Look at the placement of the orthocenter, point *X*, in the following examples.

Acute triangle	Obtuse triangle	Right triangle
		
X is inside the triangle.	*X* is outside the triangle.	*X* is at the vertex of the right angle.

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Instruction

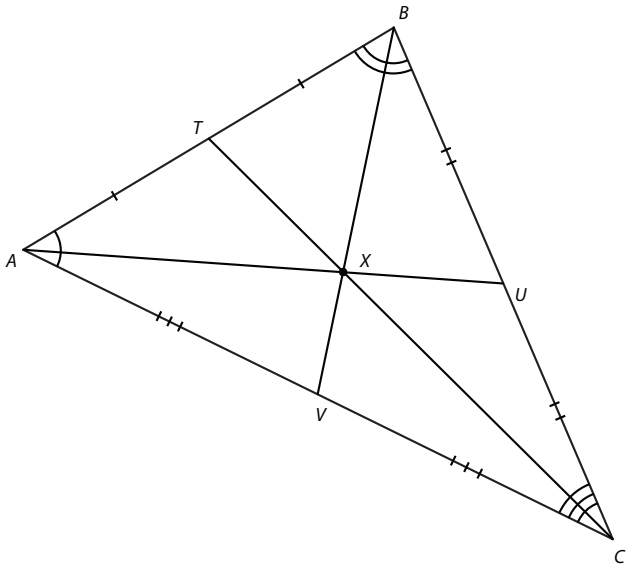
Centroids

- The **median of a triangle** is a segment that joins a vertex of the triangle to the midpoint of the opposite side.
- Every triangle has three medians.
- The three medians of a triangle are also concurrent.
- This point of concurrency is called the **centroid** of the triangle.
- The centroid is always located inside the triangle $\frac{2}{3}$ the distance from each vertex to the midpoint of the opposite side. This is known as the Centroid Theorem.

Theorem

Centroid Theorem

The centroid of a triangle is $\frac{2}{3}$ the distance from each vertex to the midpoint of the opposite side.



The centroid of this triangle is at point X.

$$AX = \frac{2}{3}AU;\ BX = \frac{2}{3}BV;\ CX = \frac{2}{3}CT$$

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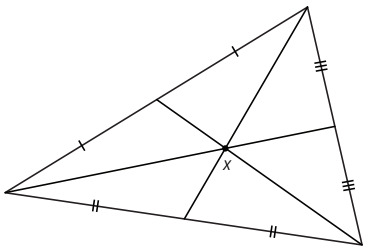
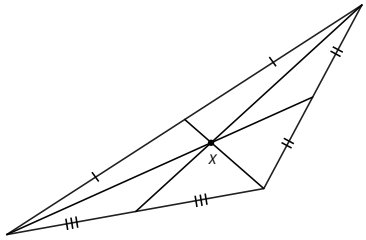
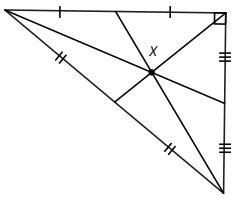
Instruction

- The centroid is always located inside the triangle.

Acute triangle	Obtuse triangle	Right triangle
		

- The centroid is also called the center of gravity of a triangle because a physical triangle will always balance at this point.
- Each point of concurrency discussed is considered a center of the triangle.
- Each center serves its own purpose in design, planning, and construction.

Center of triangle	Intersection of…
Circumcenter	Perpendicular bisectors
Incenter	Angle bisectors
Orthocenter	Altitudes
Centroid	Medians

Common Errors/Misconceptions

- not recognizing that the circumcenter and orthocenter are outside of obtuse triangles
- incorrectly assuming that the perpendicular bisector of the side of a triangle will pass through the opposite vertex
- interchanging circumcenter, incenter, orthocenter, and centroid
- confusing medians with midsegments
- misidentifying the height of the triangle

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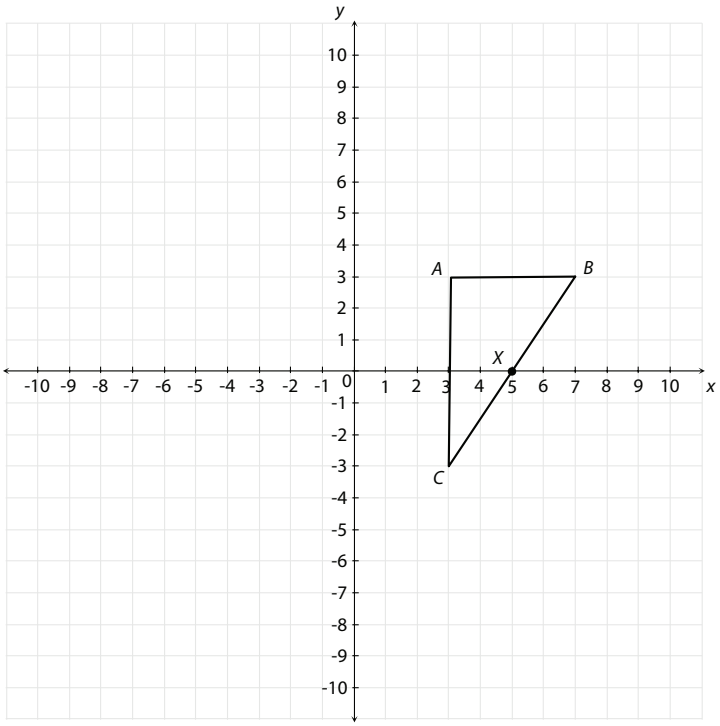
Instruction

Guided Practice 5.6.4

Example 1

$\triangle ABC$ has vertices A (3, 3), B (7, 3), and C (3, –3). Justify that (5, 0) is the circumcenter of $\triangle ABC$.



1. Identify the known information.

   $\triangle ABC$ has vertices A (3, 3), B (7, 3), and C (3, –3).

   The circumcenter is X (5, 0).

   The circumcenter of a triangle is equidistant from the vertices of the triangle.

SWB p. 195; SRB p. U5-233

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2. Determine the distance from the circumcenter to each of the vertices.

Use the distance formula to calculate the distance from X to A.

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	Distance formula
$d=\sqrt{[(3)-(5)]^2+[(3)-(0)]^2}$	Substitute (5, 0) and (3, 3) for (x_1, y_1) and (x_2, y_2).
$d=\sqrt{(-2)^2+(-3)^2}$	Simplify.
$d=\sqrt{4+9}$	
$d=\sqrt{13}$	

The distance from X to A is $\sqrt{13}$ units.

Use the distance formula to calculate the distance from X to B.

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	Distance formula
$d=\sqrt{[(7)-(5)]^2+[(3)-(0)]^2}$	Substitute (5, 0) and (7, 3) for (x_1, y_1) and (x_2, y_2).
$d=\sqrt{(2)^2+(3)^2}$	Simplify.
$d=\sqrt{4+9}$	
$d=\sqrt{13}$	

The distance from X to B is $\sqrt{13}$ units.

(*continued*)

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Instruction

Use the distance formula to calculate the distance from X to C.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance formula

$d = \sqrt{[(3) - (5)]^2 + [(-3) - (0)]^2}$ Substitute (5, 0) and (3, –3) for (x_1, y_1) and (x_2, y_2).

$d = \sqrt{(-2)^2 + (3)^2}$ Simplify.

$d = \sqrt{4 + 9}$

$d = \sqrt{13}$

The distance from X to C is $\sqrt{13}$ units.

3. State your conclusion.

X (5, 0) is the circumcenter of $\triangle ABC$ with vertices A (3, 3), B (7, 3), and C (3, –3) because the distance from X to each of the vertices is $\sqrt{13}$ units.

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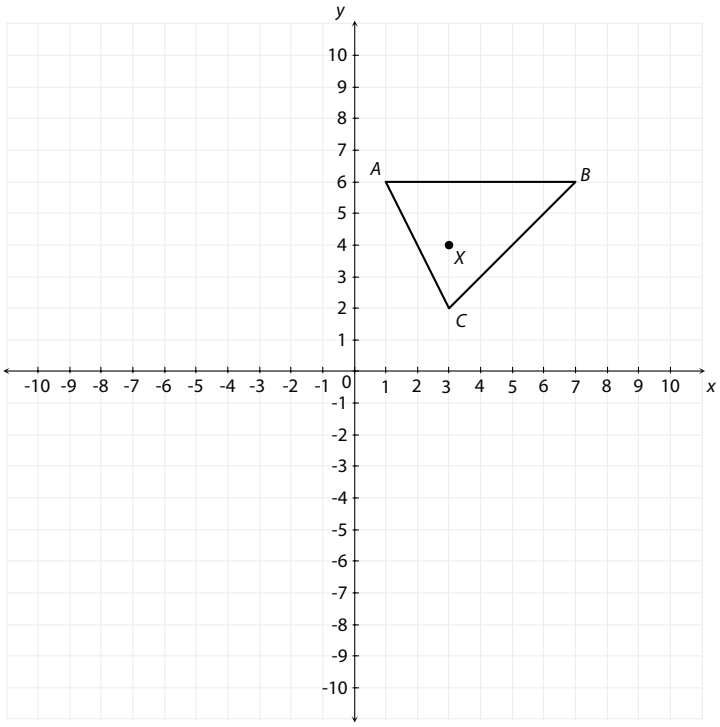
Instruction

Example 2

$\triangle ABC$ has vertices A (1, 6), B (7, 6), and C (3, 2). Find the equation of each altitude of $\triangle ABC$ to verify that (3, 4) is the orthocenter of $\triangle ABC$.



1. Identify known information.

   $\triangle ABC$ has vertices A (1, 6), B (7, 6), and C (3, 2).

   The orthocenter is X (3, 4).

   The orthocenter of a triangle is the intersection of the altitudes of the triangle.

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2. Determine the altitudes of $\triangle ABC$.

Find an equation of the line that is perpendicular to $\overline{BC}$ and passes through A.

Use the slope formula to calculate the slope of $\overline{BC}$.

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope formula

$m = \frac{(2) - (6)}{(3) - (7)}$ Substitute (7, 6) and (3, 2) for (x_1, y_1) and (x_2, y_2).

$m = \frac{-4}{-4} = 1$ Simplify.

The slope of $\overline{BC}$ is 1.

The slope of the line that is perpendicular to $\overline{BC}$ has the opposite reciprocal of the slope of $\overline{BC}$.

The slope of the perpendicular line is –1.

Find the equation of the altitude from point A.

$y - y_1 = m(x - x_1)$ Point-slope form of a line

$y - 6 = -1(x - 1)$ Substitute (1, 6) for (x_1, y_1) and –1 for m.

$y - 6 = -1x + 1$ Simplify.

$y = -x + 7$

The equation of the altitude from point A to $\overline{BC}$ is $y = -x + 7$.

Find an equation of the line that is perpendicular to $\overline{AC}$ and passes through B.

Use the slope formula to calculate the slope of $\overline{AC}$.

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope formula

$m = \frac{(2) - (6)}{(3) - (1)}$ Substitute (1, 6) and (3, 2) for (x_1, y_1) and (x_2, y_2).

$m = \frac{-4}{2} = -2$ Simplify.

The slope of $\overline{AC}$ is –2.

(*continued*)

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Instruction

The slope of the line that is perpendicular to $\overline{AC}$ has the opposite reciprocal of the slope of $\overline{AB}$.

The slope of the perpendicular line is $\frac{1}{2}$.

Find the equation of the altitude from point *B*.

$y - y_1 = m(x - x_1)$	Point-slope form of a line
$y - 6 = \frac{1}{2}(x - 7)$	Substitute (7, 6) for (x_1, y_1) and $\frac{1}{2}$ for m.
$y - 6 = \frac{1}{2}x - \frac{7}{2}$	Simplify.
$y = \frac{1}{2}x + \frac{5}{2}$	

The equation of the altitude from point *B* to $\overline{AC}$ is $y = \frac{1}{2}x + \frac{5}{2}$.

Find an equation of the line that is perpendicular to $\overline{AB}$ and passes through *C*.

Use the slope formula to calculate the slope of $\overline{AB}$.

$m = \frac{y_2 - y_1}{x_2 - x_1}$	Slope formula
$m = \frac{(6) - (6)}{(7) - (1)}$	Substitute (1, 6) and (7, 6) for (x_1, y_1) and (x_2, y_2).
$m = \frac{0}{6} = 0$	Simplify.

The slope of $\overline{AB}$ is 0.

$\overline{AB}$ is a horizontal line; therefore, the altitude is vertical.

The vertical line that passes through point *C* is $x = 3$.

The equation of the altitude from point *C* to $\overline{AB}$ is $x = 3$.

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Instruction

3. Verify that X (3, 4) is the intersection of the three altitudes.

For (3, 4) to be the intersection of the three altitudes, the point must satisfy each of the equations: $y = -x + 7$, $y = \frac{1}{2}x + \frac{5}{2}$, and $x = 3$.

$y = -x + 7$	Equation of the altitude from A through $\overline{BC}$
$(4) = -(3) + 7$	Substitute X (3, 4) for (x, y).
$4 = 4$	Simplify.

(3, 4) satisfies the equation of the altitude from A through $\overline{BC}$.

$y = \frac{1}{2}x + \frac{5}{2}$	Equation of the altitude from B through $\overline{AC}$
$(4) = \frac{1}{2}(3) + \frac{5}{2}$	Substitute X (3, 4) for (x, y).
$4 = \frac{3}{2} + \frac{5}{2}$	Simplify.
$4 = \frac{8}{2}$	
$4 = 4$	

(3, 4) satisfies the equation of the altitude from B through $\overline{AC}$.

$x = 3$	Equation of altitude from C through $\overline{AB}$.
$3 = 3$	Substitute X (3, 4) for x.

(3, 4) satisfies the equation of the altitude from C through $\overline{AB}$.

4. State your conclusion.

X (3, 4) is the orthocenter of $\triangle ABC$ with vertices A (1, 6), B (7, 6), and C (3, 2) because X satisfies each of the equations of the altitudes of the triangle.

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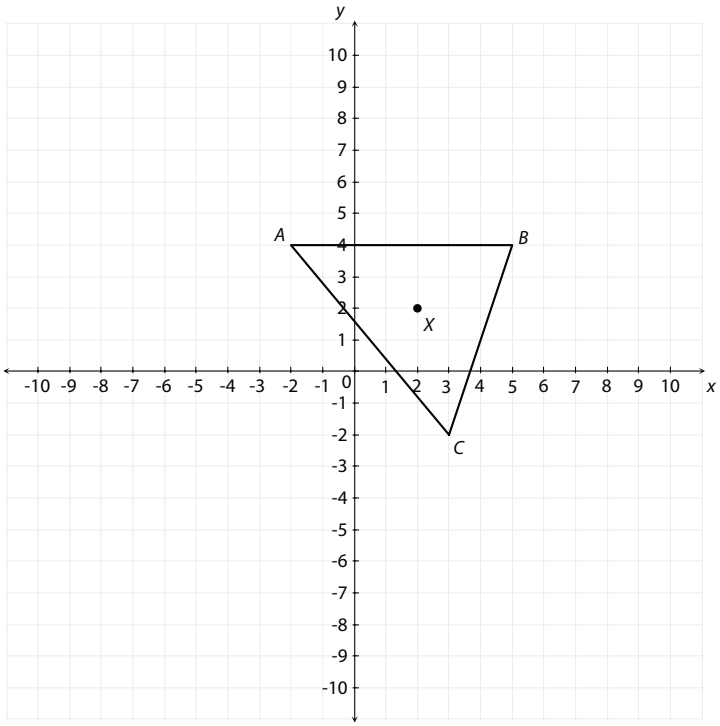
Instruction

Example 3

$\triangle ABC$ has vertices A (–2, 4), B (5, 4), and C (3, –2). Find the equation of each median of $\triangle ABC$ to verify that (2, 2) is the centroid of $\triangle ABC$.



1. Identify known information.

   $\triangle ABC$ has vertices A (–2, 4), B (5, 4), and C (3, –2).

   The centroid is X (2, 2).

   The centroid of a triangle is the intersection of the medians of the triangle.

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Instruction

2. Determine the midpoint of each side of the triangle.

Use the midpoint formula to find the midpoint of $\overline{AB}$.

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ Midpoint formula

$\left(\frac{(-2)+(5)}{2}, \frac{(4)+(4)}{2}\right)$ Substitute (–2, 4) and (5, 4) for (x_1, y_1) and (x_2, y_2).

$\left(\frac{3}{2}, \frac{8}{2}\right)$ Simplify.

$\left(\frac{3}{2}, 4\right)$

The midpoint of $\overline{AB}$ is $\left(\frac{3}{2}, 4\right)$.

Use the midpoint formula to find the midpoint of $\overline{BC}$.

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ Midpoint formula

$\left(\frac{(5)+(3)}{2}, \frac{(4)+(-2)}{2}\right)$ Substitute (5, 4) and (3, –2) for (x_1, y_1) and (x_2, y_2).

$\left(\frac{8}{2}, \frac{2}{2}\right)$ Simplify.

(4, 1)

The midpoint of $\overline{BC}$ is (4, 1).

(*continued*)

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Instruction

Use the midpoint formula to find the midpoint of $\overline{AC}$.

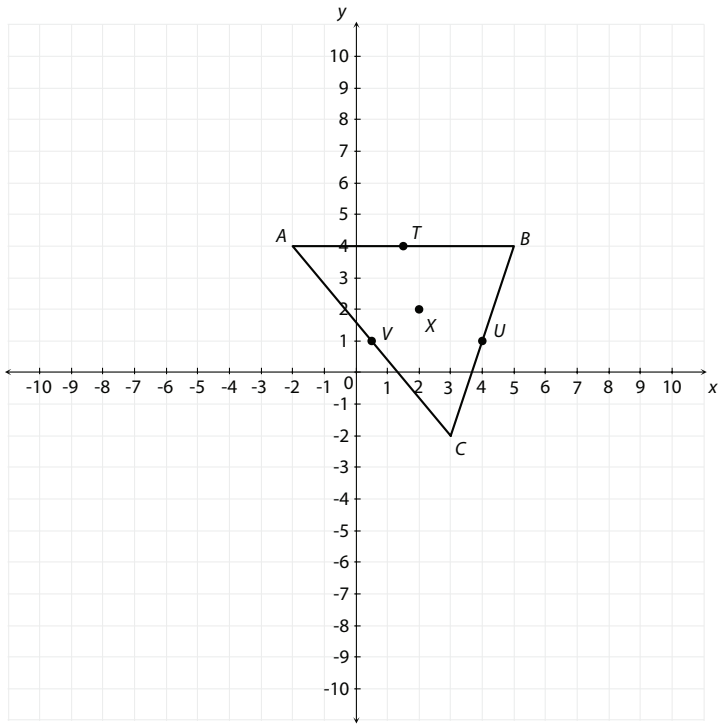
$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ Midpoint formula

$\left(\frac{(-2)+(3)}{2}, \frac{(4)+(-2)}{2}\right)$ Substitute (–2, 4) and (3, –2) for (x_1, y_1) and (x_2, y_2).

$\left(\frac{1}{2}, \frac{2}{2}\right)$ Simplify.

$\left(\frac{1}{2}, 1\right)$

The midpoint of $\overline{AC}$ is $\left(\frac{1}{2}, 1\right)$.



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3. Determine the medians of the triangle.

Find the equation of $\overline{AU}$, which is the line that passes through A and the midpoint of $\overline{BC}$.

Use the slope formula to calculate the slope of $\overline{AU}$.

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope formula

$m = \frac{(1)-(4)}{(4)-(-2)}$ Substitute (–2, 4) and (4, 1) for (x_1, y_1) and (x_2, y_2).

$m = \frac{-3}{6}$ Simplify.

$m = -\frac{1}{2}$

The slope of $\overline{AU}$ is $-\frac{1}{2}$.

Find the equation of $\overline{AU}$.

$y - y_1 = m(x - x_1)$ Point-slope form of a line

$y - 4 = -\frac{1}{2}[x-(-2)]$ Substitute (–2, 4) for (x_1, y_1) and $-\frac{1}{2}$ for m.

$y - 4 = -\frac{1}{2}x - 1$ Simplify.

$y = -\frac{1}{2}x + 3$

The equation of $\overline{AU}$ that passes through A and the midpoint of $\overline{BC}$ is $y = -\frac{1}{2}x + 3$.

(*continued*)

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Instruction

Find the equation of $\overline{BV}$, which is the line that passes through B and the midpoint of $\overline{AC}$.

Use the slope formula to calculate the slope of $\overline{BV}$.

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope formula

$m = \frac{(1)-(4)}{\left(\frac{1}{2}\right)-(5)}$ Substitute (5, 4) and $\left(\frac{1}{2}, 1\right)$ for (x_1, y_1) and (x_2, y_2).

$m = \frac{-3}{-\frac{9}{2}}$ Simplify.

$m = \frac{2}{3}$

The slope of $\overline{BV}$ is $\frac{2}{3}$.

Find the equation of $\overline{BV}$.

$y - y_1 = m(x - x_1)$ Point-slope form of a line

$y - 4 = \frac{2}{3}(x - 5)$ Substitute (5, 4) for (x_1, y_1) and $\frac{2}{3}$ for m.

$y - 4 = \frac{2}{3}x - \frac{10}{3}$ Simplify.

$y = \frac{2}{3}x + \frac{2}{3}$

The equation of $\overline{BV}$ that passes through B and the midpoint of $\overline{AC}$ is $y = \frac{2}{3}x + \frac{2}{3}$.

(*continued*)

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Find the equation of $\overline{CT}$, which is the line that passes through C and the midpoint of $\overline{AB}$.

Use the slope formula to calculate the slope of $\overline{CT}$.

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope formula

$m = \frac{(4) - (-2)}{\left(\frac{3}{2}\right) - (3)}$ Substitute (3, –2) and $\left(\frac{3}{2}, 4\right)$ for (x_1, y_1) and (x_2, y_2).

$m = \frac{6}{-\frac{3}{2}}$ Simplify.

$m = -4$

The slope of $\overline{CT}$ is –4.

Find the equation of $\overline{CT}$.

$y - y_1 = m(x - x_1)$ Point-slope form of a line

$y - (-2) = -4(x - 3)$ Substitute (3, –2) for (x_1, y_1) and –4 for m.

$y + 2 = -4(x - 3)$ Simplify.

$y + 2 = -4x + 12$

$y = -4x + 10$

The equation of $\overline{CT}$ that passes through C and the midpoint of $\overline{AC}$ is $y = -4x + 10$.

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Instruction

4. Verify that X (2, 2) is the intersection of the three medians.

For (2, 2) to be the intersection of the three medians, the point must satisfy each of the equations: $y=-\frac{1}{2}x+3$, $y=\frac{2}{3}x+\frac{2}{3}$, and $y = -4x + 10$.

$y=-\frac{1}{2}x+3$	Equation of the median from A to the midpoint of $\overline{BC}$
$(2)=-\frac{1}{2}(2)+3$	Substitute X (2, 2) for (x, y).
$2 = -1 + 3$	Simplify.
$2 = 2$	

(2, 2) satisfies the equation of the median from A to the midpoint of $\overline{BC}$.

$y=\frac{2}{3}x+\frac{2}{3}$	Equation of the median from B to the midpoint of $\overline{AC}$
$(2)=\frac{2}{3}(2)+\frac{2}{3}$	Substitute X (2, 2) for (x, y).
$2=\frac{4}{3}+\frac{2}{3}$	Simplify.
$2=\frac{6}{3}$	
$2 = 2$	

(2, 2) satisfies the equation of the median from B to the midpoint of $\overline{AC}$.

$y = -4x + 10$	Equation of the median from C to the midpoint of $\overline{AB}$
$(2) = -4(2) + 10$	Substitute X (2, 2) for (x, y).
$(2) = -8 + 10$	Simplify.
$2 = 2$	

(2, 2) satisfies the equation of the median from C to the midpoint of $\overline{AB}$.

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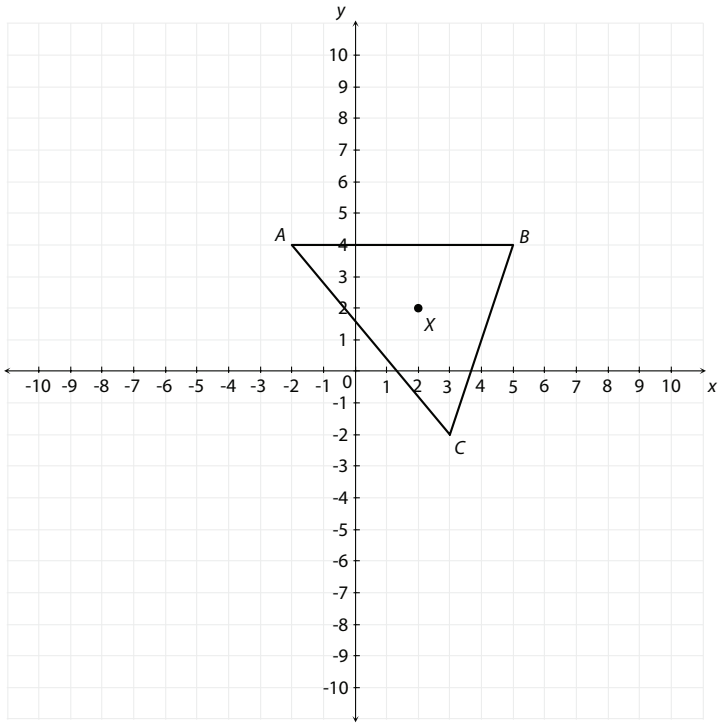
Instruction

5. State your conclusion.

   X (2, 2) is the centroid of $\triangle ABC$ with vertices A (–2, 4), B (5, 4), and C (3, –2) because X satisfies each of the equations of the medians of the triangle.

Example 4

Using $\triangle ABC$ from Example 3, which has vertices A (–2, 4), B (5, 4), and C (3, –2), verify that the centroid is $\frac{2}{3}$ the distance from each vertex.



1. Identify the known information.

   $\triangle ABC$ has vertices A (–2, 4), B (5, 4), and C (3, –2).

   The centroid is X (2, 2).

   The midpoints of $\triangle ABC$ are $T\left(\frac{3}{2}, 4\right)$, U (4, 1), and $V\left(\frac{1}{2}, 1\right)$.

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2. Use the distance formula to show that point X (2, 2) is $\frac{2}{3}$ the distance from each vertex.

Use the distance formula to calculate the distance from A to U.

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ Distance formula

$d=\sqrt{[(4)-(-2)]^2+[(1)-(4)]^2}$ Substitute (–2, 4) and (4, 1) for (x_1, y_1) and (x_2, y_2).

$d=\sqrt{(6)^2+(-3)^2}$ Simplify.

$d=\sqrt{36+9}$

$d=\sqrt{45}=\sqrt{9\bullet 5}=3\sqrt{5}$

The distance from A to U is $3\sqrt{5}$ units.

Calculate the distance from X to A.

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ Distance formula

$d=\sqrt{[(-2)-(2)]^2+[(4)-(2)]^2}$ Substitute (2, 2) and (–2, 4) for (x_1, y_1) and (x_2, y_2).

$d=\sqrt{(-4)^2+(2)^2}$ Simplify.

$d=\sqrt{16+4}$

$d=\sqrt{20}=\sqrt{4\bullet 5}=2\sqrt{5}$

The distance from X to A is $2\sqrt{5}$ units.

$\frac{2}{3}AU=XA$ Centroid Theorem

$\frac{2}{3}\left(3\sqrt{5}\right)=2\sqrt{5}$ Substitute the distances found for AU and XA.

$2\sqrt{5}=2\sqrt{5}$ Simplify.

X is $\frac{2}{3}$ the distance from A.

(*continued*)

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Instruction

Use the distance formula to calculate the distance from B to V.

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ Distance formula

$d=\sqrt{\left[\left(\frac{1}{2}\right)-(5)\right]^2+[(1)-(4)]^2}$ Substitute (5, 4) and $\left(\frac{1}{2},1\right)$ for (x_1, y_1) and (x_2, y_2).

$d=\sqrt{\left(-\frac{9}{2}\right)^2+(-3)^2}$ Simplify.

$d=\sqrt{\frac{81}{4}+9}$

$d=\sqrt{\frac{117}{4}}=\frac{\sqrt{9\bullet 13}}{\sqrt{4}}=\frac{3\sqrt{13}}{2}$

The distance from B to V is $\frac{3\sqrt{13}}{2}$ units.

Calculate the distance from X to B.

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ Distance formula

$d=\sqrt{[(5)-(2)]^2+[(4)-(2)]^2}$ Substitute (2, 2) and (5, 4) for (x_1, y_1) and (x_2, y_2).

$d=\sqrt{(3)^2+(2)^2}$ Simplify.

$d=\sqrt{9+4}$

$d=\sqrt{13}$

The distance from X to B is $\sqrt{13}$ units.

$\frac{2}{3}BV=XB$ Centroid Theorem

$\frac{2}{3}\left(\frac{3\sqrt{13}}{2}\right)=\sqrt{13}$ Substitute the distances found for BV and XB.

$\sqrt{13}=\sqrt{13}$ Simplify.

X is $\frac{2}{3}$ the distance from B.

(*continued*)

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Instruction

Use the distance formula to calculate the distance from C to T.

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ Distance formula

$d=\sqrt{\left[\left(\frac{3}{2}\right)-(3)\right]^2+[(4)-(-2)]^2}$ Substitute (3, –2) and $\left(\frac{3}{2},4\right)$ for (x_1, y_1) and (x_2, y_2).

$d=\sqrt{\left(-\frac{3}{2}\right)^2+(6)^2}$ Simplify.

$d=\sqrt{\frac{9}{4}+36}$

$d=\sqrt{\frac{153}{4}}=\frac{\sqrt{9\bullet 17}}{\sqrt{4}}=\frac{3\sqrt{17}}{2}$

The distance from C to T is $\frac{3\sqrt{17}}{2}$ units.

Calculate the distance from X to C.

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ Distance formula

$d=\sqrt{[(3)-(2)]^2+[(-2)-(2)]^2}$ Substitute (2, 2) and (3, –2) for (x_1, y_1) and (x_2, y_2).

$d=\sqrt{(1)^2+(-4)^2}$ Simplify.

$d=\sqrt{1+16}$

$d=\sqrt{17}$

The distance from X to C is $\sqrt{17}$ units.

$\frac{2}{3}CT=XC$ Centroid Theorem

$\frac{2}{3}\left(\frac{3\sqrt{17}}{2}\right)=\sqrt{17}$ Substitute the distances found for CT and XC.

$\sqrt{17}=\sqrt{17}$ Simplify.

X is $\frac{2}{3}$ the distance from C.

The centroid, X (2, 2), is $\frac{2}{3}$ the distance from each vertex.

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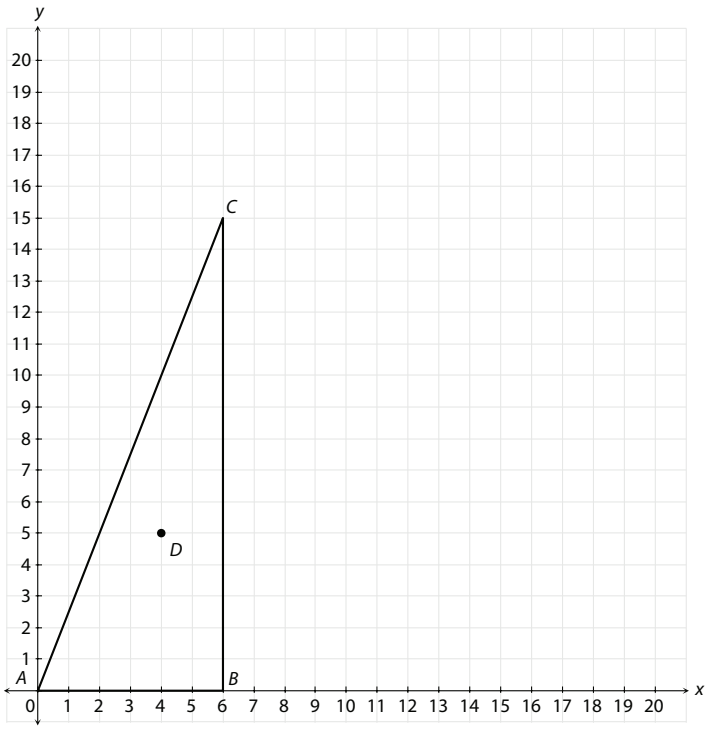
Problem-Based Task 5.6.4: Sailing Centroid

SMP	
1	2 ✓
3	4
5	6
7 ✓	8 ✓

When sailing, the boat must remain balanced at all times. There are many factors that can affect the stability and balance of a sailboat. One such factor is the determination of the correct center of effort, or the middle of the sail area. This center is found by determining the centroid of the sail. Donzel's sail is a right triangle with a base of 6 feet and a height of 15 feet. The sail is represented on the graph. Donzel has found what he thinks is the centroid of the sail, marked by point *D* on the graph. Has Donzel correctly identified the centroid?



Has Donzel correctly identified the centroid?

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Problem-Based Task 5.6.4: Sailing Centroid

Coaching

a. What is the definition of a centroid?

b. What are the vertices of the triangle representing the sail?

c. What is the midpoint of $\overline{AB}$?

d. What is the midpoint of $\overline{BC}$?

e. What is the midpoint of $\overline{AC}$?

f. What is the equation of the median that passes through $\angle A$?

g. What is the equation of the median that passes through $\angle B$?

h. What is the equation of the median that passes through $\angle C$?

i. Does the point (4, 5) satisfy the equation of each median?

j. Has Donzel correctly identified the centroid?

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Instruction

Problem-Based Task 5.6.4: Sailing Centroid

Coaching Sample Responses

a. What is the definition of a centroid?

The centroid of a triangle is the intersection of the medians of the triangle.

b. What are the vertices of the triangle representing the sail?

The vertices of the triangle representing the sail are A (0, 0), B (6, 0), and C (6, 15).

c. What is the midpoint of $\overline{AB}$?

Use the midpoint formula to find the midpoint of $\overline{AB}$.

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ Midpoint formula

$\left(\frac{(0)+(6)}{2}, \frac{(0)+(0)}{2}\right)$ Substitute (0, 0) and (6, 0) for (x_1, y_1) and (x_2, y_2).

$\left(\frac{6}{2}, \frac{0}{2}\right)$ Simplify.

(3, 0)

The midpoint of $\overline{AB}$ is (3, 0).

d. What is the midpoint of $\overline{BC}$?

Use the midpoint formula to find the midpoint of $\overline{BC}$.

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ Midpoint formula

$\left(\frac{(6)+(6)}{2}, \frac{(0)+(15)}{2}\right)$ Substitute (6, 0) and (6, 15) for (x_1, y_1) and (x_2, y_2).

$\left(\frac{12}{2}, \frac{15}{2}\right)$ Simplify.

(6, 7.5)

The midpoint of $\overline{BC}$ is (6, 7.5).

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e. What is the midpoint of $\overline{AC}$?

Use the midpoint formula to find the midpoint of $\overline{AC}$.

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ Midpoint formula

$\left(\frac{(0)+(6)}{2}, \frac{(0)+(15)}{2}\right)$ Substitute (0, 0) and (6, 15) for (x_1, y_1) and (x_2, y_2).

$\left(\frac{6}{2}, \frac{15}{2}\right)$ Simplify.

(3, 7.5)

The midpoint of $\overline{AC}$ is (3, 7.5).

f. What is the equation of the median that passes through A?

Find the equation of the line that passes through A and the midpoint of $\overline{BC}$.

Use the slope formula to calculate the slope of this equation.

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope formula

$m = \frac{(7.5)-(0)}{(6)-(0)}$ Substitute (0, 0) and (6, 7.5) for (x_1, y_1) and (x_2, y_2).

$m = \frac{7.5}{6}$ Simplify.

$m = 1.25$

The slope of the line that passes through A and the midpoint of $\overline{BC}$ is 1.25.

Find the equation of the line that passes through A and the midpoint of $\overline{BC}$.

$y - y_1 = m(x - x_1)$ Point-slope form of a line

$y - 0 = 1.25(x - 0)$ Substitute (0, 0) for (x_1, y_1) and 1.25 for m.

$y = 1.25x$ Simplify.

The equation of the line that passes through A and the midpoint of $\overline{BC}$ is $y = 1.25x$.

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g. What is the equation of the median that passes through B?

Find the equation of the line that passes through B and the midpoint of $\overline{AC}$.

Use the slope formula to calculate the slope of this equation.

$m=\frac{y_2-y_1}{x_2-x_1}$	Slope formula
$m=\frac{(7.5)-(0)}{(3)-(6)}$	Substitute (6, 0) and (3, 7.5) for (x_1, y_1) and (x_2, y_2).
$m=\frac{7.5}{-3}$	Simplify.
$m = -2.5$	

The slope of the line that passes through B and the midpoint of $\overline{AC}$ is –2.5.

Find the equation of the line that passes through B and the midpoint of $\overline{AC}$.

$y - y_1 = m(x - x_1)$	Point-slope form of a line
$y - 0 = -2.5(x - 6)$	Substitute (6, 0) for (x_1, y_1) and –2.5 for m.
$y - 0 = -2.5x + 15$	Simplify.
$y = -2.5x + 15$	

The equation of the line that passes through B and the midpoint of $\overline{AC}$ is $y = -2.5x + 15$.

h. What is the equation of the median that passes through C?

Find the equation of the line that passes through C and the midpoint of $\overline{AB}$.

Use the slope formula to calculate the slope of this equation.

$m=\frac{y_2-y_1}{x_2-x_1}$	Slope formula
$m=\frac{(0)-(15)}{(3)-(6)}$	Substitute (6, 15) and (3, 0) for (x_1, y_1) and (x_2, y_2).
$m=\frac{-15}{-3}$	Simplify.
$m = 5$	

The slope of the line that passes through C and the midpoint of $\overline{AB}$ is 5.

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Instruction

Find the equation of the line that passes through C and the midpoint of $\overline{AB}$.

$y - y_1 = m(x - x_1)$	Point-slope form of a line
$y - 15 = 5(x - 6)$	Substitute (6, 15) for (x_1, y_1) and 5 for m.
$y - 15 = 5x - 30$	Simplify.
$y = 5x - 15$	

The equation of the line that passes through C and the midpoint of $\overline{AB}$ is $y = 5x - 15$.

i. Does the point (4, 5) satisfy the equation of each median?

For (4, 5) to be the intersection of the three medians, the point must satisfy each of the equations: $y = 1.25x$, $y = -2.5x + 15$, and $y = 5x - 15$.

$y = 1.25x$	Equation of the median from A to the midpoint of $\overline{BC}$
$5 = 1.25(4)$	Substitute (4, 5) for (x, y).
$5 = 5$	Simplify.

(4, 5) satisfies the equation of the median from A to the midpoint of $\overline{BC}$.

$y = -2.5x + 15$	Equation of the median from B to the midpoint of $\overline{AC}$
$5 = -2.5(4) + 15$	Substitute (4, 5) for (x, y).
$5 = -10 + 15$	Simplify.
$5 = 5$	

(4, 5) satisfies the equation of the median from B to the midpoint of $\overline{AC}$.

$y = 5x - 15$	Equation of the median from C to the midpoint of $\overline{AB}$
$5 = 5(4) - 15$	Substitute (4, 5) for (x, y).
$5 = 20 - 10$	Simplify.
$5 = 5$	

(4, 5) satisfies the equation of the median from C to the midpoint of $\overline{AB}$.

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j. Has Donzel correctly identified the centroid?

Yes; D (4, 5) is the centroid of $\triangle ABC$ with vertices A (0, 0), B (3, 0), and C (6, 15) because (4, 5) satisfies each of the equations of the medians of the triangle.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 5.6.4: Proving Centers of Triangles

Use what you know about centers of triangles to complete each problem.

1. $\triangle ABC$ has vertices A (0, 0), B (0, –8), and C (4, –8). Justify that (2, –4) is the circumcenter of $\triangle ABC$.

2. $\triangle ABC$ has vertices A (–4, 10), B (8, –2), and C (12, 10). Justify that (8, 6) is the orthocenter of $\triangle ABC$.

3. $\triangle ABC$ has vertices A (6, 9), B (0, 0), and C (–15, 0). Justify that (–3, 3) is the centroid of $\triangle ABC$.

4. Verify that the centroid, (–3, 3), of $\triangle ABC$ with vertices A (6, 9), B (0, 0), and C (–15, 0) is $\frac{2}{3}$ the distance from each vertex to the midpoint of the opposite side.

5. $\triangle ABC$ has vertices A (–3, 2), B (4, 7), and C (0, 6). Will the incenter be inside, outside, or on a side of $\triangle ABC$? Explain your answer.

6. $\triangle ABC$ has vertices A (–4, 6), B (1, 6), and C (4, 9). Will the orthocenter be inside, outside, or on a side of $\triangle ABC$? Explain your answer.

continued

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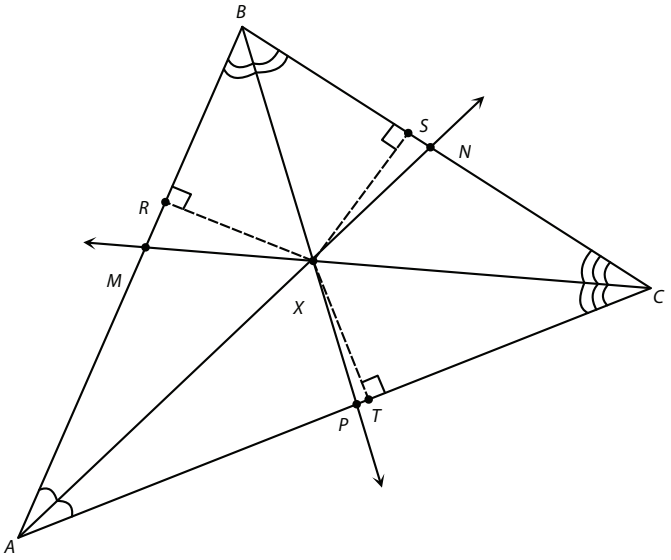
7. The Incenter Theorem states that the incenter of a triangle is equidistant from the sides of a triangle. Prove this theorem using the information provided.

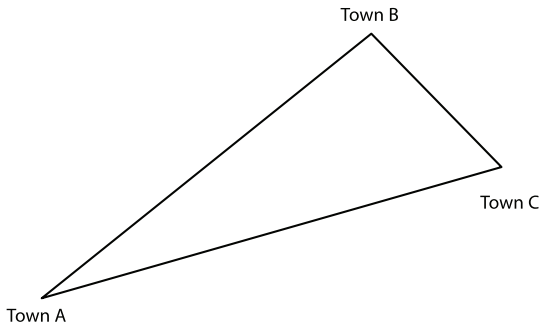
   Given: $\triangle ABC$ has angle bisectors $\overline{AN}$, $\overline{BP}$, and $\overline{CM}$.

   $\overline{XR} \perp \overline{AB}$, $\overline{XS} \perp \overline{BC}$, and $\overline{XT} \perp \overline{AC}$.

   Prove: $XR = XS = XT$



8. A fire station is to be built to assist three towns. The relative locations of the towns and the concurrent roads connecting the towns are shown in the diagram. If the fire station cannot be built outside the area of the triangle, which point(s) of concurrency cannot be used to determine the location of the fire station?



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9. A circular pond is to be constructed in a triangular park. Which center of the park should be determined to create the largest possible pond? Explain your answer.

10. The park's recreation director is determining the location of a new water fountain to be equidistant from the swings, basketball court, and gazebo. Which center of the triangle created between each location should be determined? Explain your answer.

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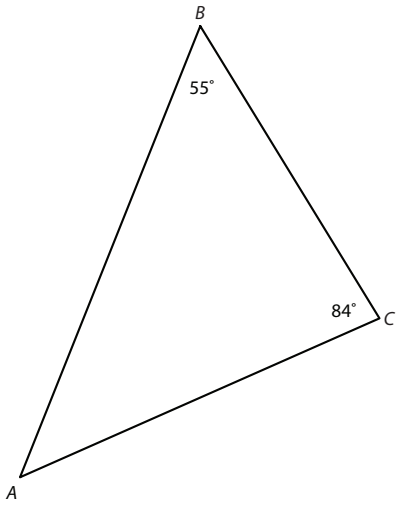
Progress Assessment

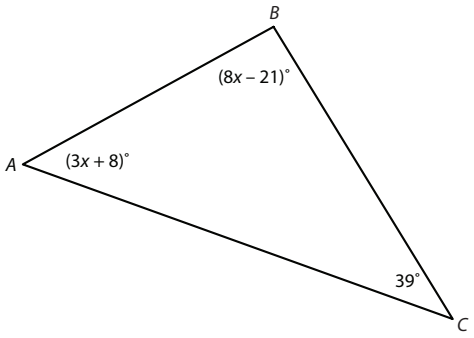
Circle the letter of the best answer.

1. What is the measure of $\angle A$?



a. 135°
b. 41°
c. 20.5°
d. 96°

2. What is the measure of $\angle B$?



a. 14°
b. 70.5°
c. 50°
d. 91°

continued

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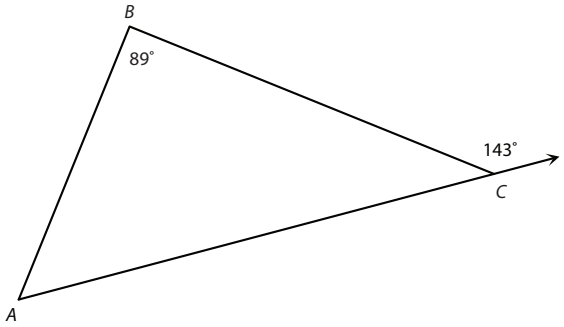
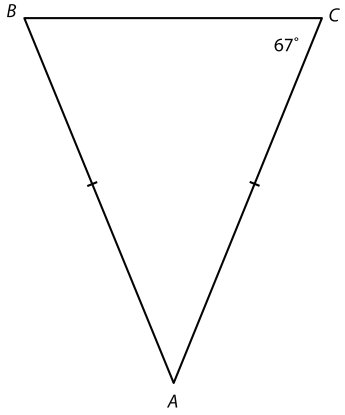
3. What is the measure of $\angle A$?



a. 126°
b. 37°
c. 54°
d. 20°

4. What is the measure of $\angle A$?



a. 46°
b. 113°
c. 67°
d. 134°

continued

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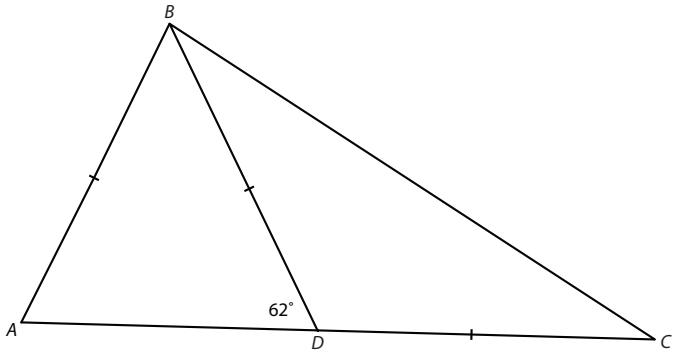
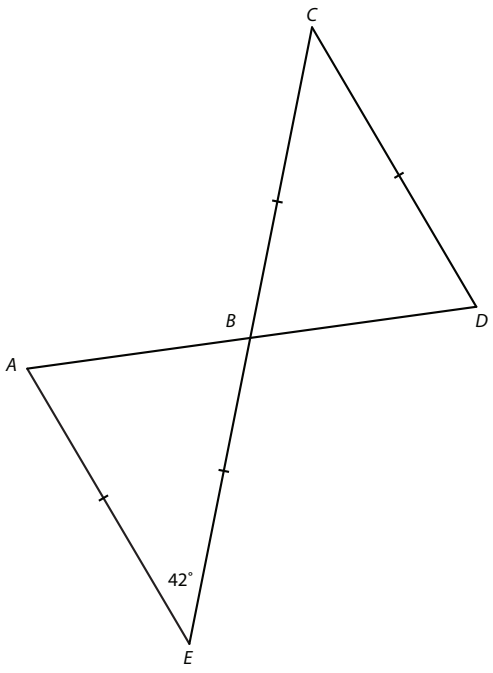
5. What is the measure of $\angle C$?



a. 31°
b. 87°
c. 62°
d. 56°

6. What is the measure of $\angle C$?



a. 138°
b. 111°
c. 69°
d. 42°

continued

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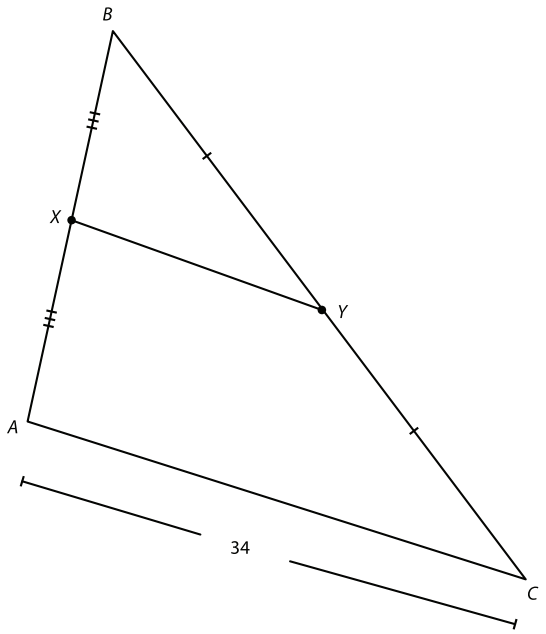
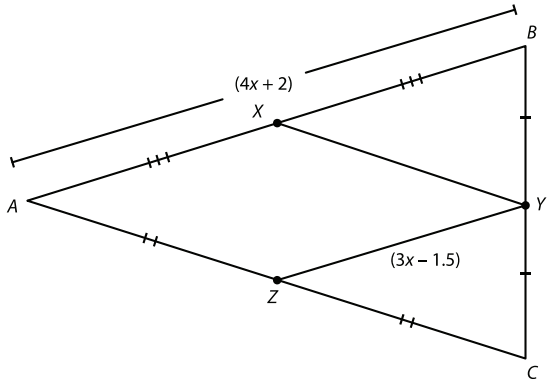
7. If $AC = 34$ units, what is the length of XY?



a. 68 units
b. 17 units
c. 8.5 units
d. 51 units

8. If $AB = 4x + 2$ and $ZY = 3x - 1.5$, what is the length of AB?



a. 6 units
b. 2.5 units
c. 12 units
d. 24 units

continued

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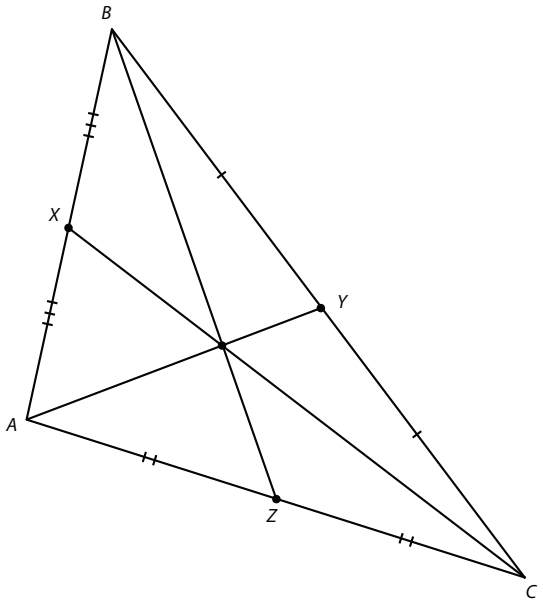
9. Which centers of a triangle could be located outside of a triangle?

   a. centroid and incenter
   b. orthocenter and incenter
   c. circumcenter and centroid
   d. orthocenter and circumcenter

10. Which center is represented in $\triangle ABC$?



   a. incenter
   b. circumcenter
   c. centroid
   d. orthocenter

continued

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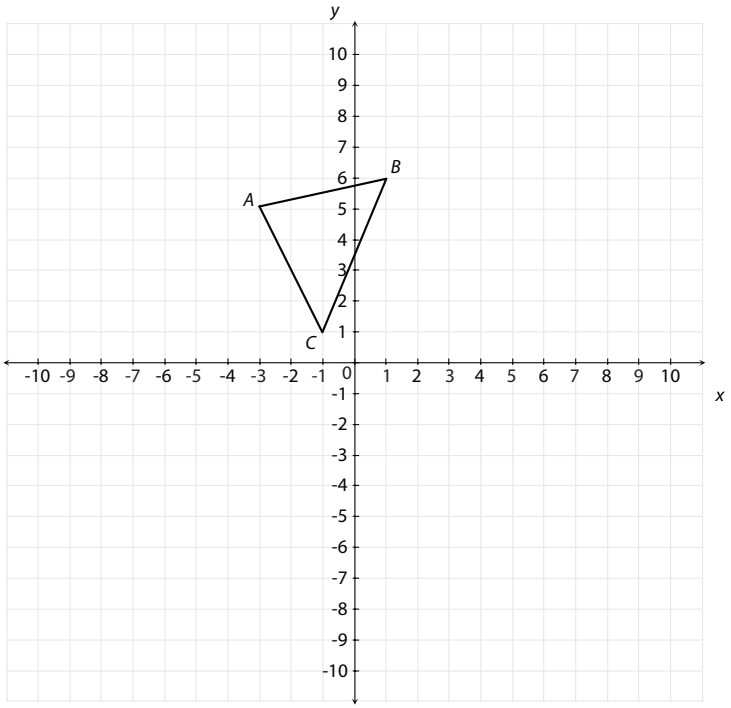
Assessment

Use what you have learned about triangles to solve the problem.

11. $\triangle ABC$ has vertices A (–3, 5), B (1, 6), and C (–1, 1).



a. What are the midpoints of $\triangle ABC$?

b. How does the length of each midsegment compare to the lengths of the sides of $\triangle ABC$?

c. What are the coordinates of the centroid of $\triangle ABC$?

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Determine whether these four vertices form a parallelogram: A (–3, 0), B (6, 0), C (1, 1), D (–2, –2).
   a. No, because both pairs of opposite sides are not parallel.
   b. Yes, because both pairs of opposite sides are parallel.
   c. No, because the diagonals bisect each other.
   d. Yes, because the diagonals bisect each other.

2. Determine whether these four vertices form a parallelogram: S (–7, 3), T (–1, 3), U (–2, 1), V (–8, 1).
   a. No, because both pairs of opposite sides are not parallel.
   b. Yes, because both pairs of opposite sides are parallel.
   c. No, because both pairs of opposite sides are not congruent.
   d. Yes, because both pairs of opposite sides are not congruent.

3. Classify a quadrilateral as precisely as possible given four vertices: P (–6, –3), Q (–3, –7), R (–7, –10), S (–10, –6).
   a. square
   b. rhombus
   c. trapezoid
   d. kite

4. Classify a quadrilateral as precisely as possible given four vertices: D (–5, 2), E (–2, –3), F (8, 3), G (5, 8).
   a. square
   b. kite
   c. isosceles trapezoid
   d. rectangle

5. What is always true about rhombuses?
   a. The diagonals are not congruent.
   b. Two pairs of adjacent distinct sides are congruent.
   c. The diagonals intersect at right angles.
   d. Every angle measures 90°.

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Lesson 7: Proving Theorems About Parallelograms

Instruction

Common Core State Standard

G–CO.11 Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

SMP	
1 ✓	2 ✓
3 ✓	4 ✓
5 ✓	6 ✓
7	8

Essential Questions

1. What makes a quadrilateral a parallelogram?
2. What is the hierarchy of quadrilaterals?
3. What theorems are used for parallelograms?
4. How are rectangles, rhombuses, squares, kites, and trapezoids alike and different?

WORDS TO KNOW

concave polygon	a polygon with at least one interior angle greater than 180° and at least one diagonal that does not lie entirely inside the polygon
consecutive angles	angles of a polygon that share a side
convex polygon	a polygon with no interior angle greater than 180°; all diagonals lie inside the polygon
diagonal	a line segment that connects nonconsecutive vertices of a polygon
isosceles trapezoid	a trapezoid with one pair of opposite parallel sides and congruent legs
kite	a quadrilateral with two distinct pairs of congruent sides that are adjacent
parallelogram	a special type of quadrilateral with two pairs of opposite sides that are parallel; denoted by the symbol $\square$
quadrilateral	a polygon with four sides
rectangle	a special parallelogram with four right angles
rhombus	a special parallelogram with all four sides congruent

SRB p. U5-254

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square	a special parallelogram with four congruent sides and four right angles
trapezoid	a quadrilateral with exactly one pair of opposite parallel sides

Recommended Resources

- MathIsFun.com. "Interactive Quadrilaterals."

  http://www.walch.com/rr/00037

  Review different quadrilaterals with this interactive site. Click the name of a quadrilateral to view its shape and definition, then select and drag the vertices to change or rotate the shape. Options to view the angle measures and/or diagonals of each given quadrilateral are also included.

- Math Warehouse. "Parallelograms."

  http://www.walch.com/rr/00038

  This website gives a brief overview of the properties of parallelograms. Users can examine the relationships among sides and angles with an interactive parallelogram. Each section offers three test questions, with answers provided when you click on the "Answer" button.

- Math Warehouse. "Rhombus: Properties and Shape."

  http://www.walch.com/rr/00039

  Rhombuses are defined and explained with examples at this site. Clickable practice questions are provided to test understanding.

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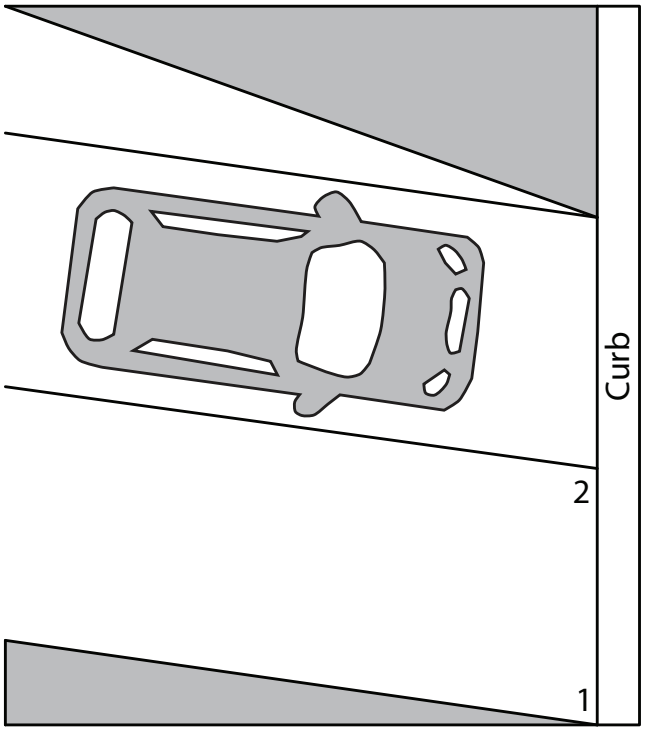
Lesson 5.7.1: Proving Properties of Parallelograms

Warm-Up 5.7.1

Parking lots have standard measures for the width of each space, the backout distance required, and the angle measure. There are several acceptable angles for parking spaces. The desired angle is marked as $\angle 1$ in the illustration below.



1. Find the measure of $\angle 2$ given the desired angle. Assume that the width of the parking spaces is constant, meaning the lines of the spaces are parallel.

$m\angle 1$	$m\angle 2$
45°	
60°	
75°	
90°	

2. Explain how you determined your angle measures.

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Instruction

Lesson 5.7.1: Proving Properties of Parallelograms

Common Core State Standard

G–CO.11 Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

Warm-Up 5.7.1 Debrief

1. Find the measure of $\angle 2$ given the desired angle. Assume that the width of the parking spaces is constant, meaning the lines of the spaces are parallel.

$m\angle 1$	$m\angle 2$
45°	135°
60°	120°
75°	105°
90°	90°

2. Explain how you determined your angle measures.

   $\angle 1$ and $\angle 2$ are same-side interior angles. Same-side interior angles are supplementary. Therefore, use the equation as follows:

   $m\angle 1 + m\angle 2 = 180$

   $m\angle 2 = 180 - m\angle 1$

   Substitute each given measure of $\angle 1$ into the equation in order to solve for the measure of $\angle 2$.

Connection to the Lesson

- Students will be using supplementary angles when finding measures of consecutive angles in a parallelogram.
- Students will extend their understanding of parallel lines intersected by a transversal to two sets of parallel lines that ultimately intersect to form a parallelogram.
- Students will further investigate the relationships of these angles in two intersecting sets of parallel lines.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

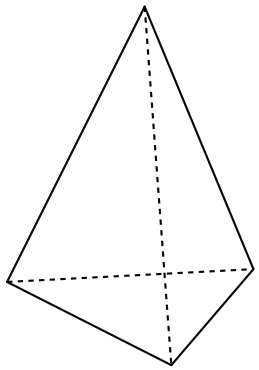
- applying angle relationships in parallel lines intersected by a transversal (G–CO.9)
- applying triangle congruence postulates (G–SRT.5)
- applying triangle similarity postulates (G–SRT.5)
- setting up and solving linear equations (8.EE.7b)
- writing proofs (G–CO.11)

Introduction

What does it mean to be opposite? What does it mean to be consecutive? Think about a rectangular room. If you put your back against one corner of that room and looked directly across the room, you would be looking at the opposite corner. If you looked to your right, that corner would be a consecutive corner. If you looked to your left, that corner would also be a consecutive corner. The walls of the room could also be described similarly. If you were to stand with your back at the center of one wall, the wall straight across from you would be the opposite wall. The walls next to you would be consecutive walls. There are two pairs of opposite walls in a rectangular room, and there are two pairs of opposite angles. Before looking at the properties of parallelograms, it is important to understand what the terms opposite and consecutive mean.

Key Concepts

- A **quadrilateral** is a polygon with four sides.
- A **convex polygon** is a polygon with no interior angle greater than 180° and where all diagonals lie inside the polygon.
- A **diagonal** of a polygon is a line that connects nonconsecutive vertices.



Convex polygon

SRB p. U5-256

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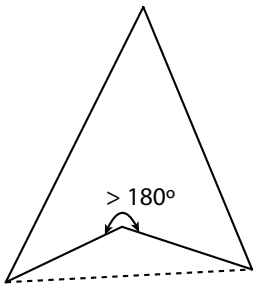
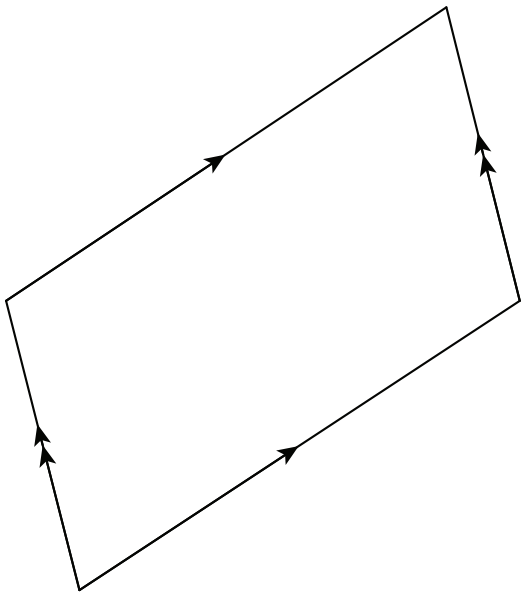
Instruction

- Convex polygons are contrasted with concave polygons.
- A **concave polygon** is a polygon with at least one interior angle greater than 180° and at least one diagonal that does not lie entirely inside the polygon.



Concave polygon

- A **parallelogram** is a special type of quadrilateral with two pairs of opposite sides that are parallel.
- By definition, if a quadrilateral has two pairs of opposite sides that are parallel, then the quadrilateral is a parallelogram.
- Parallelograms are denoted by the symbol $\square$.



Parallelogram

- If a polygon is a parallelogram, there are five theorems associated with it.

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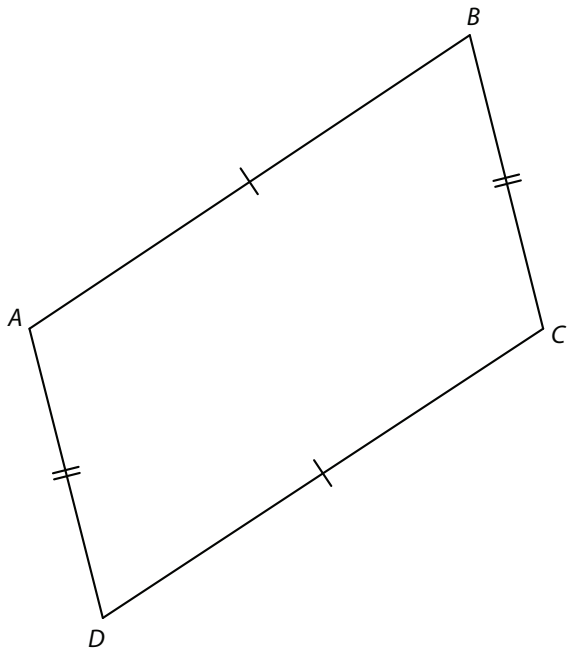
Instruction

- In a parallelogram, both pairs of opposite sides are congruent.

Theorem

If a quadrilateral is a parallelogram, opposite sides are congruent.



$\overline{AB} \cong \overline{DC}$

$\overline{AD} \cong \overline{BC}$

The converse is also true. If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

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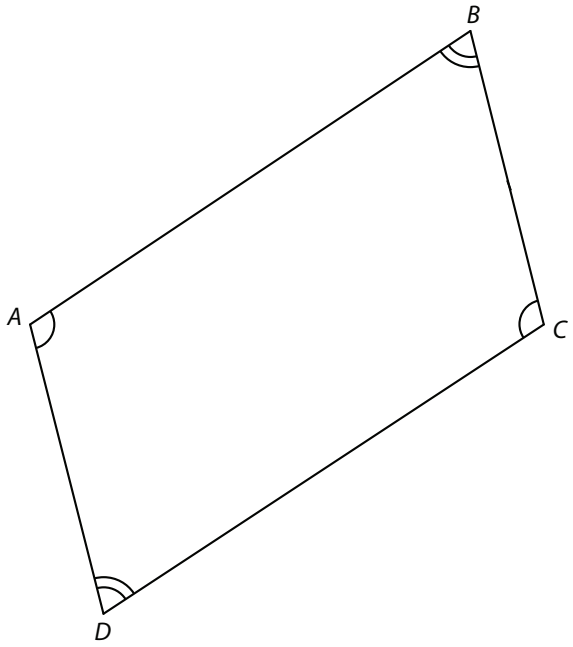
Instruction

- Parallelograms also have two pairs of opposite angles that are congruent.

Theorem

If a quadrilateral is a parallelogram, opposite angles are congruent.



$\angle A \cong \angle C$

$\angle B \cong \angle D$

The converse is also true. If the opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

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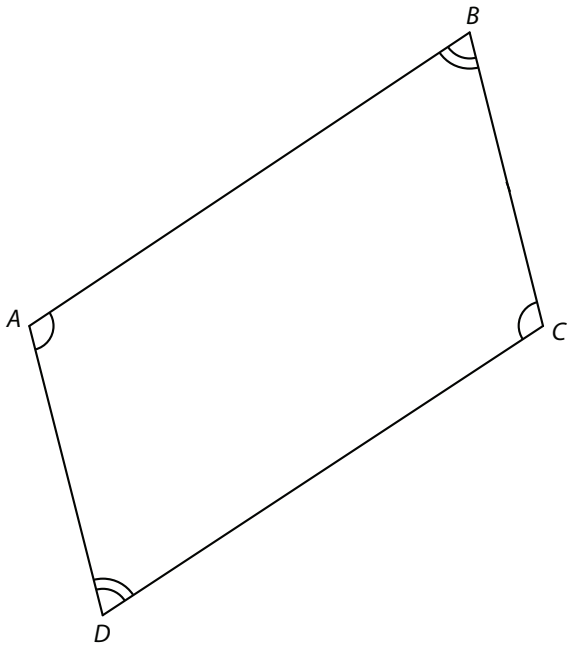
Instruction

- **Consecutive angles** are angles of a polygon that share a side.
- In a parallelogram, consecutive angles are supplementary; that is, they sum to 180°.

Theorem

If a quadrilateral is a parallelogram, then consecutive angles are supplementary.



$m\angle A + m\angle B = 180$

$m\angle B + m\angle C = 180$

$m\angle C + m\angle D = 180$

$m\angle D + m\angle A = 180$

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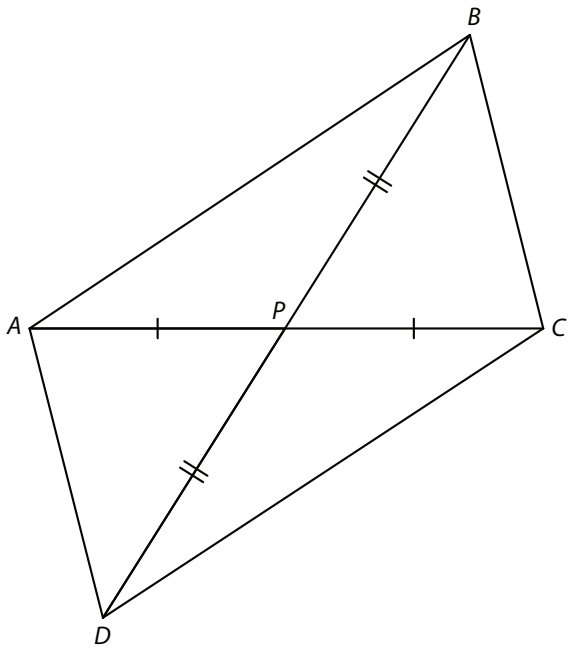
Instruction

- The diagonals of a parallelogram have a relationship. They bisect each other.

Theorem

The diagonals of a parallelogram bisect each other.



$\overline{AP} \cong \overline{PC}$

$\overline{BP} \cong \overline{PD}$

The converse is also true. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

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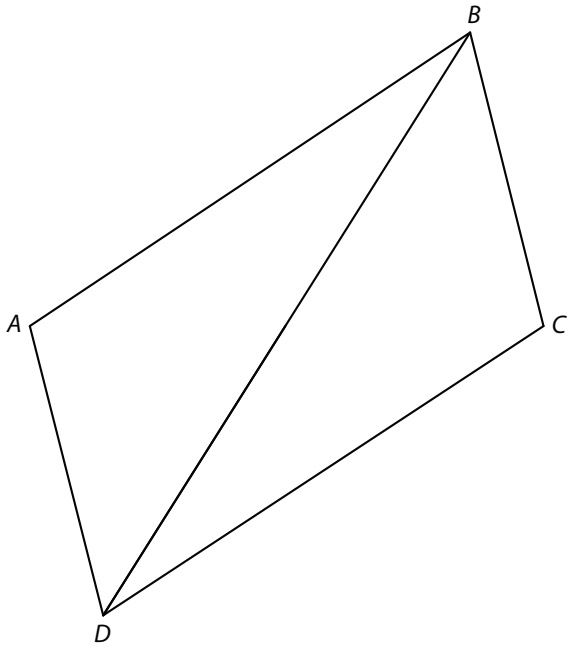
Instruction

- Notice that each diagonal divides the parallelogram into two triangles. Those two triangles are congruent.

Theorem

The diagonal of a parallelogram forms two congruent triangles.



$\triangle BAD \cong \triangle DCB$

Common Errors/Misconceptions

- thinking that all angles in a parallelogram are congruent even if the parallelogram isn't a rectangle or square
- misidentifying opposite pairs of sides
- misidentifying opposite pairs of angles and consecutive angles

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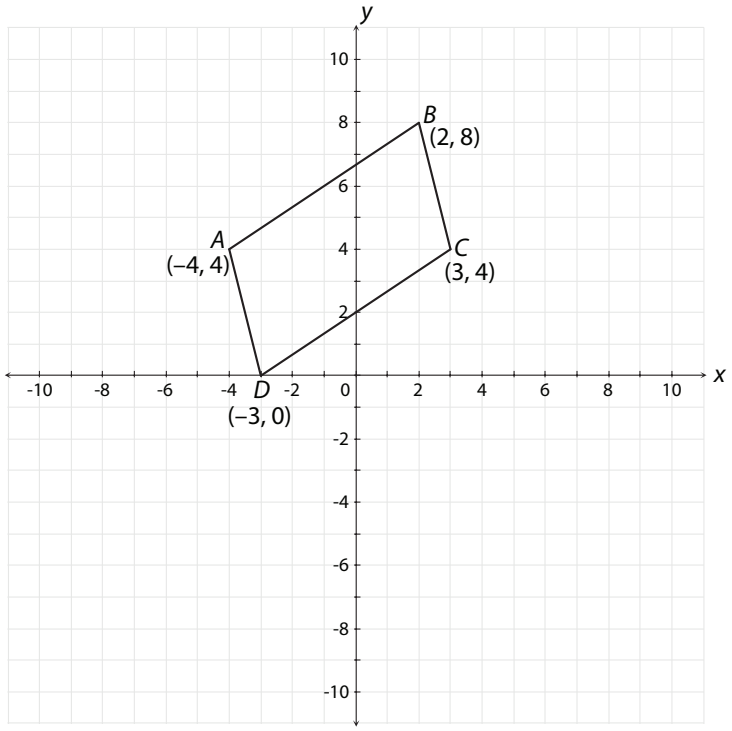
Guided Practice 5.7.1

Example 1

Quadrilateral *ABCD* has the following vertices: *A* (–4, 4), *B* (2, 8), *C* (3, 4), and *D* (–3, 0). Determine whether the quadrilateral is a parallelogram. Verify your answer using slope and distance to prove or disprove that opposite sides are parallel and opposite sides are congruent.

1. Graph the figure.



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Instruction

2. Determine whether opposite pairs of lines are parallel.

Calculate the slope of each line segment.

$\overline{AB}$ is opposite $\overline{DC}$; $\overline{BC}$ is opposite $\overline{AD}$.

$$m_{\overline{AB}}=\frac{\Delta y}{\Delta x}=\frac{(8-4)}{[2-(-4)]}=\frac{4}{6}=\frac{2}{3} \qquad m_{\overline{BC}}=\frac{\Delta y}{\Delta x}=\frac{(4-8)}{(3-2)}=\frac{-4}{1}=-4$$

$$m_{\overline{DC}}=\frac{\Delta y}{\Delta x}=\frac{(4-0)}{[3-(-3)]}=\frac{4}{6}=\frac{2}{3} \qquad m_{\overline{AD}}=\frac{\Delta y}{\Delta x}=\frac{(0-4)}{[-3-(-4)]}=\frac{-4}{1}=-4$$

Calculating the slopes, we can see that the opposite sides are parallel because the slopes of the opposite sides are equal. By the definition of a parallelogram, quadrilateral *ABCD* is a parallelogram.

3. Verify that the opposite sides are congruent.

Calculate the distance of each segment using the distance formula.

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$AB=\sqrt{[2-(-4)]^2+(8-4)^2} \qquad BC=\sqrt{(3-2)^2+(4-8)^2}$$

$$AB=\sqrt{(6)^2+(4)^2} \qquad BC=\sqrt{(1)^2+(-4)^2}$$

$$AB=\sqrt{36+16} \qquad BC=\sqrt{1+16}$$

$$AB=\sqrt{52}=2\sqrt{13} \qquad BC=\sqrt{17}$$

$$DC=\sqrt{[3-(-3)]^2+(4-0)^2} \qquad AD=\sqrt{[-3-(-4)]^2+(0-4)^2}$$

$$DC=\sqrt{(6)^2+(4)^2} \qquad AD=\sqrt{(1)^2+(-4)^2}$$

$$DC=\sqrt{36+16} \qquad AD=\sqrt{1+16}$$

$$DC=\sqrt{52}=2\sqrt{13} \qquad AD=\sqrt{17}$$

From the distance formula, we can see that opposite sides are congruent. Because of the definition of congruence and since $AB = DC$ and $BC = AD$, then $\overline{AB}\cong\overline{DC}$ and $\overline{BC}\cong\overline{AD}$.

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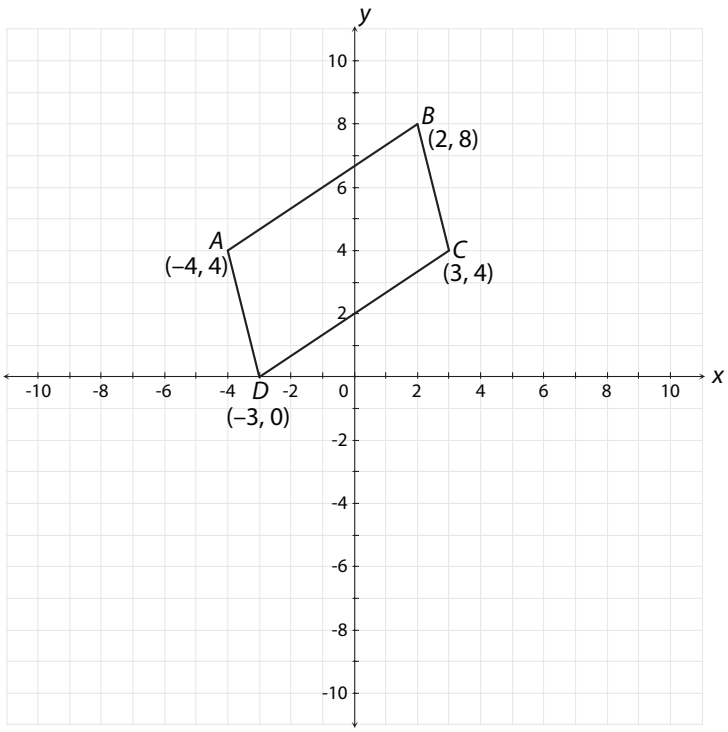
Instruction

Example 2

Use the parallelogram from Example 1 to verify that the opposite angles in a parallelogram are congruent and consecutive angles are supplementary given that $\overline{AD} \| \overline{BC}$ and $\overline{AB} \| \overline{DC}$.



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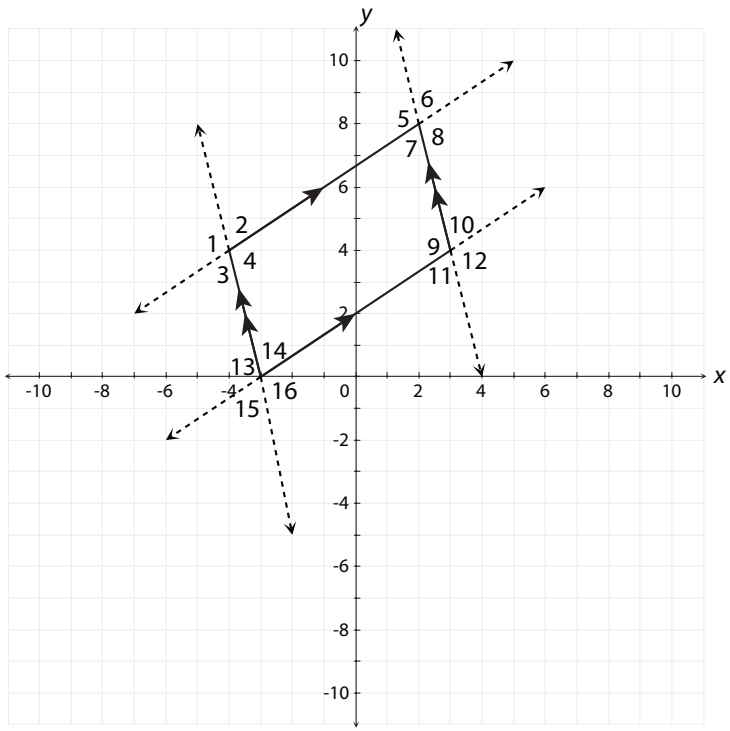
Instruction

1. Extend the lines in the parallelogram to show two pairs of intersecting lines and label the angles with numbers.



2. Prove $\angle 4 \cong \angle 9$.

$\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$	Given
$\angle 4 \cong \angle 13$	Alternate Interior Angles Theorem
$\angle 13 \cong \angle 16$	Vertical Angles Theorem
$\angle 16 \cong \angle 9$	Alternate Interior Angles Theorem
$\angle 4 \cong \angle 9$	Transitive Property

We have proven that one pair of opposite angles in a parallelogram is congruent.

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Instruction

3. Prove $\angle 7 \cong \angle 14$.

$\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$	Given
$\angle 7 \cong \angle 10$	Alternate Interior Angles Theorem
$\angle 10 \cong \angle 11$	Vertical Angles Theorem
$\angle 11 \cong \angle 14$	Alternate Interior Angles Theorem
$\angle 7 \cong \angle 14$	Transitive Property

We have proven that both pairs of opposite angles in a parallelogram are congruent.

4. Prove that consecutive angles of a parallelogram are supplementary.

$\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$	Given
$\angle 4$ and $\angle 14$ are supplementary.	Same-Side Interior Angles Theorem
$\angle 14$ and $\angle 9$ are supplementary.	Same-Side Interior Angles Theorem
$\angle 9$ and $\angle 7$ are supplementary.	Same-Side Interior Angles Theorem
$\angle 7$ and $\angle 4$ are supplementary.	Same-Side Interior Angles Theorem

We have proven consecutive angles in a parallelogram are supplementary using the Same-Side Interior Angles Theorem of a set of parallel lines intersected by a transversal.

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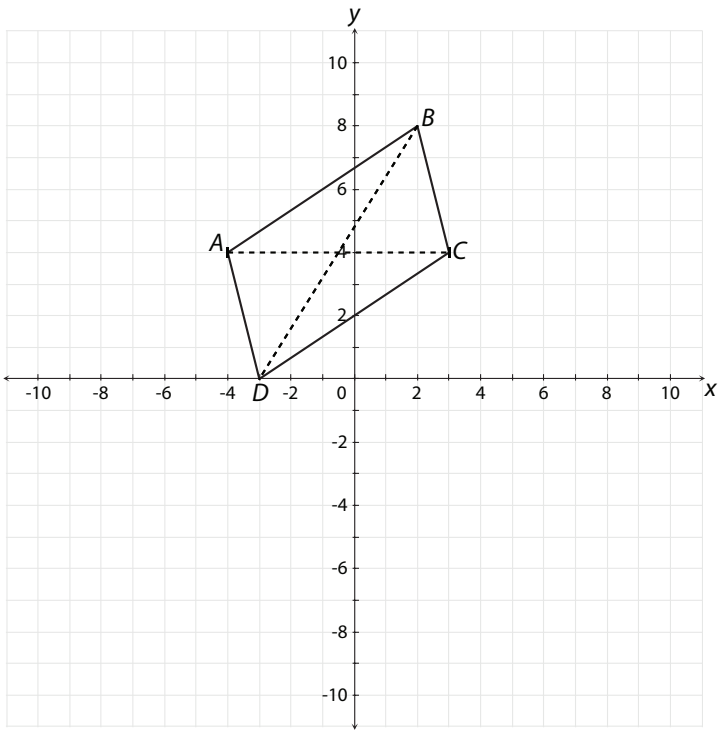
Instruction

Example 3

Use the parallelogram from Example 1 to prove that diagonals of a parallelogram bisect each other.



1. Find the midpoint of $\overline{AC}$, where M stands for midpoint.

   By definition, the midpoint is the point on a segment that divides the segment into two congruent parts.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$ Midpoint formula

$$M_{\overline{AC}} = \left(\frac{-4+3}{2}, \frac{4+4}{2} \right) = \left(\frac{-1}{2}, \frac{8}{2} \right) = \left(-\frac{1}{2}, 4 \right)$$ Substitute values for x_1, x_2, y_1, and y_2, then solve.

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Instruction

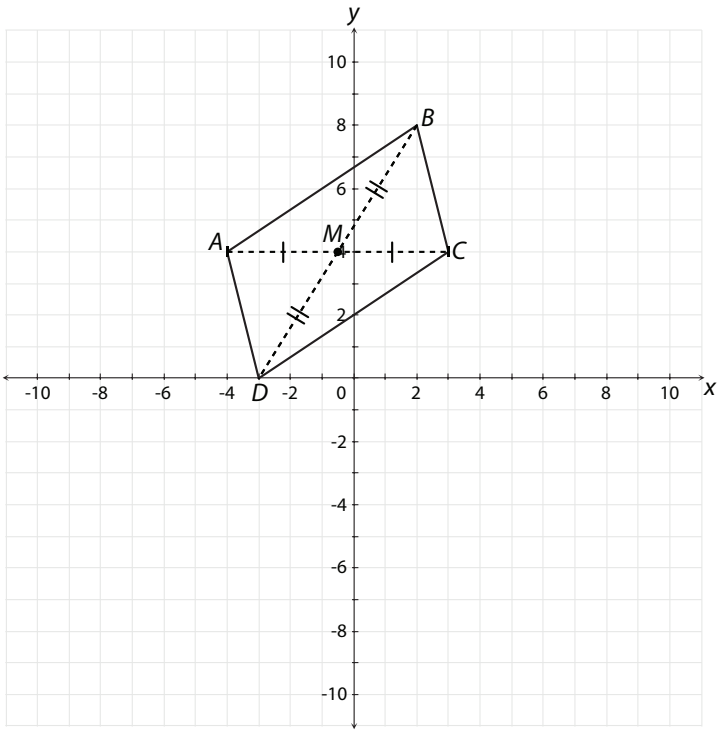
2. Find the midpoint of $\overline{DB}$.

$$M=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$ Midpoint formula

$$M_{\overline{DB}}=\left(\frac{-3+2}{2},\frac{0+8}{2}\right)=\left(\frac{-1}{2},\frac{8}{2}\right)=\left(-\frac{1}{2},4\right)$$ Substitute values for x_1, x_2, y_1, and y_2, then solve.

3. Mark the midpoint of each segment on the graph.

Notice that the midpoint of $\overline{AC}$ and the midpoint of $\overline{DB}$ are the same point.



4. Write statements that prove the diagonals bisect each other.

Since M is the midpoint of $\overline{AC}$, $\overline{AM}\cong\overline{MC}$. M is also a point on $\overline{DB}$. Therefore, $\overline{DB}$ is the bisector of $\overline{AC}$. The midpoint of $\overline{DB}$ is M. This means that $\overline{DM}\cong\overline{DB}$. Since M is a point on $\overline{AC}$, $\overline{AC}$ is the bisector of $\overline{DB}$. The diagonals bisect each other.

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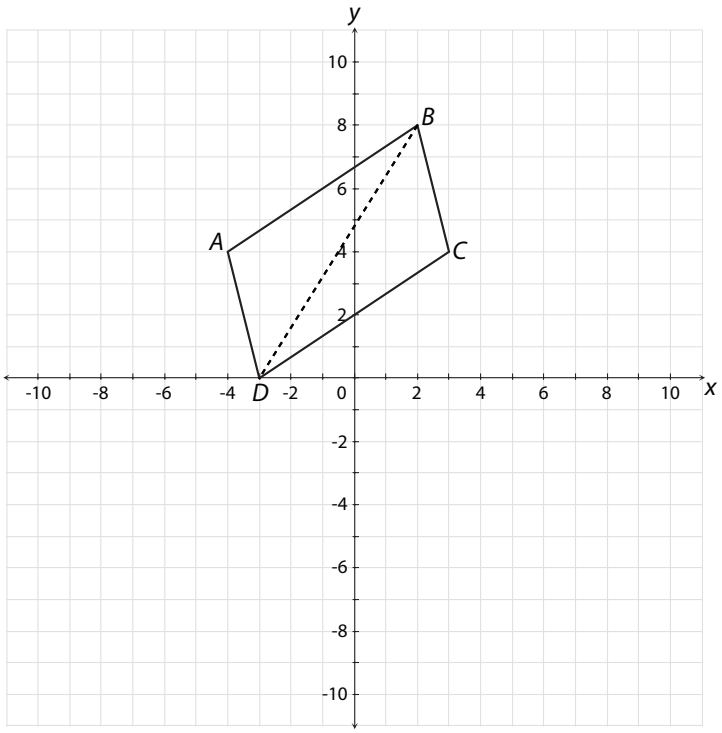
Instruction

Example 4

Use the parallelogram from Example 1 and the diagonal $\overline{DB}$ to prove that a diagonal of a parallelogram divides the parallelogram into two congruent triangles.



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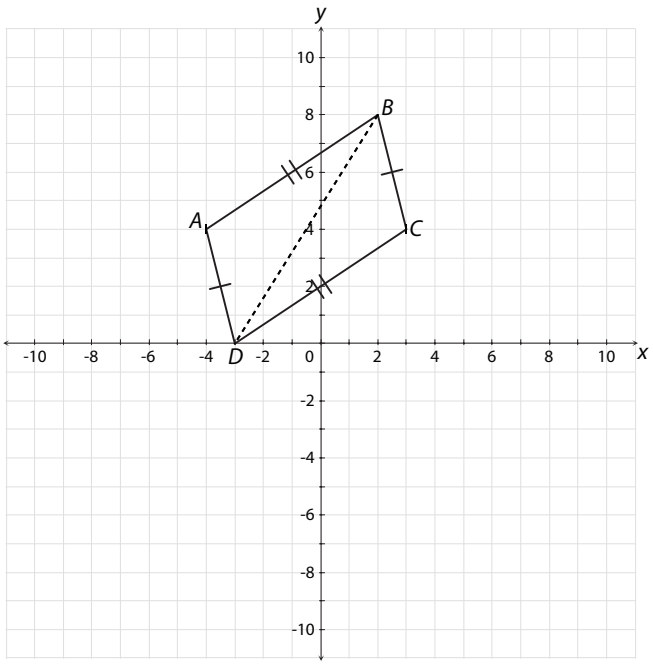
Instruction

1. Use theorems about parallelograms to mark congruent sides.

   Opposite sides of a parallelogram are congruent, as proven in Example 1.

   $\overline{AB} \cong \overline{DC}$ and $\overline{BC} \cong \overline{AD}$ Opposite sides of a parallelogram are congruent.

   So far, we know that the triangles each have two sides that are congruent to the corresponding sides of the other triangle. To prove triangles congruent, we could use ASA, SAS, or SSS. From the information we have, we could either try to find the third side congruent or the included angles congruent.



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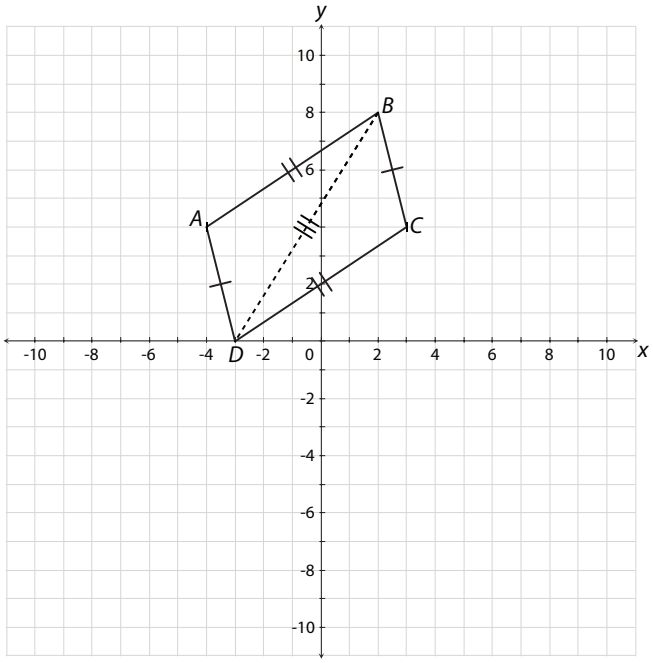
Instruction

2. Use the Reflexive Property to identify a third side of the triangle that is congruent.

$\overline{DB} \cong \overline{DB}$ by the Reflexive Property.



Now, all three sides of the triangles are congruent.

3. State the congruent triangles.

Using SSS, we verified that $\triangle DAB \cong \triangle BCD$. Therefore, the diagonal divides the parallelogram into two congruent triangles.

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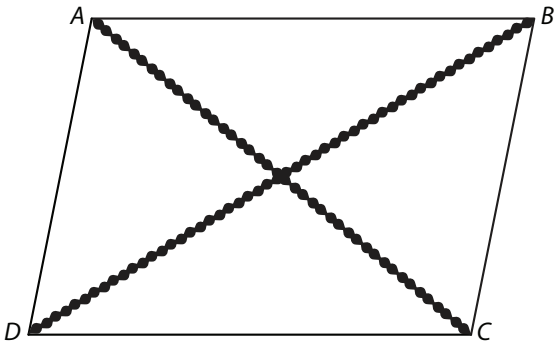
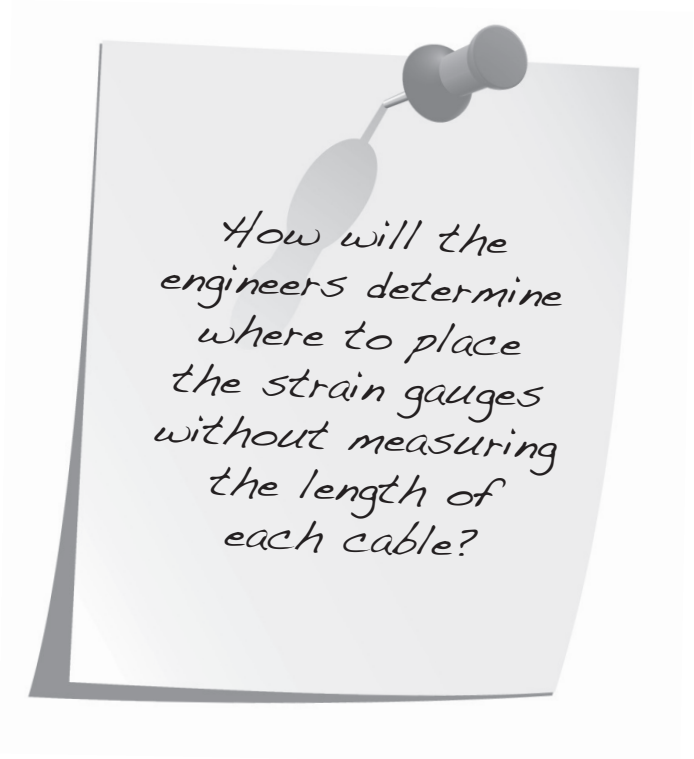
Problem-Based Task 5.7.1: Measuring Bridge Cable Tension

SMP	
1 ✓	2 ✓
3 ✓	4
5	6
7	8

A bridge truss is in the shape of a parallelogram as pictured below. To support the truss, steel cables are placed on each of the diagonals of the parallelogram. Engineers are concerned with the tension in one pair of cables, and want to put a strain gauge in the center of the two cables to measure the tension. If the tension is too high, the cables could snap. How will the engineers determine where to place the strain gauges without measuring the length of each cable? Prove that the engineers' decision will work. Use the diagram below for reference in your proof.





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Problem-Based Task 5.7.1: Measuring Bridge Cable Tension

Coaching

a. What is the property of the diagonals of a parallelogram?

b. How can you use this property?

c. Which lines are parallel?

d. What are the transversals?

e. What angles are congruent and how do you know?

f. What sides are congruent and how do you know?

g. What triangles are congruent and how do you know?

h. What segments are congruent that you have not already identified? How do you know?

i. Summarize your findings and write your proof.

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Instruction

Problem-Based Task 5.7.1: Measuring Bridge Cable Tension

Coaching Sample Responses

a. What is the property of the diagonals of a parallelogram?

The diagonals of a parallelogram bisect each other.

b. How can you use this property?

Where the cables intersect is the midpoint of each cable and, therefore, the center of the cable. That is where the strain gauge should be located.

c. Which lines are parallel?

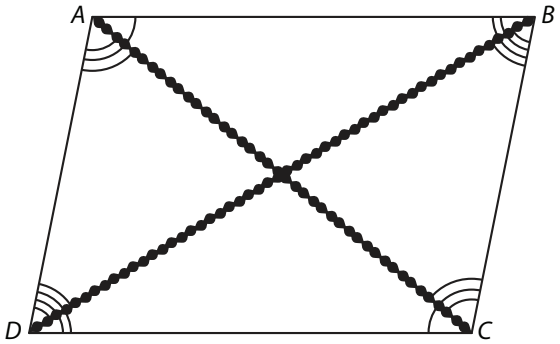
Since the truss *ABCD* is a parallelogram, opposite sides are parallel. This means that the top is parallel to the bottom and the left side is parallel to the right side; or, stated mathematically, $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$.

d. What are the transversals?

Each cable acts as a transversal to both sets of parallel lines. This means that $\overline{AC}$ is a transversal of $\overline{AB}$ and $\overline{DC}$, as well as a transversal of $\overline{AD}$ and $\overline{BC}$. This also means that $\overline{DB}$ is a transversal of $\overline{AB}$ and $\overline{DC}$, as well as a transversal of $\overline{AD}$ and $\overline{BC}$.

e. What angles are congruent and how do you know?

Because alternate interior angles of a set of parallel lines intersected by a transversal are congruent, $\angle BAC \cong \angle DCA$, $\angle BDC \cong \angle DBA$, $\angle DAC \cong \angle ACB$, and $\angle ADB \cong \angle DBC$.



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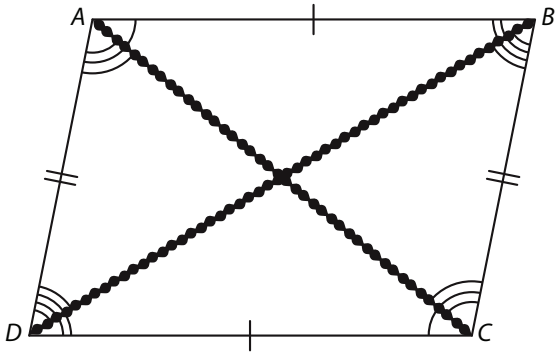
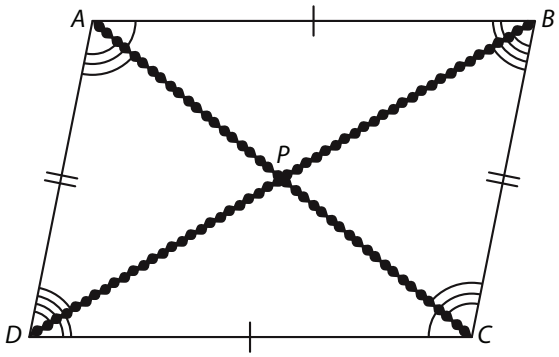
f. What sides are congruent and how do you know?

Opposite sides of a parallelogram are congruent. Therefore, $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$.



g. What triangles are congruent and how do you know?

Label the intersection point of the diagonals with the letter *P*.



By ASA, $\triangle ABP \cong \triangle CDP$ and $\triangle ADP \cong \triangle CBP$.

h. What segments are congruent that you have not already identified? How do you know?

Because of CPCTC, $\overline{AP} \cong \overline{CP}$ and $\overline{DP} \cong \overline{BP}$. By the definition of bisection, $\overline{AC}$ bisects $\overline{DB}$ and $\overline{DB}$ bisects $\overline{AC}$.

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i. Summarize your findings and write your proof.

Given that $ABCD$ is a parallelogram, opposite sides are parallel, so $\overline{AB} \| \overline{DC}$ and $\overline{AD} \| \overline{BC}$. Since alternate interior angles of a set of parallel lines intersected by a transversal are congruent, $\angle BAC \cong \angle DCA$, $\angle BDC \cong \angle DBA$, $\angle DAC \cong \angle ACB$, and $\angle ADB \cong \angle DBC$. Opposite sides of a parallelogram are congruent. Therefore, $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$. By SAS, $\triangle ABP \cong \triangle CDP$ and $\triangle ADP \cong \triangle CBP$. Since corresponding parts of congruent triangles are congruent, $\overline{AP} \cong \overline{CP}$ and $\overline{DP} \cong \overline{BP}$. By the definition of bisection, $\overline{AC}$ bisects $\overline{DB}$ and $\overline{DB}$ bisects $\overline{AC}$.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 5.7.1: Proving Properties of Parallelograms

Use slope to determine whether the given vertices form a parallelogram.

1. A (–2, –1), B (–1, 3), C (4, 3), and D (3, –1)

2. F (–1, 1), G (1, 3), H (4, –2), and I (2, –4)

Use the distance formula to determine whether the given vertices form a parallelogram.

3. J (0, 0), K (1, 5), L (4, 6), and M (3, 0)

4. M (0, 8), N (–1, 2), O (5, 6), and P (6, 12)

Use the midpoint formula to determine whether the given vertices form a parallelogram.

5. P (–2, 2), Q (1, 8), R (4, 5), and S (3, –2)

6. D (2, 2), E (4, 5), F (10, 6), and G (8, 3)

continued

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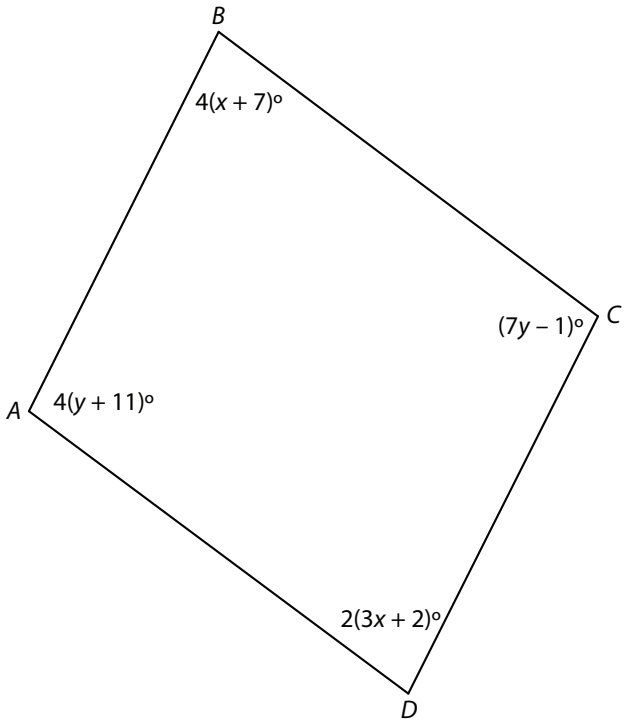
Determine the unknown angle measures and the values of x and y that make quadrilateral $ABCD$ a parallelogram.

7.



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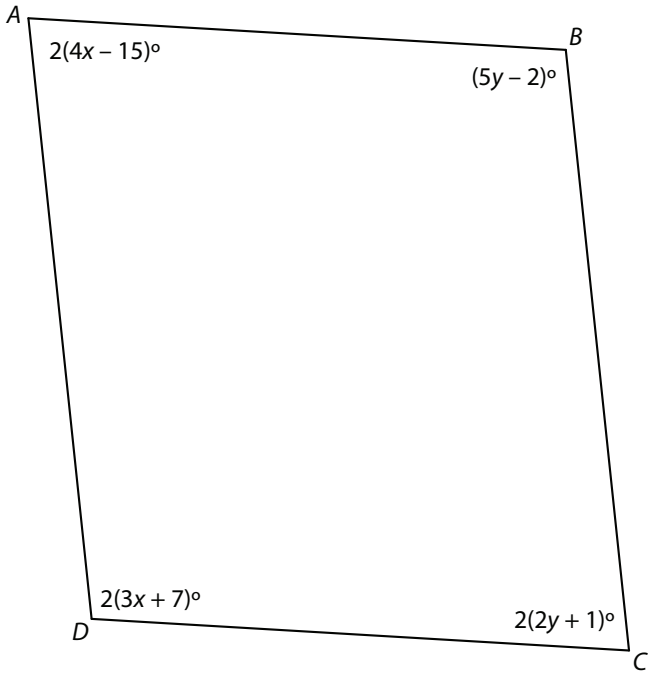

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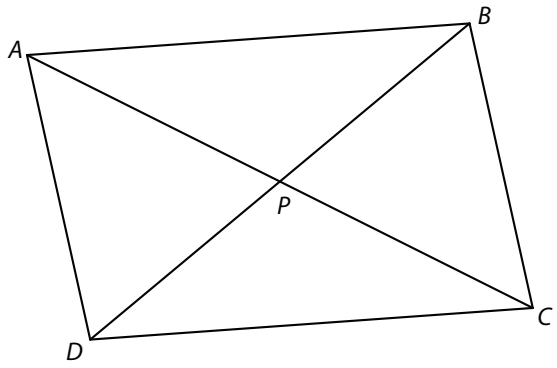
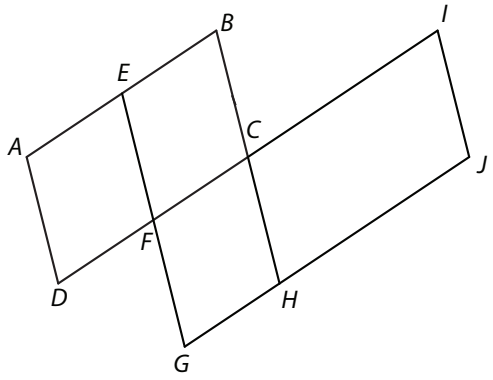
Prove the following using the proof method of your choosing.

9. Given $\square ABCD$, prove that $\triangle DPA \cong \triangle BPC$.



10. Given $\square ABCD$, $\square EBHG$, and $\square FIJG$ prove $\angle D \cong \angle I$.



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Lesson 5.7.2: Proving Properties of Special Quadrilaterals

Warm-Up 5.7.2

You are taking a road trip from Carrollton, Georgia, to Campton, Georgia, traveling through Atlanta. Your best efforts to avoid traffic will take you around the city of Atlanta, but you want to know how far it is "as the crow flies" (moving in a straight line) between Carrollton and Campton.

1. When looking at a map with a grid, if Carrollton lies at (2, 2) and Campton lies at (10, 4.5), calculate the distance between those two cities if 1 unit on the grid is approximately 10 miles.

2. If Atlanta is the midpoint between the two cities, where does Atlanta lie on the grid?

3. What is the distance between Atlanta and each of the cities?

4. What is the slope of the line that passes through these three cities?

5. What is the slope of a line perpendicular to the line through the cities?

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Instruction

Lesson 5.7.2: Proving Properties of Special Quadrilaterals

Common Core State Standard

G–CO.11 Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

Warm-Up 5.7.2 Debrief

1. When looking at a map with a grid, if Carrollton lies at (2, 2) and Campton lies at (10, 4.5), calculate the distance between those two cities if 1 unit on the grid is approximately 10 miles.

   Use the distance formula: $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

   $d=\sqrt{(10-2)^2+(4.5-2)^2}$

   $d=\sqrt{(8)^2+(2.5)^2}$

   $d=\sqrt{64+6.25}$

   $d=\sqrt{70.25}$

   $d\approx 8.4$

   The distance between Carrollton and Campton is about 8.4 units. Multiply that by 10 miles.

   The distance between Carrollton and Campton is about 84 miles.

2. If Atlanta is the midpoint between the two cities, where does Atlanta lie on the grid?

   Use the midpoint formula to find Atlanta's location on the grid.

   $M=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

   $M=\left(\frac{2+10}{2},\frac{2+4.5}{2}\right)$

   $M=\left(\frac{12}{2},\frac{6.5}{2}\right)=(6,3.25)$

   Atlanta is at (6, 3.25) on the grid.

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Instruction

3. What is the distance between Atlanta and each of the cities?

   By the definition of a midpoint, each shorter segment is half the distance of the total segment.

   $\frac{84}{2} = 42$ miles

   Each city is about 42 miles from Atlanta.

4. What is the slope of the line that passes through these three cities?

   Using the endpoints, calculate the slope of the line.

   The endpoints are the grid locations of Carrollton and Campton: (2, 2) and (10, 4.5).

   $$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{(4.5-2)}{(10-2)} = \frac{2.5}{8} = \frac{\frac{5}{2}}{8} = \frac{5}{2} \bullet \frac{1}{8} = \frac{5}{16}$$

   The slope is $\frac{5}{16}$.

5. What is the slope of a line perpendicular to the line through the cities?

   Perpendicular lines have opposite reciprocal slopes.

   Since the slope of the original line is $\frac{5}{16}$, a line perpendicular to that line would have a slope of $-\frac{16}{5}$.

Connection to the Lesson

- Students will be using distance, midpoints, and slopes to prove properties of special quadrilaterals.
- Students will also be using distance, midpoints, and slopes to classify special quadrilaterals.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- calculating distance, midpoint, and slope (8.G.8, G–GPE.6, 8.F.4)
- applying angle relationships of parallel lines intersected by a transversal (G–CO.9)
- applying triangle congruency postulates (G–SRT.5)
- writing proofs (G–CO.11)

Introduction

There are many kinds of quadrilaterals. Some quadrilaterals are parallelograms; some are not. For example, trapezoids and kites are special quadrilaterals, but they are not parallelograms.

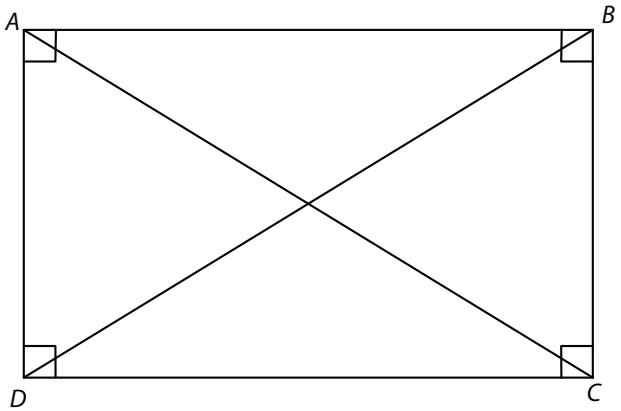
Some parallelograms are known as special parallelograms. What makes a parallelogram a more specialized parallelogram? Rectangles, rhombuses, and squares are all special parallelograms with special properties. They have all the same characteristics that parallelograms have, plus more.

Key Concepts

- A **rectangle** has four sides and four right angles.
- A rectangle is a parallelogram, so opposite sides are parallel, opposite angles are congruent, and consecutive angles are supplementary.
- The diagonals of a rectangle bisect each other and are also congruent.

Theorem

If a parallelogram is a rectangle, then the diagonals are congruent.



$\overline{AC} \cong \overline{DB}$

SRB p. U5-276

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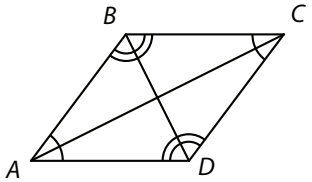
Instruction

- A **rhombus** is a special parallelogram with all four sides congruent.
- Since a rhombus is a parallelogram, opposite sides are parallel, opposite angles are congruent, and consecutive angles are supplementary.
- The diagonals bisect each other; additionally, they also bisect the opposite pairs of angles within the rhombus.

Theorem

If a parallelogram is a rhombus, the diagonals of the rhombus bisect the opposite pairs of angles.



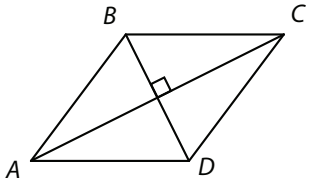
$\angle BAC \cong \angle CAD \cong \angle BCA \cong \angle DCA$

$\angle CBD \cong \angle ABD \cong \angle ADB \cong \angle CDB$

- The diagonals of a rhombus also form four right angles where they intersect.

Theorem

If a parallelogram is a rhombus, the diagonals are perpendicular.



$\overline{BD} \perp \overline{AC}$

The converse is also true. If the diagonals of a parallelogram intersect at a right angle, then the parallelogram is a rhombus.

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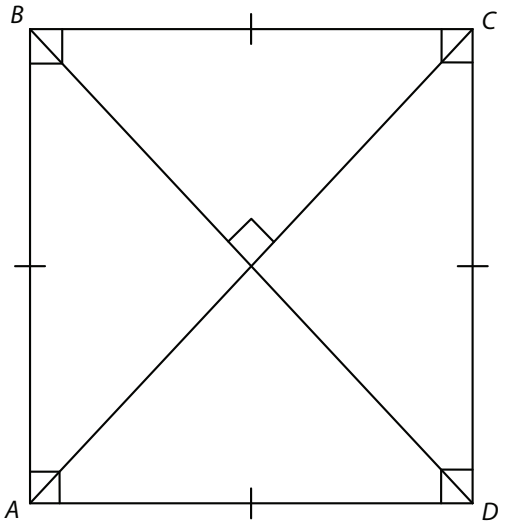
Instruction

- A **square** has all the properties of a rectangle and a rhombus.
- Squares have four congruent sides and four right angles.
- The diagonals of a square bisect each other, are congruent, and bisect opposite pairs of angles.
- The diagonals are also perpendicular.

Properties of Squares



$\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$

$\overline{BD} \cong \overline{AC}$

$\overline{BD} \perp \overline{AC}$

$m\angle A = m\angle B = m\angle C = m\angle D = 90^\circ$

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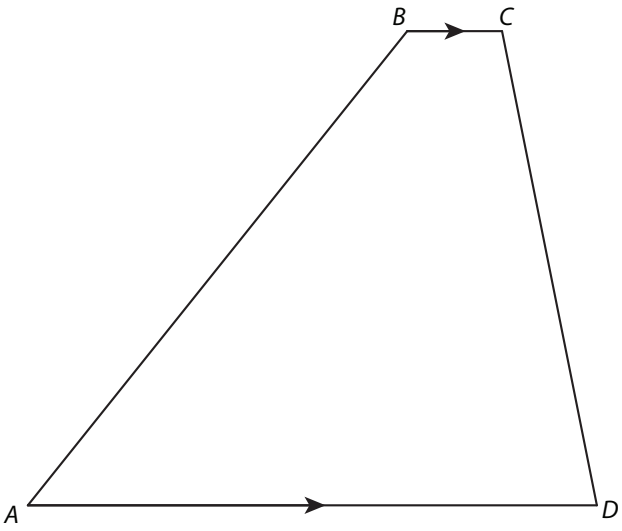
Instruction

- **Trapezoids** are quadrilaterals with exactly one pair of opposite parallel lines.
- Trapezoids are not parallelograms because they do not have two pairs of opposite lines that are parallel.
- The lines in a trapezoid that are parallel are called the bases, and the lines that are not parallel are called the legs.

Properties of Trapezoids



$\overline{BA}$ and $\overline{CD}$ are the legs.

$\overline{BC}$ and $\overline{AD}$ are the bases.

$\overline{BC} \parallel \overline{AD}$

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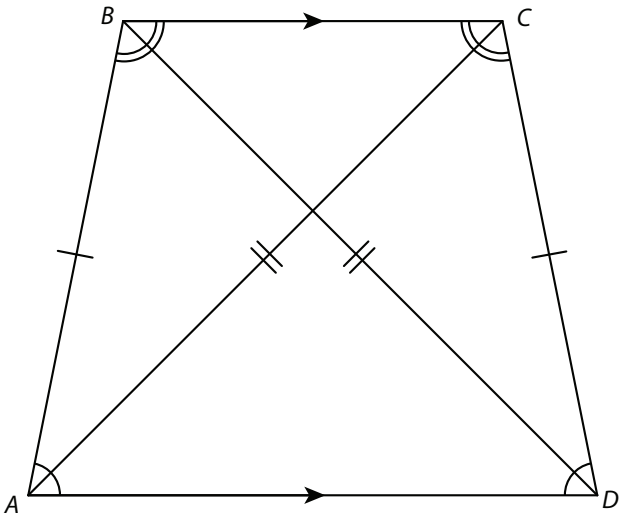
Instruction

- **Isosceles trapezoids** have one pair of opposite parallel sides, and congruent legs.
- Since the legs are congruent, both pairs of base angles are also congruent, similar to the legs and base angles in an isosceles triangle.
- The diagonals of an isosceles trapezoid are congruent.

Properties of Isosceles Trapezoids



$\overline{BA}$ and $\overline{CD}$ are the legs.

$\overline{BC}$ and $\overline{AD}$ are the bases.

$\overline{BC} \parallel \overline{AD}$

$\overline{BA} \cong \overline{CD}$ and $\overline{AC} \cong \overline{BD}$

$\angle BAD \cong \angle ADC$ and $\angle ABC \cong \angle BCD$

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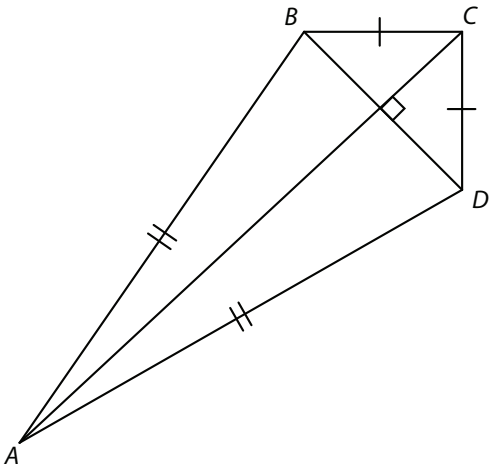
Instruction

- A **kite** is a quadrilateral with two distinct pairs of congruent sides that are adjacent.
- Kites are not parallelograms because opposite sides are not parallel.
- The diagonals of a kite are perpendicular.

Properties of Kites



$\overline{CD} \cong \overline{CB}$ and $\overline{AB} \cong \overline{AD}$

$\overline{CA} \perp \overline{BD}$

- Quadrilaterals can be grouped according to their properties. This kind of grouping is called a hierarchy.
- In the following hierarchy of quadrilaterals, you can see that all quadrilaterals are polygons but that not all polygons are quadrilaterals.
- The arrows connecting the types of quadrilaterals indicate a special version of the category above each quadrilateral type. For example, parallelograms are special quadrilaterals. Rectangles and rhombuses are special parallelograms, and squares have all the properties of rectangles and rhombuses.

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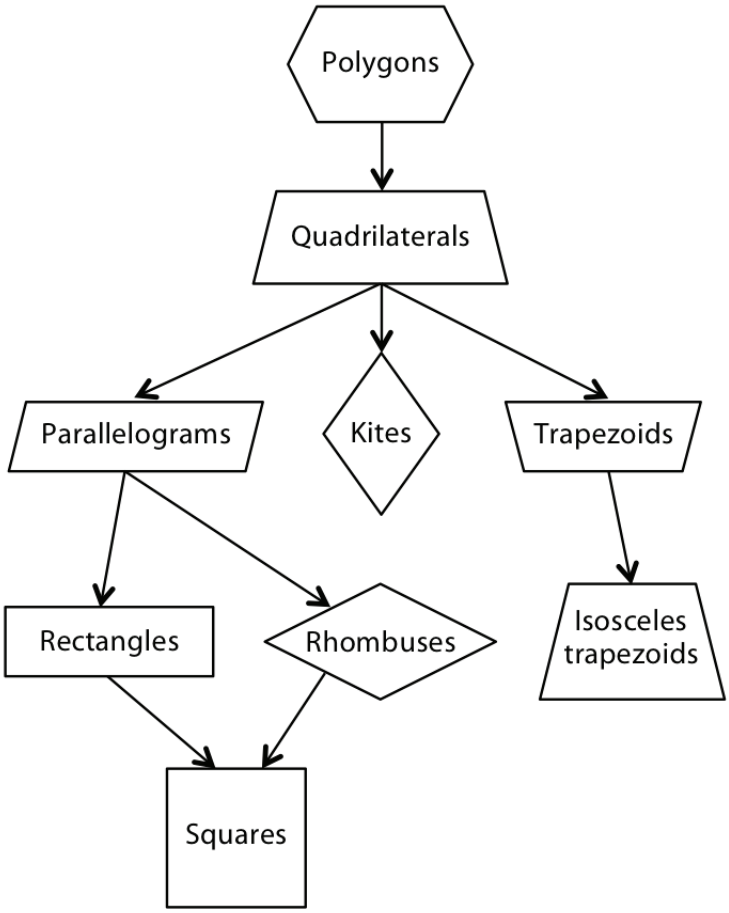
Hierarchy of Quadrilaterals



Common Errors/Misconceptions

- assuming both pairs of opposite sides are parallel after determining that one pair of opposite sides is parallel
- mistakenly classifying a rhombus as a square since all the sides are congruent
- confusing the properties among the special quadrilaterals
- not understanding that rectangles have the same properties as a parallelogram plus additional properties
- not understanding the hierarchy of quadrilaterals

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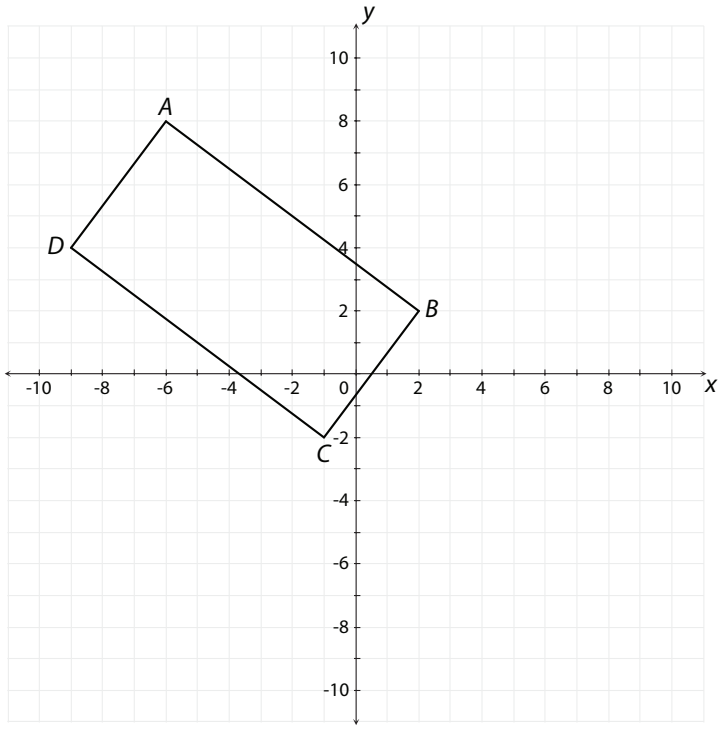
Instruction

Guided Practice 5.7.2

Example 1

Quadrilateral *ABCD* has vertices *A* (–6, 8), *B* (2, 2), *C* (–1, –2), and *D* (–9, 4). Using slope, distance, and/or midpoints, classify quadrilateral *ABCD* as a rectangle, rhombus, square, trapezoid, isosceles trapezoid, or kite.

1. Graph the quadrilateral.



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Instruction

2. Calculate the slopes of the sides to determine if opposite sides are parallel.

   If opposite sides are parallel, the quadrilateral is a parallelogram.

$$m_{\overline{AB}} = \frac{\Delta y}{\Delta x} = \frac{(2-8)}{[2-(-6)]} = \frac{-6}{8} = -\frac{3}{4}$$

$$m_{\overline{DC}} = \frac{\Delta y}{\Delta x} = \frac{(-2-4)}{[-1-(-9)]} = \frac{-6}{8} = -\frac{3}{4}$$

   The first pair of opposite sides is parallel: $\overline{AB} \parallel \overline{DC}$.

$$m_{\overline{AD}} = \frac{\Delta y}{\Delta x} = \frac{(4-8)}{[-9-(-6)]} = \frac{-4}{-3} = \frac{4}{3}$$

$$m_{\overline{BC}} = \frac{\Delta y}{\Delta x} = \frac{(-2-2)}{(-1-2)} = \frac{-4}{-3} = \frac{4}{3}$$

   The second pair of opposite sides is parallel: $\overline{AD} \parallel \overline{BC}$. Therefore, the quadrilateral is a parallelogram.

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3. Examine the slopes of the consecutive sides to determine if they intersect at right angles.

   If the slopes are opposite reciprocals, the lines are perpendicular and therefore form right angles. If there are four right angles, the quadrilateral is a rectangle or a square.

   $m_{\overline{AB}} = m_{\overline{DC}} = -\frac{3}{4}$

   $m_{\overline{AD}} = m_{\overline{BC}} = \frac{4}{3}$

   $-\frac{3}{4}$ is the opposite reciprocal of $\frac{4}{3}$.

   The slopes of the consecutive sides are perpendicular: $\overline{AB} \perp \overline{AD}$ and $\overline{DC} \perp \overline{BC}$. There are four right angles at the vertices. The parallelogram is a rectangle or a square.

   You could also determine if the diagonals are congruent by calculating the length of each diagonal using the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. If the diagonals are congruent, then the parallelogram is a rectangle or square.

   $AC = \sqrt{[-1-(-6)]^2 + (-2-8)^2}$ $\qquad$ $DB = \sqrt{[2-(-9)]^2 + (2-4)^2}$

   $AC = \sqrt{(5)^2 + (-10)^2}$ $\qquad$ $DB = \sqrt{(11)^2 + (-2)^2}$

   $AC = \sqrt{25 + 100}$ $\qquad$ $DB = \sqrt{121 + 4}$

   $AC = \sqrt{125}$ $\qquad$ $DB = \sqrt{125}$

   $AC = 5\sqrt{5}$ $\qquad$ $DB = 5\sqrt{5}$

   The diagonals are congruent: $\overline{AC} \cong \overline{DB}$. The parallelogram is a rectangle.

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4. Calculate the lengths of the sides.

If all sides are congruent, the parallelogram is a rhombus or a square. Since we established that the angles are right angles, the rectangle can be more precisely classified as a square if the sides are congruent. If the sides are not congruent, the parallelogram is a rectangle.

Use the distance formula to calculate the lengths of the sides.

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$AB=\sqrt{[2-(-6)]^2+(2-8)^2} \qquad AD=\sqrt{[-9-(-6)]^2+(4-8)^2}$$

$$AB=\sqrt{(8)^2+(-6)^2} \qquad AD=\sqrt{(-3)^2+(-4)^2}$$

$$AB=\sqrt{64+36} \qquad AD=\sqrt{9+16}$$

$$AB=\sqrt{100} \qquad AD=\sqrt{25}$$

$$AB=10 \qquad AD=5$$

$$DC=\sqrt{[-1-(-9)]^2+(-2-4)^2} \qquad BC=\sqrt{(-1-2)^2+(-2-2)^2}$$

$$DC=\sqrt{(8)^2+(-6)^2} \qquad BC=\sqrt{(-3)^2+(-4)^2}$$

$$DC=\sqrt{64+36} \qquad BC=\sqrt{9+16}$$

$$DC=\sqrt{100} \qquad BC=\sqrt{25}$$

$$DC=10 \qquad BC=5$$

Opposite sides are congruent, which is consistent with a parallelogram, but all sides are not congruent.

5. Summarize your findings.

The quadrilateral has opposite sides that are parallel and four right angles, but not four congruent sides. This makes the quadrilateral a parallelogram and a rectangle.

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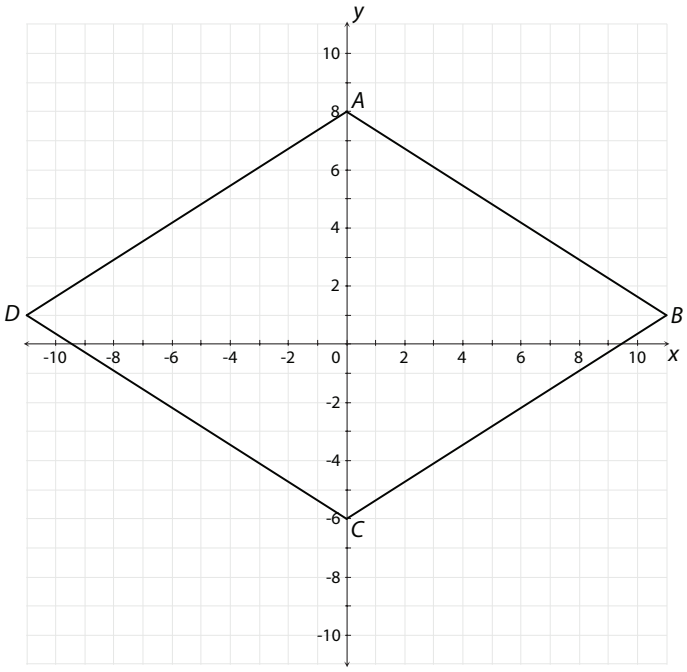
Instruction

Example 2

Quadrilateral *ABCD* has vertices *A* (0, 8), *B* (11, 1), *C* (0, –6), and *D* (–11, 1). Using slope, distance, and/or midpoints, classify quadrilateral *ABCD* as a rectangle, rhombus, square, trapezoid, isosceles trapezoid, or kite.

1. Graph the quadrilateral.



2. Calculate the slopes of the sides to determine if the quadrilateral is a parallelogram.

   If opposite sides are parallel, the quadrilateral is a parallelogram.

   $\overline{AB}$ is opposite $\overline{DC}$. $\overline{BC}$ is opposite $\overline{AD}$.

$$m_{\overline{AB}}=\frac{\Delta y}{\Delta x}=\frac{(1-8)}{(11-0)}=-\frac{7}{11} \qquad m_{\overline{BC}}=\frac{\Delta y}{\Delta x}=\frac{(-6-1)}{(0-11)}=\frac{-7}{-11}=\frac{7}{11}$$

$$m_{\overline{DC}}=\frac{\Delta y}{\Delta x}=\frac{(-6-1)}{[0-(-11)]}=-\frac{7}{11} \qquad m_{\overline{AD}}=\frac{\Delta y}{\Delta x}=\frac{(1-8)}{(-11-0)}=\frac{-7}{-11}=\frac{7}{11}$$

   The opposite sides are parallel: $\overline{AB}\,||\,\overline{DC}$ and $\overline{BC}\,||\,\overline{AD}$. Therefore, the quadrilateral is a parallelogram.

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Instruction

3. Examine the slopes of consecutive sides to determine if the sides are perpendicular.

   If the slopes of consecutive sides are opposite reciprocals of each other, then the sides intersect at right angles.

   If the sides intersect at right angles, then the parallelogram is a rhombus or square.

   Let's use consecutive sides $\overline{AB}$ and $\overline{BC}$.

   $$m_{\overline{AB}} = -\frac{7}{11} \qquad m_{\overline{BC}} = \frac{7}{11}$$

   The slopes are not opposite reciprocals, so the parallelogram is not a rectangle or a square.

4. Calculate the lengths of the sides.

   In a rhombus, the sides are congruent.

   Use the distance formula to calculate the lengths of the sides.

   $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

   $$AB = \sqrt{(11-0)^2 + (1-8)^2} \qquad BC = \sqrt{(0-11)^2 + (-6-1)^2}$$
   $$AB = \sqrt{(11)^2 + (-7)^2} \qquad BC = \sqrt{(-11)^2 + (-7)^2}$$
   $$AB = \sqrt{121+49} \qquad BC = \sqrt{121+49}$$
   $$AB = \sqrt{170} \qquad BC = \sqrt{170}$$

   $$DC = \sqrt{[0-(-11)]^2 + (-6-1)^2} \qquad AD = \sqrt{(-11-0)^2 + (1-8)^2}$$
   $$DC = \sqrt{(11)^2 + (-7)^2} \qquad AD = \sqrt{(-11)^2 + (-7)^2}$$
   $$DC = \sqrt{121+49} \qquad AD = \sqrt{121+49}$$
   $$DC = \sqrt{170} \qquad AD = \sqrt{170}$$

   The sides are all congruent.

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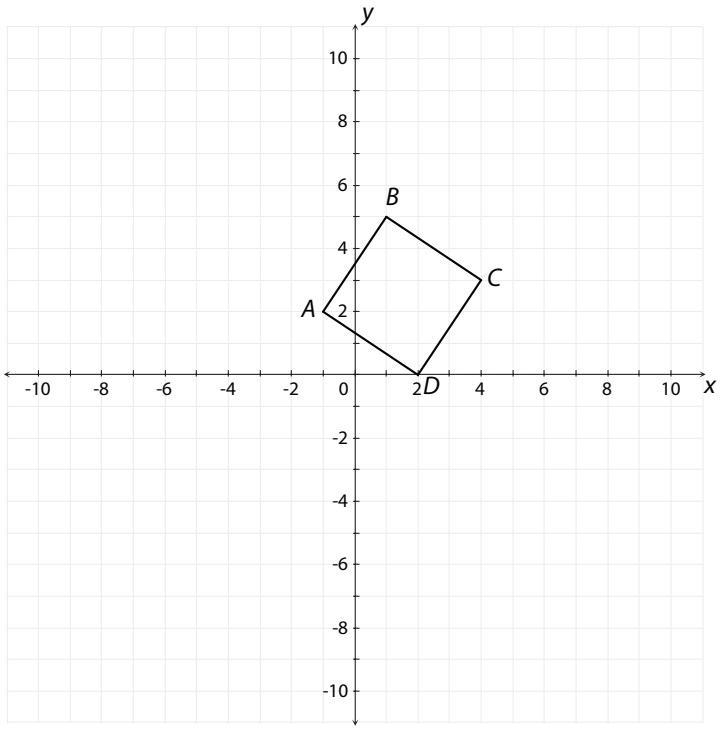
5. Summarize your findings.

   The quadrilateral has opposite sides that are parallel and all four sides are congruent, but the sides are not perpendicular. Therefore, the quadrilateral is a parallelogram and a rhombus, but not a square.

Example 3

Quadrilateral *ABCD* has vertices *A* (–1, 2), *B* (1, 5), *C* (4, 3), and *D* (2, 0). Using slope, distance, and/or midpoints, classify quadrilateral *ABCD* as a rectangle, rhombus, square, trapezoid, or kite.

1. Graph the quadrilateral.



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Instruction

2. Calculate the slopes of the sides to determine if the quadrilateral is a parallelogram.

   If opposites sides are parallel, the quadrilateral is a parallelogram.

   $\overline{AB}$ is opposite $\overline{DC}$. $\overline{BC}$ is opposite $\overline{AD}$.

   $$m_{\overline{AB}} = \frac{\Delta y}{\Delta x} = \frac{(5-2)}{[1-(-1)]} = \frac{3}{2} \qquad m_{\overline{BC}} = \frac{\Delta y}{\Delta x} = \frac{(3-5)}{(4-1)} = -\frac{2}{3}$$

   $$m_{\overline{DC}} = \frac{\Delta y}{\Delta x} = \frac{(3-0)}{(4-2)} = \frac{3}{2} \qquad m_{\overline{AD}} = \frac{\Delta y}{\Delta x} = \frac{(0-2)}{[2-(-1)]} = -\frac{2}{3}$$

   The opposite sides are parallel: $\overline{AB} \parallel \overline{DC}$ and $\overline{BC} \parallel \overline{AD}$. The quadrilateral is a parallelogram.

3. Examine the slopes of consecutive sides.

   If consecutive sides are perpendicular, the angles at the vertices are right angles and the parallelogram is a rectangle or a square.

   The slopes are opposite reciprocals; therefore, $\overline{AB} \perp \overline{BC}$ and $\overline{DC} \perp \overline{AD}$, and the angles at the vertices are right angles, indicating a rectangle or square.

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Instruction

4. Calculate the lengths of the sides.

If the sides are congruent, the parallelogram with four right angles is a square.

Use the distance formula to calculate the lengths of the sides.

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$AB=\sqrt{[1-(-1)]^2+(5-2)^2} \qquad BC=\sqrt{(4-1)^2+(3-5)^2}$$

$$AB=\sqrt{(2)^2+(3)^2} \qquad BC=\sqrt{(3)^2+(-2)^2}$$

$$AB=\sqrt{4+9} \qquad BC=\sqrt{9+4}$$

$$AB=\sqrt{13} \qquad BC=\sqrt{13}$$

$$DC=\sqrt{(4-2)^2+(3-0)^2} \qquad AD=\sqrt{[2-(-1)]^2+(0-2)^2}$$

$$DC=\sqrt{(2)^2+(3)^2} \qquad AD=\sqrt{(3)^2+(-2)^2}$$

$$DC=\sqrt{4+9} \qquad AD=\sqrt{9+4}$$

$$DC=\sqrt{13} \qquad AD=\sqrt{13}$$

The sides are all congruent.

5. Summarize your findings.

The quadrilateral has opposite sides that are parallel, four right angles, and four congruent sides, making the quadrilateral a parallelogram, a rectangle, a rhombus, and most specifically, a square.

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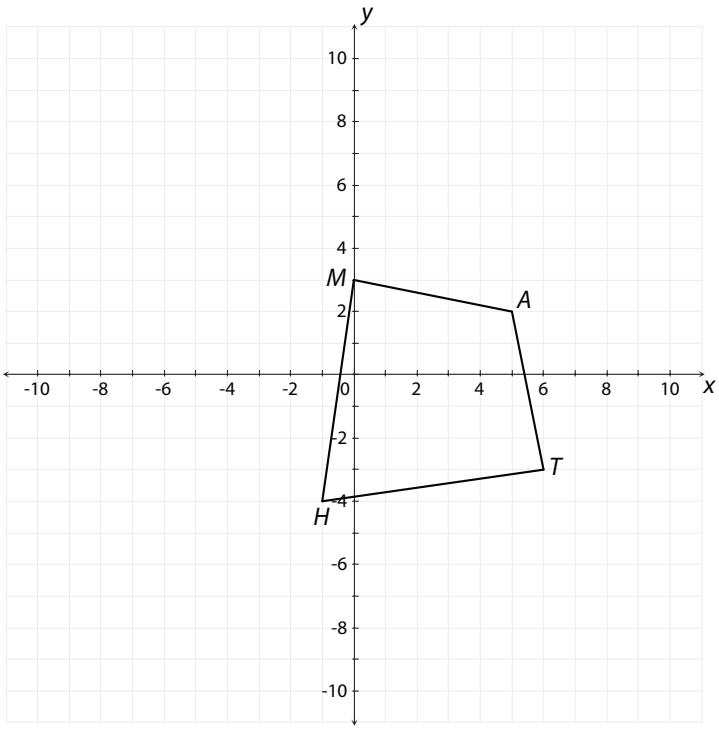
Instruction

Example 4

Use what you know about the diagonals of rectangles, rhombuses, squares, kites, and trapezoids to classify the quadrilateral given the vertices M (0, 3), A (5, 2), T (6, –3), and H (–1, –4).

1. Graph the quadrilateral.



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Instruction

2. Determine if the diagonals bisect each other.

If the diagonals bisect each other, then the quadrilateral is a parallelogram.

Find the midpoints of the diagonals using the midpoint formula.

Let's start with the midpoint of the diagonal $\overline{MT}$.

$$M=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$ Midpoint formula

$$M_{\overline{MT}}=\left(\frac{0+6}{2},\frac{3+(-3)}{2}\right)=\left(\frac{6}{2},\frac{0}{2}\right)=(3,0)$$ Substitute values for x_1, x_2, y_1, and y_2, then solve.

Now find the midpoint of the diagonal $\overline{AH}$.

$$M_{\overline{AH}}=\left(\frac{5+(-1)}{2},\frac{2+(-4)}{2}\right)=\left(\frac{4}{2},\frac{-2}{2}\right)=(2,-1)$$ Substitute values for x_1, x_2, y_1, and y_2, then solve.

The midpoints are not the same, so the diagonals do not bisect each other. This rules out the quadrilateral being any type of parallelogram, including a rectangle, rhombus, or square.

3. Calculate the slopes of the diagonals.

If the diagonals are perpendicular, then the quadrilateral could be a rhombus or a kite.

$$m_{\overline{MT}}=\frac{\Delta y}{\Delta x}=\frac{(-3-3)}{(6-0)}=\frac{-6}{6}=-1 \qquad m_{\overline{AH}}=\frac{\Delta y}{\Delta x}=\frac{(-4-2)}{(-1-5)}=\frac{-6}{-6}=1$$

The slopes are opposite reciprocals, so the diagonals are perpendicular: $\overline{MT}\perp\overline{AH}$. Therefore, the quadrilateral could be a rhombus or a kite.

However, since we established earlier that the diagonals do not bisect each other, the quadrilateral cannot be a rhombus.

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Instruction

4. Calculate the lengths of the sides of the quadrilateral.

Use the distance formula: $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$MA=\sqrt{(5-0)^2+(2-3)^2}$ $\quad$ $AT=\sqrt{(6-5)^2+(-3-2)^2}$

$MA=\sqrt{(5)^2+(-1)^2}$ $\quad$ $AT=\sqrt{(1)^2+(-5)^2}$

$MA=\sqrt{25+1}$ $\quad$ $AT=\sqrt{1+25}$

$MA=\sqrt{26}$ $\quad$ $AT=\sqrt{26}$

$MH=\sqrt{(-1-0)^2+(-4-3)^2}$ $\quad$ $TH=\sqrt{(-1-6)^2+[-4-(-3)]^2}$

$MH=\sqrt{(-1)^2+(-7)^2}$ $\quad$ $TH=\sqrt{(-7)^2+(-1)^2}$

$MH=\sqrt{1+49}$ $\quad$ $TH=\sqrt{49+1}$

$MH=\sqrt{50}=5\sqrt{2}$ $\quad$ $TH=\sqrt{50}=5\sqrt{2}$

The adjacent pairs of sides are congruent.

5. Summarize your findings.

The diagonals do not bisect each other but are perpendicular. Since the diagonals do not bisect each other, the quadrilateral is not a rectangle, rhombus, or square. Since the diagonals are perpendicular and two distinct pairs of adjacent sides are congruent, the quadrilateral is a kite.

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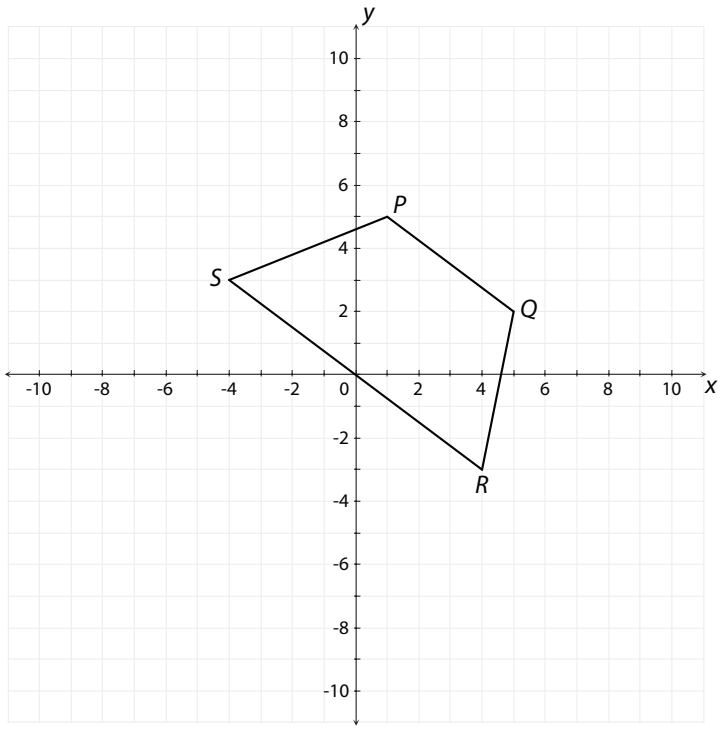
Instruction

Example 5

Use what you know about the diagonals of rectangles, rhombuses, squares, kites, and trapezoids to classify the quadrilateral given vertices P (1, 5), Q (5, 2), R (4, –3), and S (–4, 3).

1. Graph the quadrilateral.



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Instruction

2. Determine if the diagonals bisect each other.

If the diagonals bisect each other, then the quadrilateral is a parallelogram.

Find the midpoints of the diagonals using the midpoint formula.

Let's start with the midpoint of the diagonal $\overline{PR}$.

$$M=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$ Midpoint formula

$$M_{\overline{PR}}=\left(\frac{1+4}{2},\frac{5+(-3)}{2}\right)=\left(\frac{5}{2},\frac{2}{2}\right)=\left(\frac{5}{2},1\right)$$ Substitute values for x_1, x_2, y_1, and y_2, then solve.

Now find the midpoint of the diagonal $\overline{QS}$.

$$M_{\overline{QS}}=\left(\frac{5+(-4)}{2},\frac{2+3}{2}\right)=\left(\frac{1}{2},\frac{5}{2}\right)$$ Substitute values for x_1, x_2, y_1, and y_2, then solve.

The midpoints are not the same, so the diagonals do not bisect each other. This rules out the quadrilateral being any type of parallelogram, including a rectangle, rhombus, or square.

3. Determine the slopes of the diagonals.

If the slopes are opposite reciprocals, then the diagonals are perpendicular and the quadrilateral could be a kite.

$$m_{\overline{PR}}=\frac{\Delta y}{\Delta x}=\frac{(-3-5)}{(4-1)}=-\frac{8}{3} \qquad m_{\overline{QS}}=\frac{\Delta y}{\Delta x}=\frac{(3-2)}{(-4-5)}=-\frac{1}{9}$$

The slopes are not opposite reciprocals. Therefore, the diagonals are not perpendicular. The quadrilateral is neither a parallelogram nor a kite.

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Instruction

4. Determine the lengths of the diagonals.

If the diagonals are congruent, the quadrilateral could be an isosceles trapezoid.

Use the distance formula, $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$, to determine if the diagonals are congruent.

$PR=\sqrt{(4-1)^2+(-3-5)^2}$ $\quad$ $QS=\sqrt{(-4-5)^2+(3-2)^2}$

$PR=\sqrt{(3)^2+(-8)^2}$ $\quad$ $QS=\sqrt{(-9)^2+(1)^2}$

$PR=\sqrt{9+64}$ $\quad$ $QS=\sqrt{81+1}$

$PR=\sqrt{73}$ $\quad$ $QS=\sqrt{82}$

The diagonals are not congruent. Therefore, the quadrilateral is not an isosceles trapezoid.

5. Calculate the slopes of opposite pairs of sides.

If one pair of opposite sides is parallel, then the quadrilateral is a trapezoid.

$m_{\overline{PS}}=\frac{\Delta y}{\Delta x}=\frac{(3-5)}{(-4-1)}=\frac{-2}{-5}=\frac{2}{5}$ $\quad$ $m_{\overline{PQ}}=\frac{\Delta y}{\Delta x}=\frac{(2-5)}{(5-1)}=-\frac{3}{4}$

$m_{\overline{QR}}=\frac{\Delta y}{\Delta x}=\frac{(-3-2)}{(4-5)}=\frac{-5}{-1}=5$ $\quad$ $m_{\overline{SR}}=\frac{\Delta y}{\Delta x}=\frac{(-3-3)}{[4-(-4)]}=-\frac{6}{8}=-\frac{3}{4}$

One pair of opposite sides, $\overline{PQ}$ and $\overline{SR}$, is parallel since the slopes are equal: $\overline{PQ}\,||\,\overline{SR}$. Therefore, the quadrilateral is a trapezoid.

6. Summarize your findings.

The diagonals do not bisect each other, ruling out the quadrilateral being a parallelogram; therefore, it cannot be a rectangle, rhombus, or square. The diagonals are not perpendicular, ruling out a kite. However, one pair of opposite sides is parallel, indicating that the quadrilateral is a trapezoid. The diagonals are not congruent. Therefore, the quadrilateral is not an isosceles trapezoid.

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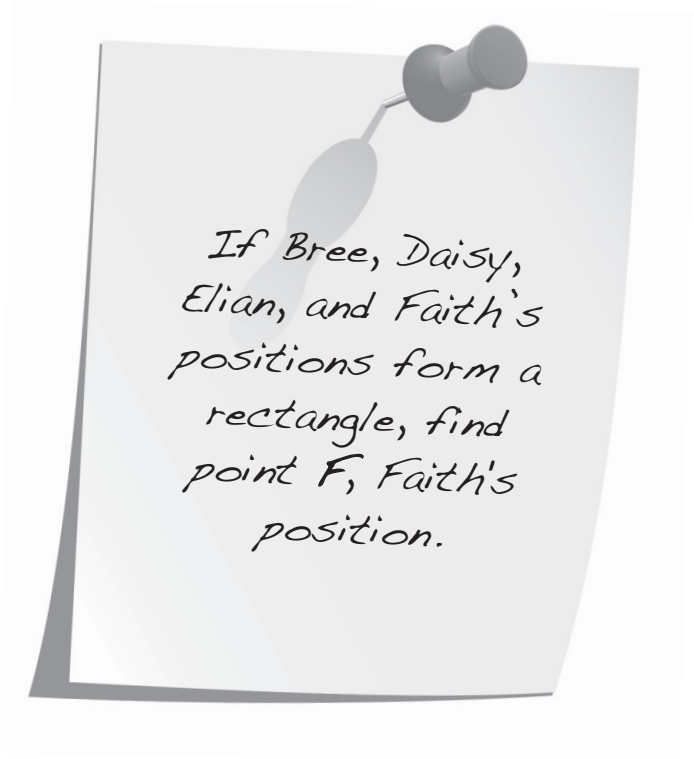
Problem-Based Task 5.7.2: Where Are the Catchers?

SMP	
1	2
3	4 ✓
5 ✓	6 ✓
7	8

Aric, Carl, Bree, and Daisy play Little League baseball. At catching practice, the coach positions each player according to his or her skill level. The four players' positions form a rhombus. Aric, Bree, and Carl can be represented on a coordinate plane by points A (0, 0), B (–5, 0), and C (–1, 3). Find Daisy's location, point D.

Elian and Faith are the best catchers on the team, so the coach has them practice separately. The coach places Elian in a spot represented by point E (7, –6) on the coordinate plane. The coach places Faith so that she's as far away from Elian as Bree is from Daisy. If Bree, Daisy, Elian, and Faith's positions form a rectangle, find point F, Faith's position. Graph the positions of all the players.



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Problem-Based Task 5.7.2: Where Are the Catchers?

Coaching

a. What does the graph of the given points look like?

b. What do you know about the diagonals of a rhombus?

c. What is the slope of $\overline{CA}$?

d. What is the slope of $\overline{BD}$ and why?

e. What is the equation of the line that encompasses $\overline{BD}$?

f. What other line intersects $\overline{BD}$ where we know the slope and one point?

g. What is the slope of $\overline{AD}$, and what given point is on that line?

h. What is the equation of $\overline{AD}$?

i. How can you use these two equations to find point D?

j. What is point D?

k. What point is still unknown?

l. What do we know about the slopes of the sides of a rectangle?

m. What is the slope of $\overline{BD}$?

n. What is the slope of $\overline{EF}$?

o. What is the equation of $\overline{EF}$?

p. What other line intersects $\overline{EF}$ where we know the slope and one point?

q. What is the equation of this line?

r. How can you use the two equations to find point F?

s. What is point F?

t. What is the graph of the final positions of all the catchers?

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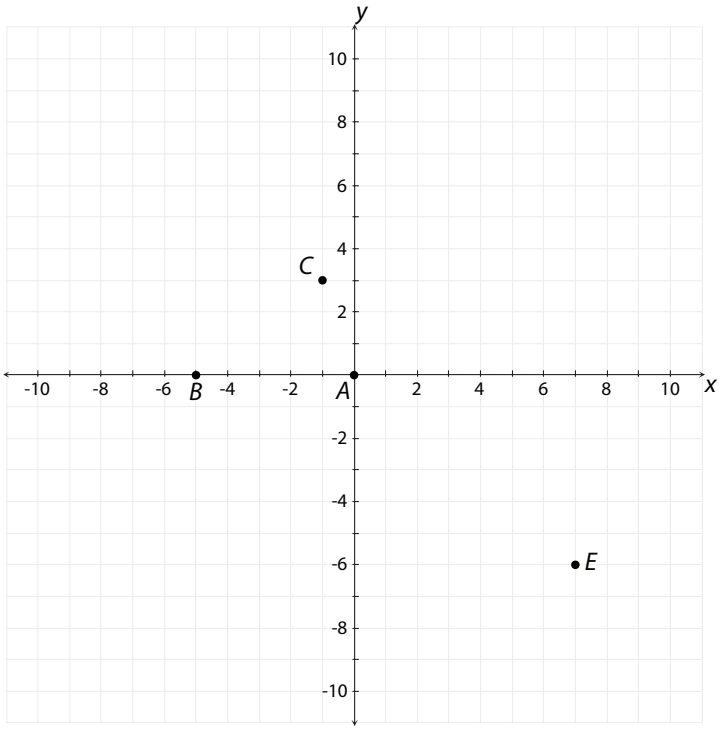
Instruction

Problem-Based Task 5.7.2: Where Are the Catchers?

Coaching Sample Responses

a. What does the graph of the given points look like?



b. What do you know about the diagonals of a rhombus?

They intersect at right angles, so the lines are perpendicular, and the slopes are opposite reciprocals.

c. What is the slope of $\overline{CA}$?

$$m_{\overline{CA}} = \frac{\Delta y}{\Delta x} = \frac{0-3}{0-(-1)} = -\frac{3}{1} = -3$$

d. What is the slope of $\overline{BD}$ and why?

$m_{\overline{BD}} = \frac{1}{3}$ because it is the opposite reciprocal of $\overline{CA}$, since $\overline{CA} \perp \overline{BD}$.

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e. What is the equation of the line that encompasses $\overline{BD}$?

We know that point B (–5, 0) is on the line and the slope is $\frac{1}{3}$. Use the slope-intercept form of the equation to find the equation of the line.

$y = mx + b$ $\qquad$ $y = mx + b$

$0 = \frac{1}{3}(-5) + b$ $\qquad$ $y = \frac{1}{3}x + \frac{5}{3}$

$0 = -\frac{5}{3} + b$

$b = \frac{5}{3}$

f. What other line intersects $\overline{BD}$ where we know the slope and one point?

There are two other lines: $\overline{CD}$ and $\overline{AD}$.

g. What is the slope of $\overline{AD}$, and what given point is on that line?

$\overline{AD}$ is parallel to $\overline{BC}$ because opposite pairs of sides in a parallelogram are parallel and a rhombus is a parallelogram. When lines are parallel, their slopes are equal.

$$m_{\overline{BC}} = \frac{\Delta y}{\Delta x} = \frac{3-0}{-1-(-5)} = \frac{3}{4}$$

$$m_{\overline{AD}} = \frac{3}{4}$$

Point A is on the line and is given, A (0, 0).

h. What is the equation of $\overline{AD}$?

Insert the slope of $\overline{AD}$ and the point A (0, 0) into the equation $y = mx + b$.

$y = mx + b$ $\qquad$ $y = mx + b$

$0 = \frac{3}{4}(0) + b$ $\qquad$ $y = \frac{3}{4}x + 0$

$0 = 0 + b$ $\qquad$ $y = \frac{3}{4}x$

$b = 0$

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Instruction

i. How can you use these two equations to find point D?

Solve the two equations as a system of equations. The solution will be the intersection point of the two lines, point D.

j. What is point D?

Set the two equations equal to each other since they are both equal to y.

$\overline{BD}: y = \frac{1}{3}x + \frac{5}{3}$

$\overline{AD}: y = \frac{3}{4}x$

Find x.

$\frac{1}{3}x + \frac{5}{3} = \frac{3}{4}x$

$\frac{1}{3}x - \frac{3}{4}x = -\frac{5}{3}$

$\frac{4}{12}x - \frac{9}{12}x = -\frac{5}{3}$

$-\frac{5}{12}x = -\frac{5}{3}$

$x = -\frac{5}{3} \bullet -\frac{12}{5}$

$x = \frac{60}{15}$

$x = 4$

Use x to find y.

$y = \frac{3}{4}x$

$y = \frac{3}{4}(4)$

$y = 3$

Point D is at (4, 3).

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Instruction

k. What point is still unknown?

Point F is the second unknown point.

l. What do we know about the slopes of the sides of a rectangle?

The opposite sides are parallel and consecutive sides are perpendicular. The opposite sides have the same slope and the consecutive sides have slopes that are opposite reciprocals.

m. What is the slope of $\overline{BD}$?

From part c, we know $m_{\overline{BD}} = \frac{1}{3}$.

n. What is the slope of $\overline{EF}$?

$\overline{EF}$ is parallel to $\overline{BD}$, so the slopes are equal.

$$m_{\overline{EF}} = \frac{1}{3}$$

o. What is the equation of $\overline{EF}$?

We can determine the equation of the line if we know the slope and a point on the line.

We know the slope, $\frac{1}{3}$, and a point on the line, E (7, –6). Substitute these into $y = mx + b$.

$$y = mx + b$$
$$-6 = \frac{1}{3}(7) + b$$
$$-6 = \frac{7}{3} + b$$
$$-6 - \frac{7}{3} = b$$
$$-\frac{18}{3} - \frac{7}{3} = b$$
$$b = -\frac{25}{3}$$

$$y = mx + b$$
$$y = \frac{1}{3}x - \frac{25}{3}$$

p. What other line intersects $\overline{EF}$ where we know the slope and one point?

$\overline{BF}$ intersects $\overline{EF}$.

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q. What is the equation of the line?

We know the slope of $\overline{BF}$ is perpendicular to $\overline{EF}$. Therefore, the slopes are opposite reciprocals. The slope of $\overline{EF}$ is $\frac{1}{3}$. This means the slope of $\overline{BF}$ is –3. We know point B (–5, 0). Substitute this information into the slope-intercept form of the equation.

$y = mx + b$
$0 = -3(-5) + b$
$0 = 15 + b$
$b = -15$

$y = mx + b$
$y = -3x - 15$

r. How can you use the two equations to find point F?

Solve the two equations as a system. The solution will be the intersection point, F.

s. What is point F?

Set the equations equal to each other since they are both equal to y.

$\overline{EF}: y = \frac{1}{3}x - \frac{25}{3}$; $\overline{BF}: y = -3x - 15$

Find x.

$\frac{1}{3}x - \frac{25}{3} = -3x - 15$
$\frac{1}{3}x + 3x = -15 + \frac{25}{3}$
$\frac{1}{3}x + \frac{9}{3}x = -\frac{45}{3} + \frac{25}{3}$
$\frac{10}{3}x = -\frac{20}{3}$
$x = -\frac{20}{3} \bullet \frac{3}{10}$
$x = -\frac{60}{30}$
$x = -2$

Find y.

$y = -3x - 15$
$y = -3(-2) - 15$
$y = 6 - 15$
$y = -9$

Point F is at (–2, –9).

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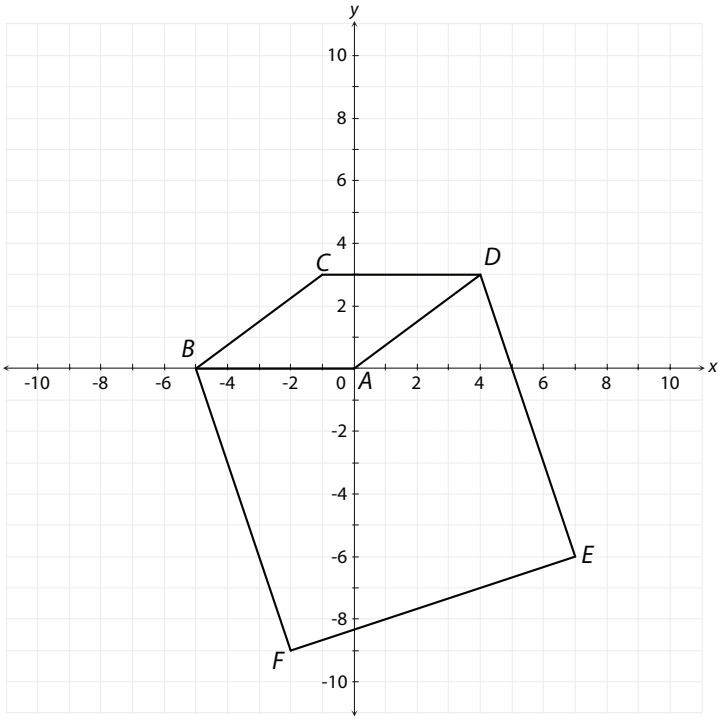
t. What is the graph of the final positions of all the catchers?



Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 5.7.2: Proving Properties of Special Quadrilaterals

For problems 1–8, use the given coordinates as well as slope, distance, midpoints, and/or diagonals to classify each quadrilateral in as many ways as possible (parallelogram, rectangle, rhombus, square, kite, trapezoid, or isosceles trapezoid). Justify your answers.

1. A (–5, 6), B (3, –3), C (0, –6), D (–8, 3)

2. E (0, 2), F (4, 2), G (4, –2), H (–1, –3)

3. I (–6, 7), J (–3, 4), K (–6, 1), L (–9, 4)

4. M (–3, 8), N (2, 5), O (–1, 0), P (–6, 3)

5. P (1, 5), Q (5, 2), R (3, –1), S (–1, 2)

6. T (–6, –4), U (6, –4) V (3, –8), W (–3, –8)

7. W (3, 3), X (8, 1), Y (4, –9), Z (–1, –7)

8. A (2, –2), B (9, –2), C (9, –9), D (2, –9)

continued

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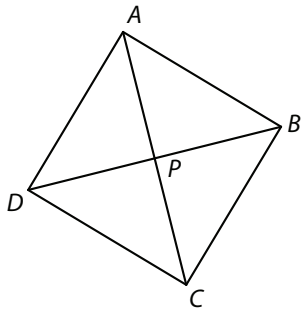
Use the information given in each problem that follows to write proofs.

9. Given that quadrilateral *ABCD* below is a rhombus, prove that the diagonals form four congruent triangles.



10. Given the vertices of a rhombus, A (0, 0), B (b, c), $C\left(b+\sqrt{b^2+c^2}, c\right)$, and $D\left(\sqrt{b^2+c^2}, 0\right)$, prove that the diagonals form four right angles. (*Hint*: Think about the product of the slopes of perpendicular lines.)

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Progress Assessment

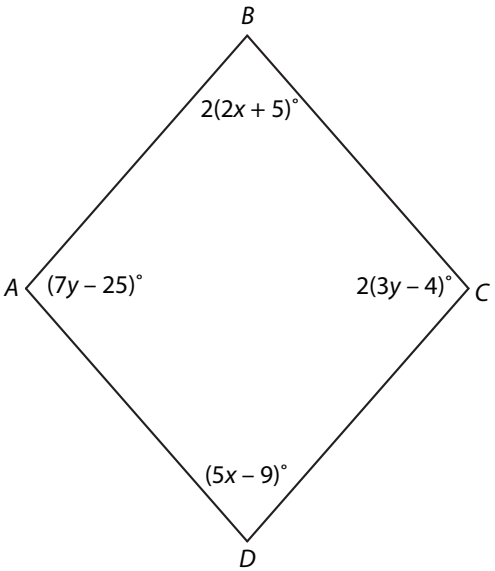
Circle the letter of the best answer.

1. Which of the following is NOT always a property of parallelograms?
   a. Opposite angles are congruent.
   b. The diagonals bisect each other.
   c. The diagonals are perpendicular.
   d. Consecutive angles are supplementary.

2. Determine whether the four vertices form a parallelogram: P (3, 3), Q (7, 4), R (5, 2), S (1, 1).
   a. No, because the opposite sides are not congruent.
   b. Yes, because the diagonals bisect each other.
   c. No, because only one pair of opposite sides is congruent.
   d. Yes, because the diagonals are congruent.

3. Determine whether the four vertices form a parallelogram: D (0, 4), E (3, 5), F (2, 2), G (–1, 2).
   a. No, because the opposite sides are not parallel.
   b. Yes, because opposite sides are parallel.
   c. No, because the diagonals bisect each other.
   d. Yes, because the diagonals are congruent.

4. Find the values of x and y if quadrilateral $ABCD$ is a parallelogram.



   a. $x = 90, y = 90$
   b. $x = 26, y = 16$
   c. $x = 86, y = 94$
   d. $x = 19, y = 17$

continued

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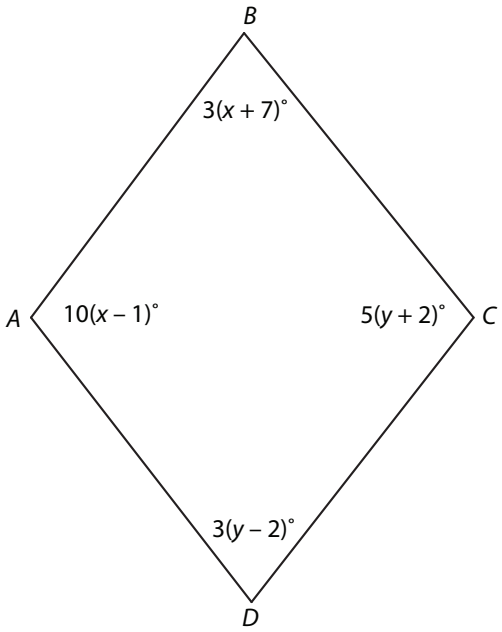
5. Find the values of x and y if quadrilateral $ABCD$ is a parallelogram.



a. $x = 90, y = 90$

b. $x = 4, y = 176$

c. $x = 60, y = 120$

d. $x = 13, y = 22$

6. If the diagonals of a given quadrilateral are perpendicular, how could the quadrilateral be classified?

a. rhombus

b. square

c. kite

d. all of the above

7. If the diagonals of a given quadrilateral are congruent, the quadrilateral could be classified as all of the following EXCEPT:

a. rectangle

b. isosceles trapezoid

c. kite

d. square

8. Classify a quadrilateral as precisely as possible given four vertices: E (0, –1), F (3, –5), G (–2, –5), and H (–5, –1).

a. rhombus

b. square

c. kite

d. rectangle

continued

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9. Classify a quadrilateral as precisely as possible given four vertices: A (1, 1), B (6, 6), C (11, 1), and D (6, –4).

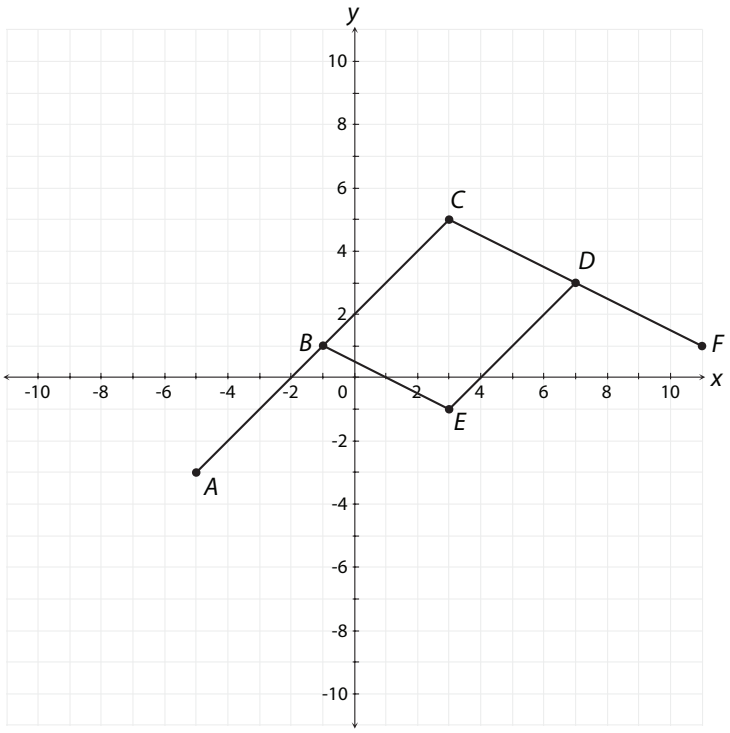
   a. rhombus
   b. square
   c. kite
   d. rectangle

10. Classify a quadrilateral as precisely as possible given four vertices: A (3, –5), B (3, 3), C (9, 1), and D (9, –3).

   a. trapezoid
   b. rectangle
   c. parallelogram
   d. isosceles trapezoid

Use the diagram and the given information to complete the problem that follows.

11. A pantograph is a drawing tool used to enlarge or reduce figures, such as photographs. Pantographs were used before computers were available. The graph below represents the positions of the joints of a pantograph. Point A is a fixed point and attaches to the top of the desk. It is able to pivot or turn, but it remains in place. Points B, C, D, and E are hinged. This means the angles of these segments can change. However, the lengths always remain the same. Point E is the tracing point. The user traces the photograph or drawing at this point. Point F is the pen, where the new enlarged figure is drawn. What is the shape of quadrilateral $BCDE$? Justify your answer by proving at least two theorems, and show your work.



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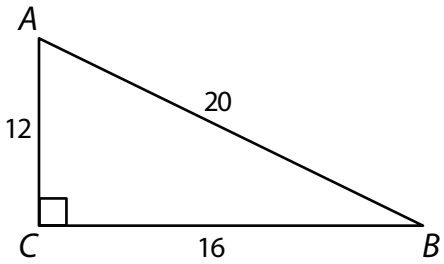
Pre-Assessment

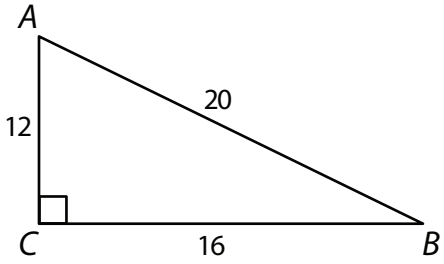
Circle the letter of the best answer.

1. Find the sine of $\angle A$.



   a. 0.8
   b. 0.6
   c. 1.333
   d. 0.75

2. Find the cosine of $\angle A$.



   a. 0.8
   b. 0.6
   c. 1.333
   d. 0.75

continued

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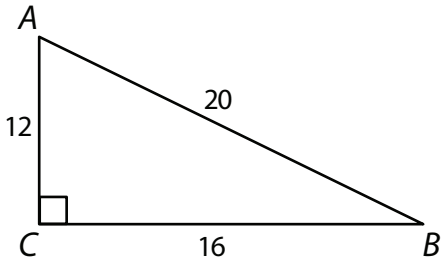
3. Find the tangent of $\angle A$.



   a. 0.8
   b. 0.6
   c. 1.333
   d. 0.75

4. If the sine of 30° = 0.5, what is the cosine of 60°?

   a. 2
   b. 0.866
   c. 0.5
   d. 1

5. Find a value of θ for which $\sin \theta = \cos 25°$ is true.

   a. 25°
   b. 45°
   c. 55°
   d. 65°

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Lesson 8: Exploring Trigonometric Ratios

Instruction

Common Core State Standards

SMP	
1 ✓	2 ✓
3 ✓	4
5 ✓	6
7	8 ✓

G–SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

G–SRT.7 Explain and use the relationship between the sine and cosine of complementary angles.

Essential Questions

1. How are the properties of similar triangles used to create trigonometric ratios?
2. What are the relationships between the sides and angles of right triangles?
3. On what variable inputs are the ratios in trigonometry dependent?
4. What is the relationship between the sine and cosine ratios for the two acute angles in a right triangle?

WORDS TO KNOW

adjacent side the leg next to an acute angle in a right triangle that is not the hypotenuse

cofunction a trigonometric function whose ratios have the same values when applied to the two acute angles in a right triangle. The sine of one acute angle in a right triangle is equal to the cosine of the other acute angle, so sine and cosine are cofunctions.

complementary angles two angles whose sum is 90°

cosecant the reciprocal of the sine ratio;

$$\csc\theta = \frac{\text{length of hypotenuse}}{\text{length of opposite side}} \text{ or } \csc\theta = \frac{1}{\sin\theta}$$

cosine a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the side adjacent to the length of the hypotenuse;

$$\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$$

SRB p. U5-300

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Instruction

cotangent — the reciprocal of tangent;

$$\cot\theta = \frac{\text{length of adjacent side}}{\text{length of opposite side}} \quad \text{or} \quad \cot\theta = \frac{1}{\tan\theta}$$

hypotenuse — the side opposite the 90° angle in a right triangle

identity — an equation that is true regardless of what values are chosen for the variables

opposite side — the side across from an angle in a triangle

***phi* (ϕ)** — a Greek letter sometimes used to refer to an unknown angle measure

ratio — the relation between two quantities; can be expressed in words, fractions, decimals, or as a percentage

reciprocal — a number that, when multiplied by the original number, has a product of 1

right triangle — a triangle with one angle that measures 90°

scale factor — a multiple of the lengths of the sides from one figure to the transformed figure. If the scale factor is larger than 1, then the figure is enlarged. If the scale factor is between 0 and 1, then the figure is reduced.

secant — the reciprocal of cosine;

$$\sec\theta = \frac{\text{length of hypotenuse}}{\text{length of adjacent side}}$$

similar — two figures that are the same shape but not necessarily the same size. Corresponding angles must be congruent and sides must have the same ratio. The symbol for representing similarity is $\sim$.

sine — a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the hypotenuse;

$$\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$$

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tangent	a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the adjacent side; $\tan\theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$
***theta* (θ)**	a Greek letter commonly used to refer to unknown angle measures
trigonometry	the study of triangles and the relationships between their sides and angles

Recommended Resources

- AJ Design Software. "Triangle Equations Formulas Calculator."

  http://www.walch.com/rr/00041

  This excellent website provides links to interactive calculators that help users solve for the various attributes of different types of triangles. For the Pythagorean Theorem, users can plug in known values for the legs of a right triangle, and the site will calculate the length of the hypotenuse.

- Keisan: High Accuracy Calculation. "Solar elevation angle (for a day)."

  http://www.walch.com/rr/00042

  This website features a solar calculator that will return the angle of elevation of the sun for any time of day at any latitude. The results are shown in a table and a graph.

- MathIsFun.com. "Sine, Cosine and Tangent."

  http://www.walch.com/rr/00043

  Scroll down to the bottom of this site to find a tool for changing an angle and seeing the three main ratios of trigonometry change.

- Teach Engineering. "Hands-on Activity: Stay in Shape."

  http://www.walch.com/rr/00044

  This site provides a real-life example of how the equations in basic geometry and trigonometry are used in navigation.

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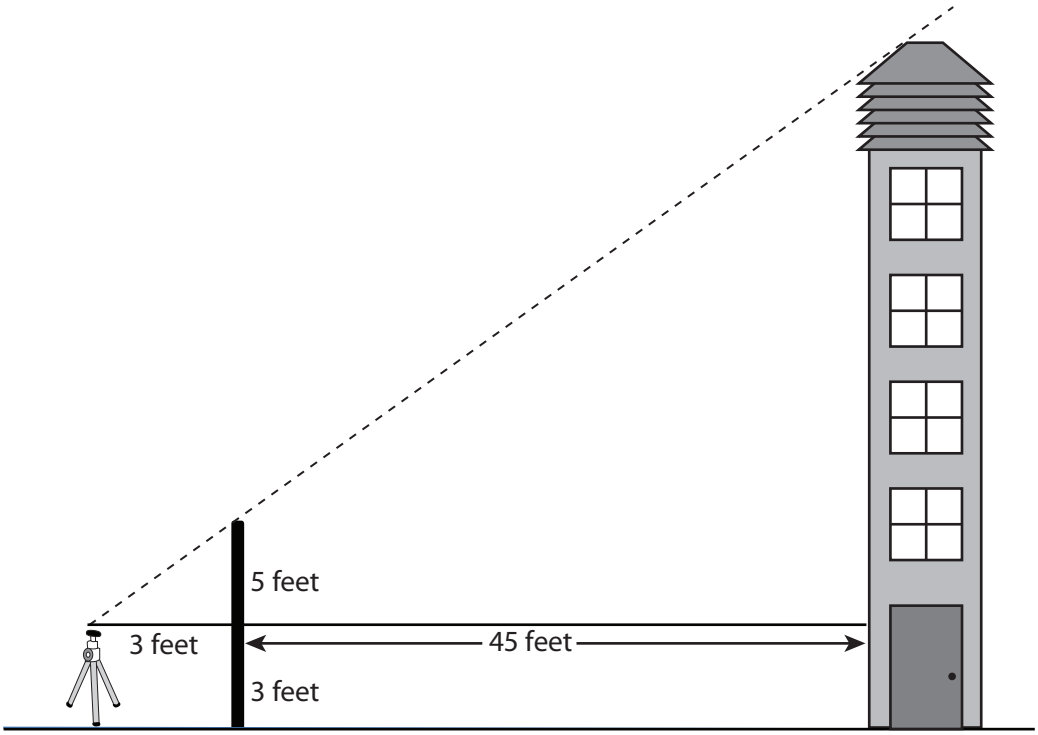
Lesson 5.8.1: Defining Trigonometric Ratios

Warm-Up 5.8.1

Contractors are repairing and painting the tower on the county courthouse. They need to calculate the height of the tower to know what equipment they will need. The diagram below shows how they measured the height of the tower using a laser light, a post, and a tape measure. They put the post in the ground 3 feet in front of a 3-foot-tall tripod that has a laser on it. The height of the post is 8 feet, and the post is perpendicular to the ground. The laser is aimed just above the post so that the light beam hits the top of the tower. The post is 45 feet from the base of the courthouse tower.



1. How many triangles are shown in the diagram? Draw and label each one.

2. Which angle do the triangles share in common?

continued

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3. Which angles are congruent?

4. Are the triangles similar?

5. What is the scale factor, k, of the triangles?

6. How can you use the scale factor, k, to find the height, h, of the tower?

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Instruction

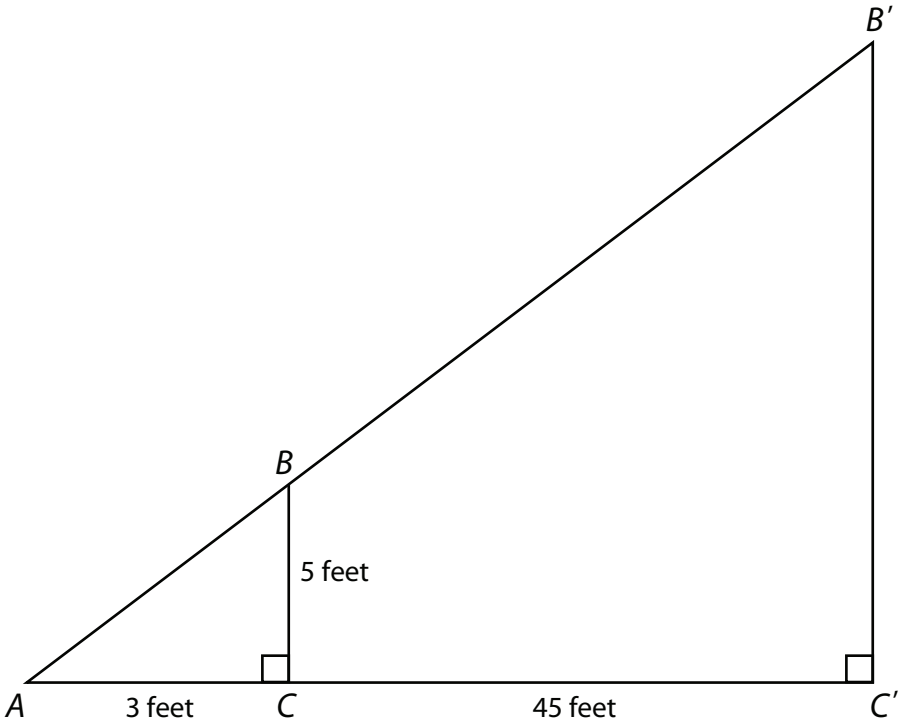
Lesson 5.8.1: Defining Trigonometric Ratios

Common Core State Standard

G–SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

Warm-Up 5.8.1 Debrief

1. How many triangles are shown in the diagram? Draw and label each one.



There are two triangles in the drawing: $\triangle ABC$ and $\triangle AB'C'$.

2. Which angle do the triangles share in common?

The triangles share $\angle A$.

3. Which angles are congruent?

Both triangles are right triangles with a 90° angle and a shared angle. Because the triangles have two congruent angles, their third angles must also be congruent.

4. Are the triangles similar?

Yes, the triangles are similar by the Angle-Angle Similarity Statement.

$\triangle ABC \sim \triangle AB'C'$ because they have two congruent angles ($\angle A \cong \angle A$ and $\angle C \cong \angle C'$).

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5. What is the scale factor, k, of the similar triangles?

$\frac{\text{length of image side}}{\text{length of preimage side}} = \text{scale factor}$	Equation for scale factor
$\frac{AC'}{AC} = k$	Divide the image side by the preimage side.
$\frac{48}{3} = 16$	Substitute values and solve.

6. How can you use the scale factor, k, to find the height, h, of the tower?

   Divide the height of the tower ($B'C'$) by the height of the post (BC).

   $$\frac{B'C'}{BC} = k$$
   $$B'C' = BC \bullet k$$
   $$B'C' = 5 \bullet 16 = 80$$

   Add 3 feet to account for the height of the laser tripod.

   $h = 80 + 3 = 83$ feet

Connection to the Lesson

- Students will use the properties of similar triangles to solve real-life problems involving side lengths.
- Students will use trigonometry to find unknown angles and lengths in right triangles.
- Students will learn that the applications of trigonometry can be even more powerful than the use of similar triangles or the Pythagorean Theorem.
- Students will discover the ratios found in right triangles.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- measuring angles with a protractor (7.G.2)
- understanding how to label angles and sides in triangles (8.G.1b)
- converting fractions into decimals (7.EE.3)
- solving for one unknown number in a ratio or proportion (7.RP.3)
- understanding the properties of similar triangles (G–SRT.2)
- understanding and applying the properties of dilations (G–SRT.1a)
- using the Pythagorean Theorem (8.G.7)
- understanding how to set up and use ratios (7.RP.3)
- converting measurement units within the same system (e.g., from inches to feet) (4.MD.1)

Introduction

Navigators and surveyors use the properties of similar right triangles. Designers and builders use right triangles in constructing structures and objects. Cell phones and Global Positioning Systems (GPS) use the mathematical principles of algebra, geometry, and trigonometry. **Trigonometry** is the study of triangles and the relationships between their sides and angles. In this lesson, we will learn about the ratios between angles and side lengths in right triangles. A **ratio** is the relation between two quantities; it can be expressed in words, fractions, decimals, or as a percentage.

Key Concepts

- Two triangles are similar if they have congruent angles.
- Remember that two figures are similar when they are the same shape but not necessarily the same size; the symbol for representing similarity is $\sim$.
- Recall that the **hypotenuse** is the side opposite the vertex of the 90° angle in a right triangle. Every right triangle has one 90° angle. The hypotenuse is the longest side of a right triangle.
- If two right triangles have a second angle that is congruent, the two triangles are similar.
- Similar triangles have proportional side lengths. The side lengths are related to each other by a scale factor.
- Examine the proportional relationships between similar triangles $\triangle ABC$ and $\triangle DEF$ in the diagram that follows. The scale factor is $k = 2$. Notice how the ratios of corresponding side lengths are the same as the scale factor.

SRB p. U5-303

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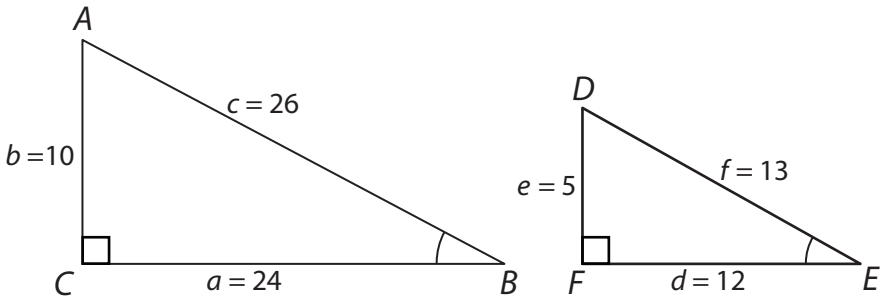
Instruction

Proportional Relationships in Similar Triangles



Corresponding sides

$$\frac{a}{d}=\frac{b}{e}=\frac{c}{f}$$

Side lengths

$$\frac{24}{12}=\frac{10}{5}=\frac{26}{13}$$

Examine the three ratios of side lengths in $\triangle ABC$. Notice how these ratios are equal to the same ratios in $\triangle DEF$.

Corresponding sides

$$\frac{a}{c}=\frac{d}{f}$$

$$\frac{b}{c}=\frac{e}{f}$$

$$\frac{a}{b}=\frac{d}{e}$$

Side lengths

$$\frac{24}{26}=\frac{12}{13}$$

$$\frac{10}{26}=\frac{5}{13}$$

$$\frac{24}{10}=\frac{12}{5}$$

- The ratio of the lengths of two sides of a triangle is the same as the ratio of the corresponding sides of any similar triangle.
- The three main ratios in a right triangle are the sine, the cosine, and the tangent. These ratios are based on the side lengths relative to one of the acute angles.
- The **sine** of an acute angle in a right triangle is the ratio of the length of the opposite side to the length of the hypotenuse; sine of $\theta = \sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$.
- The **cosine** of an acute angle in a right triangle is the ratio of the length of the side adjacent to the length of the hypotenuse; cosine of $\theta = \cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$.

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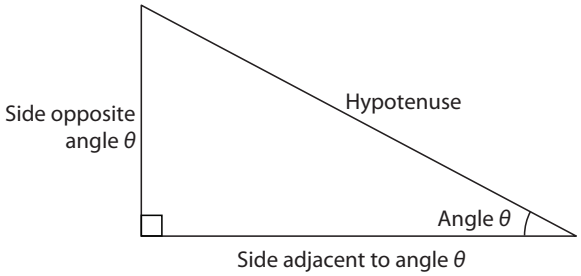
Instruction

- The **tangent** of an acute angle in a right triangle is the ratio of the length of the opposite side to the length of the adjacent side; tangent of $\theta = \tan\theta = \dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$.
- The acute angle that is being used for the ratio can be called the angle of interest. It is commonly marked with the symbol θ (*theta*).
- ***Theta* (θ)** is a Greek letter commonly used to label an unknown angle measure.



- See the following examples of the ratios for sine, cosine, and tangent.

$$\text{sine of } \theta = \sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}} \qquad \text{Abbreviation: } \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\text{cosine of } \theta = \cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}} \qquad \text{Abbreviation: } \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\text{tangent of } \theta = \tan\theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}} \qquad \text{Abbreviation: } \frac{\text{opposite}}{\text{adjacent}}$$

- Unknown angle measures can also be written using the Greek letter ***phi* (ϕ).**
- The three main ratios can also be shown as reciprocals.
- The **reciprocal** is a number that when multiplied by the original number the product is 1.
- The reciprocal of sine is **cosecant**. The reciprocal of cosine is **secant**, and the reciprocal of tangent is **cotangent**.

$$\text{cosecant of } \theta = \csc\theta = \frac{\text{length of hypotenuse}}{\text{length of opposite side}}$$

$$\text{secant of } \theta = \sec\theta = \frac{\text{length of hypotenuse}}{\text{length of adjacent side}}$$

$$\text{cotangent of } \theta = \cot\theta = \frac{\text{length of adjacent side}}{\text{length of opposite side}}$$

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Instruction

- Each acute angle in a right triangle has different ratios of sine, cosine, and tangent.
- The length of the hypotenuse remains the same, but the sides that are opposite or adjacent for each acute angle will be different for different angles of interest.
- The two segments that make up each acute angle in a right triangle are a leg and the hypotenuse. The leg is called the **adjacent side** to the angle. Adjacent means "next to."
- In a right triangle, the side of the triangle opposite the angle of interest is called the **opposite side**.
- Calculations in trigonometry will vary due to the variations that come from measuring angles and distances.
- A final calculation in trigonometry is frequently expressed as a decimal.
- A calculation can be made more accurate by including more decimal places.
- The context of the problem will determine the number of decimals places to which to round. Examples:
  - A surveyor usually measures tracts of land to the nearest tenth of a foot.
  - A computer manufacturer needs to measure a microchip component to a size smaller than an atom.
  - A carpenter often measures angles in whole degrees.
  - An astronomer measures angles to $\frac{1}{3600}$ of a degree or smaller.

Common Errors/Misconceptions

- confusing the differences between the trigonometric ratios
- forgetting to change the adjacent and opposite sides when working with the two acute angles
- mistakenly trying to use sine, cosine, and tangent ratios for triangles that are not right triangles
- mistakenly thinking that trigonometry will always find the exact length of a side or the exact measure of an angle

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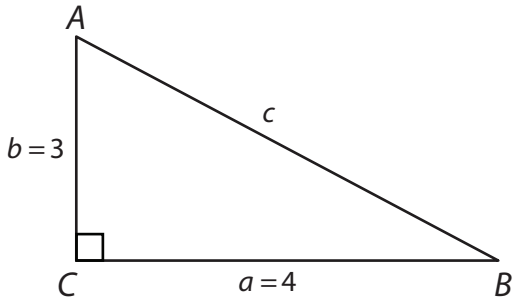
Instruction

Guided Practice 5.8.1

Example 1

Find the sine, cosine, and tangent ratios for $\angle A$ and $\angle B$ in $\triangle ABC$. Convert the ratios to decimal equivalents.



1. Find the length of the hypotenuse using the Pythagorean Theorem.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$4^2 + 3^2 = c^2$	Substitute values for a and b.
$16 + 9 = c^2$	Simplify.
$25 = c^2$	
$\pm\sqrt{25} = \sqrt{c^2}$	
$c = \pm 5$	

Since c is a length, use the positive value, $c = 5$.

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Instruction

2. Find the sine, cosine, and tangent of $\angle A$.

Set up the ratios using the lengths of the sides and hypotenuse, then convert to decimal form.

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} = 0.8$$

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5} = 0.6$$

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3} = 1.33\overline{3}$$

3. Find the sine, cosine, and tangent of $\angle B$.

Set up the ratios using the lengths of the sides and the hypotenuse, and then convert to decimal form.

$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5} = 0.6$$

$$\cos B = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5} = 0.8$$

$$\tan B = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4} = 0.75$$

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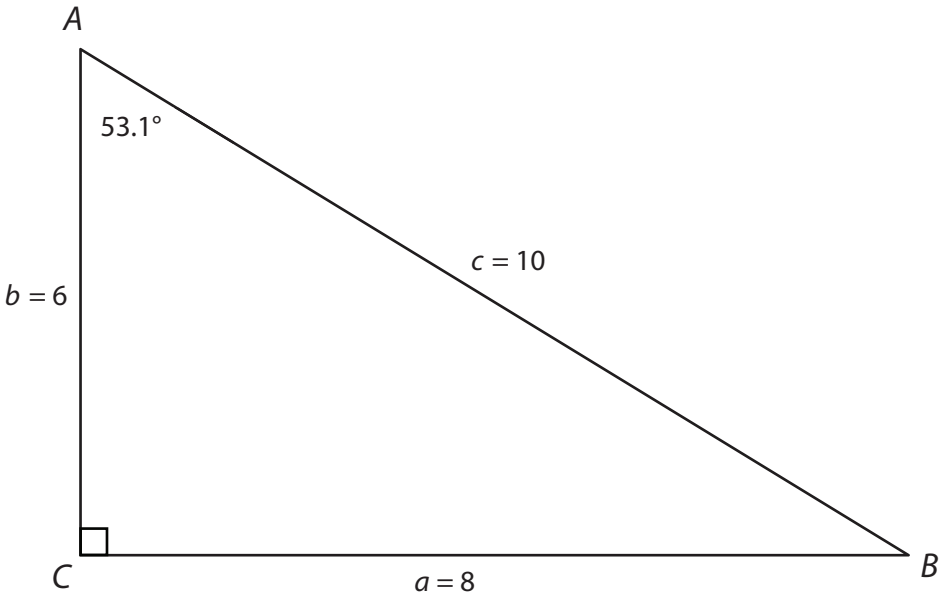
Example 2

Given the following triangle, set up the three trigonometric ratios of sine, cosine, and tangent for the angle given. Compare these ratios to the trigonometric functions using your calculator.



1. Set up the ratios for sin A, cos A, and tan A and calculate the decimal equivalents.

$$\sin 53.1^\circ = \frac{8}{10} = 0.8$$

$$\cos 53.1^\circ = \frac{6}{10} = 0.6$$

$$\tan 53.1^\circ = \frac{8}{6} = 1.33\overline{3}$$

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2. Use your graphing calculator to find sin 53.1°.

On a TI-83/84:

First, make sure your calculator is in Degree mode.

Step 1: Press [MODE].

Step 2: Arrow down to RADIAN.

Step 3: Arrow over to DEGREE.

Step 4: Press [ENTER]. The word DEGREE should be highlighted inside a black rectangle.

Step 5: Press [2ND].

Step 6: Press [MODE] to QUIT.

Important: You will not have to change to Degree mode again unless you have changed your calculator to Radian mode.

Now you can find the value of sin 53.1°:

Step 1: Press [SIN].

Step 2: Enter the angle measurement: 53.1.

Step 3: Press [ )].

Step 4: Press [ENTER].

On a TI-Nspire:

First, make sure the calculator is in Degree mode.

Step 1: From the home screen, select 5: Settings & Status, then 2: Settings, then 2: Graphs & Geometry.

Step 2: Press [tab] twice to move to the Geometry Angle field, and then select Degree from the drop-down menu.

Step 3: Press [tab] to move to the "OK" option and select it using the center button on the navigation pad.

Next, calculate the sine of 53.1°.

Step 1: From the home screen, select a new Calculate window.

Step 2: Press [trig] to bring up the menu of trigonometric functions. Use the keypad to select "sin," then type 53.1 and press [enter].

$\sin 53.1° \approx 0.7996846585$

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3. Use your graphing calculator to find cos 53.1°.

On a TI-83/84:

Step 1: Press [COS].

Step 2: Enter the angle measurement: 53.1.

Step 3: Press [ )].

Step 4: Press [ENTER].

On a TI-Nspire:

Step 1: From the home screen, select a new Calculate window.

Step 2: Press [trig] to bring up the menu of trigonometric functions. Use the keypad to select "cos," then type 53.1 and press [enter].

$\cos 53.1° \approx 0.6004202253$

4. Use your graphing calculator to find tan 53.1°.

On a TI-83/84:

Step 1: Press [TAN].

Step 2: Enter the angle measurement: 53.1.

Step 3: Press [ )].

Step 4: Press [ENTER].

On a TI-Nspire:

Step 1: From the home screen, select a new Calculate window.

Step 2: Press [trig] to bring up the menu of trigonometric functions. Use the keypad to select "$\tan^{-1}$," then type 53.1 and press [enter].

$\tan 53.1° \approx 1.331874952$

5. Compare the calculator values to the ratios. Explain the difference.

The values are very close to the ratios of the side lengths. The differences are due to the angle measurement of 53.1° being an approximation.

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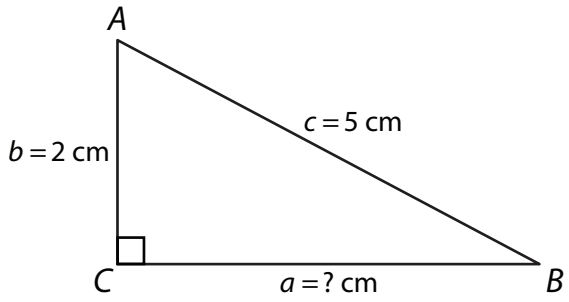
Instruction

Example 3

A right triangle has a hypotenuse of 5 cm and a side length of 2 cm. Use a protractor to find the angle measurements and the unknown side length. Find the sine, cosine, and tangent for both angles. Without drawing another triangle, compare the trigonometric ratios of $\triangle ABC$ with those of a triangle that has been dilated by a factor of $k = 3$.

1. First, draw the triangle with a ruler, and label the side lengths and angles.



2. Find a by using the Pythagorean Theorem.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$a^2 + 2^2 = 5^2$	Substitute values for b and c.
$a^2 + 4 = 25$	Simplify.
$a^2 = 21$	
$a \approx 4.5826$ centimeters	

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Instruction

3. Use a protractor to measure one of the acute angles, and then use that measurement to find the other acute angle.

$m\angle A \approx 66.5$

We know that $m\angle C = 90$ by the definition of right angles.

The measures of the angles of a triangle sum to 180.

Subtract $m\angle A$ and $m\angle C$ from 180 to find $m\angle B$.

$m\angle B = 180 - m\angle A - m\angle C$

$m\angle B = 180 - (66.5) - (90)$

$m\angle B = 180 - 156.5$

$m\angle B \approx 23.5$

4. Find the sine, cosine, and tangent ratios for both acute angles. Express your answer in decimal form to the nearest thousandth.

$\angle A$	$\angle B$
$\sin 66.5^\circ \approx \frac{4.5826}{5} \approx 0.916$	$\sin 23.5^\circ \approx \frac{2}{5} \approx 0.4$
$\cos 66.5^\circ \approx \frac{2}{5} \approx 0.4$	$\cos 23.5^\circ \approx \frac{4.5826}{5} \approx 0.916$
$\tan 66.5^\circ \approx \frac{4.5826}{2} \approx 2.291$	$\tan 23.5^\circ \approx \frac{2}{4.5826} \approx 0.436$

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5. Without drawing a triangle, find the sine, cosine, and tangent for a triangle that has a scale factor of 3 to $\triangle ABC$. Compare the trigonometric ratios for the two triangles.

   Multiply each side length (a, b, and c) by 3 to find a', b', and c'.

   $a' = 3 \bullet a = 3 \bullet (4.5826) = 13.7478$

   $b' = 3 \bullet b = 3 \bullet (2) = 6$

   $c' = 3 \bullet c = 3 \bullet (5) = 15$

   Set up the ratios using the side lengths of the dilated triangle.

   $\sin 66.5^\circ \approx \frac{13.7478}{15} \approx 0.916$ $\quad$ $\sin 23.5^\circ \approx \frac{6}{15} \approx 0.4$

   $\cos 66.5^\circ \approx \frac{6}{15} \approx 0.4$ $\quad$ $\cos 23.5^\circ \approx \frac{13.7478}{15} \approx 0.916$

   $\tan 66.5^\circ \approx \frac{13.7478}{6} \approx 2.291$ $\quad$ $\tan 23.5^\circ \approx \frac{6}{13.7478} \approx 0.436$

   The sine, cosine, and tangent are the same in the larger triangle as they were in the smaller triangle.. Similar triangles have identical side length ratios and, therefore, identical trigonometric ratios.

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Example 4

What are the secant (sec), cosecant (csc), and cotangent (cot) ratios for an isosceles right triangle with legs that are 1 unit long?

1. An isosceles right triangle has two angles that measure 45°. The two side lengths have equal lengths of 1 unit.

2. Use the Pythagorean Theorem to find the hypotenuse.

$1^2 + 1^2 = c^2$

$1 + 1 = c^2$

$2 = c^2$

$\sqrt{2} = c$

3. Substitute the side lengths into the ratios.

$$\csc 45^\circ = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{\sqrt{2}}{1} \approx 1.414$$

$$\sec 45^\circ = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\sqrt{2}}{1} \approx 1.414$$

$$\cot 45^\circ = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{1} = 1$$

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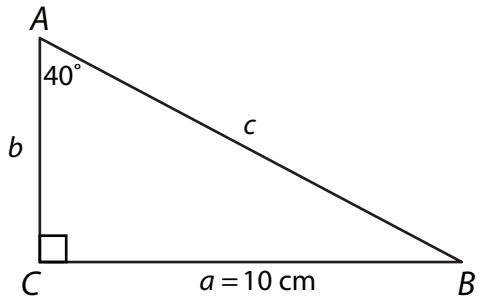
Example 5

Triangle ABC is a right triangle where $m\angle A = 40^\circ$ and side a = 10 centimeters. What is the sine of $\angle A$? Check your work with the sin function on your graphing calculator.

1. Draw $\triangle ABC$ with a ruler and protractor.



2. Measure the length of hypotenuse c.

   $c \approx 15.6$ centimeters

   Substitute the side lengths into the ratios to determine the sine of $\angle A$.

$$\sin 40^\circ \approx \frac{10}{15.6} \approx 0.641$$

3. Use your calculator to check the answer.

   Follow the steps outlined in Example 2.

   $\sin 40^\circ \approx 0.6427876097$

   The two answers are fairly close. The difference is due to the imprecise nature of manually drawing and measuring a triangle.

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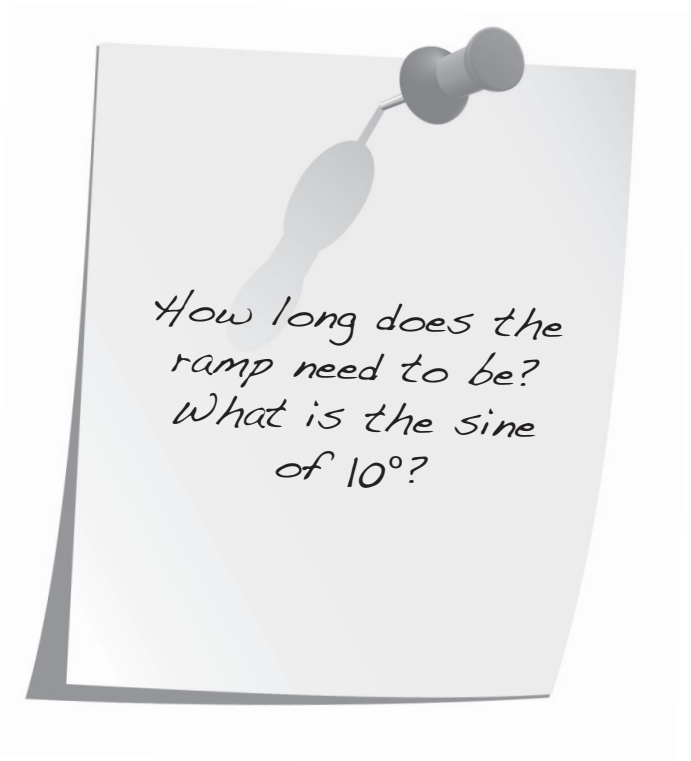
Problem-Based Task 5.8.1: Building a Porch Ramp

SMP	
1 ✓	2 ✓
3	4
5 ✓	6
7	8

Bill's father is building a ramp to reach the front porch of their house. The ramp will rise from the flat ground to the top of the porch. The porch is 2 feet tall. The ramp will rise at a 10° angle from the ground. How long does the ramp need to be? What is the sine of 10°?



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Problem-Based Task 5.8.1: Building a Porch Ramp

Coaching

a. Create a drawing of the problem.

b. Is there a triangle in this problem?

c. Do you have enough information to calculate the third angle of the triangle?

d. Can you make a precise drawing to scale that shows the known length and angles?

e. Is your small triangle similar to the actual ramp?

f. What is the scale factor between the height of the porch in your drawing and the porch's actual height?

g. Which length in your drawing corresponds to the ramp length?

h. How do you use the length in the drawing and the scale factor to find the length of the ramp?

i. What is the ratio for the sine of 10°?

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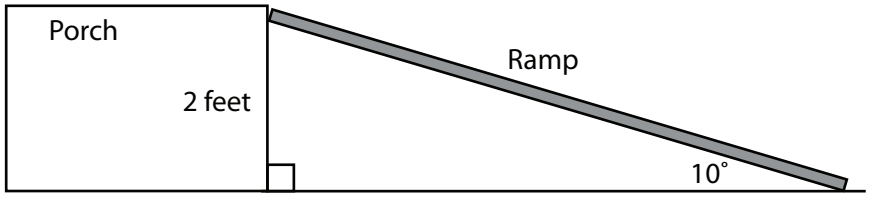
Instruction

Problem-Based Task 5.8.1: Building a Porch Ramp

Coaching Sample Responses

a. Create a drawing of the problem.



b. Is there a triangle in this problem?

Yes; the triangle is formed by the ramp (the hypotenuse), the height of the porch, and the flat ground.

c. Do you have enough information to calculate the third angle of the triangle?

Yes. There is information for two angles: an angle of 10° is given, and the angle the ground makes with the porch is 90°. The third angle is $180° - 90° - 10° = 80°$.

d. Can you make a precise drawing to scale that shows the known length and angle?

Yes. You can use a protractor to create a 10° angle and a 90° angle, and a ruler to draw a side length that corresponds in units to the porch height (suggested scale: 1/2 inch = 1 foot). Scale factors of 1 inch : 1 foot will require paper that is at least 12 inches long.

e. Is your small triangle similar to the actual ramp?

Yes. This is a scaled drawing that includes the 10° angle, the 90° angle, and the height of the porch.

f. What is the scale factor between the height of the porch in your drawing and the porch's actual height?

The scale factor will vary depending on how large the triangle is in the drawing. If students use the ratio 1/2 : 1, the scale factor is 1 : 24, making $k = 24$.

g. Which length in your drawing corresponds to the ramp length?

The length is the hypotenuse on the drawing.

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h. How do you use the length in the drawing and the scale factor to find the length of the ramp?

Multiply the length of the hypotenuse by the scale factor.

First, measure the hypotenuse. If the scale on the drawing is 1/2 : 1, the length of the ramp in the drawing would equal approximately 5.75 inches.

Multiply the length of the drawn ramp by the scale factor to determine the length of the actual ramp.

5.75 • 24 = 138 inches = 11.5 feet

Compare students' answers and see the variation in measurement data. Answers should range from 10.75 feet to 12.25 feet.

i. What is the sine of 10°?

$$\sin 10^\circ = \frac{2}{11.5} \approx 0.174$$

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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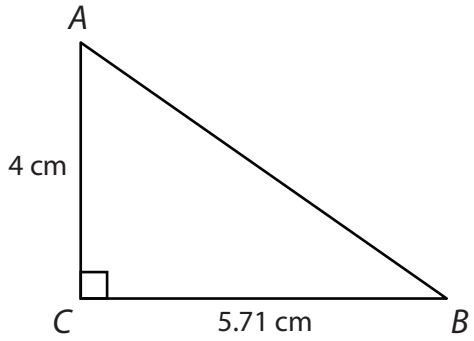
Practice 5.8.1: Defining Trigonometric Ratios

Use $\triangle ABC$ to complete problems 1–4. Round your answers to the nearest thousandth.



1. Set up and calculate the trigonometric ratios for the sine, cosine, and tangent of $\angle A$.

2. Set up and calculate the trigonometric ratios for the cosecant, secant, and cotangent of $\angle A$.

3. Set up and calculate the trigonometric ratios for the sine, cosine, and tangent of $\angle B$.

4. Set up and calculate the trigonometric ratios for the cosecant, secant, and cotangent of $\angle B$.

continued

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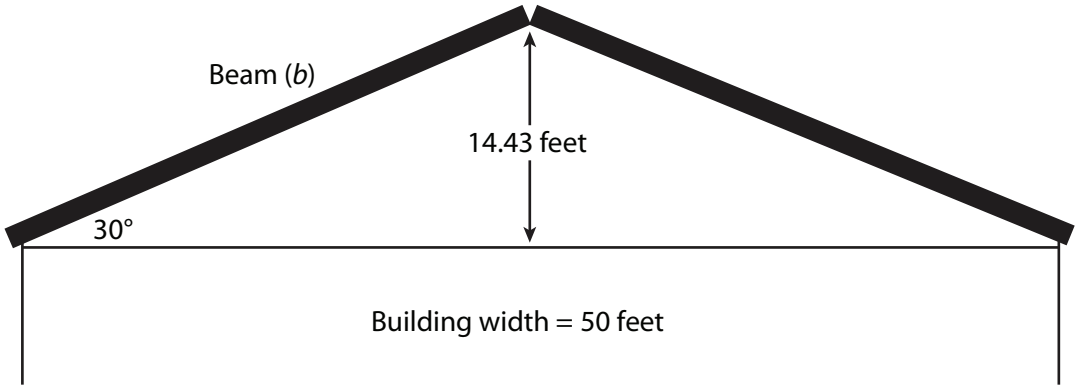
Draw triangles using the given information to complete problems 5 and 6.

5. The legs of a right triangle have lengths of $a = 8$ and $b = 4$. Find the angle measurements and the length of the hypotenuse. Then, find the sine, cosine, and tangent for angles A and B (across from sides a and b respectively).

6. A right triangle has one leg with a length of $a = 4$ and a hypotenuse with a length of $c = 8$. Find the angle measurements of A and B. Then, find the sine, cosine, and tangent for both acute angles.

Use the given information to complete problems 7–10.

7. A carpenter needs to measure the length of the beams that will support a roof. The building is 50 feet wide. The roof will rise at an angle of 30° from the top of the walls. The peak of the roof is 14.43 feet above the top of the walls. The side adjacent to the 30° angle is half the width of the building. How long is each supporting beam, *b,* to the nearest thousandth? Add 2 feet to the beam length so that the roof can extend 2 feet past the walls. What is the cosine of 30°?



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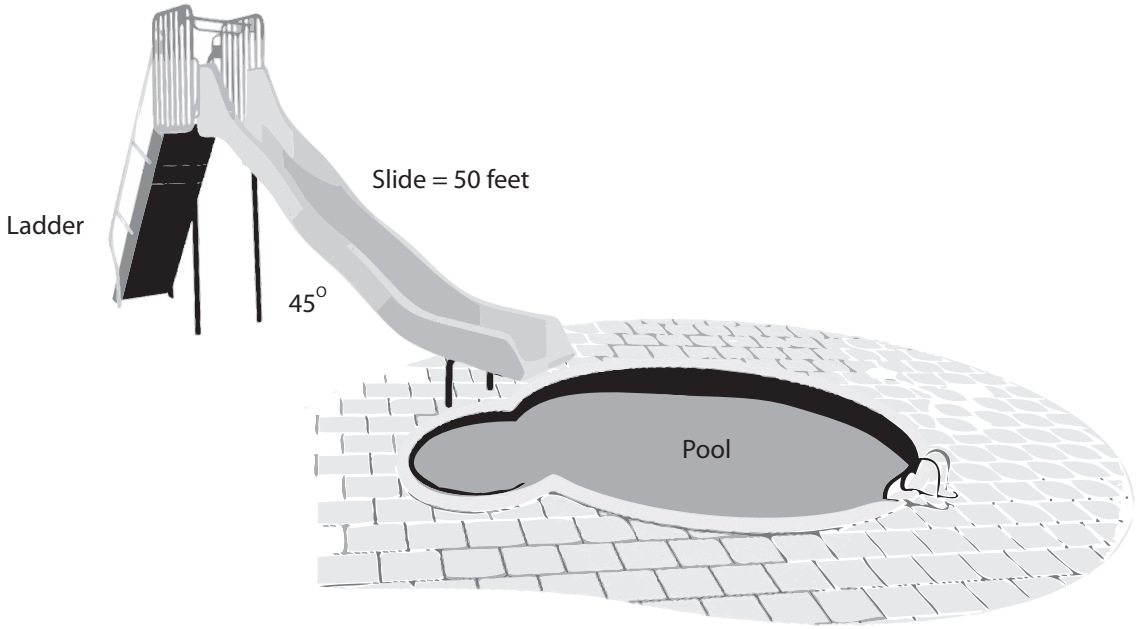
8. A water park has a straight slide into a deep pool. The slide is 50 feet long. It rises from the pool at an angle of 45°. How tall do the vertical supports of the platform need to be to support the platform at the top of the slide? The bottom of the slide is 35.355 feet from the vertical supports. What is the sine of 45°? Round to the nearest thousandth.



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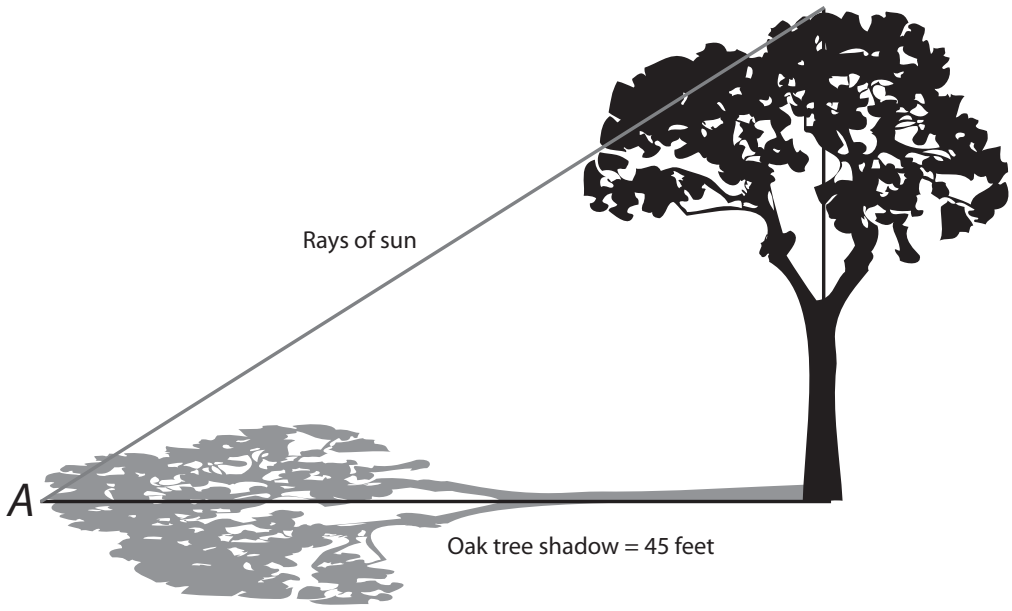
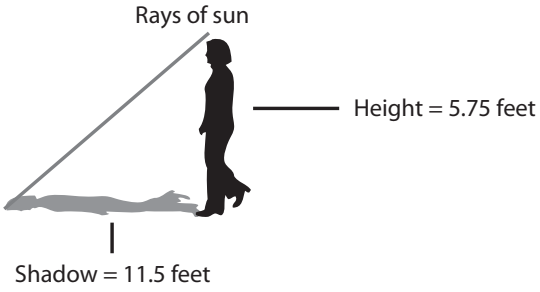
9. Students are having a contest to see who can find the tallest tree in a park. To win, a student must measure the height of the tree without climbing the tree. Martha locates a very tall oak tree. She measures that the tree's shadow is 45 feet long. Martha has a shadow that is 11.5 feet long. She is 5.75 feet tall. How tall is the oak tree? What is the tangent of $\angle A$?





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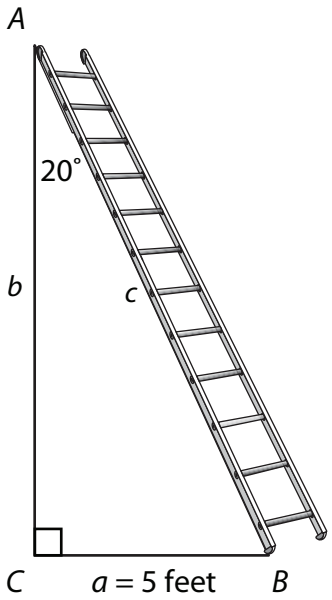
10. The drawing below shows a ladder against a wall. How high on the wall does the ladder reach if the ladder is 14.62 feet long? Use your answer to find the sine of $\angle B$.



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Lesson 5.8.2: Exploring Sine and Cosine As Complements

Warm-Up 5.8.2

The high school basketball team has won the division title, and a work crew is going to put a celebratory banner up on the gymnasium wall. The top of the banner will be 20 feet off the floor. The crew leader knows that the base of an extension ladder should be 7 feet away from the wall to form an angle of approximately 75° with the floor. He is not sure if a 25-foot ladder will be long enough.

1. Create a drawing of this situation.

2. What side length needs to be found in order to determine if the ladder will be long enough to hang the banner so that the top of the banner is 20 feet off the floor?

3. Use the Pythagorean Theorem to find the height the ladder will reach on the wall. Will the ladder be long enough?

4. What is the measurement of $\angle A$?

5. How does the sine of 75° compare to the cosine of 15°?

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Instruction

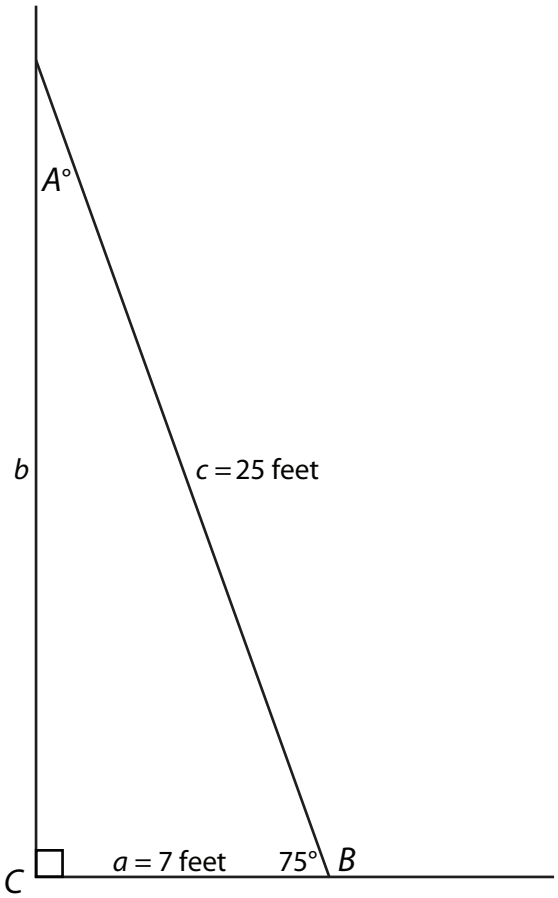
Lesson 5.8.2: Exploring Sine and Cosine As Complements

Common Core State Standard

G–SRT.7 Explain and use the relationship between the sine and cosine of complementary angles.

Warm-Up 5.8.2 Debrief

1. Create a drawing of this situation.



2. What side length needs to be found in order to determine if the ladder will be long enough to hang the banner so that the top of the banner is 20 feet off the floor?

   Length *b* needs to be found (the height on the wall).

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3. Use the Pythagorean Theorem to find the height the ladder will reach on the wall. Will the ladder be long enough?

   $a = 7$ feet

   $c = 25$ feet

   $7^2 + b^2 = 25^2$

   $b = \sqrt{576}$

   $b = 24$ feet

   The ladder will be long enough to reach the height of the banner on the wall.

4. What is the measurement of $\angle A$?

   $m\angle A = 90 - 75 = 15$

5. How does the sine of 75° compare with the cosine of 15°?

   Using your calculator, find sin 75°.

   $\sin 75° \approx 0.9659258263$

   Using your calculator, find cos 15°.

   $\cos 15° \approx 0.9659258263$

   The sine of 75° and the cosine of 15° have exactly the same values.

Connection to the Lesson

- Students will explore further the relationship between the sine and cosine of complementary angles.
- Students will apply the Pythagorean Theorem and sine and cosine ratios.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

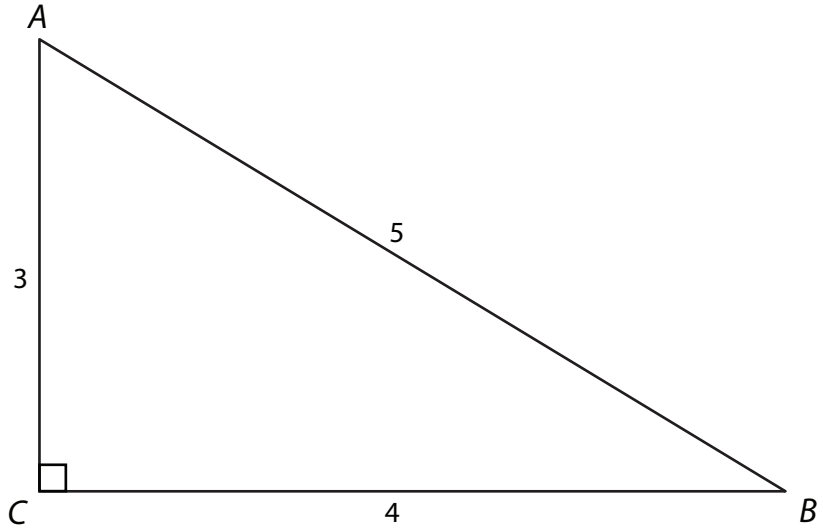
- knowing how to find the sine and cosine ratios in a right triangle (G–SRT.6)
- solving proportions that have an unknown (7.RP.3)
- dividing with decimals (6.NS.3)

Introduction

In the previous lesson, we applied the properties of similar triangles to find unknown side lengths. We discovered that the side ratios of similar triangles are always the same. As a preparation to using trigonometry to solve problems, we will look more deeply into the relationship between sine and cosine in this lesson.

Key Concepts

- Sine and cosine are side length ratios in right triangles.
- The ratio for the sine of an angle is as follows: $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$.
- The ratio for the cosine of an angle is as follows: $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$.
- Examine $\triangle ABC$.



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Instruction

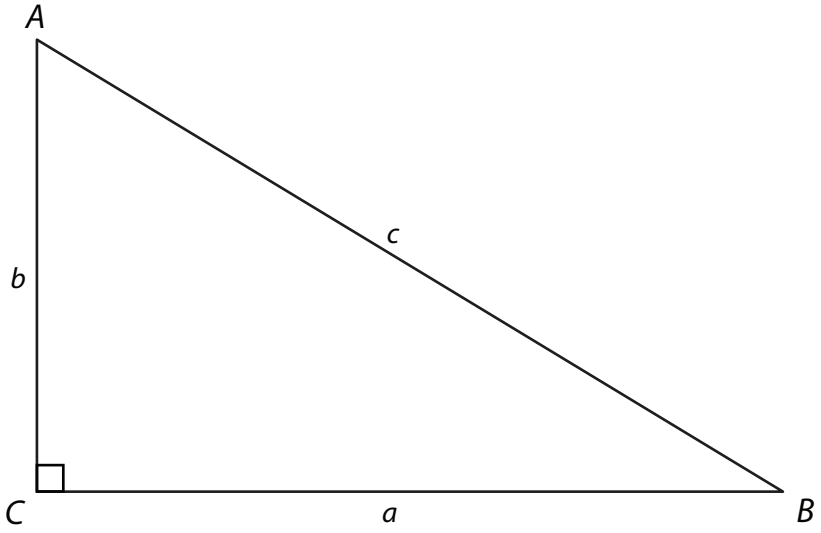
- Determine the sine of $\angle A$.

$$\sin A = \frac{4}{5}$$

- Determine the cosine of $\angle B$.

$$\cos B = \frac{4}{5}$$

- This shows $\sin A = \cos B$.
- You can also see from the diagram that $\sin B = \frac{3}{5}$ and $\cos A = \frac{3}{5}$.
- Show that this relationship will work for any right triangle.



$$\sin A = \frac{a}{c} = \cos B \qquad \sin B = \frac{b}{c} = \cos A$$

- In $\triangle ABC$, $\sin A = \cos B$, and $\sin B = \cos A$.
- This relationship between sine and cosine is known as an identity. An equation is an **identity** if it is true for every value that is used in the equation.
- Sine and cosine are called **cofunctions** because the value of one ratio for one angle is the same as the value of the other ratio for the other angle.
- The two acute angles in a right triangle have a sum of 90°. They are complementary angles. If one acute angle has a measure of x, the other angle has a measure of $90° - x$.

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Instruction

- For example, if one acute angle x has a measure of 70°, the other acute angle must measure $90 - x$.

  $90 - 70 = 20$, or 20°

- The sine-cosine cofunction identities can be written as:

  $\sin \theta = \cos (90° - \theta)$

  $\cos \theta = \sin (90° - \theta)$

- In other words, you can use the sine of one acute angle to find the cosine of its complementary angle.
- Also, you can use the cosine of one acute angle to find the sine of its complementary angle.
- This identity relationship makes sense because the same side lengths are being used in the ratios for the different angles.
- Cofunctions such as sine-cosine give you flexibility in solving problems, particularly if several ratios of trigonometry are used in the same problem.

Postulate
Sine and cosine are cofunction identities. $\sin \theta = \cos (90° - \theta)$ $\cos \theta = \sin (90° - \theta)$

Common Errors/Misconceptions

- losing track of which side length or angle is being solved for
- forgetting to take the complement of the angle when using the sine-cosine cofunction

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Guided Practice 5.8.2

Example 1

Find sin 28° if cos 62° ≈ 0.469.

1. Set up the cofunction identity.

   $\sin \theta = \cos (90° - \theta)$

2. Substitute the values of the angles into the identity and simplify.

   $\sin 28° = \cos (90° - 28°)$

   $\sin 28° = \cos 62°$

3. Verify the identity by calculating the sine of 28° and the cosine of 62° using a scientific calculator.

   $\sin 28° \approx 0.469$

   $\cos 62° \approx 0.469$

Example 2

Complete the following table using the sine and cosine identities.

Angle	Sine	Cosine
10°	0.174	0.985
80°		

1. Determine the relationship between the two given angles.

   10° and 80° are complementary angles.

   $10° + 80° = 90°$

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Instruction

2. Apply the cofunction identity for sine.

   $\sin \theta = \cos (90^\circ - \theta)$

   $\sin 10^\circ = \cos 80^\circ$

3. Use the given value of $\sin 10^\circ$ from the table to find $\sin 80^\circ$.

   The sine of $10^\circ = 0.174$; therefore, the cosine of $80^\circ = 0.174$.

4. Apply the cofunction identity for cosine.

   $\cos \theta = \sin (90^\circ - \theta)$

   $\cos 10^\circ = \sin 80^\circ$

5. Use the given value of $\cos 10^\circ$ from the table to find $\sin 80^\circ$.

   The cosine of $10^\circ = 0.985$; therefore, the sine of $80^\circ = 0.985$.

6. Fill in the table.

Angle	Sine	Cosine
10°	0.174	0.985
80°	0.985	0.174

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Example 3

Find a value of θ for which $\sin \theta = \cos 15^\circ$ is true.

1. Determine which identity to use.

   The cosine was given, so use the cofunction identity for cosine.

   Since θ is used as the variable in the problem, use the variable *phi* (ϕ) for the identity.

   $\cos \phi = \sin (90^\circ - \phi)$

   $\phi = 15^\circ$

   $\cos 15^\circ = \sin (90^\circ - 15^\circ)$

   The cosine of 15° is equal to the sine of its complement.

2. Find the complement of 15°.

   $90^\circ - 15^\circ = 75^\circ$

   The complement of 15° is 75°.

3. Substitute the complement of 15° into the identity.

   $\cos 15^\circ = \sin 75^\circ$ or $\sin 75^\circ = \cos 15^\circ$

4. Write the value of θ.

   $\theta = 75^\circ$

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Example 4

Complete the following table using the sine and cosine identities.

Angle	Sine	Cosine
28°	0.470	
	0.883	0.470

1. Use a graphing calculator to find the cosine of 28°.

   $\cos 28° = 0.883$

2. Analyze the two sets of values for the sine and cosine.

   The values are switched, suggesting that the second angle of interest is the complement of the first angle of interest, 28°.

   Find the complement of 28°.

   $90° - 28° = 62°$

   The complement of 28° is 62°.

   The second angle must be 62°. Therefore, $\sin 62° = \cos 28°$.

3. Verify that the second angle of interest is correct using a scientific calculator.

   $\sin 62° = 0.883$

   $\cos 28° = 0.883$

   The second angle of interest is correct.

4. Use your findings to fill in the table.

Angle	Sine	Cosine
28°	0.470	0.883
62°	0.883	0.470

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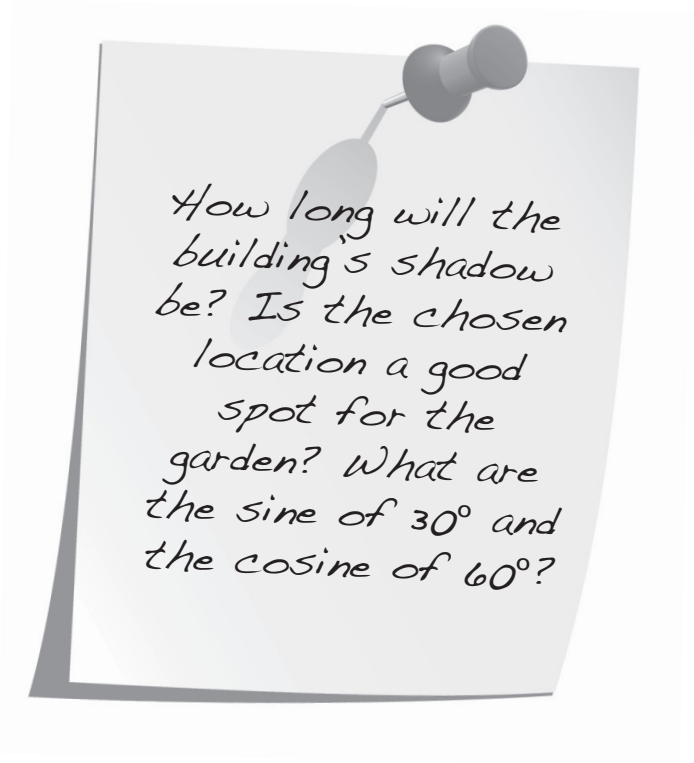
Problem-Based Task 5.8.2: School Vegetable Garden

SMP	
1	2 ✓
3 ✓	4
5	6
7	8 ✓

A high school biology class is planning a vegetable garden. The garden will be on the west side of the school building. The students are concerned that the garden will not get enough direct sunlight in the morning if the building's shadow shades the garden too much. Starting May 1, the garden needs direct sunlight by 10 A.M. at the latest. The school building is 8 meters high. The class is planning to put the garden 6 meters from the building. The students find on the Internet that the angle of the sun's rays at 10 A.M. on May 1 will be 60° above the horizon. How long will the building's shadow be? Is the chosen location a good spot for the garden? What are the sine of 30° and the cosine of 60°?



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Problem-Based Task 5.8.2: School Vegetable Garden

Coaching

a. Draw a picture of the problem.

b. What are the measures of the angles in the drawing's triangle?

c. What is the ratio between the shortest side length and the hypotenuse in this triangle?

d. What is the length of the hypotenuse if the shortest side length of this triangle is 1 inch?

e. Find the length of the longer side by using the Pythagorean Theorem.

f. What are the side lengths of such a triangle if the shortest side length is 1 unit?

g. What is the scale factor of the triangle in the drawing for the problem?

h. How can you use the scale factor to find the length of the building's shadow?

i. Is the chosen location a good spot for the garden? Explain.

j. What are the sine of 30° and the cosine of 60°?

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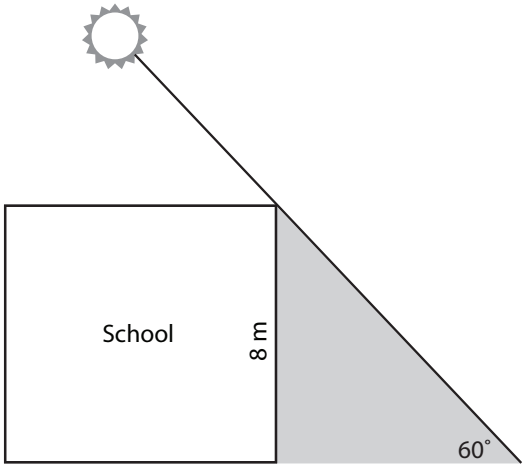
Instruction

Problem-Based Task 5.8.2: School Vegetable Garden

Coaching Sample Responses

a. Draw a picture of the problem.



b. What are the measures of the angles in the drawing's triangle?

The angles measure 30°, 60°, and 90°.

c. What is the ratio between the shortest side length and the hypotenuse in this triangle?

Draw a triangle that has angles of 30°, 60°, and 90°.

Measure the shortest side length, a.

Measure the hypotenuse, c.

Divide the hypotenuse length by the shortest side length, a.

The ratio is 1 : 2. The hypotenuse is twice as long as the shortest leg.

d. What is the length of the hypotenuse if the shortest side length of this triangle is 1 inch?

Create a similar 30°–60°–90° triangle with the shortest side length 1 unit.

The length of the hypotenuse = 2 units.

e. Find the length of the longer side by using the Pythagorean Theorem.

Substitute values for a and c.

$1^2 + b^2 = 2^2$

$b^2 = 3$

$b = \sqrt{3}$ units

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Instruction

f. What are the side lengths of a 30°–60°–90° triangle if the shortest side length is 1 unit?

The proportional lengths opposite the angles in a 30°–60°–90° triangle are:

opposite side length of 30° angle (shortest leg): 1 unit

opposite side length of 60° angle (longest leg): $\sqrt{3}$ units

opposite side length of 90° angle (hypotenuse): 2 units

g. What is the scale factor, k, of the 30°–60°–90° triangle in the drawing for the problem?

The only known side length (the longer leg opposite the 60° angle) is 8 meters.

The length of the longer leg in a 30°–60°–90° triangle with the shortest side 1 unit is $\sqrt{3}$ units.

$$\sqrt{3} \cdot k = 8 \text{ meters}$$
$$k = \frac{8}{\sqrt{3}}$$
$$k \approx 4.619 \text{ meters}$$

h. How can you use the scale factor to find the length of the building's shadow?

The shadow is opposite the 30° angle in the drawing. This makes it the shortest side length.

The length of the shortest side in the scaled down 30°–60°–90° triangle is 1 unit.

Multiply the scale factor by the side length of the triangle that corresponds with the building's shadow. This is the short side of the triangle, and the length is 1. Therefore, the length of the building's shadow is approximately 4.619 meters.

i. Is the chosen location a good spot for the garden? Explain.

Yes, because the garden will not have shade at 10 A.M. on May 1. Since the garden will be placed 6 meters from the building, and the building's shadow will be about 4.619 meters long, the garden will be outside of the shade.

j. What are the sine of 30° and the cosine of 60°?

$\sin 30° = 0.5$

$\cos 60° = 0.5$

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 5.8.2: Exploring Sine and Cosine As Complements

Use what you have learned about the sine and cosine identities to complete the following problems.

1. $\sin 40^\circ \approx 0.643$. What is $\cos 50^\circ$?

2. $\cos 25^\circ \approx 0.906$. Find the sine of the complementary angle.

3. Find a value of θ for which $\sin \theta = \cos 22^\circ$ is true.

4. Find a value of θ for which $\cos \theta = \sin 59^\circ$ is true.

5. $\sin 5^\circ \approx 0.087$. Use this information to write an equation for the cosine.

6. $\cos 34^\circ \approx 0.829$. Use this information to write an equation for the sine.

continued

SWB p. 249; SRB Practice 5.8.2 p. U5-329

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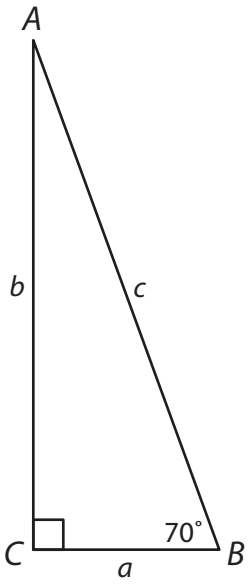
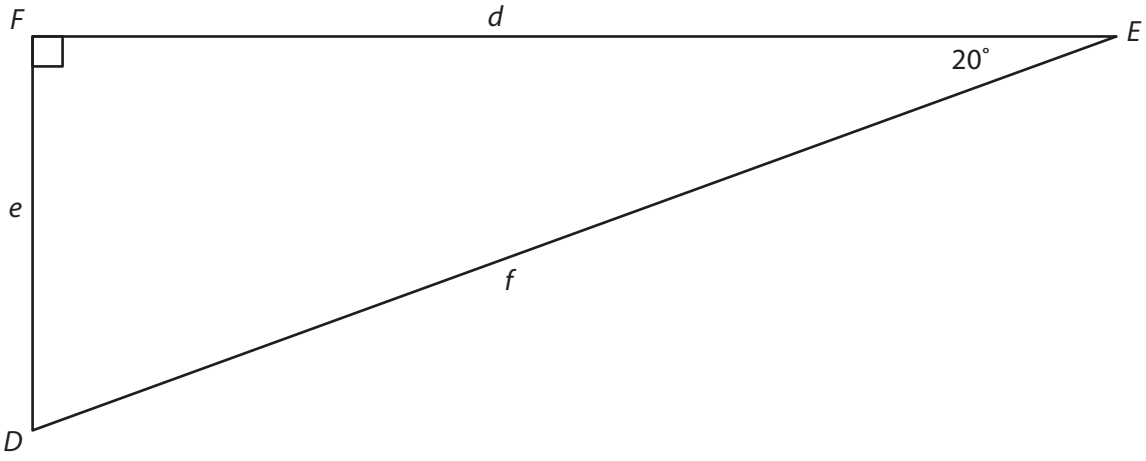
7. $\sin 20^\circ + \sin 10^\circ \approx 0.517$. Find θ if $\cos \theta + \cos 80^\circ \approx 0.517$.

8. Use the diagrams of $\triangle ABC$ and $\triangle DEF$ to fill in the empty boxes for the following equations.



$$\sin A = \frac{a}{c} = \cos \square = \frac{\square}{f}$$

$$\cos B = \frac{\square}{c} = \sin \square = \frac{\square}{f}$$



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9. Lucia needs to buy a ladder to reach her roof. The edge of the roof is 21.125 feet high. To use a ladder safely, the base of a ladder should be 1 foot away from a building for every 3.25 feet of building height. The ladder should extend 3 feet longer than the edge of the roof. How long does the ladder need to be?

10. Use the information you found in problem 9 to draw the triangle created by leaning the ladder against the house at the recommended distance. Label the angles so that $\angle A$ is where the ladder meets the building, $\angle B$ is where the ladder meets the ground, and $\angle C$ is where the building meets the ground. Then, calculate the sine and cosine values for angles A and B.

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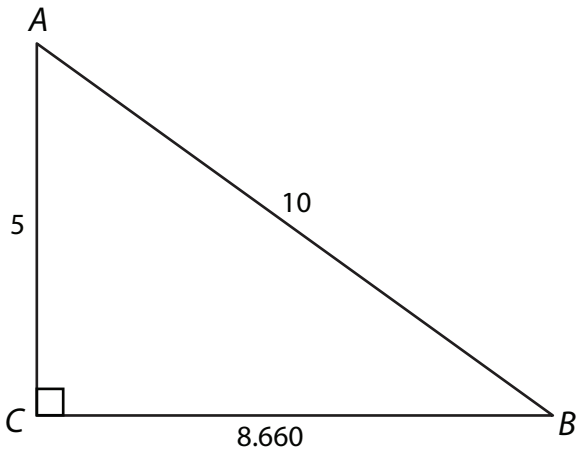
Progress Assessment

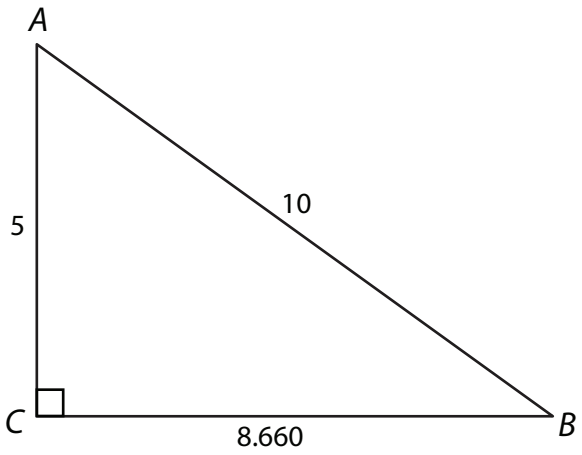
Circle the letter of the best answer.

1. Find sin A.



a. 0.5
b. 0.866
c. 1.732
d. 1.155

2. Find cos A.



a. 0.5
b. 0.866
c. 1.732
d. 1.155

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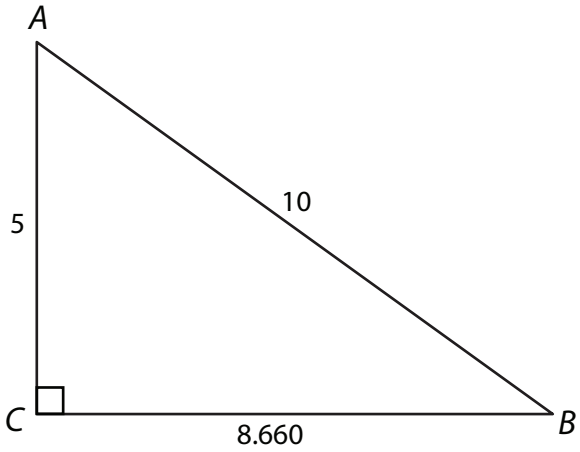
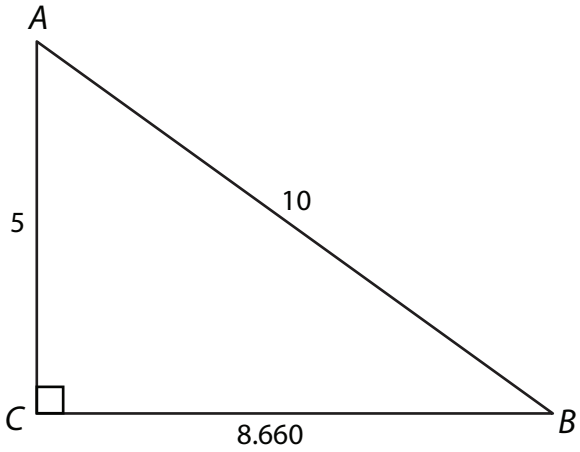
3. Find tan *B*.



a. 0.5
b. 0.866
c. 1.732
d. 0.577

4. Find csc *B*.



a. 2
b. 0.5
c. 1.155
d. 0.866

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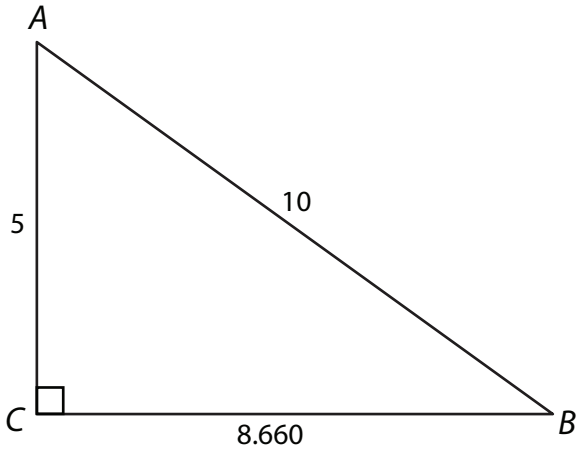
5. Find cos *B*.



a. 0.5
b. 0.866
c. 1.732
d. 1.155

6. $\triangle ABC$ is a right triangle. One of the acute angles is 38°. What is the cosine of the other acute angle?

a. 0.6157
b. 0.3843
c. 1.624
d. 0.3079

7. For which value of θ is $\cos\theta = \sin 73°$?

a. 0.956
b. 73°
c. 17°
d. There is not enough information given.

8. Use your calculator to find the sine of 61°.

a. $\sin 61° \approx 0.4851$
b. $\sin 61° \approx 0.875$
c. $\sin 61° \approx 1.804$
d. $\sin 61° \approx 1.143$

9. For which value of θ is $\sin\theta = \cos 64°$?

a. 0.899
b. 26°
c. 0.4384
d. 64°

continued

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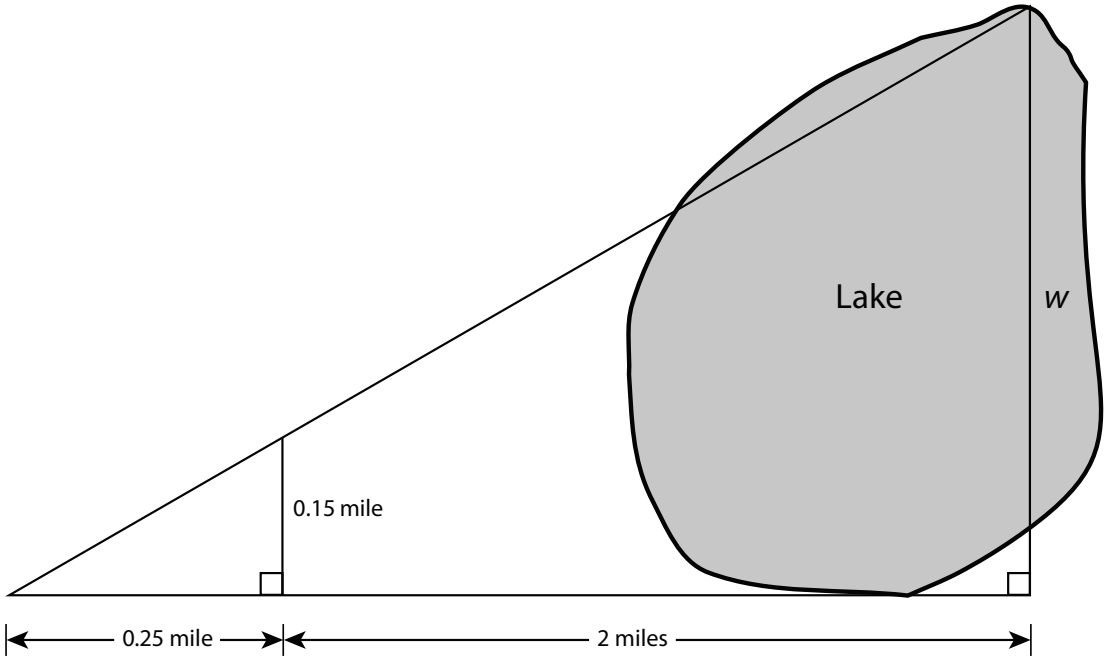
10. What is the width, *w*, of the lake shown below?



a. 1.2 miles
b. 1.35 miles
c. 1.75 miles
d. 2.25 miles

Use your knowledge of trigonometric ratios to complete the problem that follows.

11. How can you measure the height, *h*, of a flagpole without measuring the pole directly?
    a. Describe your method.
    b. Make a drawing of the method you chose. Label the known and unknown measurements in your drawing.
    c. Write the equation(s) that need to be solved. Be sure to define the parts of the equation(s) that correspond with your drawing's labels.

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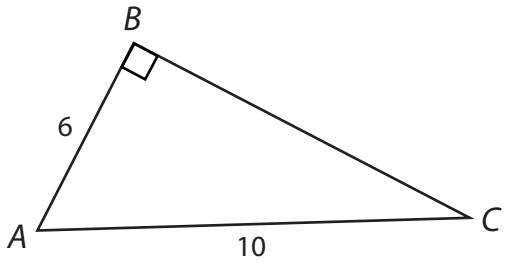
Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Solve the right triangle below. Round sides to the nearest thousandth and angles to the nearest degree.



   a. $BC = 11.662,\ m\angle A = 53$, and $m\angle C = 37$

   b. $BC = 8,\ m\angle A = 53$, and $m\angle C = 37$

   c. $BC = 11.662,\ m\angle A = 37$, and $m\angle C = 53$

   d. $BC = 8,\ m\angle A = 37$, and $m\angle C = 53$

2. A 15-foot ladder is placed against a wall. The ladder is on level ground at an angle of 68.5° to the horizontal. About how far up the wall will the top of the ladder be located?

   a. 5.498 ft

   b. 38.1 ft

   c. 16.122 ft

   d. 13.956 ft

continued

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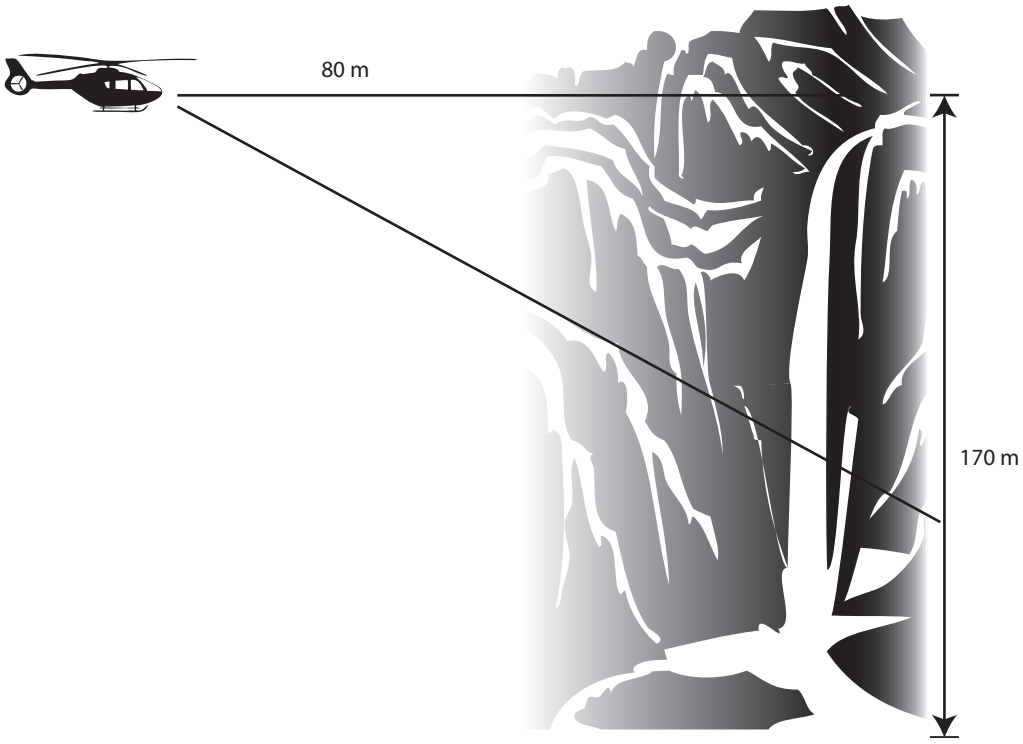
Assessment

3. A landscape photographer in a helicopter is taking pictures of a waterfall that is 170 meters tall. The helicopter is hovering 80 meters away from the waterfall and is level with the waterfall's tallest point. The photographer is focusing her camera on a point halfway down the waterfall. Approximate the angle of depression of the camera lens.



a. 25°
b. 47°
c. 65°
d. 43°

4. If $0° < \theta < 90°$, and $\cos\theta = \frac{\sqrt{3}}{2}$, use a Pythagorean identity to find the value of $\sin\theta$.

a. $\frac{1}{4}$
b. $\frac{1}{2}$
c. $\frac{2-\sqrt{3}}{2}$
d. $\frac{\sqrt{3}}{2}$

5. Simplify the expression $\sin^2\theta + 2 + \cos^2\theta$.

a. 1
b. 2
c. 3
d. 4

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Lesson 9: Applying Trigonometric Ratios

Instruction

Common Core State Standards

G–SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.★

F–TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

SMP
1 ✓ 2 ✓
3 ✓ 4 ✓
5 ✓ 6 ✓
7 ✓ 8

Essential Questions

1. How could you use sine, cosine, tangent, and the Pythagorean Theorem to find missing angles and side lengths of a right triangle?
2. How could you use cosecant, secant, cotangent, and the Pythagorean Theorem to find missing angles and side lengths of a right triangle?
3. How are the angle of depression and angle of elevation used to solve real-world right triangle problems?
4. How do you determine which trigonometric function to use when solving real-world triangle problems?
5. How can the Pythagorean Theorem be used to find a relationship between sine and cosine?

WORDS TO KNOW

altitude	the perpendicular line from a vertex of a figure to its opposite side; height
angle of depression	the angle created by a horizontal line and a downward line of sight to an object that is below the observer
angle of elevation	the angle created by a horizontal line and an upward line of sight to an object that is above the observer
arccosine	the inverse of the cosine function, written $\cos^{-1}\theta$ or $\arccos\theta$
arcsine	the inverse of the sine function, written $\sin^{-1}\theta$ or $\arcsin\theta$
arctangent	the inverse of the tangent function, written $\tan^{-1}\theta$ or $\arctan\theta$
cosecant	the reciprocal of the sine ratio; $\csc\theta = \frac{1}{\sin\theta}$

SRB p. U5-332

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Instruction

cotangent the reciprocal of the tangent ratio; $\cot\theta = \frac{1}{\tan\theta}$

identity an equation that is true regardless of what values are chosen for the variables

Pythagorean identities trigonometric identities that are derived from the Pythagorean Theorem: $\sin^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$, and $1 + \cot^2\theta = \csc^2\theta$

Pythagorean Theorem a theorem that relates the length of the hypotenuse of a right triangle (c) to the lengths of its legs (a and b). The theorem states that $a^2 + b^2 = c^2$.

ratio identities identities that define tangent and cotangent in terms of sine and cosine; the following two identities are ratio identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \text{ and } \cot\theta = \frac{\cos\theta}{\sin\theta}$$

reciprocal identities trigonometric identities that define cosecant, secant, and cotangent in terms of sine, cosine, and tangent:

$$\csc\theta = \frac{1}{\sin\theta},\ \sec\theta = \frac{1}{\cos\theta},\ \cot\theta = \frac{1}{\tan\theta},$$
$$\sin\theta = \frac{1}{\csc\theta},\ \cos\theta = \frac{1}{\sec\theta}, \text{ and } \tan\theta = \frac{1}{\cot\theta}$$

secant the reciprocal of the cosine ratio; $\sec\theta = \frac{1}{\cos\theta}$

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Recommended Resources

- Coolmath.com. "The Pythagorean Identities."

  http://www.walch.com/rr/00130

  This site illustrates the derivation of the Pythagorean identities through the use of a unit circle, and can serve as an informal introduction to the unit circle.

- MathIsFun.com. "Sine, Cosine, and Tangent."

  http://www.walch.com/rr/00064

  This website gives a brief overview of three trigonometric functions and shows some applications of the uses of the ratios. Toward the bottom of the page, a manipulative allows the user to drag the vertex of a triangle to change the acute angle and observe the relationships between the legs of the triangle and the trigonometric ratios when the hypotenuse is one unit.

- Oswego City School District Regents Exam Prep Center. "Practice with Pythagorean Identities."

  http://www.walch.com/rr/00131

  This site offers practice problems to check users' understanding of the application of trigonometric identities.

- ProProfs Flashcards. "Reciprocal, Quotient, and Pythagorean Identities."

  http://www.walch.com/rr/00132

  This site provides interactive flash cards for various identities. The first screen previews all the cards; click on "view flashcards" to flip through the cards from front to back. Users who create an account or log in with a Facebook or Twitter account can make their own flash cards and add missed terms to an online review list. Users can create up to 10 flash cards per set for free.

- Purplemath. "Angles of Elevation / Inclination and Angles of Depression / Declination."

  http://www.walch.com/rr/00045

  This website boasts a variety of resources regarding basic trigonometry. Specifically, the angles of depression and elevation links offer common misconceptions and additional tricks to mastering proper trigonometric equation setups.

- Wolfram MathWorld. "Trigonometry."

  http://www.walch.com/rr/00046

  This online math encyclopedia contains definitions for terms as well as visuals to support continued terminology mastery.

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Lesson 5.9.1: Calculating Sine, Cosine, and Tangent

Warm-Up 5.9.1

After seeing a great deal advertised online, your family bought a refurbished television. The television was advertised as having a 60″ screen, but you doubt this after you unpack the actual TV. You measure the length of the screen to be 48″ and the height of the screen to be 36″, as shown in the diagram below. Given that television screen sizes are a measure of the diagonal distance across the screen, use the Pythagorean Theorem to check the validity of your family's purchase. Is the screen 60″?

1. Draw a sketch of the scenario, indicating the measurements as given.

2. Use the Pythagorean Theorem to solve for the diagonal distance.

3. Does the television have a 60″ screen? Why or why not?

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Instruction

Lesson 5.9.1: Calculating Sine, Cosine, and Tangent

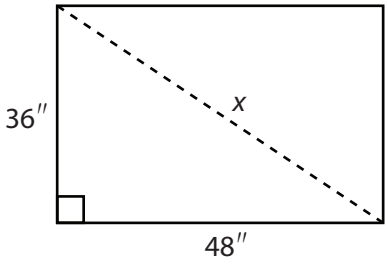
Common Core State Standard

G–SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.★

Warm-Up 5.9.1 Debrief

1. Draw a sketch of the scenario, indicating the measurements as given.

   Since the television is a rectangle, drawing either diagonal will yield two right triangles with legs that measure 36″ and 48″. The diagonal will be the same in both triangles and should be marked with a variable.



2. Use the Pythagorean Theorem to solve for the diagonal distance.

   $36^2 + 48^2 = x^2$

   $1296 + 2304 = x^2$

   $3600 = x^2$

   $\sqrt{3600} = x$

   $x = 60$

   The diagonal distance is 60 inches.

3. Does the television have a 60″ screen? Why or why not?

   Since television sizes are a measure of the diagonal distance across a rectangular screen, you can use the Pythagorean Theorem on the subsequent right triangles created by the diagonal. In this case, we know that the screen is 60″ because the sum of the squares of the length and width of the television yields the square of the diagonal distance.

Connection to the Lesson

- Students will extend their understanding of the Pythagorean Theorem.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- manipulating the Pythagorean Theorem given any two sides of a right triangle (8.G.7)
- defining the three basic trigonometric ratios (sine, cosine, and tangent) given an acute angle of a right triangle (G–SRT.6)

Introduction

In the real world, if you needed to verify the size of a television, you could get a measuring tool and hold it up to the television to determine that the TV was advertised at the correct size. Imagine, however, that you are a fact-checker for the Guinness World Records organization. It is your job to verify that the tallest building in the world is in fact Burj Khalifa, located in Dubai. Could you use measuring tools to determine the size of a building so large? It would be extremely difficult and impractical to do so. You can use measuring tools for direct measurements of distance, but you can use trigonometry to find indirect measurements. First, though, you must be able to calculate the three basic trigonometric functions that will be applied to finding those larger distances. Specifically, we are going study and practice calculating the sine, cosine, and tangent functions of right triangles as preparation for measuring indirect distances.

Key Concepts

- The three basic trigonometric ratios are ratios of the side lengths of a right triangle with respect to one of its acute angles.
- As you learned previously:
  - Given the angle θ, $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$.
  - Given the angle θ, $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$.
  - Given the angle θ, $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$.
- Notice that the trigonometric ratios contain three unknowns: the angle measure and two side lengths.
- Given an acute angle of a right triangle and the measure of one of its side lengths, you can use sine, cosine, or tangent to find another side.

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Instruction

- Given the lengths of two sides of a right triangle, you can use inverse trigonometric functions to find the measures of the acute angles.
- The inverse trigonometric functions are arcsine, arccosine, and arctangent.
- **Arcsine** is the inverse of the sine function, written $\sin^{-1}\theta$ or $\arcsin\theta$.
- **Arccosine** is the inverse of the cosine function, written $\cos^{-1}\theta$ or $\arccos\theta$.
- **Arctangent** is the inverse of the tangent function, written $\tan^{-1}\theta$ or $\arctan\theta$.
- There are two different types of notation for these functions:
  - $\sin^{-1}\theta = \arcsin\theta$; if $\sin\theta = \frac{2}{3}$, then $\arcsin\left(\frac{2}{3}\right) = \theta$.
  - $\cos^{-1}\theta = \arccos\theta$; if $\cos\theta = \frac{2}{3}$, then $\arccos\left(\frac{2}{3}\right) = \theta$.
  - $\tan^{-1}\theta = \arctan\theta$; if $\tan\theta = \frac{2}{3}$, then $\arctan\left(\frac{2}{3}\right) = \theta$.
- Note that "–1" is not an exponent; it is simply the notation for an inverse trigonometric function. Because this notation can lead to confusion, the "arc-" notation is frequently used instead.

  $\sin^{-1}\theta = \arcsin\theta$ but $(\sin\theta)^{-1} = \left(\sin\theta\right)^{-1} = \frac{1}{\sin\theta}$
- To calculate inverse trigonometric functions on your graphing calculator:

On a TI-83/84:

Step 1: Press [2ND][SIN].

Step 2: Type in the ratio.

Step 3: Press [ENTER].

On a TI-Nspire:

Step 1: In the calculate window from the home screen, press [trig] to bring up the menu of trigonometric functions. Use the keypad to select "$\sin^{-1}$."

Step 2: Type in the ratio.

Step 3: Press [enter].

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Instruction

Common Errors/Misconceptions

- not using the given angle as the guide in determining which side is opposite or adjacent
- forgetting to ensure the calculator is in degree mode before completing the operations
- using the trigonometric function instead of the inverse trigonometric function when calculating the acute angle measure
- confusing inverse notation ($\sin^{-1}\theta$) with reciprocal notation $(\sin\theta)^{-1}$

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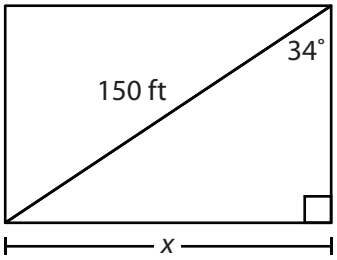
Instruction

Guided Practice 5.9.1

Example 1

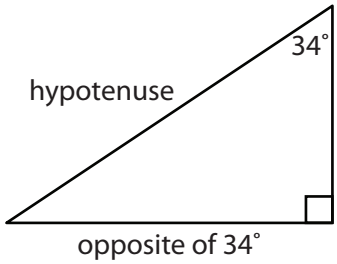
Leo is building a concrete pathway 150 feet long across a rectangular courtyard, as shown in the following figure. What is the length of the courtyard, x, to the nearest foot?



1. Determine which trigonometric function to use by identifying the given information.

   Given an angle of 34°, the length of the courtyard, x, is opposite the angle.

   The pathway is the hypotenuse since it is opposite the right angle of the triangle.



   Sine is the trigonometric function that uses opposite and hypotenuse, $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$, so we will use it to calculate the length of the courtyard.

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Instruction

2. Set up an equation using the sine function and the given measurements.

   $\theta = 34^\circ$

   opposite side = x

   hypotenuse = 150 ft

   Therefore, $\sin 34^\circ = \frac{x}{150}$.

3. Solve for x by multiplying both sides of the equation by 150.

   $150 \bullet \sin 34^\circ = x$

4. Use a calculator to determine the value of x.

   On a TI-83/84:

   First, make sure your calculator is in Degree mode.

   Step 1: Press [MODE].
   Step 2: Arrow down twice to RADIAN.
   Step 3: Arrow right to DEGREE.
   Step 4: Press [ENTER]. The word "DEGREE" should be highlighted inside a black rectangle.
   Step 5: Press [2ND].
   Step 6: Press [MODE] to QUIT.
   Note: You will not have to change to Degree mode again unless you have changed your calculator to Radian mode.

   Next, perform the calculation.

   Step 1: Enter [150][×][SIN][34][)].
   Step 2: Press [ENTER].
   $x = 83.879$

(*continued*)

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On a TI-Nspire:

First, make sure the calculator is in Degree mode.

Step 1: Choose 5: Settings & Status, then 2: Settings, and 2: Graphs and Geometry.
Step 2: Move to the Geometry Angle field and choose "Degree".
Step 3: Press [tab] to "ok" and press [enter].

Then, if necessary, set the Scratchpad in Degree mode.

Step 1: In the calculate window from the home screen, press [doc].

Step 2: Select 7: Settings and Status, then 2: Settings, and 1: General.

Step 3: Move to the Angle field and choose "Degree".

Step 4: Press [tab] to "Make Default" and press [enter] twice to apply this as the new default setting.

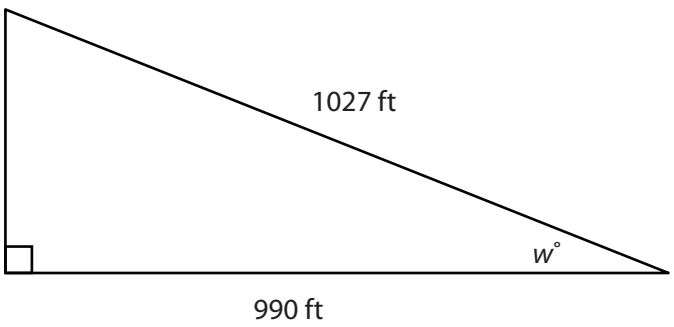
Next, perform the calculation.

Step 1: In the calculate window from the home screen, enter (150), then press [×][trig]. Use the keypad to select "sin," then type 34.
Step 2: Press [enter].
$x = 83.879$

The length of Leo's courtyard is about 84 feet.

Example 2

A trucker drives 1,027 feet up a hill that has a constant slope. When the trucker reaches the top of the hill, he has traveled a horizontal distance of 990 feet. At what angle did the trucker drive to reach the top? Round your answer to the nearest degree.



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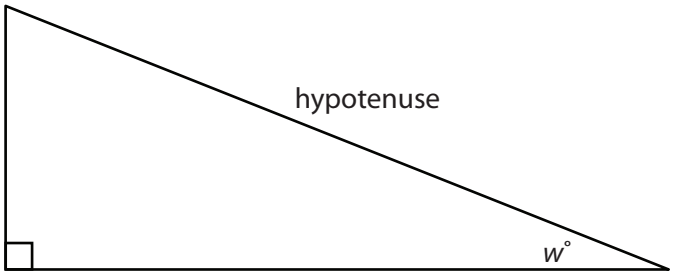
Instruction

1. Determine which trigonometric function to use by identifying the given information.

   Given an angle of w^o, the horizontal distance, 990 feet, is adjacent to the angle.

   The distance traveled by the trucker is the hypotenuse since it is opposite the right angle of the triangle.



   Cosine is the trigonometric function that uses adjacent and hypotenuse, $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, so we will use it to calculate the angle the truck drove to reach the bottom of the road.

   Set up an equation using the cosine function and the given measurements.

   $\theta = w^o$

   adjacent leg = 990 ft

   hypotenuse = 1027 ft

   Therefore, $\cos w = \frac{990}{1027}$.

   Solve for w.

   Solve for w by using the inverse cosine since we are finding an angle instead of a side length.

$$\cos^{-1}\left(\frac{990}{1027}\right) = w$$

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Instruction

2. Use a calculator to calculate the value of w.

On a TI-83/84:

Check to make sure your calculator is in Degree mode first. Refer to the directions in Example 1.

Step 1: Press [2ND][COS][990][÷][1027][)].
Step 2: Press [ENTER].
$w = 15.426$, or about 15°.

On a TI-Nspire:

Check to make sure your calculator is in Degree mode first. Refer to the directions in Example 1.

Step 1: In the calculate window from the home screen, press [trig] to bring up the menu of trigonometric functions. Use the keypad to select "$\cos^{-1}$." Enter 990, then press [÷] and enter 1027.
Step 2: Press [enter].
$w = 15.426$, or about 15°.

The trucker drove at an angle of approximately 15°.

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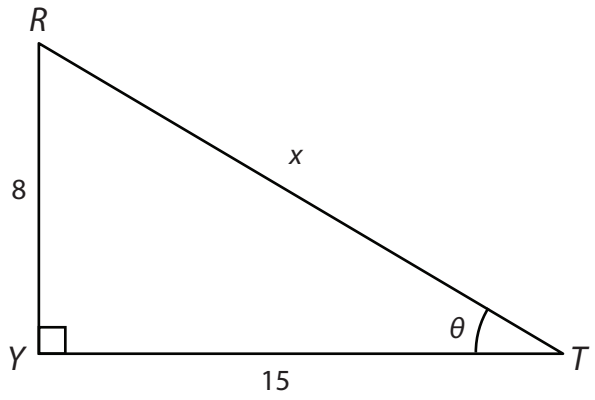
Instruction

Example 3

In $\triangle TRY$, $\angle Y$ is a right angle and $\tan T = \frac{8}{15}$. What is $\sin R$? Express the answer as a fraction and as a decimal.

1. Draw a diagram.



2. Use the Pythagorean Theorem to find the hypotenuse.

$8^2 + 15^2 = x^2$

$64 + 225 = x^2$

$\sqrt{289} = x$

$x = 17$

3. Calculate $\sin R$.

$$\sin R = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{15}{x} = \frac{15}{17} \approx 0.882$$

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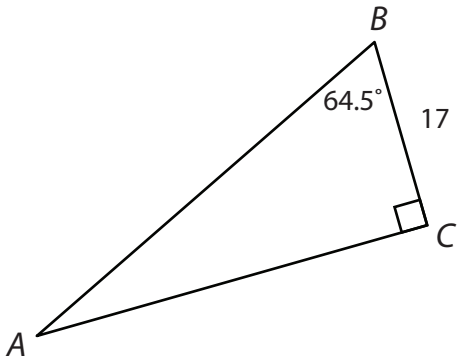
Instruction

Example 4

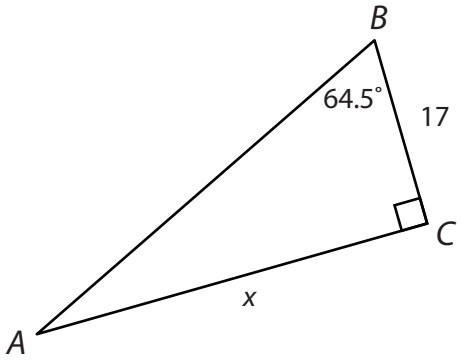
Solve the following right triangle. Round sides to the nearest thousandth.



1. Find the measures of $\overline{AC}$ and $\overline{AB}$.

   Solving a right triangle means to find all the missing angle measures and all the missing side lengths. The given angle is 64.5° and 17 is the length of the adjacent side. With this information, we could either use cosine or tangent since both functions' ratios include the adjacent side of a right triangle. Start by using the tangent function to find $\overline{AC}$.

   Recall that $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$.



$$\tan 64.5^\circ = \frac{x}{17}$$

$$17 \cdot \tan 64.5^\circ = x$$

(*continued*)

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On a TI-83/84:

Step 1: Press [17][TAN][64.5][)].
Step 2: Press [ENTER].
$x \approx 35.641$

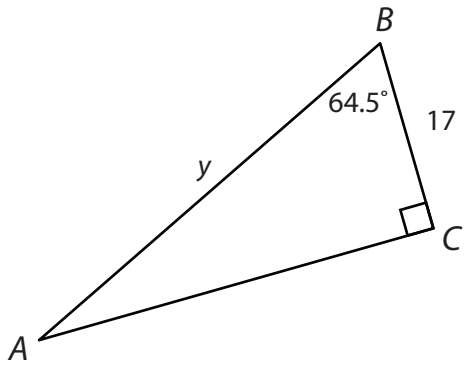
On a TI-Nspire:

Step 1: In the calculate window from the home screen, enter 17, then press [trig] to bring up the menu of trigonometric functions. Use the keypad to select "$\tan^{-1}$," then enter 64.5.
Step 2: Press [enter].
$x \approx 35.641$

The measure of $AC \approx 35.641$.

To find the measure of $\overline{AB}$, either acute angle may be used as an angle of interest. Since two side lengths are known, the Pythagorean Theorem may be used as well.

Note: It is more precise to use the given values instead of approximated values.



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Instruction

2. Use the cosine function based on the given information.

Recall that $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$.

$\theta = 64.5^\circ$

adjacent leg = 17

hypotenuse = y

$\cos 64.5^\circ = \frac{17}{y}$

$y \cdot \cos 64.5^\circ = 17$

$y = \frac{17}{\cos 64.5^\circ}$

On a TI-83/84:

Check to make sure your calculator is in Degree mode first. Refer to the directions in Example 1.

Step 1: Press [17][÷][COS][64.5][)].
Step 2: Press [ENTER].
$y \approx 39.488$

On a TI-Nspire:

Check to make sure your calculator is in Degree mode first. Refer to the directions in Example 1.

Step 1: In the calculate window from the home screen, enter 17, then press [÷][trig]. Use the keypad to select "cos," and then enter 64.5.
Step 2: Press [enter].
$y \approx 39.488$

The measure of $AB \approx 39.488$.

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Instruction

3. Use the Pythagorean Theorem to check your trigonometry calculations.

$17^2 + 35.641^2 = y^2$

$289 + 1267.36 = y^2$

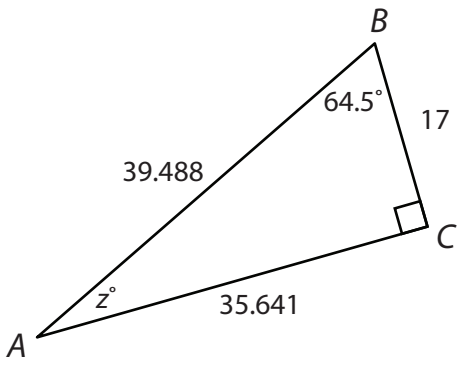
$1559.281 = y^2$

$\sqrt{1559.281} = y$

$y \approx 39.488$ The answer checks out.

$AC \approx 35.641$ and $AB \approx 39.488$.

4. Find the value of $\angle A$.



Using trigonometry, you could choose any of the three functions since you have solved for all three side lengths. In an attempt to be as precise as possible, let's choose the given side length and one of the approximate side lengths.

$$\sin z = \frac{17}{39.488}$$

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Instruction

5. Use the inverse trigonometric function since you are solving for an angle measure.

$$z = \arcsin\left(\frac{17}{39.488}\right)$$

On a TI-83/84:

Step 1: Press [2ND][SIN][17][÷][39.488][)].
Step 2: Press [ENTER].
$z \approx 25.500°$

On a TI-Nspire:

Step 1: In the calculate window from the home screen, press [trig] to bring up the menu of trigonometric functions. Use the keypad to select "$\sin^{-1}$," and then enter 17, press [÷], and enter 39.488.
Step 2: Press [enter].
$z \approx 25.500°$

Check your angle measure by using the Triangle Sum Theorem.

$m\angle A + 64.5 + 90 = 180$

$m\angle A + 154.5 = 180$

$m\angle A = 25.5$ The answer checks out.

$\angle A$ is 25.5°.

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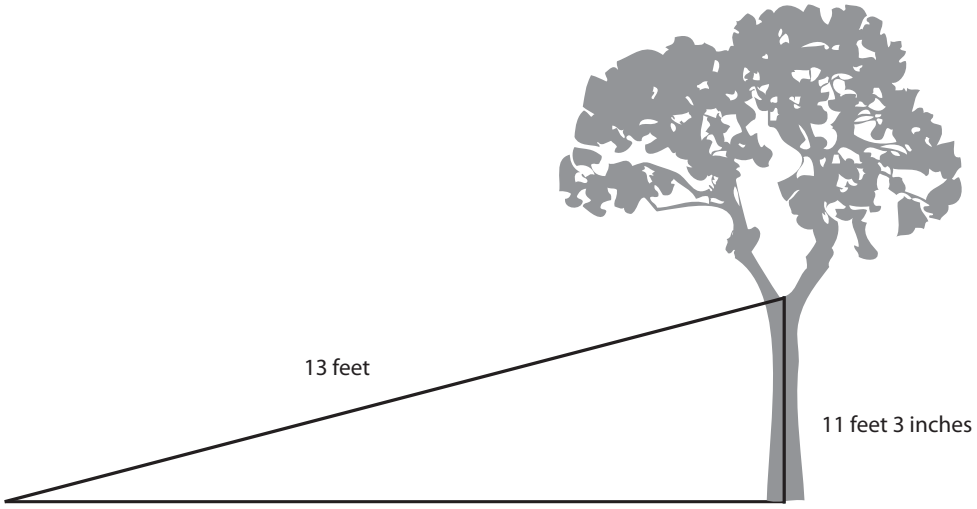
Problem-Based Task 5.9.1: Replanting Project

SMP	
1	2 ✓
3	4
5	6 ✓
7	8

The environmental club at your school has taken on the task of planting a garden to provide fresh fruit, herbs, and vegetables to the school cafeteria. The approved area has several trees that need to be moved and replanted so that you can use the garden space more efficiently. A building supply company donates several pieces of wire, each 13 feet in length, to help brace the replanted trees.

As the president of the environmental club, you must provide the specifications to the club members and volunteers before they can begin the replanting process. At most, the ladders you have can reach 11 feet 3 inches up the trunk of a tree. Given this information, what angle should the wires make with the ground, and at what horizontal distance from the tree trunk should the wires be staked to the ground?



What angle should the wires make with the ground, and at what horizontal distance from the tree trunk should the wires be staked to the ground?

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Problem-Based Task 5.9.1: Replanting Project

Coaching

a. What is the maximum height, in inches, each wire can be secured to a tree?

b. What is the length of each wire in inches?

c. Based on the given information, which trigonometric function is required to find the angle at which each wire should be staked to the ground? What is the angle measure?

d. What right triangle property should you use to most accurately find the distance from the trunk of the tree to the spot where the wire will be staked to the ground?

e. What is the distance, in inches, from the trunk of the tree to the spot where the wire will be staked to the ground?

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Instruction

Problem-Based Task 5.9.1: Replanting Project

Coaching Sample Responses

a. What is the maximum height, in inches, that each wire can be secured to a tree?

The wires can be secured as high up as the ladders will reach. Since this is known to be 11 feet 3 inches, start by converting 11 feet to inches.

$$(11\text{ ft})\left(\frac{12\text{ in}}{1\text{ ft}}\right)=132\text{ in}$$

Add 3 inches to determine the total height in inches.

$132 + 3 = 135$

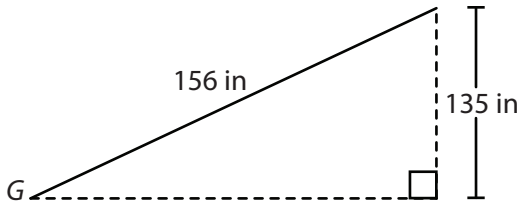
The maximum height each wire can be secured is 135 inches.

b. What is the length of each wire in inches?

Convert the known length of the wires (13 feet) to inches.

$$(13\text{ ft})\left(\frac{12\text{ in}}{1\text{ ft}}\right)=156\text{ in}$$

c. Based on the given information, which trigonometric function is required to find the angle at which each wire should be staked to the ground? What is the angle measure?



The opposite side of the angle and the hypotenuse are known, so sine should be used.

$$\sin G=\frac{135}{156}$$
$$G=\sin^{-1}\left(\frac{135}{156}\right)$$

$G = 59.926°$. The angle of elevation is approximately 60°.

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d. What right triangle property should you use to most accurately find the distance from the trunk of the tree to the spot where the wire will be staked to the ground?

Since two side lengths were given, you could use either trigonometry or the Pythagorean Theorem. However, in an effort to be as precise as possible, the best choice is the Pythagorean Theorem. With trigonometry, you would have to use the acute angle *G* that we just approximated in part c.

e. What is the distance, in inches, from the trunk of the tree to the spot where the wire will be staked to the ground?

From part d we determined that the Pythagorean Theorem is the most accurate method for determining the distance.

d = unknown

leg = 135 in

hypotenuse = 156 in

$d^2 + 135^2 = 156^2$

$d^2 + 18{,}225 = 24{,}336$

$d^2 = 6111$

$d = \pm\sqrt{6111}$

$d \approx 78.173$

The distance from the tree trunk to the staked wire is about 78 inches.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 5.9.1: Calculating Sine, Cosine, and Tangent

Draw a sketch of the triangle relationship for all problems that do not have a diagram. Unless otherwise stated, round all side lengths to the nearest thousandth and round angle measures to the nearest degree.

1. In $\triangle TRY$, $\angle Y$ is a right angle and $\sin R = \frac{3}{5}$. What are the cosine and tangent of $\angle T$? Write your answers as fractions and as decimals.

2. In $\triangle RAY$, $\angle Y$ is a right angle and $\tan R = \frac{11}{60}$. What are the cosine and sine of $\angle A$? Write your answers as fractions and as decimals.

3. A ladder manufacturer recommends that its ladders be used on level ground at an angle of 72.5° to the horizontal. At that angle, how far up the side of a building will the top of a 14-foot ladder reach?

4. At a certain time of day, an 18-foot pole casts a shadow that forms an angle of 50° with the ground. How long is the shadow?

5. A right triangle has legs that each measure 6 inches. Use a trigonometric function to find the measure of the triangle's acute angles to the nearest degree. Why does the solution make sense in this situation?

continued

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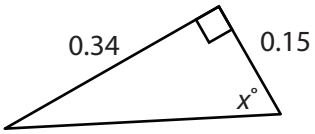
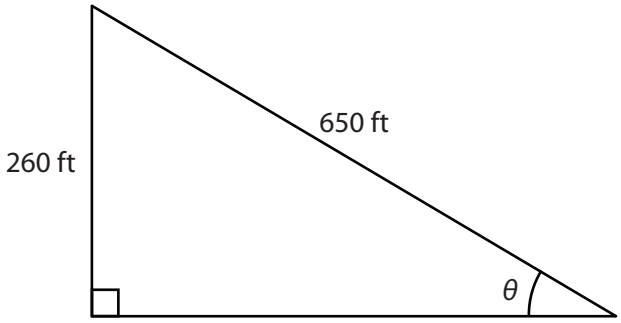
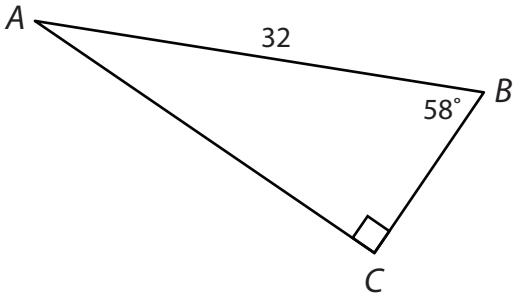
6. What is the value of x?



7. One section of a ski run is 650 feet long and falls 260 feet in elevation at a constant slope. To the nearest degree, what angle does the ski run form with the horizontal?



8. Solve the right triangle below.



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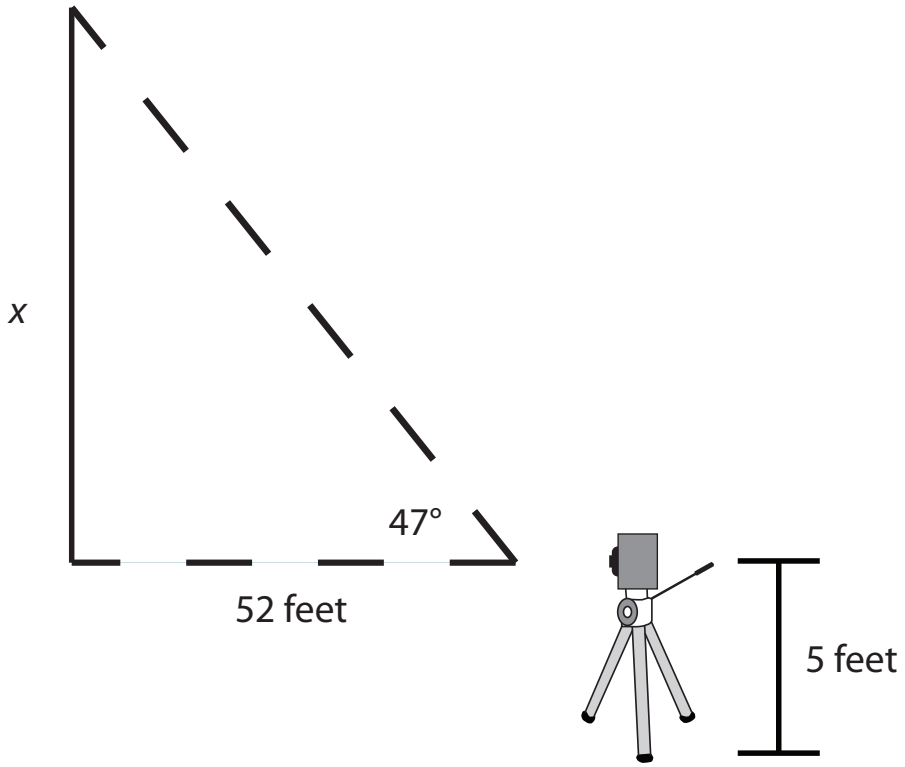
9. Solve the right triangle XYZ, where $\angle Y$ is a right angle, $XZ = 12$ inches, and $XY = 8$ inches.

10. A photographer shines a camera light at a particular painting, forming an angle of 47° from the camera's horizontal line of sight. If the light is 52 feet from the wall where the painting hangs and the camera lens is 5 feet from the floor, how high above the floor is the painting?



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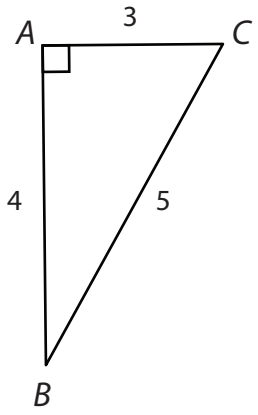
Lesson 5.9.2: Calculating Cosecant, Secant, and Cotangent

Warm-Up 5.9.2

Your class is reviewing for the midterm exam with a series of activities. One activity features a bag with slips of paper folded separately inside it. Each slip of paper has a different side length of the right triangle below written on it. Your directions are to pull two numbers from the bag and set up a ratio, using the first number as a numerator and the second number as a denominator.



1. What is the probability that, given the three basic trigonometric ratios, the ratio of the numbers will be the cosine of $\angle B$? Explain.

2. What is the measure of $\angle B$ to the nearest degree? Explain.

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Instruction

Lesson 5.9.2: Calculating Cosecant, Secant, and Cotangent

Common Core State Standard

G–SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.★

Warm-Up 5.9.2 Debrief

1. What is the probability that, given the three basic trigonometric ratios, the ratio of the numbers will be the cosine of $\angle B$? Explain.

   In order to determine the probability of pulling out the numbers that create the cosine of $\angle B$, you must first know all of the possible ratios.

   $$\frac{1}{\cos B}=\frac{5}{4} \qquad \cos B=\frac{4}{5} \qquad \tan B=\frac{3}{4}$$
   $$\frac{1}{\cos C}=\frac{5}{3} \qquad \cos C=\frac{3}{5} \qquad \tan C=\frac{4}{3}$$

   There are six trigonometric ratios possible, and one of them is the cosine of $\angle B$. As such, the probability of pulling two numbers that form the cosine of $\angle B$ is $\frac{1}{6}$.

2. What is the measure of $\angle B$ to the nearest degree? Explain.

   To find the measure of an angle using trigonometry, you have to determine the correct trigonometric function based on the given information and then use its inverse. Since all three sides of the triangle are given and we've listed all possibilities for $m\angle B$ in question 1, there are three options to choose from. Let's use cosine. Make sure the calculator is in degree mode before performing the calculation.

   $$\cos B=\frac{4}{5}$$
   $$B=\arccos\left(\frac{4}{5}\right)$$
   $$m\angle B=37^\circ$$

Connection to the Lesson

- Students will extend their knowledge of the sine, cosine, and tangent ratios to apply them to the reciprocal ratios.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- performing calculations using the sine, cosine, and tangent functions (G–SRT.6, G–SRT.8)

Introduction

In previous lessons, you defined and calculated using the three basic trigonometric functions, sine, cosine, and tangent. In this lesson, you will extend your working definitions of their reciprocal functions and use them to determine the unknown sides and angles of triangles.

Key Concepts

- Remember:
  - Cosecant is the reciprocal of sine. Given the angle θ, $\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$.
  - Secant is the reciprocal of cosine. Given the angle θ, $\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$.
  - Cotangent is the reciprocal of tangent. Given the angle θ, $\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$.

Common Errors/Misconceptions

- not using the proper order of operations when calculating inverse reciprocal functions with the calculator

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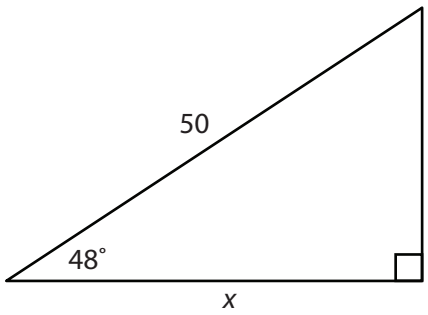
Instruction

Guided Practice 5.9.2

Example 1

Determine the correct reciprocal trigonometric function to solve for x in the following triangle, and calculate the value of x to the nearest thousandth.



1. Identify the given information.

   $\theta = 48°$, the hypotenuse is 50, and we are trying to calculate the adjacent side, x.

2. Set up the correct reciprocal function.

   Recall that:

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

   Based on the given information, we must use the secant function, the reciprocal of cosine.

$$\sec 48° = \frac{50}{x}$$

$$x \bullet \sec 48° = 50$$

$$x = \frac{50}{\sec 48°}$$

SWB p. 269; SRB p. U5-352

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3. Use the calculator to determine the value of x.

Secant is the reciprocal of cosine. This means that $\sec 48^\circ = \frac{1}{\cos 48^\circ}$. Since most calculators do not have buttons for the reciprocal functions, you will have to substitute this value in the expression in order to correctly calculate the value of x.

$$x = \frac{50}{\sec 48^\circ}$$

$$x = \frac{50}{\frac{1}{\cos 48^\circ}}$$

On a TI-83/84:

Be sure that your calculator is in Degree mode.

Step 1: Press [50][÷][(][1][÷][COS][48][)][)].
Step 2: Press [ENTER].
$x \approx 33.457$

On a TI-Nspire:

Be sure that your calculator is in Degree mode.

Step 1: In the calculate window from the home screen, enter 50, then press [÷][(][trig]. Use the keypad to select "sec," then enter 48 and press [)].
Step 2: Press [enter].
$x \approx 33.457$

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4. Check your solution by using the reciprocal function of the function you chose earlier.

   We solved for x by using the secant function, which is the reciprocal of cosine. To check the answer, use cosine to see if you get the same solution.

   $$\cos 48^\circ = \frac{x}{50}$$

   $$50 \bullet \cos 48^\circ = x$$

   On a TI-83/84:

   Step 1: Press [50][COS][48][)].
   Step 2: Press [ENTER].
   $x \approx 33.457$ The answer checks out.

   On a TI-Nspire:

   Step 1: In the calculate window from the home screen, enter 50, then press [×][trig]. Use the keypad to select "cos," and then enter 48.
   Step 2: Press [enter].
   $x \approx 33.457$ The answer checks out.

   Note: You can check your solutions when using reciprocal functions by following the steps in this example.

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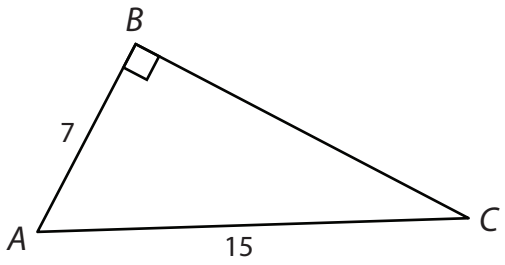
Instruction

Example 2

Use cosecant, secant, or cotangent to find $m\angle C$.



1. Identify the given information.

   The opposite side is given, $AB = 7$, as well as the hypotenuse, $AC = 15$. We are solving for $m\angle C$.

2. Set up the correct reciprocal function.

   Based on the given information, use the cosecant function, the reciprocal of sine.

   $$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

   $$\csc C = \frac{15}{7}$$

   Use the inverse of this function to find the desired angle measure.

   $$C = \csc^{-1}\left(\frac{15}{7}\right)$$

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3. Use the calculator to calculate the measure of $\angle C$.

Cosecant is the reciprocal of sine. This means that $\csc^{-1}\left(\frac{15}{7}\right)=\sin^{-1}\left(\frac{7}{15}\right)$. Because there is no arccosecant function on the calculator, you must use the arcsine function.

On a TI-83/84:

Step 1: Press [2ND][SIN][7][÷][15][)].
Step 2: Press [ENTER].
$m\angle C \approx 28^\circ$

On a TI-Nspire:

Step 1: In the calculate window from the home screen, press [trig] to bring up the menu of trigonometric functions. Use the keypad to select "$\sin^{-1}$," and then enter 7, press [÷], and enter 15.
Step 2: Press [enter].
$m\angle C \approx 28^\circ$

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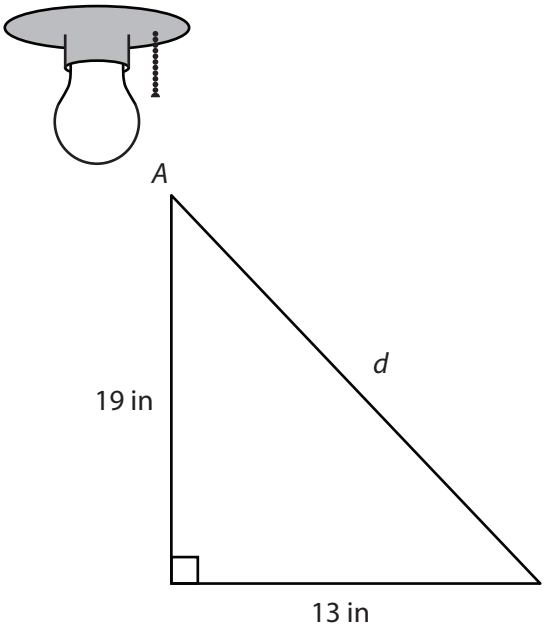
Instruction

Example 3

The light from a 19-inch-tall egg incubator casts a 13-inch shadow across a shelf. Use reciprocal functions and the Pythagorean Theorem as necessary to determine the distance from the top of the incubator to the farthest part of the shadow. What is the angle at which the incubator light casts its shadow?

1. Create a drawing of the scenario. Label the distance d and the angle A.



2. Identify the given information.

   Two side lengths are given: 19 inches and 13 inches.

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3. Determine which trigonometric functions are necessary to find the unknown values.

   Since the given information includes two side lengths, use the Pythagorean Theorem to find the third side length.

   $19^2 + 13^2 = d^2$

   $361 + 169 = d^2$

   $530 = d^2$

   $\sqrt{530} = d$

   $d \approx 23.022$

   The distance from the top of the incubator to the farthest tip of the shadow is about 23.022 inches.

   Find the measure of $\angle A$.

   To find the measure of $\angle A$, use the two side lengths that were given to produce the most precise answer possible. Since those sides are adjacent to the angle and opposite from the angle, use the cotangent function, which is the reciprocal of tangent.

   $$\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

   $$\cot A = \frac{19}{13}$$

   $$A = \cot^{-1}\left(\frac{19}{13}\right)$$

(*continued*)

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Instruction

To enter this into the calculator, first write it in its reciprocal format, $A = \tan^{-1}\left(\frac{13}{19}\right)$.

On a TI-83/84:

Step 1: Press [2ND][TAN][13][÷][19][)].
Step 2: Press [ENTER].
$m\angle A \approx 34°$

On a TI-Nspire:

Step 1: In the calculate window from the home screen, press [trig]. Use the keypad to select "$\tan^{-1}$," then enter 13, press [÷], and enter 19.
Step 2: Press [enter].
$m\angle A \approx 34°$

The incubator casts a shadow at an angle of 34°.

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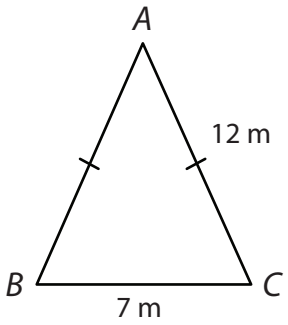
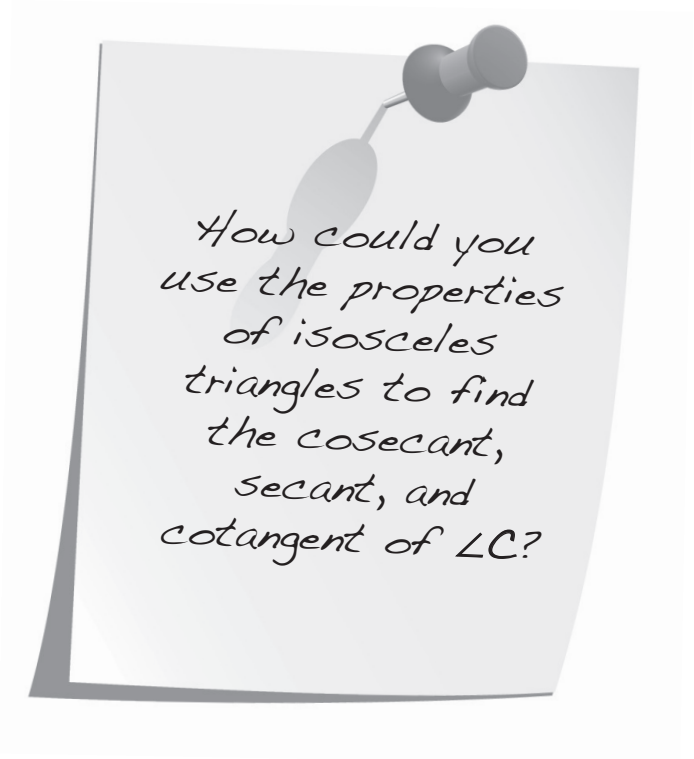
Problem-Based Task 5.9.2: A Baseball Triangle

SMP	
1	2
3	4
5	6 ✓
7 ✓	8

Three baseball players are in the outfield at locations A, B, and C, as shown in the diagram. The distance between A and B is the same as the distance between A and C. Their positions form an isosceles triangle. How could you use the properties of isosceles triangles to find the cosecant, secant, and cotangent of $\angle C$? Calculate the cosecant, secant, and cotangent of $\angle C$, and then find the angle measures.





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Problem-Based Task 5.9.2: A Baseball Triangle

Coaching

a. What type of triangle is needed in order to use the trigonometry you've learned so far?

b. How could you create a right triangle using the given isosceles triangle?

c. What does the altitude, or height, of an isosceles triangle create with the base of the triangle? What does that mean about the base of the isosceles triangle?

d. What information do you need to calculate the cosecant, secant, and cotangent of $\angle C$?

e. Calculate the cosecant, secant, and cotangent of $\angle C$.

f. How did you calculate the cosecant, secant, and cotangent of $\angle C$?

g. What buttons do you need to use on your calculator to find the angle measures given a trigonometric ratio?

h. What is the relationship between the buttons you have to use and the trigonometric ratios you calculated?

i. What are the measures of $\angle A$, $\angle B$, and $\angle C$?

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Instruction

Problem-Based Task 5.9.2: A Baseball Triangle

Coaching Sample Responses

a. What type of triangle is needed in order to use the trigonometry you've learned so far?

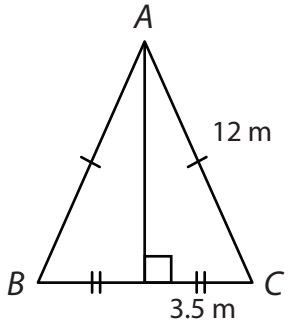
A right triangle is needed in order to use the trigonometric functions learned so far.

b. How could you create a right triangle using the given isosceles triangle?

Draw the height, or the altitude, of the triangle.

c. What does the altitude of an isosceles triangle create with the base of the triangle? What does that mean about the base of the isosceles triangle?

The altitude of an isosceles triangle is the perpendicular bisector of the base. That means the right triangles created will each have a short leg of 3.5 meters since the base of the isosceles triangle is 7 meters.



d. What information do you need to calculate the cosecant, secant, and cotangent of $\angle C$?

Recall that:

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Since the side opposite from $\angle C$ is not known and is needed in order to calculate both cosecant and cotangent, use the Pythagorean Theorem to determine the opposite side. Let *h*, height, stand for the opposite side $\angle C$.

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$3.5^2 + h^2 = 12^2$

$12.25 + h^2 = 144$

$h^2 = 131.75$

$h = \sqrt{131.75}$

$h \approx 11.478$

The side opposite from $\angle C$, or the height, h, is approximately 11.478 meters.

e. Calculate the cosecant, secant, and cotangent of $\angle C$.

$$\csc C = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{12}{11.478} \approx 1.045$$

$$\sec C = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{12}{3.5} \approx 3.429$$

$$\cot C = \frac{\text{adjacent}}{\text{opposite}} = \frac{3.5}{11.478} \approx 0.305$$

f. How did you calculate the cosecant, secant, and cotangent of $\angle C$?

After creating a right triangle using the altitude of the isosceles triangle, I used the Pythagorean Theorem to find the measure of the side opposite from $\angle C$. Given that side length, I set up the trigonometric ratio for each reciprocal function and divided their respective ratios to get the decimal approximation.

g. What buttons do you need to use on your calculator to find the angle measures given a trigonometric ratio?

You must use the sine (SIN), cosine (COS), and tangent (TAN) buttons because there are no buttons for cosecant (CSC), secant (SEC), and cotangent (COT).

h. What is the relationship between the functions of the buttons you have to use and the trigonometric ratios you calculated?

The cosecant is the reciprocal of sine, the secant is the reciprocal of cosine, and the cotangent is the reciprocal of tangent.

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i. What are the measures of $\angle A$, $\angle B$, and $\angle C$?

Use the following relationships and find the reciprocal of each trigonometric ratio.

$$\csc C = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{12}{11.478} \approx 1.045 \text{ becomes } \sin C = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{11.478}{12} \approx 0.957.$$

$$\sec C = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{12}{3.5} \approx 3.429 \text{ becomes } \cos C = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3.5}{12} \approx 0.292.$$

$$\cot C = \frac{\text{adjacent}}{\text{opposite}} = \frac{3.5}{11.478} \approx 0.305 \text{ becomes } \tan C = \frac{\text{opposite}}{\text{adjacent}} = \frac{11.478}{3.5} \approx 3.279.$$

Now, take the inverse of at least one of the trigonometric functions to find the measure of $\angle C$. Verify with the other two trigonometric ratios that the angle measure is the same for any trigonometric ratio used.

$$\sin C = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{11.478}{12} \approx 0.957$$

$$\sin C \approx 0.957$$

$$C \approx \sin^{-1}(0.957) \approx 73^\circ$$

$$\cos C = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3.5}{12} \approx 0.292$$

$$\cos C \approx 0.292$$

$$C \approx \cos^{-1}(0.292) \approx 73^\circ$$

$$\tan C = \frac{\text{opposite}}{\text{adjacent}} = \frac{11.478}{3.5} \approx 3.279$$

$$\tan C \approx 3.279$$

$$C \approx \tan^{-1}(3.279) \approx 73^\circ$$

We know that $\triangle ABC$ is an isosceles triangle and the angles opposite the congruent sides are congruent. Therefore, $\angle C \cong \angle B$, which means that the measure of $\angle B$ is also 73°. Use the Triangle Sum Theorem to find the measure of the third angle: 180 – (73 + 73) = 34. The measure of $\angle A$ is 34°.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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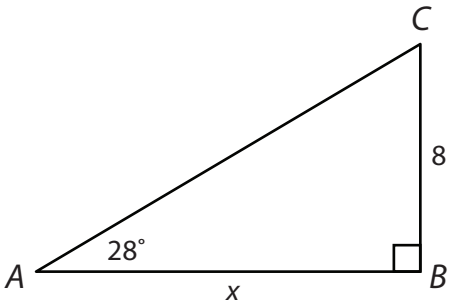
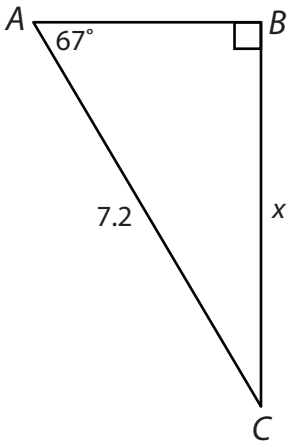
Practice 5.9.2: Calculating Cosecant, Secant, and Cotangent

Use what you've learned about cosecant, secant, and cotangent to solve the following problems. Unless otherwise stated, round your answers to the nearest thousandth.

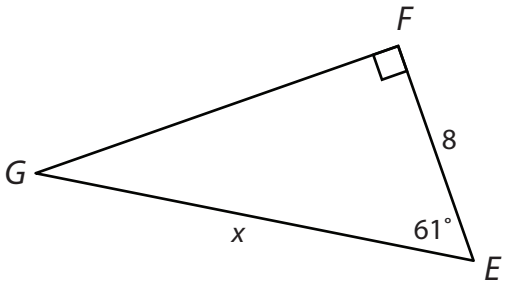
1. Use cosecant, secant, or cotangent to find the value of x.



2. Use cosecant, secant, or cotangent to find the value of x.



3. Use cosecant, secant, or cotangent to find the value of x.



continued

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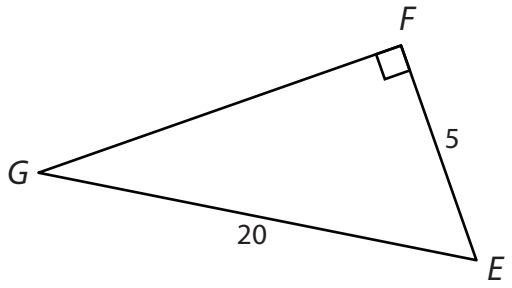
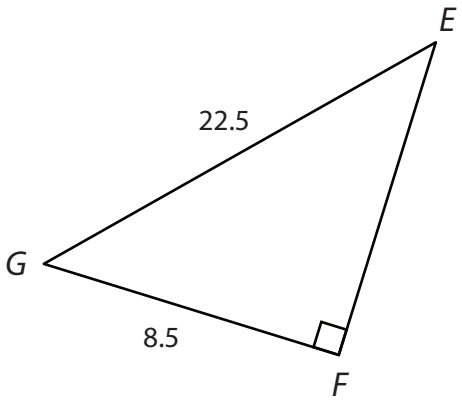
4. Use cosecant, secant, or cotangent to find $m\angle E$.



5. Use cosecant, secant, or cotangent to find $m\angle E$.



6. In $\triangle EFG$, $\angle F$ is a right angle. Given $\tan E = \frac{7}{24}$, find the cosecant, secant, and cotangent of $\angle E$. Write your answers as fractions and as decimals.

7. A 30-foot-tall street lamp casts a 16.5-foot-long shadow. Use the reciprocal functions and the Pythagorean Theorem, as necessary, to determine the distance from the top of the lamppost to the farthest part of the shadow. What is the angle at which the street lamp casts its shadow?

continued

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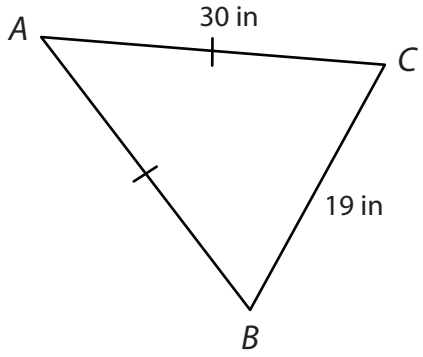
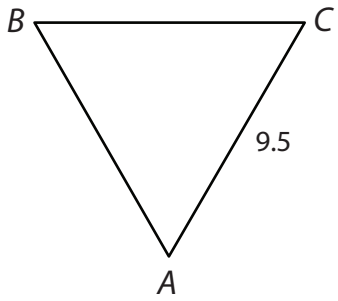
8. A plane is departing from the airport an angle of 33° with the level ground. When it reaches 10,000 feet above the ground, the pilot is no longer flying at an angle and cruises at that altitude. Use a reciprocal function to determine the horizontal distance the airplane is from the airport when it reaches 10,000 feet.

9. The triangle below is isosceles. Calculate the cosecant, secant, and cotangent of $\angle B$.



10. The triangle below is equilateral. Calculate the cosecant, secant, and cotangent of $\angle B$.



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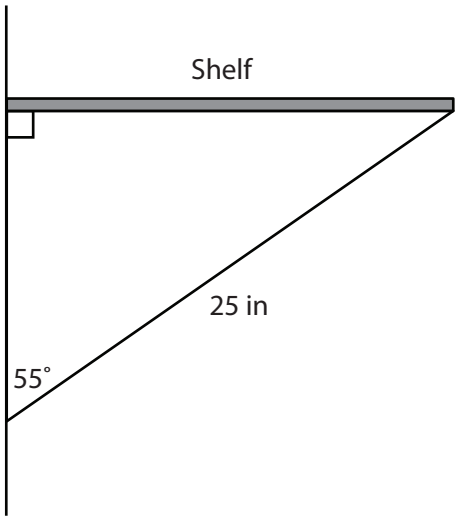
Lesson 5.9.3: Problem Solving with the Pythagorean Theorem and Trigonometry

Warm-Up 5.9.3

Chris is mounting a shelf on a wall to hold his awards. He is attaching the shelf using mounting brackets that are shaped like right triangles. He wants the width of the shelf to fit the length of the bracket perfectly, as shown in the diagram below. If the width of the shelf is 21 inches, will the shelf fit the length of the bracket exactly?



1. Draw the right triangle represented and assign a variable to the length of the bracket.

2. Use trigonometry to calculate the length of the bracket. Round to the nearest tenth of an inch.

3. Will the shelf fit exactly? Why or why not?

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Instruction

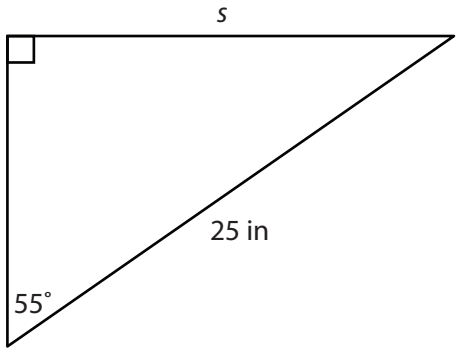
Lesson 5.9.3: Problem Solving with the Pythagorean Theorem and Trigonometry

Common Core State Standard

G–SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.★

Warm-Up 5.9.3 Debrief

1. Draw the right triangle represented and assign a variable to the length of the bracket.



2. Use trigonometry to calculate the length of the bracket. Round to the nearest tenth of an inch.

   s is the side opposite the given angle, and the hypotenuse is given. Use sine to calculate the length of the bracket.

   $$\sin 55^\circ = \frac{s}{25}$$

   $$25 \sin 55^\circ = s$$

   $$s \approx 20.479$$

   The length of the bracket, s, is about 20.5 inches.

3. Will the shelf fit exactly? Why or why not?

   The shelf has a width of 21 inches but the brackets have a length of about 20.5 inches. The shelf will not fit exactly; it will be slightly wider than the brackets.

Connection to the Lesson

- Students will extend their knowledge of using sine, cosine, and tangent to solve right triangle problems using angles of elevation and depression.

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Instruction

Prerequisite Skills

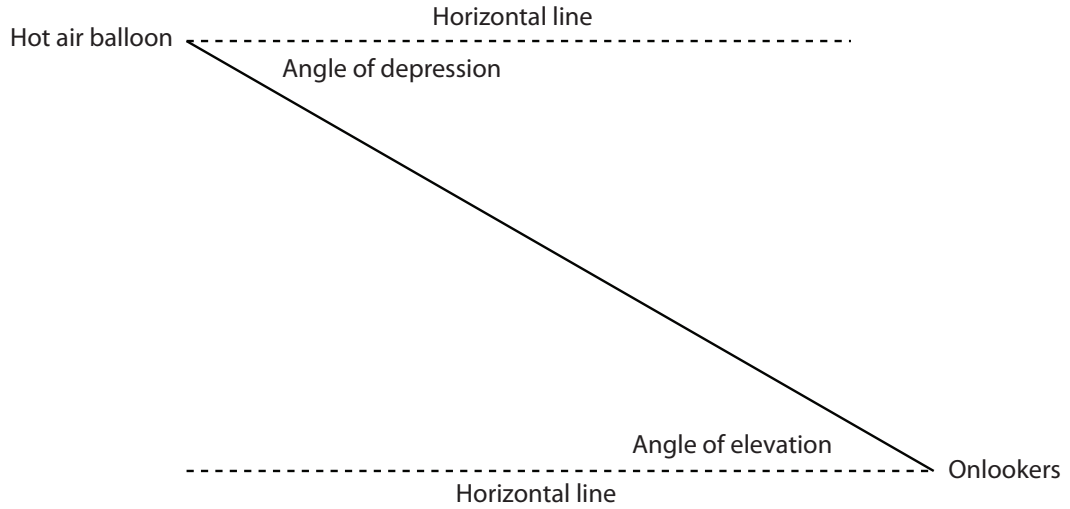
This lesson requires the use of the following skills:

- defining and calculating sine, cosine, and tangent (G–SRT.6)
- setting up and solving problems using the Pythagorean Theorem (8.G.7)
- identifying the appropriate situations in which to use sine, cosine, or tangent based on the given information (G–SRT.7)
- having proper knowledge of how to use a scientific calculator to calculate sine, cosine, and tangent (no standard)

Introduction

Imagine it is a beautiful afternoon in Utah. Your class has raised enough money to take a ride in a hot air balloon that overlooks Antelope Island, Great Salt Lake, the Rocky Mountains, and the Salt Lake City skyline, among other sights. As you fly high in the sky, you observe the geography below, and onlookers below marvel at the beauty of the hot air balloon. The angles of observation from both parties, downward toward the sights and upward toward the balloon, have specific names.

As you look down toward the onlookers, you view the landscape at an **angle of depression**, which is the angle created by a horizontal line and a downward line of sight. The onlookers view the hot air balloon at an **angle of elevation**, which is the angle created by a horizontal line and an upward line of sight. In the following diagram, notice the labeled angles given the horizontals.



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Instruction

These two angles are very important to understand and will be the basis for our study and practice during this lesson. Specifically, you will study and practice calculating angles of depression and elevation and use these angles to calculate distances that are not easily measured by common devices.

It is important to note that the horizontal lines are parallel to each other, and therefore the line of sight behaves as a transversal to the parallel lines. As such, the angle of elevation and angle of depression are alternate interior angles and, therefore, their angle measures are congruent.

Key Concepts

- The angle of depression is the angle created by a horizontal line and a downward line of sight to an object below the observer.
- The angle of elevation is the angle created by a horizontal line and an upward line of sight to an object above the observer.
- Remember:
  - Given the angle θ, $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$.
  - Given the angle θ, $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$.
  - Given the angle θ, $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$.
  - Given right triangle ABC, $a^2 + b^2 = c^2$, where a and b are the legs and c is the hypotenuse.

Common Errors/Misconceptions

- marking an angle of depression or elevation in reference to the vertical instead of the horizontal
- not recognizing the pair of alternate interior angles formed in angles of elevation and depression

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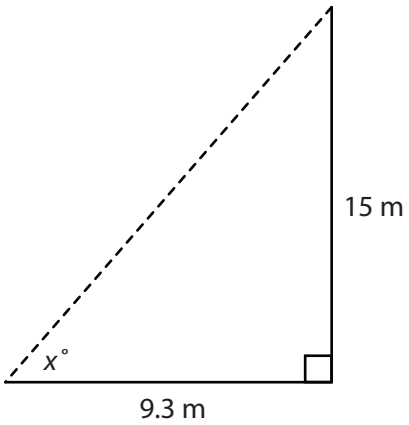
Guided Practice 5.9.3

Example 1

The height of a tree is 15 meters. To the nearest whole degree, what is the angle of elevation of the sun when the tree casts a shadow that is 9.3 meters long on level ground?

1. Make a drawing of the scenario.

   Make sure to correctly identify the angle of elevation as the measure from the ground, above the horizontal.



2. Identify the trigonometric function that you must use in order to find the angle of elevation.

   In this case, the height of the tree represents the side that is opposite the angle of elevation, x. The length of the shadow represents the adjacent side to x. Therefore, use the tangent ratio.

   Use the inverse since you are finding an angle and not a side length.

$$\tan x = \frac{\text{opposite}}{\text{adjacent}}\text{; opposite leg = 15; adjacent leg = 9.3}$$

$$\tan x = \frac{15}{9.3}$$

$$x = \tan^{-1}\left(\frac{15}{9.3}\right)$$

(*continued*)

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On a TI-83/84:

Step 1: Press [2ND][TAN][15][÷][9.3].
Step 2: Press [ENTER].
$x \approx 58.2°$

On a TI-Nspire:

Step 1: In the calculate window from the home screen, press [trig]. Use the keypad to select "$\tan^{-1}$," then enter 15, press [÷], and enter 9.3.
Step 2: Press [enter].
$x \approx 58.2°$

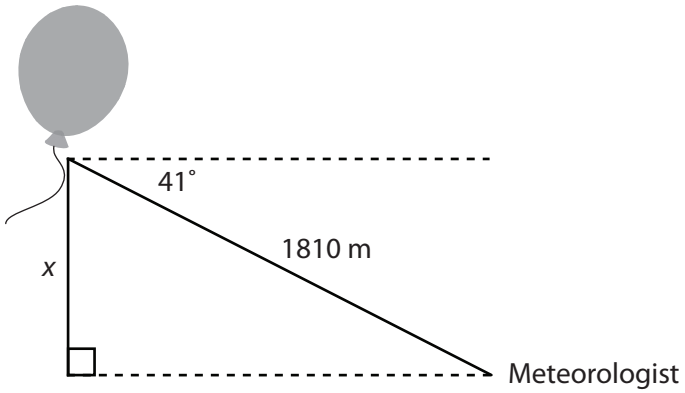
The angle of elevation of the sun is about 58°.

Example 2

A meteorologist reads radio signals to get information from a weather balloon. The last alert indicated that the angle of depression of the weather balloon to the meteorologist was 41° and the balloon was 1,810 meters away from his location on the diagonal. To the nearest meter, how high above the ground was the balloon?

1. Make a drawing of the scenario.

   Remember, the angle of depression is between the diagonal line of sight and the horizontal.



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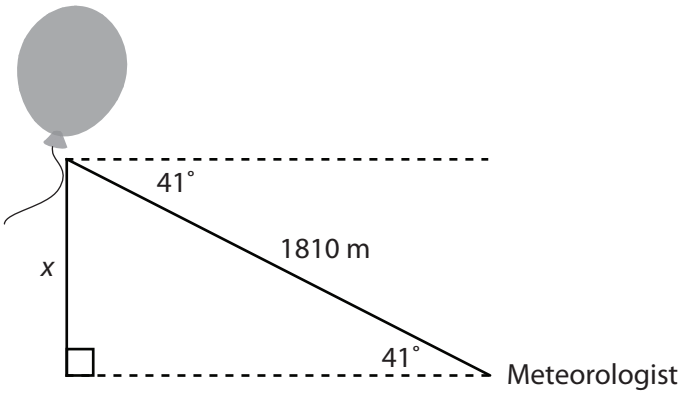
Instruction

2. Identify the correct trigonometric function to use given the angle of depression.

   Recall that the angles of depression and elevation are congruent, so we can use this to determine the trigonometric function.



   Since x is opposite the angle of elevation (which yields the same information as the angle of depression), and the distance from the weather balloon to the meteorologist is represented by the hypotenuse of the right triangle, use sine to find the vertical height.

3. Set up the function and solve for the height.

$$\sin 41^\circ = \frac{x}{1810}$$

$$1810 \sin 41^\circ = x$$

On a TI-83/84:

Step 1: Press [1810][SIN][41].
Step 2: Press [ENTER].
$x \approx 1187.467$ m

On a TI-Nspire:

Step 1: In the calculate window from the home screen, enter 1810 and then press [trig]. Use the keypad to select "sin," then enter 41.
Step 2: Press [enter].
$x \approx 1187.467$ m

The weather balloon was about 1,187 meters above the ground.

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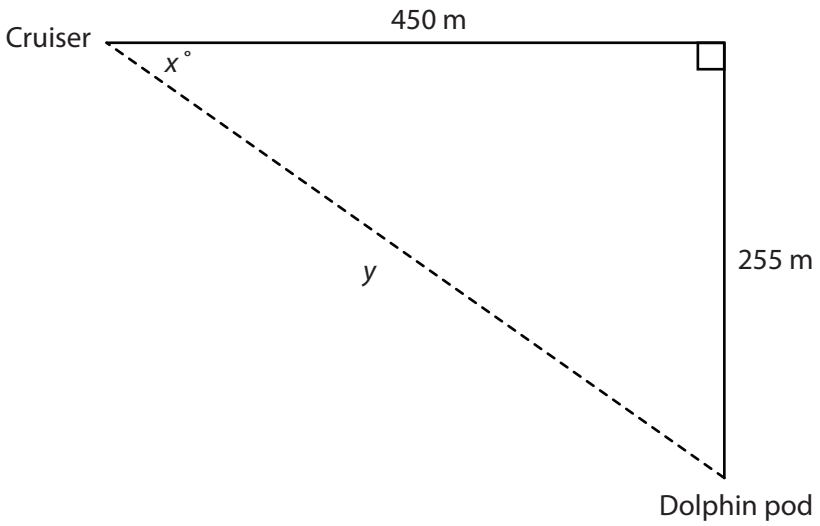
Instruction

Example 3

A sonar operator on an anchored cruiser detects a pod of dolphins feeding at a depth of about 255 meters directly below. If the cruiser travels 450 meters west and the dolphins remain at the same depth to feed, what is the angle of depression, x, from the cruiser to the pod? What is the distance, y, between the cruiser and the pod? Round your answers to the nearest whole number.

1. Make a drawing of the scenario.



2. Find the angle of depression. Identify the given information to determine which trigonometric function to use.

   Since you are calculating an angle, use the inverse of the trigonometric function. Notice that because of the orientation of the triangle and the horizontal side, the angle of depression lies above the diagonal.

   We are given the distances opposite and adjacent to the angle of depression. Therefore, use the tangent function.

$$\tan x = \frac{255}{450}$$

$$x = \tan^{-1}\left(\frac{255}{450}\right)$$

(*continued*)

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On a TI-83/84:

Step 1: Press [2ND][TAN][255][÷][450].
Step 2: Press [ENTER].
$x \approx 29.539$

On a TI-Nspire:

Step 1: In the calculate window from the home screen, press [trig]. Use the keypad to select "$\tan^{-1}$," then enter 255, press [÷], and enter 450.
Step 2: Press [enter].
$x \approx 29.539$

The angle of depression from the cruiser is about 30°.

3. Determine the distance, y.

Since two side lengths were given, to determine the distance between the cruiser and the pod, there is the option of using either a trigonometric ratio or the Pythagorean Theorem. However, since the value of the angle of depression was not given and had to be approximated, using the Pythagorean Theorem because the two given distances will yield a more precise answer.

$450^2 + 255^2 = y^2$

$202{,}500 + 65{,}025 = y^2$

$267{,}525 = y^2$

$\pm\sqrt{267{,}525} = y$

$y \approx 517$

The distance from the cruiser to the dolphin pod after travelling 450 meters west of the vertical is about 517 meters.

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Instruction

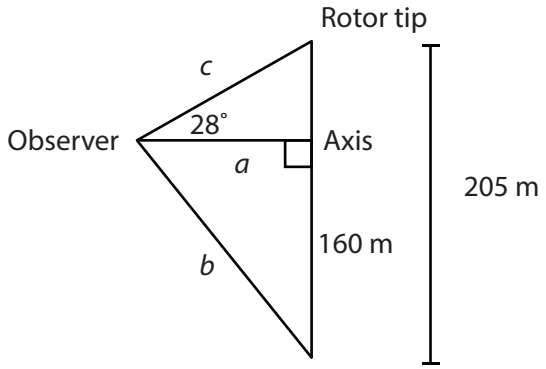
Example 4

You are on a steep hillside directly across from the axis of a wind turbine. The information post on the hill indicates that the wind turbines are 160 meters from the ground to their axis (or center of rotation) and that the rotor tips reach a height of 205 meters when they are in line, or at 180°, with the wind turbine pole. The information post also says that for an observer who is 6 feet tall, the angle of elevation to the rotor tip is 28° and the observer's eyes are level with the axis of the wind turbine. Use this information to answer the following questions.

- If an observer is 6 feet tall, to the nearest meter, what is the distance from the observer's eye level to the wind turbine's axis?
- What is the distance, to the nearest meter, from the observer's eye level to the base of the wind turbine?
- What is the distance, to the nearest meter, from the observer's eye level to the tip of a rotor when it reaches its highest point?

1. Make a drawing of the scenario.

   Label the given information accordingly.



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Instruction

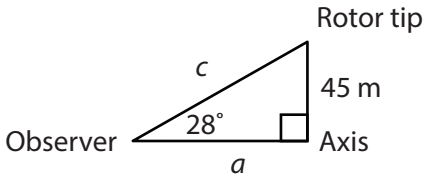
2. Use the given information to solve for the base of the top triangle to determine the distance from the observer's eye level to the wind turbine's axis.

   Subtract 160 meters from 205 meters to get the height from the axis to the rotor tip, which is also a leg of this triangle. Calculate the distance from the axis to the tip of a rotor at its highest point. Knowing this side will help you to find the distance, a, from the observer's eye level to the wind turbine's axis.

   $205 - 160 = 45$

   The vertical distance from the axis to the rotor tip is 45 meters.

   Now use the angle of elevation and the distance from the axis to the rotor tip at its highest point to find a.



   Given that the opposite leg is 45 meters and the adjacent leg is the distance a from the given angle 28°, use the tangent function.

$$\tan 28^\circ = \frac{45}{a}$$

$$a \tan 28^\circ = 45$$

$$a = \frac{45}{\tan 28^\circ}$$

(*continued*)

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On a TI-83/84:

Step 1: Press [45][÷][TAN][28][)].
Step 2: Press [ENTER].
$a \approx 84.633$

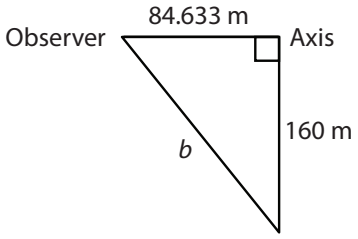
On a TI-Nspire:

Step 1: In the calculate window from the home screen, enter 45, then press [÷][trig]. Use the keypad to select "$\tan^{-1}$," then enter 28.
Step 2: Press [enter].
$a \approx 84.633$

The distance from the observer's eye level to the axis of the wind turbine is about 85 meters.

3. Find the length of the hypotenuse in the bottom triangle, b, to determine the distance from the observer's eye level to the base of the wind turbine.

   Now that we know a, and since a represents the length of the same segment for both triangles, we can use a and the distance from the ground to the axis to find the distance from the observer to the base of the wind turbine. Use the Pythagorean Theorem.



$84.633^2 + 160^2 = b^2$

$7162.692 + 25{,}600 = b^2$

$32{,}762.692 = b^2$

$\pm\sqrt{32762.692} = b$

$b \approx 181.004$

The distance from the observer's eye level to the base of the wind turbine is about 181 meters.

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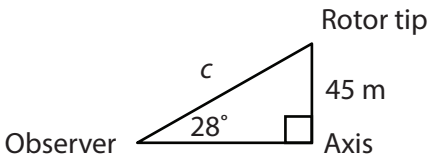
Instruction

4. Solve for the hypotenuse of the top triangle, c, to determine the distance from the observer's eye level to the tallest tip of a rotor.

   Use the top triangle to find c.



   Use sine to find c.

$$\sin 28^\circ = \frac{45}{c}$$

$$c \sin 28^\circ = 45$$

$$c = \frac{45}{\sin 28^\circ}$$

$$c \approx 95.852$$

   The distance from the observer's line of sight to the tallest tip of a rotor is about 96 meters.

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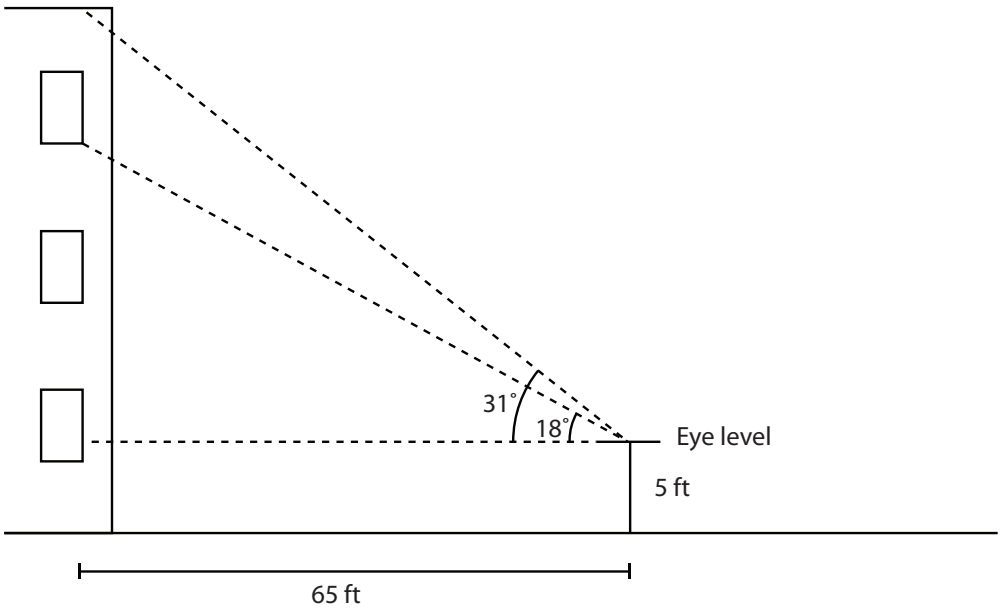
Problem-Based Task 5.9.3: Fighting Flames from a Distance

SMP	
1 ✓	2
3	4 ✓
5 ✓	6
7	8

You are a firefighter on call at a burning building. Your colleagues are on the roof preparing to help put out the blaze by entering the building through the third-floor window. They need you to find the distance from the roof to the windowsill and then determine if the firefighters on the ground are close enough to the building for the water to reach the flames through the window on the third floor.

You observe from below. The angle of elevation to the windowsill is 18° and the angle of elevation to the top of the building is 31°. You are standing 65 feet away from the building and your eyes are 5 feet above the ground, as shown in the diagram. You hold the hose at eye level in order to take aim at the third-floor window.



To the nearest foot, what is the distance from the roof to the windowsill that the firefighters will need to descend by rope to enter the building? If the hoses can spray water at a distance of 100 feet, are the firefighters standing close enough to the building to put the flames out at the third-floor window?

What is the distance from the roof to the windowsill that the firefighters will need to descend by rope to enter the building? Are the firefighters standing close enough to the building?

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Problem-Based Task 5.9.3: Fighting Flames from a Distance

Coaching

a. In order to determine the distance from the windowsill to the roof, what other information do you need to find first?

b. What is the vertical height from the roof to your eye level? Sketch the triangle that provides the information necessary to answer the question.

c. What is the vertical height from the windowsill to your eye level?

d. What is the distance from the roof to the third-floor windowsill?

e. How can you use the information you've already found to determine the distance the water has to travel?

f. What distance does the water have to travel to reach the window?

g. Are the firefighters on the ground close enough to the building?

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Instruction

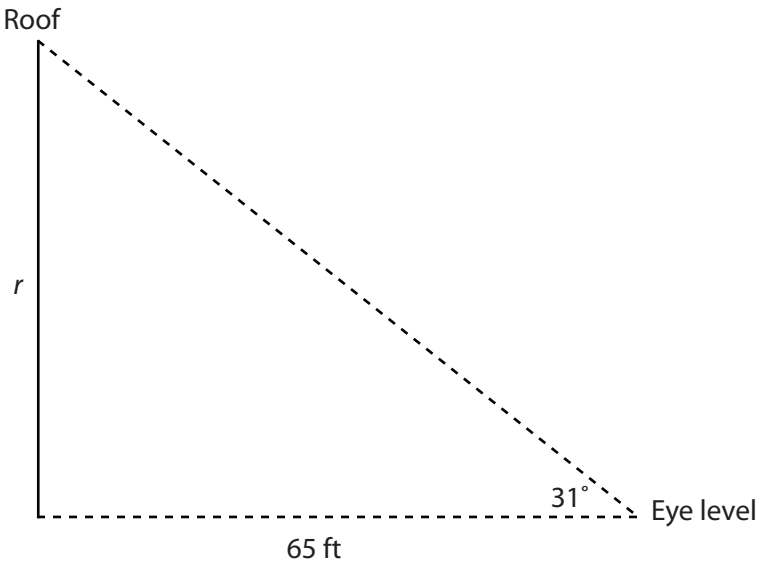
Problem-Based Task 5.9.3: Fighting Flames from a Distance

Coaching Sample Responses

a. In order to determine the distance from the windowsill to the roof, what other information do you need to find first?

Since the vertical height from the roof to the windowsill is not a side of a right triangle, you have to use the two right triangles that are given to find the vertical distance from your eye level to the windowsill and the vertical distance from your eye level to the roof. Subtracting the two values will yield the distance from the roof to the windowsill.

b. What is the vertical height from the roof to your eye level? Sketch the triangle that provides the information necessary to answer the question.



Let r represent the vertical height from the roof to your eye level. Given an angle of elevation of 31°, the leg adjacent to the angle of elevation is 65 feet. To calculate the leg opposite the angle of elevation, the tangent function is required.

$$\tan 31^\circ = \frac{r}{65}$$

$$65 \cdot \tan 31^\circ = r$$

On a TI-83/84:

Step 1: Press [65][TAN][31][)].
Step 2: Press [ENTER].
$r \approx 39.056$

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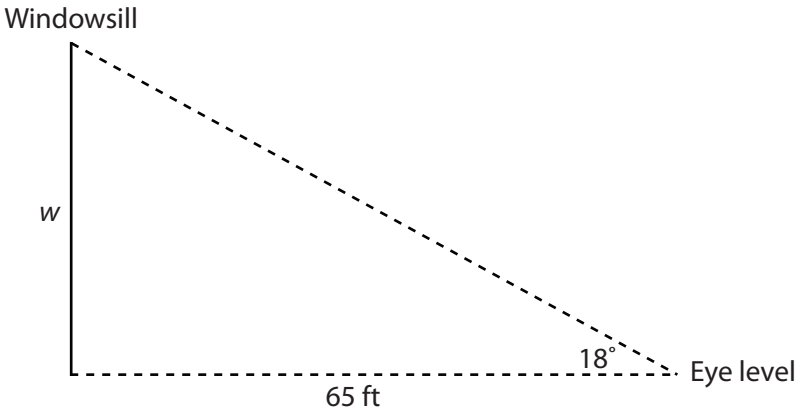
On a TI-Nspire:

Step 1: In the calculate window from the home screen, enter 65, then press [trig]. Use the keypad to select "$\tan^{-1}$," then enter 31.
Step 2: Press [enter].
$r \approx 39.056$

The vertical height from the roof to your eye level is about 39 meters.

c. What is the vertical height from the windowsill to your eye level?



Let w represent the vertical height from the windowsill to your eye level. Given an angle of elevation of 18°, the leg adjacent to the angle of elevation is 65 feet long. To calculate the leg opposite the angle of elevation, the tangent function is required.

$$\tan 18^\circ = \frac{w}{65}$$

$65 \cdot \tan 18^\circ = w$

On a TI-83/84:

Step 1: Press [65][TAN][18][)].
Step 2: Press [ENTER].
$w \approx 21.120$

On a TI-Nspire:

Step 1: In the calculate window from the home screen, enter 65, then press [trig]. Use the keypad to select "$\tan^{-1}$," then enter 18.
Step 2: Press [enter].
$w \approx 21.120$

The vertical height from the windowsill to your eye level is about 21 feet.

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Instruction

d. What is the distance from the roof to the third-floor windowsill?

To find the distance from the roof to the windowsill, subtract 21.120 from 39.056.

$39.056 - 21.120 = 17.936$

The distance is approximately 18 feet.

e. How can you use the information you've already found to determine the distance the water has to travel?

Using the diagram in part c, use the Pythagorean Theorem to find the hypotenuse, or the distance the water will travel from the spout of the hose at eye level to the windowsill.

f. What distance does the water have to travel to reach the window?

$21.120^2 + 65^2 = d^2$

$446.054 + 4225 = d^2$

$4671.054 = d^2$

$\pm\sqrt{4671.054} = d$

$d \approx 68.345$

The distance the water has to travel to reach the window is about 68 feet.

g. Are the firefighters on the ground close enough to the building?

Yes, the firefighters on the ground are close enough to the building because the water sprays 100 feet and the distance the water needs to travel is about 68 feet.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 5.9.3: Problem Solving with the Pythagorean Theorem and Trigonometry

Use the given information to solve each problem. Round your answers to the nearest whole number.

1. A 6-foot-tall man is standing 50 feet from a flagpole. When he looks at the top of the flagpole, the angle of elevation is 39°. Find the height of the flagpole to the nearest foot.

2. A boy flies a kite with a 100-foot-long string. The angle of elevation of the string is 48°. Find the vertical distance between the kite and the point at which the boy is holding the string.

3. A 14-foot ladder is being used to get to the top of a 12-foot-tall wall. At what angle of elevation must the ladder be positioned in order to reach the top of the wall?

4. A mother gazes out a second-floor window at her son playing at the playground. If the mother's eye level is 12.6 meters off of level ground and the playground is 20 meters from the base of the building, what is the angle of depression from the mother's line of sight to the playground?

5. A little girl is watching planes take off of the runway from a building's rooftop 400 meters away from the airport. If the height of the building is 40 meters and the girl snaps a photo of a plane at a 24° angle of elevation, what is the altitude, or vertical height, of the plane when the photo is taken?

continued

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6. From a hot air balloon 2,500 feet above the ground, you see a clearing whose angle of depression is 25°. To the nearest foot, find your horizontal distance from the clearing.

7. A slide at a water park with a constant slope sends riders traveling a distance of 45 feet to the pool at the bottom of the slide. If the depth of the pool is 12 feet and the angle of depression from the top of the slide is 45°, what is the vertical distance from the top of the slide to the bottom of the pool?

8. Tourists marvel at Niagara Falls from two sightseeing boats, *A* and *B*. The boats are 100 feet and 150 feet away from the base of the falls, respectively. Given that Niagara Falls is 167 feet high, what is the angle of elevation from both boats to the top of the falls?

9. A hot air balloon is flying at an altitude of 800 feet. A passenger takes a picture of the top of a tree and estimates that the angle of depression to the top of the tree is about 43°. He estimates the angle of depression to the base of the tree to be 46°. What is the height of the tree?

10. Burj Khalifa in Dubai is the tallest building in the world, standing at 828 meters. An adjacent building, 100 meters away, stands at 550 meters tall. What is the angle of depression from Burj Khalifa to the adjacent building?

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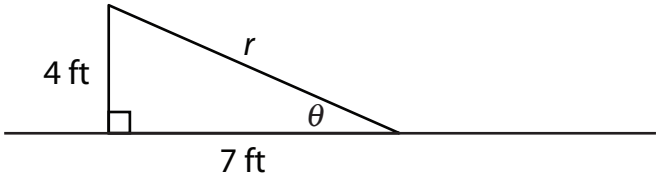
Lesson 5.9.4: Proving the Fundamental Pythagorean Identity

Warm-Up 5.9.4

Rodrigo is practicing for a bicycle motocross tournament. He wants to make sure he challenges himself each week with steeper angles on his practice ramp. However, he's not sure of the ramp's current angle of incline. He knows the wooden ramp is 4 feet high at the highest point, and the base of the ramp is 7 feet long, as shown in the diagram below.



1. How long is the incline of the ramp, labeled r in the diagram?

2. Find $\sin\theta$ and $\cos\theta$.

3. What is the measure of angle θ?

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Instruction

Lesson 5.9.4: Proving the Fundamental Pythagorean Identity

Common Core State Standard

F–TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Warm-Up 5.9.4 Debrief

1. How long is the incline of the ramp, labeled r in the diagram?

   Use the Pythagorean Theorem to find the length of r, the hypotenuse of this right triangle.

   $x^2 + y^2 = r^2$

   $7^2 + 4^2 = r^2$

   $49 + 16 = r^2$

   $65 = r^2$

   $r = \sqrt{65} \approx 8.06$

2. Find $\sin\theta$ and $\cos\theta$.

   Use the trigonometric ratios: $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ and $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$.

   $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ Trigonometric ratio for sine

   $\sin\theta = \dfrac{4}{8.06} \approx 0.5$ Substitute values for the opposite leg (4) and the hypotenuse (8.06).

   $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ Trigonometric ratio for cosine

   $\cos\theta = \dfrac{7}{8.06} \approx 0.87$ Substitute values for the adjacent leg (7) and the hypotenuse (8.06).

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3. What is the measure of angle θ?

   The sine of θ is approximately 0.5. To find the measure of angle θ, find the inverse sine (also called the arcsine) of 0.5.

   $\sin\theta \approx 0.5$

   $\sin^{-1}(0.5) \approx \theta$

   $\theta \approx 30°$

Connection to the Lesson

- Students will learn how to calculate the values of trigonometric functions in a new way, using the Pythagorean identity.
- Students will connect the relationship between the Pythagorean Theorem used here and the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$.

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Prerequisite Skills

This lesson requires the use of the following skills:

- calculating the ratios for sine, cosine, and tangent (G–SRT.8★)
- identifying and applying the reciprocal trigonometric identities (G–SRT.8★)
- finding the values of the trigonometric functions if given an acute angle of a right triangle (G–SRT.6)
- finding the measure of an acute angle of a right triangle if given the side lengths (G–SRT.8★)
- simplifying complex fractions (A–APR.6)

Introduction

Previously in this unit, you learned how to calculate the values of trigonometric functions of an angle θ by using ratios involving the adjacent side, the opposite side, and the hypotenuse of a right triangle. In this lesson, you will discover another way to calculate these values by using what is known as a Pythagorean identity. Such identities are most often used to simplify expressions that contain trigonometric functions.

Key Concepts

- An identity is an equation that is true regardless of what values are chosen for the variables.
- The **Pythagorean Theorem** is a theorem that relates the length of the hypotenuse of a right triangle (c) to the lengths of its legs (a and b). The theorem states that $a^2 + b^2 = c^2$.
- There are three trigonometric identities, known as the **Pythagorean identities,** which are derived from the Pythagorean Theorem. In this lesson, we will take a look at the primary Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$. This identity describes the relationship between sine and cosine, which is true for any angle value for which the function is defined.
- In this lesson, we will only consider θ values between 0° and 90°.
- Note that $\sin^2\theta$ is the same as $(\sin\theta)^2$. It is customary to write the exponent after the name of the trigonometric function to avoid confusing it with $\sin\theta^2$. To enter this into your calculator, enter $(\sin(\theta))^2$, as shown:

On a TI-83/84:

Step 1: Open a set of parentheses by typing [(].

Step 2: Press the [SIN] button followed by [X, T, θ, n].

Step 3: Close both sets of parentheses by pressing [)] twice.

Step 4: Press the [x^2] button to square the expression.

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On a TI-Nspire:

Step 1: In the calculate window, open a set of parentheses by pressing [(].

Step 2: Press [trig] to bring up the menu of trigonometric functions.

Step 3: Use the keypad to select "sin." Press [X].

Step 4: Close both sets of parentheses by pressing [)] twice.

Step 5: Press [x^2] to square the expression.

- Algebraic manipulations of the terms in this identity produce equivalent identities. For example, the identity could be written $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$.
- A Pythagorean identity can be used to help simplify trigonometric expressions and equations. For example, if an equation includes the expression $\sin^2\theta + \cos^2\theta$, this could be replaced with 1, and if an equation includes the expression $\sin^2\theta - 1$, this can be replaced with $-\cos^2\theta$.
- The Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ can be used to find the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$. If you know one of these values, you can find the other two.
- You can solve this Pythagorean identity for one of the trigonometric functions. For example:

  $\sin^2\theta + \cos^2\theta = 1$

  $\sin^2\theta = 1 - \cos^2\theta$

  $\sin\theta = \sqrt{1-\cos^2\theta}$

- Note that this is similar to the distance formula, which is also derived from the Pythagorean Theorem.
- **Reciprocal identities** are trigonometric identities that define cosecant, secant, and cotangent in terms of sine, cosine, and tangent.
- In the previous lesson, you learned the following reciprocal identities:

$$\csc\theta = \frac{1}{\sin\theta} \quad \sec\theta = \frac{1}{\cos\theta} \quad \cot\theta = \frac{1}{\tan\theta}$$

- There are two more identities called **ratio identities** that are useful in simplifying trigonometric expressions:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{and} \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

- When solving an application problem, it is helpful to draw a diagram first to see the relationship among the sides and the given angle.

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Common Errors/Misconceptions

- confusing which trigonometric functions are used in which of the Pythagorean identities
- entering $\sin^2\theta$ incorrectly into the calculator as $\sin(\theta)^2$ instead of $(\sin\theta)^2$
- incorrectly substituting Pythagorean identities when simplifying expressions
- incorrectly labeling the sides and/or angles in a right triangle diagram

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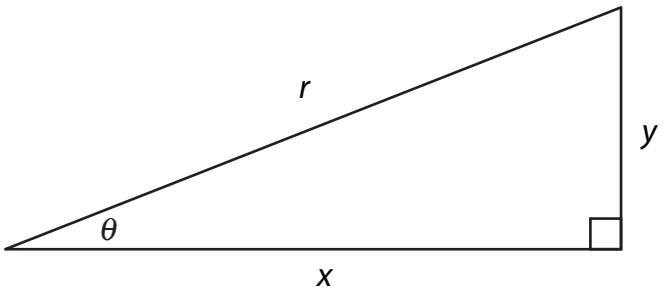
Guided Practice 5.9.4

Example 1

Prove the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$.

1. Draw a labeled diagram of a right triangle.



2. Apply the Pythagorean Theorem to this triangle.

   The Pythagorean Theorem is $a^2 + b^2 = c^2$.

   Replace the variables in the theorem with the values from the diagram.

   $x^2 + y^2 = r^2$

3. Divide both sides of this equation by r^2.

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

4. Rewrite this equation using the laws of exponents.

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

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5. Find $\sin\theta$ and $\cos\theta$ by looking at the diagram.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}$$

6. Substitute $\sin\theta$ and $\cos\theta$ into the equation from step 4.

$(\cos\theta)^2 + (\sin\theta)^2 = 1$

7. Rewrite the equation using the standard notation for trigonometric functions with exponents.

$\cos^2\theta + \sin^2\theta = 1$

$\sin^2\theta + \cos^2\theta = 1$

Example 2

If $\sin\theta = \frac{2}{5}$ and $0 < \theta < 90$, use the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ to find the values of $\cos\theta$ and $\tan\theta$.

1. Substitute the given value of sine into the identity.

$\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{2}{5}\right)^2 + \cos^2\theta = 1$

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2. Solve for $\cos\theta$.

$\left(\frac{2}{5}\right)^2 + \cos^2\theta = 1$	Equation with substituted value of $\sin\theta$
$\frac{4}{25} + \cos^2\theta = 1$	Square the term.
$\cos^2\theta = 1 - \frac{4}{25}$	Isolate $\cos^2\theta$.
$\cos^2\theta = \frac{25}{25} - \frac{4}{25}$	Find the common denominator.
$\cos^2\theta = \frac{21}{25}$	Simplify.
$\cos\theta = \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{5} \approx 0.92$	Take the square root of both sides of the equation.

The cosine of θ is approximately 0.92.

3. To find $\tan\theta$, use the ratio identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

$\tan\theta = \frac{\sin\theta}{\cos\theta}$	Tangent ratio identity
$\tan\theta = \frac{\frac{2}{5}}{\frac{\sqrt{21}}{5}}$	Substitute the values for $\sin\theta$ and $\cos\theta$.
$\tan\theta \approx 0.44$	Simplify.

The value of $\tan\theta$ is approximately 0.44.

4. Summarize your findings.

$\cos\theta \approx 0.92$ and $\tan\theta \approx 0.44$

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Example 3

Simplify the expression $\sin^2\theta - \sin^2\theta\cos^2\theta$.

1. Factor the expression.

   Both terms have a common factor of $\sin^2\theta$.

   Factor out $\sin^2\theta$.

   $\sin^2\theta - \sin^2\theta\cos^2\theta$

   $\sin^2\theta\,(1 - \cos^2\theta)$

2. Analyze the expression for any substitutions that can be made using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$.

   Notice that the identity can be rewritten to solve for $\sin^2\theta$ by subtracting $\cos^2\theta$ from both sides.

   $\sin^2\theta + \cos^2\theta = 1$

   $\sin^2\theta = 1 - \cos^2\theta$

   Also, notice that when the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ is rearranged and solved for $\sin^2\theta$, the right side of the equation is equal to the expression inside the parentheses of the factored expression in step 1. Using substitution, the expression can be simplified to write the expression in terms of sine only.

3. Substitute the rearranged Pythagorean identity into the factored expression.

   Since we have determined $\sin^2\theta = 1 - \cos^2\theta$, we can substitute $\sin^2\theta$ into the factored expression, $\sin^2\theta\,(1 - \cos^2\theta)$.

$\sin^2\theta\,(1 - \cos^2\theta)$	Factored expression from step 1
$\sin^2\theta\,(\sin^2\theta)$	Substitute $\sin^2\theta$ for $1 - \cos^2\theta$.
$\sin^4\theta$	Simplify.

   The expression $\sin^2\theta - \sin^2\theta\cos^2\theta$ can be simplified to $\sin^4\theta$ using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$.

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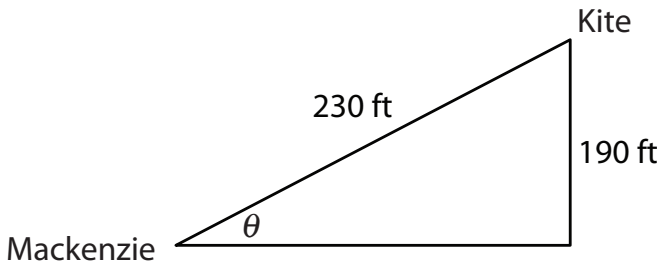
Example 4

Mackenzie is flying a kite. She is standing on the handle of the kite string for a minute while she ties her shoelace. The angle that the kite string makes with the ground is θ. If the length of the kite string is 230 feet when the kite is 190 feet off the ground, find the cosine of θ.

1. Draw a diagram of this situation.



2. Determine which trigonometric ratio to use based on the information in the diagram.

   Because the two sides given in the diagram are the side opposite θ and the hypotenuse, the correct trigonometric ratio to use in order to determine $\cos\theta$ is sine.

   Substitute the values from the diagram into the ratio for sine.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{190}{230} = \frac{19}{23}$$

3. Use the value for $\sin\theta$ to find $\cos\theta$.

   Substitute the value from step 2 for sine into the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ and solve for $\cos\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\left(\frac{19}{23}\right)^2 + \cos^2\theta = 1$$

$$0.682 + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - 0.682$$

$$\cos^2\theta = 0.318$$

$$\cos\theta \approx 0.564$$

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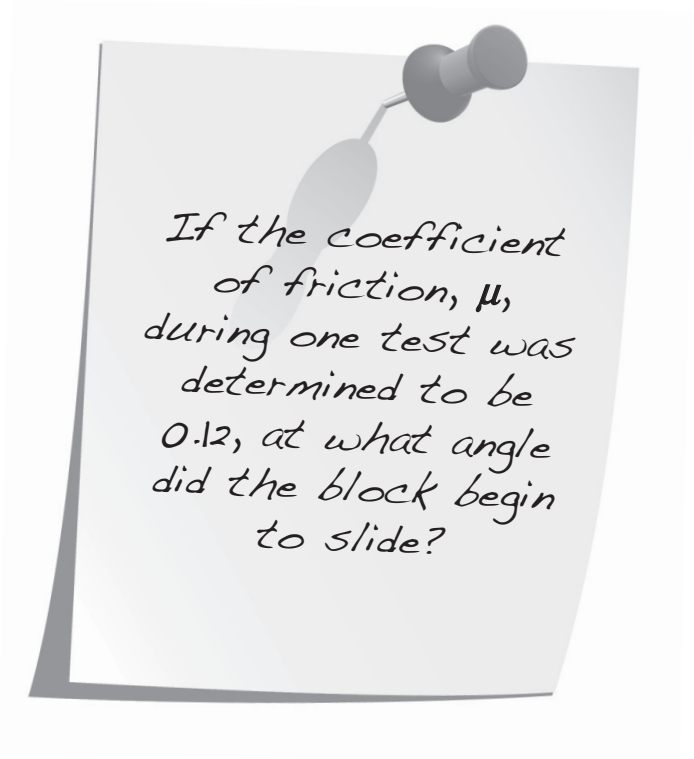
Problem-Based Task 5.9.4: Slick Surfboards

SMP	
1 ✓	2 ✓
3 ✓	4 ✓
5	6 ✓
7 ✓	8

Sloane makes custom surfboards. He applies surfboard wax to each board to reduce the amount of friction created when the board is in the water, so that the board slides over the water more quickly. Sloane is testing several different surfboard waxes to determine which is best. He tests each wax by first applying it to a surfboard that's lying horizontally. Then, he places a block on the board and slowly lifts one end of the board up at an angle. Sloane's assistant records the angle at which the block begins to slide. The sooner the block slides, the better the wax.

The equation that relates the force of the block to the resistive force of friction is $mg\sin\theta = \mu mg\cos\theta$, where θ = the angle of incline, m = the mass of the block, g = the block's acceleration due to gravity, and μ = the coefficient of friction. The coefficient of friction is a constant that represents the force the object needs to overcome in order to move. If the coefficient of friction, μ, during one test was determined to be 0.12, at what angle did the block begin to slide?



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Problem-Based Task 5.9.4: Slick Surfboards

Coaching

a. What is the equation that relates the force of the block to the resistive force of friction?

b. What value(s) are given and can be substituted into the equation?

c. How can you rearrange the equation to isolate the unknowns?

d. What trigonometric identity can you use to simplify the equation?

e. How can you solve for the angle of incline? What is the angle of incline?

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Problem-Based Task 5.9.4: Slick Surfboards

Coaching Sample Responses

a. What is the equation that relates the force of the block to the resistive force of friction?

$mg\sin\theta = \mu mg\cos\theta$

b. What value(s) are given and can be substituted into the equation?

We are given that the coefficient of friction is 0.12. Therefore, $\mu = 0.12$. No other values are given.

c. How can you rearrange the equation to isolate the unknowns?

First substitute in the known quantity, $\mu = 0.12$. Then, rearrange to solve for θ.

$mg\sin\theta = \mu mg\cos\theta$

$mg\sin\theta = 0.12mg\cos\theta$

$\frac{\cancel{mg}\sin\theta}{\cancel{mg}} = \frac{0.12\cancel{mg}\cos\theta}{\cancel{mg}}$

$\sin\theta = 0.12\cos\theta$

$\frac{\sin\theta}{\cos\theta} = \frac{0.12\cancel{\cos\theta}}{\cancel{\cos\theta}}$

$\frac{\sin\theta}{\cos\theta} = 0.12$

d. What trigonometric identity can you use to simplify the equation?

The tangent ratio identity can be used to replace $\frac{\sin\theta}{\cos\theta}$.

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

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e. How can you solve for the angle of incline? What is the angle of incline?

After substituting, you are left with the following:

$\tan\theta = 0.12$

The ratio is known but the angle is unknown. Use the inverse tangent to find the angle of incline.

$\theta = \tan^{-1}(0.12)$

$\theta \approx 6.8$

The angle of incline is about 7°.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 5.9.4: Proving the Fundamental Pythagorean Identity

Find the value of each given trigonometric function using the Pythagorean identity.

1. Find $\sin\theta$ if $0° < \theta < 90°$ and $\cos\theta = \frac{1}{3}$.

2. Find $\cos\theta$ if $0° < \theta < 90°$ and $\sin\theta = \frac{\sqrt{3}}{2}$.

3. Find $\sin\theta$ if $0° < \theta < 90°$ and $\cos\theta = \frac{12}{13}$.

For problems 4 and 5, one trigonometric function value is given, and $0° < \theta < 90°$. Find the values of the remaining 5 trigonometric functions using identities, then find the value of θ.

4. $\sin\theta = \frac{\sqrt{2}}{2}$

5. $\cos\theta = \frac{3}{5}$

Use identities to simplify the following expressions.

6. $\cos\theta\sin^2\theta - \cos\theta$

7. $\frac{1-\cos^2\theta}{1-\sin^2\theta}$

continued

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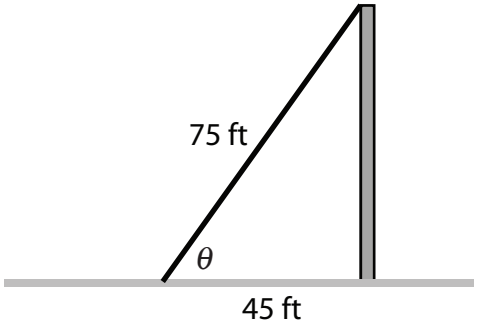
Use the given information and identities to solve the following problems.

8. A 75-foot-long wire extends from the top of a telephone pole to a stake in the ground such that it meets the ground at an angle of θ. The stake is 45 feet from the base of the pole. Find $\sin\theta$ using the Pythagorean identity.



9. A 12-foot ladder is leaning against a house such that it touches the house 8 feet above the ground. The angle that the ladder makes with the ground is θ. Use a Pythagorean identity to find $\cos\theta$.

10. How can you prove or disprove the following identity by manipulating the left side of the equation only?

    $2(\sin^2\theta + \cos^2\theta) + 3 = 5$

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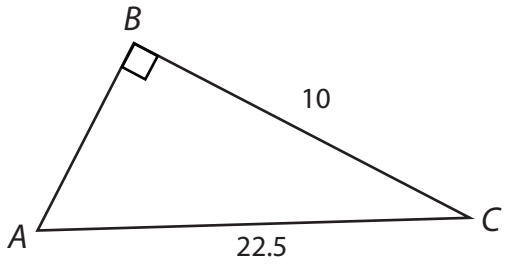
Progress Assessment

Circle the letter of the best answer.

1. Solve the right triangle below.



   a. $\overline{AB} = 20.2$, $m\angle A = 26°$, and $m\angle C = 64°$

   b. $\overline{AB} = 24.6$, $m\angle A = 26°$, and $m\angle C = 64°$

   c. $\overline{AB} = 20.2$, $m\angle A = 64°$, and $m\angle C = 26°$

   d. $\overline{AB} = 24.6$, $m\angle A = 64°$, and $m\angle C = 26°$

2. You see a hiker sitting on a bench taking a water break at the top of a hill at a 60° angle of elevation. Your eye level is 5 feet off the ground and you are standing 100 feet from the base of the hill. At what altitude is the hiker sitting on the bench?

   a. 173 ft

   b. 178 ft

   c. 58 ft

   d. 63 ft

3. A right triangle has one leg that measures 12 feet and another that measures 14 feet. Use sine, cosine, and/or tangent to find the measures of its acute angles.

   a. 36° and 54°

   b. 27° and 63°

   c. 41° and 49°

   d. 31° and 59°

continued

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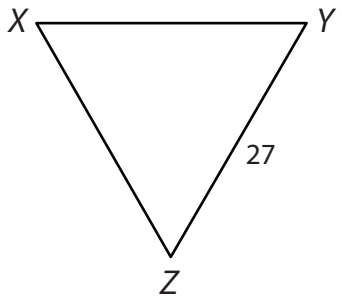
4. The triangle below is equilateral. Calculate the cosecant, secant, and cotangent of $\angle Z$.



   a. csc $Z = 0.5$, sec $Z = 0.9$, and cot $Z = 0.6$

   b. csc $Z = 0.6$, sec $Z = 0.9$, and cot $Z = 0.5$

   c. csc $Z = 1.2$, sec $Z = 0.6$, and cot $Z = 1.2$

   d. csc $Z = 1.7$, sec $Z = 2.0$, and cot $Z = 1.7$

5. The height of a building is 75 meters. To the nearest degree, what is the angle of elevation of the sun when the building casts a shadow that is 60 meters long?

   a. It is not possible to determine the answer with the information given.

   b. 23°

   c. 39°

   d. 51°

6. A naval ship is stationed in calm waters. The sonar detects a submarine at a depth of about 400 meters and a horizontal distance of 600 meters. What is the distance between the ship and the submarine? What is the angle of elevation of the submarine?

   a. 447 m and 42°

   b. 721 m and 34°

   c. 500 m and 51°

   d. 387 m and 48°

continued

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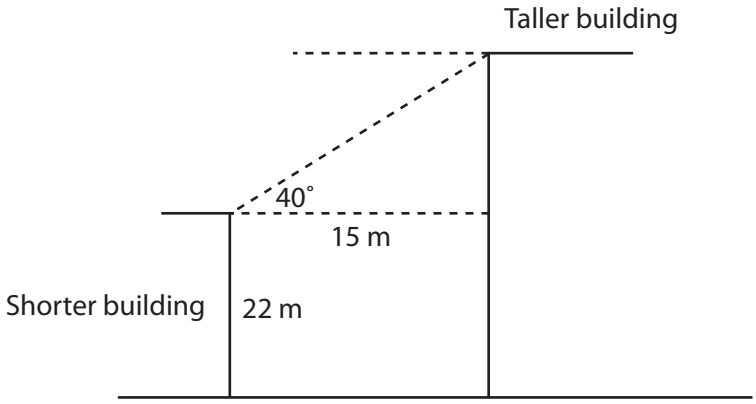
7. Two buildings are 15 meters apart. The height of the shorter building is 22 meters. The angle of elevation from the shorter building to the taller building is 40°. What is the height of the taller building?



a. 13 m
c. 32 m
b. 35 m
d. 25 m

8. If $0° < \theta < 90°$ and $\cos\theta = \frac{\sqrt{2}}{5}$, find $\sin\theta$ using a Pythagorean identity.

a. $\frac{23}{25}$
c. $\frac{\sqrt{23}}{5}$
b. $\frac{\sqrt{23}}{25}$
d. $\frac{\sqrt{2}}{5}$

9. Simplify the expression $\frac{1-\sin^2\theta}{\cos\theta}$.

a. $\frac{1}{\cos\theta}$
c. $\sin\theta$
b. $\cos^3\theta$
d. $\cos\theta$

10. All of the following are variations of the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ except:

a. $1 - \cos^2\theta = \sin^2\theta$
c. $\cos^2\theta = 1 - \sin^2\theta$
b. $\sin^2\theta + \cos^2\theta + 1 = 0$
d. $\sin^2\theta - 1 = -\cos^2\theta$

continued

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Assessment

Use the given information to answer the question.

11. You are standing 20 feet away from the base of a tree when you discover there are two kittens stuck at different levels on the tree's branches. You approximate that the angle of elevation to the first kitten is 36° and the angle of elevation to the second kitten is 40°. Your eye level is 6 feet off of the ground. To the nearest foot, what is the distance between both kittens? Justify your answer.

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Unit 5 Assessment

Circle the letter of the best answer.

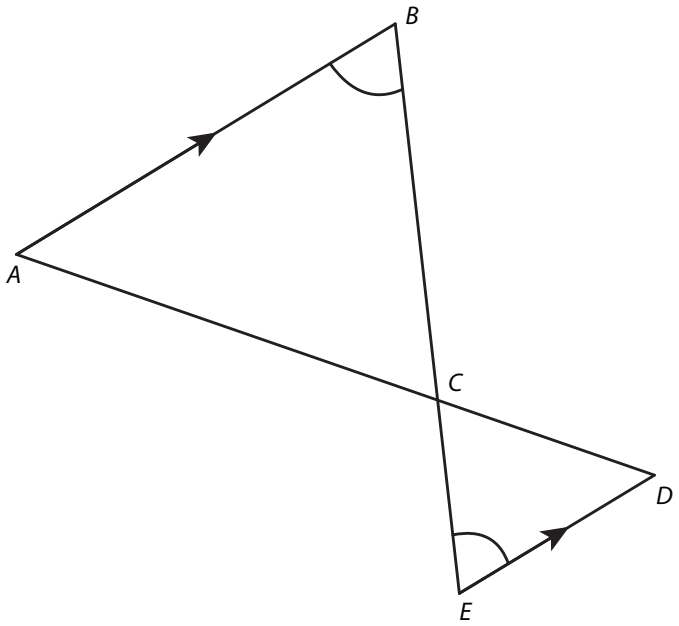
1. A line segment has endpoints at (–4, 5) and (4, –11). A point on the segment has a distance that is $\frac{1}{4}$ of the length of the segment from endpoint (–4, 5). What are the coordinates of the point?

   a. (2, 7)
   b. (–4, –8)
   c. (–2, 1)
   d. (0, 3)

2. If $\triangle DEF$ has vertices with the coordinates D (9, –3), E (–6, 3), and F (3, 6), what are the vertices of $\triangle D'E'F'$ under a dilation with a scale factor of $\frac{2}{3}$ and the center at the origin, (0, 0)?

   a. D' (13.5, –4.5), E' (–9, 4.5), F' (4.5, 9)
   b. D' (6, –2), E' (–4, 2), F' (2, 4)
   c. D' (–2, 6), E' (2, –4), F' (4, 2)
   d. D' (–4.5, 13.5), E' (4.5, –9), F' (9, 4.5)

3. Are the two triangles similar? Why or why not?



   a. Yes, they are similar because of the AA Similarity Statement.
   b. Yes, they are similar because of the ASA Congruence Statement.
   c. No, they are not similar because congruence is not preserved.
   d. It cannot be determined if the triangles are similar.

continued

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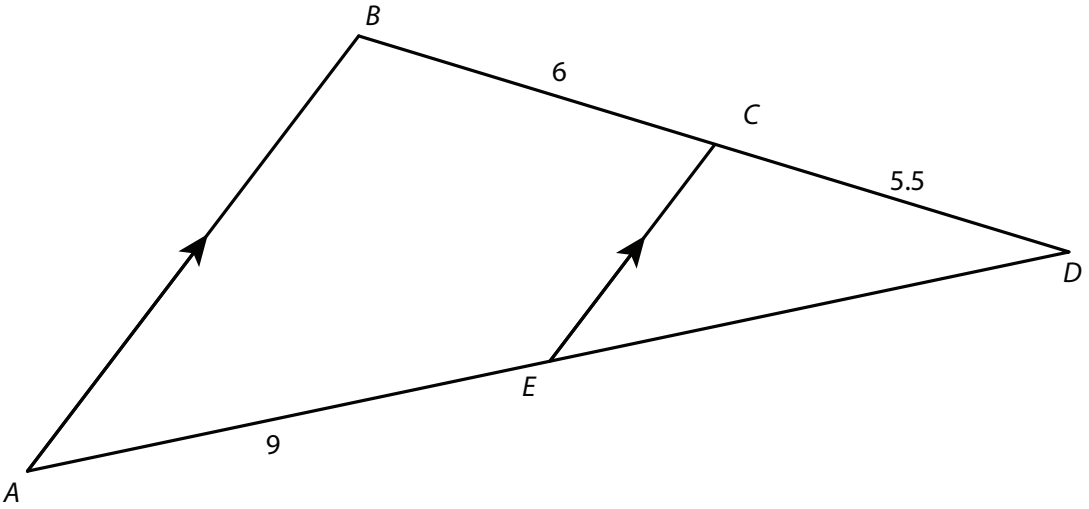
4. $\triangle ABD \sim \triangle ECD$. What is the length of $\overline{DE}$?



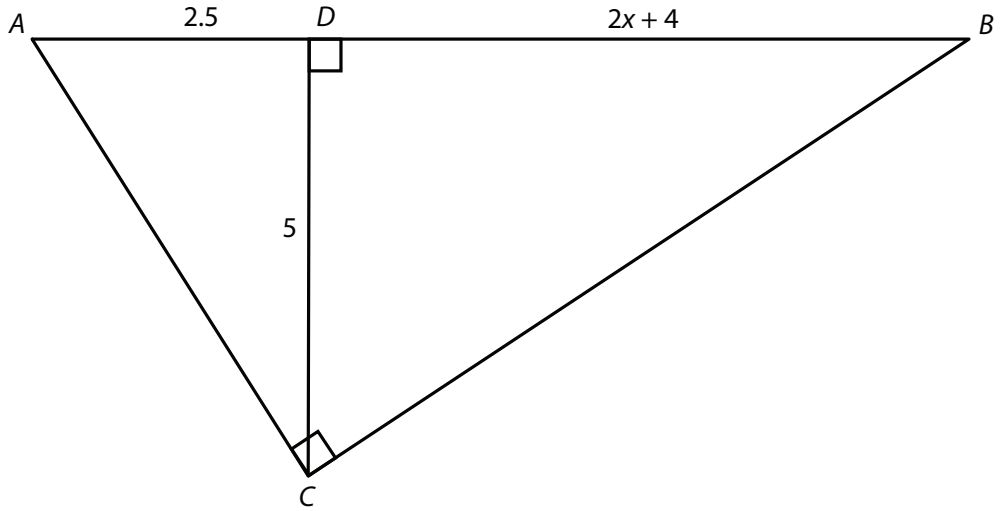
a. 8.25 units

b. 8.5 units

c. 9.8 units

d. There is not enough information to determine the length of $\overline{DE}$.

5. $\triangle ABC$ is a right triangle. Find the length of $\overline{BD}$.



a. 3 units

b. 7.5 units

c. 10 units

d. There is not enough information to determine the length of $\overline{BD}$.

continued

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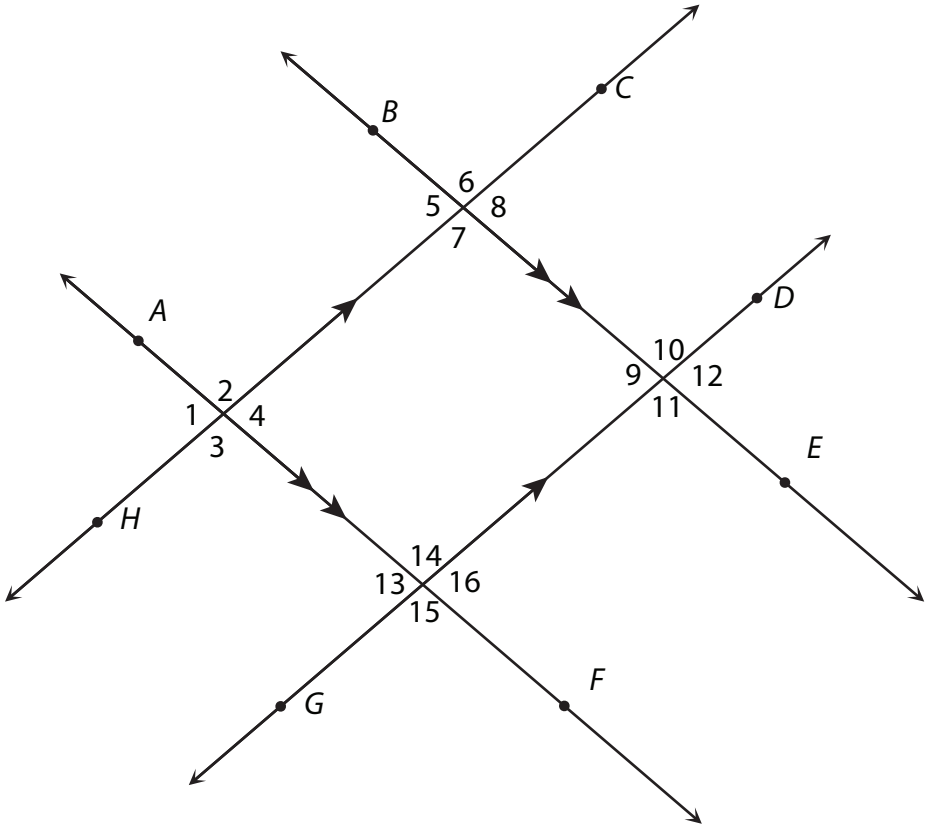
6. Given two sets of parallel lines in the diagram below, what is the relationship between $\angle 4$ and $\angle 10$?



a. No relationship exists.

b. $\angle 4 \cong \angle 10$ because $\angle 4 \cong \angle 13$ by the Alternate Interior Angles Theorem. Then $\angle 13 \cong \angle 10$ by the Alternate Exterior Angles Theorem. Therefore, by the Transitive Property, $\angle 4 \cong \angle 10$.

c. $\angle 4$ and $\angle 10$ are congruent right angles because all right angles are congruent and two pairs of intersecting parallel lines are always perpendicular.

d. $\angle 4$ is supplementary to $\angle 10$ because $\angle 4 \cong \angle 13$ by the Alternate Interior Angles Theorem. Then $\angle 13$ is supplementary to $\angle 10$ by the Alternate Exterior Angles Theorem. This means that $m\angle 13 + m\angle 10 = 180$ and, by substitution, $m\angle 4 + m\angle 10 = 180$. Therefore, by the definition of supplementary angles, $\angle 4$ is supplementary to $\angle 10$.

continued

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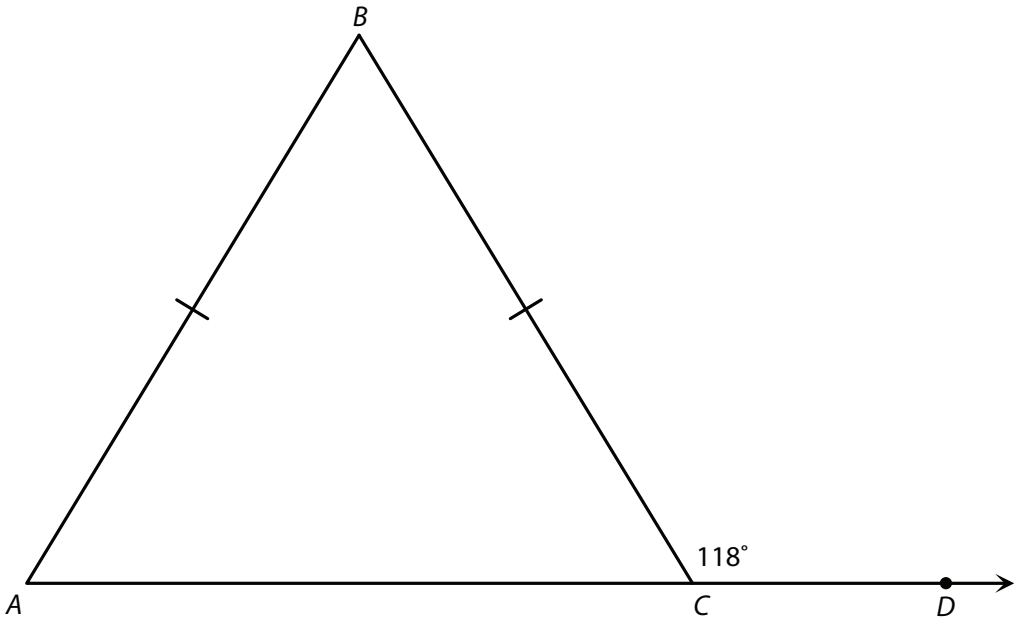
7. What is the measure of $\angle A$?



a. 31°

b. 62°

c. 56°

d. There is not enough information to determine the measure of $\angle A$.

continued

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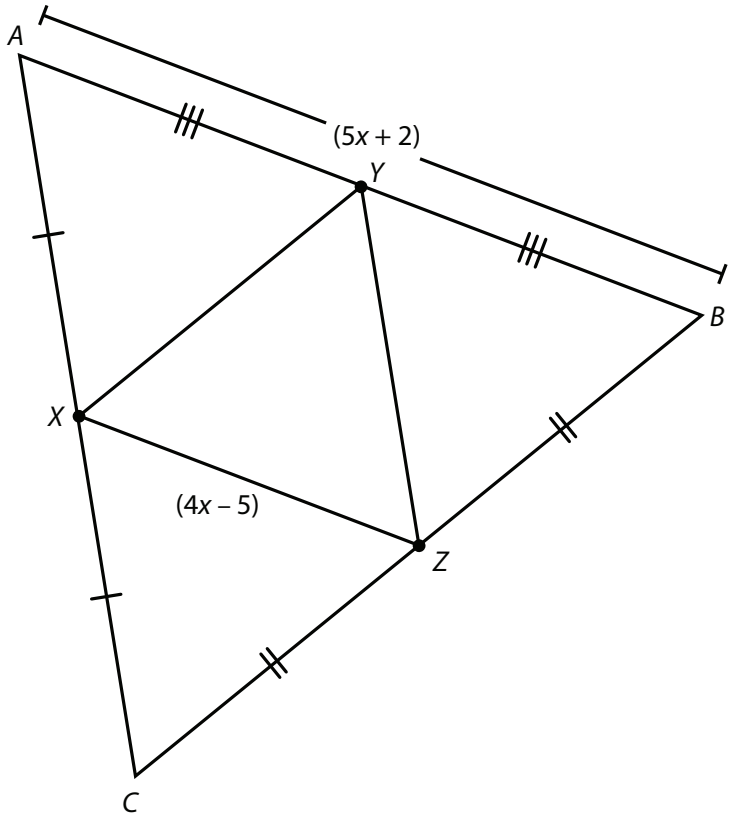
8. If $AB = 5x + 2$ and $XZ = 4x - 5$, what is the actual length of AB?



a. 1.5 units
b. 4 units
c. 11 units
d. 22 units

9. Classify a quadrilateral as precisely as possible given four vertices: C (–4, –3), D (5, 0), E (–6, 3), and F (3, 6).

a. rhombus
b. square
c. rectangle
d. kite

continued

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10. $\triangle ABC$ is a right triangle. One of the acute angles measures 24°. What is the cosine of the other acute angle?

    a. 0.914
    b. 2.246
    c. 0.445
    d. 0.407

11. A monument is 75 meters high. At an information booth, an observer notices the angle of elevation to the top of the monument is 36°. How far is the observer from the base of the building?

    a. about 128 m
    b. about 103 m
    c. about 93 m
    d. about 55 m

12. If $0° < \theta < 90°$ and $\sin\theta = \frac{2}{9}$, find $\cos\theta$ using a Pythagorean identity.

    a. $\frac{77}{81}$
    b. $\frac{4}{81}$
    c. $\frac{\sqrt{77}}{9}$
    d. $\frac{\sqrt{85}}{9}$

Use the information given in each problem to complete problems 13 and 14.

13. A bluebird on a 30-meter-tall flagpole spots a worm on the ground below at an angle of depression of 25°.

    a. What equation can be written to determine the distance of the worm from the base of the flagpole?

    b. How far away from the base of the flagpole is the worm?

continued

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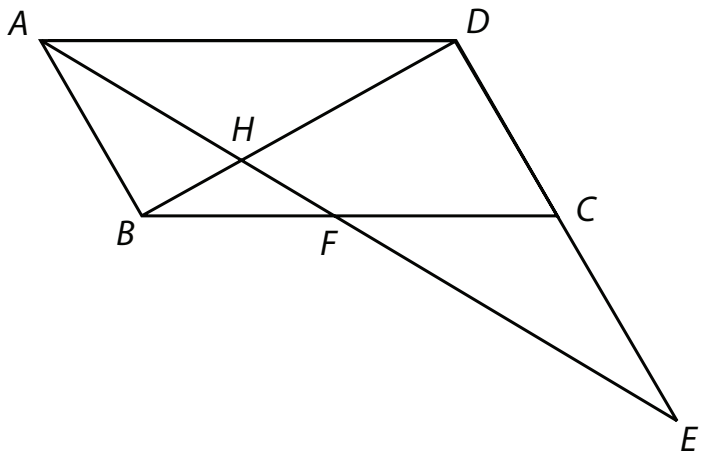
14. Given: *ABCD* is a parallelogram.

    Prove: $\frac{BH}{HA} = \frac{DH}{HE}$



Use Pythagorean identities, reciprocal identities, and the Pythagorean Theorem to complete the problem that follows.

15. A recent storm downed a tree in your backyard. The top of the tree is now leaning on the roof of your shed. The shed's roof is 8 feet tall and the tree is 24 feet long. The angle that the tree makes with the ground is θ.

    a. Find all of the trigonometric functions of θ, using Pythagorean and reciprocal identities.

    b. Find the measure of angle θ.

    c. How far is the base of the tree from the base of the shed?

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Answer Key

Lesson 1: Line Segments

Pre-Assessment, p. U5-1

1. d
2. b
3. a
4. b
5. c

Warm-Up 5.1.1, p. U5-3

1. The distance between the cars is increasing by 78.1 miles each hour.
2. The eastbound car is traveling at a speed of 30 mph.

Practice 5.1.1: Midpoints and Other Points on Line Segments, p. U5-24

1. (–1, –14)
2. (24, –2.5)
3. (1, 10)
4. (17, 5)
5. (–8, 16)
6. (13, 12); 13th Avenue and 12th Street
7. (7, 5); 7th Avenue and 5th Street
8. (4, 10); 4th Avenue and 10th Street
9. (3, 13); 3rd Avenue and 13th Street
10. (7, 11); 7th Avenue and 11th Street

Progress Assessment, p. U5-26

1. b
2. c
3. d
4. a
5. a
6. b
7. c
8. d
9. a
10. b
11. The midpoint is (2, 12). The distance from Johann to the midpoint is $3\sqrt{5}$ units. The distance from Tia to the midpoint is also $3\sqrt{5}$ units.

Lesson 2: Investigating Properties of Dilations

Pre-Assessment, p. U5-28

1. d
2. c
3. b
4. a
5. b

Warm-Up 5.2.1, p. U5-34

1. Reflect the garden over the line $y = x$: $A'(3, -7)$, $B'(7, -3)$, $C'(3, -3)$, or rotate the garden about the origin 180° counterclockwise: $A'(7, -3)$, $B'(3, -7)$, $C'(3, -3)$.
2. Reflections and rotations are rigid motions. They preserve angle and length measures. The result of either transformation is a congruent figure.

Practice 5.2.1: Investigating Properties of Parallelism and the Center, p. U5-51

1. The pentagon has been dilated. The corresponding sides of the pentagon are parallel and the scale factor is consistent ($k = 2$). Additionally, the preimage points and image points are collinear with the center of dilation.
2. The triangle has been dilated. The corresponding sides of the triangle are parallel and the scale factor is consistent ($k = 1/3$). Additionally, the preimage points and image points are collinear with the center of dilation.
3. The triangle has not been dilated. The scale factor is inconsistent between corresponding sides.
4. The rectangle has not been dilated. The scale factor is inconsistent between corresponding sides.
5. $k = 2.5$; enlargement
6. $k = 1.25$; enlargement
7. $k = 0.4$; reduction
8. $k = 0.5$; reduction
9. No, it is not a dilation. The scale factors are inconsistent. The scale factor of the vertical sides is 1.5. The scale factor of the horizontal sides is 4/3.
10. $k = 3/5$; reduction

Warm-Up 5.2.2, p. U5-56

1. 0.85
2. 85%
3. 6/5.1
4. 118%
5. Hideki might have pushed his sister again to cause her to swing higher.

Practice 5.2.2: Investigating Scale Factors, p. U5-69

1. 7.2
2. 10.2
3. 12.3
4. 20
5. $H'(-21, -9)$, $J'(-15, -18)$, $K'(-18, -24)$
6. $P'(-3, 2)$, $Q'(2.5, 4.5)$, $R'(-1.5, -2)$
7. $M'(-3.75, 6)$, $N'(5.25, -2.25)$, $O'(-7.5, -3)$
8. $A'(8.4, 7)$, $B'(2.8, 2.8)$, $D'(-4.2, 5.6)$
9. $D''(12, 8)$, $E''(24, 8)$, $F''(-6, 16)$; $k = 4$
10. $S'(-4.5, 3)$, $T'(4.5, 3)$, $U'(4.5, -3)$, $V'(-4.5, -3)$; the new countertop is longer by a factor of 1.5 feet.

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Progress Assessment, p. U5-70

1. c
2. d
3. a
4. b
5. b
6. c
7. d
8. a
9. a
10. b

11. a. Enlargement; the scale factor when converted to a decimal is 1.8. Since 1.8 is greater than 1, the scale factor creates an enlargement.
    b. $G'(0, 0)$, $H'(0, 14.4)$, $I'(9, 14.4)$, and $J'(9, 0)$
    c. The perimeter of the original book's front cover is 26 inches. The perimeter of the dilated book is 46.8 inches. The ratio 46.8/26 is equal to the scale factor of the dilation, which is 1.8 or 180%.
    d. The area of the original book's front cover (the preimage) is 40 in^2. The area of the dilated book cover (the image) is 129.6 in^2. The ratio of the areas is 129.6/40 and is equal to the square of the scale factor of the dilation, 1.8^2 or 3.24.

Lesson 3: Defining and Applying Similarity

Pre-Assessment, p. U5-75

1. d
2. a
3. a
4. a
5. c

Warm-Up 5.3.1, p. U5-80

1. 2,722.5 feet
2. 1,440 feet
3. 1,282.5 feet

Practice 5.3.1: Defining Similarity, p. U5-98

1. $\angle C = 133^\circ$, $\angle D = 37^\circ$, $\angle E = 10^\circ$, $EF = 4.5$, $DE = 5.5$
2. $\angle M = 26^\circ$, $\angle N = 14^\circ$, $\angle P = 140^\circ$, $NP = 9.8$, $PM = 5.6$, $KL = 4$
3. Similarity transformations preserve angle measures. The triangles are not similar because the angle measures in each triangle are different.
4. The triangles are similar. $\triangle ABC$ can be dilated by a scale factor of 2 with the center at (0, 0) and then reflected over the line $x = 6$ to obtain $\triangle DEF$.
5. The triangles are similar. $\triangle ABC$ can be dilated by a scale factor of 2.5 with center at (0, 0) to obtain $\triangle DEF$.
6. The triangles are similar. $\triangle ABC$ can be dilated by a scale factor of 1 with the center at (0, 0), translated horizontally –4 units and vertically –4 units, and then reflected over the line $y = -4$ to obtain $\triangle DEF$.
7. The triangles are similar. $\triangle ABC$ can be dilated by a scale factor of 1/3 with the center at (0, 0) and then rotated 90° clockwise about the origin to obtain $\triangle DEF$.
8. The triangles are similar. $\triangle ABC$ can be dilated by a scale factor of 1 with the center at (0, 0) and then rotated 180° clockwise about the origin to obtain $\triangle DEF$.
9. Similarity transformations preserve angle measures. The triangles are not similar because the angle measures in each triangle are different.
10. The triangles are similar. $\triangle ABC$ can be dilated by a scale factor of 1/2 with the center at (0, 0) and then translated vertically –6 units to obtain $\triangle DEF$.

Warm-Up 5.3.2, p. U5-103

1. 1/12
2. 3.875 meters

Practice 5.3.2: Applying Similarity Using the Angle-Angle (AA) Criterion, p. U5-115

1. Yes, $\triangle ABC \sim \triangle ZYX$ because of the AA Similarity Statement.
2. Yes, $\triangle ABD \sim \triangle ECD$ because of the AA Similarity Statement.
3. There is not enough information to determine similarity.
4. $\triangle ABC \sim \triangle YZX$; $x = 3\ 1/3$
5. $\triangle ABC \sim \triangle DEA$; $x = 4$; $BC = 3$; $AE = 6$
6. $\triangle AED \sim \triangle CBD$; $x = 1$; $ED = 2$; $DB = 6$
7. 6 feet 8 inches
8. 1063 1/3 feet = 1063 feet 4 inches
9. 36 feet
10. 2.56 feet

Progress Assessment, p. U5-119

1. d
2. c
3. b
4. c
5. a
6. d
7. b
8. c
9. a
10. b

11. a. 96.25 feet
    b. approximately 2.18 feet
    c. Triangles can be formed using the height of the object and the length of the shadow as two sides. The height of the object and the length of the shadow create a 90° angle. Connecting these two sides creates the third side of the triangle and an acute angle. This acute angle is congruent to the angle created by a nearby object and its shadow at the same time of day. Because two angles are known to be congruent, the Angle-Angle Similarity Statement affirms the triangles are similar and the side lengths are proportional.

Lesson 4: Proving Similarity

Pre-Assessment, p. U5-126

1. a
2. c
3. b
4. a
5. b

Warm-Up 5.4.1, p. U5-130

1. Two angles in each triangle are congruent, as noted by the arc marks. According to the Angle-Angle Similarity Statement, if two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.
2. $x = 5.5$ cm; $y = 8.75$ cm; $z = 3.5$ cm

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Practice 5.4.1: Proving Triangle Similarity Using Side-Angle-Side (SAS) and Side-Side-Side (SSS) Similarity, p. U5-144

1. $\triangle ABC \sim \triangle DEF$ by SAS
2. $\triangle ABC \sim \triangle EDF$ by SSS
3. $\triangle ABD \sim \triangle ECD$ by SAS
4. $\triangle ABC \sim \triangle FED$; SSS
5. not similar; corresponding sides are not proportional
6. $\triangle ABC \sim \triangle DEF$; SAS
7. not similar; corresponding sides are not proportional
8. $x = 9$
9. $x = 2\ 2/3$
10. $x = 5$

Warm-Up 5.4.2, p. U5-147

1. $x = 3$
2. 7 units
3. $y = 4$
4. 25 units

Practice 5.4.2: Working with Ratio Segments, p. U5-163

1. $BD = 3.8$ units
2. $BE = 11\ 2/3$ units
3. $EC = 9$ units
4. $BD = 4$ units
5. $CD = 3.5$ units; $BD = 2.5$ units
6. $CD = 7.5$ units; $CB = 9$ units
7. No; the sides are not proportional.
8. Yes; the sides are proportional.
9. No; the sides are not proportional.
10. Yes; the sides are proportional.

Warm-Up 5.4.3, p. U5-167

1. $\sqrt{44{,}980} \approx 212.1$ cm
2. No, the diagonals are not congruent.
3. To be "square," the lengths of each diagonal must be congruent. The lengths are not congruent. The door, as assembled, is not square.

Practice 5.4.3: Proving the Pythagorean Theorem Using Similarity, p. U5-182

1. $x = 6$ units
2. $x = 10$ units
3. $x = 8\sqrt{3} \approx 13.9$ units
4. $x = 12$ units; $f = 28.8$ units
5. $b = 65$ units; $f = 144$ units
6. $x = 6$ units; $a = 7.5$ units; $b = 10$ units
7. $a = 4\sqrt{5} \approx 8.9$ units ; $f = 4$ units
8. $x = 4\sqrt{6} \approx 9.8$ units; $a = \sqrt{33} \approx 11.5$ units; $b = 4\sqrt{22} \approx 18.8$ units
9. $a = 20$ units; $b = 15$ units; $c = 25$ units
10.

Statements	Reasons
1. $\triangle ABC$ with sides of length a, b, and c, where $c^2 = a^2 + b^2$ Draw $\overline{DE}$ equal in length to a on line ℓ. Draw line m perpendicular to $\overline{DE}$. Locate a point, F, on m so that $\overline{DF}$ is equal in length to b. Connect F and E. Call this segment x.	1. Given
2. $\triangle FED$ is a right triangle.	2. Line m is perpendicular to line ℓ.
3. $a^2 + b^2 = x^2$	3. Pythagorean Theorem
4. $x^2 = c^2$, or $x = c$	4. Segments are congruent.
5. $\triangle ABC \cong \triangle FED$	5. Side-Side-Side Congruence Statement
6. $\angle C \cong \angle D$ $\angle C$ is a right angle.	6. Corresponding angles of congruent triangles are congruent.
7. $\triangle ABC$ is a right triangle	7. $\angle C$ is a right angle.

Warm-Up 5.4.4, p. U5-187

1. $\triangle ACE \sim \triangle ABD$
2. 238 feet
3. 176 sections

Practice 5.4.4: Solving Problems Using Similarity and Congruence, p. U5-205

1. 6 m
2. 3 ft
3. 252 ft
4. 700 ft
5. 54 m
6. 5.7 m
7. 27.5 m
8. 33.8 ft
9. 3.2 m
10. 84 m

Progress Assessment, p. U5-210

1. c
2. b
3. d
4. a
5. c
6. b
7. c
8. b
9. a
10. c
11. a. $\triangle ABC \sim \triangle ADE$ because of the Side-Angle-Side (SAS) Similarity Statement. $\angle BAC \cong \angle DAE$. $\frac{AD}{DB} = \frac{14}{21} = \frac{2}{3}$ and $\frac{AE}{EC} = \frac{17.5}{26.25} = \frac{2}{3}$; therefore, the side lengths are proportional.

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b. $\overline{DE} \parallel \overline{BC}$ because of the converse to the Triangle Proportionality Theorem, which states that if a line divides two sides proportionally, then the line is parallel to the third side.

Lesson 5: Proving Theorems About Lines and Angles

Pre-Assessment, p. U5-216

1. d
2. c
3. b
4. a
5. d

Warm-Up 5.5.1, p. U5-222

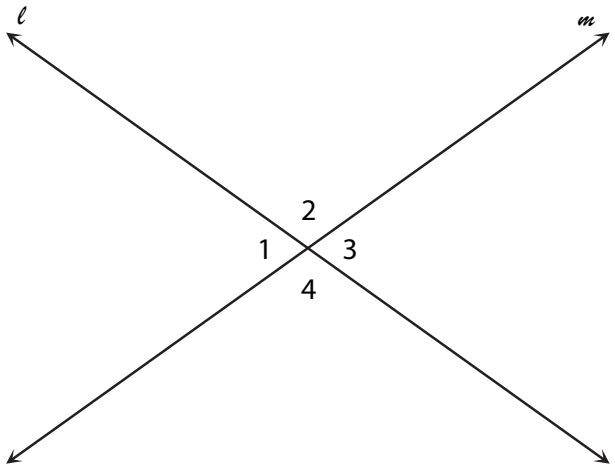
1. $m\angle 1$ must be between 75° and 105°.
2. $m\angle 2$ must be between 75° and 105°.
3. $m\angle 3$ must be between 75° and 105°.
4. $m\angle 4$ must be between 75° and 105°.
5. Completed table:

	Based on lower boundary of $\angle 1$	Based on upper boundary of $\angle 1$
$m\angle 1$	75°	105°
$m\angle 2$	105°	75°
$m\angle 3$	75°	105°
$m\angle 4$	105°	75°

6. The angles that aren't next to each other but are opposite the point of intersection are congruent. $\angle 1$ and $\angle 3$ are congruent. $\angle 2$ and $\angle 4$ are congruent. If the angles are next to each other and form a line, they add up to 180°.

Practice 5.5.1: Proving the Vertical Angles Theorem, p. U5-247

1. Answers may vary. Sample answer: Adjacent angles are $\angle 2$ and $\angle 3$ as well as $\angle 3$ and $\angle 4$. Nonadjacent angles are $\angle 1$ and $\angle 4$ as well as $\angle 5$ and $\angle 2$.
2. Supplementary angles are $\angle 1$ and $\angle 2$ as well as $\angle 1$ and $\angle 5$. Statement: $m\angle 1 + m\angle 2 = 180$; $m\angle 1 + m\angle 5 = 180$.
3. $\angle 2$ and $\angle 5$ are vertical angles. Statement: $\angle 2 \cong \angle 5$.
4. $\angle 2$ and $\angle 3$ are complementary angles. Statement: $m\angle 2 + m\angle 3 = 90$.
5. 68°
6. 98°
7. 30°
8. 37°
9. Answers may vary. Sample answer:
   Given that $\angle 1$ and $\angle 2$ form a linear pair and $\angle 2$ and $\angle 3$ form a linear pair, prove that $\angle 1 \cong \angle 3$.



   $\angle 1$ and $\angle 2$ are supplementary and $\angle 2$ and $\angle 3$ are supplementary because of the definition of a linear pair. $\angle 1 \cong \angle 3$ because angles supplementary to the same or congruent angles are congruent.
10. Answers may vary. Sample answer:

Statements	Reasons
1. $\overline{DB}$ is the perpendicular bisector of $\overline{AC}$. E is a point on $\overline{DB}$.	1. Given
2. B is the midpoint of $\overline{AC}$.	2. Definition of perpendicular bisector
3. $\overline{AB} \cong \overline{BC}$	3. Definition of midpoint
4. $\angle EBA$ and $\angle EBC$ are right angles.	4. Definition of perpendicular bisector
5. $\overline{EB} \cong \overline{EB}$	5. Reflexive Property
6. $\triangle EBA \cong \triangle EBC$	6. SAS
7. $\overline{EA} \cong \overline{EC}$	7. CPCTC
8. $EA = EC$	8. Definition of congruence

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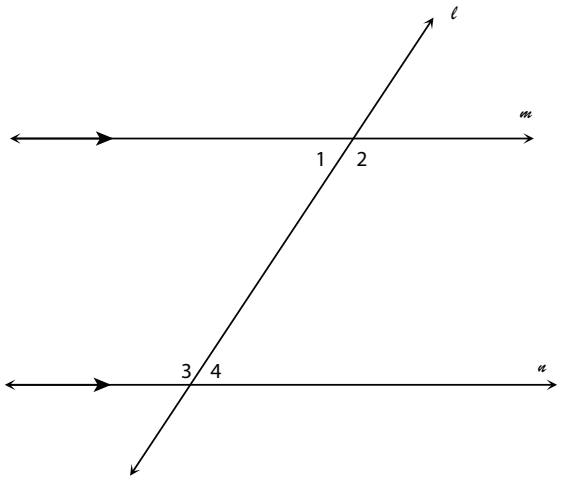
Warm-Up 5.5.2, p. U5-251

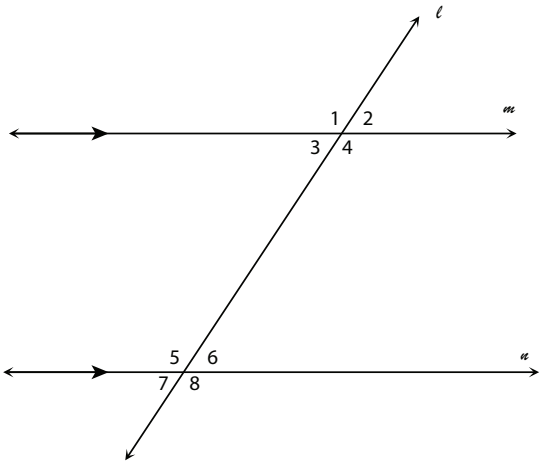
1. $m\angle 2 = 70$
2. Completed table:

Angle	Measure	Angle relationship used to determine measure
1	110°	Linear pairs are supplementary. $\angle 1$ and $\angle 2$ are a linear pair.
2	70°	The sum of the interior angles of a triangle equals 180°.
3	110°	Vertical angles are congruent. $\angle 1$ and $\angle 3$ are vertical angles.
4	80°	Corresponding angles of similar triangles are congruent. $\angle 4$ and $\angle 7$ are corresponding angles in similar triangles.
5	100°	Linear pairs are supplementary. $\angle 4$ and $\angle 5$ are a linear pair.
6	100°	Vertical angles are congruent. $\angle 5$ and $\angle 6$ are vertical angles.
7	80°	Given
8	30°	Given

Practice 5.5.2: Proving Theorems About Angles in Parallel Lines Cut by a Transversal, p. U5-275

1. 74°, because the given angles are corresponding angles and corresponding angles of a set of parallel lines intersected by a transversal are congruent.
2. 67° because alternate interior angles in a set of parallel lines intersected by a transversal are congruent.
3. 139° because same-side interior angles in a set of parallel lines intersected by a transversal are supplementary.
4. 42° because the given angle is a vertical angle with $\angle 7$, which is a corresponding angle with $\angle 5$, and in parallel lines intersected by a transversal, corresponding angles are congruent.
5. 167° because same-side exterior angles in a set of parallel lines intersected by a transversal are supplementary.
6. 115°
7. 94°
8. $m\angle 1 = 79$, $m\angle 2 = m\angle 3 = 101$, $x = 19$, and $y = 11$; same-side exterior angles are supplementary in a set of parallel lines intersected by a transversal.
9. 

Statements	Reasons
1. $m \parallel n$ and ℓ is the transversal.	1. Given
2. $\angle 1$ and $\angle 2$ are a linear pair. $\angle 3$ and $\angle 4$ are a linear pair.	2. Definition of linear pair
3. $\angle 1$ and $\angle 2$ are supplementary. $\angle 3$ and $\angle 4$ are supplementary.	3. If two angles form a linear pair, then they are supplementary.
4. $\angle 1 \cong \angle 4$ $\angle 2 \cong \angle 3$	4. Alternate interior angles
5. $\angle 1$ and $\angle 3$ are supplementary. $\angle 2$ and $\angle 4$ are supplementary.	5. Substitution

10. 

Line ℓ is a transversal and $m \parallel n$. $\angle 1 \cong \angle 5$ and $\angle 2 \cong \angle 6$ because they are corresponding angles. $\angle 5 \cong \angle 8$ and $\angle 6 \cong \angle 7$ because of vertical angles. Therefore, $\angle 1 \cong \angle 8$ and $\angle 2 \cong \angle 7$ by the Transitive Property.

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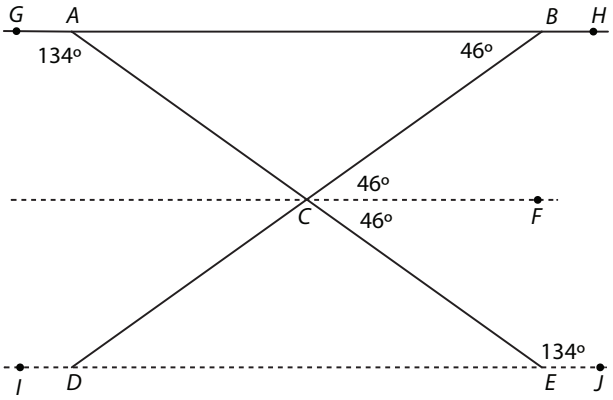
Progress Assessment, p. U5-278

1. a
2. c
3. d
4. b
5. b
6. c
7. b
8. d
9. a
10. c
11. $m\angle 3 = 92$; justifications may vary. Sample answer: First label the vertices and lines in the diagram with points. Then, draw in another line that passes through vertex C so that it is parallel to the two given parallel lines. Determine angle relationships with the angles that were given. $\angle 2$ can now be labeled with $\angle ABC$. $m\angle ABC = 46$. $\angle ABC$ is an alternate interior angle with $\angle BCF$. Therefore, they are congruent. By congruence of angles, the measures are equal. This means that $m\angle BCF = 46$. From the Angle Addition Postulate, $m\angle 3 = m\angle BCF + m\angle FCE$. Find $m\angle FCE$. Using $\overleftrightarrow{AE}$ as the transversal to parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$, $m\angle GAC = m\angle 2 = 134$ and this angle is an alternate interior angle with $\angle JEC$. Therefore, these angles are congruent and have a measure of 134°. Now, using $\overleftrightarrow{FC} \parallel \overleftrightarrow{DE}$ and $\overleftrightarrow{AE}$ as the transversal, $\angle FCE$ is a same-side interior angle with $\angle JEC$. Same-side interior angles are supplementary. This means that $m\angle FCE + m\angle JEC = 180$. By substituting in $m\angle JEC = 134$ and following with the Subtraction Property, $m\angle FCE = 46$. $m\angle 3 = m\angle BCF + m\angle FCE$. Therefore, $m\angle 3 = 92$.



Lesson 6: Proving Theorems About Triangles

Pre-Assessment, p. U5-286

1. b
2. c
3. a
4. a
5. d

Warm-Up 5.6.1, p. U5-291

1. 70°
2. 20°
3. Sample response: The angle created by the mirror and the reflected ray is congruent to the angle created by the mirror and the incident ray. This is true because we are told that the angle of incidence is congruent to the angle of reflection. The angle created by the mirror and the flashlight is the complement of the angle of incidence. The angle created by the mirror and the reflected ray is also the complement to the angle of reflection. By subtracting the angle of reflection from 90°, we are able to determine that the angle measures 20°.

Practice 5.6.1: Proving the Interior Angle Sum Theorem, p. U5-310

1. $m\angle B = 88$
2. $m\angle C = 37$
3. $m\angle A = 22$; $m\angle B = 140$
4. $m\angle A = m\angle B = m\angle C = 60$
5. $m\angle A = 62$; $m\angle B = 37$
6. $m\angle A = 156$; $m\angle B = 24$
7. $m\angle CAB = 80$; $m\angle ABC = 46$
8. $m\angle CAB = 55$; $m\angle ABC = 35$
9. $m\angle CAB = 14$; $m\angle ABC = 155$
10. A line drawn parallel to $\overline{AB}$ of $\triangle ABC$ through C creates two exterior angles, $\angle 4$ and $\angle 5$. The measures of $\angle 4$, $\angle 3$, and $\angle 5$ are equal to 180° because of the Angle Addition Postulate and the definition of a straight angle. $\angle 1$ is congruent to $\angle 4$ and $\angle 2$ is congruent to $\angle 5$ because of the Alternate Interior Angles Theorem. It follows that the measures of $\angle 1$ and $\angle 4$ are equal and the measures of $\angle 2$ and $\angle 5$ are equal because of the definition of congruent angles. By substitution, the sum of the measures of $\angle 1$, $\angle 2$, and $\angle 3$ is 180°.

Warm-Up 5.6.2, p. U5-313

1. $\triangle ABC$ has two congruent sides, so by definition, $\triangle ABC$ is an isosceles triangle.
2. 47°
3. 94°

Practice 5.6.2: Proving Theorems About Isosceles Triangles, p. U5-331

1. $m\angle B = 80$; $m\angle C = 50$
2. $m\angle B = 57.5$; $m\angle C = 65$; $m\angle D = 57.5$
3. $m\angle A = m\angle C = 30$; $m\angle B = 120$
4. $m\angle A = m\angle B = 24$; $m\angle C = 132$
5. $x = 2$
6. $x = 90$; $y = 135$
7. $x = 17$; $y = 15$
8. $\triangle ABC$ is isosceles; $\angle A \cong \angle B$.
9. $\triangle ABC$ is not isosceles.

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10.

Statements	Reasons
1. Draw a line AD perpendicular to $\overline{BC}$.	1. Given a point and line, there is only one line perpendicular though the point.
2. $\angle ADB$ and $\angle ADC$ are right angles and are congruent.	2. Perpendicular lines form right angles; right angles are congruent.
3. $\triangle ADB$ and $\triangle ADC$ are right triangles.	3. Definition of right triangles
4. $\overline{AD} \cong \overline{AD}$	4. Reflexive Property
5. $\angle B \cong \angle C$	5. Given
6. $\triangle ADB \cong \triangle ADC$	6. AAS Congruence Statement
7. $\overline{AB} \cong \overline{AC}$	7. Congruent Parts of Congruent Triangles are Congruent
8. $\triangle ABC$ is isosceles.	8. Definition of isosceles triangle

Warm-Up 5.6.3, p. U5-334

1. approximately 3,780 feet 2. (2, 1)

Practice 5.6.3: Proving the Midsegment of a Triangle, p. U5-359

1. $BC = 13.6$; $XZ = 7.1$; $m\angle BZX = 20$
2. $AC = 15$; $YZ = 6.25$; $m\angle XZY = 43$
3. $YZ = 18$
4. $BC = 7$
5. (–13, –4), (3, –4), and (7, 8)
6. (–2, 3), (2, –2), and (5, 5)
7. Use the slope formula to show that $\overline{EF}$ and $\overline{BC}$ have the same slope equal to $-\frac{1}{4}$. Therefore, $\overline{EF} \parallel \overline{BC}$. Use the distance formula to find $EF = \sqrt{17}$ and $BC = 2\sqrt{17}$. So, $EF = \frac{1}{2}BC$.
8. Use the slope formula to show that $\overline{EF}$ and $\overline{AB}$ have the same slope equal to $-\frac{5}{2}$. Therefore, $\overline{EF} \parallel \overline{AB}$. Use the distance formula to find $EF = \sqrt{29}$ and $BC = 2\sqrt{29}$. So, $EF = \frac{1}{2}AB$.
9. The midpoint of $\overline{AC}$ = (–1, –2); the midpoint of $\overline{BC}$ = (7, –1); use the slope formula to show that $\overline{EF}$ and $\overline{AB}$ have the same slope equal to $\frac{1}{8}$. Therefore, $\overline{EF} \parallel \overline{AB}$. Use the distance formula to find $EF = \sqrt{65}$ and $BC = 2\sqrt{65}$. So, $EF = \frac{1}{2}AB$.
10. The midpoint of $\overline{AC}$ = (0, –5); the midpoint of $\overline{BC}$ = (3, 0); use the slope formula to show that $\overline{EF}$ and $\overline{AB}$ have the same slope equal to $\frac{5}{3}$. Therefore, $\overline{EF} \parallel \overline{AB}$. Use the distance formula to find $EF = \sqrt{34}$ and $BC = 2\sqrt{34}$. So, $EF = \frac{1}{2}AB$.

Warm-Up 5.6.4, p. U5-363

1. $y = -\frac{1}{2}x + 6$
2. approximately 178.9 yards

Practice 5.6.4: Proving Centers of Triangles, p. U5-399

1. (2, –4) is the circumcenter of $\triangle ABC$ because the distance from this point to each of the vertices is $2\sqrt{5}$.
2. (8, 6) is the orthocenter of $\triangle ABC$ because this point is a solution to the equation of each altitude: $y = -\frac{1}{3}x + \frac{26}{3}$, $x = 8$, and $y = x - 2$.
3. The midpoints of $\triangle ABC$ are T (3, 4.5), U (–7.5, 0), and V (–4.5, 4.5). The equations of each of the medians of the triangle are $y = \frac{2}{3}x + 5$, $y = -x$, and $y = \frac{1}{4}x + \frac{15}{4}$. (–3, 3) is a solution to the equation of each median.
4. The distance from A to U is approximately 16.23 units. The distance from (–3, 3) to A is 10.82. (–3, 3) is $\frac{2}{3}$ the distance from A. The distance from B to V is approximately 6.36 units. The distance from (–3, 3) to B is approximately 4.24 units. (–3, 3) is $\frac{2}{3}$ the distance from B. The distance from C to U is approximately 18.55 units. The distance from (–3, 3) to C is approximately 12.37 units. (–3, 3) is $\frac{2}{3}$ the distance from C.
5. $\triangle ABC$ is an obtuse triangle. The incenter is the intersection of the angle bisectors. The incenter of a triangle is always inside the triangle.
6. $\triangle ABC$ is an obtuse triangle. The orthocenter of an obtuse triangle is always outside of the triangle.
7. It is given that $\triangle ABC$ has angle bisectors $\overline{AN}$, $\overline{BP}$, and $\overline{CM}$, and that $\overline{XR} \perp \overline{AB}$, $\overline{XS} \perp \overline{BC}$, and $\overline{XT} \perp \overline{AC}$. Because any point on the angle bisector is equidistant from the sides of the angle, $XR = XT$, $XT = XS$, and $XR = XS$. By the Transitive Property, $XR = XT = XS$.
8. The circumcenter and orthocenter cannot be used to determine the location of the fire station. The triangle created when the towns are connected is obtuse. The circumcenter and orthocenter would fall outside the triangle created.
9. The incenter should be determined to achieve the largest possible pond. Once the incenter is found, a circle could be inscribed for the area of the pond.
10. The incenter should be determined. This center of the triangle is equidistant to the swings, basketball court, and gazebo. All other locations for the fountain would create different distances between each attraction.

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Progress Assessment, p. U5-402

1. b
2. d
3. c
4. a
5. a
6. d
7. b
8. c
9. d
10. c

11. a. The midpoint of $\overline{AB}$ is (–1, 5.5), the midpoint of $\overline{BC}$ is (0, 3.5), and the midpoint of $\overline{AC}$ is (–2, 3).
    b. The length of each midsegment is one-half the length of the side it is parallel to.
    c. The centroid is located at (–1, 4).

Lesson 7: Proving Theorems About Parallelograms

Pre-Assessment, p. U5-408

1. a
2. b
3. a
4. d
5. c

Warm-Up 5.7.1, p. U5-411

1.

$m\angle 1$	$m\angle 2$
45°	135°
60°	120°
75°	105°
90°	90°

2. $\angle 1$ and $\angle 2$ are same-side interior angles. Same-side interior angles are supplementary. Therefore, use the equation as follows:

   $m\angle 1 + m\angle 2 = 180$

   $m\angle 2 = 180 - m\angle 1$

   Substitute each given measure of $\angle 1$ into the equation in order to solve for the measure of $\angle 2$.

Practice 5.7.1: Proving Properties of Parallelograms, p. U5-435

1. Yes, it's a parallelogram because opposite sides are parallel; $m_{\overline{AB}} = m_{\overline{CD}} = 4$ and $m_{\overline{BC}} = m_{\overline{DA}} = 0$.
2. Yes, it's a parallelogram because opposite sides are parallel; $m_{\overline{FG}} = m_{\overline{HI}} = 1$ and $m_{\overline{GH}} = m_{\overline{IF}} = -\frac{5}{3}$.
3. No, it's not a parallelogram because opposite sides are not congruent: $JK = \sqrt{26}$, $LM = \sqrt{37}$, $KL = \sqrt{10}$, and $MJ = 3$.
4. Yes, it's a parallelogram because opposite sides are congruent: $MN = OP = \sqrt{37}$ and $NO = PM = 2\sqrt{13}$.
5. No, it's not a parallelogram because the midpoints of the diagonals are not the same, indicating that the diagonals do not bisect each other: $M_{\overline{PR}} = \left(1, \frac{7}{2}\right)$ and $M_{\overline{QS}} = (2, 3)$.
6. Yes, it's a parallelogram because the midpoints of the diagonals are the same, indicating that the diagonals bisect each other: $M = (6, 4)$.
7. $m\angle A = m\angle C = 104$ and $m\angle B = m\angle D = 76$; $x = 12$ and $y = 15$
8. $m\angle A = m\angle C = 82$ and $m\angle B = m\angle D = 98$; $x = 14$ and $y = 20$
9. Given that quadrilateral *ABCD* is a parallelogram, by definition opposite sides are parallel and opposite sides are congruent. This means that $\overline{AD} \parallel \overline{BC}$, with the diagonals acting as transversals and $\overline{AD} \cong \overline{BC}$. By alternate interior angles, $\angle BDA \cong \angle DBC$ and $\angle DAC \cong \angle ACB$. By ASA, $\triangle DPA \cong \triangle BPC$.
10.

Statements	Reasons
1. $\square ABCD, \square EBHG, \square FIJG$	1. Given
2. $\angle D \cong \angle B$ $\angle B \cong \angle G$ $\angle G \cong \angle I$	2. Opposite angles in a parallelogram are congruent.
3. $\angle D \cong \angle I$	3. Transitive Property

Warm-Up 5.7.2, p. U5-438

1. The distance between Carrollton and Campton is about 84 miles.
2. Atlanta is at (6, 3.25) on the grid.
3. Each city is about 42 miles from Atlanta.
4. The slope is $\frac{5}{16}$.
5. $-\frac{16}{5}$

Practice 5.7.2: Proving Properties of Special Quadrilaterals, p. U5-471

1. Quadrilateral *ABCD* is a parallelogram and a rectangle. Justification: opposite sides are parallel, adjacent sides are perpendicular, the diagonals bisect each other, and not all four sides are congruent.
2. Quadrilateral *EFGH* is a kite. Justification: adjacent sides are congruent and the diagonals intersect at a right angle.
3. Quadrilateral *IJKL* is a parallelogram, a rectangle, a rhombus, and a square. Justification: opposite sides are parallel, consecutive sides are perpendicular, the diagonals bisect each other, and all four sides are congruent.
4. Quadrilateral *MNOP* is a parallelogram, a rhombus, and a square. Justification: opposite sides are parallel, adjacent sides are perpendicular, the diagonals are perpendicular, and all four sides are congruent.
5. Quadrilateral *PQRS* is a parallelogram. Justification: opposite sides are parallel, plus the diagonals are congruent and bisect each other.
6. Quadrilateral *TUVW* is an isosceles trapezoid. Justification: one pair of opposite sides is parallel and the other pair of sides is congruent.

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7. Quadrilateral *WXYZ* is a parallelogram and a rectangle. Justification: opposite sides are parallel, adjacent sides are perpendicular, the diagonals bisect each other, and not all four sides are congruent.
8. Quadrilateral *ABCD* is a parallelogram, a rectangle, a rhombus, and a square. Justification: opposite sides are parallel, consecutive sides are perpendicular, the diagonals bisect each other, and all four sides are congruent.
9. Given that quadrilateral *ABCD* is a rhombus, the diagonals of a rhombus bisect each other and are perpendicular. Therefore, $\overline{DP} \cong \overline{PB}$ and $\overline{AP} \cong \overline{PC}$. The diagonals of a rhombus bisect the opposite pairs of angles in the rhombus so that $\angle DAP \cong \angle PAB \cong \angle DCP \cong \angle BCP$. The diagonals of a rhombus are perpendicular. By the definition of perpendicular lines, four right and congruent triangles are formed. Therefore, $\angle DPA \cong \angle APB \cong \angle BPC \cong \angle CPD$. By ASA, $\triangle APD \cong \triangle APB \cong \triangle CPB \cong \triangle CPD$.
10. The slopes are opposite reciprocals of each other: $m_{\overline{AC}} = \frac{c}{b+\sqrt{b^2+c^2}}$ and $m_{\overline{BD}} = \frac{-c}{\sqrt{b^2+c^2}-b}$. The product of the slopes is –1, showing that they are opposite reciprocals of each other.

Progress Assessment, p. U5-473

1. a
2. b
3. a
4. d
5. d
6. d
7. c
8. a
9. b
10. d

11. Quadrilateral *BCDE* is a parallelogram. Justifications may vary. Students should have at least two of the following justifications: Opposite sides are parallel $\left(m_{\overline{BC}} = m_{\overline{DE}} = 1 \text{ and } m_{\overline{CD}} = m_{\overline{BE}} = \frac{1}{2}\right)$; opposite sides are congruent $\left(ED = BC = 4\sqrt{2} \text{ and } CD = BE = 2\sqrt{5}\right)$; and/or diagonals bisect each other because the midpoints of the diagonals are the same ($M = (3, 2)$).

Lesson 8: Exploring Trigonometric Ratios

Pre-Assessment, p. U5-476

1. a
2. b
3. c
4. c
5. d

Warm-Up 5.8.1, p. U5-481

1. 2
2. A
3. $A \cong A$; $C' \cong C$; $B' \cong B$
4. Yes, the two triangles are similar by the Angle-Angle Similarity Statement.
5. $k = 16$
6. $h = 83$ feet

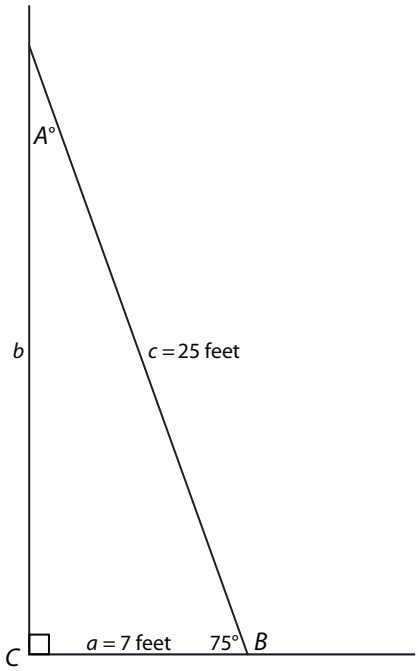
Practice 5.8.1: Defining Trigonometric Ratios, p. U5-503

Answers will vary due to the variation that comes when drawing and measuring or rounding.

1. $\sin A \approx 0.819$; $\cos A \approx 0.574$; $\tan A \approx 1.428$
2. $\csc A \approx 1.221$; $\sec A \approx 1.743$; $\cot A \approx 0.701$
3. $\sin B \approx 0.574$; $\cos B \approx 0.819$; $\tan B \approx 0.701$
4. $\csc B \approx 1.74$; $\sec B \approx 1.221$; $\cot B \approx 1.428$
5. hypotenuse ≈ 8.944; $m\angle A \approx 63°$; $m\angle B \approx 27°$; $\sin A \approx 0.894$; $\cos A \approx 0.447$; $\tan A = 2$; $\sin B \approx 0.447$; $\cos B \approx 0.894$; $\tan B = 0.5$
6. $b \approx 6.928$; $m\angle A = 30°$; $m\angle B = 60°$; $\sin A = 0.5$; $\cos A \approx 0.866$; $\tan A \approx 0.577$; $\sin B \approx 0.866$; $\cos B = 0.5$; $\tan B = 1.732$
7. Each beam ≈ 28.866 feet as shown in the diagram, but adding 2 feet in order for each beam to extend past the walls yields a length of ≈ 30.866 feet; $\cos 30° \approx 0.866$.
8. The ladder needs to be about 35.356 feet tall, or 36 feet tall with rounding; $\sin 45° \approx 0.707$.
9. The tree is 22.5 feet tall; $\tan A = 0.5$.
10. The ladder reaches about 13.737 feet high on the wall; $\sin B \approx 0.940$.

Warm-Up 5.8.2, p. U5-508

1.



2. Length b needs to be found (the height of the ladder on the wall).
3. $c = 25$ feet; $a = 7$ feet; $7^2 + b^2 = 25^2$; $b = \sqrt{576}$; $b = 24$ feet. The ladder will be long enough to reach the height of the banner on the wall.
4. 15°
5. $\sin 75° \approx 0.9659258263$; $\cos 15° \approx 0.9659258263$; the sine of 75° and the cosine of 15° have exactly the same values.

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Practice 5.8.2: Exploring Sine and Cosine As Complements, p. U5-522

1. approximately 0.643
2. $\sin 65^\circ \approx 0.906$
3. $\theta = 68^\circ$
4. $\theta = 31^\circ$
5. $\cos 85^\circ \approx 0.087$
6. $\sin 56^\circ \approx 0.829$
7. $\theta = 70^\circ$
8. $\sin A = \frac{a}{c} = \cos D = \frac{e}{f}$; $\cos B = \frac{a}{c} = \sin E = \frac{e}{f}$
9. approximately 25.1 feet
10. $\sin A \approx 0.294$; $\cos A \approx 0.956$; $\sin B \approx 0.956$; $\cos B \approx 0.294$

Progress Assessment, p. U5-525

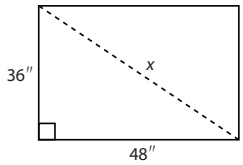
1. b
2. a
3. d
4. a
5. b
6. a
7. c
8. b
9. b
10. b
11. Methods will vary; check that drawings and equations suit the chosen method. Possible answers:
    - Method 1: Measure the shadow of the flagpole and one's own shadow, then set up a ratio to solve for the unknown height of the pole. Drawings should show the flagpole and its shadow as well as a person and his/her shadow. Equations should be similar to this: $\frac{\text{length of my shadow}}{\text{my height}} = \frac{\text{length of flagpole's shadow}}{\text{height of flagpole}}$
    - Method 2: Place a shorter post in front of the flagpole. Aim a laser light to create a sight line from the laser to the top of the object/post to the top of the flagpole. Measure the distance a from the laser to the post (height = p). Measure the distance b from the post to the flagpole. Find a scale factor, k. Set up and solve a ratio for the two similar triangles. The drawing should show the similar triangles. Equations should be similar to this: $\frac{a+b}{a} = k$; $k \bullet p = h$
    - Method 3: Make a scale drawing on paper showing the angle of measurement looking up at the flagpole from a specific location on the ground. Use that angle to draw a similar triangle. Measure the distance on the ground to the flagpole. Find the scale factor, k, of the model drawing to the actual distance. Multiply k by the corresponding length on the model drawing.
    - Method 4: Use the properties of the special triangles 45°–45°–90° or 30°–60°–90°. This method is similar to Method 3, but the scale drawing would not be required because the scale factor k is already known. In this case, the angle of inclination (not formally defined in this lesson) would be 45°, 30°, or 60°, and the appropriate scale factor of the side lengths should be listed, respectively, for those three angles as either 1, $\frac{1}{\sqrt{3}}$, or $\sqrt{3}$. The student should show that the measurement on the ground to the flagpole is multiplied by the scale factor.

Lesson 9: Applying Trigonometric Ratios

Pre-Assessment, p. U5-529

1. b
2. d
3. b
4. b
5. c

Warm-Up 5.9.1, p. U5-534

1. 

2. $x = 60''$
3. Since television sizes are a measure of the diagonal distance across a rectangular or square screen, you can use the Pythagorean Theorem on the subsequent right triangles created by the diagonal. In this case, the screen is 60″ because the sum of the squares of the length and width of the television yields the square of the diagonal distance.

Practice 5.9.1: Calculating Sine, Cosine, and Tangent, p. U5-554

1. $\cos T = \frac{3}{5} = 0.6$; $\tan T = \frac{4}{3} \approx 1.333$
2. $\sin A = \frac{60}{61} \approx 0.983$; $\cos A = \frac{11}{61} \approx 0.180$
3. 13.352 ft
4. 15.104 ft
5. 45°. The triangle is isosceles right and by definition its acute angles are 45°.
6. 66°
7. 24°
8. $AC = 27.138$; $BC = 16.957$; $m\angle A = 32^\circ$
9. $m\angle X \approx 48^\circ$; $m\angle Z \approx 42^\circ$; $YZ = 8.944$ in
10. 60.763 ft

Warm-Up 5.9.2, p. U5-557

1. 1/6
2. 37°

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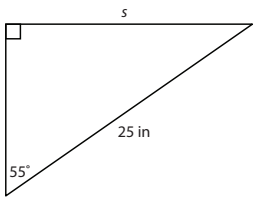
Practice 5.9.2: Calculating Cosecant, Secant, and Cotangent, p. U5-573

1. 15.046
2. 6.628
3. 16.501
4. 76°
5. 22°
6. $\csc E = \frac{25}{7} \approx 3.571$; $\sec E = \frac{25}{24} \approx 1.042$; $\cot E = \frac{24}{7} \approx 3.429$
7. 34.238 ft; 29°
8. 15,398.650 ft or about 15,399 ft
9. $\csc B = 1.054$; $\sec B = 3.158$; $\cot B = 0.334$
10. $\csc B = 1.155$; $\sec B = 2$; $\cot B = 0.577$

Warm-Up 5.9.3, p. U5-576

1. 

2. $s = 20.5$ in
3. The shelves have a width of 21 inches but the brackets have a length of 20.5 inches. The shelves will not fit exactly; they will be slightly wider.

Practice 5.9.3: Problem Solving with the Pythagorean Theorem and Trigonometry, p. U5-594

1. 46 ft
2. 74 ft
3. 59°
4. 32°
5. 218 m
6. 5,361 ft
7. 44 ft
8. 59°; 48°
9. 80 ft
10. 70°

Warm-Up 5.9.4, p. U5-596

1. 8.06
2. $\sin\theta \approx 0.5$; $\cos\theta \approx 0.87$
3. $\theta \approx 30°$

Practice 5.9.4: Proving the Fundamental Pythagorean Identity, p. U5-611

1. $\frac{2\sqrt{2}}{3}$
2. $\frac{1}{2}$
3. $\frac{5}{13}$
4. $\cos\theta = \frac{\sqrt{2}}{2}$; $\tan\theta = 1$; $\sec\theta = \sqrt{2}$; $\csc\theta = \sqrt{2}$; $\cot\theta = 1$; $\theta = 45°$
5. $\sin\theta = \frac{4}{5}$; $\tan\theta = \frac{4}{3}$; $\sec\theta = \frac{5}{3}$; $\csc\theta = \frac{5}{4}$; $\cot\theta = \frac{3}{4}$; $\theta \approx 53.13°$
6. $-\cos^3\theta$
7. 1
8. $\frac{4}{5}$
9. $\frac{\sqrt{5}}{3}$
10. Substituting 1 for $\sin^2\theta + \cos^2\theta$ yields the following:
    $2(\sin^2\theta + \cos^2\theta) + 3 = 5$
    $2(1) + 3 = 5$
    $5 = 5$

Progress Assessment, p. U5-613

1. a
2. b
3. c
4. c
5. d
6. b
7. b
8. c
9. d
10. b
11. The distance to the first kitten is 21 feet, found by using the equation $\tan 36° = \frac{x}{20}$. The height of the second kitten can be found using the equation $\tan 40° = \frac{x}{20}$. The difference between the two heights is 2 feet.

Unit Assessment
p. U5-617

1. c
2. b
3. a
4. a
5. c
6. d
7. b
8. d
9. c
10. d
11. b
12. c
13. a. $\tan 25° = \frac{30}{x}$
    b. The worm is approximately 64 meters away from the base of the flagpole.
14. Because they are vertical angles, $m\angle AHB = m\angle EHD$. Because $\overline{AB} \| \overline{DE}$, $m\angle ABH = m\angle EDH$. These two pairs of equal angles indicate that the AA Similarity Postulate can be used: $\triangle HAB \sim \triangle HED$. Because they are corresponding sides of similar triangles, $\frac{HA}{HE} = \frac{BH}{DH}$. Using the property of proportions, we conclude that $\frac{BH}{HA} = \frac{DH}{HE}$.
15. a. $\sin\theta = \frac{1}{3}$; $\cos\theta = \frac{2\sqrt{2}}{3}$; $\tan\theta = \frac{\sqrt{2}}{4}$; $\csc\theta = 3$; $\sec\theta = \frac{3\sqrt{2}}{4}$; $\cot\theta = 2\sqrt{2}$
    b. $\theta \approx 19.48°$
    c. approximately 22.6 feet

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Student Book Answer Key

Lesson 1: Line Segments

Practice 5.1.1: Midpoints and Other Points on Line Segments, p. U5-15

1. (7, 8)
2. (–1, –8)
3. (–6, 4)
4. (20, 20)
5. (4, 10)
6. (11, 8)
7. (8, 7)
8. (16, 14)
9. (7, 5)
10. (19, 1)

Lesson 2: Investigating Properties of Dilations

Practice 5.2.1: Investigating Properties of Parallelism and the Center, p. U5-30

1. The triangle has been dilated. The corresponding sides of the triangle are parallel and the scale factor is consistent ($k = 1/2$). Additionally, the preimage points and image points are collinear with the center of dilation.
2. The quadrilateral has not been dilated. The scale factor is inconsistent between corresponding sides.
3. The rectangle has not been dilated. The scale factor is inconsistent between corresponding sides.
4. The rectangle has been dilated. The corresponding sides of the rectangle are parallel and the scale factor is consistent ($k = 1.6$). Additionally, the preimage points and image points are collinear with the center of dilation.
5. $k = 1/3$; reduction
6. $k = 0.4$; reduction
7. $k = 1$; congruency transformation
8. $k = 1.25$; enlargement; the scale factor is greater than 1; therefore, it is an enlargement.
9. No, because the scale factors of corresponding sides are inconsistent.
10. $k = 1.5$; enlargement; the scale factor is greater than 1; therefore, it is an enlargement.

Practice 5.2.2: Investigating Scale Factors, p. U5-40

1. 10.125
2. 28.5
3. 4.6
4. 3/5
5. T' (–3, –1), U' (–2, –2), V' (–2/3, –1)
6. B' (–2, 0), D' (–10, –12), E' (6, –8)
7. N' (–9.6, –3.2), O' (4.8, 8), P' (6.4, –12.8)
8. E' (1.2, 2.7), F' (1.5, 0.9), G' (2.7, 3)
9. I'' (3.375, 2.8125), J'' (1.125, 1.125), K'' (–1.6875, 2.25); $k = 9/16$ or 0.5625
10. 6 feet by 8 feet

Lesson 3: Defining and Applying Similarity

Practice 5.3.1: Defining Similarity, p. U5-53

1. $\angle C = 44^\circ$, $\angle D = \angle E = 68^\circ$, $CB = 4.5$, $DF = 2.1$
2. $\angle L = \angle M = \angle P = 45^\circ$, $MP = 9$, $NP = 6$
3. The triangles are similar. $\triangle ABC$ can be dilated by a scale factor of 2 with the center at (0, 0) to obtain $\triangle DEF$.
4. The triangles are similar. $\triangle ABC$ can be dilated by a scale factor of 1 with center at (0, 0) and then translated –3 units vertically and 2 units horizontally to obtain $\triangle DEF$.
5. Similarity transformations preserve angle measures. The triangles are not similar because the angle measures in each triangle are different.
6. The triangles are similar. $\triangle ABC$ can be dilated by a scale factor of 2.5 with the center at (0, 0) and then translated 3 units vertically and 5 units horizontally to obtain $\triangle DEF$.
7. Similarity transformations preserve angle measures. The triangles are not similar because the angle measures in each triangle are different.
8. The triangles are similar. $\triangle ABC$ can be dilated by a scale factor of 3/2 with the center at (0, 0) and then reflected over the line $x = 0$ to obtain $\triangle DEF$.
9. Similarity transformations preserve angle measures. The triangles are not similar because the angle measures in each triangle are different.
10. The triangles are similar. $\triangle ABC$ can be dilated by a scale factor of 5 with the center at (0, 0) and then rotated 90° counterclockwise about the origin to obtain $\triangle DEF$.

Practice 5.3.2: Applying Similarity Using the Angle-Angle (AA) Criterion, p. U5-64

1. There is not enough information to determine similarity.
2. Yes, $\triangle ABC \sim \triangle XYZ$ because of the AA Similarity Statement.
3. Yes, $\triangle ABC \sim \triangle YXZ$ because of the AA Similarity Statement.
4. $\triangle ABC \sim \triangle YXZ$; $x = 3\ 1/3$
5. $\triangle ABC \sim \triangle ZYX$; $x = 2$
6. $\triangle BCD \sim \triangle BDA$; $x = 10\ 2/3$
7. 3 feet
8. 9 feet
9. 2.16 meters
10. 60 feet

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Lesson 4: Proving Similarity

Practice 5.4.1: Proving Triangle Similarity Using Side-Angle-Side (SAS) and Side-Side-Side (SSS) Similarity, p. U5-79

1. $\triangle ABC \sim \triangle FDE$ by SSS
2. $\triangle ACE \sim \triangle BCD$ by SAS
3. $\triangle ABC \sim \triangle DEF$ by SSS
4. not similar; corresponding sides are not proportional
5. $\triangle ABC \sim \triangle EFD$; SAS
6. not similar; corresponding sides are not proportional
7. $\triangle ABC \sim \triangle DEF$; SSS
8. $x = 7$
9. $x = 5$
10. $x = 6$

Practice 5.4.2: Working with Ratio Segments, p. U5-92

1. CD = 12 3/8 units
2. BC = 3 1/3 units
3. DE = 8.4 units
4. CD = 6 units
5. BC = 10 units; CD = 12 units
6. CB = 21 units; CD = 35 units
7. Yes; the sides are proportional.
8. No; the sides are not proportional.
9. No; the sides are not proportional.
10. Yes; the sides are proportional.

Practice 5.4.3: Proving the Pythagorean Theorem Using Similarity, p. U5-105

1. $x = 4\sqrt{2} \approx 5.7$ units
2. $x = \frac{3\sqrt{26}}{2} \approx 7.6$ units
3. $x = \sqrt{3} \approx 1.73$ units
4. $x = \frac{5\sqrt{6}}{2} \approx 19.4$ units
5. $a = 2\sqrt{6} \approx 4.9$ units; e = 1/5; f = 4 4/5
6. a = 600; b = 175; x = 168
7. $c = 4\sqrt{2} \approx 5.7$ units; $e = f = 2\sqrt{2} \approx 2.8$ units
8. $a = \sqrt{6} \approx 2.4$ units; $b = \sqrt{3} \approx 1.7$ units; $x = \sqrt{2} \approx 1.4$ units
9. e = 3.6; f = 6.4
10. A segment, $\overline{DE}$, drawn on line ℓ is equal in length to a. Locate a point F on a line m equal in length to b, which is perpendicular to $\overline{DE}$. Connect points F and E. Call this segment x. $\triangle FED$ is a right triangle, so $a^2 + b^2 = x^2$. It is also true that $a^2 + b^2 = c^2$, so $x^2 = c^2$, or $x = c$. $\triangle ABC \cong \triangle FED$ by the Side-Side-Side Congruence Statement. For this reason, $\angle C \cong \angle D$ and $\angle C$ must be a right angle, making $\triangle ABC$ a right triangle.

Practice 5.4.4: Solving Problems Using Similarity and Congruence, p. U5-119

1. 33 ft
2. 8.75 ft
3. 58 1/3 ft
4. 649.6 m
5. 23 m
6. 3.2 ft
7. 33.8 ft
8. 22.3 m
9. 7.8 m
10. 30 m

Lesson 5: Proving Theorems About Lines and Angles

Practice 5.5.1: Proving the Vertical Angles Theorem, p. U5-145

1. Answers may vary. Sample answer: Adjacent angles are $\angle 2$ and $\angle 3$ as well as $\angle 3$ and $\angle 4$. Nonadjacent angles are $\angle 1$ and $\angle 4$ as well as $\angle 5$ and $\angle 2$.
2. Supplementary angles are $\angle 7$, $\angle 1$, and $\angle 2$. Statement: $m\angle 7 + m\angle 1 + m\angle 2 = 180$.
3. $\angle 1$ and $\angle 4$ are vertical angles. Statement: $\angle 1 \cong \angle 4$.
4. $\angle 4$ and $\angle 5$ are complementary angles. Statement: $m\angle 4 + m\angle 5 = 90$.
5. 131°
6. 79°
7. 102°
8. 48°
9. Since $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ as this was given, $\angle 1$ and $\angle 2$, $\angle 2$ and $\angle 3$, and $\angle 1$ and $\angle 4$ all form linear pairs. This means that those pairs of angles are also supplementary by the Supplement Theorem. Therefore, $m\angle 1 + m\angle 2 = 180$, $m\angle 2 + m\angle 3 = 180$, and $m\angle 1 + m\angle 4 = 180$. Given that $\angle 1$ is a right angle, by the definition of right angles, $m\angle 1 = 90$. Use substitution so that $90 + m\angle 2 = 180$, and by the Subtraction Property, $m\angle 2 = 90$. By definition, $\angle 2$ is a right angle. Use substitution so that $90 + m\angle 3 = 180$, and by the Subtraction Property $m\angle 3 = 90$. By definition, $\angle 3$ is a right angle. Use substitution so that $90 + m\angle 4 = 180$, and by the Subtraction Property $m\angle 4 = 90$. By definition, $\angle 4$ is a right angle. Therefore, $\angle 2$, $\angle 3$, and $\angle 4$ are right angles.
10. x = 13

Practice 5.5.2: Proving Theorems About Angles in Parallel Lines Cut by a Transversal, p. U5-165

1. 88°, because alternate interior angles in a set of parallel lines intersected by a transversal are congruent.
2. 90°, because same-side interior angles in a set of lines intersected by a transversal are supplementary.
3. 86°, because alternate interior angles in a set of lines intersected by a transversal are congruent.
4. 57°, because same-side exterior angles in a set of parallel lines intersected by a transversal are supplementary.
5. 144°, because corresponding angles in a set of lines intersected by a transversal are congruent.
6. 111°

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7. 95°
8. $m\angle 1 = 100$, $m\angle 2 = m\angle 3 = 80$, $x = 11$, and $y = 12$
9.

Statements	Reasons
1. $m \parallel n$ and ℓ is the transversal.	1. Given
2. $\angle 2 \cong \angle 6$	2. Corresponding Angles Postulate
3. $\angle 6$ and $\angle 8$ are a linear pair.	3. Definition of a linear pair
4. $\angle 6$ and $\angle 8$ are supplementary.	4. If two angles form a linear pair, then they are supplementary.
5. $m\angle 6 + m\angle 8 = 180$	5. Supplement Theorem
6. $m\angle 2 + m\angle 8 = 180$	6. Substitution
7. $\angle 2$ and $\angle 8$ are supplementary.	7. Supplement Theorem

10. Since ℓ is a transversal and $\ell \perp m$, $\angle 1$ is a right angle. $m\angle 1 = 90$ because of the definition of a right angle. Since lines m and n are parallel, $\angle 1 \cong \angle 2$ because of the Corresponding Angles Postulate. $m\angle 1 = m\angle 2$ because of the definition of congruent angles. $m\angle 2 = 90$ by substitution. $\angle 2$ is a right angle. Therefore, $\ell \perp m$ because of the definition of perpendicular lines.

Lesson 6: Proving Theorems About Triangles

Practice 5.6.1: Proving the Interior Angle Sum Theorem, p. U5-185

1. $m\angle B = 20$
2. $m\angle B = 56$
3. $m\angle B = 81$; $m\angle C = 21$
4. $m\angle A = 85$; $m\angle B = 52$; $m\angle C = 43$
5. $m\angle A = 92$; $m\angle B = 29$
6. $m\angle A = 128$; $m\angle B = 52$
7. $m\angle CAB = 26$; $m\angle ABC = 24$
8. $m\angle CAB = 25$; $m\angle ABC = 65$
9. $m\angle CAB = 10$; $m\angle ABC = 125$
10. $\triangle ABC$ is a triangle. By the Triangle Sum Theorem, the sum of the measures of $\angle 1$, $\angle 2$, and $\angle 3$ is equal to 180°. If two angles form a linear pair, they are supplementary, so $\angle 3$ and $\angle 4$ are also supplementary. The sum of the measures of $\angle 3$ and $\angle 4$ is equal to 180° by the definition of supplementary angles. The sum of the measures of $\angle 3$ and $\angle 4$ is equal to the sum of the measures of $\angle 1$, $\angle 2$, and $\angle 3$ by substitution. The measure of $\angle 4$ is equal to the sum of the measures of $\angle 1$ and $\angle 2$ by the Subtraction Property.

Practice 5.6.2: Proving Theorems About Isosceles Triangles, p. U5-200

1. $m\angle A = 40$; $m\angle C = 70$
2. $m\angle ADB = 108$; $m\angle DCB = 72$; $m\angle DBC = 36$
3. $m\angle A = 120$; $m\angle B = m\angle C = 30$
4. $m\angle A = m\angle C = 45$; $m\angle B = 90$
5. $x = 3$
6. $x = 78$; $y = 51$
7. $x = 13$
8. $\triangle ABC$ is not isosceles.
9. $\triangle ABC$ is isosceles; $\angle B \cong \angle C$.
10. Given $\triangle ABC$ is equiangular, prove $\triangle ABC$ is equilateral. Since $\triangle ABC$ is equiangular, $\angle A \cong \angle B$ and $\angle B \cong \angle C$. By the converse of the Isosceles Triangle Theorem, $\overline{AB} \cong \overline{BC}$ and $\overline{AC} \cong \overline{BC}$, so by the Transitive Property, $\overline{AB} \cong \overline{BC} \cong \overline{AC}$. Therefore, $\triangle ABC$ is equilateral.

Practice 5.6.3: Proving the Midsegment of a Triangle, p. U5-221

1. $BC = 12$; $XZ = 7.5$; $m\angle BZX = 55$
2. $BC = 21$; $YZ = 5.625$; $m\angle AXY = 72$
3. $XY = 5$
4. $AB = 42$
5. (–9, –2), (5, 8), (5, –2)
6. (2, 6), (4, 3), (8, 5)
7. Use the slope formula to show that $\overline{EF}$ and $\overline{BC}$ have the same slope equal to $-\frac{3}{4}$. Therefore, $\overline{EF} \parallel \overline{BC}$. Use the distance formula to find $EF = 5$ and $BC = 10$. So, $EF = \frac{1}{2}BC$.
8. Use the slope formula to show that $\overline{EF}$ and $\overline{BC}$ have the same slope equal to –1. Therefore, $\overline{EF} \parallel \overline{BC}$. Use the distance formula to find $EF = \sqrt{2}$ and $BC = 2\sqrt{2}$. So, $EF = \frac{1}{2}BC$.
9. The midpoint of $\overline{AC}$ = (1, 1); the midpoint of $\overline{BC}$ = (4, 2); use the slope formula to show that $\overline{EF}$ and $\overline{AB}$ have the same slope equal to $\frac{1}{3}$. Therefore, $\overline{EF} \parallel \overline{AB}$. Use the distance formula to find $EF = \sqrt{10}$ and $AB = 2\sqrt{10}$. So, $EF = \frac{1}{2}AB$.
10. The midpoint of $\overline{AC}$ = (1, 1); the midpoint of $\overline{BC}$ = (–1, 6); use the slope formula to show that $\overline{EF}$ and $\overline{AB}$ have the same slope equal to $-\frac{5}{2}$. Therefore, $\overline{EF} \parallel \overline{AB}$. Use the distance formula to find $EF = \sqrt{29}$ and $AB = 2\sqrt{29}$. So, $EF = \frac{1}{2}AB$.

Practice 5.6.4: Proving Centers of Triangles, p. U5-251

1. (–1, –3) is the circumcenter of $\triangle ABC$ because the distance from this point to each of the vertices is $\sqrt{10}$.
2. (–2, 1) is the orthocenter of $\triangle ABC$ because this point is a solution to the equation of each altitude: $x = -2$ and $y = 1$. $\triangle ABC$ is a right triangle, so the orthocenter is the vertex B.

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3. The midpoints of $\triangle ABC$ are T (7.5, –0.5), U (4, –3), and V (3.5, 0.5). The equations of each of the medians of the triangle are $y = 2x - 11$, $y = -x + 4$, and $y = \frac{1}{5}x - 2$. (5, –1) is a solution to the equation of each median.
4. The distance from A to U is approximately 6.71 units. The distance from (5, –1) to A is 4.47 units. (5, –1) is $\frac{2}{3}$ the distance from A. The distance from B to V is approximately 6.36 units. The distance from (5, –1) to B is approximately 4.24 units. (5, –1) is $\frac{2}{3}$ the distance from B. The distance from C to T is approximately 7.65 units. The distance from (5, –1) to C is approximately 5.10 units. (5, –1) is $\frac{2}{3}$ the distance from C.
5. $\triangle ABC$ is an obtuse triangle. The incenter is the intersection of the angle bisectors. The incenter of a triangle is always inside the triangle.
6. $\triangle ABC$ is a right triangle. The orthocenter of a right triangle is always on a vertex of the triangle.
7. It is given that $\triangle ABC$ has perpendicular bisectors [illegible], and [illegible] of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. X is on the perpendicular bisector of $\overline{AB}$, so it is equidistant from A and B. $AX = BX$ by the definition of equidistant. The perpendicular bisector of $\overline{BC}$ also has the point X, so $BX = CX$. $AX = CX$ by the Transitive Property of Equality; therefore, $AX = BX = CX$.
8. The circumcenter and orthocenter cannot be used to determine the location of the trauma center. The triangle created when the towns are connected is obtuse. The circumcenter and orthocenter would fall outside the triangle created.
9. The incenter should be determined. This center of the triangle is equidistant to the each location within the park. All other locations for the first aid station would create different distances between each location.
10. The incenter should be determined to achieve the maximum amount of space for the dog. Once the incenter is found, the dog's leash can be staked in the ground.

Lesson 7: Proving Theorems About Parallelograms

Practice 5.7.1: Proving Properties of Parallelograms, p. U5-273

1. No, it's not a parallelogram because opposite sides are not parallel: $m_{\overline{TU}} = 5$, $m_{\overline{VW}} = 2$, $m_{\overline{UV}} = -\frac{2}{3}$, $m_{\overline{WT}} = \frac{1}{3}$.
2. Yes, it's a parallelogram because opposite sides are parallel: $m_{\overline{WX}} = m_{\overline{YZ}} = 5$ and $m_{\overline{XY}} = m_{\overline{ZW}} = \frac{2}{5}$.
3. Yes, it's a parallelogram because opposite sides are congruent: $GH = IJ = \sqrt{10}$ and $HI = JG = \sqrt{26}$.
4. No, it's not a parallelogram because opposite sides are not congruent: $AB = \sqrt{13}$, $CD = \sqrt{2}$, $BC = \sqrt{29}$, and $DA = 3\sqrt{2}$.
5. Yes, it's a parallelogram because the midpoints of the diagonals are the same, indicating the diagonals bisect each other: $M = \left(3, -\frac{3}{2}\right)$.
6. No, the midpoints are not the same, indicating the diagonals do not bisect each other: $M_{\overline{MO}} = (1, 1)$ and $M_{\overline{NP}} = \left(\frac{3}{2}, \frac{3}{2}\right)$.
7. $m\angle A = m\angle C = 165$ and $m\angle B = m\angle D = 15$; $x = 15$ and $y = 21$
8. $m\angle A = m\angle C = 92$ and $m\angle B = m\angle D = 88$; $x = 18$ and $y = 24$
9. Given that $\overline{AB}$ is parallel to $\overline{DE}$, we can use the Alternate Interior Angles Theorem to show that $m\angle BAH = m\angle DEH$. Given that $\overline{AD}$ is parallel to $\overline{BC}$, the same theorem indicates that $m\angle ABF = m\angle EAD$. From these pairs of angles, it can be seen that $\triangle ABF \sim \triangle EDA$. Using the full meaning of the triangles being similar, we can conclude that corresponding sides are in a constant ratio; or, in other words, $\frac{AB}{ED} = \frac{BF}{DA}$.
10. Opposite sides are parallel: $m_{\overline{AB}} = m_{\overline{CD}} = \frac{b}{a}$ and $m_{\overline{BC}} = m_{\overline{DA}} = 0$. The diagonals bisect each other because they have the same midpoint: $M = \left(\frac{a+c}{2}, \frac{b}{2}\right)$.

Practice 5.7.2: Proving Properties of Special Quadrilaterals, p. U5-298

1. Quadrilateral $ABCD$ is a parallelogram, a rectangle, a rhombus, and a square. Justification: opposite sides are parallel, consecutive sides are perpendicular, the diagonals bisect each other, and all four sides are congruent.
2. Quadrilateral $EFGH$ is a parallelogram and a rectangle. Justification: opposite sides are parallel, adjacent sides are perpendicular, the diagonals bisect each other, and not all four sides are congruent.
3. Quadrilateral $JKLM$ is an isosceles trapezoid. Justification: one pair of opposite sides is parallel and the other pair of sides is congruent.
4. Quadrilateral $NOPQ$ is a parallelogram. Justification: opposite sides are parallel, plus the diagonals are congruent and bisect each other.

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5. Quadrilateral *STUV* is a parallelogram and a rhombus. Justification: opposite sides are parallel, adjacent sides are not perpendicular, the diagonals are perpendicular, and all four sides are congruent.
6. Quadrilateral *WXYZ* is a parallelogram, a rhombus, and a square. Justification: opposite sides are parallel, adjacent sides are perpendicular, the diagonals are perpendicular, and all four sides are congruent.
7. Quadrilateral *ABCD* is a kite. Justification: adjacent sides are congruent and the diagonals intersect at a right angle.
8. Quadrilateral *FGHJ* is a parallelogram and a rectangle. Justification: opposite sides are parallel, adjacent sides are perpendicular, the diagonals bisect each other, and not all four sides are congruent.
9.

Statements	Reasons
1. Quadrilateral *ABCD* is a square.	1. Given
2. $\overline{DP} \cong \overline{PB}$ $\overline{AP} \cong \overline{PC}$	2. The diagonals are congruent and bisect each other.
3. $\overline{AC} \perp \overline{DB}$	3. The diagonals of a square are perpendicular.
4. $\angle APD \cong \angle APB \cong \angle BPC \cong \angle CPD$, and they are right angles.	4. Definition of perpendicular lines
5. $\angle APD \cong \angle APB \cong \angle BPC \cong \angle CPD$	5. All right angles are congruent.
6. $\triangle APD \cong \triangle APB \cong \triangle CPB \cong \triangle CPD$	6. SAS Congruence Statement

10. Given $\triangle APD \cong \triangle APB \cong \triangle CPB \cong \triangle CPD$, Corresponding Parts of Congruent Triangles are Congruent. This means that $\overline{AD} \cong \overline{AB} \cong \overline{CB} \cong \overline{CD}$. Also by CPCTC, $\angle PAD \cong \angle PCB$. This means that $\overline{BC} \parallel \overline{AD}$ since these are alternate interior angles. Then, by CPCTC again, $\angle BAP \cong \angle PCD$. This means that $\overline{AB} \parallel \overline{CD}$ because these angles are alternate interior angles. Since opposite sides are parallel and the sides are all congruent, by definition, quadrilateral *ABCD* is a rhombus.

Lesson 8: Exploring Trigonometric Ratios

Practice 5.8.1: Defining Trigonometric Ratios, p. U5-317

Answers will vary due to the variation that comes when drawing and measuring or rounding.

1. $\sin A \approx 0.866$; $\cos A = 0.5$; $\tan A \approx 1.732$
2. $\csc A \approx 1.155$; $\sec A = 2$; $\cot A \approx 0.577$
3. $\sin B = 0.5$; $\cos B \approx 0.866$; $\tan B \approx 0.577$
4. $\csc B = 2$; $\sec B = 1.155$; $\cot B \approx 1.732$
5. hypotenuse = 5; $m\angle A \approx 37°$; $m\angle B \approx 53°$; $\sin A = 0.6$; $\cos A = 0.8$; $\tan A = 0.75$; $\sin B = 0.8$; $\cos B \approx 0.6$; $\tan B \approx 1.333$
6. $b \approx 3.317$; $m\angle A \approx 56°$; $m\angle B \approx 34°$; $\sin A \approx 0.833$; $\cos A \approx 0.553$; $\tan A \approx 1.507$; $\sin B \approx 0.553$; $\cos B \approx 0.833$; $\tan B \approx 0.663$
7. The object is 0.288 miles deep; $\tan 30° \approx 0.577$.
8. perimeter ≈ 19.314 meters; $\cos 45° \approx 0.707$
9. distance ≈ 0.898 miles; $\tan 50° \approx 1.192$
10. ramp length ≈ 40.311 feet; $\sin 7.2° \approx 0.125$

Practice 5.8.2: Exploring Sine and Cosine As Complements, p. U5-329

1. approximately 0.707
2. $\cos 5° \approx 0.996$
3. $\theta = 38°$
4. $\theta = 88°$
5. $\sin 69° \approx 0.934$
6. $\cos 16° \approx 0.961$
7. $\theta = 60°$
8. $\sin A = \frac{a}{c} = \cos D = \frac{e}{f}$; $\sin B = \frac{b}{c} = \cos E = \frac{d}{f}$
9. 640 meters
10. $\sin A = 0.8$; $\cos A = 0.6$; $\sin B = 0.6$; $\cos B = 0.8$

Lesson 9: Applying Trigonometric Ratios

Practice 5.9.1: Calculating Sine, Cosine, and Tangent, p. U5-348

1. $\cos M = \frac{21}{29} \approx 0.724$; $\tan M = \frac{20}{21} \approx 0.952$
2. $\sin T = \frac{3}{5} = 0.6$; $\tan T = \frac{3}{4} 0.75$
3. 72°
4. 4.339 ft
5. 2.812 ft
6. 73°
7. 649.721 ft
8. $m\angle B = 40°$; $AB = 70.492$ cm; $AC = 45.311$ cm
9. $YZ = 6.708$ in; $m\angle X = 48.2°$; $m\angle Z = 41.8°$
10. 238.395 ft

Practice 5.9.2: Calculating Cosecant, Secant, and Cotangent, p. U5-360

1. 13.462
2. 3.294
3. 102.609
4. 14°
5. $\csc E = \frac{5}{3} \approx 1.667$; $\sec E = \frac{5}{4} \approx 1.25$; $\cot E = \frac{4}{3} \approx 1.333$
6. $\csc E = \frac{5}{3} \approx 1.667$; $\sec E = \frac{5}{4} \approx 1.25$; $\cot E = \frac{4}{3} \approx 1.333$
7. 1190.201 ft
8. 5°
9. $\csc B = \frac{31}{29} \approx 1.069$; $\sec B = \frac{29}{11} \approx 2.636$; $\cot E = \frac{11}{29} \approx 0.379$
10. $\csc B \approx 1.155$; $\sec B \approx 2.0$; $\cot B \approx 0.577$

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Practice 5.9.3: Problem Solving with the Pythagorean Theorem and Trigonometry, p. U5-375

1. 31°
2. 61°
3. 7°
4. 4,285 ft
5. 29 ft; 41 ft
6. 184 ft
7. 19°; 15 ft
8. 212 m; 216 m
9. 11°; 226 m
10. a. 72 m
    b. 175 m
    c. 85 m

Practice 5.9.4: Proving the Fundamental Pythagorean Identity, p. U5-386

1. $\frac{\sqrt{3}}{2}$
2. $\frac{\sqrt{6}}{3}$
3. $\frac{\sqrt{7}}{4}$
4. $\cos\theta = \frac{\sqrt{7}}{3}$; $\tan\theta = \frac{\sqrt{14}}{7}$; $\sec\theta = \frac{3\sqrt{7}}{7}$; $\csc\theta = \frac{3\sqrt{2}}{2}$; $\cot\theta = \frac{\sqrt{14}}{2}$; $\theta \approx 28.13°$
5. $\sin\theta = \frac{5}{17}$; $\tan\theta = \frac{15}{8}$; $\sec\theta = \frac{17}{8}$; $\csc\theta = \frac{17}{15}$; $\cot\theta = \frac{8}{15}$; $\theta \approx 61.93°$
6. $\cot\theta$
7. $\tan^2\theta$
8. $\frac{24}{25}$
9. 0.993
10. Substituting $1 - \cos^2\theta$ for $\sin^2\theta$ yields the following:
    $\sin^2\theta + 1 = 2 - \cos^2\theta$
    $1 - \cos^2\theta + 1 = 2 - \cos^2\theta$
    $2 - \cos^2\theta = 2 - \cos^2\theta$

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Instruction

Goal: To provide opportunities for students to develop concepts and skills related to proving that triangles are similar using scale factors and angle relationships

Common Core State Standards

G–SRT.1 Verify experimentally the properties of dilations given by a center and a scale factor:

b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

G–SRT.2 Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

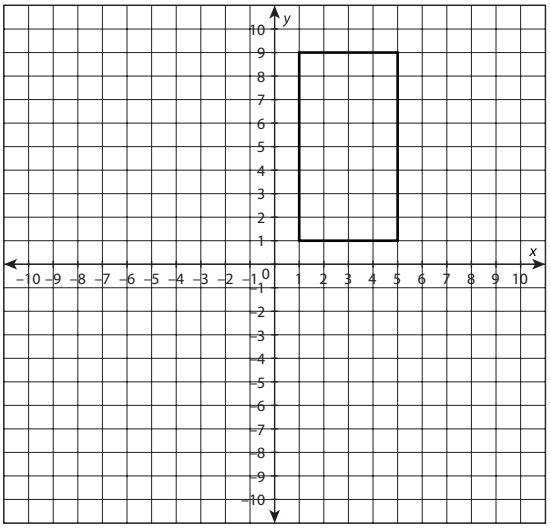
Student Activities Overview and Answer Key

Station 1

Students will be given graph paper, a ruler, a red marker, a green marker, and a blue marker. Students will construct similar and non-similar rectangles. They will determine ratios of corresponding sides to see if the rectangles are similar. Then they will find the relationship between perimeter and area with the corresponding sides of similar rectangles.

Answers

1. red rectangle:



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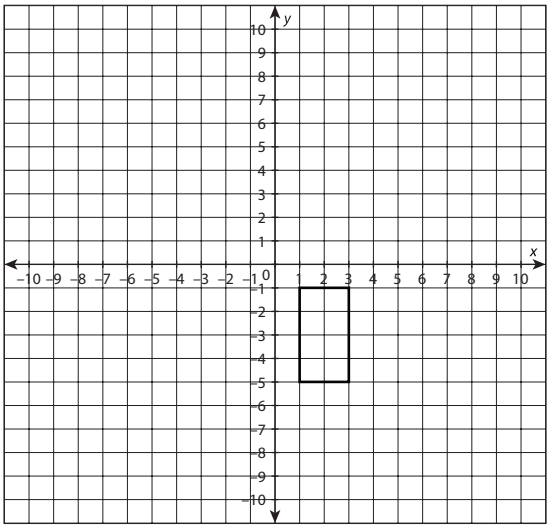
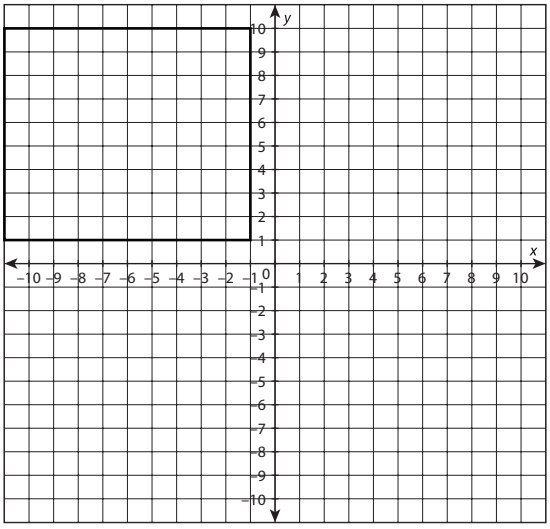
2. green rectangle:



3. blue rectangle:



4. 4/2; 8/4; 4/2 = 8/4
5. 4/9; 8/10; 4/9 ≠ 8/10
6. red and green
7. red = 24 units, green = 12 units, blue = 38 units
8. red = 32 square units, green = 8 square units, blue = 90 square units
9. The perimeter has the same ratio as the corresponding sides.
10. The area is equal to the square of the ratio of the corresponding sides.

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Instruction

Station 2

Students will be given a ruler and a protractor. They will measure the angles and lengths of two trapezoids. They will determine if the two trapezoids are similar. Students will derive the relationship between the angles in similar polygons.

Answers

1. $m\angle A = 56$, $m\angle B = 56$, $m\angle C = 124$, and $m\angle D = 124$
2. $m\angle M = 124$, $m\angle N = 124$, $m\angle O = 56$, and $m\angle P = 56$
3. No. It cannot be assumed that the trapezoids are similar based on the given angle measures because no information is given about the sides. The sides have to be proportional, in addition to the corresponding angles being congruent, in order for the trapezoids to be similar.
4. The base is 3 inches, the sides are 1.3 inches each, and the top is 1.7 inches.
5. The base is 1.5 inches, the sides are 0.65 inches each, and the top is 0.85 inches.
6. 2/1
7. Yes. 3/1.5 = 1.7/0.85 = 1.3/0.65
8. Answers will vary.
9. Answers will vary.
10. Corresponding angles are equal.

Station 3

Students will be given graph paper, a blue marker, a red marker, and a green marker. They will create similar rectangles using graph paper. Students will find the scale factors of the rectangles by physically placing them on top of each other. They will find the scale factors of other rectangles.

Answers

1. Student drawings should depict a blue rectangle that is 12 units long by 9 units wide.
2. Student drawings should depict a red rectangle that is 4 units long by 3 units wide.
3. 3; 3; 1/3; 1/3
4. 3; 1/3
5. 60 units long, 45 units wide
6. 2 units long, 1.5 units wide
7. No, because the rectangles aren't similar. No, because the rectangles aren't similar.
8. Answers will vary. Some examples include maps, diagrams, blueprints, and models.

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Instruction

Station 4

Students will find the missing side lengths of similar triangles. Students will discuss the strategy they used to determine whether triangles were similar or congruent. Students will explain why right triangles are not similar to obtuse triangles. Then they will explain why not all right triangles are similar.

Answers

1. $x = 16$
2. $x = 6$
3. $x = 3$
4. Answers will vary. Sample response: To find the missing side length, I created a proportion comparing the corresponding sides of the triangles.
5. No. Corresponding angles of similar triangles must have the same angle measures. Obtuse triangles will not have a right angle and, therefore, will not have the same angle measures as a right triangle.
6. No. Having one right angle does not guarantee that the remaining corresponding angles will be congruent and, therefore, the angles will not always be similar.

Materials List/Setup

Station 1 graph paper; ruler; red marker; green marker; blue marker

Station 2 ruler; protractor

Station 3 graph paper; scissors; blue marker; red marker; green marker

Station 4 none

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. What is the relationship between corresponding sides in similar polygons?
2. What is the relationship between corresponding angles in similar polygons?
3. What does it mean to scale up a figure by a factor of 4?
4. What does it mean to scale down a figure by a factor of 1/2?
5. How can you find the missing side length in similar triangles?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. All corresponding sides are proportional.
2. All corresponding angles are congruent.
3. Make the figure 4 times the size of the original.
4. Make the figure 1/2 the size of the original.
5. Use proportions of corresponding sides.

Possible Misunderstandings/Mistakes

- Incorrectly identifying corresponding sides of similar polygons
- Incorrectly setting up proportions between similar polygons
- Forgetting to multiply all sides of the polygon by the scale factor
- Incorrectly concluding that all right triangles are similar

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Station 1

At this station, you will find graph paper, a ruler, a red marker, a green marker, and a blue marker. Work as a group to construct the polygons and answer the questions.

1. On your graph paper, use the red marker to construct a rectangle that has vertices (1, 1), (5, 1), (1, 9), and (5, 9).

2. On your graph paper, use the green marker to construct a rectangle that has vertices (1, –1), (3, –1), (1, –5), and (3, –5).

3. On your graph paper, use the blue marker to construct a rectangle that has vertices (–1, 1), (–1, 10), (–11, 10), and (–11, 1).

4. What is the ratio between the shorter side of the red rectangle and the shorter side of the green rectangle?

   What is the ratio between the longer side of the red rectangle and the longer side of the green rectangle?

   What is the proportion between the shorter side and the longer side of the red and green triangles?

5. What is the ratio between the shorter side of the red rectangle and shorter side of the blue rectangle?

   What is the ratio between the longer side of the red rectangle and the longer side of the blue rectangle?

   What is the proportion between the shorter side and the longer side of the red and blue rectangles?

continued

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6. Which of the rectangles are similar? Write your answer as a proportion. Justify your answer by measuring and comparing the side lengths of each rectangle. Show your work and answer in the space below.

7. What is the perimeter of each rectangle? Show your work and answer in the space below.

8. What is the area of each rectangle? Show your work and answer in the space below.

9. Based on your observations in problems 1–8, what is the relationship between the perimeter and corresponding sides of similar rectangles?

10. Based on your observations in problems 1–8, what is the relationship between the area and the corresponding sides of similar rectangles?

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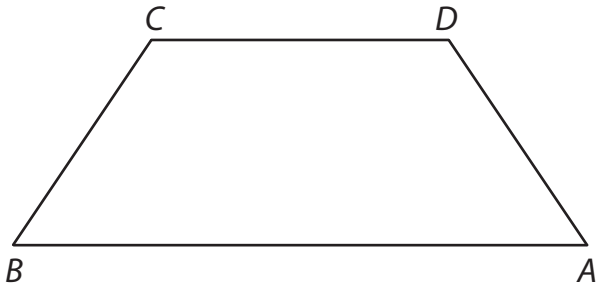
Station 2

At this station, you will find a ruler and a protractor. Work as a group to answer the questions.

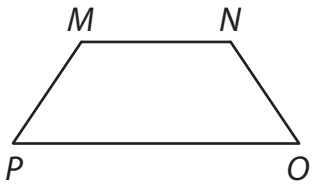
For problems 1 and 2, find the measure of each angle. Then, use your results to complete the remaining problems.

1.



$m\angle A$ = _____ $m\angle B$ = _____ $m\angle C$ = _____ $m\angle D$ = _____

2.



$m\angle M$ = _____ $m\angle N$ = _____ $m\angle O$ = _____ $m\angle P$ = _____

3. Can you say that the trapezoids in problems 1 and 2 are similar based solely on the measure of the angles? Why or why not?

4. What is the length of each side in problem 1?

5. What is the length of each side in problem 2?

continued

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6. What is the ratio of the corresponding sides between problems 1 and 2?

7. Are the trapezoids in problems 1 and 2 similar? Explain your answer by setting up a proportion.

8. In the space below, construct and label a trapezoid that is similar to the trapezoid in problem 1.

9. In the space below, construct and label a trapezoid that is NOT similar to the trapezoid in problem 1.

10. What is the relationship between the angles in similar trapezoids and other similar polygons?

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Station 3

At this station, you will find graph paper, scissors, a blue marker, a red marker, and a green marker. Work as a group to answer the questions.

1. On your graph paper, construct a rectangle that is 12 units long by 9 units wide. Shade in the rectangle with your blue marker.

2. On your graph paper, construct a rectangle that is 4 units long by 3 units wide. Shade in the rectangle with your red marker.

3. Use your scissors to cut out each rectangle from the graph paper. Place the red rectangle on the blue rectangle.

   How many red rectangles can you fit along the length of the blue rectangle?

   How many red rectangles can you fit along the width of the blue rectangle?

   Place the blue rectangle on the red rectangle.

   What fraction of the blue rectangle can you fit along the length of the red rectangle?

   What fraction of the blue rectangle can you fit along the width of the red rectangle?

4. In problem 3, you found the scale factors of the two rectangles.

   The blue rectangle was larger than the red rectangle by what scale factor?

   The red rectangle was smaller than the blue rectangle by what scale factor?

continued

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5. What would be the length and width of a rectangle that has the dimensions of the blue rectangle scaled up by a factor of 5?

6. What would be the length and width of a rectangle that has the dimensions of the red rectangle scaled down by a factor of $\frac{1}{2}$?

7. On your graph paper, construct a rectangle that is 10 units long by 2 units wide. Shade in the rectangle with your green marker.

   Use your scissors to cut this rectangle from the graph paper.

   Can you find the scale factor between the green rectangle and the blue rectangle? Why or why not?

   Can you find the scale factor between the green rectangle and the red rectangle? Why or why not?

8. What is an example of a real-world application of scale factors?

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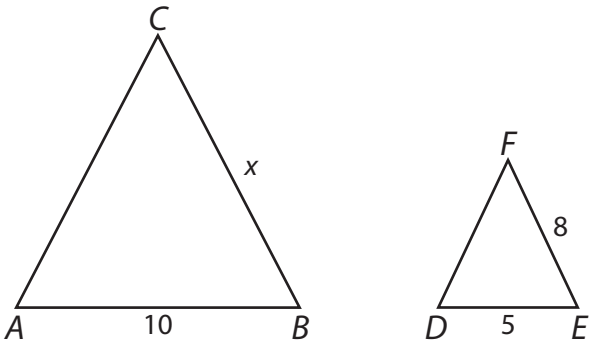
Station 4

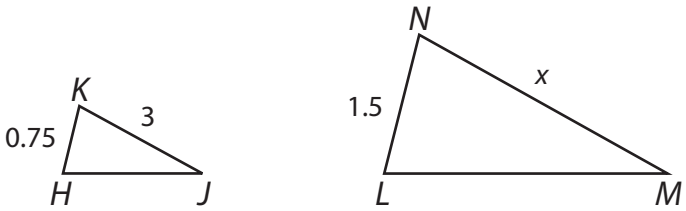
Work as a group to answer the questions.

For problems 1–3, the triangles are similar. Find the missing length, x. (Drawings are not to scale.)

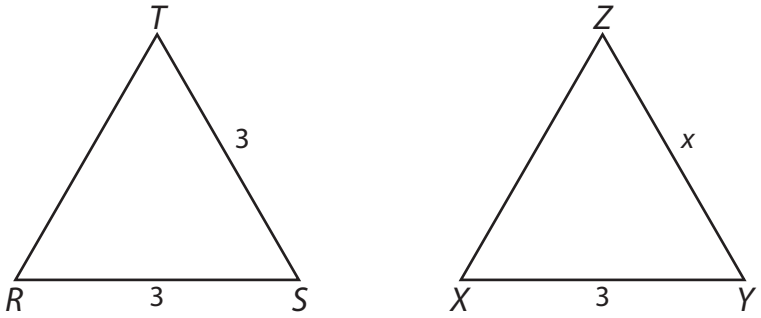
1.



2.



3.



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4. What strategy did you use to determine the missing side length, x, in each of the pairs of triangles on the previous page?

5. Is a right triangle similar to an obtuse triangle? Use your ruler and protractor to help explain your answer in the space below.

6. Are all right triangles similar? Use your ruler and protractor to help explain your answer in the space below.

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Instruction

Goal: To provide opportunities for students to develop concepts and skills related to identify and use the relationships between special pairs of angles formed by parallel lines and transversals

Common Core State Standard

G–CO.9 Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*

Student Activities Overview and Answer Key

Station 1

Students will be given graph paper, a ruler, and a protractor. Students will use the graph paper and ruler to model parallel lines cut by a transversal. They will use the protractor to find vertical angles. Then they will use the graph paper to model lines, which are not parallel, that are cut by a transversal. They will use the protractor to find vertical angles. They will realize that vertical angles can be found by both parallel and non-parallel lines cut by a transversal.

Answers

1. Answers will vary.
2. $\angle A \cong \angle C$; $\angle B \cong \angle D$
3. vertical angles
4. $\angle E \cong \angle H$; $\angle F \cong \angle G$
5. Answers will vary.
6. Answers will vary.
7. $\angle E \cong \angle H$; $\angle F \cong \angle G$
8. vertical angles
9. $\angle A \cong \angle D$; $\angle B \cong \angle C$
10. Answers will vary.
11. Both parallel and non-parallel lines have vertical angles when cut by a transversal.

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Instruction

Station 2

Students will be given graph paper, a ruler, and a protractor. Students will use the graph paper and ruler to model parallel lines cut by a transversal. They will use the protractor to find supplementary angles. Then they will use the graph paper to model lines, which are not parallel, that are cut by a transversal. They will use the protractor to find supplementary angles. They will describe which types of angles are supplementary angles when two lines are cut by a transversal.

Answers

1. $\angle A$ and $\angle B$; $\angle C$ and $\angle D$; $\angle E$ and $\angle F$; $\angle G$ and $\angle H$; $\angle A$ and $\angle C$; $\angle B$ and $\angle D$; $\angle E$ and $\angle G$; $\angle F$ and $\angle H$
2. Answers will vary.
3. $\angle A$ and $\angle B$; $\angle C$ and $\angle D$; $\angle E$ and $\angle F$; $\angle G$ and $\angle H$; $\angle A$ and $\angle C$; $\angle B$ and $\angle D$; $\angle E$ and $\angle G$; $\angle F$ and $\angle H$
4. Answers will vary.
5. adjacent angles; interior angles on the same side of the transversal (but only when the transversal intersects parallel lines)

Station 3

Students will be given uncooked spaghetti noodles, a protractor, graph paper, and a ruler. Students will use the graph paper and spaghetti noodles to model parallel lines cut by a transversal. They will use the protractor to measure the angles created by the transversal. Then they will explore and define the exterior, interior, alternate exterior, and alternate interior angles created by the transversal.

Answers

1. 8 angles
2. interior
3. exterior
4. 1, 2, 7, 8
5. 3, 4, 5, 6
6. Answers will vary.
7. $m\angle 1 = m\angle 7$; $m\angle 2 = m\angle 8$; $m\angle 3 = m\angle 5$; $m\angle 4 = m\angle 6$
8. 4 and 6; 3 and 5
9. 1 and 7; 2 and 8
10. equal; equal

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Instruction

Station 4

Students will be given uncooked spaghetti noodles, a protractor, and two parallel lines cut by a transversal. Students will use the spaghetti noodles to model the letter "F" to find corresponding angles. Then they will use the protractor to measure the angles and justify their answer.

Answers

1. 3 and 7
2. 1 and 5; 2 and 6; 3 and 7; 4 and 8
3. Answers will vary.
4. Answers will vary.

Materials List/Setup

Station 1 graph paper; ruler; protractor

Station 2 graph paper; ruler; protractor

Station 3 dry spaghetti noodles; protractor; graph paper; ruler

Station 4 dry spaghetti noodles; protractor

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. How many angles are created when parallel or non-parallel lines are cut by a transversal?
2. What are exterior angles?
3. What are interior angles?
4. What is a vertical angle?
5. What are alternate exterior angles?
6. What are alternate interior angles?
7. What two types of angles are supplementary when parallel lines are cut by a transversal?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. 8 angles
2. Exterior angles lie on the outside of the lines cut by the transversal.
3. Interior angles lie in between the two lines cut by the transversal.
4. Vertical angles are two angles formed by two intersecting lines that lie on opposite sides of the point of intersection.
5. Alternate exterior angles are pairs of angles on opposite sides of the transversal that are outside of the parallel or non-parallel lines.
6. Alternate interior angles are pairs of angles on opposite sides of the transversal that are inside the parallel or non-parallel lines.
7. Interior angles on the same side of the transversal and adjacent angles are supplementary in this situation.

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Instruction

Possible Misunderstandings/Mistakes

- Not finding corresponding angles correctly and identifying that they have the same measure if the lines cut by the transversal are parallel
- Mixing up interior and exterior angles
- Not realizing vertical angles are always equal whether or not the lines cut by the transversal are parallel
- Not realizing that alternate interior or exterior angles must be on the opposite side of the transversals

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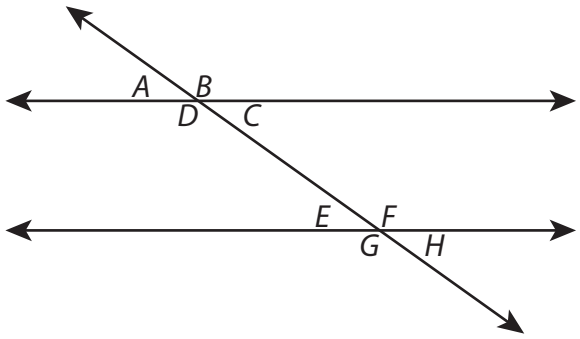
Station 1

At this station, you will find graph paper, a ruler, and a protractor. As a group, construct two parallel lines that are cut by a transversal and label the angles as shown in the diagram below.



1. Use the protractor to measure the following angles:

   $m\angle A =$ _____ $m\angle B =$ _____ $m\angle C =$ _____ $m\angle D =$ _____

2. Which pairs of angles are equal in problem 1?

3. The pairs of angles found in problem 2 have a special name. What are these angles called? (*Hint:* Think about their location in relation to each other.)

4. What pairs of angles are congruent for angles *E*, *F*, *G*, and *H*?

5. What strategy did you use to find the angle pairs in problem 4?

continued

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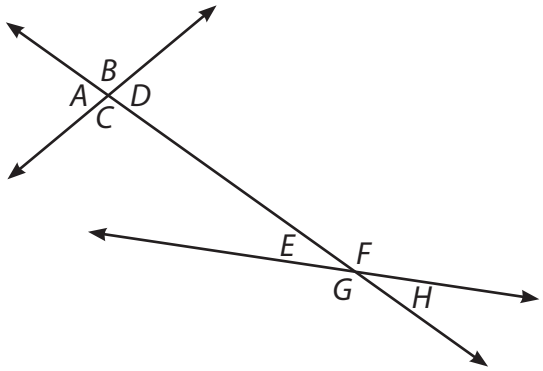
Station Activities Set 2: Parallel Lines and Transversals

On your graph paper, construct a new graph of two lines that are NOT parallel. These lines are cut by a transversal. Label the angles as shown in the diagram below.



6. Use the protractor to measure the following angles:

   $m\angle E =$ _____ $m\angle F =$ _____ $m\angle G =$ _____ $m\angle H =$ _____

7. Which pairs of angles are equal?

8. The pairs of angles found in problem 7 have a special name. What are these angles called? (*Hint:* Think about their location in relation to each other.)

9. What pairs of angles are congruent for angles *A*, *B*, *C*, and *D*?

10. What strategy did you use to find the angle pairs in problem 9?

11. Based on your observations in problems 1–10, vertical angles can be found for what types of lines cut by a transversal?

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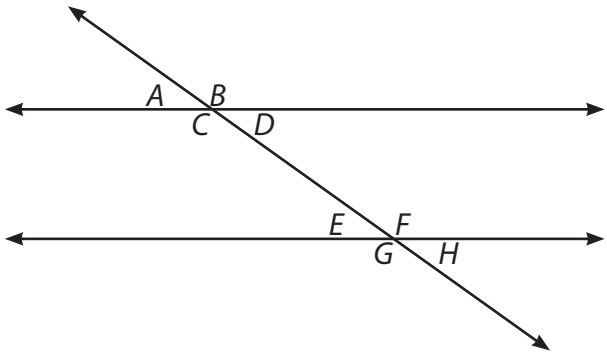
Station 2

At this station, you will find graph paper, a ruler, and a protractor. As a group, construct two parallel lines that are cut by a transversal and label the angles as shown in the diagram below.



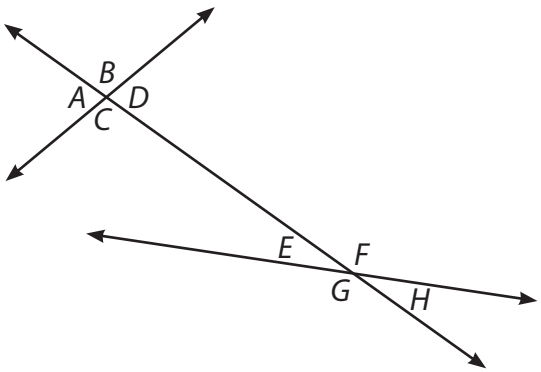
1. Which angles are supplementary angles?

2. Use your protractor to justify your answer to problem 1 by recording the measurements of each angle below.

   $m\angle A =$ _____ $m\angle C =$ _____ $m\angle E =$ _____ $m\angle G =$ _____

   $m\angle B =$ _____ $m\angle D =$ _____ $m\angle F =$ _____ $m\angle H =$ _____

On your graph paper, construct a new graph of two lines that are NOT parallel. These lines are cut by a transversal. Label the angles as shown in the diagram below.



3. Which angles are supplementary angles?

continued

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4. Use your protractor to justify your answer to problem 3 by recording the measurements of each angle below.

   $m\angle A =$ _____ $m\angle C =$ _____ $m\angle E =$ _____ $m\angle G =$ _____

   $m\angle B =$ _____ $m\angle D =$ _____ $m\angle F =$ _____ $m\angle H =$ _____

5. Based on your answers and observations in problems 1–4, which of the following types of angles are supplementary? Circle or underline any/all that apply.

   vertical angles adjacent angles alternate interior angles

   alternate exterior angles interior angles on the same side of the transversal

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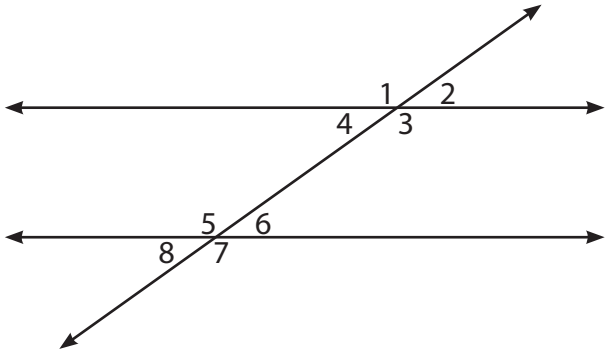
Station 3

At this station, you will find spaghetti noodles, a protractor, graph paper, and a ruler. Follow the directions below, and then answer the questions.

- On the graph paper, construct two parallel horizontal lines.
- Construct a diagonal line that passes through both parallel lines. This line is called a transversal.

1. How many angles does this transversal create with the two parallel lines?

On your graph paper, label the angles to model the parallel lines and transversal shown below.



Place a spaghetti noodle on each parallel line.

2. Are the angles between the spaghetti noodles called exterior or interior angles?

3. Are the angles above the top parallel spaghetti noodle and below the bottom spaghetti noodle called exterior or interior angles?

continued

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4. Which angles are exterior angles?

5. Which angles are interior angles?

6. Use your protractor to measure each angle. Write the angle measurements below.

   $m\angle 1 =$ _____ $m\angle 3 =$ _____ $m\angle 5 =$ _____ $m\angle 7 =$ _____

   $m\angle 2 =$ _____ $m\angle 4 =$ _____ $m\angle 6 =$ _____ $m\angle 8 =$ _____

7. Based on your answers in problem 6, which pairs of interior and which pairs of exterior angles are equal?

8. Based on your answers in problems 6 and 7, which angles are alternate interior angles?

9. Based on your answers in problems 6 and 7, which angles are alternate exterior angles?

10. Based on your observations, alternate interior angles have _____ measure.

    Alternate exterior angles have _____ measure.

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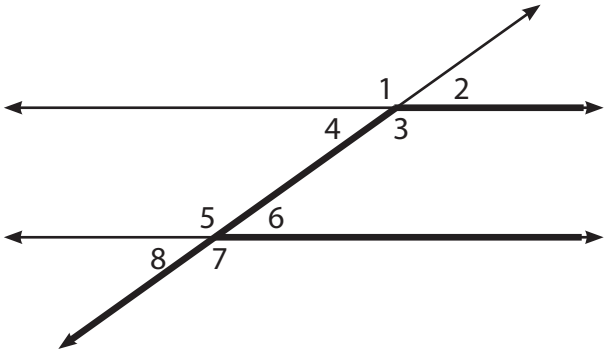
Station 4

At this station, you will find spaghetti noodles, a protractor, and parallel lines cut by a transversal.



The letter "F" has been drawn on the diagram above in a bold line. Place spaghetti noodles on the "F".

1. What angles are at the inside corners of the "F"?

   These are called corresponding angles.

2. Move the spaghetti noodles around to create more "F's" that help you find the corresponding angles. Write the pairs of corresponding angles in the space below.

3. What strategy did you use to move the letter "F" around to help you find the corresponding angles?

4. Use your protractor to measure each angle to justify your answers for problem 2.

   $m\angle 1 =$ _____ $m\angle 3 =$ _____ $m\angle 5 =$ _____ $m\angle 7 =$ _____

   $m\angle 2 =$ _____ $m\angle 4 =$ _____ $m\angle 6 =$ _____ $m\angle 8 =$ _____

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Instruction

Goal: To provide opportunities for students to develop concepts and skills related to describing and comparing relationships among quadrilaterals, including squares, rectangles, rhombi, parallelograms, trapezoids, and kites

Common Core State Standard

G–CO.11 Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

Student Activities Overview and Answer Key

Station 1

Students will be given four drinking straws, tape, a ruler, and a protractor. Students will construct a square. Then they will "lean" the square to create a rhombus. They describe why a rhombus is a parallelogram. Then they relate a square to a rhombus.

Answers

1. 90°
2. yes
3. yes
4. yes
5. yes
6. rhombus
7. A rhombus is like a square because all four sides are congruent.

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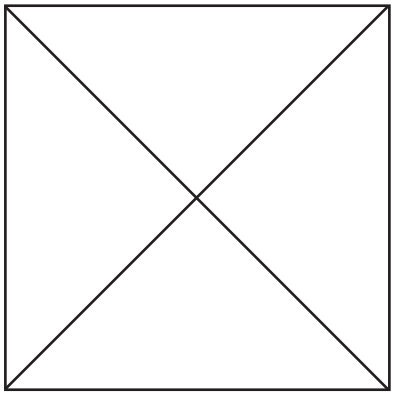
Instruction

Station 2

Students will be given a ruler and a protractor. Students will construct a square. They will find the diagonals of the square. Then they will relate the square to a rectangle, a rhombus, and a parallelogram.

Answers

1. 
2. by using the Pythagorean Theorem
3. $2\sqrt{2}$
4. A square is always a rectangle. A square is a rectangle with four equal sides.
5. A square is a rhombus with four equal angles.
6. A square is a parallelogram with equal diagonals that bisect the angles.

Station 3

Students will be given three drinking straws, a ruler, tape, and scissors. Students will create a kite from the drinking straws, and describe the sides, angles, and diagonals in the kite. They will determine if a kite is a parallelogram.

Answers

1. Answers will vary.
2. Answers will vary.
3. Answers will vary.
4. Two pairs of adjacent sides are congruent.
5. Draw the horizontal diagonal to make the base of each isosceles triangle.
6. A kite is not a parallelogram because opposite sides are not congruent.
7. Diagonals are perpendicular.

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Instruction

Station 4

Students will be given tracing paper, a ruler, and scissors. They will construct and describe a trapezoid, and explain if the trapezoid can be turned into a parallelogram. They will construct and describe an isosceles trapezoid. They will explain if the trapezoid can be turned into a parallelogram.

Answers

1. Yes, because it has four sides.
2. No, because opposite sides are not congruent.
3. No, because the sides of the trapezoid consist of two non-congruent triangles and a rectangle.
4. Yes, because it has four sides.
5. No, because opposite sides are not congruent.
6. Yes, because the trapezoid consists of two congruent isosceles triangles and a rectangle; rectangle

Materials List/Setup

Station 1 four drinking straws; tape; ruler; protractor

Station 2 ruler; protractor

Station 3 three drinking straws; ruler; tape; scissors

Station 4 tracing paper; ruler; scissors

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. What is a rhombus? Is it a parallelogram?
2. How does a square relate to a rhombus?
3. What is a kite? Is it a parallelogram?
4. What is a trapezoid? Is it a parallelogram?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. A rhombus is a quadrilateral with all four sides equal and opposite angles equal. Yes, it is a parallelogram.
2. A square is a rhombus with four right angles.
3. A kite is a quadrilateral that has two pairs of adjacent sides that are equal. No, it is not a parallelogram.
4. A trapezoid is a quadrilateral that has one pair of parallel sides. No, it is not a parallelogram.

Possible Misunderstandings/Mistakes

- Not recognizing that a square is a type of rhombus
- Not recognizing that all sides are equal in a rhombus
- Not realizing that all squares are rectangles, but not all rectangles are squares
- Not recognizing that pairs of adjacent sides in a kite have equal length

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Station 1

At this station, you will find four drinking straws, tape, a ruler, and a protractor. Work together to answer the questions.

1. Tape the four drinking straws together to create a square. What are the measures of each interior angle?

2. "Lean" the straws so that you create interior angles of 120°, 60°, 120°, and 60°. Are the lengths of the sides still equal? Why or why not?

3. Are pairs of opposite sides congruent?

4. Are pairs of opposite angles congruent?

5. Is this figure a parallelogram? Why or why not?

6. What is the special name for this quadrilateral?

7. How does this quadrilateral relate to a square?

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Station 2

At this station, you will find a ruler and a protractor. Work together to construct the figures and answer the questions.

1. In the space below, construct a square with side lengths of 2 inches. Construct the diagonals on the square.

2. How can you find the length of each diagonal without using your ruler?

3. What is the length of each diagonal? Verify your answer with your ruler.

4. Is a square always a rectangle or is a rectangle always a square? Justify your answer.

continued

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5. How can you relate a square to a rhombus?

In the space below, draw a square and rhombus to show how they relate to each other.

6. How can you relate a square to a parallelogram?

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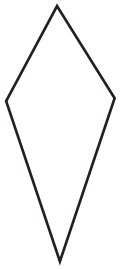
Station 3

At this station, you will find three drinking straws, a ruler, tape, and scissors. Work as a group to create a kite.

1. Cut one straw exactly in half. Tape the four straws together to create a kite similar to the image below.



2. What is the length of each side?

3. What is the measure of each angle?

4. What is the relationship between adjacent sides of a kite?

5. How can you create two isosceles triangles from the kite?

6. Is a kite a parallelogram? Why or why not?

continued

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7. Place your straw kite in the space below. Draw the diagonals of the kite. What relationship do the diagonals have to each other?

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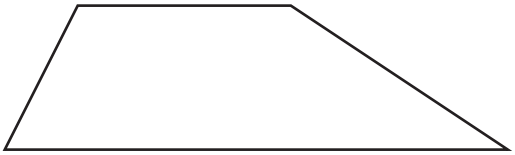
Station 4

At this station, you will find tracing paper, a ruler, and scissors. Work as a group to answer the questions.

Given the following trapezoid:



Use the tracing paper to trace the trapezoid above. Cut the trapezoid out of the tracing paper.

1. Is this trapezoid a quadrilateral? Why or why not?

2. Is this trapezoid a parallelogram? Why or why not?

   If so, what type of parallelogram did you create?

3. Can you cut the trapezoid into two pieces and create a parallelogram? Why or why not?

continued

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Given the following trapezoid:

Use the tracing paper to trace the trapezoid above. Cut the trapezoid out of the tracing paper.

4. Is this trapezoid a quadrilateral? Why or why not?

5. Is this trapezoid a parallelogram? Why or why not?

6. Can you cut the trapezoid into two pieces and create a parallelogram? Why or why not?

If so, what type of parallelogram did you create?

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Station Activities Set 4: Sine, Cosine, and Tangent Ratios, and Angles of Elevation and Depression

Instruction

Goal: To provide opportunities for students to develop concepts and skills related to trigonometric ratios for right triangles and angles of elevation and depression

Common Core State Standards

G–SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

G–SRT.7 Explain and use the relationship between the sine and cosine of complementary angles.

G–SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.★

Student Activities Overview and Answer Key

Station 1

Students will be given a graphing calculator, a ruler, and a protractor. Students will find the sine, cosine, and tangent ratios in a right triangle. Then they will find the measure of each angle in the triangle using these trigonometric ratios.

Answers

1. $\frac{3}{5}$
2. $\frac{4}{5}$
3. $\frac{3}{4}$
4. $\frac{4}{5}$
5. $\frac{3}{5}$
6. $\frac{4}{3}$
7. $\alpha = 36.9^\circ$; $\beta = 53.1^\circ$

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Instruction

Station 2

Students will be given a graphing calculator. Students will work together to find the length of a missing side given the length of one side and an angle. Then they will use the trigonometric ratios to find the length of the missing side.

Answers

1. $\sin 50^\circ = \frac{x}{12}$
2. $\sin 50^\circ = {}^{x}\!/\!_{12}$; $0.766 = {}^{x}\!/\!_{12}$; $x = 9.19$
3. $\sin 45^\circ = \frac{1}{\sqrt{2}}$
4. $x = 12.85$
5. $x = 20.31$
6. $x = 2.13$

Station 3

Students will be given three drinking straws, a graphing calculator, scissors, tape, and a ruler. Students will construct a right triangle out of the drinking straws. They will use it to model an angle of elevation problem. They will use trigonometric ratios of right triangles to find the angle of elevation.

Answers

1. Answers will vary. Sample answer: 5 and 7 inches
2. Answers will vary. Sample answer: approximately 8.6 inches
3. Answers will vary; leg
4. Answers will vary. Sample answer: 8.6 inches; hypotenuse
5. Answers will vary. Sample answer: 7 inches; leg
6. Answers will vary. Possible method: $\tan(x) = 5/7$; $x = 35.5$

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Station Activities Set 4: Sine, Cosine, and Tangent Ratios, and Angles of Elevation and Depression

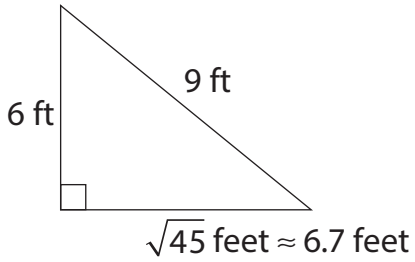
Instruction

Station 4

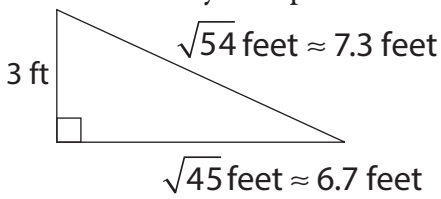
Students will be given a tape measure and tape. Students physically model an angle of depression. They use trigonometric ratios of right triangles to find the angle of depression.

Answers

1. Answers will vary. Sample answer:



2. Answers will vary. Sample answer: $\cos(x) = 6/9$; $\cos^{-1} = \cos^{-1}(6/9)$; $x \approx 48.2$
3. Answers will vary; Sample answer: It is the angle that is "looking downward."
4. Answers will vary. Sample answer:



5. Answers will vary. Sample answer: $\cos(x) = 3/7$; $\cos^{-1} = \cos^{-1}(3/7)$; $x \approx 64.6$
6. The angle of depression found in problem 2 is less than the one found in problem 5.

Materials List/Setup

Station 1 graphing calculator, ruler, protractor

Station 2 graphing calculator

Station 3 three drinking straws, graphing calculator, scissors, tape, ruler

Station 4 tape measure, tape

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Station Activities Set 4: Sine, Cosine, and Tangent Ratios, and Angles of Elevation and Depression

Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. What are the sine, cosine, and tangent trigonometric ratios for right triangles?
2. How do you find the length of a side of a right triangle given one side and an angle?
3. What is the angle of elevation?
4. How do you find the angle of elevation?
5. What is the angle of depression?
6. How do you find the angle of depression?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}; \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}; \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$
2. Use the appropriate trigonometric ratio to write an equation and solve the equation for the missing side.
3. The angle of elevation is the angle between the horizontal and the line of sight of an object above the horizontal.
4. Use the information to draw a right triangle. Then use the appropriate trigonometric ratio to write an equation. Solve the equation for the missing value.
5. The angle of depression is the angle between the horizontal and the line of sight of an object below the horizontal.
6. Use the information to draw a right triangle. Then use the appropriate trigonometric ratio to write an equation. Solve the equation for the missing value.

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Instruction

Possible Misunderstandings/Mistakes

- Incorrectly identifying the opposite side and adjacent side when writing trigonometric ratios
- Incorrectly writing the trigonometric ratios
- Mixing up the angle of elevation with the angle of depression
- Mixing up when to use the trigonometric function and when to use the inverse of the function when finding missing side lengths or angles

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UNIT 5 • SIMILARITY, RIGHT TRIANGLE TRIGONOMETRY, AND PROOF
Station Activities Set 4: Sine, Cosine, and Tangent Ratios, and Angles of Elevation and Depression

Station 1

You will be given a graphing calculator, a ruler, and a protractor. Work together to answer the questions.

In the space below, construct a right triangle with legs that are 3 inches and 4 inches in length, and a hypotenuse that is 5 inches in length. Label the angle opposite the shorter leg as α (*alpha*). Label the angle opposite the longer leg as β (*beta*).

The triangle is a right triangle; you can write trigonometric ratios to represent the values of α and β. Review the trigonometric functions:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}};\ \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}};\ \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

1. Find $\sin(\alpha)$.
2. Find $\cos(\alpha)$.
3. Find $\tan(\alpha)$.
4. Find $\sin(\beta)$.
5. Find $\cos(\beta)$.
6. Find $\tan(\beta)$.

You can use the trigonometric ratios and your graphing calculator to find the measure of angle α.

7. Find the measures of α and β in the triangle you drew.

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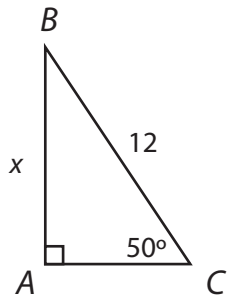
Station Activities Set 4: Sine, Cosine, and Tangent Ratios, and Angles of Elevation and Depression

Station 2

You will be given a graphing calculator. Work as a group to answer the questions.



1. How can you write the sine ratio for the 50° angle in the right triangle above?

2. Find the value of x using this sine ratio. (*Hint*: First, use your calculator to find sin 50°.) Show your work and answer in the space below.

3. In the space below, construct a 45°–45°–90° triangle. Use your knowledge of the 45°–45°–90° right triangles and trigonometric ratios to prove that $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$.

continued

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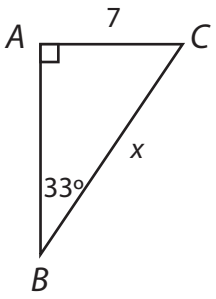
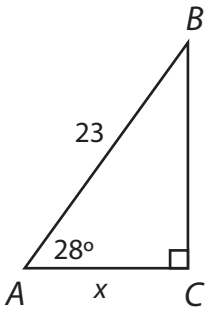
Station Activities Set 4: Sine, Cosine, and Tangent Ratios, and Angles of Elevation and Depression

For problems 4–6, find the value of x in each right triangle. Round your answer to the nearest hundredth.

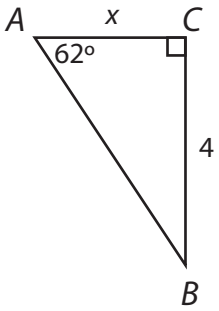
4.



5.



6.



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Station Activities Set 4: Sine, Cosine, and Tangent Ratios, and Angles of Elevation and Depression

Station 3

You will be given three drinking straws, a graphing calculator, scissors, tape, and a ruler. Work as a group to solve the following problem.

Construct a small right triangle out of the three drinking straws. Use the tape to secure the straws together. Then trim off the excess straws not in the right triangle.

1. What are the lengths of the legs of the triangle?

2. What is the length of the hypotenuse of the triangle?

Imagine that one of the straws in your triangle is a tree. This tree makes a 90° angle with the ground. The tree casts a shadow on the ground. Tape your triangle to your table so it remains upright to represent this situation.

3. What is the height of the straw that represents your tree? Is this a leg or hypotenuse of the triangle? Explain your answer.

4. What is the length of the straw that represents the shadow cast from the top of the tree to the ground? Is this a leg or hypotenuse of the triangle? Explain your answer.

5. What is the length of the straw that represents the distance between the tree and the point on the ground where the shadow ends? Is this a leg or hypotenuse of the triangle? Explain your answer.

6. What is the angle of elevation from the end of the shadow to the top of the tree? Show your work and answer on a separate sheet of paper. (*Hint*: Use trigonometric ratios.)

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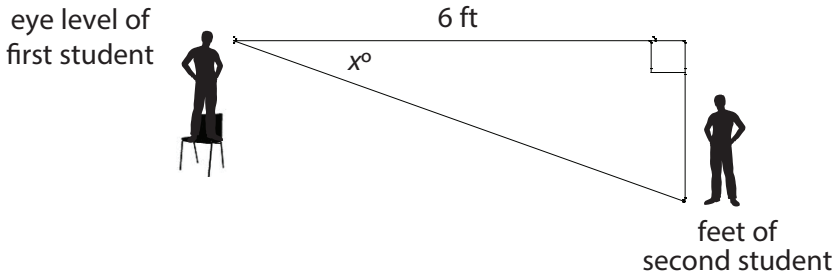
Station 4

You will be given a tape measure and tape. Work together to solve the following problem.

Have two students stand 6 feet away from each other. Have the first student stand on a chair. Model the diagram below.



1. Measure the distance between the eye level of the first student and the feet of the second student. What is this distance?

2. Find the angle of depression, represented by x in the diagram, using trigonometric ratios. Show your work and answer in the space below.

3. The angle represented by x in the diagram represents the angle of depression. Why do you think the angle has this name?

continued

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Repeat the procedure used in problems 1 and 2 by having two students stand 3 feet apart.

4. Measure the distance between the eye level of the first student and the feet of the second student. What is this distance?

5. Find the angle of depression, represented by x in the diagram, using trigonometric ratios. Show your work and answer in the space below.

6. How does your answer in problem 2 compare with your answer in problem 5?

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APPENDIX

Problem-Based Task Implementation Guides

Introduction

This addendum contains a collection of Implementation Guides (IGs) designed to support the use of the Problem-Based Tasks found throughout this unit.

Each Implementation Guide includes an overview and suggestions for introducing and facilitating the task. These contain specific recommendations for selected Standards for Mathematical Practice (SMPs) to highlight, guidance on addressing common errors and misconceptions associated with the task, and ideas for monitoring and coaching students as they work through the task. Finally, there is guidance for debriefing the task, with special attention to the SMPs and relevant Key Concepts from the instruction, as well as information on suitable extensions and alternate strategies.

Throughout the Implementation Guides, suggestions are provided in a question/prompt format to support your efforts to elicit student thinking and facilitate classroom discourse.

We developed these Implementation Guides for our Problem-Based Tasks at the request of math educators, and with advice and feedback from mathematics supervisors and teachers. Please let us know if they are helpful. Visit us at www.walch.com, follow us on Twitter (@WalchEd), or e-mail suggestions to customerservice@walch.com.

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Problem-Based Task 5.1.1: Points of Shade

Common Core State Standard

G–GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

Task Overview

Focus

How can the location of an object be found in relation to other objects, by using coordinates to represent the objects? How can mathematical formulas be applied to determine midpoints and specific distances between two objects? In this task, students will model a real-world scenario by using coordinates to represent the locations of objects, and they will use the midpoint and distance formulas to determine distances between these objects.

This activity will provide practice with:

- translating verbal information
- defining everyday terms and correlating them to mathematical terminology
- identifying the correct formulas to use based on the context of a scenario
- applying the midpoint and distance formulas
- calculating distances and midpoints between points
- explaining results in the context of a scenario

Introduction

This task should be used to explore or to apply the skill of finding the point on a directed line segment between two given points that partitions the segment in a given ratio. Students should already be familiar with plotting points on a coordinate plane and calculating distances in the coordinate plane.

Begin by reading the problem and clarifying the meaning of *distance formula, halfway*, *midpoint*, and *midpoint formula*.

distance formula formula that states the distance between (x_1, y_1) and (x_2, y_2) is equal to $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

halfway the midpoint of a segment

midpoint a point on a line segment that divides the segment into two equal parts

midpoint formula formula that states the midpoint of a segment created by connecting (x_1, y_1) and (x_2, y_2) is given by the formula $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine the location of the acacia tree in the park, as well as the distance from the tree to various lion prides located in the park. They will recognize that the location of the tree can be found by applying the midpoint formula, since the tree is located halfway between two watering holes, whose locations are represented by coordinates. They will also recognize that the distance formula can be used to determine the distance between the tree and two prides of lions. They will convert the results of the distance to feet, and they will draw conclusions about the location of the tree in the context of the scenario.

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they make sense of the goal of the task by first recognizing that mathematical formulas can be used to determine the locations and distances of the acacia tree in regards to the specified prides of lions. They will examine the given coordinates and determine which points represent the correct coordinates to substitute into the midpoint and distance formulas. They will reason that the distance found is the same from both prides to the tree, based on their calculations. They will reason quantitatively as they substitute coordinates into the midpoint and distance formulas, and they will simplify the calculations and then convert the results to units of feet.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled by using a coordinate plane and by applying the midpoint and distance formulas. They will analyze the given information and recognize that the coordinates represent the locations of the lion prides, the watering holes, and the acacia tree in the park. They will apply their knowledge of the formulas and use them to determine the locations of the objects, in terms of coordinates, and the distances between the locations of the objects.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- misidentifying x_1, x_2 and y_1, y_2

  Have students write down each set of coordinates used in the task. For each set that is used in a formula, remind them to label each point above the numbers with x_1, x_2 and y_1, y_2. Remind them to be careful when substituting each coordinate into a formula and double-check their work.

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- attempting to use the distance formula to locate the midpoint

  Have students write both the midpoint and the distance formulas on their papers. Review the definitions and meaning of both, including in which cases each formula would apply.

- incorrectly choosing a different point for the distance formula when calculating the distance between the acacia tree and the lion pride located at (6, 2)

  Have students make a list of each set of coordinates given in the stem of the task and write down what each point represents in the context of the scenario.

- incorrectly representing the distance between objects

  Remind students that in the stem of the task, it is stated that each unit represents 1,000 feet. Discuss how the results of the calculations for the distance between objects will need to be converted to feet by multiplying the values by 1,000.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "What other words can you think of that have the same meaning as "halfway"?" (**Answer:** Answers will vary. Some examples include the following: middle, center, mid, central, midway.)
- After students have read through the stem of the task, ask them, "What is a way you can visually represent the information given in the problem before you start your calculations? (**Answer:** Since the locations of the lion prides and the watering holes are given as coordinates, these points can be graphed on a coordinate plane. This might help students have a better understanding of the problem before they begin.)
- Once students have found the midpoint between the two watering holes, ask them, "Based on the calculations from the midpoint formula, the midpoint of the two watering holes is located at $\left(\frac{9}{2}, -\frac{5}{2}\right)$. What other formats could you change the coordinates of this point to in order to make it easier to plot on the coordinate plane? What would the new points be in these other formats?" (**Answer:** The coordinates are stated as improper fractions, and they can be changed to mixed numbers or decimals. The two equivalent formats are $\left(4\frac{1}{2}, -2\frac{1}{2}\right)$ and (4.5, –2.5) .)

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- Once students have simplified their calculations in the distance formula and have a result of $\frac{3\sqrt{10}}{2}$, ask them, "What are the steps for simplifying the radical so that you have an answer that better represents the context of the scenario? How is this answer converted to feet?

  (**Answer:** The first step is to simplify the numerator by finding the product of 3 and $\sqrt{10}$. That product is then divided by 2: $\frac{3 \bullet \sqrt{10}}{2} = \frac{3 \bullet 3.16228}{2} = \frac{9.48684}{2} = 4.74342$. Since 1 unit represents 1,000 feet, the result is multiplied by 1,000 for a final result of 4,743.4, rounded to the nearest tenth of a foot.)
- Ask students, "What conclusion can you make about the location of the acacia tree in terms of the locations of the prides at points (6, 2) and (3, –7), and why?" (**Answer:** The acacia tree is located the same distance from each of the prides at the two points, so therefore the acacia tree has to be the midpoint between the prides.)
- Ask students to consider the reality of the scenario in terms of the objects in the task. Ask them, "What is the only way to ensure that the lions from each pride travel the same distance as the lions from other prides to the tree? Is this realistic or not? (**Answer:** The only way to ensure that the lions from different prides travel the same distance to the tree is if they all walked in a straight line, representing the line segment from the location of their pride to the location of the tree. It is not very realistic to assume that lions will always walk in a straight line and not roam around.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts and ideas about how to proceed on the task, given the scenario that the locations of the objects were represented by coordinates. Discuss how graphing the points on a coordinate plane can help visual the problem and can help in understanding the goal of the task. Also discuss their choices for coordinates when applying the midpoint and distance formulas. Ask them to explain how the resulting coordinate of the location of the acacia tree apply to the context of the scenario, specifically using it to determine distances. Also discuss the necessity of converting their results to units of feet. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.

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- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and procedure for identifying the correct coordinates and substituting the correct values into the midpoint and distance formulas. Ask them how they applied their knowledge of the definition of midpoint and how this knowledge translated to determining locations of the acacia trees. Focus on the use of precise mathematical language and clarity, specifically when describing the midpoints and distances between objects, as well as how the coordinates are substituted into the appropriate formulas.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The length of a line segment can be found using the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

  In this task, students will apply the distance formula when determining the distance between the acacia tree and the lion prides located at given points. They will substitute the values for x_1, x_2 and y_1, y_2 into the formula and perform the calculations.

- The **midpoint** of a line segment is the point on the segment that divides it into two equal parts.

  Students will translate the term given in the task – "halfway" – and recognize that the halfway point is the midpoint of a segment, where the endpoints of the segment represent two of the objects in the task.

- The **midpoint formula** is used to find the midpoint of a line segment. The formula is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

  Students will apply the midpoint formula by substituting in values for x_1, x_2 and y_1, y_2, which represent the locations of the two watering holes, and they will perform the calculations in order to get the resulting coordinate representing the midpoint.

Extending the Task

- To extend the task, ask students to determine the distance of the acacia tree from the other two prides, whose locations are represented by the (–4, –1) and (1, 5). Ask them to follow the same procedures when applying the distance formula, carefully substituting in the correct values for the four coordinates, x_1, x_2 and y_1, y_2. Have them compare their answers with the another classmate's answers, and ask them to draw conclusions about the distance of these prides from the acacia tree, to the distance of the other two prides in the task. Discuss with students the context of the scenario and which prides are closer to the watering hole, as well as which prides are closer to the trees. Ask students to suggest alternate locations of the watering hole and the trees that could possibly benefit all four prides as equally as possible.

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- Another option for extending the task is to ask students to work with a partner to create their own real life scenario, in which locations of four objects can be represented with coordinates. Ask them to create and plot their coordinates on a coordinate plane such that the points form a rectangle when the line segments are drawn. Ask them to find the midpoints between each set of opposite corners of the rectangle, first making predictions about the location of the midpoint of each line segment which joins the opposite corners. Discuss their results and conclusions, as well as how well their results matched their predictions. (As a further extension, ask students to create four points which make a different quadrilateral, such as a rhombus or trapezoid, and discuss predictions and results about the midpoint of the quadrilateral.)

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by recognizing that the goal of the task is to determine the location of the acacia tree in the park, as well as the distance from the tree to various lion prides located in the park. I recognized that the location of the tree could be found by applying the midpoint formula, since the tree is located halfway between two watering holes, whose locations are represented by coordinates. I also recognized that the distance formula could be used to determine the distance between the tree and two prides of lions. I converted the results of the distance to feet, and I analyzed these results and made conclusions about the location of the tree in the context of the scenario.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first made sense of the goal of the task by recognizing that mathematical formulas could be used to determine the locations and distances of the acacia tree in regards to the specified prides of lions. I examined the given coordinates and determined which points represented the correct coordinates to substitute into the midpoint and distance formulas. I reached the conclusion that the distance found is the same from both prides to the tree, based on my calculations.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning when I substituted coordinates into the midpoint and distance formulas, and simplified the calculations, and finally converted the results to units of feet.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this scenario by recognizing that the scenario could be modeled by using a coordinate plane and by applying the midpoint and distance formulas. I analyzed the given information and recognized that the coordinates represented the locations of the lion prides, the watering holes, and the acacia tree in the park. I applied my knowledge of the formulas and used them to determine the locations of the objects, in terms of coordinates, and the distances between locations of these objects.)

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Alternate Strategies or Solutions

Some students may choose to initially graph the given coordinates for the locations of the lion prides and watering holes on a coordinate plane. They may choose to create a key depicting what each point on the graph represents. Some students may use a ruler to get an estimate of the midpoints between locations. Remind students that while a ruler can provide a reasonable estimate of the midpoint, only calculating the midpoint with a formula will provide accurate results. Discuss how the graph can help students get a sense of the reasonableness of the results of their calculations.

Technology

Students can use scientific calculators in order to simplify the radical answers.

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Problem-Based Task 5.2.1: Prettying Up the Pentagon

Common Core State Standard

G–SRT.1 Verify experimentally the properties of dilations given by a center and a scale factor:

a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.

Task Overview

Focus

How do the scale factors for the side lengths of similar figures compare to the scale factor for the perimeters of those figures? Students will use measurements from the outer and inner walls, as well as the walkway at the pentagon to find how they can use the ratio of the side lengths to find the ratio of the perimeters.

This activity will provide practice with:

- examining similar figures
- comparing the ratios of side lengths and perimeters for similar figures
- interpreting the results of calculations

Introduction

This task should be used to explore the relationship between the scale factor and the perimeters of similar figures. It is best to implement this task while students are learning about the properties of dilations.

Begin by reading the problem and explaining the concept of updating the courtyard. Review the following terms:

borders	the parts or edges of a surface or area that form its outer boundary
courtyard	an enclosed area open to the sky
landscapers	people who design outside spaces, including the plants
temporary	not permanent

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will analyze the diagram to understand how to determine the ratios of the side lengths and perimeters.

- **SMP 2:** Reason abstractly and quantitatively.

  Using the ratios of side lengths and perimeters, students will draw a conclusion about the perimeters of similar figures.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  After looking at the ratios of side lengths and perimeters of more than one pair of similar figures, students generalize their findings as a property of similar figures.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- inconsistently setting up the ratio of side lengths

  Encourage students to write word ratios before working with the actual dimensions of the figures. A simple word ratio, such as *small : big*, can help them keep their calculations consistent.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students cannot find the lengths or perimeters, ask, "What is the scale factor? How can it help you?" (**Answer:** The scale factor from the inner wall to the walkway is 0.25. Multiply the inner wall by 0.25 to find the length of the walkway.) Ask, "How can you find the perimeter?" (**Answer:** Multiply the side lengths by 5.)
- If students struggle to find the ratio of the perimeters, ask, "How could you find the ratio of the perimeters?" (**Answer:** Divide the perimeter of the walkway by the perimeter of the inner wall or the perimeter of the inner wall by the perimeter of the outer wall.)
- If students struggle to draw a conclusion, ask, "How do the ratios of the side lengths and the ratio of the corresponding perimeters compare?" (**Answer:** They are equal.) Provide students with a couple of example questions, like "If the ratio of the side lengths is 0.4, what is the ratio of the perimeters?" or "If the ratio of the perimeters is 1.2, what is the ratio of the side lengths?"
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

Ask students to discuss their conclusions. Have them expand upon their conclusions by explaining why it holds true.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Use a ratio of corresponding sides to find the scale factor.

  Students use a ratio of the sides of the walls or walkway to find the scale factor.

- All sides of similar figures are changed by a single scale factor, k.

  Knowing that each side of the similar figures will be changed by the same scale factor, students better understand that the perimeter will have the same scale factor.

Extending the Task

To extend the task, change the parameters of the task. For example, have students use different shapes. Ask them to use the ratio of the perimeters to solve a problem, and find the ratio of the areas of figures.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I found the ratios of the sides and then the perimeters for more than one figure.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I drew a conclusion from my data.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I found the ratios of side lengths in similar figures as well as their perimeters.)
- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** Based on my repeated results, I drew a conclusion about the ratio of the perimeters of similar figures.)

Alternate Strategies or Solutions

Students could have reasoned more abstractly about the ratios of the side lengths and the perimeters. Since the perimeter is just the sum of the side lengths, the ratios should be the same. Then, based on that conclusion, found the distances requested.

Technology

Students can use a calculator to find the ratios of the side lengths and the perimeters.

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Problem-Based Task 5.3.1: Video Game Transformations

Common Core State Standard

G–SRT.2 Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

Task Overview

Focus

Throughout the course of a video game, characters move, requiring the programmer to track their movements. Students will use similarity transformations to identify the position of a character in a video game.

This activity will provide practice with:

- comparing lengths of line segments after transformations
- analyzing movements on a coordinate plane
- interpreting the results of the transformations
- examining the results to find the location of the third point

Introduction

This task should be used to explore similarity transformations. It is best to implement this task as students learn to apply the properties of similar triangles.

Begin by reading the problem and clarifying the meaning of creating a video game. Review the following terms:

creators	makers
design stages	steps in the process of making something
setting	the background image(s) of the game, where the action occurs

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will take the idea of character movement in a video game and apply it to graphing similar triangles on a coordinate plane. They will analyze lengths of segments to find the location of a point after the transformations.

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- **SMP 5:** Use appropriate tools strategically.

  While finding the location of the third point, students may choose to graph the points on a coordinate grid. They can also use calculators to find the lengths of sides of the triangles.

- **SMP 7:** Look for and make use of structure.

  In order to find the lengths of the sides of the triangles, students will apply the structure of the Distance Formula. Then, they will use what they know about similarity and proportions to find some side lengths.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- assuming corresponding parts indicate congruent parts

  Encourage students to graph the figures and label the parts. After calculating lengths, they should add the lengths to the diagram.

- changing the order of named triangles, causing parts to be incorrectly interpreted

  Remind students that when identifying congruencies and similarities, the order of the labels has meaning. Have them watch their graphs carefully to reinforce the correct order. If necessary, some students may want to color code corresponding parts so that they are easy to identify upon glance.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students struggle to find the length of segments:
  - Ask, What formula should you use?" (**Answer:** the distance formula)
  - Ask, "What is the distance formula?" (**Answer:** $d = \sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$) Next, ask, "What points will you use?" (**Answer:** the endpoints, or vertices, for each side)
- If students do not know how to analyze the transformations:
  - Ask, "Are corresponding sides the same length?" (**Answer:** no)
  - Ask them,"What transformation is this?" (**Answer:** dilation)
  - Finally, ask, "How can you find the scale factor for the dilation?" (**Answer:** Divide the length of the new triangle's side by the length of the corresponding side in the old triangle.)

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- If students cannot find the lengths of the other sides:
  - Ask, "What do you know about the ratio of corresponding sides of similar triangles?" (**Answer:** They are the same.)
  - Follow up by asking, "How can you apply that to find the length of another side of the new triangle?" (**Answer:** Multiply the lengths of the sides of the first triangle by the scale factor.)
- If students do not understand how to find the location of B':
  - Ask, "How far is B' from A'? From C'?" (**Answer:** 2 units, 2 units)
  - Ask, "What else do you know about the triangle?" (**Answer:** It's a right triangle.)
  - Finally, ask, "How does that help you find B'?" (**Answer:** B' is the right angle, so segments AB and BC must form a right angle. On a grid this would mean that the points are directly horizontal or vertical from B'.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Discuss whether everyone found the same locations. Ask students to identify which location they found or if they found both possibilities.

Connecting to Key Concepts

Make explicit connections to key concepts:

- If two triangles are similar, then their corresponding sides are proportional.

  In order to find the lengths of the sides in the similar triangles, students could use a proportion.
- The ratio of the corresponding sides is known as the ratio of similitude.
- By finding the ratio of similitude, students find the scale factor for the dilation.

Extending the Task

- To extend the task, include other elements of the task. For example, change the coordinates of A' and C' to create a different ratio of similitude.
- Ask students to place another character on the "screen" and then use the same transformations to find its new location. You may also ask students to identify the coordinates of $A''B''C''$ so that the figure decreases in size by a given scale factor.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I had to visualize the character in a video game as a triangle.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I had to use distances and scale factors to find the location of a point.)
- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** Rather than just imagine what the triangles look like, I used a coordinate grid to help me find B'.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I knew that the triangle was a right triangle, so that limited where the right angle, B', could be.)

Alternate Strategies or Solutions

Students could guess and check different locations for B' using a coordinate grid. They could choose a location and then use the distance formula to check.

Technology

Students can use a scientific calculator to calculate the distance between the two points. They could also use a program online where the user inputs the coordinates and the program calculates the distance.

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Problem-Based Task 5.4.1: Down, Down, Down

Common Core State Standard

G–SRT.4 Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two sides proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.*

Task Overview

Focus

How can Triangle Similarity Statements be used to prove that two triangles are similar? What components of the triangles must be proven similar? In this task, students will apply the Angle-Angle Triangle Similarity Statement to a real-world scenario and create a proportion to calculate the number of sections of downspout needed for the gutters of a house.

This activity will provide practice with:

- converting units of measurement (feet to inches/inches to feet)
- defining and explaining similar triangles
- identifying corresponding and congruent parts of triangles
- applying triangle similarity statements to a real-world scenario
- creating proportions to solve for unknown triangle lengths
- solving proportions
- analyzing results and applying them to the context of a scenario

Introduction

This task should be used to explore or to apply the skill of proving theorems about triangles, including triangle similarity. Students should already be familiar with creating ratios, solving proportions, and identifying corresponding and congruent parts of triangles.

Begin by reading the problem and clarifying the meaning of *Angle-Angle (AA) Similarity Statement*, *congruent*, and *similar triangles*.

Angle-Angle (AA) Similarity Statement	statement that says if two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar
congruent	having the same measure
similar triangles	two figures that are the same shape but not necessarily the same size; similar triangles have corresponding angles that are congruent, and the measures of their corresponding sides are proportional, or have a constant ratio

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to calculate how many sections of downspout will be needed for gutters on a house. They will interpret the verbal information given in the problem and create two similar triangles. They will apply their knowledge about Triangle Similarity Statements and prove that the triangles are similar because two pairs of corresponding angles are congruent (AA Triangle Similarity Statement). They will create a proportion in order to solve for a missing length of a triangle, which represents the height of the house, and they will use this length to determine the number of sections of downspout that will be needed.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for determining that the two triangles in the scenario are similar. They will explain the components of the AA Triangle Similarity Statement and relate them to the triangles representing the scenario. They will also justify their results by explaining the proportion that was created in order to find a missing value in a triangle, which represents the height of the house. When presenting their arguments to others, they will present their calculations and their analysis of the results, and explain how these results are interpreted in the context of the scenario.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled by using similar triangles and Triangle Similarity Statements. They will interpret the given verbal information and draw two triangles representing the scenario. They will apply their knowledge of similar triangles and identify the corresponding angles and corresponding sides of the two triangles. In order to meet the goal of the task, they will create and solve a proportion, and they will apply their results to the context of the scenario.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- misidentifying congruent parts because of the orientation of the triangles

  Remind students to carefully consider the current orientation of the triangles, ensuring that corresponding sides and angles match. Encourage them to redraw the figures in order to get a better representation of the corresponding parts, if necessary.

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- misreading similarity statements as congruency statements

  Remind students of the distinctions between "similar" and "congruent" in triangles. Have them write on their papers that congruent triangles have exactly the same angle measures and corresponding sides are exactly the same length, while similar triangles have corresponding angles that are congruent, but corresponding sides are proportional.

- incorrectly creating proportions between corresponding sides

  Before students create the proportion needed in the task, remind them to ensure that they have the correct corresponding sides. Ask them to list the pairs of corresponding sides prior to setting up the proportion.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ensure that they understand the distinction between congruent figures and similar figures. Ask them, "What is the difference between figures that are congruent and figures that are similar?" (**Answer**: When two figures are congruent, they are the same shape and exactly the same size, and therefore all corresponding angles are congruent, and all corresponding side lengths are congruent. When two figures are similar, their corresponding angles are congruent, but their corresponding sides are proportional in length.)
- As students are beginning the task, ask them, "Why do you think it is necessary to convert the height of Blair's eyes from the ground from feet and inches to only inches?" (**Answer**: Since the distance from Blair to the mirror and the distance from the mirror to the foundation of the house are given in inches, then it is necessary to convert the height of Blair's eyes from the ground to inches, so that the units are consistent when comparing side lengths.)
- After students have read through the task initially, discuss the components of the diagram with them. Ask them, "How do we know that the two triangles drawn have two pairs of corresponding angles?" (**Answer**: Since Blair is using a mirror, angles of reflection from a mirror are congruent. The other pair of congruent angles is the right angles. It can be assumed that the angle a house makes with the ground is 90°, and it can also be assumed that the angle Blair makes when standing on the ground is also 90°.)
- Discuss students' prior knowledge about triangle theorems regarding the interior angles in a triangle. Ask them, "Since you know that two pairs of corresponding angles of the triangles are congruent, why does that automatically ensure that the third pair of corresponding angles is congruent? What triangle theorem applies?" (**Answer:** The Triangle Angle-Sum Theorem states that the interior angles of a triangle sum to 180°. If two pairs of corresponding angles are congruent, then the sum of the two angles in each triangle is the same, so the third angle must be the same measure in each triangle in order to reach a sum of 180°.)

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- After discussing that the two triangles are similar by the Angle-Angle Similarity Statement, ask students to consider the third angle in the context of the scenario. Ask them, "Why is it not practical to measure the third angle of each triangle in the context of the scenario? What does the third angle represent in each triangle?" (**Answer:** The third angle in the large triangle represents the angle from the top of the house to the ground. The third angle in the smaller triangle represents the angle from Blair's eyes to the ground. These angles are more difficult to measure given the size of the house and practicality of measuring the angle of a person's eyes looking towards an object.)
- As students are creating the proportion using the known side lengths, ask them, "Given the context of the scenario, why would it be difficult to find the length of the hypotenuse of the larger triangle?" (**Answer:** It would be difficult to measure the length of the hypotenuse of the large triangle, because it would have to be measured from the top of the house to the mirror located on the ground. It is possible, but it would take certain measuring tools and would be more difficult than measuring distance on the ground.)
- After students have solved the proportion and determined that the height of the house from the foundation to the gutters is 300 inches, ask them, "Why is it necessary to convert 300 inches into a value with units of feet?" (**Answer:** Since the goal of the task is to determine how many sections of downspout are needed in 5-foot sections, inches must be converted to feet in order to determine the number of sections.)
- Ask students, "Suppose the height of the house from the foundation to the gutters was 312 inches. How would this affect the total number of 5-foot sections of downspout that Blair would need? Explain." (**Answer:** If the height was 312 inches, this value divided by 12 is 26. So the height would be 26 feet. Since the downspout comes in 5-foot sections, Blair would need to buy 6 sections to make sure he had enough. If he only bought 5 sections, there would not be enough downspout.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their thoughts about how the two triangles were drawn, in terms of which sides and angles represent which parts of the scenario. Discuss how they know that the pairs of corresponding angles are congruent without actually measuring them. Also discuss which sides of each triangle have lengths that are known, and which sides have unknown lengths. Ask students to explain the corresponding pairs of sides for the triangles. Discuss the necessity of converting units in the different parts of the task. Also discuss how the answer makes sense in the context of the scenario. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.

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- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and procedure for each step in the task. Ask them how they translated the verbal information given in the stem of the task and created a diagram showing two similar triangles. Ask them how they were able to apply the Angle-Angle Triangle Similarity Statement to conclude that the triangles were similar. Focus on the use of precise mathematical language and clarity, specifically when describing corresponding parts of the triangles, as well as when creating and solving the proportion.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The Angle-Angle (AA) Similarity Statement is one way to show that two triangles are similar.

  In this task, students will apply the AA Similarity Statement to a real-world scenario as they create similar triangles and identify congruent corresponding angles and proportional corresponding sides.

- Similar triangles have corresponding sides that are proportional, whereas congruent triangles have corresponding sides that are of the same length.

  Students will distinguish between congruent and similar triangles, as they apply the AA Similarity Statement in order to create a proportion and solve for a missing side length in a triangle.

- The ability to prove that triangles are similar is essential to solving many problems.

  Students will analyze a drawing representing two similar triangles in a real-world scenario, and they will determine that the triangles are similar based on the information given in the problem.

Extending the Task

- To extend the task, ask students to determine the values of the third missing side length on both triangles. Discuss with them that since the triangles are right triangles, the missing length is the hypotenuse. Ask them to apply two different methods to find the missing lengths, and then justify their results by showing that both methods result in the same values. Ask for volunteers to discuss their methods, specifically how the Pythagorean Theorem can be used to find the hypotenuse, as well as how creating proportions can also be used to find the missing side lengths. Discuss their calculations and analysis of which method is more efficient.

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- Another option for extending the task is to ask students to work with a partner to create their own real-world scenario, similar to the one in the original task. Ask them to explain each step and calculation in written form. Ask for volunteers to share their scenarios and explain their processes for determining that two triangles are similar, and how this was used to solve for missing values. Discuss the validity of each scenario shared and brainstorm ideas about how parts of the scenario could be revised, if necessary.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the task is to calculate how many sections of downspout will be needed for gutters on a house. I persevered in solving the problem by drawing two triangles to represent the scenario. Then I applied my knowledge of similar triangles and proved that the triangles were similar because two pairs of corresponding angles are congruent (AA Triangle Similarity Statement). Then I created a proportion and solved for a missing length of a triangle, which represented the height of the house. This value represented the amount of feet of downspout that was needed, so I calculated the number of sections of downspout needed by converting inches to feet, and then dividing that value by 5-foot sections.)
- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining my choices for the Triangle Similarity Statement used to prove that the two triangles in the problem were similar. Specifically, I explained how two pairs of corresponding angles were congruent, as well as how a proportion representing the missing height of the house was created and solved. Then, I presented my calculations and interpreted the results in the context of the scenario.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this scenario by recognizing that this scenario can be modeled by using similar triangles and Triangle Similarity Statements. I interpreted the given verbal information and drew two triangles representing the scenario. I applied my knowledge of similar triangles and identified the pairs of corresponding angles and corresponding sides of the two triangles. Then I created and solved a proportion, and I applied this result and determined the solution to the task.)

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Alternate Strategies or Solutions

Students may choose to recreate the drawing of the triangles in an alternate way than what is shown in the task. They may draw a more realistic version of the components (such as a house, a person, a mirror, etc.). They may also choose to draw the similar triangles in a different orientation than shown in the task. Encourage students to ensure that their drawings represent correct corresponding parts of the two triangles.

Technology

Students can use scientific calculators.

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Problem-Based Task 5.4.3: Towering Heights

Common Core State Standard

G–SRT.4 Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two sides proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.*

Task Overview

Focus

How can the altitude of a right triangle be applied to create similar triangles? What characteristics of similar triangles can be used to solve problems for missing side lengths? In this task, students will apply characteristics of similar triangles to a real-world scenario and calculate the height of a cell tower in order to determine whether or not the height meets the requirement for adding lighting onto the tower.

This activity will provide practice with:

- translating given verbal information into a mathematical model
- defining and explaining similar triangles
- identifying corresponding parts of similar triangles
- creating proportions to solve for unknown triangle lengths
- solving proportions
- analyzing results and applying them to the context of a scenario

Introduction

This task should be used to explore or to apply the skill of proving theorems about triangles, including triangle similarity. Students should already be familiar with creating ratios, solving proportions, and identifying similar triangles, as well as their corresponding and congruent parts.

Begin by reading the problem and clarifying the meaning of *altitude* and *similar triangles*.

altitude	the perpendicular line segment from a vertex of a figure to its opposite side; height
similar triangles	two figures that are the same shape but not necessarily the same size; similar triangles have corresponding angles that are congruent, and the measures of their corresponding sides are proportional, or have a constant ratio

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for identifying the similar triangles in the scenario. They will explain the characteristics of the similar triangles and relate them to the given values which represent the distances in the problem. They will also justify their results by explaining the proportion that was created to find the missing side length of a triangle, which represents the height of the cell tower. When presenting their arguments to others, they will present their calculations and their analysis of the results, and explain how these results are interpreted in the context of the scenario, as well as how they reached their conclusion.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled by applying similar triangles and creating proportions used to determine missing side lengths of triangles. They will interpret the given verbal information and create a diagram of similar triangles, to represent the scenario. They will identify the corresponding angles and corresponding sides of the triangles and determine which side lengths can be used to create and solve a proportion.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure and procedures involved in identifying and listing corresponding sides of similar triangles. They will recognize the structure of three similar triangles as they determine which corresponding sides are necessary to solve for a missing side length. They will also apply the structure of creating and solving a proportion.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- misidentifying congruent parts because of the orientation of the triangles

  Remind students to carefully consider the current orientation of the triangles, ensuring that corresponding sides and angles match. Encourage them to redraw the figures to get a better representation of the corresponding parts, if necessary.

- misidentifying the altitudes of triangles

  Remind students of the definition of an altitude, in that the segment drawn from a vertex of the triangle intersects the opposite side of the triangle at a right angle. Have them draw a picture of a right triangle with an altitude, so that they can see the new triangles created by the perpendicular segment.

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- incorrectly creating proportions between corresponding sides

  Before students create the proportion needed in the task, remind them to ensure that they have the correct corresponding sides. Ask them to list the pairs of corresponding sides prior to setting up the proportion.

- forgetting to add the additional measurements of 1 meter and 5 meters to determine the final solution of the task

  Remind students that once the proportion is solved and the height of the tower is found, the distance of 1 meter and 5 meters must be added to this value because the distance found is segment *AD*, and the height of the entire tower is segment *AB*. Then, to determine if additional lighting will be needed, 5 meters must be added on to the height of the tower.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ensure that they understand the goal of the task. Ask them, "How is the goal of the task going to be accomplished? What value will you need to find to accomplish this goal?" (**Answer**: The goal of the task is to determine if a cell phone company will need to add lighting to the top of their cell tower. This will be determined by finding the height of the tower, and then adding the height of the antenna to see if the overall height is taller than 61 meters.)
- Before students begin the task, ask them, "What inequality statement can be used to represent the range of values that will cause the cell phone company to have to add lighting to the top of their tower, and what inequality statement would represent the range of values for NOT having to add lighting?" (**Answer**: Since lighting must be added to a tower that is taller than 61 meters, any value greater than 61 would necessitate lighting. An inequality statement representing this is $h > 61$, where h represents the height of the tower. An inequality statement representing values in which lighting would NOT have to be added is $h \leq 61$.)
- Ask students, "If the tower were exactly 61 meters tall, do you think the FAA and the FCC would require lighting to be added to the tower?" (**Answer**: Answers will vary. Some students may think that lighting would be added if the tower were exactly 61 meters, just to be on the safe side. Some students may reason that the requirements state that the tower has to be taller than 61 meters, so an actual height of 61 meters would not require lighting to be added.)
- After students have read through the information given in the task, ask them, "What is an alternate way to represent the three similar triangles shown in the given diagram? Why might this be helpful when analyzing the scenario?" (**Answer:** The three similar triangles can be redrawn separately, and they can be drawn with the same orientation. This might be helpful when identifying corresponding parts of the triangles.)

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- As students begin working on the task, remind them about the characteristics of similar triangles. Ask them, "Given that there are three similar triangles representing the scenario, what information does that provide for you about the triangles in the context of the scenario?" (**Answer:** If triangles are similar, then their corresponding pairs of angles are congruent, and their corresponding pairs of sides are proportional. This information allows proportions to be created to find missing sides of triangles, specifically the missing side representing the height of the cell tower in this task.)
- As students begin working on the task, ask them, "Which given values are necessary to solve a proportion for the missing side length representing the height of the tower?" (**Answer:** The given values in the problem which are needed in the proportion are the distance the surveyor stands from the base of the tower (7.5 m), and the distance from the lens on the surveyor's tripod to the base of the tower (1 m).)
- As students prepare to create a proportion, ask them, "How is it possible to create a proportion to find a missing value when only two values are given in the problem? Which value is used twice?" (**Answer:** Since the characteristics of similar triangles can be applied, the value of 7.5 m is used twice when setting up the proportion.)
- Once students have found the length of segment AD as 56.25 m, ask them, "What two values must be added to 56.25 to determine a final solution to the task?" (**Answer:** The value of 1 m must be added to 56.25, since that is also part of the height of the tower. Then, to see if the antenna would cause the total height to be above 61 meters, the height of the antenna, which is 5 m, must be added. The tower is 62.25 m after the three values are added: $56.25 + 1 + 5 = 62.25$ m.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their thoughts about how they first determined which value in the problem must be found to find a solution to the task. Discuss the possible values for the height of the tower which will require lighting to be added to the top of the tower. Also discuss how drawing the altitude in the original triangle resulted in three similar triangles. Ask students to explain how to identify the correct corresponding parts of the triangles, specifically recognizing which sides are necessary for creating a proportion to find the side length representing the height of the tower. Discuss why the solution to the proportion is not the final height of the tower. Also ask students to explain how to apply the value of the height of the tower to determine a final answer to the original goal of the task. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.

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- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and procedure for each step in the task. Ask them how they identified corresponding parts of the three similar triangles, and ask them how they applied their knowledge of proportional sides to determine the height of the cell tower. Focus on the use of precise mathematical language and clarity, specifically when discussing altitudes of right triangles, describing corresponding parts of the triangles, and creating and solving proportions.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The altitude of the right triangle creates two smaller, similar right triangles.

  In this task, students will recognize that when the altitude of a right triangle is drawn, two smaller right triangles are created. They will determine that the smaller triangles are similar triangles, and that these triangles are also similar to the original triangle.

- The angles of similar triangles are congruent and the sides are proportional.

  Students will apply the characteristics of similar triangles and identify corresponding congruent angles as well as corresponding proportional sides.

- Corresponding sides of similar triangles are proportional.

  Students will identify the pairs of corresponding sides of the three triangles, and they will use the appropriate sides to create a proportion to determine a missing side length.

Extending the Task

- To extend the task, ask students to first list ALL corresponding sides of all three triangles, and then ask them to determine the values for ALL sides of each triangle. Ask them to apply two different methods to find the missing lengths, and then justify their results by showing that both methods result in the same values. Ask for volunteers to discuss their methods, specifically how the Pythagorean Theorem can be used to find the hypotenuse, as well as how creating proportions can also be used to find the missing side lengths. Discuss their calculations and analysis of which method is more efficient.

- Another option for extending the task is to ask students to work with a partner to create their own real-world scenario, similar to the one in the original task. Ask them to include a drawing, as well as written information describing their scenarios. Also, ask them to explain each step and calculation of their problem in written form. Ask for volunteers to share their scenarios and explain their processes for creating three similar triangles from an original right triangle, and how similar triangles were used to solve for missing values. Discuss the validity of each scenario shared and brainstorm ideas about how parts of the scenario could be revised, if necessary.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining my choices for identifying the corresponding parts of the similar triangles in the scenario. I explained which values represent each of the distances in the problem. I also explained the proportion that was created in order to find the missing leg of a triangle, which represented the height of the cell tower. I presented and explained my calculations and my analysis of the results, as well as how these results were interpreted in the context of the scenario so that a final solution to the task was found.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this scenario by recognizing that this scenario can be modeled by applying similar triangles and creating proportions used to determine missing side lengths of triangles. I interpreted the given verbal information and created a diagram of similar triangles, in order to represent the scenario. I identified the corresponding angles and corresponding sides of the triangles and determined which side lengths were necessary to create and solve a proportion.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** When solving this problem, I made use of structure by applying the structure and procedures involved in identifying and listing corresponding sides of similar triangles. I recognized the structure of three similar triangles and determined which corresponding sides were necessary to solve for a missing side length. I also applied the structure of creating and solvinga proportion.)

Alternate Strategies or Solutions

Students may choose to redraw the diagram such that the three similar triangles are separated and have the same orientation. This may help students better visualize the corresponding parts of the similar triangles. Encourage students to ensure that the points representing the vertices of the triangles match correctly. Also encourage them to ensure the known values are labeled correctly on the appropriate triangles. They may also draw a more realistic version of the components, such as a cell tower, the ground, a man and his tripod, etc.

Technology

Students can use scientific calculators.

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Problem-Based Task 5.5.1: Cutting Kitchen Tiles

Common Core State Standard

G–CO.9 Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*

Task Overview

Focus

How can knowledge of perpendicular bisectors be used to prove that the diagonals divide the figure into four congruent triangles?

This activity will provide practice with:

- constructing formal proofs
- choosing postulates and theorems to prove triangle congruence
- interpreting the significance of information
- examining geometric figures

Introduction

This task should be used to explore or to apply different potential methods for proving triangle congruence. There may be more than one way to reach the same goal, but the most direct and efficient method should be used.

Begin by reading the problem and clarifying the meaning of each phrase, especially if any students seem lost or confused. Explain that the top of the tile is a quadrilateral, and that, although it appears to be a square, we cannot assume that it is a square. Review the following terms:

bisector	crossing a segment at its midpoint and dividing it into two congruent halves
congruent	having the same shape and size
perpendicular	meeting or crossing at right angles

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Encourage students, individually or in small groups, to identify on their own which details might be useful for proving that the triangles are congruent.

- **SMP 2:** Reason abstractly and quantitatively.

  Remind students that the concept of a line is purely theoretical. Line segments, although they may coincide with a line, are different from lines. They are geometric figures with endpoints and a specific length.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Have students warm up by listing a few reasons why they think the triangles are congruent. Encourage students to challenge the claims of others, so that they are forced to think of ways to support their claims.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- assuming that the tile is square
- assuming that the diagonals of the quadrilateral bisect its angles
- assuming that the length of AC is equal to the length of BD

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students assume the tile is a square, assume the diagonals of the quadrilateral bisect its angles, or assume that the length of AC is equal to the length of BD, ask them to make a list of all of the information given in the text of the problem. Ask for another list of all of the information given by the diagram. (**Answer:** The only significant information given in the text is that the diagonals are perpendicular bisectors of each other. The diagram shows that the tile is a quadrilateral, and that the diagonals meet at 90-degree angles. The diagram also offers convenient letter names for all of the points/vertices pertinent to the problem.)
- Once students understand what information is given/known vs. what they were assuming to be true, have them refine the list of known facts to include only those things that they think will be most useful in proving that all four triangles are congruent.

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- If students are having trouble getting started with a formal proof, ask, "What is the significance of the segments bisecting each other?" (**Answer:** E is the midpoint of segment AC and segment BD.) Ask , "What is the significance of E being the midpoint of the segments?" (**Answer:** $AE = CE$ and $BE = DE$.)
- Ask the whole class if anyone remembers what the Perpendicular Bisector Theorem states and how it might help to prove that the triangles are congruent. If no one offers an answer, remind students that the theorem states that the perpendicular bisector is the set of all points equidistant from the endpoints of the segment it bisects. Tell students that the converse of the theorem (that any point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment it bisects) will help to prove that the triangles are congruent.
- Tell students to work out an updated list of everything they know. Ask, "What does the converse of the Perpendicular Bisector Theorem tell us about the tile?" (**Answer:** $AB = BC$, $BC = CD$, $CD = DA$, and $DA = AB$.) Ask, "What property can be used to show that *all four* sides of the tile are congruent to one another?" (**Answer:** Transitive Property)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Review the steps outlined earlier in this guide and build a formal proof table on the board as you go:

Statement	Proof
$\overline{AC}$ bisects $\overline{BD}$ and $\overline{BD}$ bisects $\overline{AC}$	Given
E is the midpoint of $\overline{AC}$ and $\overline{BD}$	Definition of bisector
$\overline{AE} \cong \overline{CE}$ and $\overline{BE} \cong \overline{DE}$	Definition of midpoint
$AB = BC$, $BC = CD$, $CD = DA$, and $DA = AB$	Perpendicular Bisector Theorem
$AB = BC = CD = DA$	Transitive Property
$\triangle AEB \cong \triangle AED \cong \triangle CED \cong \triangle CEB$	SSS Congruence Postulate

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Connecting to Key Concepts

Make explicit connections to key concepts:

- The converse of the perpendicular bisector theorem can be used to prove that two segments are congruent.
- Congruent figures have congruent angles and sides. Specific combinations of congruent sides and angles are outlined in postulates and theorems as sufficient proof that the other sides and angles are also congruent.

Extending the Task

To extend the task, have students look for other ways to reach the same conclusion. For example, students could prove the triangles are congruent using the Side-Angle-Side postulate. Ask students which step in the above exercise would be left out.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I used the description of the problem to make a list of known information.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I applied the concept of segment bisectors to the problem.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I identified which segments are congruent.)
- **For SMP 3, ASK:** "Did you construct viable arguments and did you critique the reasoning of others?" (**Answer:** I constructed a viable argument for why E is the midpoint of the diagonals and extended that conclusion to show that the four triangles are congruent to one another.)

Alternate Strategies or Solutions

Side-Angle-Side congruence could be established, with the included 90° angle, instead of Side-Side-Side congruence.

Technology

Students should not have any need for technology in this exercise.

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Problem-Based Task 5.6.1: Sensing Distance

Common Core State Standard

G–CO.10 Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*

Task Overview

Focus

How can the relationship between angles formed by parallel lines cut by a transversal be applied to solve real-world problems? What theorems about triangles can be used to find the measures of unknown angles? In this task, students will identify angle relationships and calculate unknown angle measurements in triangles, in order to determine which distance-measuring sensor provides a more accurate measurement.

This activity will provide practice with:

- identifying pairs of angles formed by a transversal
- applying triangle theorems to find unknown angle measures
- calculating angle measures
- classifying types of triangles based on side lengths
- comparing angle measures of two triangles

Introduction

This task should be used to explore or to apply the skill of using triangle theorems to determine unknown angle measures in triangles. Students should be familiar with parallel lines and the types of angles formed by a transversal.

Begin by reading the problem and reviewing the meaning of the following term:

alternate interior angles angles that are on opposite sides of the transversal and lie within the interior of the two lines that the transversal intersects

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  In order to solve the problem, students must recognize that the sensors and the objects are parallel to each other, and the segments of the triangles formed between the sensor and the object are transversals of the parallel objects. Encourage students to discuss their prior knowledge of angle pairs formed by a transversal that intersects parallel lines.

- **SMP 4:** Model with mathematics.

  Students will recognize that the scenario presented can be modeled using the given diagrams, in which parallel lines and triangles are drawn. They will analyze the relationships between the angle measurements in the models for each sensor in order to draw conclusions about which sensor calculates a more accurate distance measurement.

- **SMP 6:** Attend to precision.

  Make sure students are using precise mathematical language when they discuss terminology, definitions, and triangle theorems. They should correctly use the terms *transversal*, *alternate interior angles*, *isosceles triangle*, and *congruent*. Encourage them to communicate clearly with their peers when discussing their reasoning and calculations. Also, remind students to verify the results of their calculations.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly applying theorems to determine missing angle measures

  Have students make a list of all applicable triangle theorems and angle relationships to use as a reference for this task.

- incorrectly calculating angle measures

  Remind students of the Triangle Sum Theorem, and ask them to verify that all three angles of their triangles have a sum of 180°. Also, remind them to check that two of the angles are congruent in each triangle.

- not classifying the triangles in the diagram as isosceles

  Remind students that an isosceles triangle has two congruent angles that are opposite from the two congruent sides. Ask them to observe that for each triangle, there are two congruent angles; therefore, it can be concluded that the triangles are both isosceles triangles.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students are unsure of how to begin:

  Ask students to draw two parallel lines with a transversal cutting through the lines. Have them label all angles formed by the intersection of the parallel lines and the transversal, and ask them to identify angle pairs and relationships. (**Answer:** Students should create a diagram with four angles drawn at the intersection of the first parallel line and the transversal, and four angles drawn at the intersection of the second parallel line and the transversal. They should identify pairs of vertical angles, same-side interior angles, alternate interior and exterior angles, and corresponding angles.)

  Follow up by asking students if they notice a similar situation in the task. (**Answer:** The sensor and the object are parallel, and the lines that denote the light beam represent transversals.)

- If students aren't sure about what type of triangles are shown in the diagrams:

  Ask students to examine the diagrams of the two sensors. Ask them if they can make any predictions about the type of triangles that are formed. (**Answer:** The triangles formed appear to be isosceles triangles.)

  Follow up by discussing what criteria are necessary in order to classify a triangle as isosceles. Encourage students to analyze the information given in the diagrams, and ask them to determine if there is enough information to prove that these triangles are isosceles. (**Answer:** Yes, there is enough information to prove this. It can be proven that the two base angles are equivalent, which is enough information to prove that the triangles are isosceles.)

- If students have difficulty identifying how the interior angles of the triangles have anything to do with transversals, encourage them to extend the sides of the triangle out so that they can see the transversals more clearly. Ask them, "What types of angle pairs do you see in the diagram?" (**Answer:** The diagram contains pairs of alternate interior angles.)

  After students have located the pairs of alternate interior angles, ask, "What definition, theorem, or postulate allows you to conclude that these angles are congruent?" (**Answer:** The Alternate Interior Angles Theorem states that when two parallel lines are cut by a transversal, the resulting alternate interior angles are congruent.)

  Ask students, "What theorem allows you to determine the missing angle measures in the triangles? What equation can be used?" (**Answer:** The Triangle Sum Theorem states that the sum of the interior angles of a triangle is always 180°. The equation to find the missing angle in the first triangle is $77 + 77 + c = 180$, and the equation to find the missing angle in the second triangle is $84 + 84 + y = 180$.)

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- Ask students, "Why do you think the larger bounce-back angle causes the calculated distance to become more accurate?" (**Answer:** A larger bounce-back angle allows more light to bounce back to the sensor, so more accurate distance can be calculated. With a smaller bounce-back angle, not as much light bounces back.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask volunteers to share their initial prediction about the type of triangle that is formed by the segments between the sensors and the objects in each diagram. Encourage students to discuss their reasoning for their prediction, as well as how correct or incorrect they were based on the actual results of the task.
- Compare students' strategies and explanations for analyzing the given information and finding the unknown angle measures. Focus on the use of precise mathematical language.

Connecting to Key Concepts

Make explicit connections to key concepts:

- An isosceles triangle has at least two congruent sides.

  In this task, students will determine that the triangles in the diagrams for Sensor A and Sensor B are isosceles triangles, because they each have a pair of congruent interior angles, which means they also have two congruent sides.

- The Triangle Sum Theorem can be used to determine a missing angle measure by subtracting the known measures from 180°.

  Students will apply the Triangle Sum Theorem as they create an equation that will allow them to find an unknown angle measurement in a triangle by adding the two known measures and subtracting this sum from 180°.

Extending the Task

- To extend the task, ask students to determine the unknown angle measures only by applying what they know about same-side interior angles. Ask them to explain how same-side interior angles can be used to find other angles in the diagram, which will lead to finding the unknown bounce-back angle in each triangle.

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- Another option for extending the task is to ask students to work with a partner and create their own real-life problem scenario, in which they will need to find unknown angles in a diagram by applying triangle theorems and what they know about angle relationships. After each student pair has drawn a diagram and written an accompanying scenario, ask each pair to trade scenarios with another student pair, and then solve the other pair's problem scenario. Encourage students to discuss their strategies for determining the unknown angle measures, focusing on correct terminology and use of theorems.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you demonstrate perseverance?" (**Answer:** When I didn't know how to start, I looked at my notes and figured out which theorems about angles and triangles could help with this problem.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** The scenario was modeled with the given triangles. I used this model along with my knowledge of triangle theorems, parallel lines, and transversals to solve the problem.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision by using correct math terminology in my explanations and explaining my reasoning in a mathematically correct and precise way.)

Alternate Strategies or Solutions

- Instead of using the relationship between alternate interior angles, students may use the relationship between same-side interior angles in order to find the unknown angle measures. For example, students may find the measure of the angle adjacent to the base angle of 77°, which is 103°, and then determine that the same-side interior angle has a measure of 77°. Then, they can use the straight angle addition postulate and determine that the bounce-back angle is 26°, since the three angles that form a straight line on each object must sum to 180°.
- Students may choose to extend the segments of the triangles in order to visualize the transversal lines more clearly. Discuss how they might also use other pairs of angles, such as alternate exterior angles, in order to find angle measures that will lead to finding the requested unknown measures.

Technology

No technology is needed for this task.

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Problem-Based Task 5.6.2: Calibrating Consoles

Common Core State Standard

G–CO.10 Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*

Task Overview

Focus

What information is necessary to classify a triangle as isosceles? What is true about the side lengths and the angles of an isosceles triangle? In this task, students will find missing side lengths and angle measures in triangles created as part of a real-world scenario in order to determine if the triangles are isosceles.

This activity will provide practice with:

- defining an isosceles triangle
- applying the Pythagorean Theorem and other triangle theorems
- solving equations for missing side lengths and angle measures
- interpreting the dimensions and angle measures of triangles
- classifying triangles based on side lengths and angle measures

Introduction

This task should be used to explore or apply theorems about isosceles triangles. Students should already be familiar with the Triangle Sum Theorem and how to classify triangles based on side lengths and angle measures. If necessary, review the Pythagorean Theorem and types of triangles (equilateral, isosceles, and scalene).

Begin by reading the problem and clarifying the meaning of the following terms:

calibrate to adjust a device so that it functions accurately

optimal most desirable; best

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will examine the diagrams to determine the given information and identify the unknown information. They will also perform the necessary calculations to find the missing side lengths and angle measures. Encourage students to first identify which of the triangle theorems they will need to apply in order to prove that the triangles formed are isosceles.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will make conjectures about the triangles drawn in the diagrams, and they will test their conjectures by analyzing the given information. They will construct viable arguments about the types of triangles based on their calculations. Encourage students to discuss their reasoning with each other and discuss if they do not agree.

- **SMP 7:** Look for and make use of structure.

  Students look for and use the known structures of isosceles triangles and of right triangles that can be found within the task's provided diagrams.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly identifying parts of isosceles triangles

  Have students create a generic diagram of an isosceles triangle, and label the legs, base, vertex angle, base angles, and all congruency markings on the diagram. Have them use this diagram as a reference when working through the task.

- incorrectly setting up and solving equations to find unknown measures of triangles

  Remind students that the Triangle Sum Theorem states that the three angles of a triangle have a sum of 180°. Also remind them to check their answers to verify that the three angles do indeed have a sum of 180°.

- not applying the measurement found with the Pythagorean Theorem to both smaller triangles when calculating the third side in Scarlett's diagram

  Remind students that the small unlabeled triangle will have the same corresponding side lengths as the small labeled triangle because of the congruency markings, and because the triangles share a common side.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task:

  Ask them to make a list of the triangle definitions and theorems they think will be applicable to the problem, based on the diagrams. (**Answer:** definition of an isosceles triangle, the Pythagorean Theorem, the Triangle Sum Theorem, and the Converse of the Isosceles Triangle Theorem)

  Ask students to write predictions about whether each controller is calibrated correctly. Ask them to explain their reasoning based only on the diagrams. Encourage them to use mathematics to support or prove their predictions. (**Sample answers:** "I predict that Scarlett's triangle is isosceles because there are congruency markings in the diagram. These markings will allow me to apply the Pythagorean Theorem to the smaller triangles in order to find the missing side lengths. If these sides are the same length, then my prediction is correct." Or, "I predict that Wren's triangle is isosceles because it looks like two sides are the same length. I can check my prediction by using the Triangle Sum Theorem to see if two angles are congruent. If so, then the triangle is isosceles.")
- Ask students, "What information is provided in Scarlett's diagram that allows you to use the Pythagorean Theorem to find the length of each side of the triangle Scarlett created?" (**Answer:** The angle formed by the segment labeled '12 ft' and the segment labeled '5 ft' is denoted as a right angle, so the two smaller triangles in the diagram must be right triangles. Therefore, the Pythagorean Theorem can be used.)
- If students are having difficulty understanding why the two smaller triangles in Scarlett's diagram are congruent, ask, "Which congruent parts are identified in the two smaller triangles? Which triangle congruency theorem can be used?" (**Answer:** Both triangles have sides that measure 5 feet and 12 feet, and an included angle that measures 90°. The Side-Angle-Side (SAS) congruency theorem can be used to prove that the two triangles are congruent.)
- After students have calculated the length of the legs of Scarlett's triangle, ask, "Why does knowing that the two legs of Scarlett's triangle are congruent allow you to conclude that the triangle is isosceles?" (**Answer:** The definition of an isosceles triangle states that if two sides of a triangle are congruent, then the triangle is isosceles.)
- Ask, "What theorem can be used to determine the missing angle measure in Wren's diagram? What equation will be used?" (**Answer:** The Triangle Sum Theorem states that the interior angles of a triangle sum to 180°. The equation is $68 + 44 + x = 180$.)
- To extend students' thinking beyond the task, ask, "Is either triangle equilateral? Why or why not?" (**Answer:** Neither triangle is equilateral. Scarlett's triangle is not equilateral because an equilateral triangle has three congruent sides, and her triangle has side lengths of 13, 13, and 10. Wren's triangle is not equilateral because an equilateral triangle has three congruent angles, and her triangle has angles measuring 68°, 68°, and 44°.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask volunteers to share their initial predictions about whether each controller was calibrated correctly. Encourage students to discuss their reasoning for their predictions, as well as whether they were correct based on the actual results of the task.
- Compare students' strategies and explanations for analyzing the given information and finding the unknown side length in Scarlett's triangle and the unknown angle measure in Wren's triangle. Focus on the use of precise mathematical language.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Isosceles triangles have at least two congruent sides, called legs.

  In this task, students will identify triangles that may be isosceles, and then determine the lengths of the sides that appear to be congruent to see if they are.

- The angle created by the intersection of the legs is called the vertex angle.

  Students will observe that the vertex angle for both triangles in the task is created by the intersection of the legs of each isosceles triangle.

- The Converse of the Isosceles Triangle Theorem states that if two angles of a triangle are congruent, then the sides opposite those angles are congruent.

  In this task, students will discover that in Wren's triangle, when the unknown angle is calculated, it is congruent to the other base angle. Therefore, the Converse of the Isosceles Triangle Theorem can be applied, and the triangle can be classified as isosceles.

- If the vertex angle of an isosceles triangle is bisected, the bisector is perpendicular to the base, creating two right triangles.

  In Scarlett's triangle, students will observe that two right triangles are created by the bisection of the vertex angle. Students can verify that the angle has been bisected because of the congruency markings on the base of the triangle.

Extending the Task

To extend the task, have each student create three different triangles with either side lengths or angle measures missing. Ask each student to trade triangles with a partner, and then determine whether their partner's triangles are isosceles. Encourage students to discuss their strategies for creating triangles that are isosceles and those that are not, focusing on correct terminology and use of theorems.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you demonstrate perseverance?" (**Sample Answer:** I persevered by trying several strategies, and when they didn't work, I looked at my notes to remind myself what information is needed to prove that a triangle is isosceles.)
- **For SMP 3, ASK:** "Did you construct viable arguments and did you critique the reasoning of others?" (**Answer:** I constructed viable arguments by drawing conclusions about the type of triangle based on my analysis of the sides and angles, which allowed me to classify each triangle as isosceles. When another student disagreed with my answers, we looked at each others' work together to figure out who made a mistake.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** I recognized that the diagrams appeared to show isosceles and right triangles, and this structure allowed me to determine which theorems to use to solve the problem.)

Alternate Strategies or Solutions

- For the diagram of Scarlett's triangle, students may choose to break the large triangle apart in order to examine the two smaller congruent triangles separately. Remind them that the two smaller triangles share the altitude, and that the congruency markings indicate that each of the shorter sides is 5 feet long.
- Some students might choose to redraw the diagrams such that the vertex angle is at the top. Remind them to make sure that the sides and angles are still labeled correctly if they redraw the diagrams.
- For students who have learned about trigonometric ratios, Scarlett's triangle can be shown to be isosceles by using the arctangent function to prove that the two base angles are congruent.

Technology

Students may use scientific calculators when performing calculations with the Pythagorean Theorem.

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Problem-Based Task 5.6.3: Life-Size Support

Common Core State Standard

G–CO.10 Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*

Task Overview

Focus

What information does the midsegment of a triangle provide in terms of side lengths? How can the Triangle Midsegment Theorem be applied to find unknown side lengths? In this task, students will apply the properties of a midsegment and its related theorem in order to find a missing side length of a triangle in a real-world problem scenario.

This activity will provide practice with:

- analyzing a diagram representing a real-world scenario
- defining a midsegment of a triangle
- applying the Triangle Midsegment Theorem and the Pythagorean Theorem
- solving equations for a missing side length
- rounding decimals to a specified place value

Introduction

This task should be used to explore or apply theorems about the midsegment of a triangle. Students should be familiar with the Pythagorean Theorem.

Begin by reading the problem and reviewing the meaning of the following terms:

midpoint a point on a line segment that divides the segment into two equal parts

midsegment a line segment joining the midpoints of two sides of a figure

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will make sense of the problem by first examining the diagram and then identifying the midsegment of the triangle and recognizing the relationships among the side lengths based on properties of the midsegment. Encourage students to first identify which triangle theorems they will need to apply in order to determine the missing side length.

- **SMP 4:** Model with mathematics.

  Students will use the visual model provided, and use it to model the relationships among the side lengths algebraically in order to find a missing side length.

- **SMP 6:** Attend to precision.

  As students work through the task, make sure they are using math terms and definitions clearly and accurately to explain their reasoning. Remind students to check their calculations for accuracy, and also make sure their results are rounded correctly and labeled with the proper units.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- assuming a segment that is parallel to the third side of a triangle is a midsegment

  In this task, students will identify the segment in the triangle as the midsegment because of the congruency markings, which will then allow them to conclude that the segment is parallel to the third side of the triangle. Remind them that it cannot be assumed that a parallel segment is a midsegment; this must be proven.

- incorrectly writing and solving equations to determine lengths

  Have students write down the formula for the Pythagorean Theorem, $a^2 + b^2 = c^2$, on their papers before they substitute values for the lengths. Remind them to check their answer to make sure it is a valid one. For example, the length of the hypotenuse must be larger than that of the other two legs of the triangle.

- incorrectly applying the Triangle Midsegment Theorem to solve problems

  Have students write the Triangle Midsegment Theorem on their papers, and also have them draw a generic triangle that represents the properties of the theorem, including congruent and parallel markings.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Students may see a solution path that does not use the Triangle Midsegment Theorem. If so, allow this solution method, and then ask students if they can find another way to solve the task that *does* use the Triangle Midsegment Theorem.
- Before students begin the task, ask them what side lengths of the large triangle they know from the diagram and the given information, and ask them how they know this information. (**Answer:** The congruency markings on each side of the triangle indicate that the segment in the middle of the large triangle is a midsegment. The Triangle Midsegment Theorem states that the midsegment of a triangle is half the length of the third side and parallel to it. Therefore, it can be concluded that the length of the base is twice the length of the midsegment. The midsegment measures 15.5 inches, so the base is 31 inches. In addition, if the cardboard cutout is 66 inches tall and the angled support attaches to the back of the cutout at a point that is $\frac{2}{3}$ of this length, then the vertical leg of the large triangle must measure 44 inches.)
- If students are having difficulty understanding how to find the height of the large triangle:

  Ask, "What operation is indicated by the phrase '$\frac{2}{3}$ the length of the cutout'?" (**Answer:** This phrase indicates multiplication, so "$\frac{2}{3}$ the length of the cutout" indicates that $\frac{2}{3}$ is multiplied by 66, for a result of 44 inches.)

  After students have determined that 44 inches is $\frac{2}{3}$ of 66, ask, "Which side of the triangle does this length represent?" (**Answer:** The length of 44 inches represents the vertical leg of the triangle.)

  Ask, "Why can the Pythagorean Theorem be used to solve for the length of the angled support?" (**Answer:** The given diagram shows that the cutout, the angled support, and the floor create a right triangle, and the Pythagorean Theorem can be used to find a missing side length in any right triangle.)

  Ask, "What side length in the large right triangle does the angled support represent?" (**Answer:** The angled support represents the hypotenuse of this triangle.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask volunteers to share the steps in their solution process for finding the length of the angled support. Note that the method described in the Coaching Questions is only one of several possible solution methods. Discuss any other methods students may have used to find the missing lengths in the diagram. Encourage students to justify their processes.
- As students explain their reasoning, focus on the use of precise mathematical language, specifically when discussing the use of the Triangle Midsegment Theorem and the Pythagorean Theorem.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The midpoint is the point on a line segment that divides the segment into two equal parts.

  From the diagram, students will notice congruency markings on two sides of the large triangle, indicating the midpoints of these two sides.

- A midsegment of a triangle is a line segment that joins the midpoints of two sides of a triangle.

  Students will determine that the line segment that joins the midpoints of two sides of the large triangle is a midsegment of the triangle.

- The midsegment of a triangle is parallel to the third side of the triangle and is half as long as the third side. This is known as the Triangle Midsegment Theorem.

  Students will apply the Triangle Midsegment Theorem as they determine the length of the base by multiplying the length of the midsegment by 2.

Extending the Task

To extend the task, ask students to redraw the given diagram and then draw the other two midsegments of the triangle. Ask them to label all vertices of the midpoints, such that a smaller triangle is created inside the larger triangle. Ask them, "What is the relationship between the smaller triangle and the larger triangle?" Encourage and guide a discussion about why the two triangles are similar to each other. Have students use the values of the side lengths for the larger triangle and set up proportions to find the side lengths for the smaller triangle.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by comparing the diagram given in the task to what I know about properties about triangles, and I noticed that it resembles the Triangle Midsegment Theorem. So, I used this theorem and the Pythagorean Theorem to determine missing information in the triangle.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** The visual model was given in the task. I further modeled this scenario algebraically by using the Pythagorean Theorem to solve for the missing side length.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision by using correct math terminology in my explanations and by checking that my answer made sense.)

Alternate Strategies or Solutions

Instead of using the Pythagorean Theorem to find the length of the entire angled support, students may choose to find the side lengths of the smaller triangle in the upper half of the diagram. They can use the vertical side length of 22 inches (found by dividing the length of the vertical side of the large triangle, 44 inches, by 2) and the given horizontal side length of 15.5 inches to write and solve the equation $(22)^2 + (15.5)^2 = c^2$. The result for c, which represents the length of the hypotenuse of the smaller triangle, is $\sqrt{724.25}$, or approximately 27 inches. Remind students to then double this length in order to find the length of the entire angled support, which is approximately 54 inches.

Technology

Students may use scientific calculators when performing calculations with the Pythagorean Theorem.

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Problem-Based Task 5.6.4: Sailing Centroid

Common Core State Standard

G–CO.10 Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*

Task Overview

Focus

What is the centroid of a triangle? How can the medians of a triangle be used to find the centroid? In this task, students will apply their knowledge of medians and use the slope, point-slope, and midpoint formulas to determine if a point that is said to be the centroid of a triangle is correctly located at the geometric center of the sail of a sailboat.

This activity will provide practice with:

- defining the centroid of a triangle
- calculating the midpoint of a segment in a coordinate plane
- determining the slope of a line
- writing a linear equation in point-slope and slope-intercept forms
- evaluating an equation for a given coordinate pair
- analyzing information about the coordinates of a centroid

Introduction

This task should be used to explore or apply theorems about the centroid and the medians of a triangle. Students should already be familiar with the slope formula, the midpoint formula, and how to write linear equations in point-slope and slope-intercept forms. If necessary, review these formulas.

Begin by reading the problem and clarifying the meaning of the following term:

center of effort the point on a sail at which the entire force of the wind would produce the same effect as that produced by the wind's distribution over the whole sail; the geometric center of the sail

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they make sense of the information represented in the scenario. They will recognize that the side lengths of the triangle on the coordinate plane represent the side lengths of the sail of a sailboat. Students will reason quantitatively as they calculate the midpoints and slopes of each side length of the triangle.

- **SMP 7:** Look for and make use of structure.

  Students will apply the known structures of the midpoint formula, the slope formula, and the point-slope form of a line in order to work through the task and calculate the necessary values.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  In order to determine whether the given point is the centroid of the triangle, students will calculate the midpoints of the sides and the equations of the lines that contain the three medians of the triangle. They may use repeated reasoning to quickly come up with the coordinates of the points and the equations of these lines directly from the graph, without having to do any algebraic calculations.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- interchanging circumcenter, incenter, orthocenter, and centroid

  Have students make a list of the four different centers and their corresponding properties. Also ask them to draw triangles to represent each type of center. Encourage students to use the list and drawings for reference as they work through the task.

- confusing medians with midsegments

  Remind students that a midsegment is a segment that is drawn from midpoint to midpoint in a triangle, and a median is a segment that joins a vertex to the midpoint of the opposite side.

- incorrectly applying the midpoint and slope formulas

  Have students write the midpoint and slope formulas at the top of their papers to use as a reference when working through the task. Remind them to carefully label the coordinates as x_1, y_1, x_2, and y_2 before substituting each into the formulas.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them to examine the triangle drawn on the coordinate plane.

  Ask, "Which lengths of the triangle can you determine without using any formulas?" (**Answer:** Because the base of the triangle is a horizontal distance from the origin to the coordinates (6, 0), the base can be found by counting units to get 6. Another side of the triangle is vertical, so its distance can be found by counting up from $y = 0$ to $y = 15$. The length of this side is 15 units.)

  Ask, "Is it possible to determine the midpoints of any of the sides of the triangle without using the midpoint formula?" (**Answer:** Yes, it is possible to determine the midpoints of the two legs of the triangle without using the midpoint formula, since the legs are represented with a horizontal and vertical distance on the coordinate plane.)

  Ask students if they can identify the type of triangle that the sail represents, and ask them to explain their choice. (**Answer:** The triangle is a right scalene triangle. The coordinate plane shows that one leg of the triangle is horizontal and the other leg is vertical, so the triangle must be a right triangle. It is a scalene triangle because the three sides have different lengths.)

- If students have difficulty determining how to find the equations of the lines that contain the medians of the triangle:

  Ask, "What is the point-slope form of a linear equation, and what information is necessary to write an equation of a line in point-slope form?" (**Answer:** The point-slope form is $y - y_1 = m(x - x_1)$. In order to write an equation of a line in point-slope form, it is necessary to find the slope and a point on the line.)

  Ask students how they can write the equation of the line in slope-intercept form. (**Answer:** To rewrite the equation in slope-intercept form, simplify the point-slope form and solve for y.)

  Ask, "What does it mean if the equation of a line in slope-intercept form has no value for b? What is the y-intercept in this case?" (**Answer:** The line for the median that passes through vertex A is $y = 1.25x$. Since there is no value for b, this means that the line intersects the y-axis at the origin, so the y-intercept is 0.)

  Once students have found the equations for each median, ask, "How does substituting the point (4, 5) into each equation help determine if the point (4, 5) is actually the centroid of the triangle?" (**Answer:** The centroid is the intersection of all three medians. In order for the point (4, 5) to be the centroid, this point must be a solution to all three equations. This means that when this point is substituted into each equation, the resulting statement must be true.)

- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask volunteers to share the steps in their solution process for determining if the point (4, 5) is the centroid of the sail. Discuss the different strategies and processes that students used, and encourage students to justify their processes.
- Compare students' strategies and explanations for analyzing the given information and determining the solution plan for this task. Focus on the use of precise mathematical language, specifically when discussing the use of the midpoint formula, the slope formula, the equation of the line for each median, and the centroid.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The medians of a triangle are segments that join the vertices of the triangle to the midpoint of the opposite sides.

  In this task, students will find the midpoints of each side of the triangle and then connect them to each vertex of the triangle to create the medians.

- Every triangle has three medians.

  Students will discover that each midpoint will connect to a vertex of the triangle, and because there are three midpoints, there will also be three medians.

- The centroid is always located inside the triangle.

  In this task, students will observe that the point in which the three medians intersect, which is the centroid, is located inside of the triangle.

Extending the Task

- To extend the task, ask students to redraw the triangle on a different coordinate plane, with the same vertices as the original triangle. Ask them to find the circumcenter of the triangle, drawing and labeling the perpendicular bisectors. Then, ask them to draw the triangle again on another coordinate plane and find the incenter. Ask them to draw and label the angle bisectors. Finally, ask them to draw the triangle a third time on another coordinate plane and find the orthocenter. Ask them to draw and label the altitudes. Encourage and guide a discussion about the processes and results of finding these three other types of centers.
- Another option for extending the task is to ask students to work with a partner and create a new triangle on a coordinate plane that has different vertices than those in the original task. Encourage students to draw a non-right triangle. Ask them to find the centroid and write a brief paragraph explaining the steps in their process. Ask students to share their triangle with the class and discuss the solution process.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I reasoned abstractly as I first made sense of the information given in the problem scenario, and then I used this information to create a solution plan for the task.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning when I calculated the midpoints of the sides of the triangle, as well as the slopes of the equations of the lines representing the medians.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving the problem?" (**Answer:** I applied the structure of the midpoint and slope formulas in order to calculate these values for each side of the triangle. I also used the structure of the equation of a line to create equations representing the medians of the triangle.)
- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** I noticed that I could quickly locate the midpoints of each side by analyzing the graph, locating the point that was midway between the given x-coordinates and midway between the given y-coordinates. I also found that I could repeat the process of using the point-slope form of a line to find the equations of the three lines that contain the medians.)

Alternate Strategies or Solutions

- Instead of using the midpoint formula on all three sides, students may determine the midpoints of the two legs of the triangle by counting units, since the legs have horizontal and vertical distances on the coordinate plane. Remind students that the midpoint formula can be used to verify their results.
- In order to find the slope of the medians that pass through each vertex, students may count the "rise" over the "run," instead of using the slope formula. Encourage students to still use the slope formula as a way to verify their counting results.
- Students may find the equations of the lines that contain the medians by finding the slope and the y-intercept directly from the graph. Other students may attempt to verify that the given point is the centroid by drawing the medians on the graph to see if all three medians pass through this point. Encourage these students to verify their findings algebraically.
- Students may find the centroid of the triangle by solving a system of equations comprised of any two lines containing the medians of the triangle. Solving this system of equations will prove that the lines intersect at (4, 5).

Technology

No technology is needed for this task.

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Problem-Based Task 5.7.1: Measuring Bridge Cable Tension

Common Core State Standard

G–CO.11 Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

Task Overview

Focus

How can the segments and angles in the diagram be used to verify that the point of intersection is the midpoint of both diagonals?

This activity will provide practice with:

- constructing formal proofs
- choosing postulates and theorems to reach a valid conclusion
- interpreting the significance of information
- examining geometric figures

Introduction

This task should be used to explore or to apply different potential methods for proving that the diagonals of a parallelogram bisect each other. There may be more than one way to reach the same goal, but the most direct and efficient method should be used.

Begin by reading the problem and clarifying the meaning of each phrase, especially if any students seem lost or confused. Explain that the engineers will place the gauge at the intersection of the diagonals and that, "It is our job to prove that the point of intersection is the center of both diagonals." Draw the diagram from the task on the board and label the point of intersection *E* so that students can follow along. Review the following terms:

diagonal	a line segment that connects two opposite vertices of a polygon
center	the midpoint
parallel	lines that stay the same distance apart and never meet or cross

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Encourage students, individually or in small groups, to identify on their own which details might be useful for proving that the diagonals bisect each other.

- **SMP 2:** Reason abstractly and quantitatively.

  Remind students of the concept of parallel lines cut by transversals and of the angle relationships therein.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Have students warm up by listing a few reasons why they think the diagonals bisect each other. Encourage students to challenge the claims of others, so that they are forced to think of ways to support their claims

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- assuming that the diagonals are congruent
- assuming that the diagonals of the parallelogram bisect its angles

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students are not sure where to start, ask them to make a list of all of the information given by the text and the diagram. (**Answer:** The given information is as follows: The truss is a parallelogram, and the cables are diagonals of the parallelogram.)
- Once students have completed their lists of given information, ask, "What is our goal?" (**Answer:** To prove that P is the midpoint of AC and BD) Ask students "How can we prove that P is the midpoint?" (**Answer:** By demonstrating that $AP = CP$ and that $BP = DP$.)
- Have students, either individually or in small groups, spend a few minutes discussing and writing down a plan of action. What steps will they take to prove that E is the midpoint?

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- Once students have completed the independent exercise above, tell them, "We can demonstrate that $AP = CP$ and $BP = DP$ by proving that one of the pairs of triangles is congruent." Ask students, "What do we know about angles *PAB, PCD, PBA,* and *PDC*, and how do we know it?" (**Answer:** Angle *PAB* is congruent to *PCD,* and angle *PBA* is congruent to angle *PDC*. We know because they are Alternate Interior Angles.) Ask students, "What do we know about segments *AB* and *CD* and how do we know it?" (**Answer:** They are congruent. Opposite sides of a parallelogram are congruent.)
- Ask the whole class if anyone knows how to prove that the triangles are congruent. Whether or not anyone offers the correct answer, proceed to explain that two pairs of angles and the included side are known to be congruent, so the Angle-Side-Angle Postulate can be used to prove triangle congruency. Point out that the other pair of triangles can be proved congruent by the same method, but that it is only necessary to prove one pair.
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Review the steps outlined earlier in this guide and build a formal proof table on the board as you go:

Statement	Proof
$\angle ECD \cong \angle EAB$, $\angle EDC \cong \angle EBA$, $\angle PCD \cong \angle PAB$, and $\angle PDC \cong \angle PBA$	Alternate Interior Angles are congruent
$\overline{CD} \cong \overline{AB}$	Opposite sides of a parallelogram are congruent
$\triangle AEB \cong \triangle CED$ and $APB \cong CPD$	ASA Congruence Postulate
$\overline{AE} \cong \overline{CE}$, $\overline{BE} \cong \overline{DE}$, $\overline{AP} \cong \overline{CP}$, and $\overline{BP} \cong \overline{DP}$	Corresponding parts of congruent figures are congruent
P is the midpoint of $\overline{AC}$ and $\overline{BD}$	Definition of midpoint

Connecting to Key Concepts

Make explicit connections to key concepts:

- The point where the diagonals intersect is the midpoint of both diagonals, so if the engineers place the gauge at the point of intersection, the gauge will be in the center of the cables.

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Extending the Task

To extend the task, repeat the exercise with hypothetic length values. For example, give length values to the sides of the truss and degree values to its angles. Then have students try to find another way to approach the same problem.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I translated the information given in the description onto the diagram.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I applied the concept of parallel lines and transversals to the problem.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I made conclusions regarding which angles are equal in measure and which triangles are congruent.)
- **For SMP 3, ASK:** "Did you construct viable arguments and did you critique the reasoning of others?" (**Answer:** I constructed a viable argument for why the triangles are congruent and extended that conclusion to show that the diagonals bisect each other.)

Alternate Strategies or Solutions

The other pair of triangles could be proved congruent.

Technology

Students should not have any need for technology in this exercise.

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Problem-Based Task 5.8.1: Building a Porch Ramp

Common Core State Standard

G–SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

Task Overview

Focus

How can similar triangles and trigonometry be applied to represent and solve a real-world problem? What features of right triangles can be used to determine missing angles and side lengths? In this task, students will apply characteristics of similar right triangles to a real-world scenario and calculate the length of a ramp from the ground to the top of the front porch of a house.

This activity will provide practice with:

- translating given verbal information into a mathematical drawing
- creating scale drawings using a ruler and protractor
- identifying scale factors
- measuring sides and angles of a triangle with a ruler and protractor
- calculating side lengths by applying a scale factor
- analyzing results and applying them to the context of a scenario
- determining the value of a trigonometric ratio

Introduction

This task should be used to explore or to apply the skill of understanding that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles. Students should already be familiar with measuring angles with a protractor, labeling angles and sides of a triangle, and understanding the properties of similar triangles.

Begin by reading the problem and clarifying the meanings of *hypotenuse*, *scale factor*, and *sine*.

hypotenuse the side opposite the 90° angle in a right triangle

scale factor a multiple of the lengths of the sides from one figure to the transformed figure. If the scale factor is larger than 1, then the figure is enlarged. If the scale factor is between 0 and 1, then the figure is reduced

sine a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the hypotenuse;

$$\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$$

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine the length of a ramp that will be built from the ground to the top of a porch on a house, and they will need to determine the value of a trigonometric ratio. They will recognize that they will need to use a scale drawing to represent the scenario, and they will determine an appropriate scale factor. They will also use a protractor to accurately draw a right triangle with specified dimensions. Then, they will measure the length of the ramp on their drawing and convert it to the actual measurement of the ramp by multiplying the length by the scale factor.

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they make sense of the goal of the task by first recognizing that a scale drawing will be necessary to determine the actual length of the ramp. They will examine the given information and conclude that measurement tools can be used to construct a scale drawing to accurately represent the scenario. They will apply their knowledge about right triangles and similarity to correctly label the parts of the triangle. They will also apply their knowledge about scale factors and determine an appropriate scale factor for their drawing. They will reason quantitatively as they calculate the actual length of the ramp by multiplying the length on the scale drawing by the scale factor. They will also substitute values in for the length of the opposite side of the 10° angle, and the length of the hypotenuse, to calculate the value of the sine of 10°.

- **SMP 5:** Use appropriate tools strategically.

  Students will use a protractor and a ruler for this task. They will use the protractor to construct a right triangle with a 10° acute angle and a 90° angle. Once these angles are constructed, they will use a ruler to carefully connect the three sides of the triangle. They will also use a ruler to measure the length of the hypotenuse of the right triangle.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- mistakenly labeling the side lengths or angles on the drawing

  Remind students to carefully read the information given in the problem. Suggest that they sketch the drawing first to ensure that the values placed on the triangle make sense in the context of the problem. Once they are certain that the parts are labeled correctly, have them draw a more organized drawing of the scenario.

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- confusing the differences between the trigonometric ratios

  Have students write down the ratios for sine, cosine, and tangent. Discuss the meaning of each ratio, and have them create methods to remember the differences between each of the ratios (i.e. mnemonic devices).

- reversing the location of the 10° angle with the other acute angle in the triangle

  Have students reread the problem. Remind them that the 10° angle is formed from the ground to the top of the porch.

- claiming that the total length of the ramp is 5.75 inches

  Remind students that they will need to convert the measurement they find on their drawing to the actual measurement of the ramp. Remind them to use their scale factor of 24; therefore the length on the drawing will be multiplied by 24 in order to determine the actual length of the ramp.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "What are some examples of real-world scenarios where scales and scale factors are used? Overall, why are scales necessary?" (**Answers:** Some examples of scales and scale factors are with maps, model toy kits, house plans, and diagrams. Scales and scale factors are necessary to represent scenarios that are either too small or too large to examine in a practical, real-world manner.)
- As students begin the task, ensure their understanding of similar triangles and scale factors. Ask them, "What is the connection between similar triangles and scale factors?" (**Answer:** Similar triangles are triangles that have congruent corresponding angles, but are not the same size. Since the corresponding pairs of sides of similar triangles are proportional, the scale factor is a multiple of the lengths of the sides.)
- As students begin the task, ask them, "Only one angle measure is given in the information for the task. How do you know the measure of another angle that can be used in the problem?" (**Answer:** Although it is not stated, it can be assumed that the angle that a house makes with the ground is a right angle. Therefore, it can be assumed that the triangle drawn is a right triangle, with a 90° angle.)
- As students are preparing to use their protractors to construct a triangle, ask them, "What is the measure of the third angle of the triangle, and how do you know this value since it was not given in the task?" (**Answer:** Since the triangle is a right triangle, one angle measures 90°, by definition of a right triangle. One of the acute angles given in the task is 10°. Since all three interior angles of any triangle must sum to 180°, then the third angle is found by subtracting 90° and 10° from 180°: 180 – 90 – 10 = 80. The measure of the third angle is 80°.)

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- After students have constructed the triangle by using their protractors to draw a 90° and a 10° angle, ask them, "How can you use your protractor to check the accuracy of the measurements of the three angles of the triangle?" (**Answer:** After drawing the 90° and the 10° angles, the third angle could be measured to ensure that it is 80°. If this measurement is not reasonably close to 80°, then it is possible that one of the other two angles was incorrectly drawn.)
- Check for students' understanding of scale factors. Ask them, "If you use the suggested scale of $\frac{1}{2}$ inch = 1 foot, how does this translate into a scale factor of 24 inches? Explain." (**Answer:** Since the scale should have a ratio that has the same units, if each $\frac{1}{2}$ of an inch is equal to 1 foot, then each $\frac{1}{2}$ of an inch represents 12 inches, because there are 12 inches in 1 foot. Since the scale is $\frac{1}{2}$ of an inch, then it would take 24 inches to equal the scale of $\frac{1}{2}$ of an inch.)
- Ask students, "Once the hypotenuse is measured as 5.75 inches on the drawing, how is the final length of 11.5 feet calculated?" (**Answer:** Once 5.75 inches is measured, then this value is multiplied by the value of the scale factor, which is 24. Then, since this value is in units of inches, it is divided by 12 in order to convert the final answer to feet: 5.75 • 24 = 138 inches; 138 ÷ 12 = 11.5 feet.)
- As students are completing the task, ask them, "Why do you think there are variations in the value for the length of the ramp? In other words, why didn't every person get the same result for the length?" (**Answer:** Since a protractor and ruler were used, there can be measurement error in terms of precision. Each person's measurement should be close to the value of 5.75 inches, but human measurement errors will cause inaccurate results.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts and ideas about how to create a drawing, based on the information given in the task. Discuss which values they can determine by applying their knowledge of right triangles and similar triangles. Also discuss how the scale factor was chosen, as well as how it relates to the given value of 2 feet. Ask them to explain how measurement error resulted in a variety of results for the length of the ramp, and discuss possible methods to prevent measurement errors. Ask for volunteers to discuss their calculations using the scale factor, as well as their calculations when determining the sine of 10°. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.

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- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and procedure for creating their scale drawings, as well as each step involved in the construction of the right triangle. Ask them how they applied their knowledge of the definition of right triangles, similar triangles, and scale factors in order to determine the actual length of the ramp. Focus on the use of precise mathematical language and clarity, specifically when labeling the side lengths of the triangle (hypotenuse), the angles in the triangle (acute, right), scale factor, and the sine ratio.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Recall that the **hypotenuse** is the side opposite the vertex of the 90° angle in a right triangle. Every right triangle has one 90° angle. The hypotenuse is the longest side of a right triangle.

  In this task, students will recognize that the hypotenuse of the right triangle is the length of the porch ramp. They will draw this length in their scale drawing as the side opposite of the right angle in the triangle.

- Similar triangles have proportional side lengths. The side lengths are related to each other by a scale factor.

  Students will create a drawing to represent the real-world scenario, and they will use a scale factor to represent the relationship between their drawing and the actual lengths of the triangle in the scenario.

- The **sine** of an acute angle in a right triangle is the ratio of the length of the opposite side to the length of the hypotenuse; $\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$.

  After students have calculated the length of the porch ramp, which is the hypotenuse, they will use this value, as well as the value of the side length opposite of the 10° angle, to create the sine ratio.

Extending the Task

- To extend the task, ask students to calculate the other 5 trigonometric ratios: cosine, tangent, secant, cosecant, and cotangent. Ask them to first calculate these ratios for the 10° angle, and then also calculate the ratios for the 80° angle. Ask them to list their results in a chart so that they can examine the results in an organized manner. Ask for volunteers to discuss their conclusions about the relationships between the two acute angles and the trigonometric ratios. Then, ask them to change one of the acute angles to 15° and calculate the six ratios for a 15° angle. Ask them to make predictions about the values of the six ratios for the complement angle of 75°. As a further extension, ask them to make predictions on the values of the ratios when both acute angles are 45°.

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- Another option for extending the task is to ask students to create their own real-world scenario, in which they must create a scale drawing of a right triangle that represents their scenario. Ask them to provide one angle and one measurement in their scenario. Have them model the format of their scenario similar to the format of the original task. Once they have solved their own task, ask them to present their task to a partner, so that each student is finding the solution to another student's task. Encourage pairs of students to discuss their thought processes and calculations for each step of the task. Ask for volunteers to share their scenarios, as well as discussing any difficulties they experienced during the creation or solution process of the task.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the task is to determine the length of a ramp that will be built from the ground to the top of a porch on a house, and also determine the value of a trigonometric ratio. I persevered in solving the problem by first creating a scale drawing to represent the scenario. I used a protractor and a ruler to draw a right triangle, and I labeled the triangle with the appropriate lengths and angles. Then I determined an appropriate scale factor and used this scale factor to multiply the length of the hypotenuse on my drawing in order to get the actual length of the ramp. I also found the sine of 10° by substituting the appropriate values into the ratio.)
- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I first made sense of the goal of the task by recognizing that a scale drawing was necessary in order to determine the actual length of the ramp. I analyzed the given information and concluded that measurement tools could be used to construct a scale drawing to accurately represent the scenario. I applied my knowledge about right triangles and similarity in order to correctly label the parts of the triangle. I also applied my knowledge about scale factors and determined an appropriate scale factor for my drawing.) "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning as I calculated the actual length of the ramp by multiplying the length on the scale drawing by the scale factor. I also substituted values in for the length of the opposite side of the 10° angle, and the length of the hypotenuse, in order to calculate the value of the sine of 10°.)
- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** I used a protractor and a ruler strategically by carefully constructing a right triangle given an acute angle measurement of 10°, and then I used a ruler to complete the construction of the sides of the triangle. I also used a ruler to measure the length of the hypotenuse of the right triangle.)

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Alternate Strategies or Solutions

Students may choose to draw the diagram differently than what is shown in the task. They may draw a more realistic version of the components, such as a house, a porch, the ground, and the porch ramp. Remind students to make sure the correct values are labeled on the correct parts. Students may also choose to construct the 80° instead of the 10° angle. Remind students that the 80° angle is the angle in between the shorter side of the triangle and the hypotenuse.

Technology

Students can use scientific calculators.

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Problem-Based Task 5.9.1: Replanting Project

Common Core State Standard

G–SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.★

Task Overview

Focus

Students use trigonometric ratios and the Pythagorean Theorem to solve right triangles in a real-world context.

This activity will provide practice with:

- analyzing a diagram to understand a problem
- converting between units of measure
- solving right triangles using trigonometry
- interpreting the results of a calculation

Introduction

This task should be used to explore or to apply right triangle trigonometry. It is best to implement this task after students have had practice solving right triangles in non-contextual problems. Students also should be proficient in using the Pythagorean Theorem to find side lengths of right triangles.

Begin by reading the problem and clarifying the meaning of the problem. Use the diagram as a tool to explain and clarify any misunderstandings.

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students should be encouraged to use the diagram provided to help understand the problem. Suggest that students label the right angle, the unknown angle with θ, and the unknown side length with a variable.

- **SMP 6:** Attend to precision.

  In part d, students must decide which right triangle property is more accurate. Discuss with students why the Pythagorean Theorem is more accurate in this instance.

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Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- using the wrong conversion factor to convert feet to inches

  Remind students that 1 ft = 12 in. Because they are converting feet to inches, they should use the ratio 12 in : 1 ft.

- using cosine instead of sine to find $\angle G$

  Have students label the opposite and adjacent sides of the triangle with respect to $\angle G$. Review the sine and cosine ratios.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- If students begin by working with the numbers in the problem, without converting to all inches or all feet, review the measurements they know from the problem statement. Ask what the units of measure for the wire and the ladder are? (**Answer:** The wire is measured in feet and the ladder is measured in feet and inches.) Ask why it would be better to convert everything to inches. (**Answer:** We can work with whole numbers instead of decimals or fractions.)
- If students find the sine of the ratio instead of the inverse sine, review the difference between $\sin \theta$ and $\sin^{-1} \theta$. What is the difference between these two functions? (**Answer:** The $\sin \theta$ is the ratio of the length of the side opposite θ to the length of the hypotenuse. It is a number. The $\sin^{-1} \theta$ tells you the angle measure when you know the ratio.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

Review and discuss methods that can be used to solve the problem. Reinforce that the Pythagorean Theorem could be used to find the missing side length, but not the unknown angle measure.

Connecting to Key Concepts

Make explicit connections to key concepts:

- If you know the value of a trigonometric function, you can use the inverse trigonometric function to find the measure of the angle. Arcsine is the inverse of the sine function, written $\sin^{-1} \theta$ or arcsin θ. Use the inverses of the trigonometric functions to find the acute angle measures given two sides of a right triangle.

  Students use side lengths of a right triangle to compute the inverse sine and determine the measure of an angle of the triangle.

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Extending the Task

To extend the task, analyze how the problem will change for different ladder heights. For example, choose a height less than 11 feet 3 inches. Then choose a height greater than 11 feet 3 inches. Examine how the horizontal distance from the tree changes as the height of the ladder changes.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** Using the diagram to help understand the problem) "Which of your strategies represent quantitative reasoning?" (**Answer:** Making sure answers for the angle measure and side length make sense. The side length could not be longer than the hypotenuse and the angle measure could not be greater than 90°.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** Deciding whether to use trigonometry or the Pythagorean Theorem to find the horizontal distance from the tree trunk)

Alternate Strategies or Solutions

Students can use trigonometry to not only find the missing angle measure but also to find the horizontal distance from the tree trunk.

Technology

Students can use graphing calculators to find trigonometric ratios and square roots.

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Problem-Based Task 5.9.3: Fighting Flames from a Distance

Common Core State Standard

G–SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.★

Task Overview

Focus

How can trigonometric ratios and the Pythagorean Theorem be applied to solve a real-world problem? What information is necessary to determine missing side lengths of a right triangle? In this task, students will apply their knowledge of the Pythagorean Theorem and trigonometric ratios to a real-world scenario to calculate the distance from the roof a building to a windowsill that firefighters will need to descend by rope to enter the building.

This activity will provide practice with:

- interpreting a mathematical diagram
- identifying missing information that is necessary to solve a problem
- sketching drawings which represent a scenario
- applying trigonometric functions
- calculating side lengths of right triangles
- determining the hypotenuse of a right triangle by using the Pythagorean Theorem
- analyzing results and applying them to the context of a scenario

Introduction

This task should be used to explore or to apply the skill of using trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. Students should already be familiar with calculating the sine, cosine and tangent ratios with the use of a calculator, and also setting up and solving problems using the Pythagorean Theorem.

Begin by reading the problem and clarifying the meanings of *angle of elevation*, *Pythagorean Theorem*, and *tangent*.

angle of elevation	the angle created by a horizontal line and an upward line of sight to an object that is above the observer
Pythagorean Theorem	a theorem that relates the length of the hypotenuse of a right triangle (c) to the lengths of its legs (a and b). The theorem states that $a^2 + b^2 = c^2$.

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tangent a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the adjacent side; $\tan\theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine the distance from the windowsill to a roof of a building, and to determine if the firefighters are standing close enough to the building so that the water from their hoses reaches the flames. They will recognize the necessity of using a trigonometric ratio, specifically tangent, to calculate the vertical distances represented in the scenario. They will use a calculator to find the distances, using the angle of elevation given in the problem. They will examine the scenario and draw conclusions about how to apply their calculations to solve the task.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled by applying trigonometric ratios and the Pythagorean Theorem to determine missing side lengths of right triangles which represent distances in the real-world scenario. They will interpret the given verbal information and also analyze a diagram representing the scenario in order to determine which trigonometric ratio can be used. They will make decisions and draw conclusions based on the analysis of their results.

- **SMP 5:** Use appropriate tools strategically.

  Students will use calculators and rulers for this task. They will use either a scientific calculator, or a graphing calculator, to calculate the values of trigonometric ratios. They will use a ruler to create a drawing of a right triangle which represents the scenario. They might also use a protractor to construct an accurate representation of the given angle of elevation.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- marking an angle of depression or elevation in reference to the vertical instead of the horizontal

  Remind students that both an angle of depression and an angle of elevation are created from a horizontal line, and not a vertical line.

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- reversing the order of the values in the numerator and denominator of the tangent ratio

  Have students write down the definition of the tangent ratio. Remind them that the value of the opposite side length to the angle, which is the unknown value in this task, is the numerator, and the value of the adjacent side length to the angle, is the denominator.

- forgetting to subtract the two vertical distances to get the distance between the roof and the windowsill

  Have students mark off the distance between the roof and the windowsill on the diagram, and have them label this distance with a variable. Remind them that this distance is calculated by subtracting the other two vertical distances found in the problem.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ensure their understanding of the difference between an angle of elevation and an angle of depression. Ask them, "What is the difference between an angle of elevation and an angle of depression?" (**Answer:** An angle of elevation is an angle created by a horizontal line and an upward line of sight, and an angle of depression is an angle created by a horizontal line and a downward line of sight.)
- If students have difficulty distinguishing between an angle of depression and an angle of elevation, discuss ways in which they can remember the difference. Ask them, "What do the words "elevate" and "depress" mean?" (**Answer:** The word "elevate" means to lift or raise something up, and "depress" means to push or pull something down into a lower position.)
- Although it is not marked in the diagram given, ask students, "Why can you assume the triangles constructed by the given information are right triangles?" (**Answer:** It can be assumed that the triangles are right triangles because it is assumed that the angle formed by a building and the ground is a right angle.)
- Before students begin working on the task, ask them, "To determine if the firefighters are standing close enough to the building to put the fire out at the third-floor window, which distance will you need to find? What leg of a right triangle does this distance refer to, and which right triangle will you use to find the distance?" (**Answer:** It will be necessary to find the distance from the firefighter's eye level to the windowsill. This distance is the hypotenuse of the right triangle with an 18° angle of elevation.)
- As students begin working on the task, remind them to carefully choose the values to use in the tangent ratio. Ask them, "What is the definition of "adjacent", and how can you remember which side is the adjacent side when you are working with trigonometric ratios?" (**Answer:** The word "adjacent" means "next to", or "adjoining". The adjacent side in a triangle is the side of the triangle next to the angle, or touching one of the sides that the angle is in between.)

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- As students are working on the task and finding the vertical distances from the roof and the windowsill to the firefighter's eye level, ask them, "Does it matter which of the two distances you find first? Why or why not?" (**Answer:** No, it does not matter which distance is calculated first. The shorter of the two distances, which is the distance from eye level to the windowsill, will be subtracted from the distance from eye level to the roof, since that is the greater distance.)
- As students are calculating the vertical distances, ask them, "Why do you think it is unnecessary to use the exact values of the distances in the context of the problem? Why is it acceptable to round to the nearest whole number?" (**Answer:** In the context of the problem, using distances which are measured to the nearest thousandths of a foot is unnecessary, since that distance is minimal given the real-world scenario.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts and ideas about how to accomplish the goal of the task. Discuss the given values and how they were applied to the diagram to represent the scenario. Ask them which values they knew could be determined by applying their knowledge of trigonometric ratios and the Pythagorean Theorem. Also discuss why the tangent ratio was chosen to determine the vertical distances of the firefighter's eye level to the windowsill and to the top of the roof. Ask for volunteers to discuss their calculations, as well as how they applied the results of their calculations to draw conclusions about if the firefighters were close enough to the building for the water to reach. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and procedure for determining the vertical distances, as well as how they chose the values to substitute into the tangent ratio. Ask them how they applied their knowledge of the Pythagorean Theorem in order to draw conclusions about the firefighters. Focus on the use of precise mathematical language and clarity, specifically when discussing angle of elevation, the components of the tangent ratio, and the components of the Pythagorean Theorem.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The angle of elevation is the angle created by a horizontal line and an upward line of sight to an object above the observer.

  In this task, students will apply the definition of angle of elevation and recognize where this is angle is located based on the information given in the task. They will use this angle when calculating the tangent ratio.

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- Given the angle θ, $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$.

  Students will apply their knowledge of right triangles and the trigonometric ratio tangent, as they substitute the known values for the angle of elevation and the side adjacent to the angle, to determine the value of the length of the side opposite of the angle.

- Given right triangle ABC, $a^2 + b^2 = c^2$, where a and b are the legs and c is the hypotenuse.

  Students will apply the Pythagorean Theorem, $a^2 + b^2 = c^2$, by substituting the values for the legs of the right triangle into the formula, in order to find the length of the hypotenuse of the triangle.

Extending the Task

- To extend the task, ask students to draw an angle of depression from the top of the roof to the eye level of the firefighter on the ground. Ask them to draw the horizontal line used to create the angle of depression, and then ask them to draw a line for the third side of the new triangle. Then, ask students to write a detailed explanation of why the new triangle is similar to another triangle given in the original diagram. Ask for volunteers to discuss the components of the similar triangles, specifically why the alternate interior angles are congruent.
- Another option for extending the task is to ask students to create their own real-world scenario in which an angle of depression is used. Ask them to model their scenario based on the format of the original task. Once they have created and solved their own task, ask them to present their task to a partner, so that each student is finding the solution to another student's task. Encourage pairs of students to discuss their thought processes and calculations for each step of the task. Ask for volunteers to share their scenarios, as well as discussing any difficulties they experienced during the creation or solution process of the task.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the task is to determine the distance from the windowsill to a roof of a building, and to determine if the firefighters are standing close enough to the building so that the water from their hoses reaches the flames. I recognized that I needed to use a trigonometric ratio, specifically tangent, to calculate the vertical distances represented in the scenario. I used a calculator to find the distances, using the angle of elevation given in the problem. Then I examined the scenario and drew conclusions about how my calculations could be used to solve the task and answer the questions.)

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- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this scenario by recognizing that this scenario could be modeled by applying trigonometric ratios and the Pythagorean Theorem to determine missing side lengths of right triangles which represent distances in the real-world scenario. I interpreted the given verbal information and also analyzed a diagram representing the scenario in order to determine which trigonometric ratio could be used. I made decisions and drew conclusions based on the analysis of my results.)
- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** I used a calculator and a ruler strategically by determining which values from the scenario were used in my calculations of the tangent ratio. I used a ruler to accurately create a drawing of a right triangle which represented the scenario. I also used a protractor to construct an accurate representation of the given angle of elevation in the diagram.)

Alternate Strategies or Solutions

Students may choose to draw the diagram differently than what is shown in the task. They may draw the angle of depression and create a similar triangle with a horizontal line parallel to the ground. Remind students to make sure the correct values are labeled on the correct parts of the similar triangle. Students may also choose to use the tangent ratio with the complementary angles to the 31° and the 18° angles, which are 59° and 72°, respectively. Remind students that the values in the tangent ratio are then reversed, compared to how the ratio is set up in the original task. Also remind them to be careful with their algebraic manipulations as they solve for the unknown distances.

Technology

Students can use scientific calculators that have trigonometric functions, or they can use graphing calculators.

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Common Core State Standards

Mathematics II

Integrated Pathway

Teacher Resource

Unit 6

WALCH
INTEGRATED MATH

This program was developed and reviewed by experienced math educators who have both academic and professional backgrounds in mathematics. This ensures: freedom from mathematical errors, grade level appropriateness, freedom from bias, and freedom from unnecessary language complexity.

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Standards Correlations

Each lesson in this Integrated Pathway: Mathematics II program was written specifically to address the Common Core State Standards. Each lesson lists the standards covered in all the sub-lessons, and each sub-lesson lists the standards addressed in that particular section. In this section, you'll find a comprehensive list mapping the sub-lessons to the CCSS.

Guide to Common Core State Standards Annotation

As you use this program, you will come across a symbol included with the Common Core standards for some of the lessons and activities. These symbols are explained below.

Symbol: ⋆

Denotes: Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (⋆).

From http://www.walch.com/CCSS/00003

Symbol: (+)

Denotes: College and Career Readiness Standards

Advanced mathematics standards that are required in higher-level courses such as advanced statistics may also be included in lower-level courses. These additional standards are denoted by (+). According to the Common Core State Standards Initiative, "the evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6–8."

From http://www.walch.com/CCSS/00004

Connections to Future Courses

This section provides a map between topics introduced in each unit of this course and subsequent courses where each topic is revisited and built upon.

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CCSS INTEGRATED PATHWAY: MATHEMATICS II STANDARDS CORRELATIONS

Unit 6: Circles With and Without Coordinates				
Lesson	**Sub-lesson number**	**Title**	**Standard(s)**	**Pages**
Lesson 1	**Introducing Circles**			
	6.1.1	Similar Circles and Central and Inscribed Angles	G–C.1 G–C.2	U6-5
	6.1.2	Chord Central Angles Conjecture	G–C.2	U6-27
	6.1.3	Properties of Tangents of a Circle	G–C.2	U6-41
Lesson 2	**Inscribed Polygons and Circumscribed Triangles**			
	6.2.1	Constructing Inscribed Circles	G–C.3	U6-69
	6.2.2	Constructing Circumscribed Circles	G–C.3	U6-94
	6.2.3	Proving Properties of Inscribed Quadrilaterals	G–C.3	U6-111
Lesson 3	**Constructing Tangent Lines**			
	6.3.1	Constructing Tangent Lines	G–C.4 (+)	U6-134
Lesson 4	**Finding Arc Lengths and Areas of Sectors**			
	6.4.1	Defining Radians	G–C.5	U6-167
	6.4.2	Deriving the Formula for the Area of a Sector	G–C.5	U6-180
Lesson 5	**Explaining and Applying Area and Volume Formulas**			
	6.5.1	Circumference and Area of a Circle	G–GMD.1	U6-198
	6.5.2	Volumes of Cylinders, Pyramids, Cones, and Spheres	G–GMD.1 G–GMD.3★	U6-220
Lesson 6	**Deriving Equations**			
	6.6.1	Deriving the Equation of a Circle	G–GPE.1	U6-249
	6.6.2	Deriving the Equation of a Parabola	G–GPE.2	U6-275
Lesson 7	**Using Coordinates to Prove Geometric Theorems About Circles and Parabolas**			
	6.7.1	Using Coordinates to Prove Geometric Theorems About Circles and Parabolas	G–GPE.4	U6-310

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TOPICS FROM FUTURE COURSES: CCSS INTEGRATED PATHWAY MATHEMATICS II

Math II	Topic introduced	Course/unit where addressed	How addressed
Unit 6	Circles, their properties, and special segments	Math IV	using formulas for the volumes of spheres and other solids
	Deriving the formulas for a circle and a parabola	Math IV	expanded to working with equations of other conic sections

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Conceptual Activities

Use these interactive open education and/or Desmos resources to build conceptual understanding of mathematical ideas. (*Note*: Activity links will be monitored and repaired or replaced as necessary.)

- Desmos. "Card Sort: Parabolas."

  http://www.walch.com/ca/01022

  Find the shape of a parabola by using its form to reveal its characteristics. The activity begins with a review of both the characteristics and forms of a parabola, then moves on to determine characteristics of the graph of a parabola given in standard form, vertex form, or intercept form.

- Desmos. "Equations of Circles."

  http://www.walch.com/ca/01040

  In this activity, students write equations of circles with different given information. The activity involves writing equations in both standard and general form.

- Desmos. "Marbleslides: Parabolas."

  http://www.walch.com/ca/01029

  In this activity, students will rapidly become fluent in transforming, translating, and restricting the domains of parabolas by manipulating their symbolic form to cause marbles to slide down the parabola through stars.

- Desmos. "Polygraph: Parabolas."

  http://www.walch.com/ca/01030

  In this paired game, one student picks a single parabola from a collection and the other student asks yes/no questions in an attempt to guess which parabola was chosen. The game provides students with a reason for noticing important features of parabolas, which in turn provides a need for words to name the features. "Does your graph cross the x-axis twice?" is a common question during play that leads to the naming roots, for example.

- Desmos. "Sector Area."

  http://www.walch.com/ca/01041

  In this proportional reasoning activity, students explore the relationship between circle area, sector area, and sector angle.

- Desmos. "Will It Hit the Hoop?"

  http://www.walch.com/ca/01031

  In this activity, students predict whether various basketball shots will go through the hoop, and then model these shots with parabolas to check their predictions.

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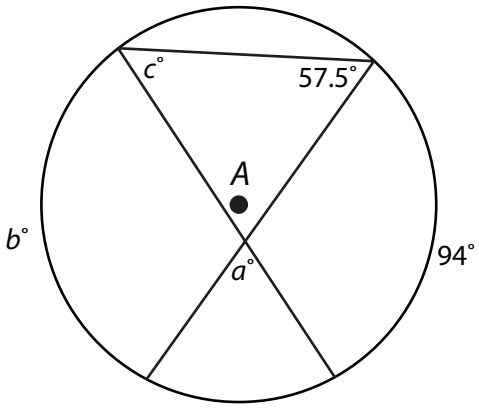
Pre-Assessment

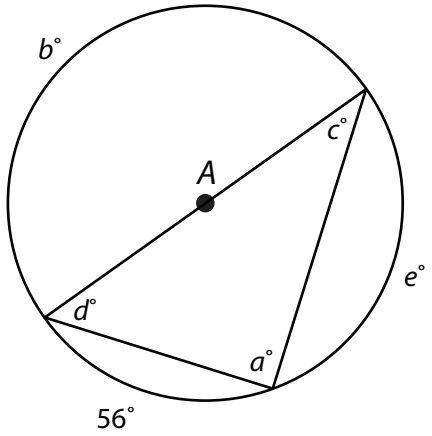
Circle the letter of the best answer.

1. Find the value of each variable.



   a. $a = 75.5°$, $b = 115°$, and $c = 47°$
   b. $a = 57.5°$, $b = 94°$, and $c = 32.5°$
   c. $a = 32.5°$, $b = 57.5°$, and $c = 94°$
   d. $a = 75.5°$, $b = 55.5°$, and $c = 94°$

2. Find the value of each variable.



   a. $a = 180°$, $b = 180°$, $c = 56°$, $d = 90°$, and $e = 90°$
   b. $a = 90°$, $b = 90°$, $c = 56°$, $d = 62°$, and $e = 62°$
   c. $a = 90°$, $b = 180°$, $c = 28°$, $d = 62°$, and $e = 124°$
   d. $a = 80°$, $b = 90°$, $c = 28°$, $d = 124°$, and $e = 124°$

continued

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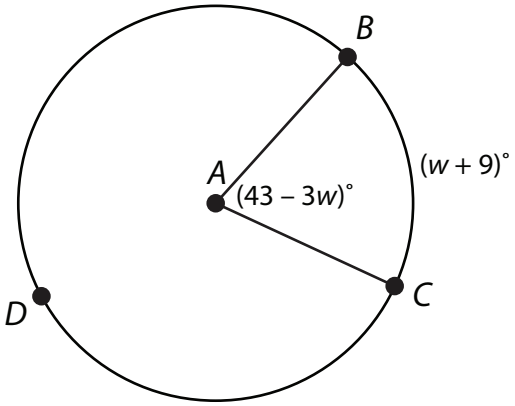
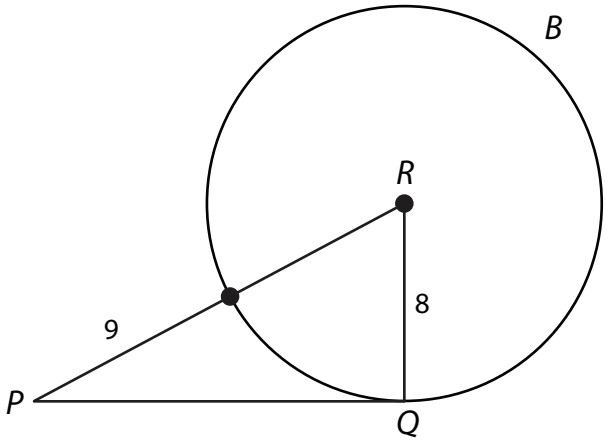
3. What is the value of w?



a. 7.5
b. 8.5
c. 9.5
d. 10.5

4. $\overline{PQ}$ is tangent to circle R at point Q in the diagram below. What is the length of $\overline{PQ}$ to the nearest whole number?



a. 4 units
b. 10 units
c. 12 units
d. 15 units

5. $\overline{XY}$ is tangent to circle Z at point Y. The slope of $\overline{XY}$ is $-\frac{1}{2}$. What is the slope of radius $\overline{ZY}$?

a. $\frac{1}{2}$
b. 2
c. $-\frac{1}{2}$
d. –2

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Lesson 1: Introducing Circles

Instruction

Common Core State Standards

SMP	
1 ✓	2
3 ✓	4 ✓
5 ✓	6 ✓
7	8 ✓

G–C.1 Prove that all circles are similar.

G–C.2 Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*

Essential Questions

1. Why are all circles similar?
2. What are the relationships among inscribed angles, radii, and chords of a circle?
3. What are the relationships among circumscribed angles, central angles, and inscribed angles?
4. What is the relationship between a tangent line and the radius of a circle?

WORDS TO KNOW

arc	part of a circle's circumference
central angle	an angle with its vertex at the center of a circle
chord	a segment whose endpoints lie on the circumference of the circle
circle	the set of all points in a plane that are equidistant from a reference point in that plane, the center
circumference	the distance around a circle; $C = 2\pi r$ or $C = \pi d$, for which C represents circumference, r represents the circle's radius, and d represents the circle's diameter
circumscribed angle	the angle formed by two lines that are tangent to a circle and whose vertex is outside of the circle
concentric circles	coplanar circles that have the same center
congruent arcs	two arcs that have the same measure and are either of the same circle or of congruent circles
diameter	a straight line passing through the center of a circle connecting two points on the circle; equal to twice the radius

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Instruction

inscribed angle	an angle formed by two chords whose vertex is on the circle
intercepted arc	an arc whose endpoints lie on the sides of an angle and whose other points are in the interior of the angle
major arc	part of a circle's circumference that is larger than its semicircle
minor arc	part of a circle's circumference that is smaller than its semicircle
***pi* (π)**	the ratio of circumference of a circle to the diameter; equal to approximately 3.14
radius	the distance from the center to a point on the circle; equal to one-half the diameter
secant line	a line that intersects a circle at two points
semicircle	an arc that is half of a circle
tangent line	a line that intersects a circle at exactly one point and is perpendicular to the radius of the circle at the point of tangency

Recommended Resources

- Math Open Reference. "Central Angle Theorem."

  http://www.walch.com/rr/00048

  This site describes the Central Angle Theorem and allows users to explore the relationship between inscribed angles and central angles.

- Math Warehouse. "What Is the Tangent of a Circle?"

  http://www.walch.com/rr/00049

  This site reviews the properties of tangent lines, and allows users to interactively explore the idea that a tangent line is perpendicular to a radius at the point of tangency. This site also provides limited practice problems, as well as solutions.

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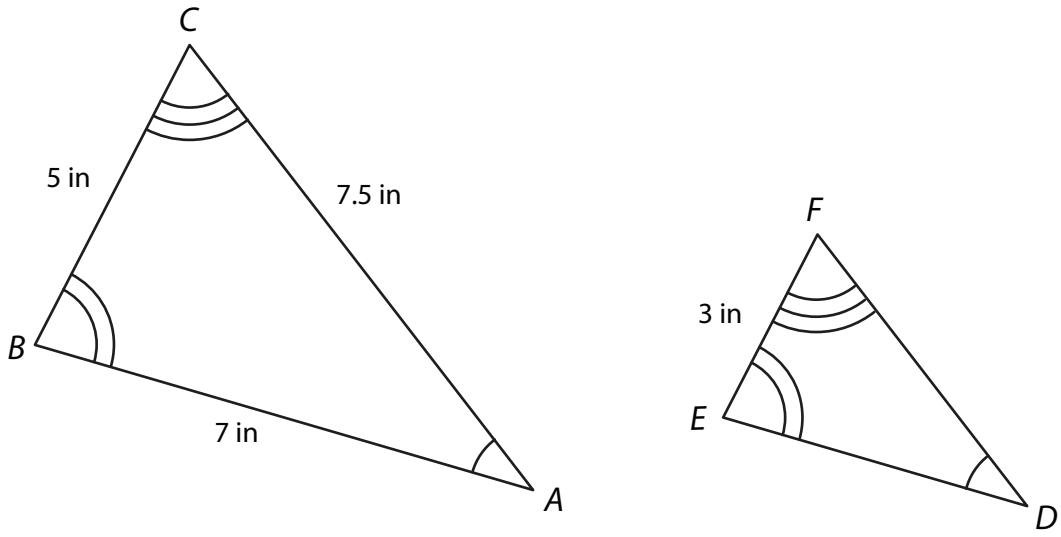
Lesson 6.1.1: Similar Circles and Central and Inscribed Angles

Warm-Up 6.1.1

Gunner is a ceramics artist. He wants to create two proportional triangular plates. The dimensions of one of the plates are given.



1. What is the scale factor Gunner used to create the smaller plate?

2. What are the lengths of the two missing sides?

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Instruction

Lesson 6.1.1: Similar Circles and Central and Inscribed Angles

Common Core State Standards

G–C.1 Prove that all circles are similar.

G–C.2 Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*

Warm-Up 6.1.1 Debrief

1. What is the scale factor Gunner used to create the smaller plate?

   $\overline{BC}$ corresponds to $\overline{EF}$ since they are each an included side between two pairs of congruent angles in the triangles.

   Since $\overline{BC}$ corresponds to $\overline{EF}$, and this is a reduction, not an enlargement, the scale factor is less than 1.

   $$\frac{EF}{BC}=\frac{3}{5}$$

   The scale factor used to create the smaller plate is $\frac{3}{5}$.

2. What are the lengths of the two missing sides?

   To find the lengths of the missing sides, multiply the lengths of the corresponding sides of $\triangle ABC$ by the scale factor, $\frac{3}{5}$.

   $\overline{CA}$ corresponds to $\overline{FD}$.

   $\overline{BA}$ corresponds to $\overline{ED}$.

   $$FD=(7.5)\frac{3}{5}=4.5$$

   $$ED=(7)\frac{3}{5}=4.2$$

   The length of FD is 4.5 inches.

   The length of ED is 4.2 inches.

Connection to the Lesson

- Students will extend their knowledge of similarity of triangles to prove similarity of circles.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

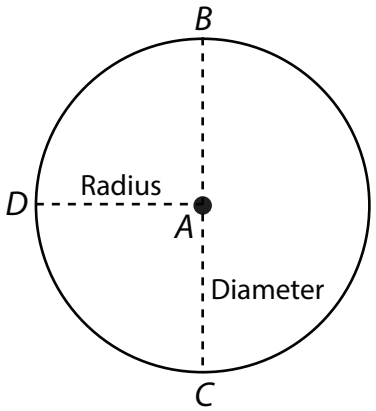
- setting up and solving ratios and proportions (7.RP.3)
- describing congruence in terms of rigid motions and similarity transformations (G–CO.6)
- understanding dilations (G–SRT.1a)

Introduction

In the third century B.C., Greek mathematician Euclid, often referred to as the "Father of Geometry," created what is known as Euclidean geometry. He took properties of shape, size, and space and postulated their unchanging relationships that cultures before understood but had not proved to always be true. Archimedes, a fellow Greek mathematician, followed that by creating the foundations for what is now known as calculus. In addition to being responsible for determining things like the area under a curve, Archimedes is credited for coming up with a method for determining the most accurate approximation of *pi*, π. In this lesson, you will explore and practice applying several properties of circles including proving that all circles are similar using a variation of Archimedes' method.

Key Concepts

- ***Pi***, π, is the ratio of the circumference to the diameter of a circle, where the **circumference** is the distance around a circle, the **diameter** is a segment with endpoints on the circle that passes through the center of the circle, and a **circle** is the set of all points that are equidistant from a fixed point (the center) and form a 2-dimensional curve.
- **Concentric circles** share the same center.
- The following diagram shows circle A $(\odot A)$ with diameter $\overline{BC}$ and radius $\overline{AD}$. The **radius** of a circle is a segment with endpoints on the circle and at the circle's center; a radius is equal to half the diameter.



SRB p. U6-3

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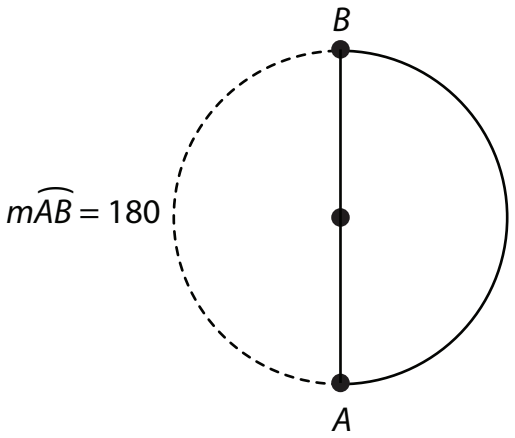
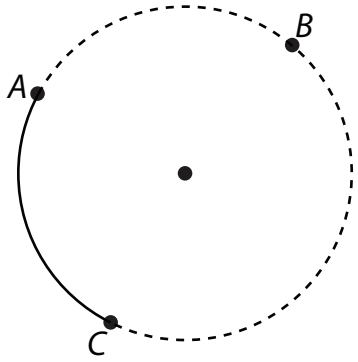
Instruction

- All circles are similar and have an angle measurement of 360°.
- A portion of a circle's circumference is called an **arc**.
- The measure of a **semicircle**, or an arc that is equal to half of a circle, is 180°.
- Arcs are named by their endpoints.
- The semicircle in the following diagram can be named $\overset{\frown}{AB}$.



- A part of the circle that is larger than a semicircle is called a **major arc**.
- It is common to identify a third point on the circle when naming major arcs.
- The major arc in the following diagram, which is indicated by the dashed arc, can be named $\overset{\frown}{ABC}$.



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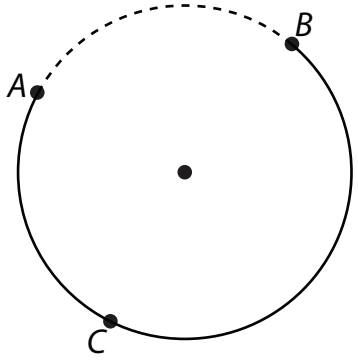
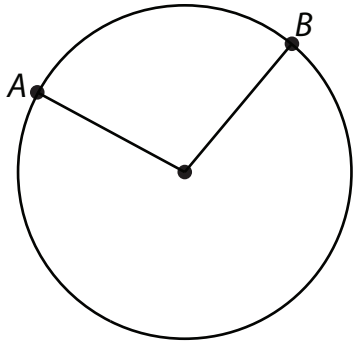
Instruction

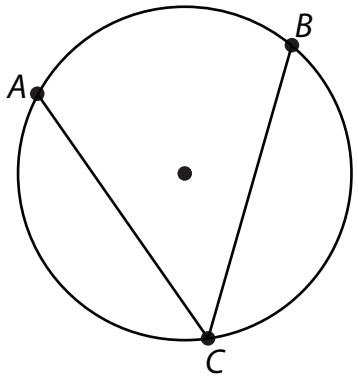
- A **minor arc** is a part of a circle that is smaller than a semicircle.
- The minor arc in the following diagram, which is indicated by the dashed arc, can be named $\overset{\frown}{AB}$.



- Two arcs of the same circle or of congruent circles are **congruent arcs** if they have the same angle measurement.
- The angle measurement of an arc is determined by the central angle.
- A **central angle** of a circle is an angle with its vertex at the center of the circle and sides that are created from two radii of the circle.



- A **chord** is a segment whose endpoints lie on the circumference of a circle.
- An **inscribed angle** of a circle is an angle formed by two chords whose vertex is on the circle.



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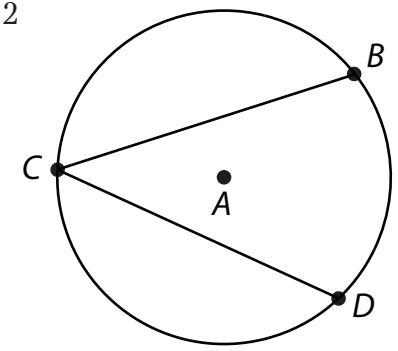
Instruction

- An **intercepted arc** is an arc whose endpoints lie on the sides of an angle and whose other points are in the interior of the angle. In the previous diagram, $\overset{\frown}{AB}$ is an intercepted arc.
- An inscribed angle is half the measure of the central angle that intercepts the same arc. Conversely, the measure of the central angle is twice the measure of the inscribed angle that intercepts the same arc. This is called the Inscribed Angle Theorem.

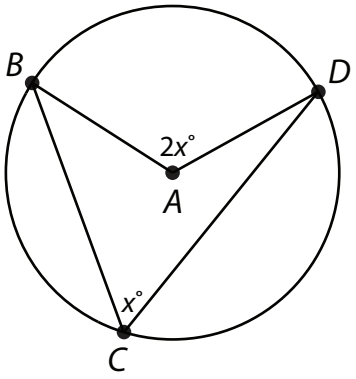
Inscribed Angle Theorem

The measure of an inscribed angle is half the measure of its intercepted arc's angle.

Given $\odot A$, $m\angle C = \frac{1}{2} m\overset{\frown}{BD}$.



- In the following diagram, $\angle BCD$ is the inscribed angle and $\angle BAD$ is the central angle. They both intercept the minor arc $\overset{\frown}{BD}$.



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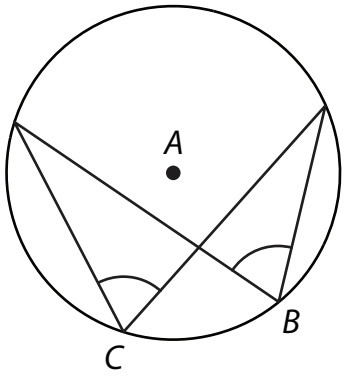
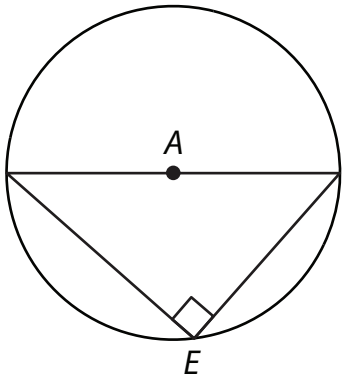
Instruction

Corollaries to the Inscribed Angle Theorem	
Corollary 1 Two inscribed angles that intercept the same arc are congruent. 	**Corollary 2** An angle inscribed in a semicircle is a right angle. 

Common Errors/Misconceptions

- confusing the measure of an arc with the length of an arc

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Instruction

Guided Practice 6.1.1

Example 1

Prove that the measure of a central angle is twice the measure of an inscribed angle that intercepts the same arc.

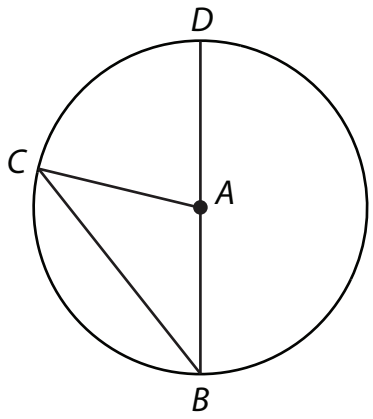
Given: $\odot A$ with inscribed $\angle B$ and central $\angle CAD$ intercepting $\overset{\frown}{CD}$.

Prove: $2m\angle B = m\angle CAD$

1. Identify the known information.

   Circle A with inscribed $\angle B$ and central $\angle CAD$ intercepts $\overset{\frown}{CD}$.

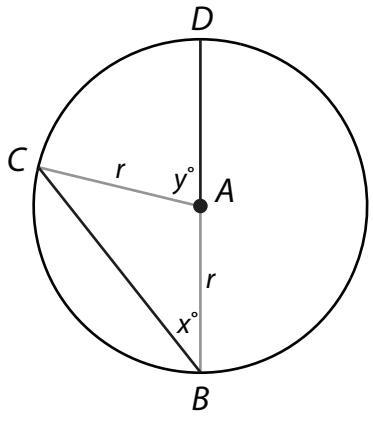
   Let $\overline{BD}$ be a diameter of the circle.



   By definition, $\overline{AB}$ and $\overline{AC}$ are radii of the circle.

   Mark them as r.

   Label $\angle B$ as x and $\angle CAD$ as y.



SWB p. 9; SRB p. U6-8

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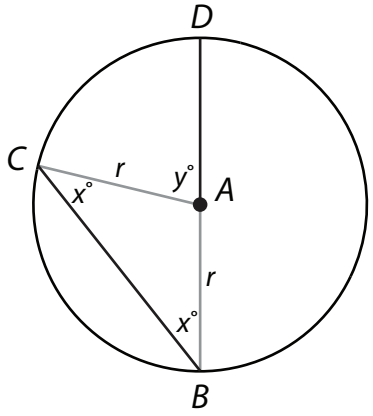
Instruction

2. Identify what information is known about the angles of the triangle.

$\triangle ABC$ is isosceles since two sides are the same length, r.

By the Isosceles Triangle Theorem, both base angles of the triangle are congruent; therefore, $\angle ABC \cong \angle ACB$, and both angles measure x°.

Also, the vertex angle of the isosceles triangle is a linear pair with y and thus $m\angle CAB = 180 - y$.



By the Triangle Sum Theorem:

$x + x + (180 - y) = 180$

$2x - y = 0$

$2x = y$

Substituting the names for x and y yields $2m\angle B = m\angle CAD$.

Therefore, the measure of a central angle is twice the measure of an inscribed angle that intercepts the same arc.

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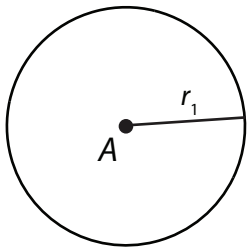
Instruction

Example 2

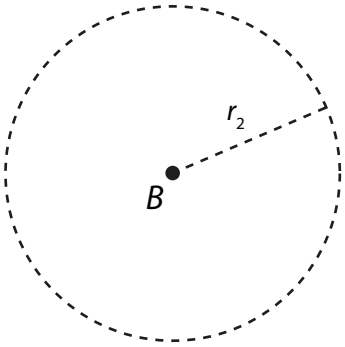
Prove that all circles are similar using the concept of similarity transformations.

1. Create a diagram to help with the proof.

   Draw $\odot A$ with radius r_1.



   Draw $\odot B$ with radius r_2 such that $r_2 > r_1$.



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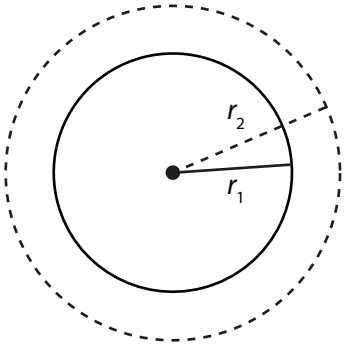
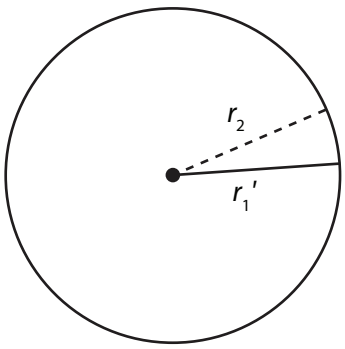
Instruction

2. Using properties of rigid motion, translate $\odot B$ so that it is concentric with $\odot A$.



3. Determine the scale factor necessary to dilate $\odot A$ so that it maps to $\odot B$.

To determine the ratio, note that each circle has a radius that by definition is the segment from the center to a point on the circle. To map $\odot A \rightarrow \odot B$, divide r_2 by r_1. The resulting ratio is the scale factor that produces the following image, where r_1' is the dilation of r_1 by the scale factor $\frac{r_2}{r_1}$.



This scale factor, $\frac{r_2}{r_1}$, is true for all circles and a similarity transformation of this sort can always be used because all circles are similar.

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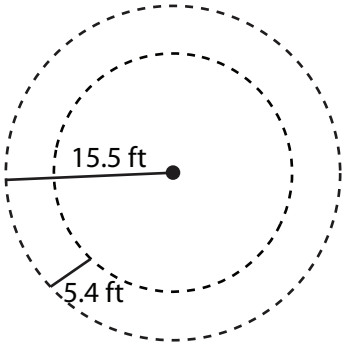
Instruction

Example 3

A car has a circular turning radius of 15.5 feet. The distance between the two front tires is 5.4 feet. To the nearest foot, how much farther does a tire on the outer edge of the turning radius travel than a tire on the inner edge if the car travels in one complete circle?



1. Calculate the circumference of the outer tire's turn.

$C = 2\pi r$	Formula for the circumference of a circle
$C = 2\pi(15.5)$	Substitute 15.5 for the radius (r).
$C = 31\pi$	Simplify.

2. Calculate the circumference of the inside tire's turn.

First, calculate the radius of the inner tire's turn.

Since all tires are similar, the radius of the inner tire's turn can be calculated by subtracting the distance between the two front wheels (the distance between each circle) from the radius of the outer tire's turn.

15.5 – 5.4 = 10.1 ft

$C = 2\pi r$	Formula for the circumference of a circle
$C = 2\pi(10.1)$	Substitute 10.1 for the radius (r).
$C = 20.2\pi$	Simplify.

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Instruction

3. Calculate the difference in the circumference of each tire's turn.

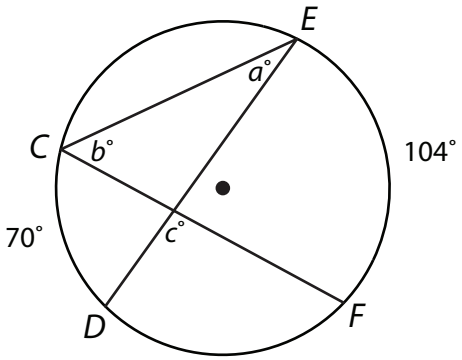
   Find the difference in the circumference of each tire's turn.

   $31\pi - 20.2\pi = 10.8\pi \approx 33.93$

   The outer tire travels approximately 34 feet farther than the inner tire.

Example 4

Find the value of each variable.



1. Identify the inscribed angle and the intercepted arc and use the proper theorem or corollary to find the value of a.

   a is the inscribed angle that intercepts arc $\overset{\frown}{CD}$.

   The measure of this intercepted arc is $70°$.

   Use the Inscribed Angle Theorem to find a.

$m\angle E = \frac{1}{2} m\overset{\frown}{CD}$	Inscribed Angle Theorem
$a = \frac{1}{2}(70)$	Substitute the given information.
$a = 35$	Solve for a.

   The value of a is $35°$.

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Instruction

2. Identify the inscribed angle and the intercepted arc and use the proper theorem or corollary to find the value of b.

   b is the inscribed angle that intercepts arc $\overset{\frown}{EF}$.

   The measure of the intercepted arc is 104°.

   Use the Inscribed Angle Theorem to find b.

$m\angle C = \frac{1}{2} m\overset{\frown}{EF}$	Inscribed Angle Theorem
$b = \frac{1}{2}(104)$	Substitute the given information.
$b = 52$	Solve for b.

   The value of b is 52°.

3. Use the Triangle Sum Theorem and the Vertical Angles Theorem to find the value of c.

   Note that a and b, in addition to being the measures of inscribed angles, are interior angles of a triangle.

   The third interior angle of the triangle is a vertical angle to c and is therefore congruent to c.

   Finding that third angle will yield the value of c.

$a + b + c = 180$	Triangle Sum Theorem and Vertical Angles Theorem
$35 + 52 + c = 180$	Substitute the measures of a and b.
$87 + c = 180$	Combine like terms.
$c = 93$	Solve for c.

   The value of c is 93°.

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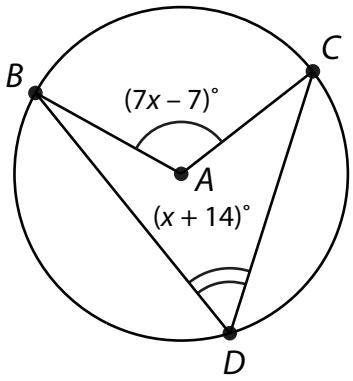
Instruction

Example 5

Find the measures of $\angle BAC$ and $\angle BDC$.



1. Set up an equation to solve for x.

   $\angle BAC$ is a central angle and $\angle BDC$ is an inscribed angle in $\odot A$.

$m\angle BAC = 2m\angle BDC$	Central Angle/Inscribed Angle Theorem
$7x - 7 = 2(x + 14)$	Substitute values for $\angle BAC$ and $\angle BDC$.
$7x - 7 = 2x + 28$	Distributive Property
$5x = 35$	Solve for x.
$x = 7$	

2. Substitute the value of x into the expression for $\angle BDC$ to find the measure of the inscribed angle.

   $m\angle BDC = (x+14)$

   $m\angle BDC = (7)+14$

   $m\angle BDC = 21$

   The measure of $\angle BDC$ is 21°.

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Instruction

3. Find the value of the central angle, $\angle BAC$.

   By the Inscribed Angle Theorem, $m\angle BAC = 2m\angle BDC$.

   $m\angle BAC = 21(2)$

   $m\angle BAC = 42$

   The measure of $\angle BAC$ is 42°.

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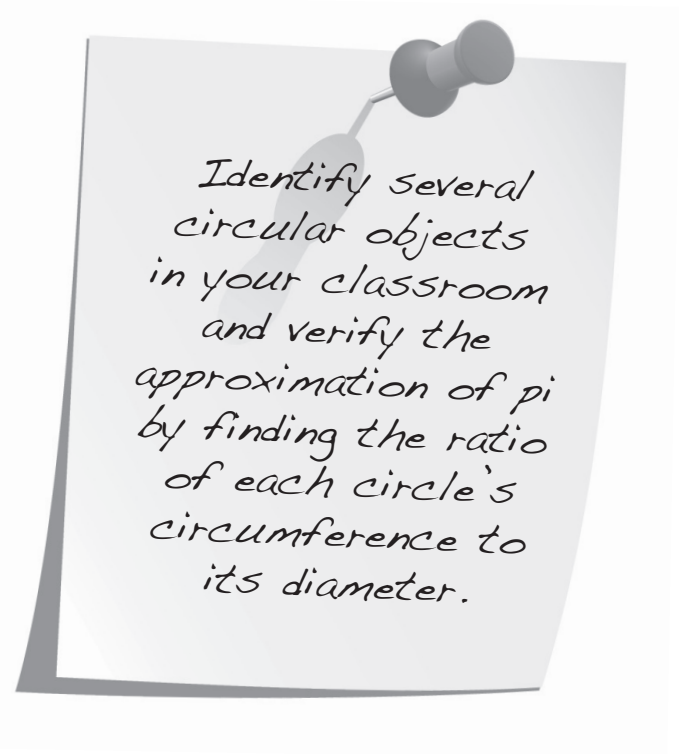
Problem-Based Task 6.1.1: Following in Archimedes' Footsteps

SMP	
1	2
3	4
5 ✓	6 ✓
7	8 ✓

The precise determination of the value of *pi* was of interest to the ancient Greeks for many years. Several approximations were calculated, but Archimedes was credited with determining a very precise approximation of *pi*. Identify several circular objects in your classroom and verify the approximation of *pi* by finding the ratio of each circle's circumference to its diameter.



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Problem-Based Task 6.1.1: Following in Archimedes' Footsteps

Coaching

a. How could you use string and a ruler to measure the circumferences of circular figures?

b. What should you do if the circular object does not have a clearly defined center?

c. If you are approximating the center, what does that tell you about the final average you will find?

d. Locate several circular objects and measure the circumference and diameter of each. Calculate the ratio of the circle's circumference to its diameter. Use the table to record your information.

Circular object	Circumference (cm)	Diameter (cm)	$\frac{C}{d}$

e. Will the number of circular objects that you choose have an effect on your approximation?

f. What could your class do to arrive at a better approximation?

g. What will the average help you find?

h. How does what you have found relate to the idea of circle similarity?

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Instruction

Problem-Based Task 6.1.1: Following in Archimedes' Footsteps

Coaching Sample Responses

a. How could you use string and a ruler to measure the circumferences of circular figures?

Wrap the string around the figure and use your finger as a marker for the start and end of the circle after one revolution. Then use the ruler to measure the length of the string that mapped the circumference of the figure.

b. What should you do if the circular object does not have a clearly defined center?

Mark a center using an agreed-upon point.

c. If you are approximating the center, what does that tell you about the final average you will find?

If you approximate, your final answer will be less precise.

d. Locate several circular objects and measure the circumference and diameter of each. Calculate the ratio of the circle's circumference to its diameter. Use the table to record your information.

Answers will vary. Check students' tables for accuracy.

e. Will the number of circular objects that you choose have an effect on your approximation?

Yes; your approximation can be found by calculating the average of all of your results. The more circular objects that are used, the more accurate the approximation.

f. What could your class do to arrive at a better approximation?

The class could find the average for the entire class.

g. What will the average help you find?

The average is a measure of the middle of a data set. In this case, it will give us one relationship of $\frac{C}{d}$ that is representative of the entire set.

h. How does what you have found relate to the idea of circle similarity?

All students should find a value close to 3.14, regardless of the size of the circle, thus demonstrating that all circles are similar.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 6.1.1: Similar Circles and Central and Inscribed Angles

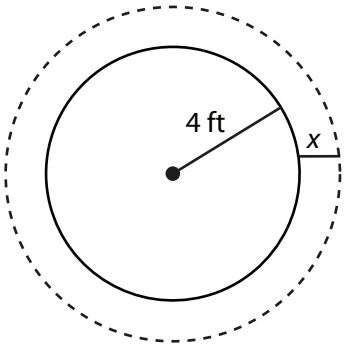
Given that all circles are similar, determine the scale factor necessary to map $\odot A \rightarrow \odot B$.

1. $\odot A$ has a radius of 4 units and $\odot B$ has a radius of 6 units.

2. $\odot A$ has a diameter of 55 units and $\odot B$ has a diameter of 75 units.

Use your knowledge of similar circles to complete problems 3 and 4.

3. A homeowner hired a landscaper to expand her circular garden. If the landscaper uses a scale factor of $\frac{4}{3}$ to expand the garden, what is the difference in the radii of the new and old garden?



4. A child's train has a circular turning radius of 12 inches. The distance between the two front wheels is 3 inches. To the nearest tenth of an inch, how much farther does the wheel on the outer edge of the turn travel than a wheel on the inner edge when the train makes one complete circle?

continued

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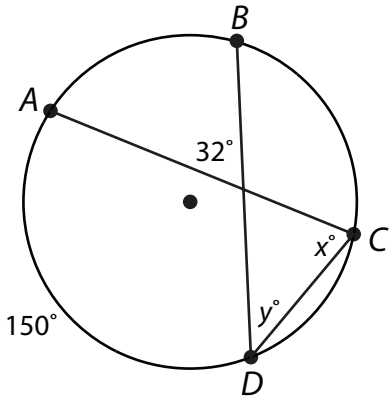
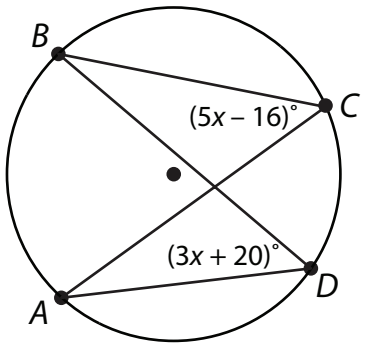
Use your knowledge of angles to complete problems 5–10.

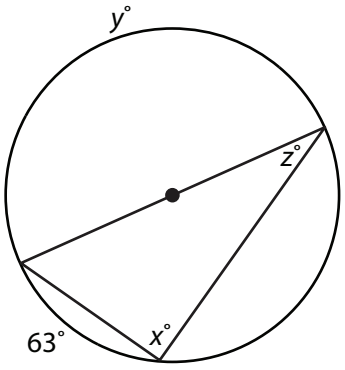
5. Find the values of x and y.



6. Find the value of x and the measure of $\overset{\frown}{AB}$.



7. Find the values of x, y, and z.



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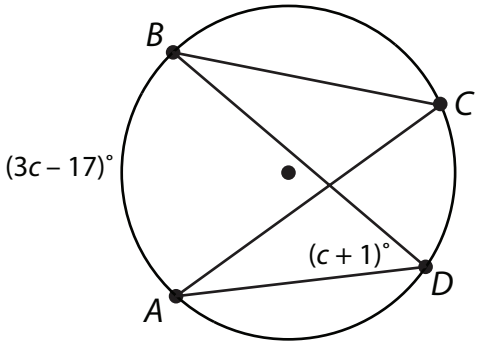
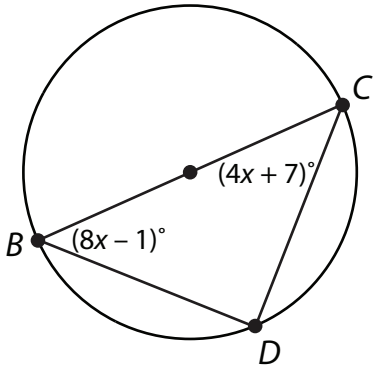
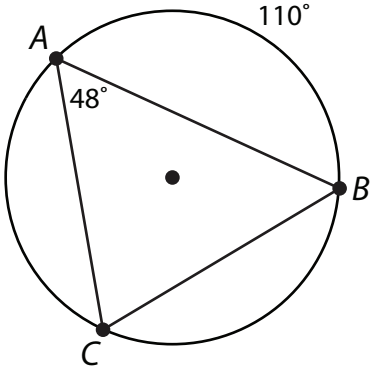
8. Find $m\angle C$ and $m\angle D$.



9. Find $m\angle B$ and $m\angle C$.



10. Find $m\overset{\frown}{BC}$ and $m\overset{\frown}{CA}$.



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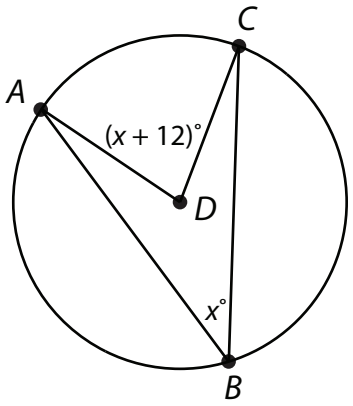
Lesson 6.1.2: Chord Central Angles Conjecture

Warm-Up 6.1.2

Your classmate Jana has been sick, and is making up several days' worth of missed homework. Your teacher has asked you to explain the homework to Jana. Use the following diagram to help her.



1. Explain to Jana the relationship between $m\angle D$ and $m\angle B$.

2. Set up an equation and show Jana how to solve for x.

3. Help Jana find $m\angle D$ and $m\angle B$.

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Lesson 6.1.2: Chord Central Angles Conjecture

Common Core State Standard

G–C.2 Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*

Warm-Up 6.1.2 Debrief

1. Explain to Jana the relationship between $m\angle D$ and $m\angle B$.

   $\angle D$ is a central angle and $\angle B$ is an inscribed angle.

   The measure of the central angle is twice that of the inscribed angle.

   Therefore, $m\angle D$ is twice that of $m\angle B$.

2. Set up an equation and show Jana how to solve for x.

   $m\angle D = 2m\angle B$

   $x + 12 = 2(x)$

   $x + 12 = 2x$

   $x = 12$

3. Help Jana find $m\angle D$ and $m\angle B$.

   Substitute the value of x into the original equation to determine the values of the central and inscribed angle.

   Since $m\angle B$, the inscribed angle, is represented by x in the diagram, $m\angle B = 12$. Also, $m\angle D = 2m\angle B = 2x$, so the measure of the central angle, D, is 24°.

   $m\angle D = 24^\circ$

   $m\angle B = 12^\circ$

Connection to the Lesson

- Students will extend their knowledge of central angles to that of the Chord Central Angles Conjecture.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

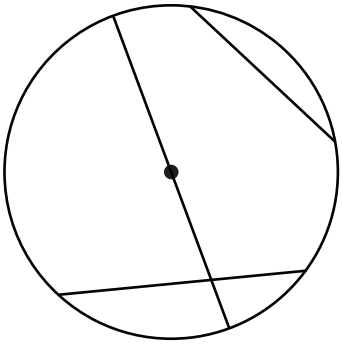
- setting up and solving equations (8.EE.7b)
- evaluating expressions (6.EE.2c)
- understanding central angles and inscribed angles (G–C.2)

Introduction

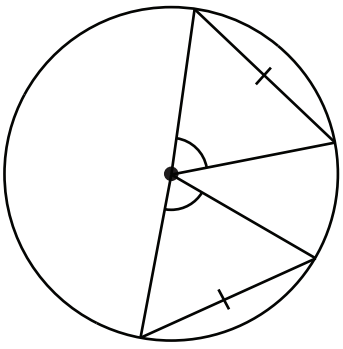
Circles have several special properties, conjectures, postulates, and theorems associated with them. This lesson focuses on the relationship between chords and the angles and arcs they create.

Key Concepts

- Chords are segments whose endpoints lie on the circumference of a circle.
- Three chords are shown on the following circle.



- Congruent chords of a circle determine congruent central angles.



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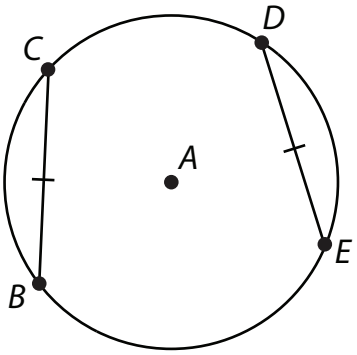
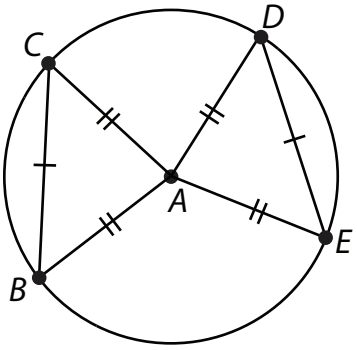
Instruction

- Congruent chords also intercept congruent arcs.
- An intercepted arc is an arc whose endpoints are on the sides of an angle and whose other points are in the interior of the angle.
- Remember that the measure of an arc is the same as the measure of its central angle.
- Also, recall that central angles are twice the measure of their inscribed angles.
- In the following circle, chords $\overline{BC}$ and $\overline{DE}$ are congruent chords of $\odot A$.



- When the radii are constructed such that each endpoint of the chord connects to the center of the circle, four central angles are created, as well as two congruent isosceles triangles by the SSS Congruence Postulate.



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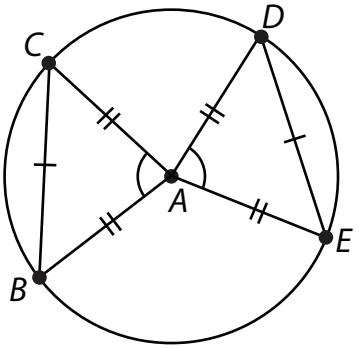
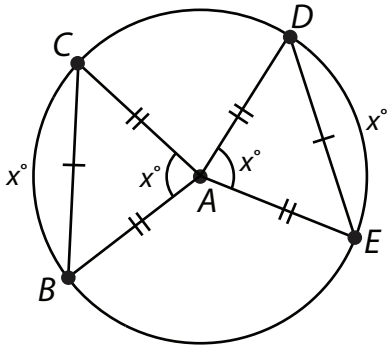
Instruction

- Since the triangles are congruent and both triangles include two central angles that are the vertex angles of the isosceles triangles, those central angles are also congruent because Corresponding Parts of Congruent Triangles are Congruent (CPCTC).



- The measure of the arcs intercepted by the chords is congruent to the measure of the central angle because arc measures are determined by their central angle.



- The previous statements about chords and their central angles can be summarized by the Chord Central Angles Conjecture.

Chord Central Angles Conjecture
• If two chords of a circle are congruent, then they determine central angles that are also congruent. • If two chords of a circle are congruent, then their intercepted arcs are congruent.

Common Errors/Misconceptions

- mistakenly believing that central angles of different circles will be congruent if their chords are congruent

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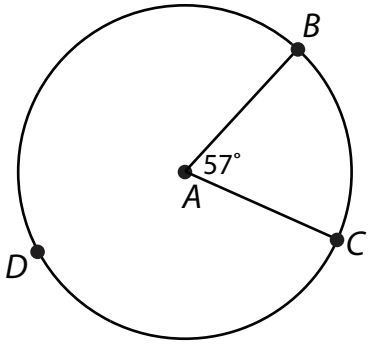
Instruction

Guided Practice 6.1.2

Example 1

In $\odot A$, $m\angle BAC = 57$. What is $m\overset{\frown}{BDC}$?



1. Find the measure of $\overset{\frown}{BC}$.

   $m\angle BAC = 57°$

   The measure of $\angle BAC$ is equal to the measure of $\overset{\frown}{BC}$ because central angles are congruent to their intercepted arc; therefore, the measure of $\overset{\frown}{BC}$ is also 57°.

2. Find the measure of $\overset{\frown}{BDC}$.

   Subtract the measure of $\overset{\frown}{BC}$ from 360°.

   $360 - 57 = 303$

   $m\overset{\frown}{BDC} = 303°$

3. State your conclusion.

   The measure of $\overset{\frown}{BDC}$ is 303°.

SWB p. 21; SRB p. U6-24

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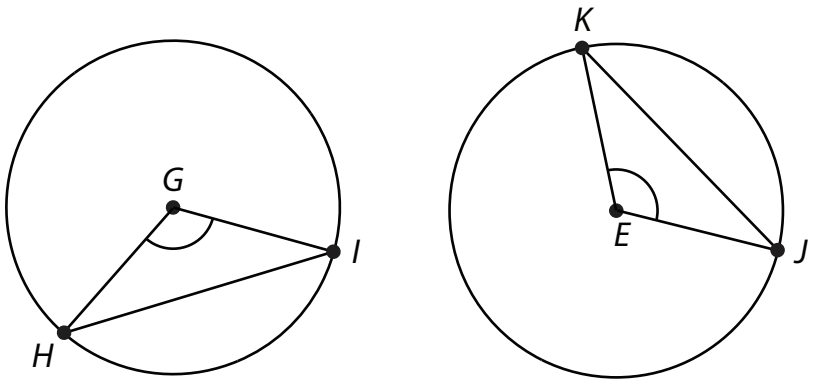
Instruction

Example 2

$\odot G \cong \odot E$. What conclusions can you make?



1. What type of angles are $\angle G$ and $\angle E$, and what does that tell you about the chords?

   $\angle G$ and $\angle E$ are congruent central angles of congruent circles, so $\overline{HI}$ and $\overline{JK}$ are congruent chords.

2. Since the central angles are congruent, what else do you know?

   The measures of the arcs intercepted by congruent chords are congruent, so minor arc *HI* is congruent to minor arc *JK*.

   Deductively, then, major arc *IH* is congruent to major arc *KJ*.

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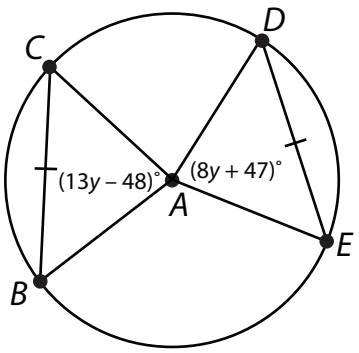
Instruction

Example 3

Find the value of y.



1. Set up an equation to solve for y.

   The angles marked are central angles created by congruent chords, and are therefore congruent.

   Set the measure of each angle equal to each other.

   $13y - 48 = 8y + 47$

2. Solve for y.

   $13y - 48 = 8y + 47$ Central angles created by congruent chords are congruent.

   $5y = 95$

   $y = 19$

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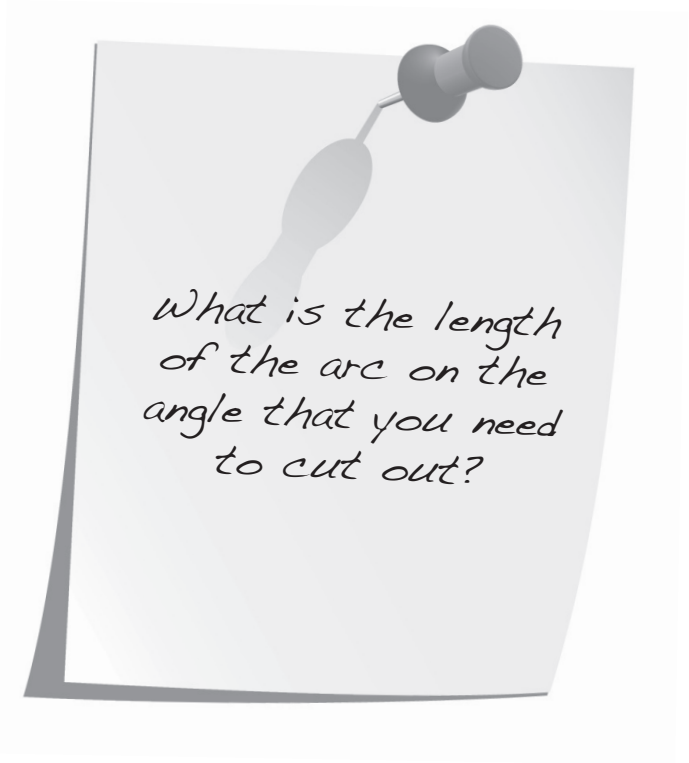
Problem-Based Task 6.1.2: Masking the Problem

SMP	
1 ✓	2
3 ✓	4 ✓
5	6
7	8

In art class, you are creating a mask from a circular plate. The circumference of the plate is 30 inches and you have been instructed to cut out the central angle such that its intercepted major arc measures 200°. What is the length of the arc on the angle that you need to cut out?



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Problem-Based Task 6.1.2: Masking the Problem

Coaching

a. What is the relationship between the central angle and its intercepted arc?

b. What is the degree measure of a circle?

c. What equation could you set up to find the length of the arc?

d. What is the length of the arc?

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Instruction

Problem-Based Task 6.1.2: Masking the Problem

Coaching Sample Responses

a. What is the relationship between the central angle and its intercepted arc?

A central angle is equal to the degree measure of its intercepted arc.

b. What is the degree measure of a circle?

A circle has a measure of 360°.

c. What equation could you set up to find the length of the arc?

You could set up a proportion relating the degrees in the arc and the degrees in the complete circle to the arc length and the circumference of the circle.

d. What is the length of the arc?

$$\frac{200}{360} = \frac{x}{30}$$

$$6000 = 360x$$

$$x = 16\frac{2}{3}$$

The length of the arc should be about 16.7 inches.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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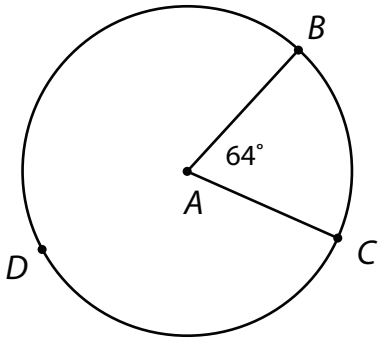
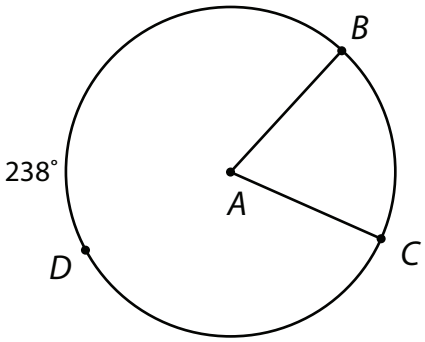
Practice 6.1.2: Chord Central Angles Conjecture

Use what you've learned about chords and central angles to solve.

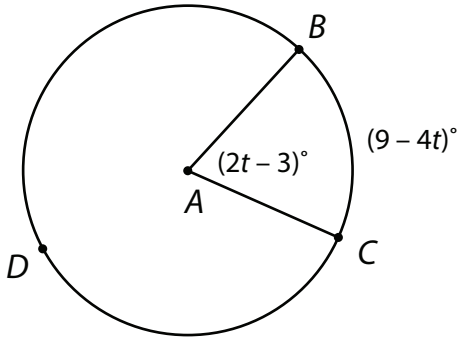
1. In $\odot A$, $m\angle BAC = 64°$. What is $m\overset{\frown}{BDC}$?



2. In $\odot A$, $m\overset{\frown}{BDC} = 238°$. What is $m\angle BAC$?



3. What is the value of t?



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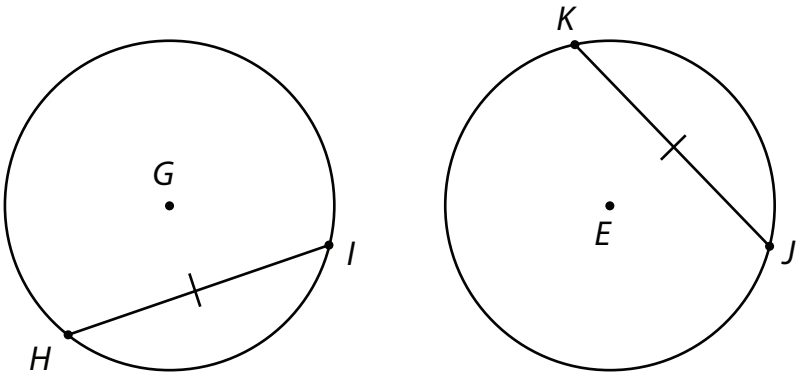
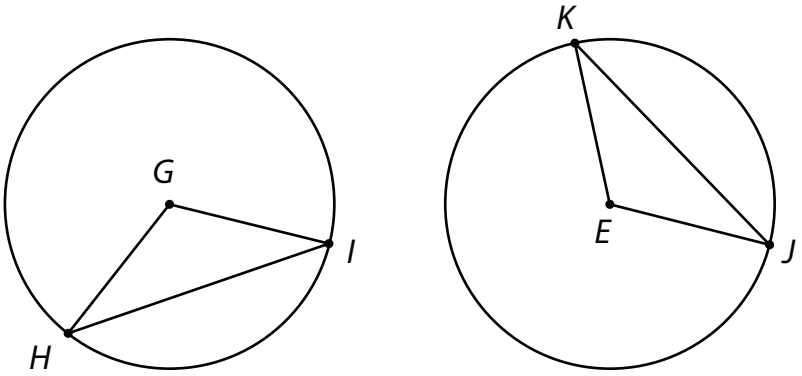
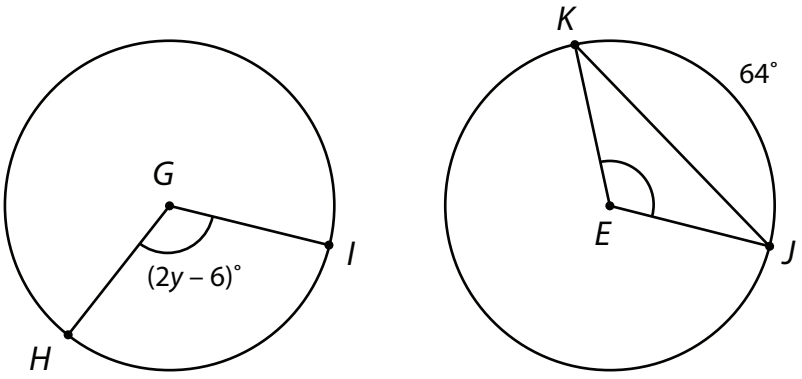
4. If circles G and E are congruent, what can you conclude about $\odot G$ and $\odot E$?



5. Is there enough information to conclude that the central angles of the chords are congruent? Explain.



6. $\odot G \cong \odot E$. What is the value of y?



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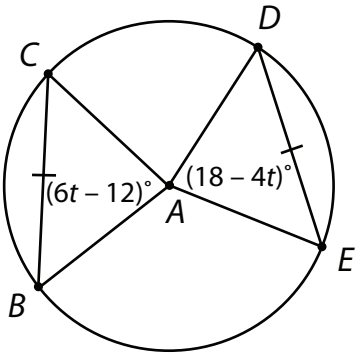
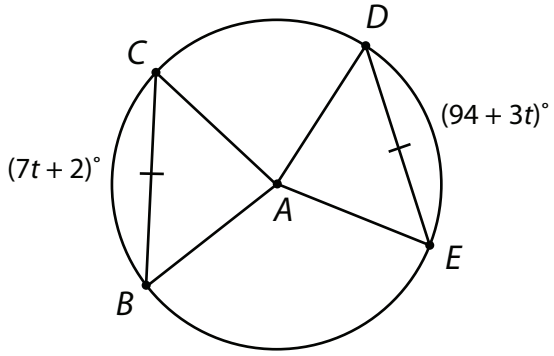
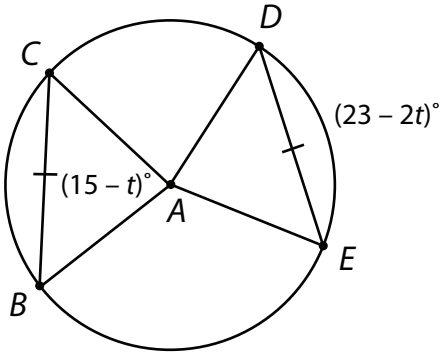
7. Find the value of t.



8. Find the value of t.



9. Find the value of t.



10. The circumference of the trunk of a tree to be decorated is 12 inches. You have 7 inches of garland to wrap partially around the tree trunk. What is the arc angle of the trunk that you will decorate?

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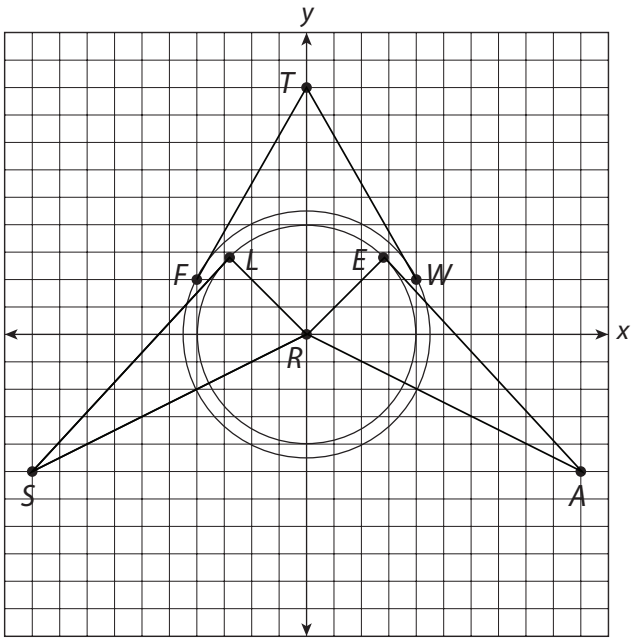
Lesson 6.1.3: Properties of Tangents of a Circle

Warm-Up 6.1.3

Lennon is an artist. He is in the process of designing a logo for the Star model rocket company and has some of the dimensions of his design. Help Lennon make sure his design is accurate by answering the following questions.



1. In order to fit in the allotted space, the length of $\overline{SL}$ must be 24 centimeters and the length of radius $\overline{RL}$ must be 7 centimeters. If both of the bottom triangles are congruent, and $\angle REA$ is equal to 90°, what must the length of $\overline{SR}$ be?

2. Based on the information discovered in your answer to the previous question, what can Lennon conclude about a line that touches a circle at exactly one point?

3. Lennon needs to figure out the lengths of $\overline{TW}$ and $\overline{TF}$. The points on the design grid are $T(0, 9)$, $F(-4, 2)$, and $W(4, 2)$. Use the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, to find the lengths of $\overline{TW}$ and $\overline{TF}$.

4. Based on the information discovered in your answer to the previous problem, what can Lennon conclude the next time he draws two lines that extend from the same point and touch the same circle, each in exactly one point?

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Lesson 6.1.3: Properties of Tangents of a Circle

Common Core State Standard

G–C.2 Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*

Warm-Up 6.1.3 Debrief

1. In order to fit in the allotted space, the length of $\overline{SL}$ must be 24 centimeters and the length of radius $\overline{RL}$ must be 7 centimeters. If both of the bottom triangles are congruent, and $\angle REA$ is equal to 90°, what must the length of $\overline{SR}$ be?

   If $\angle REA$ is a 90° angle, then $\triangle REA$ is a right triangle.

   Since $\triangle REA$ and $\triangle RLS$ are congruent, you can use the Pythagorean Theorem to find the length of $\overline{SR}$.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$7^2 + 24^2 = c^2$	Substitute values for a and b.
$49 + 576 = c^2$	Simplify.
$625 = c^2$	Solve for c.
$c = 25$	

   $\overline{SR} = 25$ cm

2. Based on the information discovered in your answer to the previous question, what can Lennon conclude about a line that touches a circle at exactly one point?

   Line segments $\overline{EA}$ and $\overline{LS}$ both touch the circle at exactly one point.

   It can be concluded that both lines form right angles with the radius of the circle, and are therefore perpendicular to the radius.

   It can also be concluded, by the Pythagorean Theorem, that both segments form right triangles with the radius and an additional line drawn from the center of the circle to a point on the segment exterior to the circle.

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Instruction

3. Lennon needs to figure out the lengths of $\overline{TW}$ and $\overline{TF}$. The points on the design grid are T (0, 9), F (–4, 2), and W (4, 2). Use the distance formula, $d=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$, to find the lengths of $\overline{TW}$ and $\overline{TF}$.

   Find the length of $\overline{TW}$.

   $d=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$ Distance formula

   $d=\sqrt{\left(4-0\right)^2+\left(2-9\right)^2}$ Substitute (0, 9) and (4, 2) for (x_1, y_1) and (x_2, y_2).

   $d=\sqrt{\left(4\right)^2+\left(-7\right)^2}$ Simplify.

   $d=\sqrt{16+49}$ Solve for d.

   $d=\sqrt{65}$

   Find the length of $\overline{TF}$.

   $d=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$ Distance formula

   $d=\sqrt{[(-4)-0]^2+\left(2-9\right)^2}$ Substitute (0, 9) and (–4, 2) for (x_1, y_1) and (x_2, y_2).

   $d=\sqrt{\left(-4\right)^2+\left(-7\right)^2}$ Simplify.

   $d=\sqrt{16+49}$ Solve for d.

   $d=\sqrt{65}$

4. Based on the information discovered in your answer to the previous problem, what can Lennon conclude the next time he draws two segments that extend from the same point and touch the same circle, each in exactly one point?

   Lennon can conclude that the two segments are congruent.

Connection to the Lesson

- Students will discover and make the connection that a tangent line is perpendicular to the radius of a circle at the point of tangency.
- Students will discover and make the connection that two segments drawn from the same exterior point and tangent to the same circle are congruent.
- Students will explore real-world applications of tangent lines.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

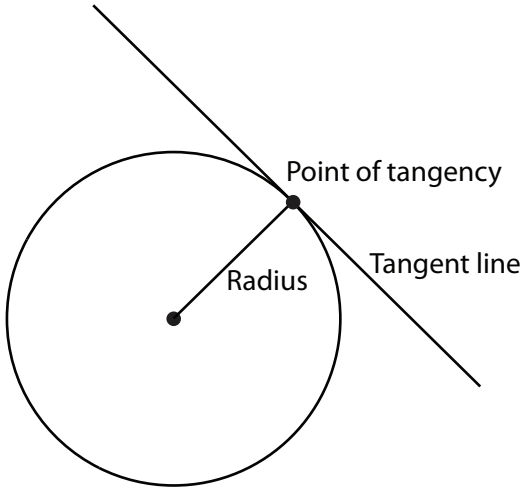
- performing operations with fractions (7.NS.1)
- understanding slope, both algebraically and graphically (8.F.4)
- understanding the relationship of the slope of perpendicular lines (G–GPE.5)
- understanding how to write an equation of a line using point-slope form (A–CED.4)
- knowing that the sum of the angles of a triangle is 180° (G–SRT.5)
- performing calculations using the Pythagorean Theorem (8.G.7)
- performing calculations using the distance formula (8.G.8)
- understanding intercepted arcs (G–C.2)

Introduction

Circles and tangent lines can be useful in many real-world applications and fields of study, such as construction, landscaping, and engineering. There are many different types of lines that touch or intersect circles. All of these lines have unique properties and relationships to a circle. Specifically, in this lesson, we will identify what a tangent line is, explore the properties of tangent lines, prove that a given line is tangent to a circle, and find the lengths of segments that are tangent to a circle. We will also identify and use secant lines, as well as discuss how they are different from tangent lines.

Key Concepts

- A **tangent line** is a line that intersects a circle at exactly one point.
- Tangent lines are perpendicular to the radius of the circle at the point of tangency.



SRB p. U6-30

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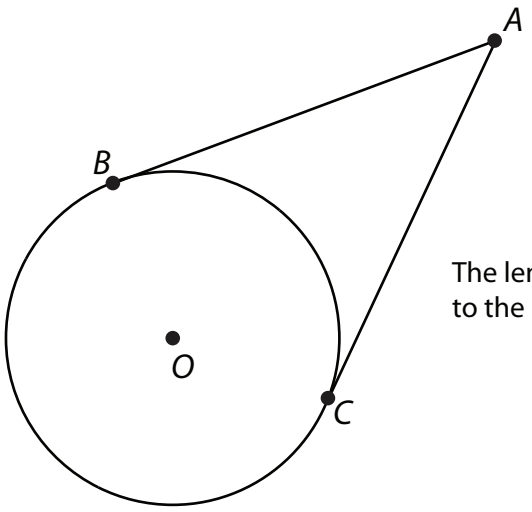
Instruction

- You can verify that a line is tangent to a circle by constructing a right triangle using the radius, and verifying that it is a right triangle by using the Pythagorean Theorem.
- The slopes of a line and a radius drawn to the possible point of tangency must be opposite reciprocals in order for the line to be a tangent.
- If two segments are tangent to the same circle, and originate from the same exterior point, then the segments are congruent.
- The angle formed by two tangent lines whose vertex is outside of the circle is called the **circumscribed angle**.
- $\angle BAC$ is a circumscribed angle.



The length of tangent $\overline{AB}$ is equal to the length of tangent $\overline{AC}$.

- The angle formed by two tangents is equal to one half the positive difference of the angle's intercepted arcs. In the previous diagram, the measure of $\angle BAC$ is one half of the difference between the angle measures of major arc $\overset{\frown}{BC}$ and minor arc $\overset{\frown}{BC}$.

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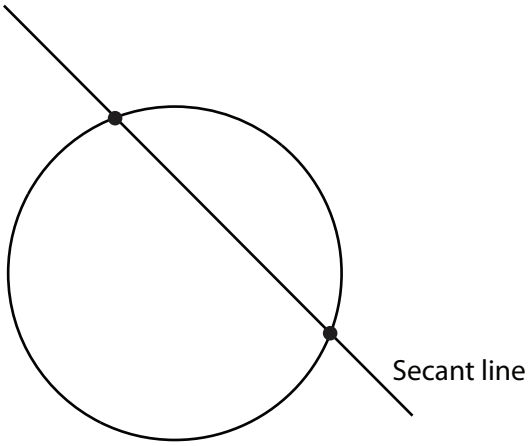
Instruction

- A **secant line** is a line that intersects a circle at two points.



- An angle formed by a secant and a tangent is equal to the positive difference of its intercepted arcs.

Common Errors/Misconceptions

- assuming that a radius and a line form right angles at the possible point of tangency simply by relying on observation
- assuming that two tangent lines are congruent by observation
- making incorrect calculations (usually sign errors) when using the slope formula
- making incorrect calculations when using formulas such as the Pythagorean Theorem and the distance formula
- confusing secant lines and tangent lines

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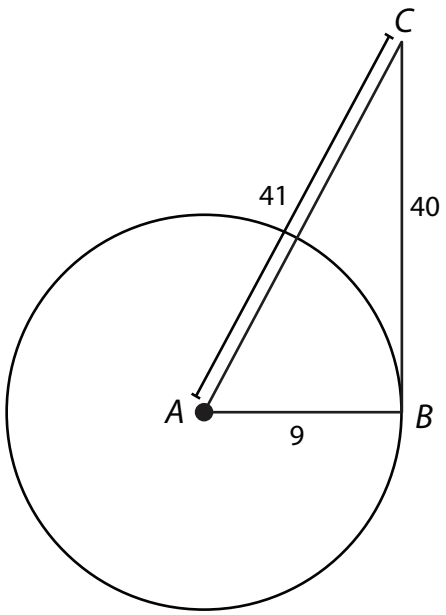
Instruction

Guided Practice 6.1.3

Example 1

Determine whether $\overline{BC}$ is tangent to $\odot A$ in the following diagram.



1. Identify the radius.

   The radius of the circle is the segment from the center of the circle to a point on the circle.

   $\overline{AB}$ is the radius of $\odot A$.

2. Determine the relationship that must exist between $\overline{AB}$ and $\overline{BC}$ at point B in order for $\overline{BC}$ to be tangent to $\odot A$.

   $\overline{AB}$ must be perpendicular to $\overline{BC}$ at point B in order for $\overline{BC}$ to be tangent to $\odot A$.

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Instruction

3. Show that $\angle ABC$ is a right angle by using the converse of the Pythagorean Theorem.

   The converse of the Pythagorean Theorem states that when the sum of the squares of two sides of a triangle is equal to the square of the third side of the triangle, the triangle is a right triangle.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$(AB)^2 + (BC)^2 = (AC)^2$	Substitute segment names for a, b, and c.
$9^2 + 40^2 = 41^2$	Substitute values for $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$.
$81 + 1600 = 1681$	Simplify, then solve.
$1681 = 1681$	

   The result is a true statement; therefore, $\angle ABC$ is a right angle.

4. State your conclusion.

   Since the converse of the Pythagorean Theorem is true, $\triangle ABC$ is a right triangle; therefore, $\angle ABC$ is a right angle. This means that $\overline{AB}$ is perpendicular to $\overline{BC}$; therefore, $\overline{BC}$ is tangent to $\odot A$.

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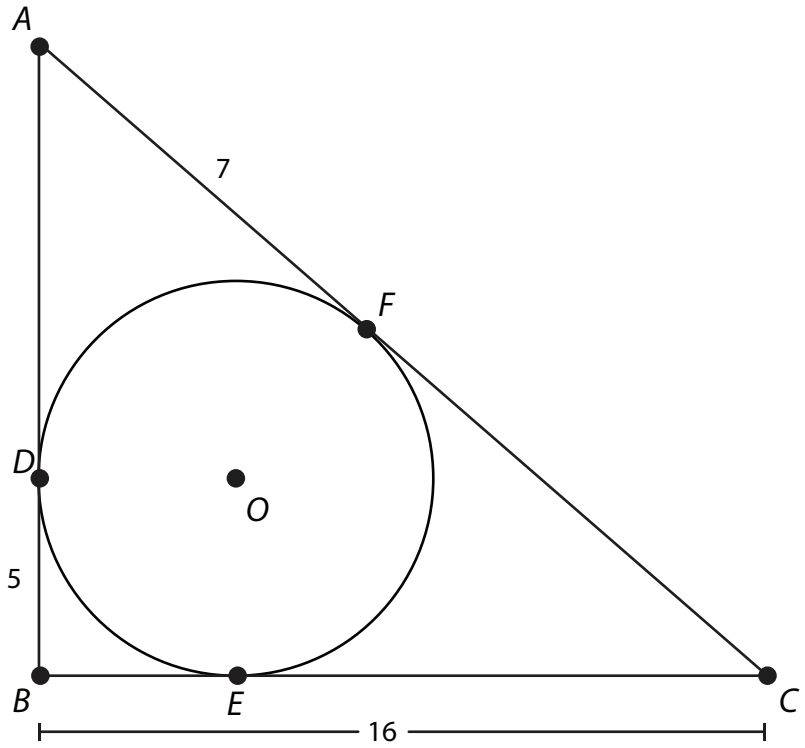
Instruction

Example 2

Each side of $\triangle ABC$ is tangent to circle O at the points D, E, and F. Find the perimeter of $\triangle ABC$.



1. Identify the lengths of each side of the triangle.

   $\overline{AD}$ is tangent to the same circle as $\overline{AF}$ and extends from the same point; therefore, the lengths are equal.

   $AD = 7$ units

   $\overline{BE}$ is tangent to the same circle as $\overline{BD}$ and extends from the same point; therefore, the lengths are equal.

   $BE = 5$ units

   To determine the length of $\overline{CE}$, subtract the length of $\overline{BE}$ from the length of $\overline{BC}$.

   $16 - 5 = 11$

   $CE = 11$ units

   $\overline{CF}$ is tangent to the same circle as $\overline{CE}$ and extends from the same point; therefore, the lengths are equal.

   $CF = 11$ units

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Instruction

2. Calculate the perimeter of $\triangle ABC$.

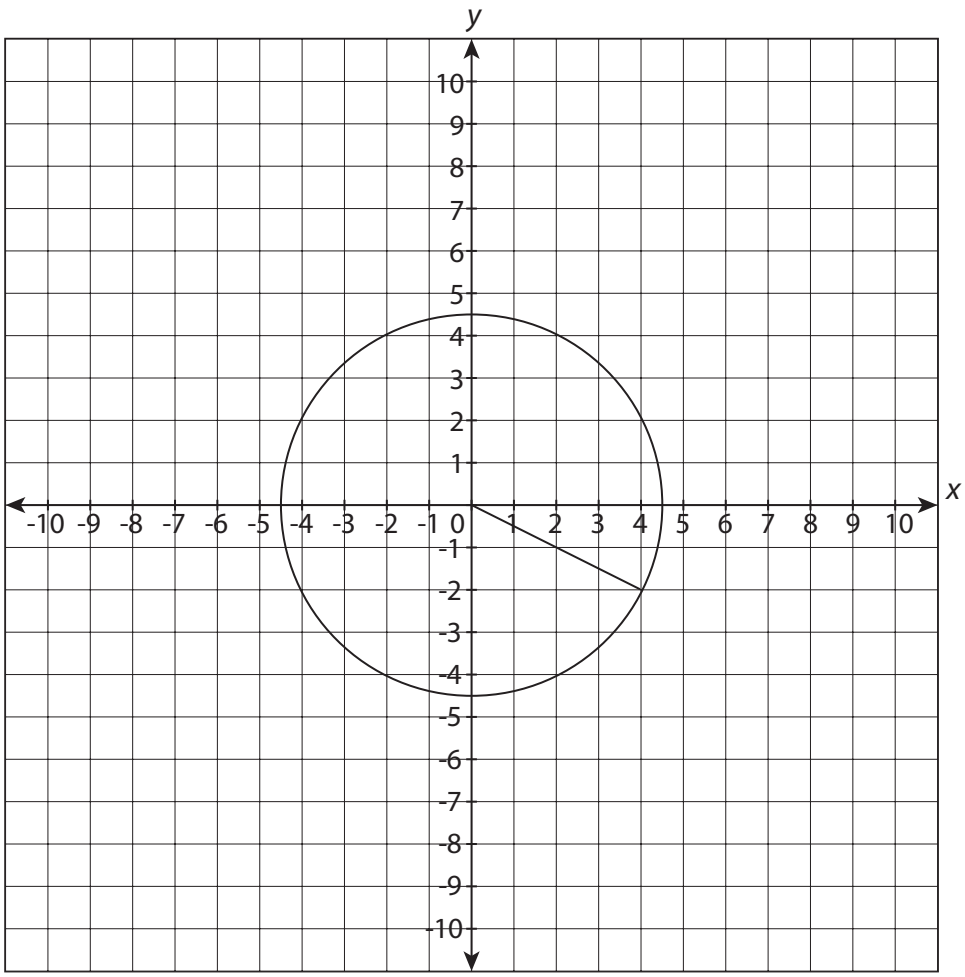
   Add the lengths of $\overline{AD}$, $\overline{AF}$, $\overline{BD}$, $\overline{BE}$, $\overline{CE}$, and $\overline{CF}$ to find the perimeter of the polygon.

   $7 + 7 + 5 + 5 + 11 + 11 = 46$ units

   The perimeter of $\triangle ABC$ is 46 units.

Example 3

A landscaper wants to build a walkway tangent to the circular park shown in the following diagram. The other walkway pictured is a radius of the circle and has a slope of $-\frac{1}{2}$ on the grid. If the walkways should intersect at (4, –2) on the grid, what equation can the landscaper use to graph the new walkway on the grid?



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Instruction

1. Determine the slope of the walkway.

   One of the properties of tangent lines states that if a line is tangent to a circle, then it is perpendicular to the radius at the point of tangency.

   Since the slope of the radius is $-\frac{1}{2}$, then the slope of the new walkway will be 2 because perpendicular lines have slopes that are opposite reciprocals.

2. Determine the equation that represents the walkway.

   Recall that the point-slope form of a line is $y-y_1=m(x-x_1)$.

   Since the new walkway is tangent to the circle at (4, –2), then the point (4, –2) is on the tangent line, so it can be used to write the equation.

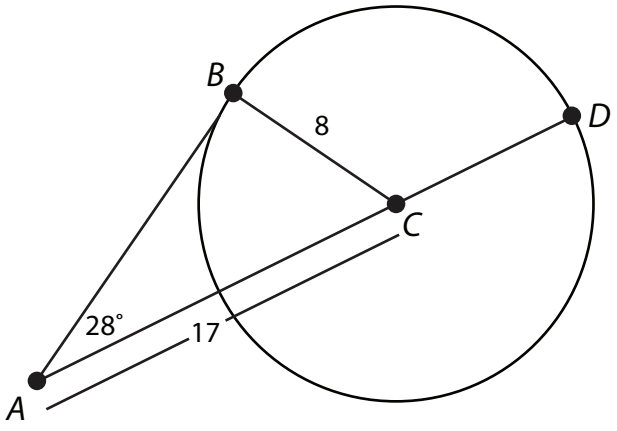
   $y-(-2)=2(x-4)$

   $y+2=2(x-4)$

   This equation can be rearranged into slope-intercept form, $y=2x-10$, for easier graphing.

Example 4

$\overline{AB}$ is tangent to $\odot C$ at point B as shown in the following circle. Find the length of $\overline{AB}$ as well as $m\overset{\frown}{BD}$.



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Instruction

1. Find the length of $\overline{AB}$.

   Since $\overline{AB}$ is tangent to $\odot C$, then $\angle ABC$ is right angle because a tangent and a radius form a right angle at the point of tangency.

   Since $\angle ABC$ is a right angle, $\triangle ABC$ is a right triangle.

   Use the Pythagorean Theorem to find the length of $\overline{AB}$.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$8^2 + (AB)^2 = 17^2$	Substitute values for a, b, and c.
$64 + (AB)^2 = 289$	Simplify.
$(AB)^2 = 225$	
$AB = 15$	

   The length of $\overline{AB}$ is 15 units.

2. Find $m\overset{\frown}{BD}$.

   First, determine the unknown measure of $\triangle ACB$.

   Recall that the sum of all three angles of a triangle is 180°.

   $\angle ABC$ is a right angle, so it is 90°.

   $\angle BAC$ is 28°, as shown in the diagram.

   Set up an equation to determine the measure of $\angle ACB$.

   $28 + 90 + m\angle ACB = 180$

   $118 + m\angle ACB = 180$

   $m\angle ACB = 62$

   Since $m\angle ACB = 62°$, then $m\angle BCD = 118°$ because $\angle ACB$ and $\angle BCD$ are a linear pair.

   $\angle BCD$ is a central angle, and recall that the measure of a central angle is the same as its intercepted arc, so $\overset{\frown}{BD}$ is 118°.

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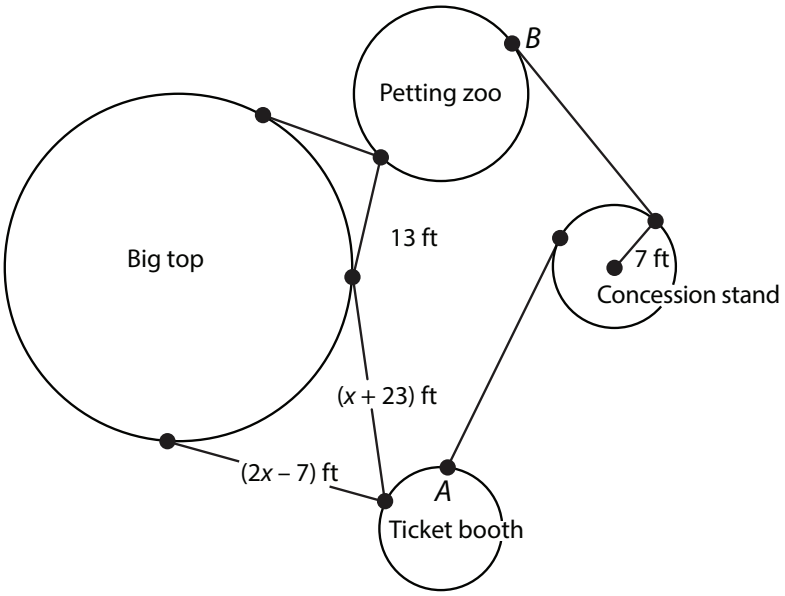
Problem-Based Task 6.1.3: The Circus Is in Town! Is It Safe?

SMP	
1 ✓	2
3 ✓	4
5	6 ✓
7	8

The circus has just arrived in town and is setting up. The fire inspector is on site to make sure that the circus meets local fire codes. The fire code requires that the circus have at least 175 feet of exterior walkways to pass inspection so that audience members are safe.

In the diagram below, all lines that appear to be tangent are tangent. The distance from the ticket booth, exit A, to the center of the concession stand is 25 feet. It is the same distance from the center of the concession stand to the petting zoo, at exit B. Use this information and the diagram to determine if the circus will pass inspection.



Use this information and the diagram to determine if the circus will pass inspection.

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Problem-Based Task 6.1.3: The Circus Is in Town! Is It Safe?

Coaching

a. What is the length of the walkway tangent to the concession stand, which starts at the ticket booth?

b. What is the length of the walkway tangent to the concession stand, which ends at the petting zoo?

c. What is the length of the walkway tangent to the big top, which begins on the edge of the petting zoo and extends from the same point as the walkway that is 13 feet long?

d. What is the value of x for the two walkways tangent to the big top, extending from the same point at the ticket booth?

e. What is the length of the lines tangent to the big top, extending from the same point at the ticket booth?

f. What is the sum of the lengths of all of the exterior walkways?

g. Will the circus pass inspection?

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Instruction

Problem-Based Task 6.1.3: The Circus Is in Town! Is It Safe?

Coaching Sample Responses

a. What is the length of the walkway tangent to the concession stand, which starts at the ticket booth?

The distance from the ticket booth to the center of the concession stand is given in the problem as 25 feet.

You can draw a radius from the point of tangency of the walkway to the center of the concession stand, which is 7 feet, as it is the same length of the other radius in the circle, which is marked on the diagram.

Use the Pythagorean Theorem to find the length of the walkway.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$7^2 + b^2 = 25^2$	Substitute values for a and c.
$49 + b^2 = 625$	Simplify, then solve for b.
$b^2 = 576$	
$b = 24$	

The length of the walkway tangent to the concession stand is 24 feet.

b. What is the length of the walkway tangent to the concession stand, which ends at the petting zoo?

The radius drawn to the point of tangency of this line is 7 feet as shown in the diagram and the distance from the center of the concession stand to the petting zoo is 25 feet; therefore, the tangent line is the same length as the one that extends from the ticket booth in part a, 24 feet.

c. What is the length of the walkway tangent to the big top, which begins on the edge of the petting zoo and extends from the same point as the walkway that is 13 feet long?

Two lines extending from the same exterior point, tangent to the same circle, are congruent.

Since one of the tangents is 13 feet long, the other is also 13 feet long.

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d. What is the value of x for the two walkways tangent to the big top, extending from the same point at the ticket booth?

Since two lines extending from the same exterior point to the same circle are congruent, set the equations for the lengths of each of the lines equal to each other to find x.

$2x - 7 = x + 23$

$2x = x + 30$

$x = 30$

e. What is the length of the lines tangent to the big top, extending from the same point at the ticket booth?

Substitute 30 into one of the equations to find the length of the tangent line.

$2x - 7$

$2(30) - 7 = 53$

Although the other tangent is 53 feet as well, it is always good to check by substituting the value of x into the other equation to verify that they are the same.

$x + 23$

$30 + 23 = 53$

The lines tangent to the big top, extending from the same point as the ticket booth, are each 53 feet long.

f. What is the sum of the lengths of all of the exterior walkways?

Add up all of the lengths of all the walkways.

$24 + 24 + 13 + 13 + 53 + 53 = 180$

The sum of the lengths of the walkways is 180 feet.

g. Will the circus pass inspection?

Yes, the circus will pass inspection; the requirement was to have at least 175 feet of exterior walkways, and it has 180 feet of walkways.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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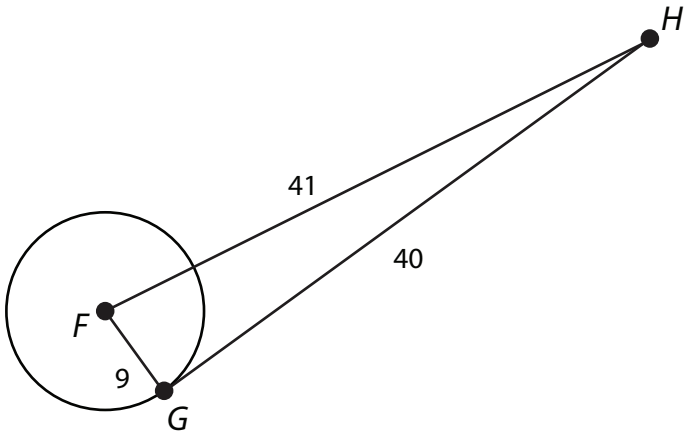
Practice 6.1.3: Properties of Tangents of a Circle

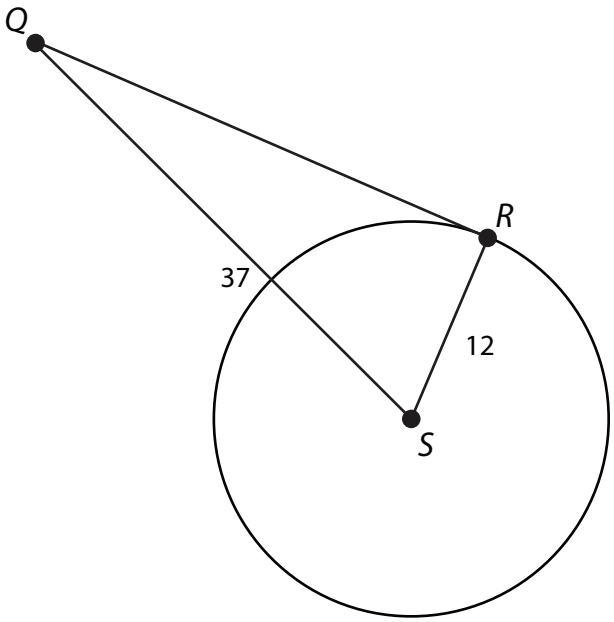
Use what you have learned about tangent lines and secant lines to answer the questions.

1. $\overline{AB}$ and $\overline{AC}$ are both tangent to circle D. If $\overline{AB}$ is 14 units and $\overline{AC}$ is $(x - 3)$ units, what is the value of x?

2. Is $\overline{GH}$ tangent to circle F in the diagram below?



3. $\overline{QR}$ is tangent to circle S at point R. What is the length of $\overline{QR}$?



continued

SWB p. 39; SRB Practice 6.1.3 p. U6-38

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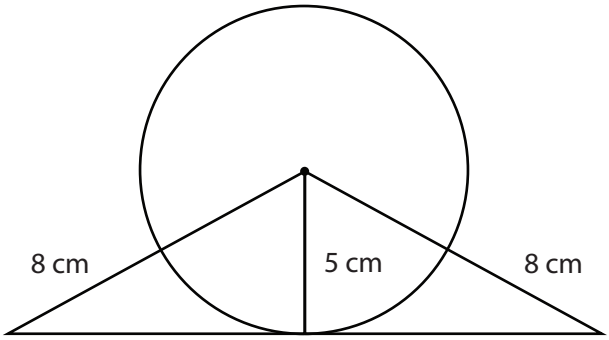
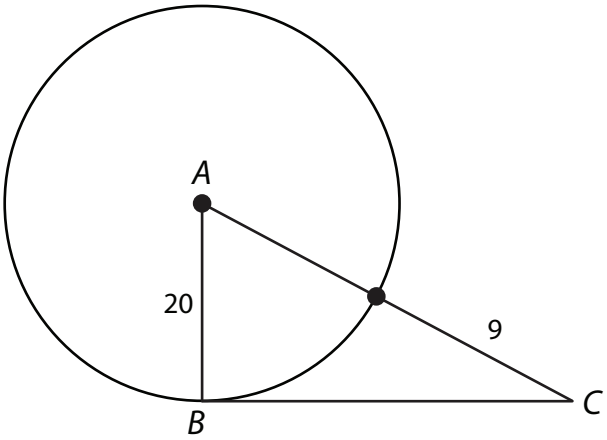
4. The class pet is a hamster. It has an exercise wheel like the one pictured below. What is the length of the base of the wheel stand?



5. $\overline{BC}$ is tangent to $\odot A$ at point B in the diagram below. What is the length of $\overline{BC}$?



6. Emma must prove that $\overline{AB}$ is tangent to a circle C at point A. She is going to show that $\triangle ABC$ is a right triangle. Why is this an important part of her proof?

continued

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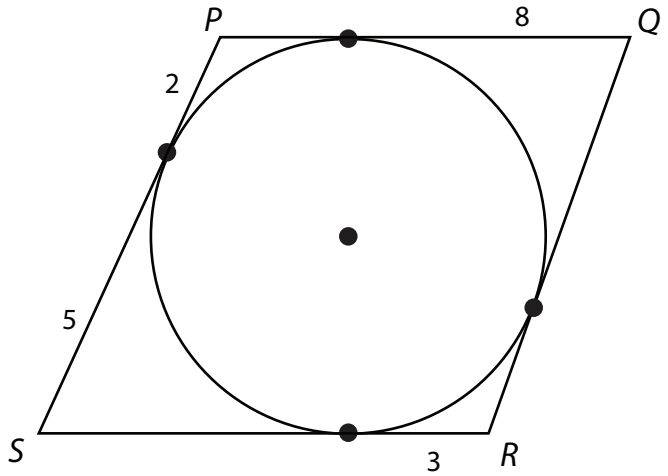
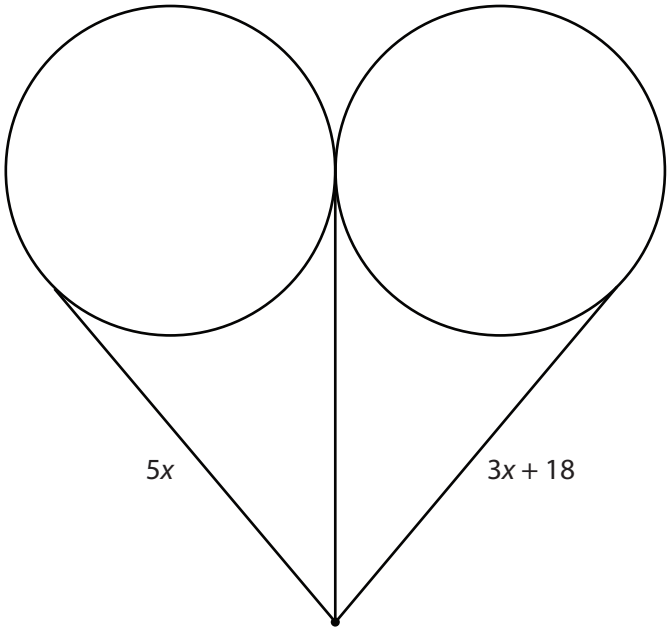
7. The sides of quadrilateral *PQRS* are tangent to the circle at the points as pictured below. What is the length of $\overline{QR}$?



8. Pictured below is the logo for a new ice cream shop. The circles are congruent. If the diameter of each circle is 10 inches, will the logo fit on a sign that is 60 inches tall? Explain.



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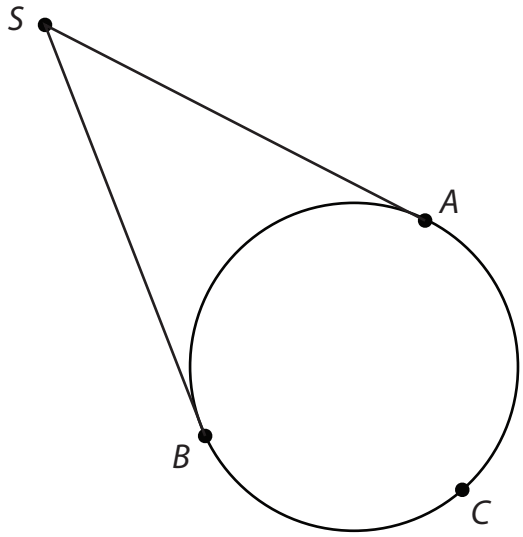
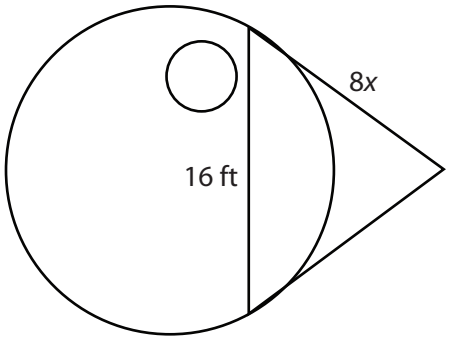
9. A satellite (S) in orbit around the Earth is sending two signals that are tangent to Earth at points A and B. If $m\overset{\frown}{ACB}$ is 208° and $m\overset{\frown}{AB}$ is 152°, what is the measure of $\angle S$?



10. Students in an algebra class are designing a new banner for the school's mascot, "The Fighting Duck." The beak is made from two tangents, connected by a vertical line at the point of tangency. If the students only have 200 feet of material to outline the duck's beak, what value of x will make the tangent sides the correct length? What is the length of each side?



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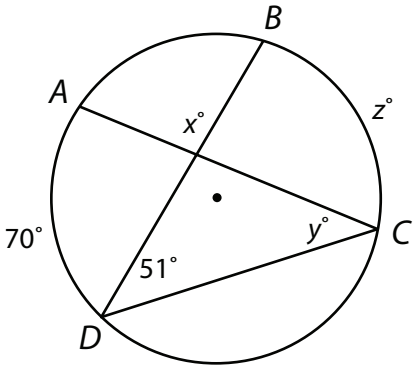
Progress Assessment

Circle the letter of the best answer.

1. A small car has a tire with a 13-inch diameter. A truck has a tire with a 29-inch diameter. How much farther than the car does the truck have to drive for its tire to complete one revolution?

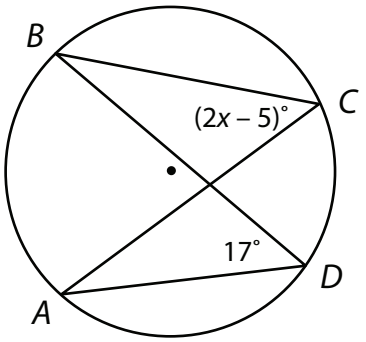
   a. 50.3 inches
   b. 42.9 inches
   c. 12 inches
   d. 28 inches

2. Find the values of x, y, and z.



   a. $x = 59°; y = 70°; z = 51°$
   b. $x = 94°; y = 35°; z = 102°$
   c. $x = 94°; y = 70°; z = 51°$
   d. $x = 59°; y = 51°; z = 70°$

3. Find the value of x and the measure of $\overset{\frown}{AB}$.



   a. $x = 11;\ m\overset{\frown}{AB} = 34$
   b. $x = 25.5;\ m\overset{\frown}{AB} = 46$
   c. $x = 17;\ m\overset{\frown}{AB} = 46$
   d. $x = 11;\ m\overset{\frown}{AB} = 17$

continued

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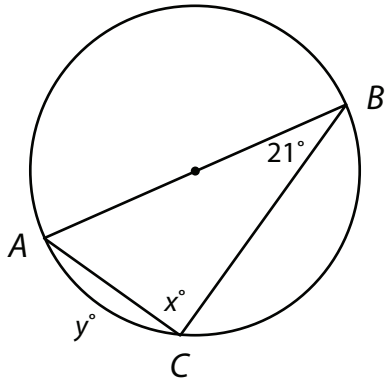
4. Find the values of x and y.



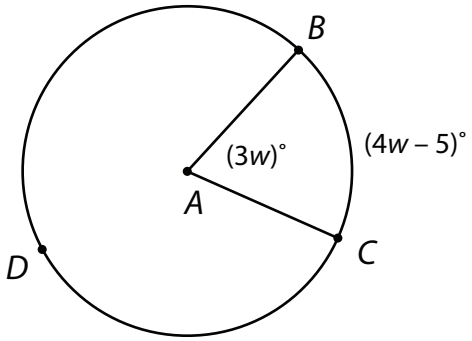
a. $x = 90°; y = 21°$

b. $x = 180°; y = 42°$

c. $x = 180°; y = 21°$

d. $x = 90°; y = 42°$

5. Find the value of w.



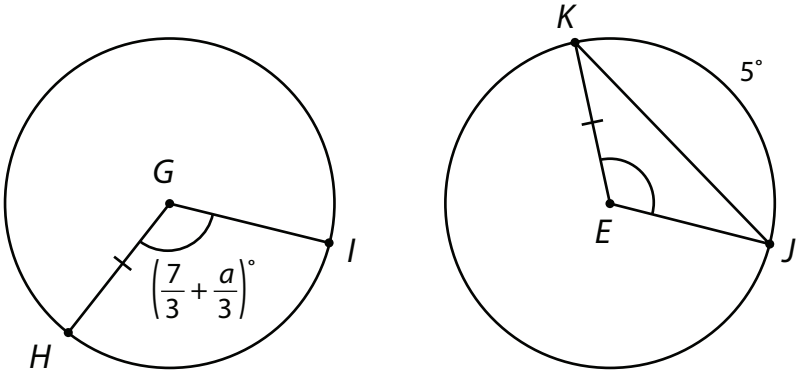
a. 3

b. 4

c. 5

d. 6

6. What is the value of a?



a. 10

b. 8

c. 12.5

d. 17.2

continued

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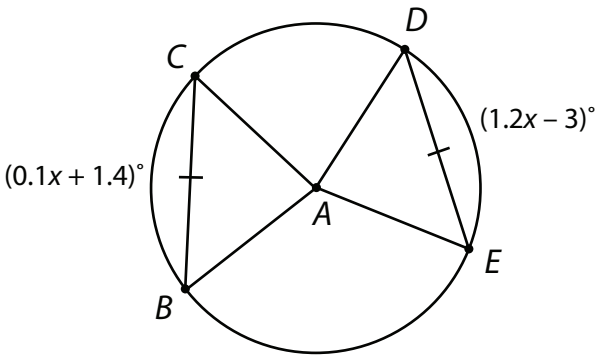
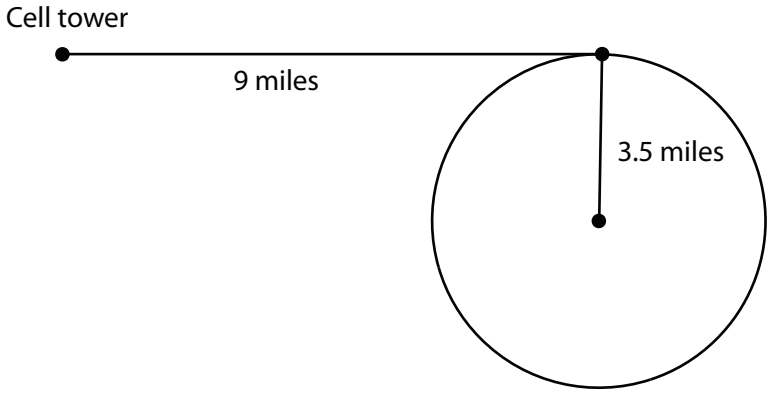
7. Find the value of x.



a. 12
b. 34
c. 4
d. 6.7

8. A cell tower, as shown below, can send a signal 10 miles. The cell tower's signal reaches the edge of a circular town, which is 9 miles away. There is a road that goes from the center of the town to the point where the signal stops. If the road is perpendicular to the path of the signal, will the signal reach the center of the town? If the signal does reach the center of the town, approximately how many miles must it travel?



a. The signal reaches the center of the town and travels approximately 8.29 miles.
b. The signal reaches the center of the town and travels exactly 10 miles.
c. The signal reaches the center of the town and travels approximately 9.66 miles.
d. The signal does not reach the center of the town.

continued

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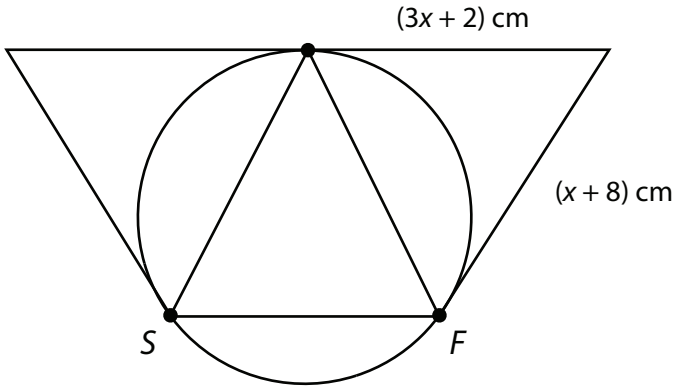
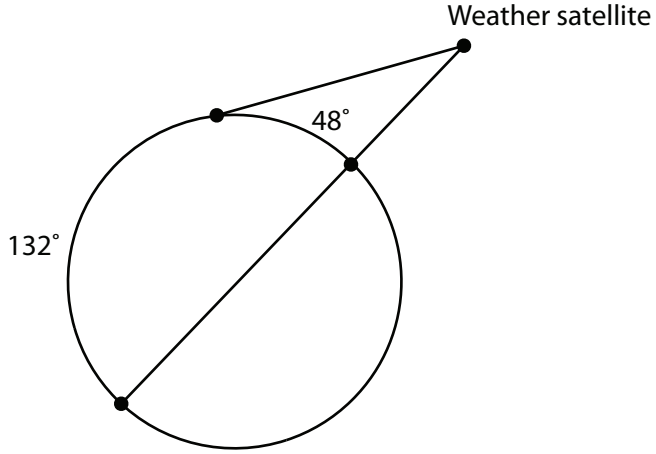
9. Organizers of an upcoming science fiction convention are designing a logo that needs to fit on their flyer. The perimeter of the logo must be 50 centimeters. The logo is made up of 3 congruent triangles and a circle as shown below. If the sides of the two outer triangles are tangent to the circle, what must the length of $\overline{SF}$ be?



a. 3 cm
b. 6 cm
c. 11 cm
d. 13 cm

10. The weather satellite shown below covers the area between the signals. One signal hits the exterior of the coverage area in exactly one point. The other signal hits the exterior of the coverage area in exactly 2 points. Based on the information in the diagram, what is the angle measure formed by the two weather signals?



a. 21°
b. 24°
c. 42°
d. 66°

continued

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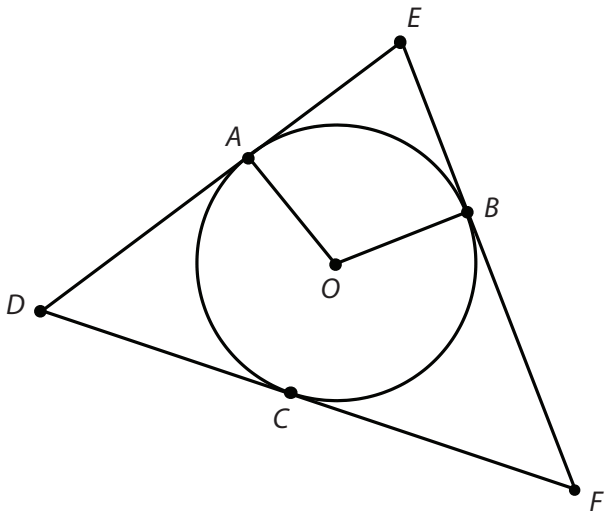
Assessment

Use what you know about circles to solve problem 11.

11. Djuna is making a design for her art project. The sides of the triangle on her design are tangent to the circle. Djuna needs to give her teacher at least 5 specific dimensions to get a good grade. Answer the following questions to help Djuna with her project.



a. If the slope of $\overline{OB}$ is $\frac{1}{2}$, what is the slope of $\overline{EB}$?

b. If $m\overset{\frown}{AB}$ is 106°, what is the measure of $\angle AEB$?

c. If $\overline{EO}$ is 9 units, and $\overline{OB}$ is approximately 4.5 units, what is the length of $\overline{EB}$ to the nearest tenth?

d. What is the length of $\overline{EA}$?

e. If the length of $\overline{EF}$ is 23 units, what is a possible way to find the distance from point *O* to point *F* given what you know about tangents?

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Pre-Assessment

Circle the letter of the best answer.

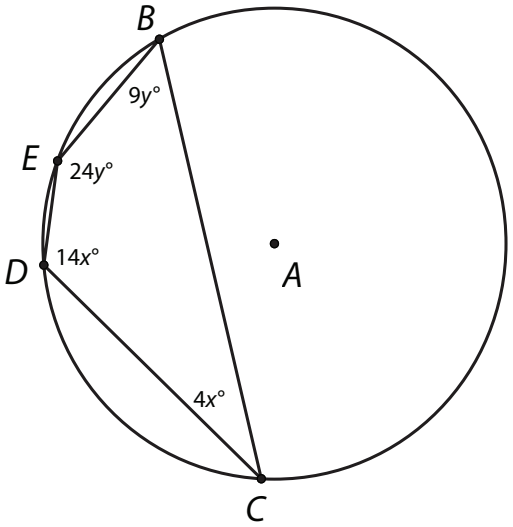
1. The incenter is the intersection of the three _____ of a triangle.
   a. altitudes
   b. angle bisectors
   c. medians
   d. perpendicular bisectors

2. The _____ is always equidistant from the three vertices of a triangle.
   a. centroid
   b. circumcenter
   c. incenter
   d. orthocenter

3. Which of the following quadrilaterals cannot be inscribed in a circle?
   a. isosceles trapezoid
   b. rectangle
   c. square
   d. parallelogram

4. If a quadrilateral is inscribed in a circle, what must be true of the quadrilateral's opposite angles?
   a. They must be complementary.
   b. They must be congruent.
   c. They must be supplementary.
   d. They have no relationship.

5. Find the value of x.



   a. 6
   b. 9
   c. 36
   d. 54

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Lesson 2: Inscribed Polygons and Circumscribed Triangles

Instruction

Common Core State Standard

G–C.3 Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

SMP
1 ✓ 2
3 ✓ 4 ✓
5 ✓ 6 ✓
7 ✓ 8 ✓

Essential Questions

1. What is the purpose of locating the incenter?
2. What is the purpose of locating the circumcenter?
3. Where are an incenter and an inscribed circle used in real life?
4. What types of quadrilaterals can be inscribed in a circle?

WORDS TO KNOW

angle bisector	a line, segment, or ray that divides an angle into two congruent angles
circumcenter	the intersection of the perpendicular bisectors of a triangle
circumscribed circle	a circle that passes through all vertices of a polygon
circumscribed triangle	a triangle whose sides are tangent to an interior circle
equidistant	the same distance from a reference point
incenter	the intersection of the angle bisectors of a triangle
inscribed circle	a circle that contains one point from each side of a triangle or other polygon
inscribed quadrilateral	a quadrilateral whose vertices are on a circle
inscribed triangle	a triangle whose vertices are on a circle
perpendicular bisector	a line that intersects a segment at its midpoint at a right angle
point of concurrency	a single point of intersection of three or more lines

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Instruction

Recommended Resources

- Beva.org. "Introduction to Sketchpad: Incenter and Circumcenter of Triangles."

  http://www.walch.com/rr/00051

  This website describes how to construct the incenter in The Geometer's Sketchpad dynamic geometry program.

- IXL Learning. "Angles in inscribed quadrilaterals."

  http://www.walch.com/rr/00052

  This interactive website gives a series of problems and scores them immediately. If the user submits a wrong answer, a description and process for arriving at the correct answer are provided. This activity is meant as practice for inscribed polygons.

- IXL Learning. "Identify medians, altitudes, angle bisectors, and perpendicular bisectors."

  http://www.walch.com/rr/00053

  This interactive website gives a series of problems and scores them immediately. If the user submits a wrong answer, a description and process for arriving at the correct answer are provided. These problems start with a triangle and a construction of a point of concurrency. Users are required to choose which point of concurrency is depicted. This activity is meant as a review of key terms for inscribed polygons and circumscribed triangles.

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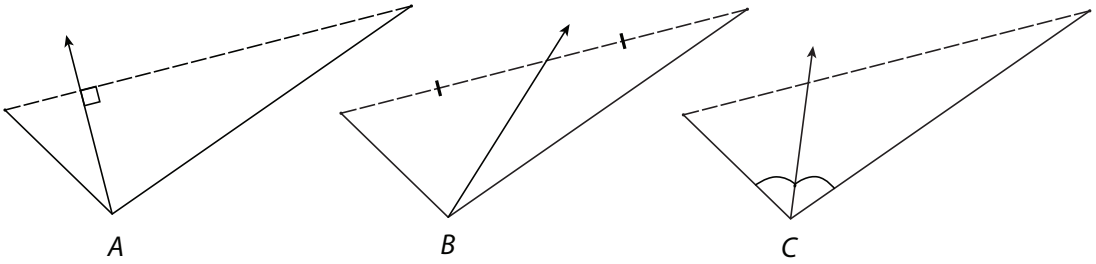
Lesson 6.2.1: Constructing Inscribed Circles

Warm-Up 6.2.1

A hiking trail with a common starting point diverges into two forks in a meadow. You notice uncommon wildflowers that you have wanted to photograph, so you decide to leave the trail.

1. Along what geometric construction would you need to walk in order to stay the same distance from either fork of the trail?

2. The following diagrams use two sides of a triangle to show the forks of the trail. Each ray represents a possible path you could take if you leave the trail. Which of the diagrams represents walking along a path that is equidistant from either trail?



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Instruction

Lesson 6.2.1: Constructing Inscribed Circles

Common Core State Standard

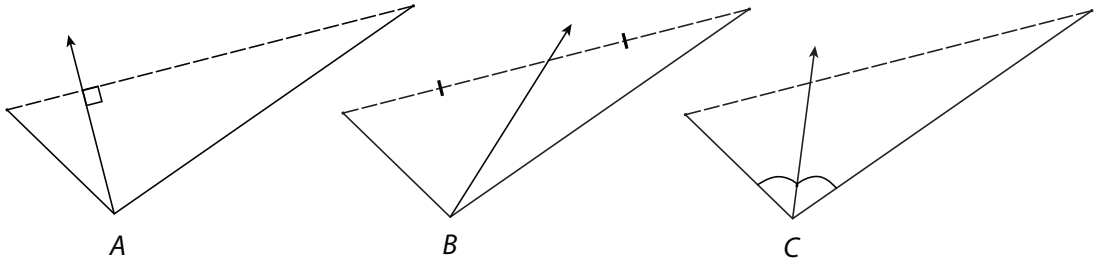
G–C.3 Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

Warm-Up 6.2.1 Debrief

1. Along what geometric construction would you need to walk in order to stay the same distance from either fork of the trail?

   To stay the same distance from either fork, you must walk along the angle bisector of the two trail forks.

2. The following diagrams use two sides of a triangle to show the forks of the trail. Each ray represents a possible path you could take if you leave the trail. Which of the diagrams represents walking along a path that is equidistant from either trail?



   As noted in the diagram, the ray with endpoint *A* is perpendicular to the segment that connects the endpoints of the paths, and therefore is not the angle bisector of the two forks of the trail.

   The ray with endpoint *B* divides the segment that connects the endpoints of the paths into two equal segments, and therefore is not the angle bisector of the two forks of the trail.

   The ray with endpoint *C* creates two equal angles; therefore, it is the angle bisector of the two forks of the trail.

Connection to the Lesson

- Students will extend their knowledge of an angle bisector to a point of concurrency.
- Students will apply this knowledge to problem-solving situations.

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Instruction

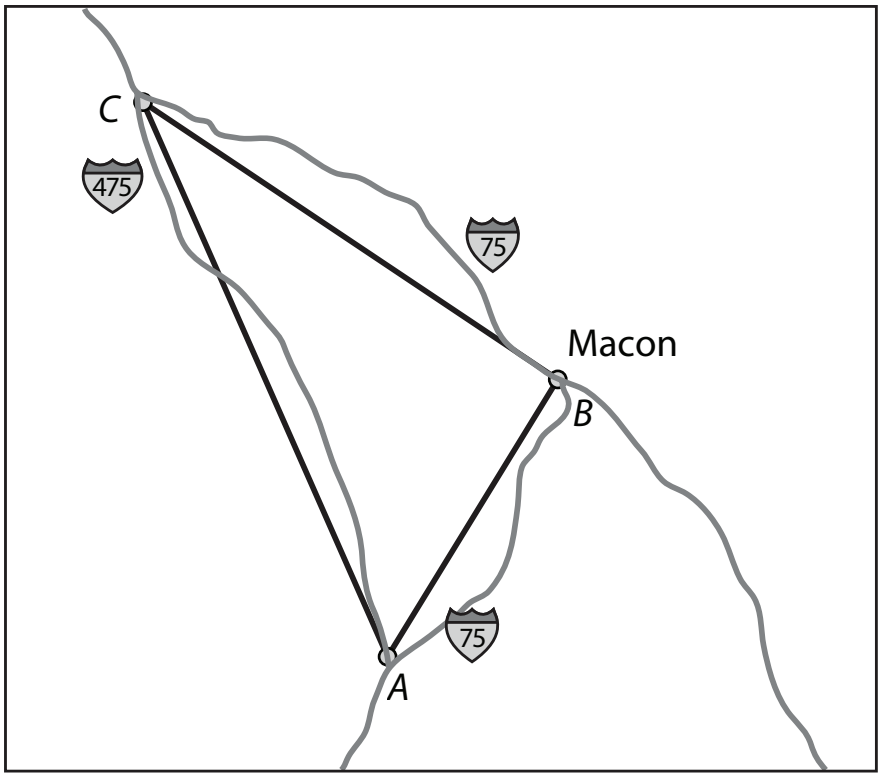
Prerequisite Skills

This lesson requires the use of the following skills:

- constructing angle bisectors (G–CO.12)
- constructing a perpendicular line from an external point to a line (G–CO.12)

Introduction

In the following map of Georgia, Interstates 475 and 75 form a triangle with Macon as one of the vertices. If a company wants to build a distribution center in the middle of that triangle so that the building will be equidistant from each interstate, where should the distribution center be built? In this lesson, we will investigate the point that solves this problem and the geometry that supports it.



Key Concepts

- A point that is **equidistant** lies the same distance away from a reference point.
- The company wants its distribution center to be equidistant from the given interstates.

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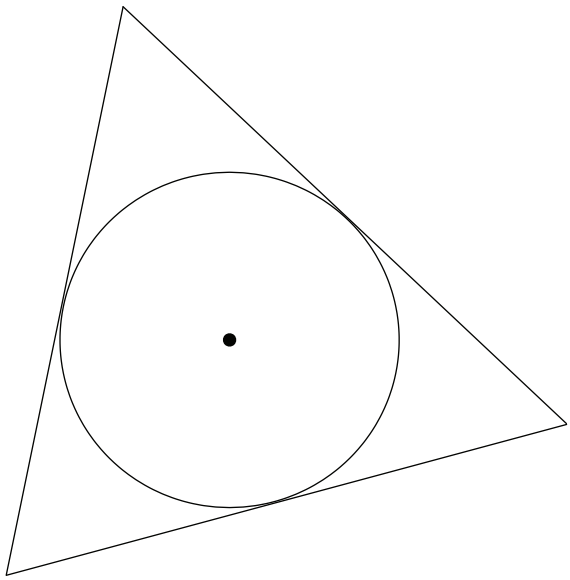
Instruction

- To determine the location of the distribution center, the company would first need to determine the point at which the distribution center would be equidistant from each of the interstates.
- To determine this point, the company would need to find the angle bisectors of the triangle created by the interstates.
- An **angle bisector** is the ray that divides an angle into two congruent angles.
- When all three angle bisectors of a triangle are constructed, the rays are said to be concurrent, which means that they intersect at one point.
- This point is called the **point of concurrency**, where three or more lines intersect.
- The point of concurrency of the angle bisectors of a triangle is called the **incenter**.
- The incenter is equidistant from the three sides of the triangle and is also the center of the circle that is inscribed in the triangle.
- A circle that contains one point from each side of a triangle or other polygon is referred to as an **inscribed circle**.
- When the inscribed circle is constructed, the triangle is referred to as a **circumscribed triangle**—a triangle whose sides are tangent to a circle.



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Instruction

- To construct an inscribed circle, determine the shortest distance from the incenter to each of the sides of the triangle.
- Remember that the shortest distance from a point to a line or line segment is a perpendicular line.
- Follow the steps outlined in the Guided Practice examples to construct a line perpendicular to a segment and passing through a point not on the line.

Common Errors/Misconceptions

- failing to construct the perpendicular from the incenter to any side of the triangle
- guessing the point of tangency and swinging a circle when finding the radius of a circle
- confusing the idea of being equidistant from the sides with the idea of being equidistant from the vertices (such as the circumcenter)

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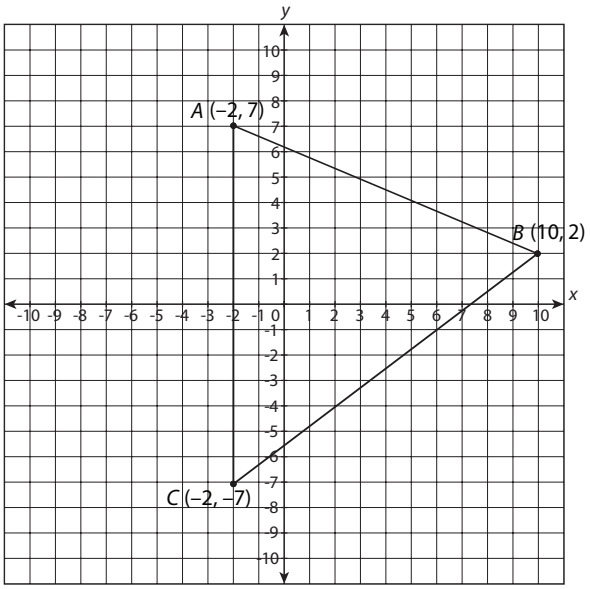
Instruction

Guided Practice 6.2.1

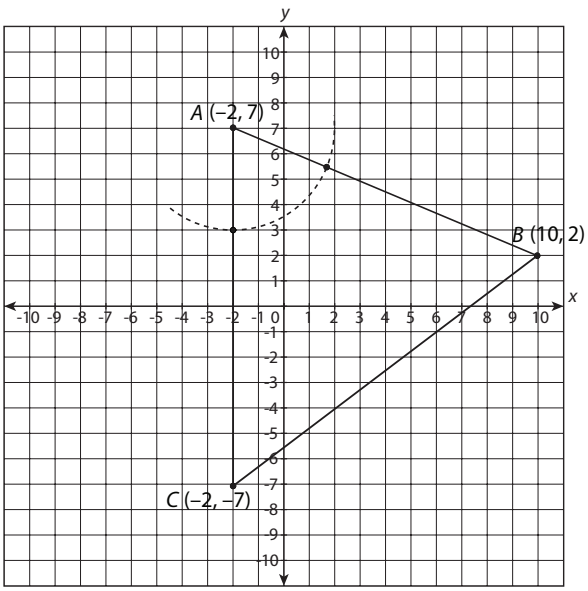
Example 1

Verify that the angle bisectors of acute $\triangle ABC$ are concurrent and that this point of concurrency is equidistant from each side.



1. Construct the angle bisector of $\angle A$.

   First, place the compass on point A and swing an arc that intersects the two sides of the angle.



(*continued*)

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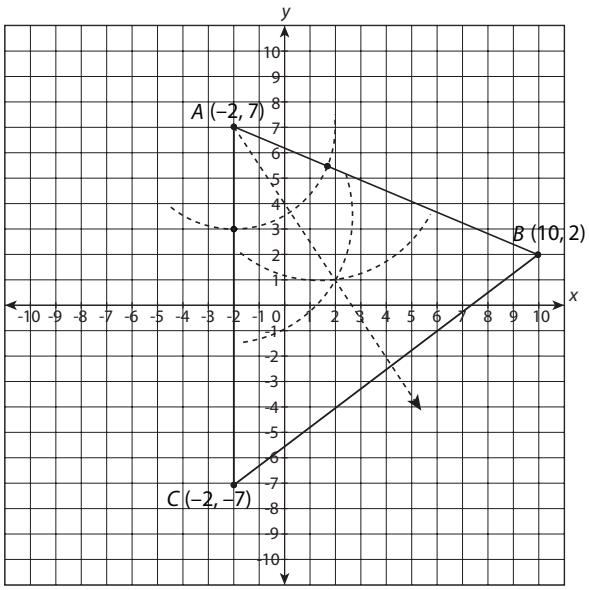
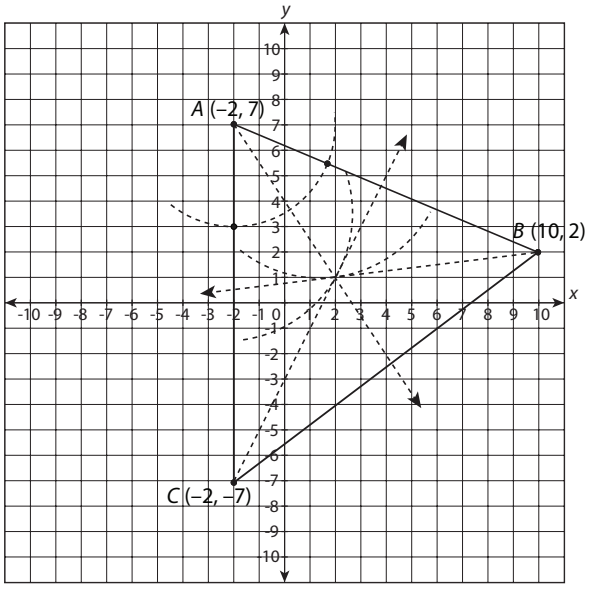
Instruction

From each point of intersection, swing an arc in the interior of the angle. These two arcs should intersect. If they do not, increase the radius of the arc. Connect A with the intersections of the two arcs to locate the angle bisector.



2. Repeat the process for $\angle B$ and $\angle C$.



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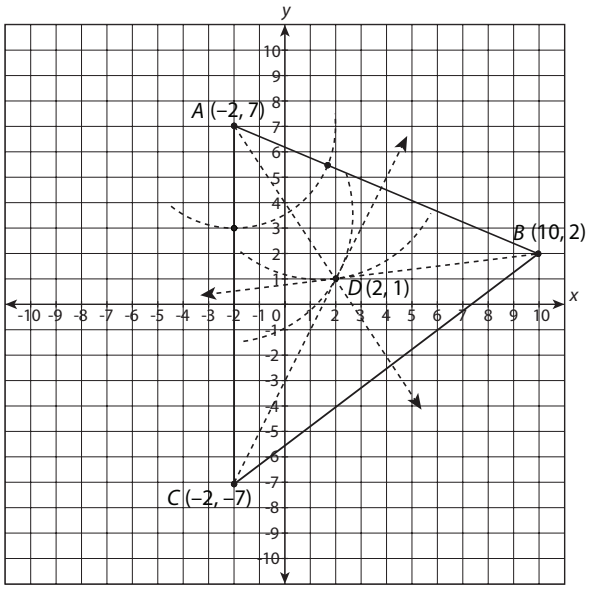
Instruction

3. Locate the point of concurrency. Label this point as D.

   The point of concurrency is where all three of the angle bisectors meet.

   As seen on the coordinate plane, the point of concurrency is (2, 1).



4. Verify that the point of concurrency is equidistant from each side.

   Determine the line that is perpendicular to $\overline{AC}$ through point D (2, 1).

   Find the equation of the line representing $\overline{AC}$.

   Use the slope formula to calculate the slope of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$ Slope formula

$$m = \frac{(-7)-(7)}{(-2)-(-2)}$$ Substitute (–2, 7) and (–2, –7) for (x_1, y_1) and (x_2, y_2).

$$m = \frac{-14}{0}$$

   The slope of $\overline{AC}$ is undefined because the line is vertical.

(*continued*)

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Instruction

The equation of a vertical line always has the form $x = a$, where a is the x-intercept. In this case, the x-intercept is –2.

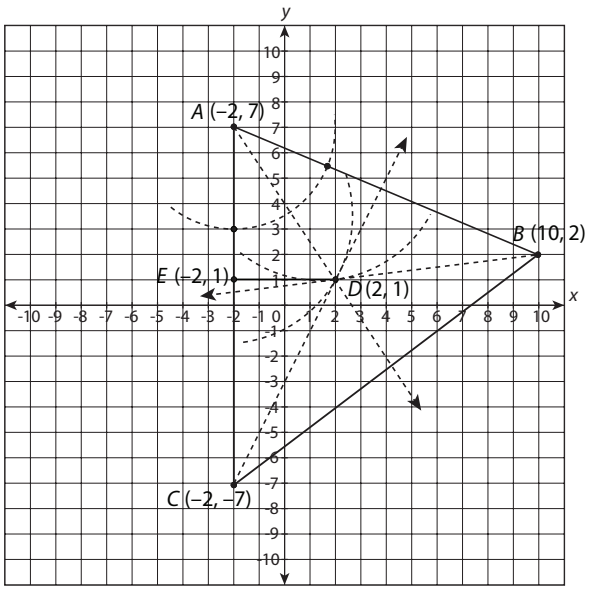
The equation of $\overline{AC}$ is $x = -2$.

The line that is perpendicular to $x = -2$ is horizontal and of the form $y = b$, where b is the y-intercept.

The line that is horizontal at (2, 1) intersects the y-intercept at 1.

The equation of the line that is perpendicular to $\overline{AC}$ is $y = 1$.

The intersection of the two lines is (–2, 1). Label this point E.



Use the distance formula to calculate the length of $\overline{DE}$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance formula
$\sqrt{[(2) - (-2)]^2 + [(1) - (1)]^2}$	Substitute (–2, 1) and (2, 1) for (x_1, y_1) and (x_2, y_2).
$DE = \sqrt{4^2 + 0^2}$	Simplify.
$DE = \sqrt{16}$	
$DE = 4$	

(*continued*)

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Instruction

Determine the line that is perpendicular to $\overline{AB}$ through point D (2, 1).

Find the equation of the line representing $\overline{AB}$.

The slope of $\overline{AB}$ is $-\frac{5}{12}$.

Use the slope-intercept form of a linear equation, $y = mx + b$, to determine the equation of $\overline{AB}$.

The equation of $\overline{AB}$ is $y = -\frac{5}{12}x + \frac{37}{6}$.

The line that is perpendicular to $\overline{AB}$ has a slope that is the opposite reciprocal of the slope of $\overline{AB}$.

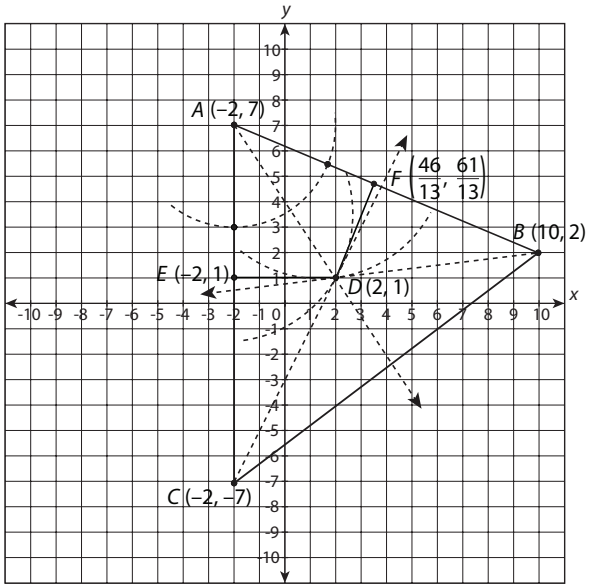
The opposite reciprocal of $-\frac{5}{12}$ is $\frac{12}{5}$.

Again, use the slope-intercept form of a linear equation to determine the perpendicular line.

The equation of the perpendicular line is $y = \frac{12}{5}x - \frac{19}{5}$.

The intersection of the two lines can be found by setting the equations equal to each other and solving for x.

The intersection of the two lines is at $\left(\frac{46}{13}, \frac{61}{13}\right)$. Label this point F.



(*continued*)

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Instruction

Use the distance formula to calculate the length of $\overline{DF}$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance formula

$DF = \sqrt{\left[(2) - \left(\frac{46}{13}\right)\right]^2 + \left[(1) - \left(\frac{61}{13}\right)\right]^2}$ Substitute $\left(\frac{46}{13}, \frac{61}{13}\right)$ and (2, 1) for (x_1, y_1) and (x_2, y_2).

$DF = \sqrt{\left[\left(\frac{26}{13}\right) - \left(\frac{46}{13}\right)\right]^2 + \left[\left(\frac{13}{13}\right) - \left(\frac{61}{13}\right)\right]^2}$ Simplify.

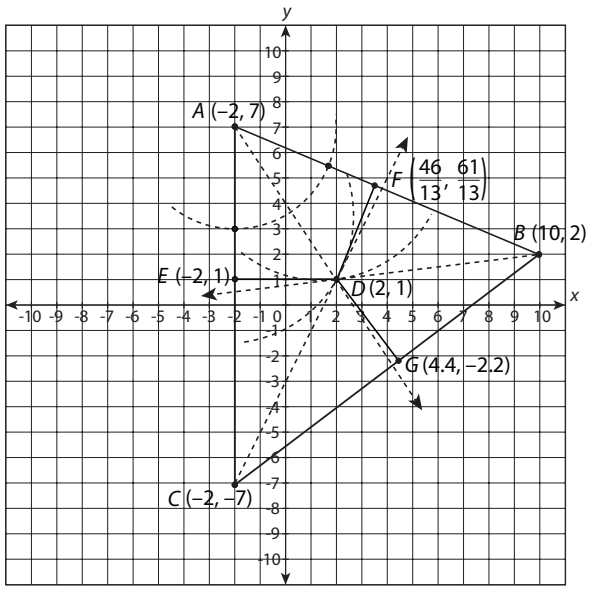
$DF = \sqrt{2.37 + 13.63}$

$DF = \sqrt{16}$

$DF = 4$

By following these same steps, it can be determined that the point of intersection of $\overline{BC}$ and the line that is perpendicular to $\overline{BC}$ through D (2, 1) is at $\left(\frac{22}{5}, -\frac{11}{5}\right)$.

The midpoint of BC is at (4.4, –2.2). Label this point G.



(*continued*)

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Instruction

Use the distance formula to calculate the length of $\overline{DG}$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance formula

$DG = \sqrt{\left[(2) - \left(\frac{22}{5}\right)\right]^2 + \left[(1) - \left(-\frac{11}{5}\right)\right]^2}$ Substitute $\left(\frac{22}{5}, -\frac{11}{5}\right)$ and (2, 1) for (x_1, y_1) and (x_2, y_2).

$DG = \sqrt{16}$ Simplify.

$DG = 4$

5. State your conclusion.

The angle bisectors of the triangle are concurrent at point D. The three segments perpendicular from point D (the incenter of the triangle) to the sides of the triangle are equal; therefore, the incenter is equidistant from the three sides of the triangle.

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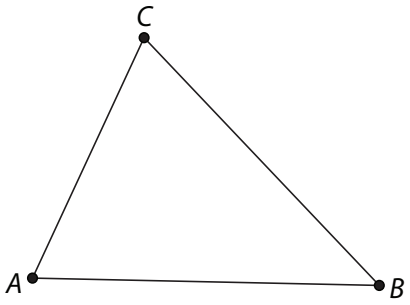
Instruction

Example 2

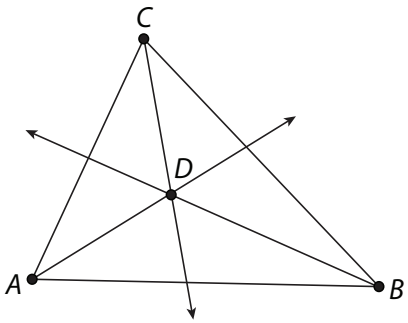
Construct a circle inscribed in acute $\triangle ABC$.



1. Construct the angle bisectors of each angle.

   Use the same method described in Example 1 to bisect each angle.

   Label the point of concurrency of the angle bisectors as point *D*.



2. Construct a perpendicular line from *D* to each side.

   To construct a perpendicular line through *D,* put the sharp point of your compass on point *D*. Open the compass until it extends farther than $\overline{AC}$.

   Make a large arc that intersects $\overline{AC}$ in exactly two places. Without changing your compass setting, put the sharp point of the compass on one of the points of intersection. Make a second arc below the given line.

   Without changing your compass setting, put the sharp point of the compass on the second point of intersection. Make a third arc below the given line. The third arc must intersect the second arc.

   Label the point of intersection *E*.

   Use your straightedge to connect points *D* and *E*.

   Repeat this process for each of the remaining sides.

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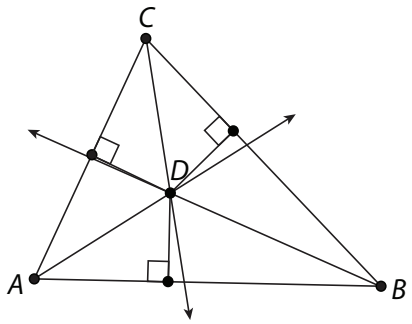
Instruction

3. Verify that the lengths of the perpendicular segments are equal.

   Use your compass and carefully measure the length of each perpendicular segment.

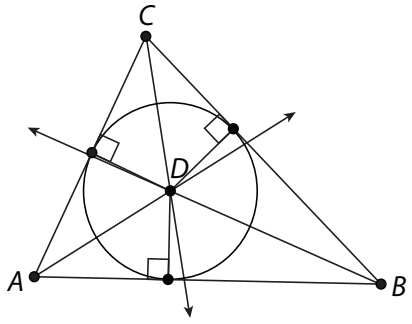
   Verify that the measurements are the same.



4. Construct the inscribed circle with center D.

   Place the compass point at D and open the compass to the length of any one of the perpendicular segments.

   Use this setting to construct a circle inside of $\triangle ABC$.



   Circle D is inscribed in $\triangle ABC$ and is tangent to each of the sides of the triangle.

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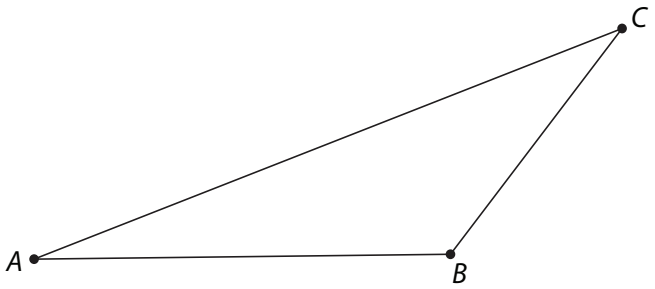
Instruction

Example 3

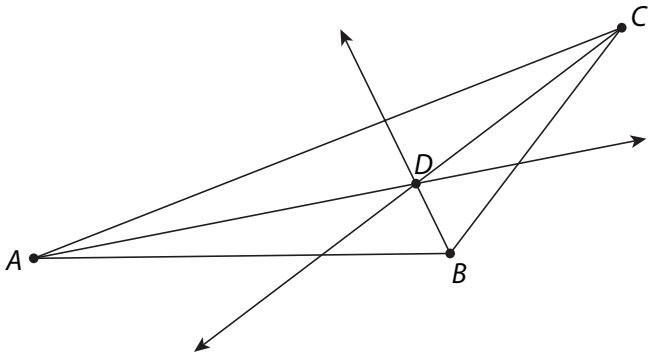
Construct a circle inscribed in obtuse $\triangle ABC$.



1. Construct the angle bisectors of each angle.

   Use the same method as described in Example 1 to bisect each angle.

   Label the point of concurrency of the angle bisectors as point D.



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Instruction

2. Construct a perpendicular line from D to each side.

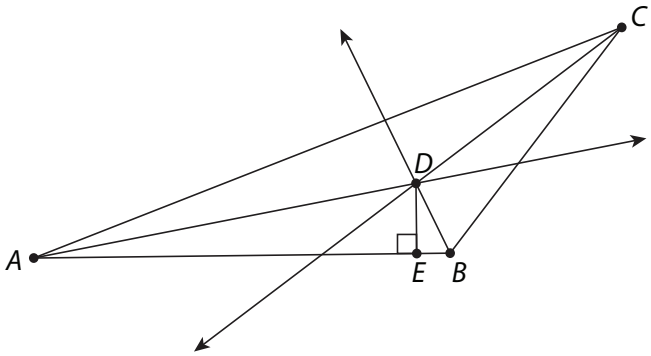
   To construct a perpendicular line through D, put the sharp point of your compass on point D. Open the compass until it extends farther than $\overline{AC}$.

   Make a large arc that intersects $\overline{AC}$ in exactly two places. Without changing your compass setting, put the sharp point of the compass on one of the points of intersection. Make a second arc below the given line.

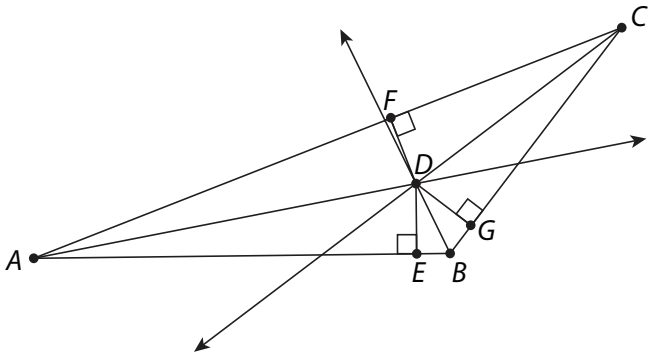
   Without changing your compass setting, put the sharp point of the compass on the second point of intersection. Make a third arc below the given line. The third arc must intersect the second arc.

   Label the point of intersection E.

   Use your straightedge to connect points D and E.



   Repeat this process for each of the remaining sides, labeling the points of intersection F and G.



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Instruction

3. Verify that the lengths of the perpendicular segments are equal.

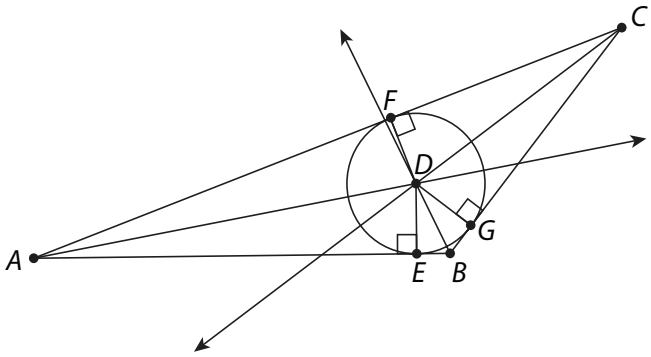
   Use your compass and carefully measure the length of each perpendicular.

   Verify that the measurements are the same.

4. Construct the inscribed circle with center D.

   Place the compass point at D and open the compass to the length of any one of the perpendicular segments.

   Use this setting to construct a circle inside of $\triangle ABC$.



   Circle D is inscribed in $\triangle ABC$ and is tangent to each of the sides of the triangle.

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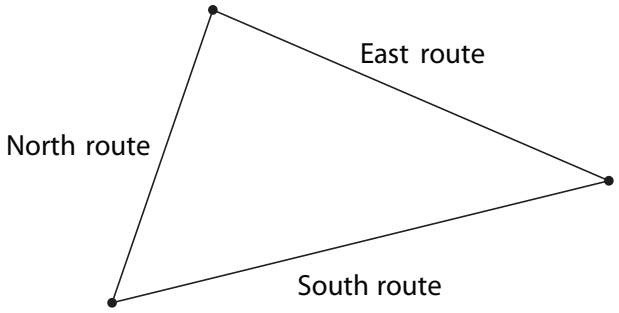
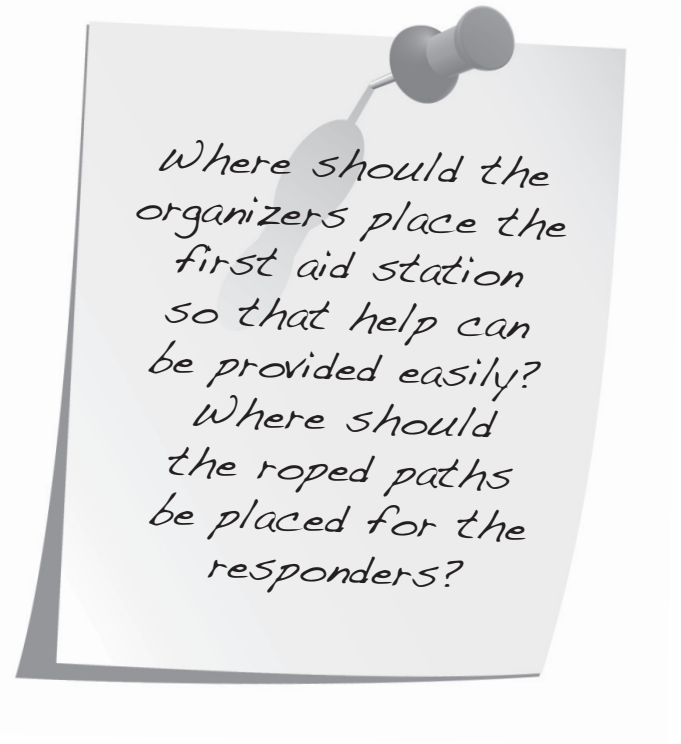
Problem-Based Task 6.2.1: First Aid Station

SMP	
1	2
3	4 ✓
5 ✓	6
7 ✓	8

Organizers are setting up a first aid station and roping off paths for a cross-country race. There are three routes that the runners will travel at least twice throughout the race. Where should the organizers place the first aid station so that help can be provided easily? Where should the roped paths be placed for the responders?





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Problem-Based Task 6.2.1: First Aid Station

Coaching

a. How many angles of the triangle need to be bisected in order to find the incenter?

b. Why is only one bisector not sufficient?

c. How many perpendicular segments need to be constructed from the incenter to a side?

d. How can the other point of intersection of the perpendicular segments be found quickly?

e. Where are the roped paths?

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Instruction

Problem-Based Task 6.2.1: First Aid Station

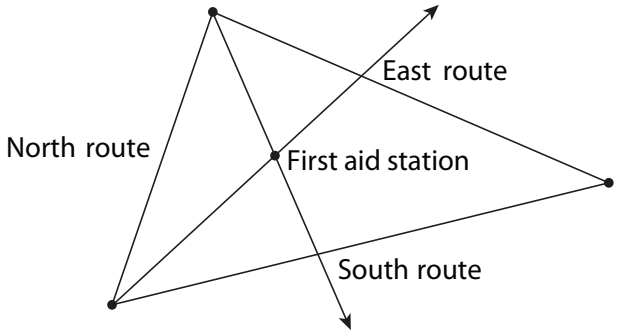
Coaching Sample Responses

a. How many angles of the triangle need to be bisected in order to find the incenter?

Two; since all three bisectors are concurrent, the third is not needed but can be used as a check.

b. Why is only one bisector not sufficient?

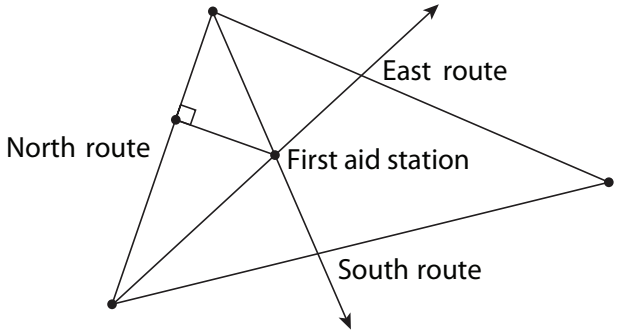
One angle bisector shows the ray on which the incenter lies, but a second bisector is needed to find the exact location.



c. How many perpendicular segments need to be constructed from the incenter to a side?

Since the incenter is equidistant from the sides, one perpendicular is sufficient to set the radius of the inscribed circle.

The other two distances will be equal.



d. How can the other point of intersection of the perpendicular segments be found quickly?

Construct a circle within the triangle; the intersections of the circle with the sides of the triangle will reveal the other points.

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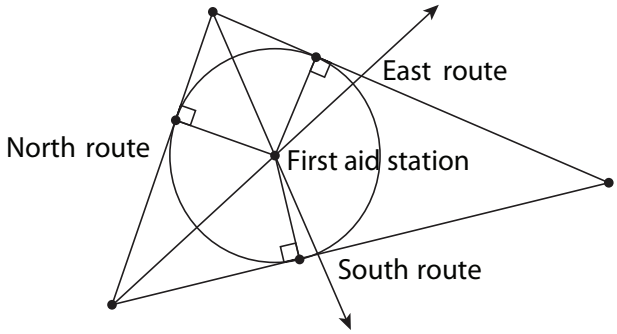
e. Where are the roped paths?

The roped paths will be located along the three perpendicular segments.



Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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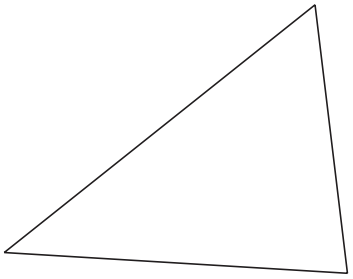
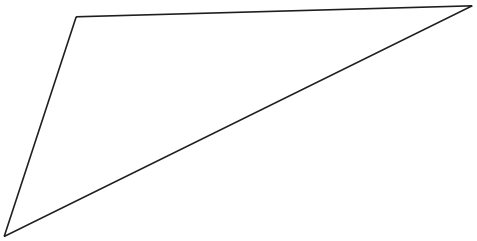
Practice 6.2.1: Constructing Inscribed Circles

Construct the inscribed circle for each of the triangles in problems 1–3.

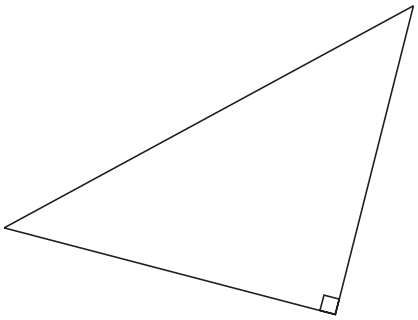
1.



2.



3.



4. Must the incenter always be found within the interior of the triangle? Why or why not? Consider the constructions you created for problems 1–3 when you answer.

continued

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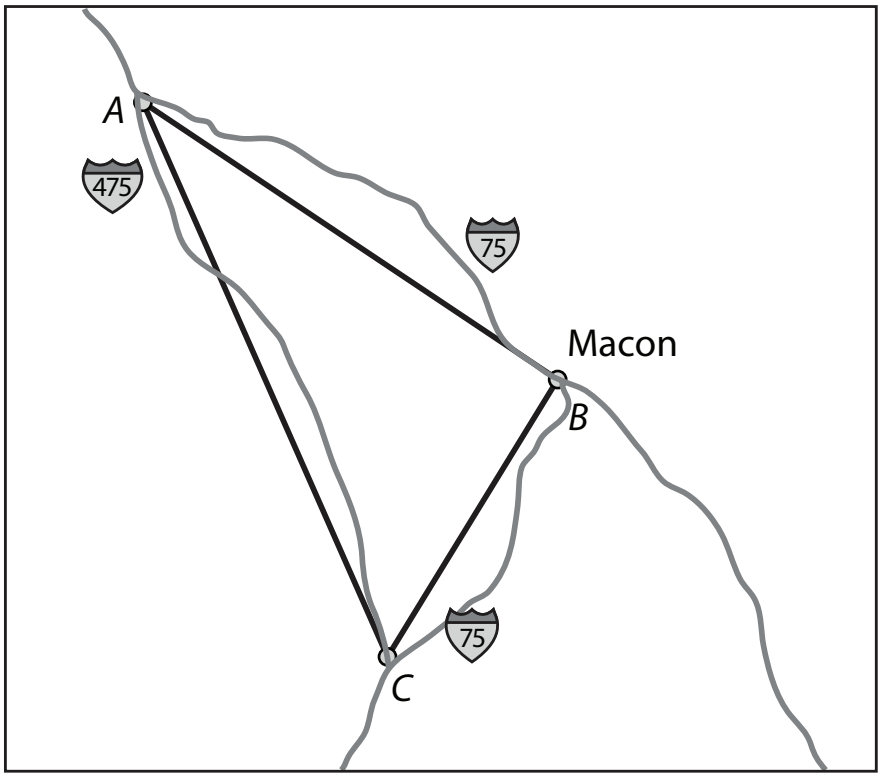
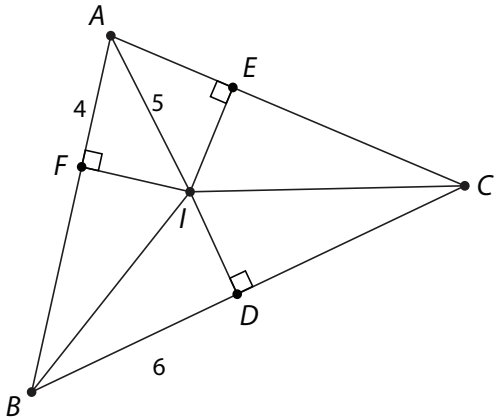
Use the diagrams provided to solve problems 5–10.

5. In this map of Georgia, Interstates 475 and 75 form a triangle with Macon as one of the vertices. A company wants to build its new headquarters in the middle of that triangle so that the building will be equidistant from each interstate. Where should the headquarters be built?



6. Find the length of *BI*. Assume that *I* is the incenter.



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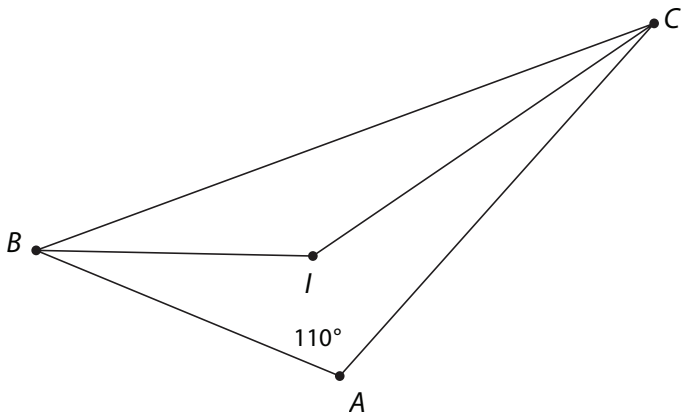
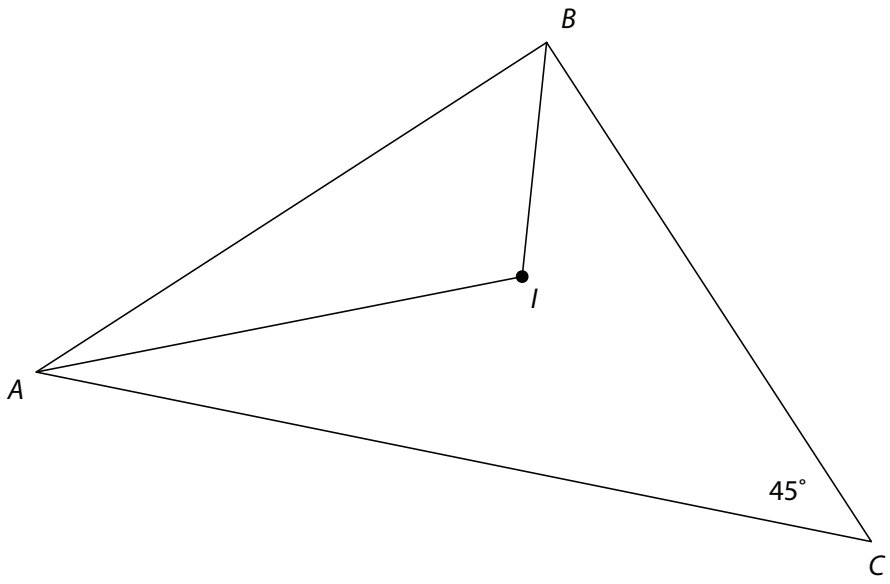
7. Find the measure of $\angle BIC$. Assume that $\overline{BI}$ and $\overline{CI}$ are angle bisectors.



8. Find the measure of $\angle AIB$. Assume that $\overline{AI}$ and $\overline{BI}$ are angle bisectors.



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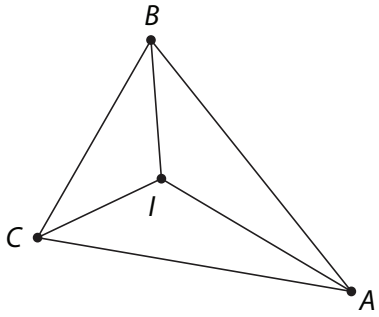
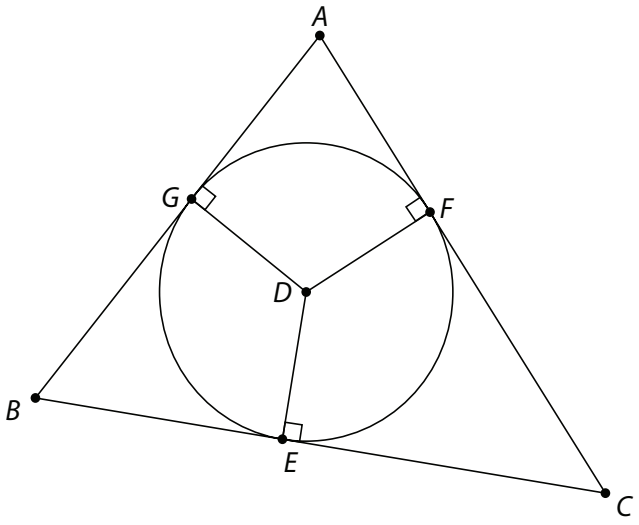
9. Suppose that $\triangle ABC$ is isosceles, $AB = AC$, and point I is the incenter. What is true about $\overline{BI}$ and $\overline{CI}$? Support your answer.



10. What type of quadrilateral is $AFDG$? Assume that D is the incenter. Support your answer.



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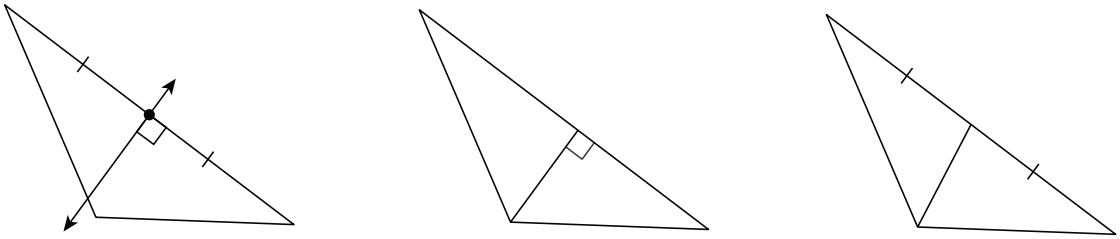
Lesson 6.2.2: Constructing Circumscribed Circles

Warm-Up 6.2.2

Luca is creating an origami boat and has folded a square piece of paper into a triangle. The next step in making the boat requires Luca to fold the triangle such that the fold becomes the perpendicular bisector of the triangle's longest side.

1. Which figure represents the relationship between the fold and the triangle?



A B C

2. Which of the following is a correct step in constructing a perpendicular bisector of a segment?
   a. Make two arcs, centered at each endpoint, that intersect the segment twice.
   b. Make two arcs, centered at each endpoint, that intersect each other twice.
   c. Use a ruler and a protractor to find the midpoint and draw a right angle.

3. In order to create the fold so that it becomes the perpendicular bisector of the triangle's longest side, what must be true about the lengths of the paper edge on either side of the fold?

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Instruction

Lesson 6.2.2: Constructing Circumscribed Circles

Common Core State Standard

G–C.3 Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

Warm-Up 6.2.2 Debrief

1. Which figure represents the relationship between the fold and the triangle?

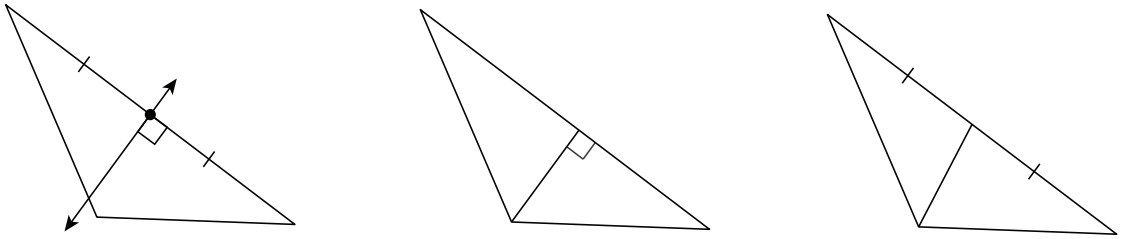

Figure A is a perpendicular bisector; therefore, it represents the relationship between the fold and the triangle when the paper is folded correctly.

2. Which of the following is a correct step in constructing a perpendicular bisector of a segment?

   a. Make two arcs, centered at each endpoint, that intersect the segment twice.

   b. Make two arcs, centered at each endpoint, that intersect each other twice.

   c. Use a ruler and a protractor to find the midpoint and draw a right angle.

   Choice b is correct.

3. In order to create the fold so that it becomes the perpendicular bisector of the triangle's longest side, what must be true about the lengths of the paper edge on either side of the fold?

   The lengths of the edge of the paper along the longest side of the original triangle on either side of the fold must be equal. The fold must lie on the perpendicular bisector of the segment formed by the ends of the longest side of the triangular paper.

Connection to the Lesson

- Students will extend their knowledge of perpendicular bisectors to a point of concurrency.
- Students will apply this knowledge to problem-solving situations.

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Instruction

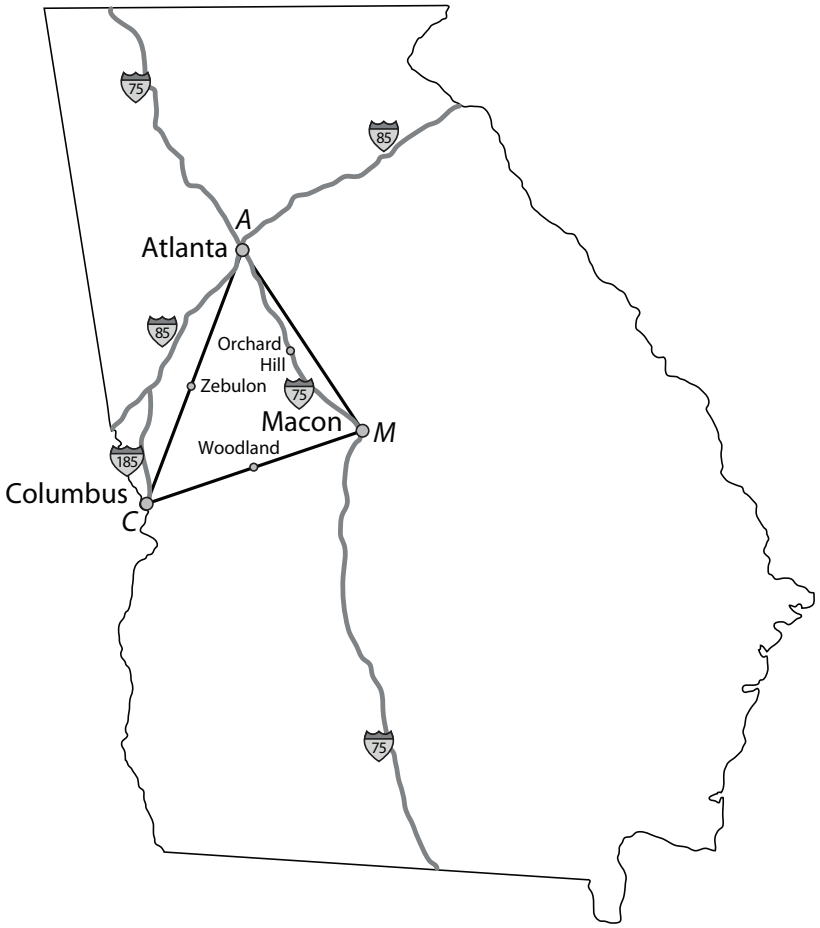
Prerequisite Skills

This lesson requires the use of the following skills:

- constructing a perpendicular bisector (G–CO.12)

Introduction

The owners of a radio station in Georgia want to build a new broadcasting building located within the triangle formed by the cities of Atlanta, Columbus, and Macon. Where should the station be built so that it is equidistant from each city? In this lesson, we will investigate the point that solves this problem and the geometry that supports it.



Key Concepts

- To determine the location of the broadcasting building, the station owners would first need to determine the point at which the building would be equidistant from each of the three cities.
- To determine this point, the owners would need to find the perpendicular bisector of each of the sides of the triangle formed by the three cities.

SRB p. U6-62

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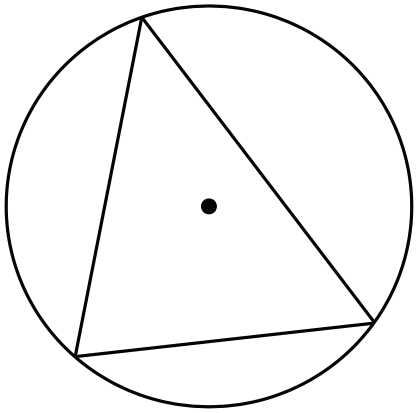
Instruction

- The **perpendicular bisector** of a segment is a line, segment, or ray that is perpendicular to a given segment and contains the midpoint of that segment.
- When all three perpendicular bisectors of a triangle are constructed, they intersect at one point.
- This point of concurrency is called the **circumcenter**.
- The circumcenter is equidistant from the three vertices of the triangle and is also the center of the circle that contains the three vertices of the triangle.
- A circle that contains all the vertices of a polygon is referred to as the **circumscribed circle**.
- When the circumscribed circle is constructed, the triangle is referred to as an **inscribed triangle**, a triangle whose vertices are on a circle.



- To construct the circumscribed circle, set the compass to the distance from the circumcenter to each of the vertices of the triangle.
- Follow the steps in the Guided Practice examples to construct the perpendicular bisectors in order to find the circumcenter of the triangle.

Common Errors/Misconceptions

- confusing the construction of the incenter with the construction of the circumcenter
- associating the incenter (angle bisectors) with being equidistant from the vertices, and associating the circumcenter with being equidistant from the sides

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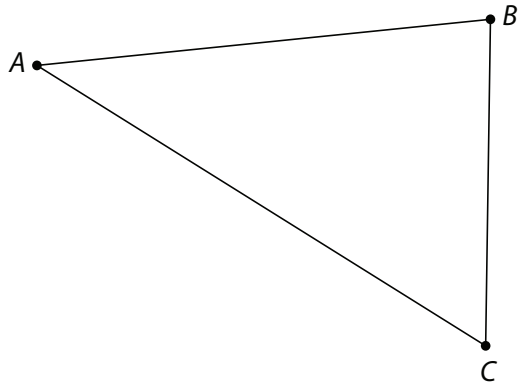
Instruction

Guided Practice 6.2.2

Example 1

Verify that the perpendicular bisectors of acute $\triangle ABC$ are concurrent and that this point of concurrency is equidistant from each vertex.



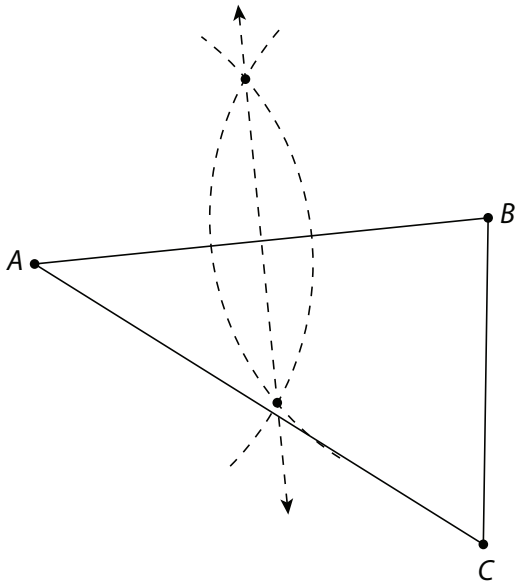
1. Construct the perpendicular bisector of $\overline{AB}$.

   Set the compass at point A and swing an arc.

   Using the same radius, swing an arc from point B so that the two arcs intersect.

   If they do not intersect, lengthen the radius for each arc.

   Connect the two intersections to locate the perpendicular bisector.



SWB p. 59; SRB p. U6-64

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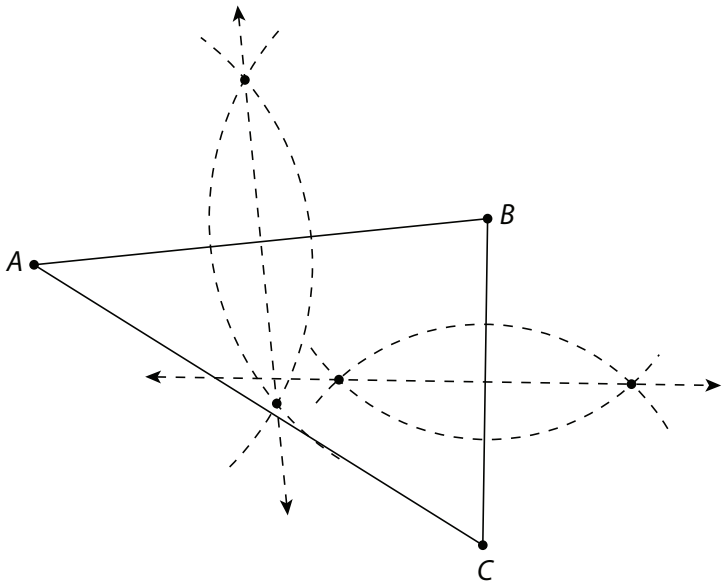
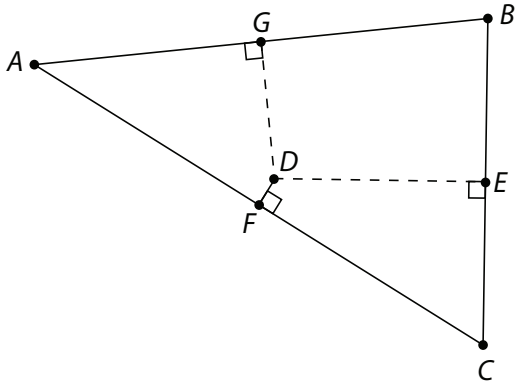
Instruction

2. Repeat the process for $\overline{BC}$ and $\overline{AC}$.



3. Locate the point of concurrency. Label this point D.

   The point of concurrency is where all three perpendicular bisectors meet.



4. Verify that the point of concurrency is equidistant from each vertex.

   Use your compass to carefully compare the distances from point D to each vertex.

   The measurements are the same.

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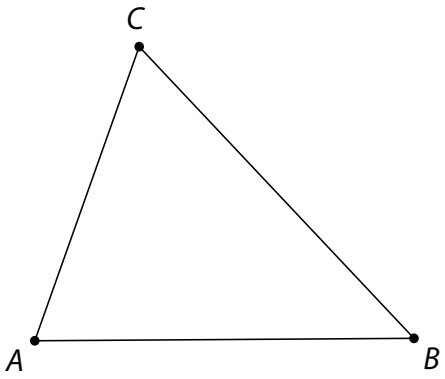
Instruction

Example 2

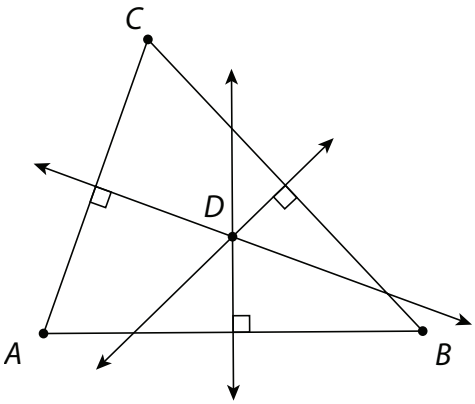
Construct a circle circumscribed about acute $\triangle ABC$.



1. Construct the perpendicular bisectors of each side.

   Use the same method as described in Example 1 to bisect each side.

   Label the point of concurrency of the perpendicular bisectors as point D.



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Instruction

2. Verify that the point of concurrency is equidistant from each of the vertices.

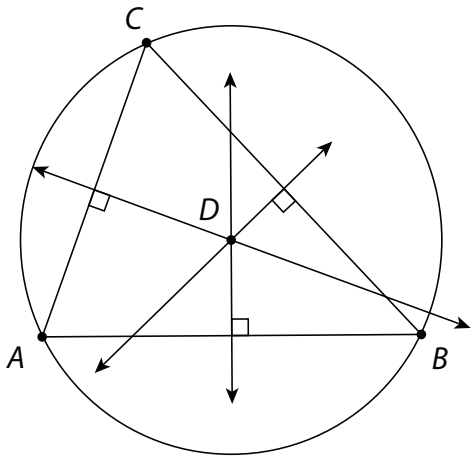
   Use your compass to carefully compare the distances from point D to each vertex.

   Verify that the measurements are the same.

3. Construct the circumscribed circle with center D.

   Place the compass point at D and open the compass to the distance of D to any vertex.

   Use this setting to construct a circle around $\triangle ABC$.



Circle D is circumscribed about $\triangle ABC$ and the circle passes through each vertex of the triangle.

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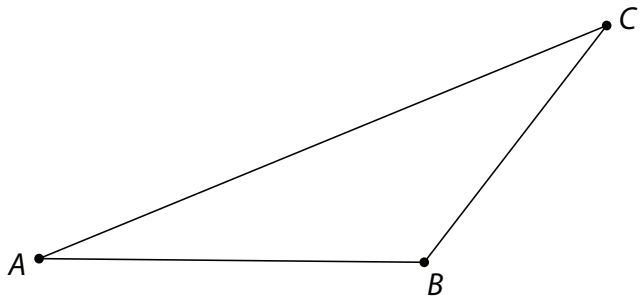
Instruction

Example 3

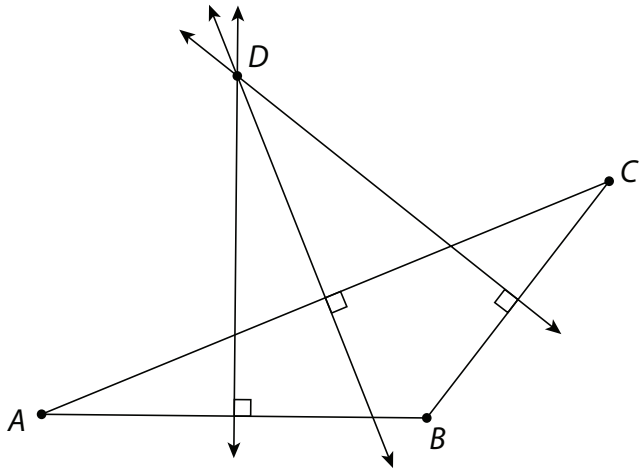
Construct a circle circumscribed about obtuse $\triangle ABC$.



1. Construct the perpendicular bisectors of each side.

   Use the same method as described in Example 1 to bisect each side.

   Label the point of concurrency of the perpendicular bisectors as point *D*.



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Instruction

2. Verify that the point of concurrency is equidistant from each of the vertices.

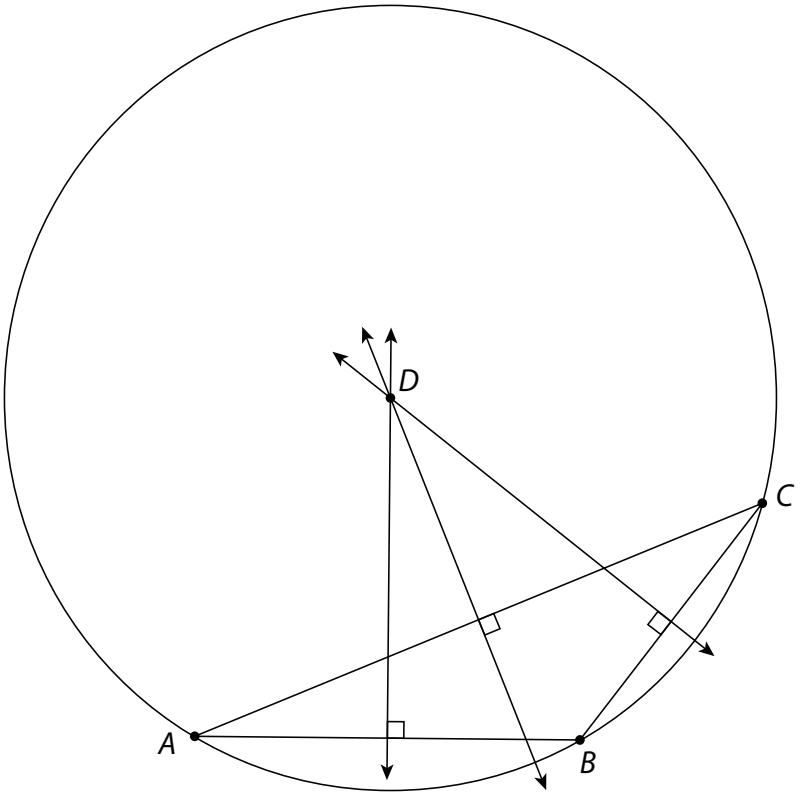
   Use your compass to carefully compare the distances from D to each vertex of the triangle.

   Verify that the measurements are the same.

3. Construct the circumscribed circle with center D.

   Place the compass point at D and open the compass to the distance of D to any vertex.

   Use this setting to construct a circle around $\triangle ABC$.



Circle D is circumscribed about $\triangle ABC$ and the circle passes through each vertex of the triangle.

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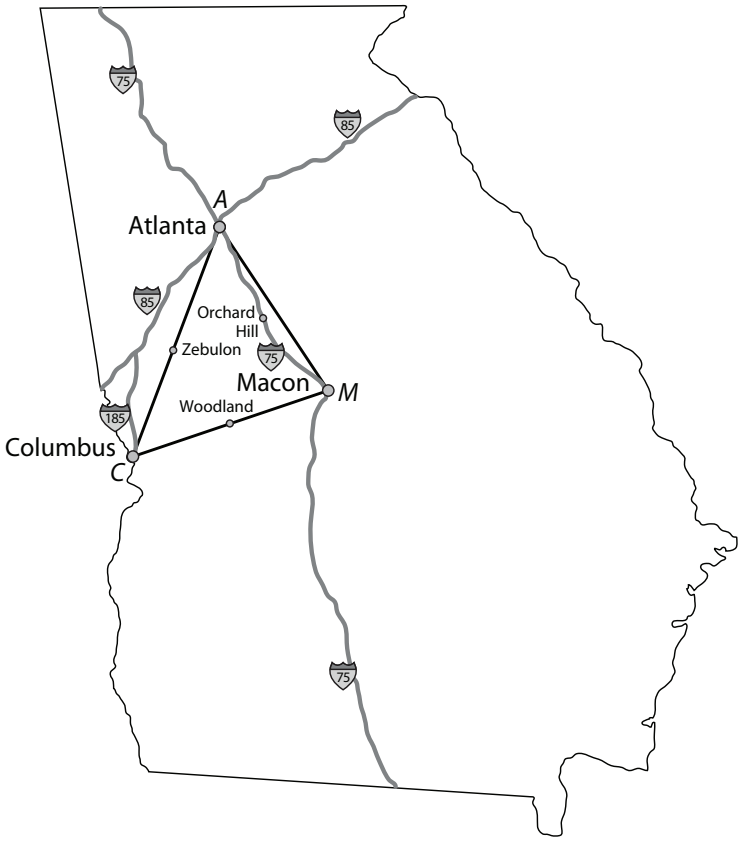
Problem-Based Task 6.2.2: Building a New Radio Station

SMP	
1 ✓	2
3 ✓	4
5	6 ✓
7	8

The owners of a radio station in Georgia want to build a new broadcasting building located within the triangle formed by Atlanta, Columbus, and Macon. Where should the station be built so that it is equidistant from each city?



Where should the station be built so that it is equidistant from each city?

SWB p. 61

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Problem-Based Task 6.2.2: Building a New Radio Station

Coaching

a. Does the new station need to be placed equidistant from the interstates or from the cities?

b. Which is needed to determine the placement of the station: the angle bisectors or the perpendicular bisectors?

c. How many bisectors need to be constructed in order to locate the point of concurrency?

d. Perform the construction. Which town is closest to the point of concurrency?

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Problem-Based Task 6.2.2: Building a New Radio Station

Coaching Sample Responses

a. Does the new station need to be placed equidistant from the interstates or from the cities?

The new station needs to be located equidistant from the cities.

b. Which is needed to determine the placement of the station: the angle bisectors or the perpendicular bisectors?

The perpendicular bisectors will locate the point that is equidistant from the cities.

c. How many bisectors need to be constructed in order to locate the point of concurrency?

Only two of the three perpendicular bisectors need to be constructed.

The third perpendicular bisector will also contain the point of intersection determined by the first two perpendicular bisectors.

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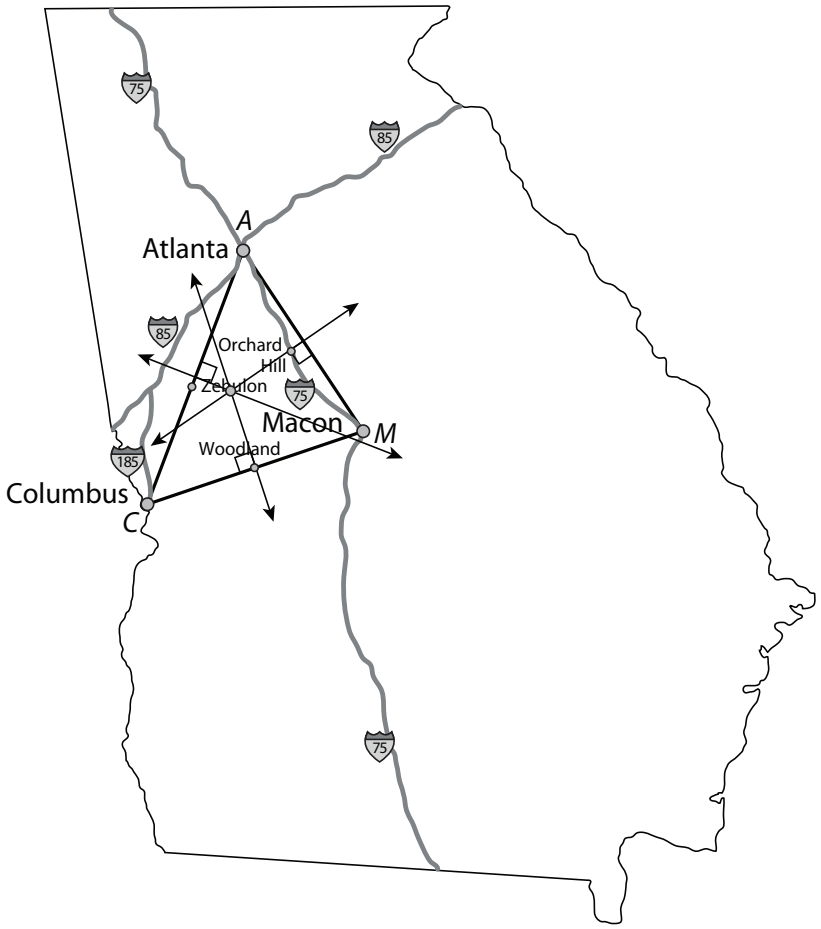
d. Perform the construction. Which town is closest to the point of concurrency?

Zebulon is closest to the point of concurrency.



Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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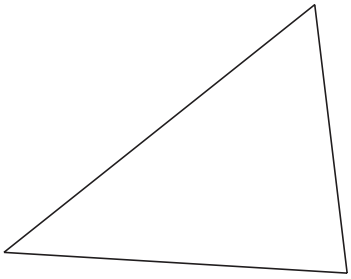
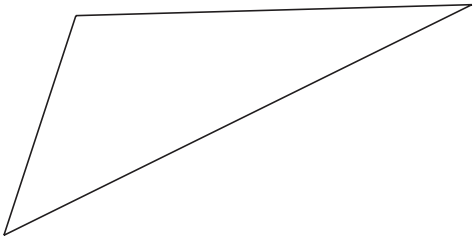
Practice 6.2.2: Constructing Circumscribed Circles

Construct the circumscribed circle for each of the triangles in problems 1–3.

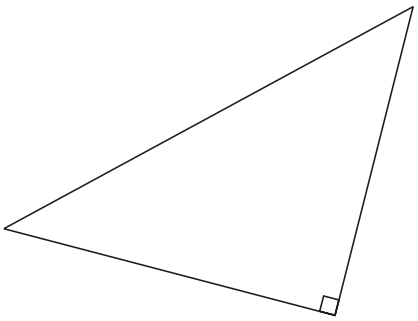
1.



2.



3.



continued

SWB p. 63; SRB Practice 6.2.2 p. U6-70

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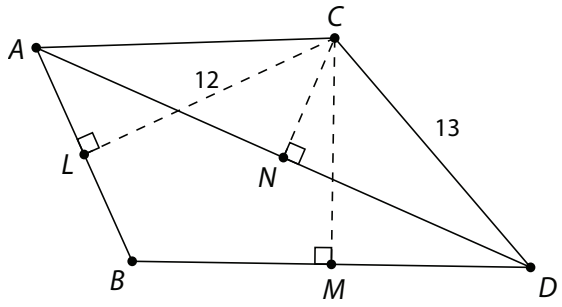
Use what you've learned and the diagrams, when provided, to complete problems 4–10.

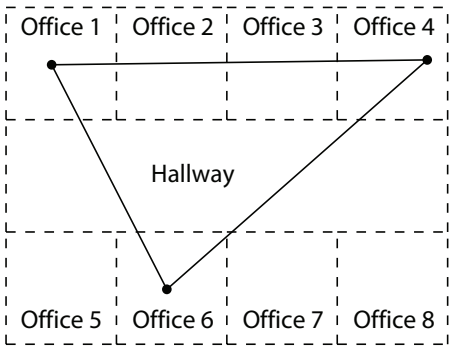
4. The producers of a cooking show on the Snackers Network are designing a new set. A food preparation station needs to be located between the refrigerator, sink, and stove. Which point of concurrency, the circumcenter or the incenter, will result in the preparation station being located in a place that is equidistant from the refrigerator, sink, and stove?

5. Assume that point C is the circumcenter for $\triangle ABD$. What is the length of $\overline{AB}$?



6. Must the circumcenter be located within the triangle? Why or why not?

7. Offices 1, 4, and 6 are located on the same floor of a renovated building. The offices must be connected by electrical wires, and all the wires will lead to a central control box. In which room should the control box be placed so that the amount of wiring is minimized?



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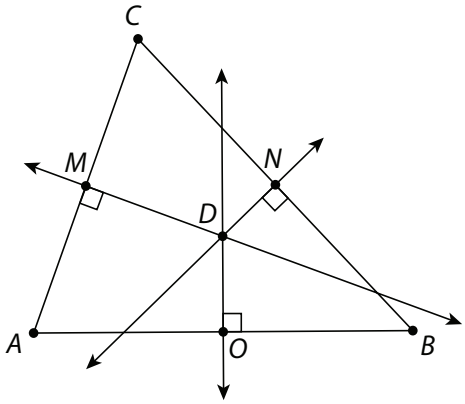
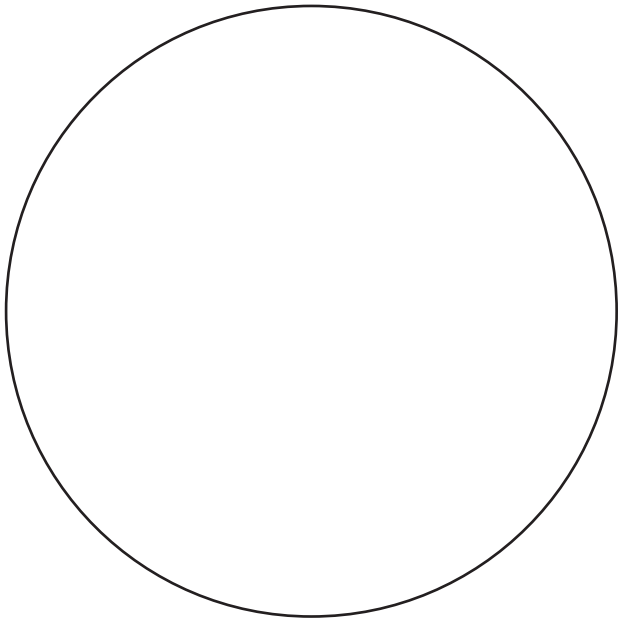
8. Using the diagram below, prove that the circumcenter is equidistant from the vertices of a triangle.



9. Is it possible for the incenter to be the same point as the circumcenter? Why or why not? If it is possible, what type(s) of triangle would meet this criterion? Consider your responses to problems 1 through 3 in determining your answer.

10. Describe a method to determine the center of the circle below. Then, carry out your plan.



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Lesson 6.2.3: Proving Properties of Inscribed Quadrilaterals

Warm-Up 6.2.3

Jordan is practicing his dunk shot for a basketball contest. He wants to show off by doing a 360° slam. If he can perfect it, he'll jump, spin in a complete circle, and slam dunk the basketball. However, in practicing, he is only turning to his left 280°.

1. How much farther must Jordan turn in order to achieve his goal of 360°?

2. What fraction of the 360° slam is Jordan currently turning?

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Instruction

Lesson 6.2.3: Proving Properties of Inscribed Quadrilaterals

Common Core State Standard

G–C.3 Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

Warm-Up 6.2.3 Debrief

1. How much farther must Jordan turn in order to achieve his goal of 360°?

   Subtract 280° from 360° to determine how much farther Jordan must turn.

   $360 - 280 = 80$

   Jordan needs to turn another 80°.

2. What fraction of the 360° slam is Jordan currently turning?

   To determine the fraction Jordan is turning, assume that 360° is the full turn and is therefore the denominator.

   The portion he is turning, 280°, is the numerator.

   $$\frac{280}{360} = \frac{7}{9}$$

   Jordan is turning $\frac{7}{9}$ of a full 360° turn.

Connection to the Lesson

- Students will connect their knowledge of properties of circles and their angles with quadrilaterals.

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Instruction

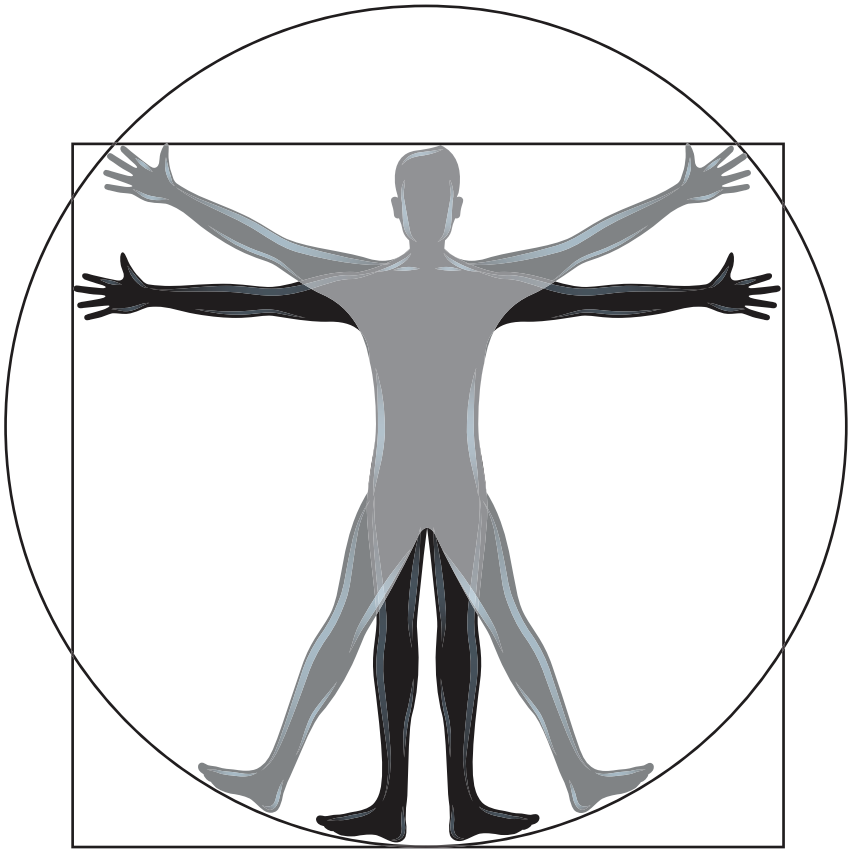
Prerequisite Skills

This lesson requires the use of the following skills:

- finding measures of inscribed angles and/or their intercepted arcs (G–C.2)
- understanding that an angle inscribed in a semicircle is a right angle (G–C.2)
- calculating the supplement of an angle (G–CO.9)
- understanding the properties of special quadrilaterals (G–CO.11)

Introduction

One of the most famous drawings of all time is Leonardo da Vinci's Vitruvian Man. Da Vinci's sketch was of a man enclosed by a circle that touched the man's feet and hands. In this lesson, we will investigate the properties of quadrilaterals inscribed in a circle.



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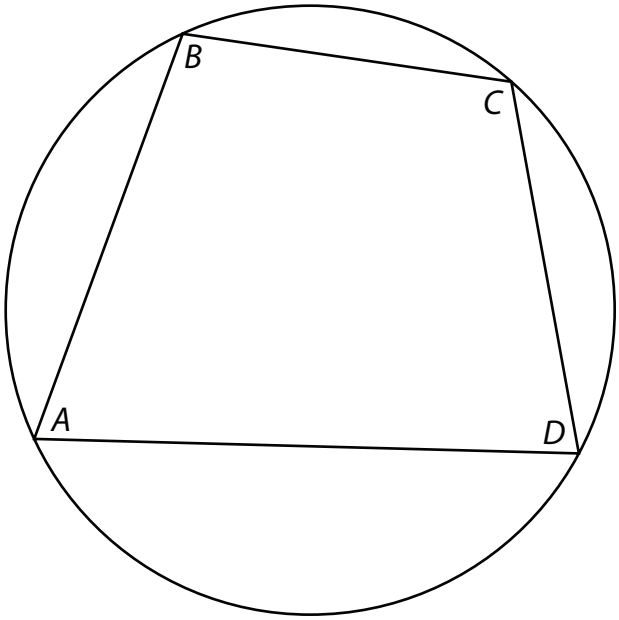
Instruction

Key Concepts

- An **inscribed quadrilateral** is a quadrilateral whose vertices are on a circle.
- The opposite angles of an inscribed quadrilateral are supplementary.



  - $m\angle A + m\angle C = 180^\circ$
  - $m\angle B + m\angle D = 180^\circ$
- Remember that the measure of an inscribed angle is half the measure of the intercepted arc.
- Rectangles and squares can always be inscribed within a circle.

Common Errors/Misconceptions

- mistakenly thinking that the diagonal of any inscribed quadrilateral is a diameter of the circle

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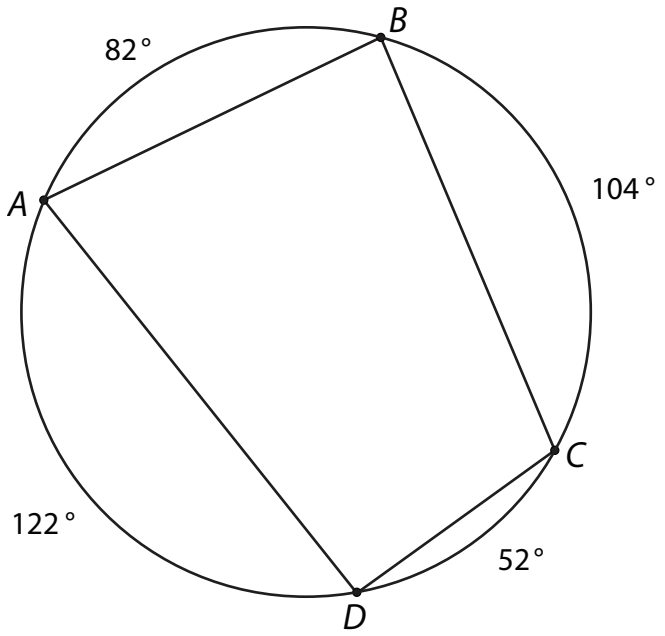
Instruction

Guided Practice 6.2.3

Example 1

Consider the inscribed quadrilateral in the following diagram. What are the relationships between the measures of the angles of an inscribed quadrilateral?



1. Find the measure of $\angle B$.

   $\angle B$ is an inscribed angle. Therefore, its measure is equal to half the measure of the intercepted arc.

   The intercepted arc $\overset{\frown}{ADC}$ has a measure of 122 + 52, or 174°.

   The measure of $\angle B$ is $\frac{1}{2}$ of 174, or 87°.

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Instruction

2. Find the measure of $\angle D$.

   The intercepted arc $\overset{\frown}{ABC}$ has a measure of 82 + 104, or 186°.

   The measure of $\angle D$ is $\frac{1}{2}$ of 186, or 93°.

3. What is the relationship between $\angle B$ and $\angle D$?

   Since the sum of the measures of $\angle B$ and $\angle D$ equals 180°, $\angle B$ and $\angle D$ are supplementary angles.

4. Does this same relationship exist between $\angle A$ and $\angle C$?

   The intercepted arc $\overset{\frown}{BCD}$ has a measure of 104 + 52, or 156°.

   The measure of $\angle A$ is $\frac{1}{2}$ of 156, or 78°.

   The intercepted arc $\overset{\frown}{BAD}$ has a measure of 82 + 122, or 204°.

   The measure of $\angle C$ is $\frac{1}{2}$ of 204, or 102°.

   The sum of the measures of $\angle A$ and $\angle C$ also equals 180°; therefore, $\angle A$ and $\angle C$ are supplementary.

5. State your conclusion.

   The opposite angles of an inscribed quadrilateral are supplementary.

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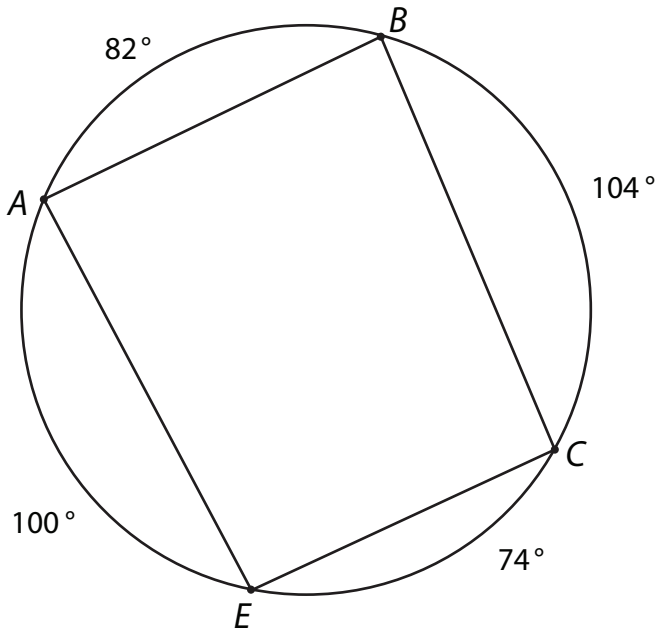
Instruction

Example 2

Consider the following inscribed quadrilateral. Do the relationships discovered between the angles in Example 1 still hold for the angles in this quadrilateral?



1. Calculate the measures of all four angles of quadrilateral $ABCE$.

   $\angle A$ intercepts $\overset{\frown}{BCE}$, so the measure of $\angle A$ is half the measure of $\overset{\frown}{BCE}$.

   $$m\angle A = \frac{1}{2}(104+74) = 89^\circ$$

   $\angle B$ intercepts $\overset{\frown}{AEC}$, so the measure of $\angle B$ is half the measure of $\overset{\frown}{AEC}$.

   $$m\angle B = \frac{1}{2}(100+74) = 87^\circ$$

   $\angle C$ intercepts $\overset{\frown}{BAE}$, so the measure of $\angle C$ is half the measure of $\overset{\frown}{BAE}$.

   $$m\angle C = \frac{1}{2}(100+82) = 91^\circ$$

   $\angle E$ intercepts $\overset{\frown}{ABC}$, so the measure of $\angle E$ is half the measure of $\overset{\frown}{ABC}$.

   $$m\angle E = \frac{1}{2}(104+82) = 93^\circ$$

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Instruction

2. Find the sum of the measures of $\angle A$ and $\angle C$.

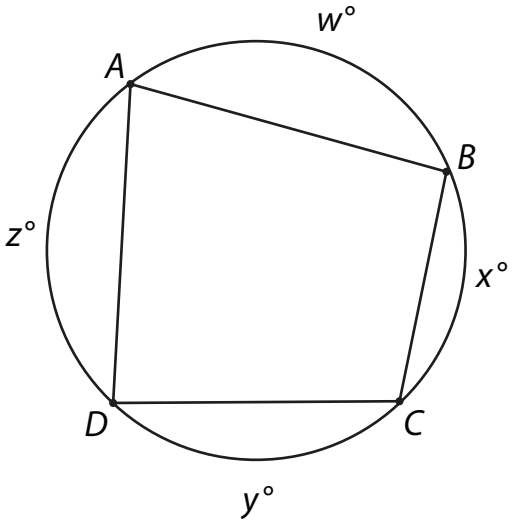
   The sum of the measures of $\angle A$ and $\angle C$ is equal to $89 + 91 = 180^\circ$.

3. State your conclusion.

   The measures of $\angle A$ and $\angle C$ sum to 180°, as do the measures of $\angle B$ and $\angle E$; therefore, it is still true that opposite angles of an inscribed quadrilateral are supplementary.

Example 3

Prove that the opposite angles of the given inscribed quadrilateral are supplementary.



1. What is the sum of $w^\circ + x^\circ + y^\circ + z^\circ$?

   Together, the arcs create a circle that measures 360°; therefore, the sum of the arc measures is 360.

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Instruction

2. Find the measure of each angle of quadrilateral $ABCD$.

$$m\angle A=\frac{1}{2}(x+y) \qquad m\angle C=\frac{1}{2}(w+z)$$
$$m\angle B=\frac{1}{2}(x+z) \qquad m\angle D=\frac{1}{2}(w+x)$$

3. Find the sum of the measures of $\angle A$ and $\angle C$.

$$m\angle A+m\angle C=\frac{1}{2}(x+y)+\frac{1}{2}(w+z)$$
$$=\frac{1}{2}(x+y+w+z)$$
$$=\frac{1}{2}(360)=180$$

4. Find the sum of the measures of $\angle B$ and $\angle D$.

$$m\angle B+m\angle D=\frac{1}{2}(y+z)+\frac{1}{2}(w+x)$$
$$=\frac{1}{2}(y+z+w+z)$$
$$=\frac{1}{2}(360)=180$$

5. State your conclusion.

$m\angle A+m\angle C=180^\circ$ and $m\angle B+m\angle D=180^\circ$. Therefore, each pair of opposite angles of an inscribed quadrilateral is supplementary.

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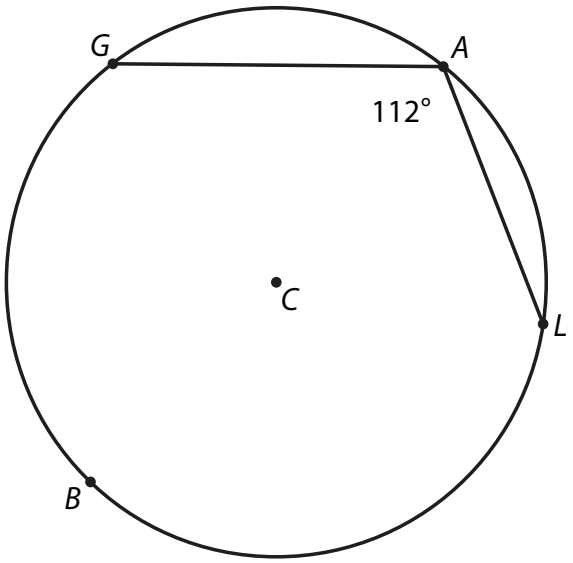
Problem-Based Task 6.2.3: King Arthur and His Round Table

SMP	
1 ✓	2
3 ✓	4
5	6
7	8 ✓

King Arthur is trying to assign seats at his Round Table for three of his knights and himself. If the angle formed by Galahad, Arthur, and Lancelot is 112°, is there enough information to determine the measure of the angle formed by Galahad, Bedivere, and Lancelot? Is there enough information to determine the measures of the angles at Galahad and Lancelot? Also, since Arthur is a geometry fan, he would like to see what types of quadrilaterals he and his knights could form.



Is there enough information to determine the measure of the angle formed by Galahad, Bedivere, and Lancelot?

SWB p. 73

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Problem-Based Task 6.2.3: King Arthur and His Round Table

Coaching

a. What is true about the opposite angles of an inscribed quadrilateral?

b. When segments $\overline{GB}$ and $\overline{LB}$ are drawn, what is the relationship between the angles at G and at L?

c. Could the knights form a rectangle? If so, how?

d. Could the knights form a square? If so, how?

e. Could the knights form a trapezoid? If so, how?

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Instruction

Problem-Based Task 6.2.3: King Arthur and His Round Table

Coaching Sample Responses

a. What is true about the opposite angles of an inscribed quadrilateral?

The opposite angles of an inscribed quadrilateral are supplementary.

b. When segments $\overline{GB}$ and $\overline{LB}$ are drawn, what is the relationship between the angles at G and at L?

These two angles are supplementary.

c. Could the knights form a rectangle? If so, how?

Yes, if all of the angles are 90°, the knights would form a rectangle.

This implies that the diagonals of the rectangle contain the center of the circle.

d. Could the knights form a square? If so, how?

Yes, if all of the angles are 90° and the distances between the knights are the same, the knights would form a square.

e. Could the knights form a trapezoid? If so, how?

Yes, if each angle of the quadrilateral is supplementary to not only the opposite angle but also to one of its adjacent angles. This would imply an isosceles trapezoid.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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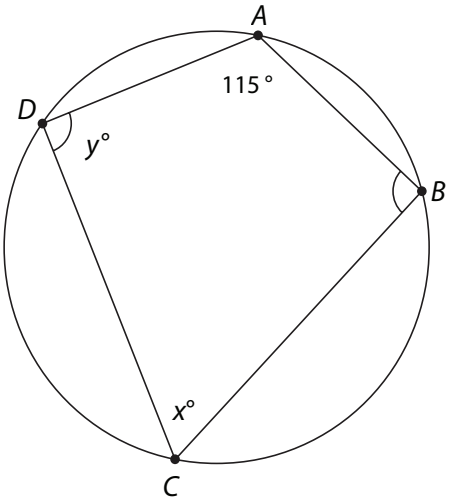
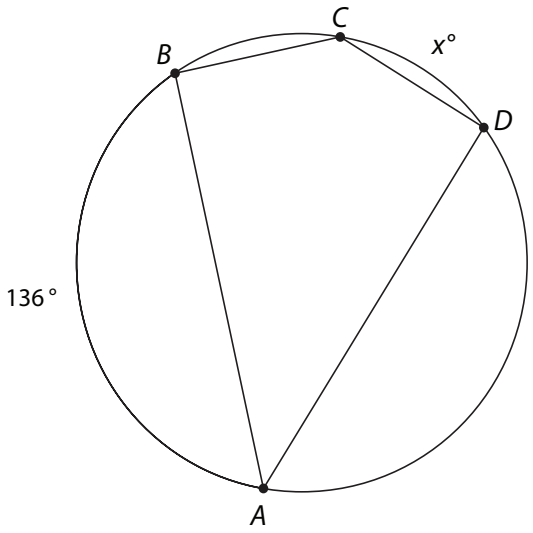
Practice 6.2.3: Proving Properties of Inscribed Quadrilaterals

Use the provided diagrams and your knowledge of the properties of inscribed quadrilaterals to complete problems 1–5.

1. Find the values of x and y.



2. Find the value of x. Assume that quadrilateral $ABCD$ is a kite.



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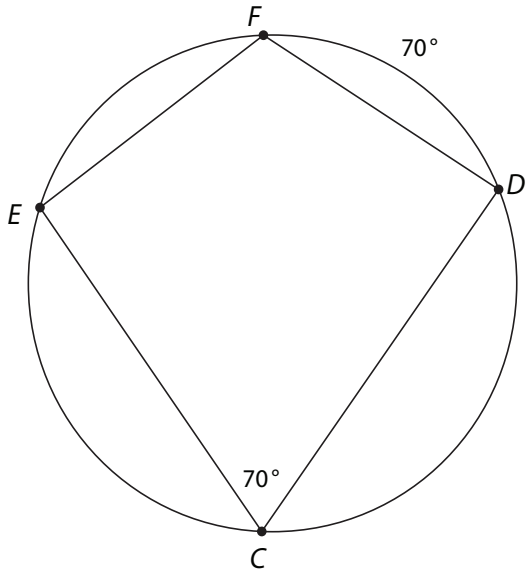
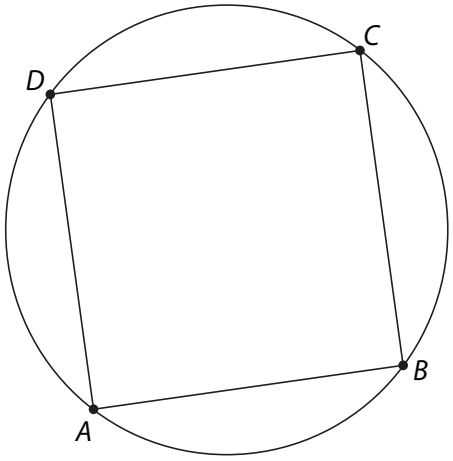
3. Is $\overline{EF} \cong \overline{FD}$? Why or why not?



4. Assume that quadrilateral *ABCD* below is a square that is inscribed in the circle. Construct a circle inscribed in the square.



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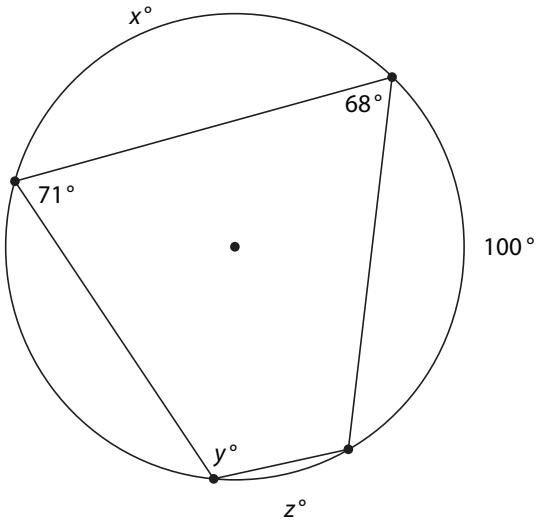
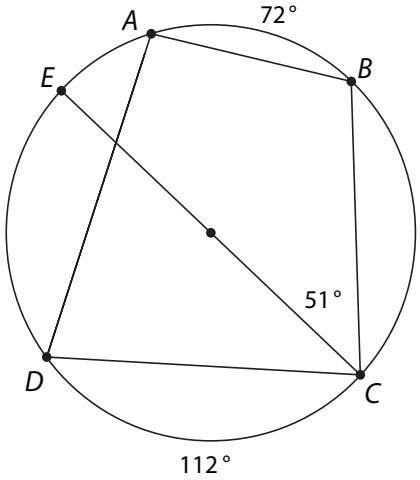
5. Find the values of x, y, and z.



Use the figure below to complete problems 6–8.



6. Find the measure of $\overset{\frown}{BC}$.

7. Find the measure of $\angle B$.

8. Find the measure of $\angle BCD$.

Use your knowledge of inscribed quadrilaterals to answer questions 9 and 10.

9. Is it possible for a trapezoid to be inscribed in a circle? Justify your answer.

10. If a rectangle is inscribed in a circle, each diagonal also serves another function. What is this function?

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Progress Assessment

Circle the letter of the best answer.

1. The circumcenter is the intersection of the three _____ of a triangle.
   a. altitudes
   b. angle bisectors
   c. medians
   d. perpendicular bisectors

2. The _____ is always equidistant from the three sides of a triangle.
   a. centroid
   b. circumcenter
   c. incenter
   d. orthocenter

3. The incenter is the intersection of the three _____ of a triangle.
   a. altitudes
   b. angle bisectors
   c. medians
   d. perpendicular bisectors

4. The _____ is equidistant from the three vertices of a triangle.
   a. centroid
   b. circumcenter
   c. incenter
   d. orthocenter

5. Which of the following quadrilaterals cannot be inscribed in a circle?
   a. square
   b. parallelogram
   c. rectangle
   d. isosceles trapezoid

6. If a quadrilateral is inscribed in a circle, what must be true of the quadrilateral's opposite angles?
   a. They are complementary.
   b. They are congruent.
   c. They are supplementary.
   d. They have no relationship.

continued

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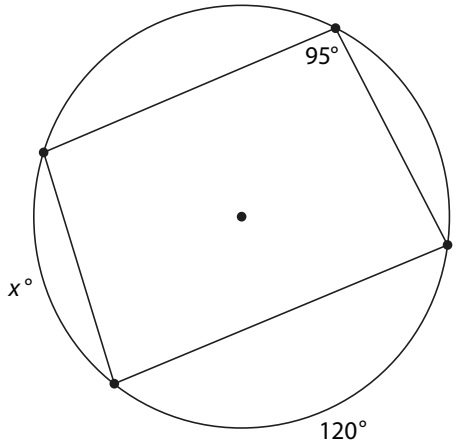
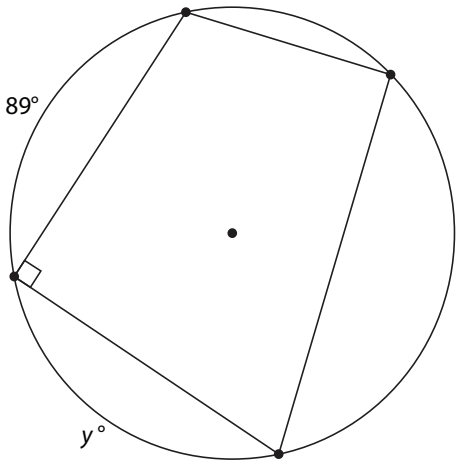
7. Find the value of x.



a. 35
b. 70
c. 85
d. 95

8. Find the value of y.



a. 45
b. 90
c. 91
d. 111

continued

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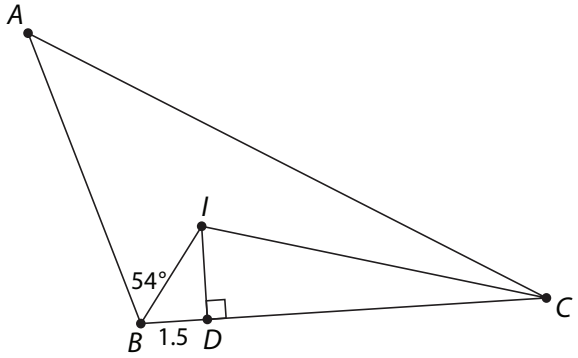
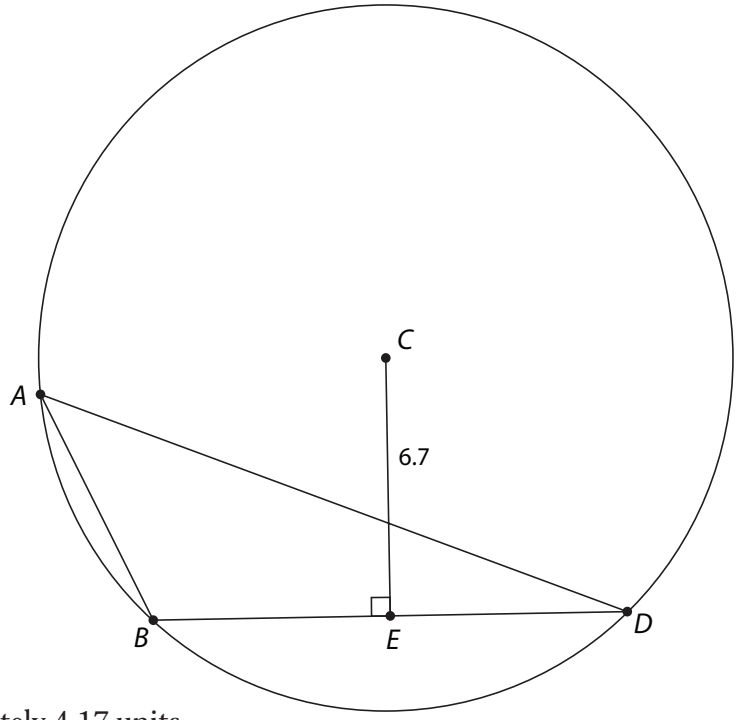
9. Point *I* is the incenter. Find the length of $\overline{DI}$.



   a. approximately 1.09 units
   b. approximately 1.5 units
   c. approximately 2.06 units
   d. There is not enough information to determine the length of $\overline{DI}$.

10. In the diagram below, point *C* is the circumcenter. If *BD* = 10.5, find the length of the radius of the circumscribed circle.



   a. approximately 4.17 units
   b. approximately 6.7 units
   c. approximately 8.51 units
   d. There is not enough information to determine the length of the radius.

continued

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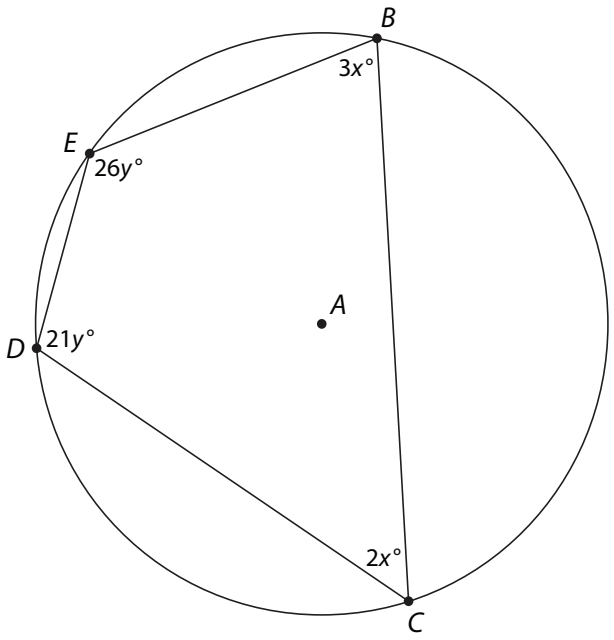
Assessment

Use your knowledge of inscribed quadrilaterals to complete the problem that follows.

11. Find the value of x. Show your work and explain your answer.



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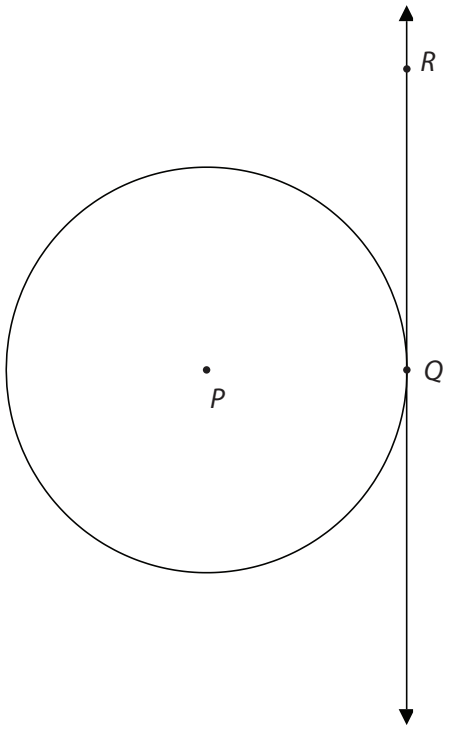
Assessment

Pre-Assessment

Circle the letter of the best answer.

1. To construct a line tangent to circle O at point B, what must be true about $\overleftrightarrow{BC}$ and radius $\overline{OB}$?
   a. $\overleftrightarrow{BC}$ intersects $\overline{OB}$ at point O.
   b. $\overleftrightarrow{BC}$ is perpendicular to $\overline{OB}$ at point B.
   c. $\overleftrightarrow{BC}$ is parallel to $\overline{OB}$ at point B.
   d. $\overleftrightarrow{BC}$ is congruent to $\overline{OB}$.

2. $\overline{RQ}$ is tangent to circle P, as shown below. Which is NOT a step in the process for constructing the tangent?



   a. Draw a straight line from center P through point Q and beyond Q.
   b. Put the compass on point Q and set it to a width less than the distance of $\overline{PQ}$.
   c. Use a protractor to measure a 90° angle at point Q.
   d. Draw an arc on either side of point Q.

continued

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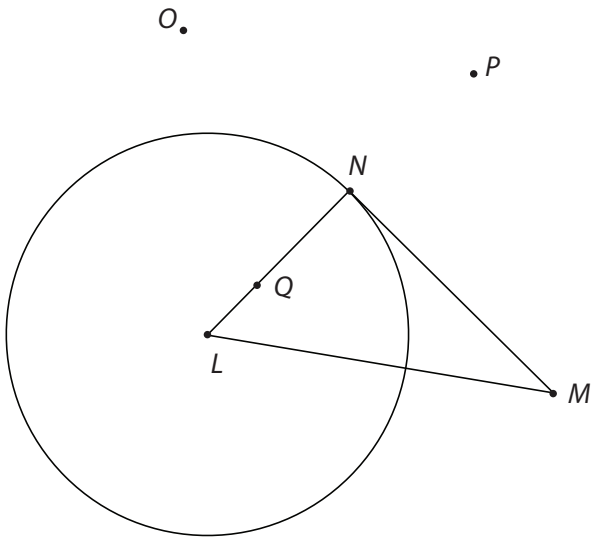
Assessment

3. $\overline{MN}$ is tangent to circle L, as shown below. $\overline{NP} \cong \overline{NQ}$ and $\overline{NO} \cong \overline{NM}$. Without changing the compass setting, how could point M have been constructed?



a. Put the compass on point P and then on point Q, and construct arcs through M.

b. Put the compass on point P and then on center L, and construct arcs through M.

c. Put the compass on point P and then on point O, and construct arcs through M.

d. Put the compass on point N and then on point Q, and construct arcs through M.

4. Given a circle and an external point, how many lines that are tangent to the given circle can be constructed through the point?

a. 0 tangent lines

b. 1 tangent line

c. 2 tangent lines

d. infinitely many tangent lines

5. How many common external tangent lines can be constructed given two non-intersecting circles with congruent radii?

a. infinitely many tangent lines

b. 4 tangent lines

c. 2 tangent lines

d. 0 tangent lines

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Lesson 3: Constructing Tangent Lines

Instruction

Common Core State Standard

G–C.4 (+) Construct a tangent line from a point outside a given circle to the circle.

SMP	
1 ✓	2
3 ✓	4
5 ✓	6
7	8

Essential Questions

1. What is the relationship between a line tangent to a circle and the radius at the point of tangency?
2. If two lines are tangent to the same circle and are drawn from the same exterior point, what must be true about the two lines?
3. What are some representations of tangent lines in the real world?

WORDS TO KNOW

common external tangent	a tangent line that is common to two circles and does not intersect the segment joining the radii of the circles
common internal tangent	a tangent line that is common to two circles and intersects the segment joining the radii of the circles
common tangent	a line that is tangent to two circles
point of tangency	the only point at which a tangent line and a circle intersect
tangent line	a line that intersects a circle at exactly one point and is perpendicular to the radius of the circle

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Recommended Resources

- IXL Learning. "Circles: Tangent lines."

  http://www.walch.com/rr/00054

  This website provides a user-friendly, interactive review of tangent properties. Immediate scoring is provided and if a wrong answer is submitted, the process for arriving at the correct answer is provided. These problems start with right triangle tangent problems, stressing the importance of knowing that a tangent must be perpendicular to a radius at the point of tangency.

- Math Open Reference. "Tangent to a circle at a point."

  http://www.walch.com/rr/00055

  This site provides step-by-step animated instructions for constructing a tangent to a circle from a point on the circle.

- Math Open Reference. "Tangents through an external point."

  http://www.walch.com/rr/00056

  This site provides step-by-step animated instructions for constructing a tangent to a circle from an external point not on the circle.

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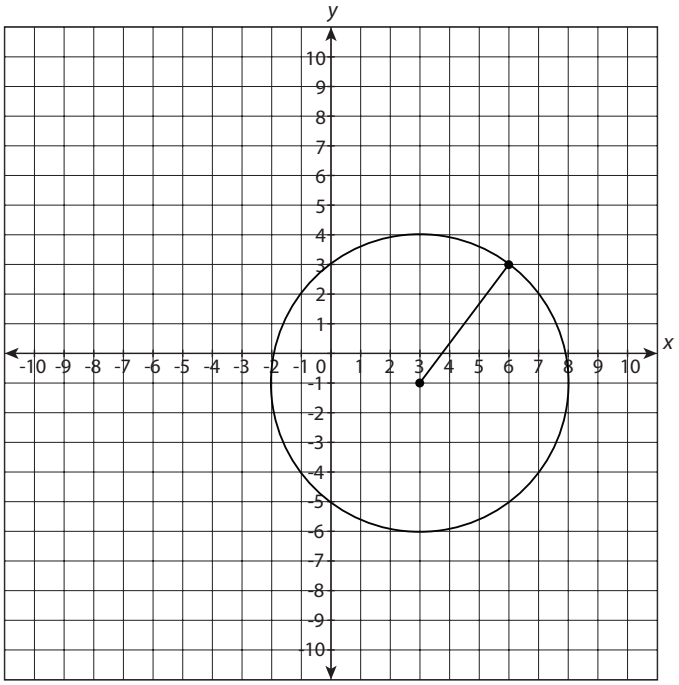
Lesson 6.3.1: Constructing Tangent Lines

Warm-Up 6.3.1

A city is planning to develop a straight trail near a circular playground. Below is a graph of the playground, with the center at (3, –1). The trail is meant to be perpendicular to the radius of the playground at the point (6, 3).



1. What must you know about the radius of the circle before you can determine the line that represents the trail? Why?

2. How would you sketch the trail at point (6, 3) so that it is perpendicular to the radius drawn to (6, 3)?

3. What is the equation of the line that represents the trail?

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Instruction

Lesson 6.3.1: Constructing Tangent Lines

Common Core State Standard

G–C.4 (+) Construct a tangent line from a point outside a given circle to the circle.

Warm-Up 6.3.1 Debrief

1. What must you know about the radius of the circle before you can determine the line that represents the trail? Why?

   The slope of the radius is needed to determine the slope of the line (representing the trail) that is tangent to the circular playground. The slope is necessary since the radius and the tangent line are perpendicular to each other, and, therefore, their slopes are opposite reciprocals.

2. How would you sketch the trail at point (6, 3) so that it is perpendicular to the radius drawn to (6, 3)?

   First, find the slope of the radius using the slope formula: $m=\frac{y_2-y_1}{x_2-x_1}=\frac{-1-3}{3-6}=\frac{4}{3}$.

   Recall that the slopes of perpendicular lines are opposite reciprocals, so the slope of the line representing the trail is $-\frac{3}{4}$.

   Using $\frac{\text{rise}}{\text{run}}$ and starting at (6, 3), locate another point on the line representing the trail using $m=-\frac{3}{4}$.

   One possibility is to move 3 units down and 4 units to the right, to the point (10, 0).

   Another possibility is to move 3 units up and 4 units to the left, to the point (2, 6).

   Use the point (6, 3) and either (10, 0) or (2, 6) to draw the line.

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3. What is the equation of the line that represents the trail?

   Recall the point-slope form of a line, $y - y_1 = m(x - x_1)$.

   Use the slope of the line that is perpendicular to the radius, $m = -\frac{3}{4}$, and the original point, (6, 3), to write the equation in point-slope form: $y - 3 = -\frac{3}{4}(x - 6)$. The equation simplifies to $y = -\frac{3}{4}x + \frac{15}{2}$.

Connection to the Lesson

- Students will extend their understanding of the properties of tangent lines to constructing tangent lines using construction tools.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

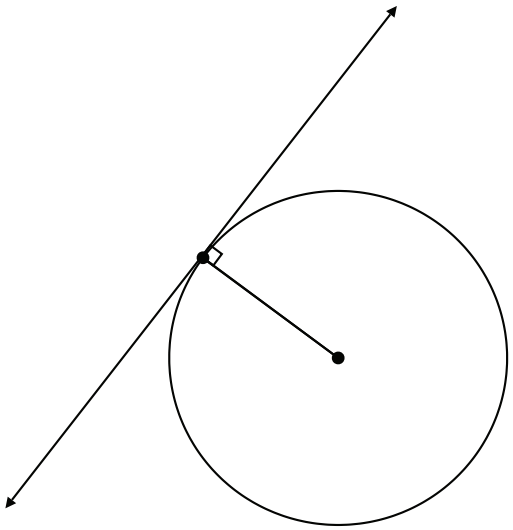
- understanding the relationship between perpendicular lines (G–GPE.5)
- using a compass and a straightedge (G–CO.12)
- constructing a perpendicular bisector of a line segment (G–CO.12)

Introduction

Tangent lines are useful in calculating distances as well as diagramming in the professions of construction, architecture, and landscaping. Geometry construction tools can be used to create lines tangent to a circle. As with other constructions, the only tools you are allowed to use are a compass and a straightedge, a reflective device and a straightedge, or patty paper and a straightedge. You may be tempted to measure angles or lengths, but remember, this is not allowed with constructions.

Key Concepts

- If a line is tangent to a circle, it is perpendicular to the radius drawn to the **point of tangency,** the only point at which a tangent line and a circle intersect.
- Exactly one tangent line can be constructed by using construction tools to create a line perpendicular to the radius at a point on the circle.



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Instruction

Constructing a Tangent at a Point on a Circle Using a Compass

1. Use a straightedge to draw a ray from center O through the given point P on the circle. Be sure the ray extends beyond point P.
2. Construct the line perpendicular to $\overrightarrow{OP}$ at point P. This is the same procedure as constructing a perpendicular line through a point on a line.
   a. Put the sharp point of the compass on P and open the compass less wide than the distance of $\overline{OP}$.
   b. Draw an arc on both sides of P on $\overrightarrow{OP}$. Label the points of intersection A and B.
   c. Set the sharp point of the compass on A. Open the compass wider than the distance of $\overline{AB}$ and make a large arc.
   d. Without changing your compass setting, put the sharp point of the compass on B. Make a second large arc. It is important that the arcs intersect each other.
3. Use your straightedge to connect the points of intersection of the arcs.
4. Label the new line m.

Do not erase any of your markings.

Line m is tangent to circle O at point P.

- It is also possible to construct a tangent line from an exterior point not on a circle.
- Exactly two lines can be constructed that are tangent to the circle through an exterior point not on the circle.

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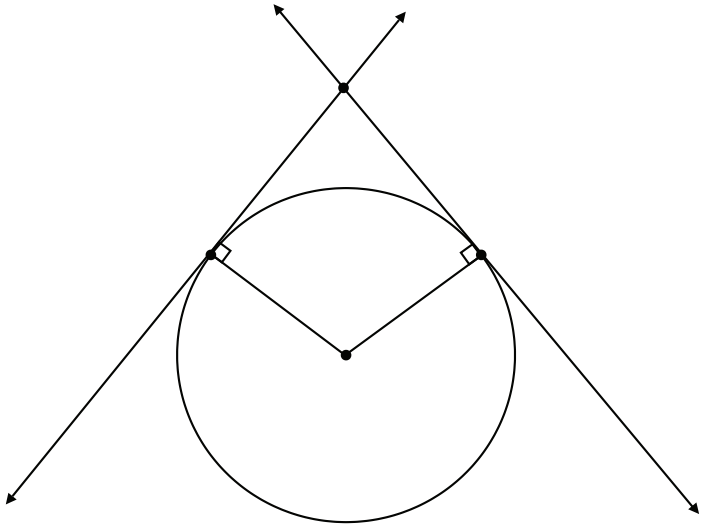
Instruction

- The two segments from the external point to each point of tangency are congruent.



Constructing a Tangent from an Exterior Point Not on a Circle Using a Compass

1. To construct a line tangent to circle O from an exterior point not on the circle, first use a straightedge to draw a ray connecting center O and the given point R.
2. Find the midpoint of $\overline{OR}$ by constructing the perpendicular bisector.
   a. Put the sharp point of your compass on point O. Open the compass wider than half the distance of $\overline{OR}$.
   b. Make a large arc intersecting $\overline{OR}$.
   c. Without changing your compass setting, put the sharp point of the compass on point R. Make a second large arc. It is important that the arcs intersect each other. Label the points of intersection of the arcs as C and D.
   d. Use your straightedge to connect points C and D.
   e. The point where $\overline{CD}$ intersects $\overline{OR}$ is the midpoint of $\overline{OR}$. Label this point F.
3. Put the sharp point of the compass on midpoint F and open the compass to point O.

(*continued*)

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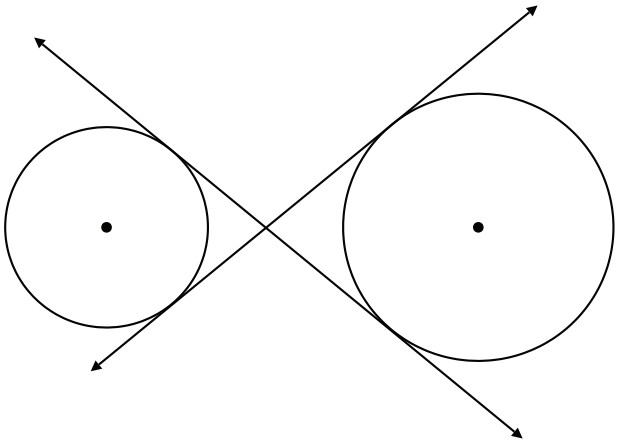
Instruction

4. Without changing the compass setting, draw an arc across the circle so it intersects the circle in two places. Label the points of intersection as G and H.
5. Use a straightedge to draw a line from point R to point G and a second line from point R to point H.

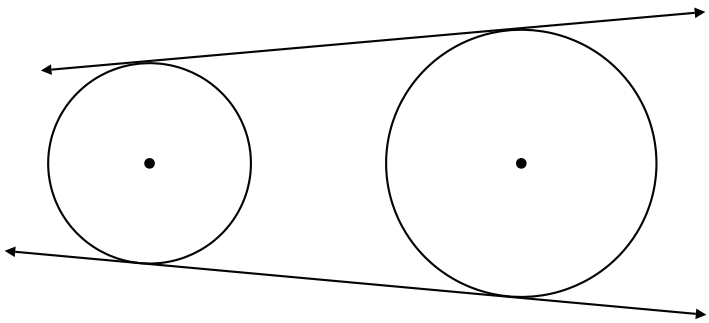
Do not erase any of your markings.

$\overleftrightarrow{RG}$ and $\overleftrightarrow{RH}$ are tangent to circle O.

- If two circles do not intersect, they can share a tangent line, called a **common tangent**.
- Two circles that do not intersect have four common tangents.
- Common tangents can be either internal or external.
- A **common internal tangent** is a tangent line that is common to two circles and intersects the segment joining the radii of the circles.



- A **common external tangent** is a tangent line that is common to two circles and does not intersect the segment joining the radii of the circles.



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Instruction

Common Errors/Misconceptions

- assuming that a radius and a line are perpendicular at the possible point of intersection simply by observation
- assuming two tangent segments are congruent by observation
- incorrectly changing the compass settings
- not making large enough arcs to find the points of intersection

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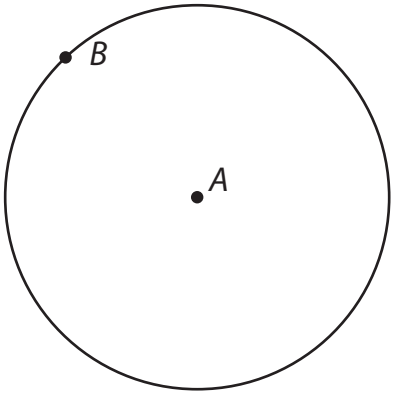
Instruction

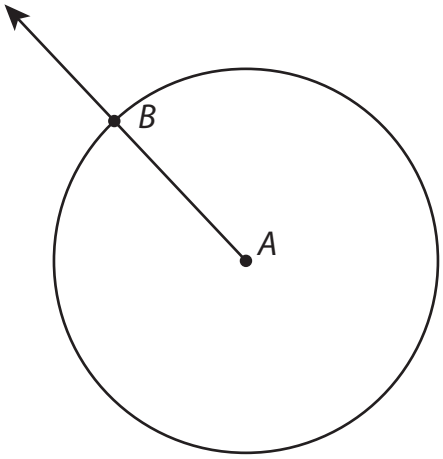
Guided Practice 6.3.1

Example 1

Use a compass and a straightedge to construct $\overline{BC}$ tangent to circle A at point B.



1. Draw a ray from center A through point B and extending beyond point B.



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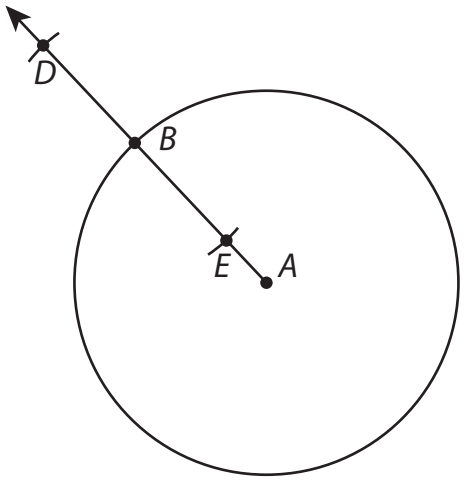
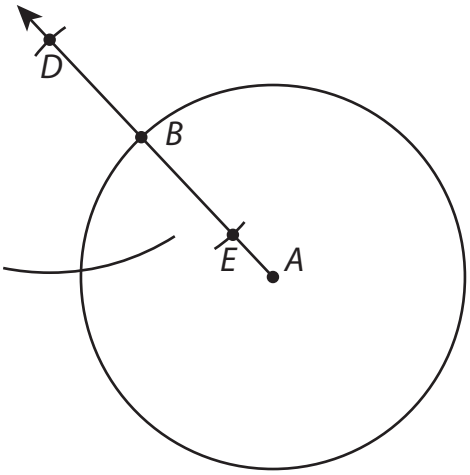
Instruction

2. Put the sharp point of the compass on point B. Set it to any setting less than the length of $\overline{AB}$, and then draw an arc on either side of B, creating points D and E.



3. Put the sharp point of the compass on point D and set it to a width greater than the distance of $\overline{DB}$. Make a large arc intersecting $\overrightarrow{AB}$.



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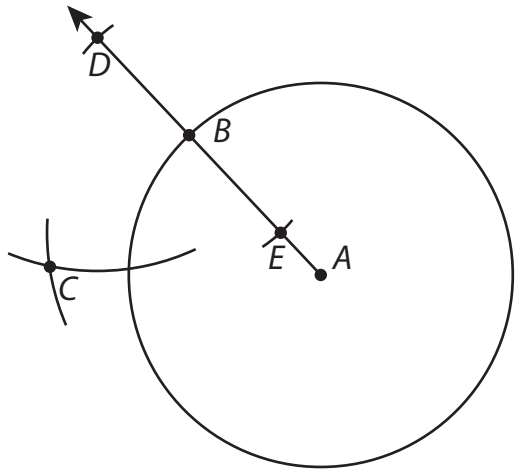
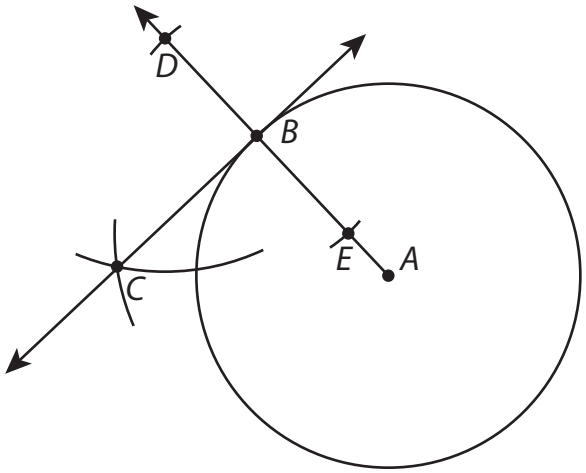
Instruction

4. Without changing the compass setting, put the sharp point of the compass on point *E* and draw a second arc that intersects the first. Label the point of intersection with the arc drawn in step 3 as point *C*.



5. Draw a line connecting points *C* and *B*, creating tangent $\overleftrightarrow{BC}$.



Do not erase any of your markings.

$\overleftrightarrow{BC}$ is tangent to circle *A* at point *B*.

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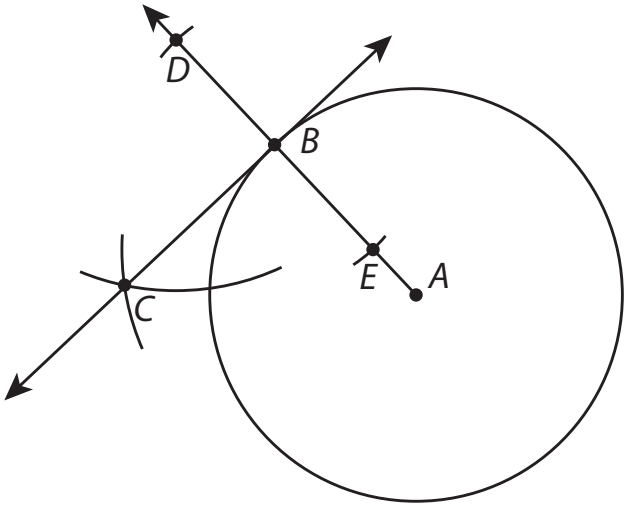
Instruction

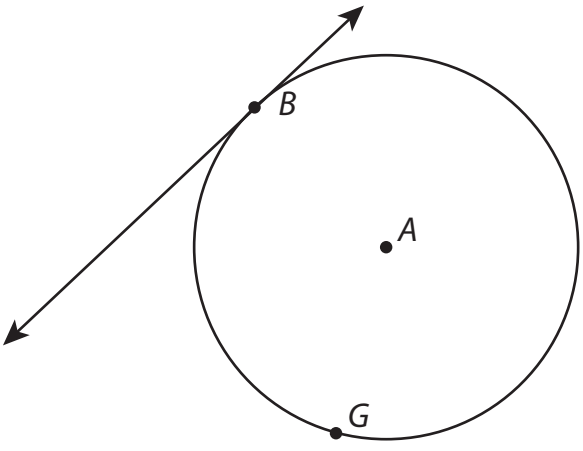
Example 2

Using the circle and tangent line from Example 1, construct two additional tangent lines, so that circle *A* below will be inscribed in a triangle.



1. Choose a point, *G*, on circle *A*.

   (*Note*: To highlight the essential ideas of this example, some features of the given diagram have been removed.)



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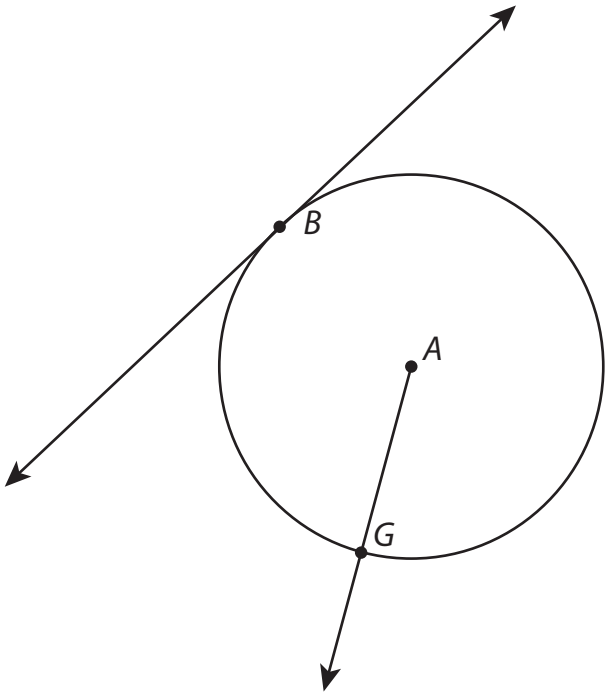
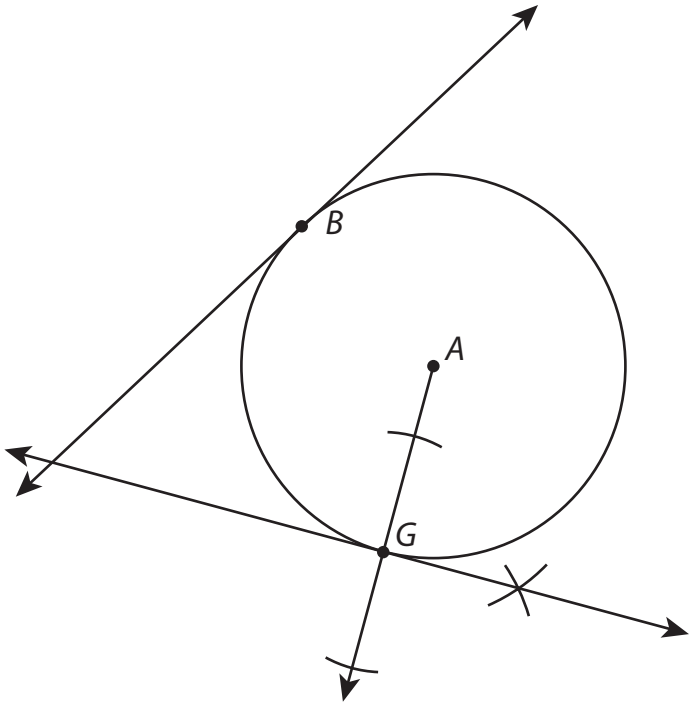
Instruction

2. Draw a ray from center A to point G.



3. Follow the process explained in Example 1 for constructing a tangent line through point G.



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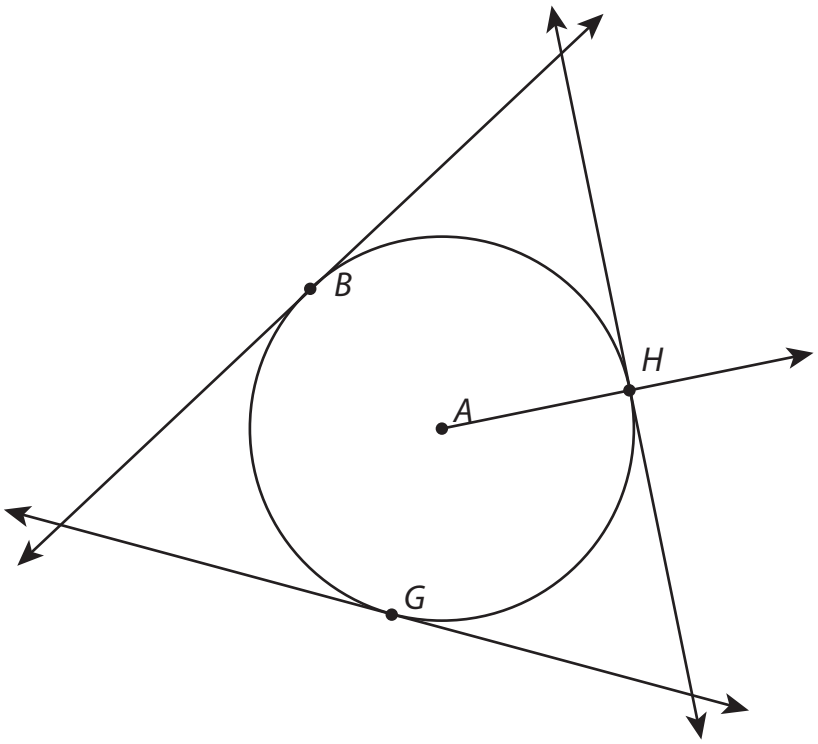
Instruction

4. Choose another point, H, on circle A. Draw a ray from center A to point H, and follow the process explained in Example 1 to construct the third tangent line. Be sure to draw the tangent lines long enough to intersect one another.



Do not erase any of your markings.

Circle A is inscribed in a triangle.

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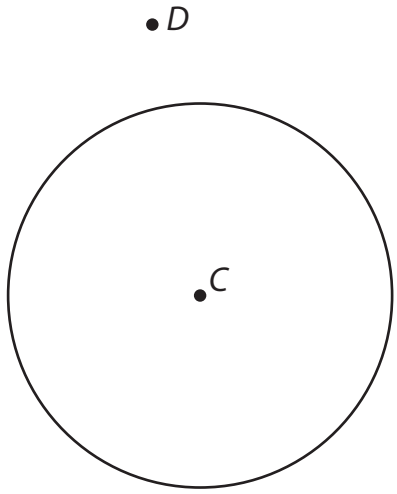
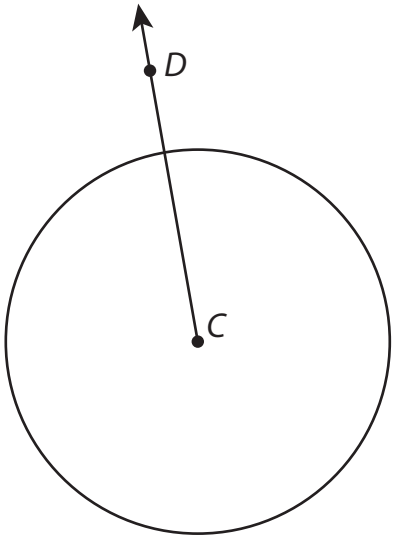
Instruction

Example 3

Use a compass and a straightedge to construct the lines tangent to circle C that pass through point D.



1. Draw a ray connecting center C and the given point D.



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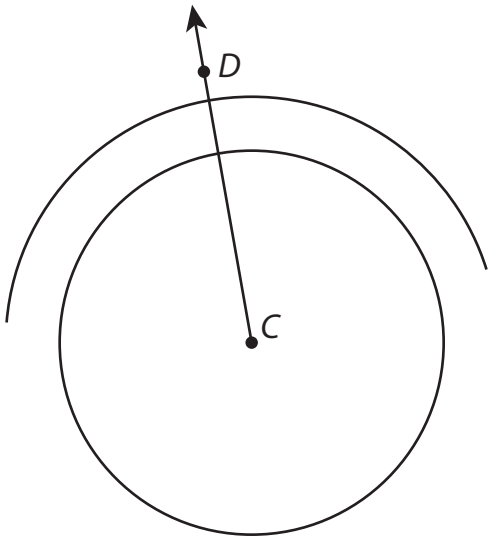
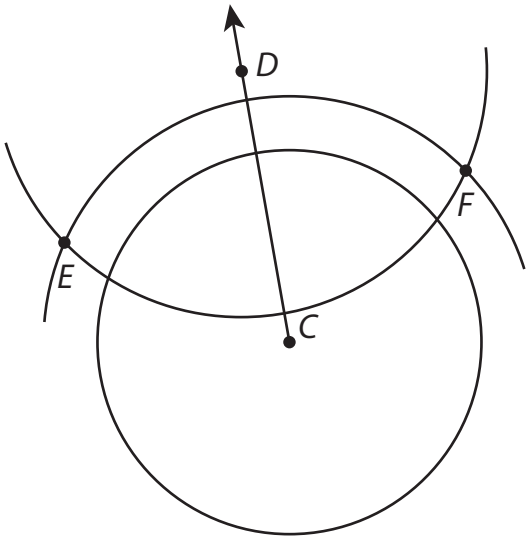
Instruction

2. Find the midpoint of $\overline{CD}$ by constructing the perpendicular bisector.

   Put the sharp point of your compass on point *C*. Open the compass wider than half the distance of $\overline{CD}$. Make a large arc intersecting $\overline{CD}$.



   Without changing your compass setting, put the sharp point of the compass on point *D*. Make a second large arc. It is important that the arcs intersect each other. Label the points of intersection of the arcs as *E* and *F*.



(*continued*)

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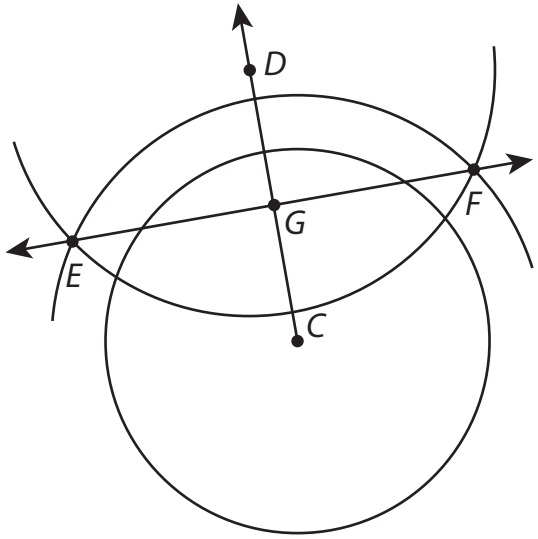
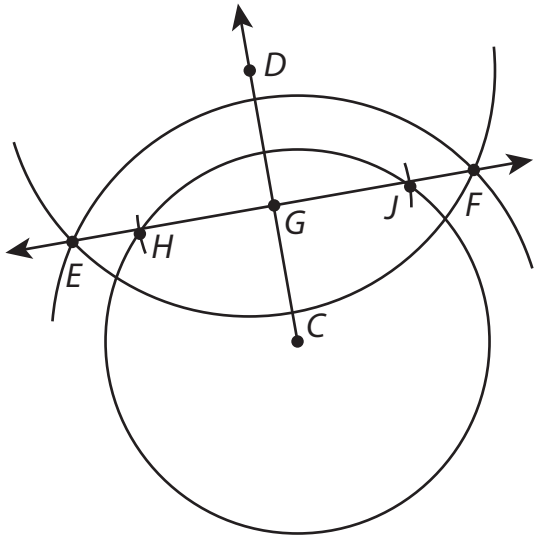
Instruction

Use your straightedge to connect points E and F. The point where $\overline{EF}$ intersects $\overline{CD}$ is the midpoint of $\overline{CD}$. Label this point G.



3. Put the sharp point of the compass on midpoint G and open the compass to point C. Without changing the compass setting, draw an arc that intersects the circle in two places. Label the points of intersection as H and J.



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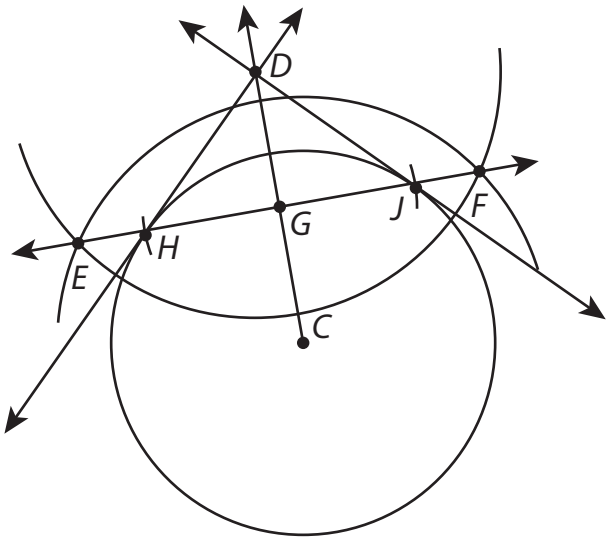
Instruction

4. Use a straightedge to draw a line from point D to point H and a second line from point D to point J.



Do not erase any of your markings.

$\overleftrightarrow{DH}$ and $\overleftrightarrow{DF}$ are both tangent to circle C.

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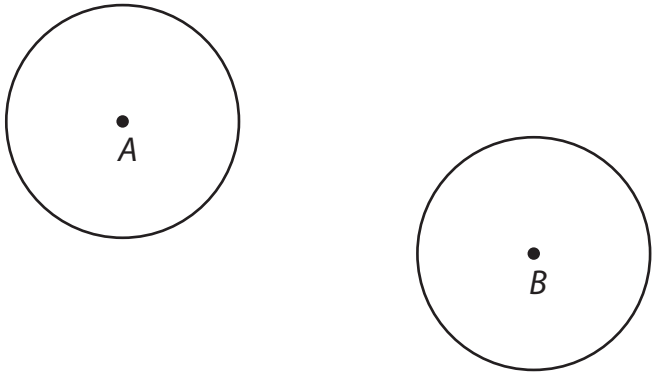
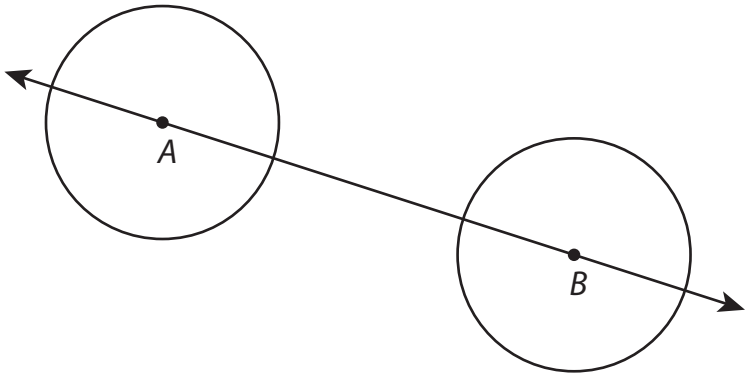
Instruction

Example 4

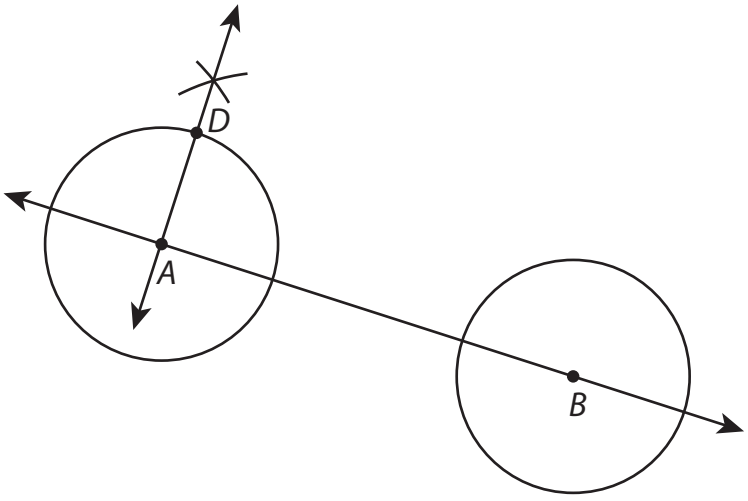
Circle A and circle B are congruent. Construct a line tangent to both circle A and circle B.



1. Use a straightedge to connect A and B, the centers of the circles.



2. At center point A, construct a line perpendicular to $\overline{AB}$. Label the point of intersection with circle A as point D.



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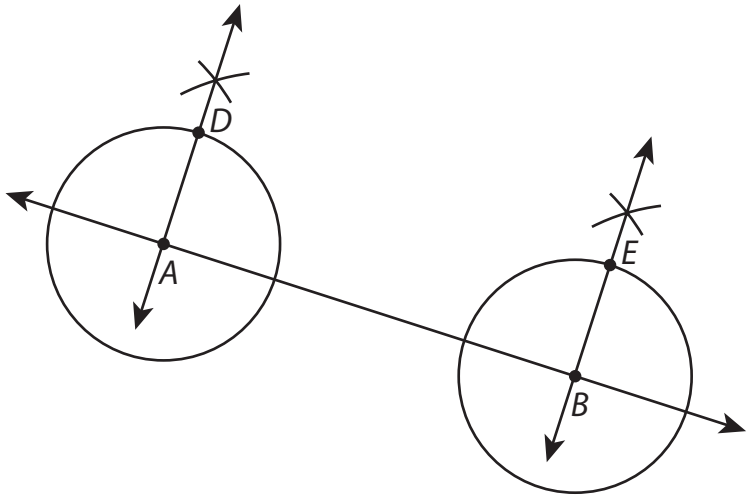
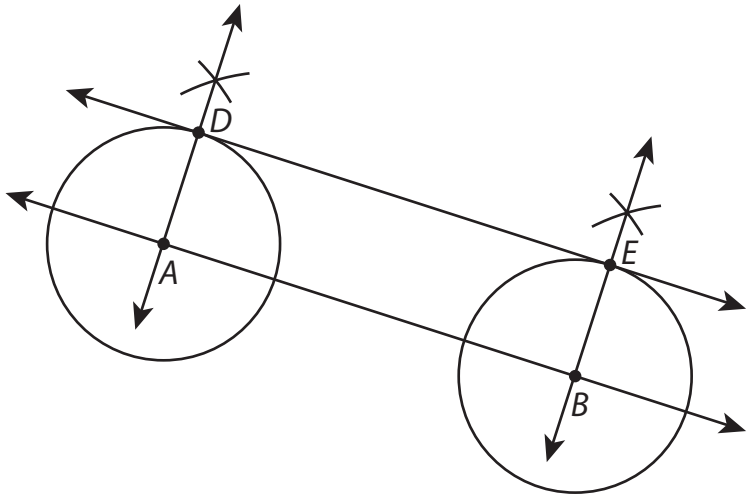
Instruction

3. At center point B, construct a line perpendicular to $\overline{AB}$. Label the point of intersection with circle B as point E.



4. Use a straightedge to connect points D and E.



Do not erase any of your markings.

$\overleftrightarrow{DE}$ is tangent to circle A and circle B.

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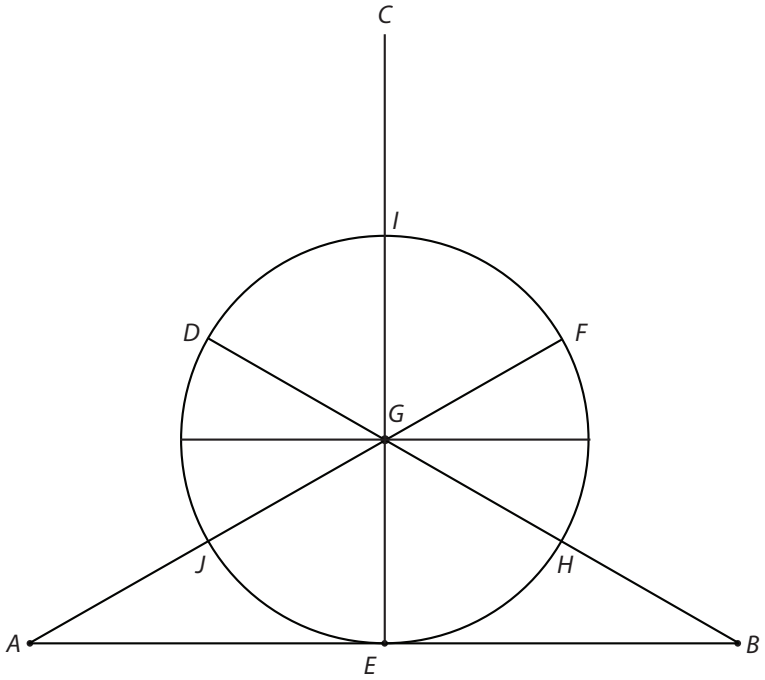
Problem-Based Task 6.3.1: Designing a T-Shirt Logo

SMP	
1 ✓	2
3 ✓	4
5 ✓	6
7	8

A local T-shirt shop is designing a logo for a company. The company's logo is a circle inscribed in a triangle. The sketch below is incomplete. Help the designer by constructing the two line segments that have an endpoint at point *C* and are tangent to the circle at points *D* and *F*, and verify that these segments will join the base of the triangle at points *A* and *B*.

Calculate the perimeter of the large triangle. The designer knows the following information about the dimensions of the design:

- The diameter of circle *G* is 8 cm.
- $\overline{CI}$ = 4 cm
- $\overline{CI}$, $\overline{BH}$, and $\overline{AJ}$ are all congruent.



Calculate the perimeter of the large triangle.

SWB p. 87

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Problem-Based Task 6.3.1: Designing a T-Shirt Logo

Coaching

a. How can the designer draw the tangents $\overline{CA}$ and $\overline{CB}$ so the design can be completed?

b. What is the length of the radius of the circle?

c. What is the length of $\overline{GC}$?

d. How do you calculate the length of $\overline{CF}$?

e. How does the length of $\overline{CF}$ help you to know the length of $\overline{CD}$?

f. If $\overline{CI}$, $\overline{BH}$, and $\overline{AJ}$ are all congruent, what are the lengths of $\overline{BF}$ and $\overline{AD}$?

g. What are the lengths of $\overline{AE}$ and $\overline{BE}$?

h. What is the perimeter of $\triangle ABC$?

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Instruction

Problem-Based Task 6.3.1: Designing a T-Shirt Logo

Coaching Sample Responses

a. How can the designer draw the tangents $\overline{CA}$ and $\overline{CB}$ so the design can be completed?

Find the midpoint of $\overline{GC}$ by constructing the perpendicular bisector of the segment.

Place the point of the compass on the midpoint and set the length of the compass to center *G* of the circle.

Without changing the compass width, draw an arc on the circle in two possible places. These are the points of tangency. Verify that these two points are indeed at points *D* and *F*.

Draw the tangent lines from *C* through points *D* and *F*, and verify that they pass through points *A* and *B*.

b. What is the length of the radius of the circle?

4 cm (half of the diameter)

c. What is the length of $\overline{GC}$?

4 + 4 = 8

d. How do you calculate the length of $\overline{CF}$?

Use the Pythagorean Theorem to calculate the length of $\overline{CF}$.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$4^2 + (CF)^2 = 8^2$	Substitute.
$16 + (CF)^2 = 64$	Simplify.
$(CF)^2 = 48$	
$CF = \sqrt{48} = 4\sqrt{3} \approx 6.9$	

The length of *CF* is $4\sqrt{3}$ or approximately 6.9 cm.

e. How does the length of $\overline{CF}$ help you to know the length of $\overline{CD}$?

Since $\overline{CF}$ and $\overline{CD}$ extend from the same exterior point and are tangent to the same circle, the line segments are congruent. *CF* = *CD*; therefore, *CD* = $4\sqrt{3}$ or approximately 6.9 cm.

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Instruction

f. If $\overline{CI}$, $\overline{BH}$, and $\overline{AJ}$ are all congruent, what are the lengths of $\overline{BF}$ and $\overline{AD}$?

The radii are all the same length regardless of the point on the circle to which they extend. $\overline{CI}$, $\overline{BH}$, and $\overline{AJ}$ are all 4 cm, so the lengths of $\overline{BF}$ and $\overline{AD}$ would be calculated exactly as the length of $\overline{CF}$ was calculated in part d. Therefore, $\overline{BF}$ and $\overline{AD}$ are both $4\sqrt{3}$ or approximately 6.9 cm.

g. What are the lengths of $\overline{AE}$ and $\overline{BE}$?

$\overline{AE}$ and $\overline{BE}$ are both $4\sqrt{3}$ or approximately 6.9 cm.

h. What is the perimeter of $\triangle ABC$?

To find the perimeter of the triangle, find the sum of the segments.

$$AE + AD + BE + BF + CD + CF = 4\sqrt{3} + 4\sqrt{3} + 4\sqrt{3} + 4\sqrt{3} + 4\sqrt{3} + 4\sqrt{3} = 24\sqrt{3} \approx 41.6$$

The perimeter of the triangle is $24\sqrt{3}$ or approximately 41.6 cm.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 6.3.1: Constructing Tangent Lines

Use your knowledge of constructions to complete each problem that follows.

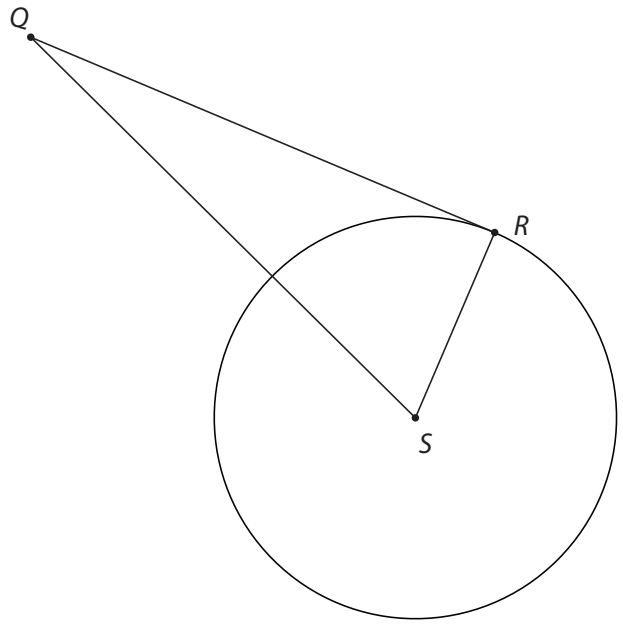
1. Construct circle A and a point B on the circle. Construct a line tangent to circle A at point B.

2. Construct circle C and a point D not on the circle. Construct a line tangent to circle C through point D.

3. Construct two non-intersecting circles with congruent radii. Construct one common exterior tangent.

4. $\overleftrightarrow{AB}$ is tangent to circle D at point B. Point A is an exterior point to circle D. What was the first step that had to be completed to construct tangent $\overleftrightarrow{AB}$?

5. $\overleftrightarrow{QR}$ is tangent to circle S at point R. Between which two points must the sharp point of the compass have been placed in order to make the arc that formed point Q?



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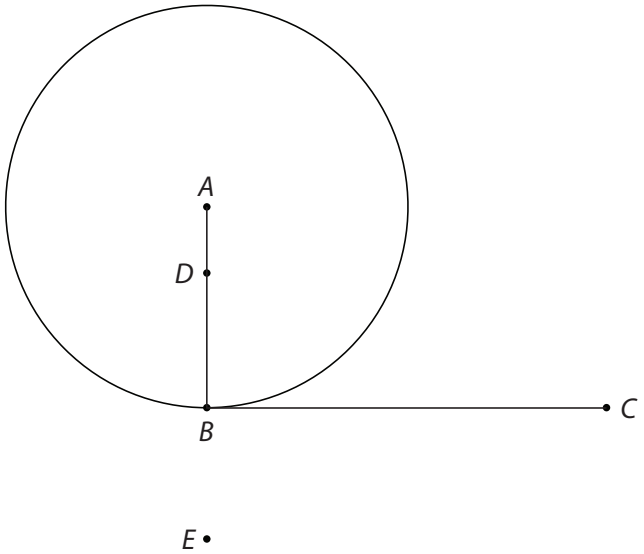
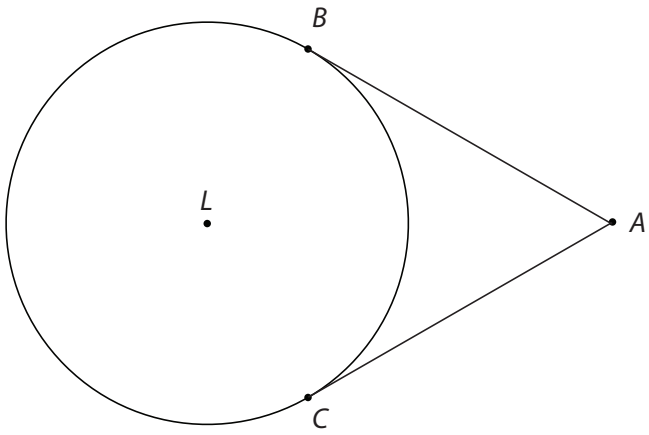
6. $\overline{BC}$ is tangent to circle A at point B. Describe the steps that were taken to construct the tangent.



7. Your friend says that if two lines are tangent to the same circle, they must intersect. Is your friend correct? Why or why not?

8. Construct circle A. Place three points on circle A and construct three tangent lines so the circle is inscribed in the triangle.

9. $\overline{AB}$ and $\overline{AC}$ are tangent to circle L in the following diagram. How can you use construction tools to verify that the segments are congruent?



10. $\overline{AB}$ is tangent to circle C at point B. What can you conclude about the construction of point A?

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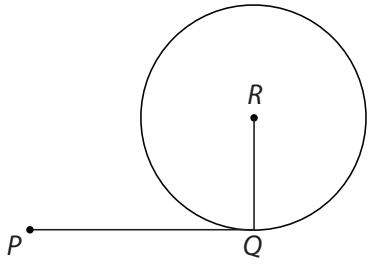
Progress Assessment

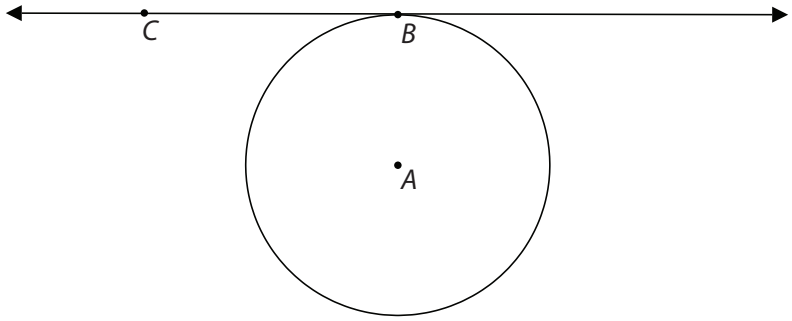
Circle the letter of the best answer.

1. $\overline{PQ}$ is tangent to circle R at point Q as shown below. What tools were necessary to construct tangent $\overline{PQ}$?



   a. a straightedge and a protractor
   b. a compass and a protractor
   c. a compass and a ruler
   d. a compass and a straightedge

2. To construct a line tangent to circle A at point B, what must be true about $\overleftrightarrow{BC}$ and the radius, $\overline{AB}$?



   a. $\overleftrightarrow{BC}$ is perpendicular to $\overline{AB}$ at point B.
   b. $\overleftrightarrow{BC}$ is parallel to $\overline{AB}$.
   c. $\overline{BC} \cong \overline{AB}$
   d. $\overline{BC} \cong \overline{AC}$

3. How many common internal tangents can be constructed given two non-intersecting circles with congruent radii?

   a. 0 tangent lines
   b. 2 tangent lines
   c. 4 tangent lines
   d. infinitely many tangent lines

continued

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Assessment

4. How many common tangent lines can be constructed given two non-intersecting circles with congruent radii?

   a. 0 tangent lines
   b. 2 tangent lines
   c. 4 tangent lines
   d. infinitely many tangent lines

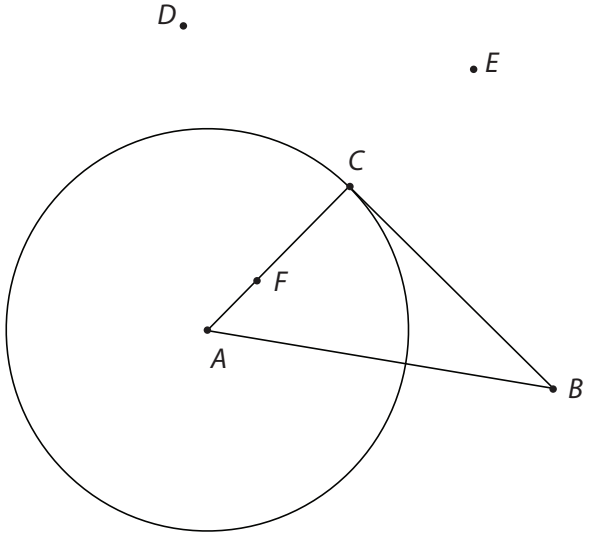
5. How many tangent lines can be constructed through one point on a given circle?

   a. 0 tangent lines
   b. 1 tangent line
   c. 2 tangent lines
   d. infinitely many tangent lines

6. Given a circle and an external point, how many lines can be constructed through the point tangent to the given circle?

   a. 0 tangent lines
   b. 1 tangent line
   c. 2 tangent lines
   d. infinitely many tangent lines

7. $\overline{BC}$ is tangent to circle A, as shown below. $\overline{CE} \cong \overline{CF}$ and $\overline{CD} \cong \overline{BC}$. Without changing the compass setting, how could point B have been constructed?



   a. Put the compass on point E and then on center A, and construct arcs through B.
   b. Put the compass on point C and then on point F, and construct arcs through B.
   c. Put the compass on point E and then on point F, and construct arcs through B.
   d. Put the compass on point E and then on point D, and construct arcs through B.

continued

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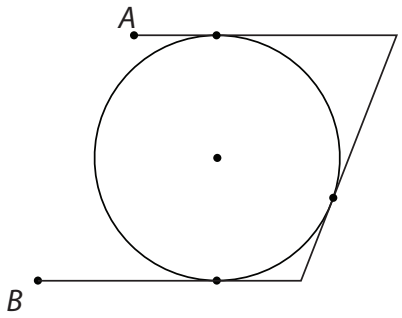
Assessment

8. A landscaper drew a diagram of a circular flower garden circumscribed by a fence in the shape of a quadrilateral. One side of the fence is missing. How can she complete this construction?



   a. Draw a line connecting *A* and *B* with a ruler.

   b. Draw a line from the center of the circle to point *A* or point *B*, and construct a perpendicular bisector to help find the tangent.

   c. Find the midpoint between *A* and *B* and draw the line tangent at that point.

   d. Draw the line so that it is parallel to the line on the opposite side of the circle.

9. An architect is designing a new office building in the shape of a circle. The center of the building is at point *A*, and a hallway leads to an exit located at point *B*. A walkway is to be constructed at point *B*, tangent to the office building. What is the relationship between the walkway and the hallway leading to the exit?

   a. The walkway is parallel to the hallway.

   b. The walkway is perpendicular to the hallway.

   c. The walkway and the hallway intersect in two places.

   d. The hallway is half the distance of the walkway.

10. What is the first step in constructing a common exterior tangent line to two congruent and non-intersecting circles?

   a. Draw a line connecting the centers of the circles.

   b. Locate a point on each circle.

   c. Construct a chord on one circle.

   d. Measure the radii of the circles.

continued

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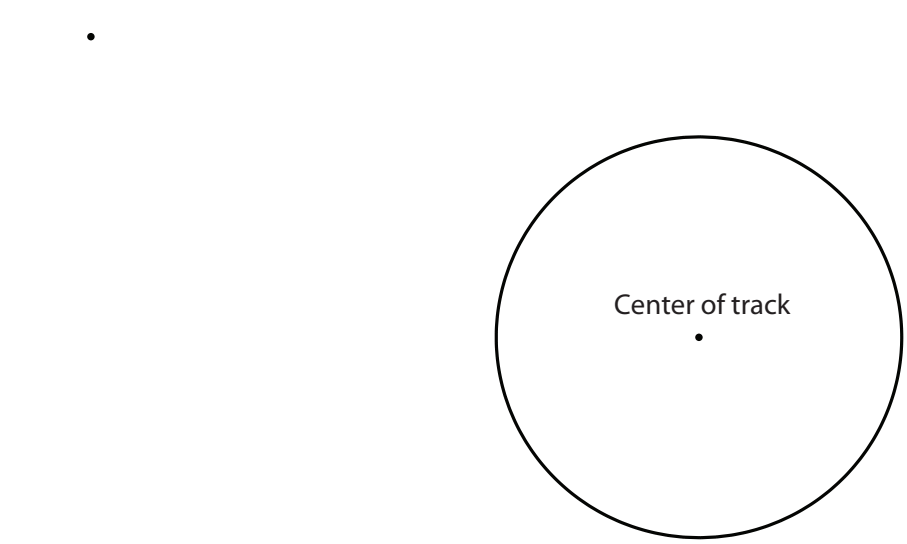
Assessment

Use what you know about tangents to solve the following problem.

11. A circular track is installed at the local recreation center. Landscapers are designing a walkway to the street entrance. The walkway from the track to the street entrance will be tangent to the track.



a. How many walkways can be created from the street entrance to the circular track?

b. Construct all possible locations of the walkway.

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Convert 45° to radians.

   a. $\frac{1}{4}$
   b. $\frac{\pi}{4}$
   c. 4
   d. 4π

2. Convert $\frac{2\pi}{3}$ radians to degrees.

   a. 60°
   b. 90°
   c. 120°
   d. 270°

3. A circle has a radius of 3 units. Find the radian measure of a central angle that intercepts an arc length of 12 units.

   a. 36 radians
   b. 8π radians
   c. 4π radians
   d. 4 radians

4. A circle has a radius of 5 units. Find the length of the arc intercepted by a central angle measuring $\frac{1}{5}$ radian.

   a. $\frac{1}{5}$ unit
   b. 1 unit
   c. 5 units
   d. 25 units

5. A circle has a radius of 12 units. Find the area of a sector with a central angle of 20°.

   a. 1,440 square units
   b. 16π square units
   c. 8π square units
   d. $\frac{36\pi}{5}$ square units

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Lesson 4: Finding Arc Lengths and Areas of Sectors

Instruction

Common Core State Standard

G–C.5 Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

SMP	
1 ✓	2
3 ✓	4 ✓
5	6 ✓
7	8

Essential Questions

1. How is radian measure related to degree measure?
2. How is an arc length related to the circumference of a circle?
3. How is the area of a sector related to the area of a circle?

WORDS TO KNOW

arc length	the distance between the endpoints of an arc; written as $m\overset{\frown}{AB}$
central angle	an angle with its vertex at the center of a circle
circumference	the distance around a circle; $C = 2\pi r$ or $C = \pi d$, for which C represents circumference, r represents the circle's radius, and d represents the circle's diameter
radian	a unit for measuring angles; one radian is equal to the measure of the central angle that intercepts an arc equal in length to the radius of the circle; π radians = 180°
radian measure	the measure of an angle in radians; the ratio of the arc intercepted by the central angle to the radius of the circle
sector	a portion of a circle bounded by two radii and their intercepted arc

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Instruction

Recommended Resources

- IXL Learning. "Circles: Arc measure and arc length."

  http://www.walch.com/rr/00057

  This interactive website gives a series of problems and scores them immediately. If the user submits a wrong answer, a description and process for arriving at the correct answer are provided. These problems start with a diagram of a circle with a given radius. Users are given an arc measure in degrees and are asked to find the length of the arc.

- IXL Learning. "Circles: Area of sectors."

  http://www.walch.com/rr/00058

  This interactive website also gives a series of problems and scores them immediately, providing instant feedback if a wrong answer is submitted. These problems start with a diagram of a circle with a given radius. Users are given a sector with a central angle measure in degrees and are asked to find the area of the sector.

- Math Open Reference. "Arc Length."

  http://www.walch.com/rr/00059

  This site gives a brief overview of how to calculate the arc length of a circle and provides a virtual manipulative to experiment with choosing different arc lengths. The equation is updated in real time, allowing users to see what changes in the equation and what remains the same.

- TeacherTube. "Geo Screencast: Arc Length."

  http://www.walch.com/rr/00061

  This video explains how to find arc length given the radius of the circle and the degree measure of the central angle.

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Lesson 6.4.1: Defining Radians

Warm-Up 6.4.1

A child is riding a bicycle around a cul-de-sac. The cul-de-sac is circular with a diameter of 60 feet.

1. If the child rides her bicycle the entire circumference of the circle, how far will she have traveled?

2. If the child gets tired halfway through the ride and stops, how far will she have traveled?

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Instruction

Lesson 6.4.1: Defining Radians

Common Core State Standard

G–C.5 Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Warm-Up 6.4.1 Debrief

1. If the child rides her bicycle the entire circumference of the circle, how far will she have traveled?

   The formula for finding the circumference of a circle is $C = 2\pi r$. Since the diameter, d, is given as 60 feet and $2r = d$, use the equation $C = \pi d$.

   $C = \pi d = 60\pi \approx 188.50$

   If the child rides the entire circumference of the circle, she will have traveled 188.50 feet.

2. If the child gets tired halfway through the ride and stops, how far will she have traveled?

   $\frac{1}{2} \bullet 60\pi = 30\pi \approx 94.25$

   If the child stops halfway through the ride, she will have traveled 94.25 feet.

Connection to the Lesson

- Students will extend their understanding of circumference by thinking about arc length as a fraction of the circumference.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

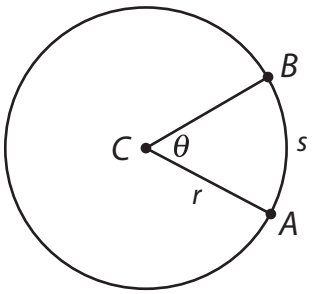
- finding the circumference of a circle (7.G.4)
- understanding scale factor in similar shapes (7.G.1)
- using ratios and proportions (7.RP.3)

Introduction

All circles are similar; thus, so are the arcs intercepting congruent angles in circles. A **central angle** is an angle with its vertex at the center of a circle. We have measured an arc in terms of the central angle that it intercepts, but we can also measure the length of an arc. **Arc length**, the distance between the endpoints of an arc, is proportional to the radius of the circle according to the central angle that the arc intercepts. The constant of proportionality is the radian measure of the angle. You already know how to measure angles in degrees. Radian measure is another way to measure angles. An angle measure given in degrees includes a degree symbol. An angle measure given in radians does not.

Key Concepts

- Arc length is the distance between the endpoints of an arc, and is commonly written as $m\overset{\frown}{AB}$.
- The **radian measure** of a central angle can be found by calculating the ratio of the length of the intercepted arc to the radius of the circle.



- This ratio leads us to a formula for the radian measure of a central angle θ in terms of the intercepted arc length, s, and the radius of the circle, r: $\theta = \frac{s}{r}$.

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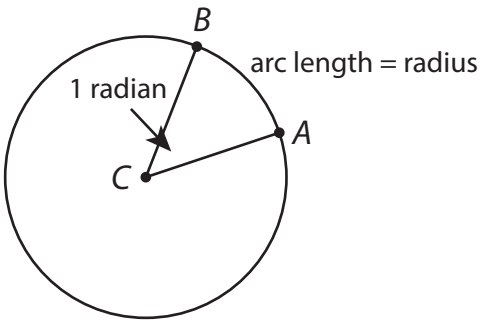
Instruction

- When the intercepted arc is equal in length to the radius of the circle, the central angle measures 1 **radian**.



- Recall that the **circumference**, or the distance around a circle, is given by $C = 2\pi r$ or $C = \pi d$, where C represents circumference, r represents radius, and d represents the circle's diameter.
- Since the ratio of the arc length of the entire circle to the radius of the circle is $\frac{2\pi r}{r} = 2\pi$, there are 2π radians in a full circle.
- We know that a circle contains 360° or 2π radians. We can convert between radian measure and degree measure by simplifying this ratio to get π radians = 180°.
- To convert between radian measure and degree measure, set up a proportion.

$$\frac{\text{radian measure}}{\pi} = \frac{\text{degree measure}}{180^\circ}$$

- To find the arc length when the central angle is given in radians, use the formula for radian measure to solve for s.
- To find arc length s when the central angle is given in degrees, we determine the fraction of the circle that we want to find using the measure of the angle. Set up a proportion with the circumference, C.

$$\frac{s}{C} = \frac{\text{degree measure}}{360^\circ}$$

Common Errors/Misconceptions

- confusing arc measure with arc length
- forgetting to check the mode on the graphing calculator
- inconsistently setting up ratios
- incorrectly simplifying ratios involving π

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Guided Practice 6.4.1

Example 1

Convert 40° to radians.

1. Set up a proportion.

$$\frac{\text{radian measure}}{\pi} = \frac{\text{degree measure}}{180^\circ}$$

$$\frac{x}{\pi} = \frac{40^\circ}{180^\circ}$$

2. Multiply both sides by π to solve for x.

$$x = \frac{40\pi}{180} = \frac{2\pi}{9} \text{ radians}$$

Example 2

Convert $\frac{3\pi}{4}$ radians to degrees.

1. Set up a proportion.

$$\frac{\text{radian measure}}{\pi} = \frac{\text{degree measure}}{180^\circ}$$

$$\frac{\frac{3\pi}{4}}{\pi} = \frac{x}{180^\circ}$$

$$\frac{3}{4} = \frac{x}{180^\circ}$$

2. Multiply both sides by 180 to solve for x.

$$x = \frac{3(180)}{4} = 135^\circ$$

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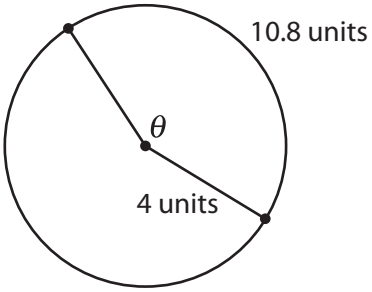
Instruction

Example 3

A circle has a radius of 4 units. Find the radian measure of a central angle that intercepts an arc of length 10.8 units.



1. Substitute the known values into the formula for radian measure.

$$\theta = \frac{\text{arc length}}{\text{radius}} = \frac{s}{r} = \frac{10.8}{4}$$

2. Simplify.

$\theta = 2.7$ radians

The radian measure is 2.7 radians.

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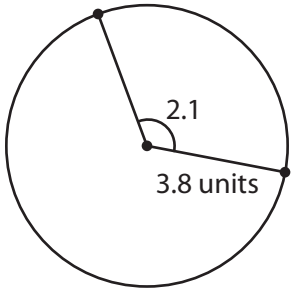
Instruction

Example 4

A circle has a radius of 3.8 units. Find the length of an arc intercepted by a central angle measuring 2.1 radians.



1. Substitute the known values into the formula for radian measure.

$$\theta = \frac{\text{arc length}}{\text{radius}} = \frac{s}{r}$$

$$2.1 = \frac{s}{3.8}$$

2. Multiply both sides by 3.8 to solve for arc length.

$s = 2.1 \bullet 3.8 = 7.98$ units

The arc length is 7.98 units.

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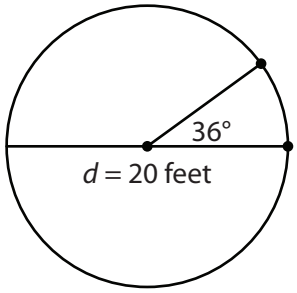
Instruction

Example 5

A circle has a diameter of 20 feet. Find the length of an arc intercepted by a central angle measuring 36°.



1. Find the circumference of the circle.

   circumference = π • diameter

   $C = \pi d = 20\pi$ feet

2. Set up a proportion.

   $$\frac{\text{arc length}}{\text{circumference}} = \frac{\text{degree measure}}{360^\circ}$$

   $$\frac{s}{C} = \frac{\text{degree measure}}{360^\circ}$$

   $$\frac{s}{20\pi} = \frac{36}{360}$$

3. Multiply both sides by 20π to find the arc length.

   $$s = \frac{36}{360} \bullet 20\pi = \frac{1}{10} \bullet 20\pi = 2\pi \text{ feet} \approx 6.28 \text{ feet}$$

   The length of the arc is approximately 6.28 feet.

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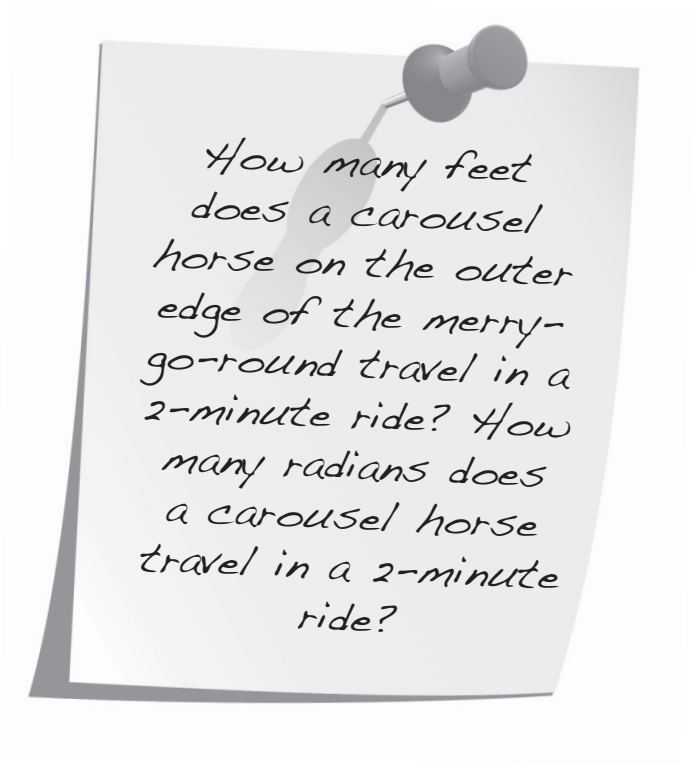
Problem-Based Task 6.4.1: Around the Merry-Go-Round

SMP	
1 ✓	2
3	4 ✓
5	6 ✓
7	8

A merry-go-round has a circumference of 160 feet and revolves at a speed of 6 miles per hour. How many feet does a carousel horse on the outer edge of the merry-go-round travel in a 2-minute ride? How many radians does a carousel horse travel in a 2-minute ride?



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Problem-Based Task 6.4.1: Around the Merry-Go-Round

Coaching

a. How many miles does the carousel horse travel in 1 minute?

b. How many feet does the carousel horse travel in 1 minute?

c. How many feet does the carousel horse travel in 2 minutes?

d. What is the radius of the merry-go-round?

e. Use the distance traveled and the radius of the merry-go-round to find the radian measure of the angle.

f. How many revolutions does the merry-go-round make in this time? Does your answer seem reasonable?

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Instruction

Problem-Based Task 6.4.1: Around the Merry-Go-Round

Coaching Sample Responses

a. How many miles does the carousel horse travel in 1 minute?

Divide the speed by 60 minutes to convert from miles per hour to miles per minute.

$$\frac{6 \text{ miles}}{1 \text{ hour}} \bullet \frac{1 \text{ hour}}{60 \text{ minutes}} = \frac{1}{10} \text{ mile per minute}$$

b. How many feet does the carousel horse travel in 1 minute?

There are 5,280 feet in 1 mile. Multiply the speed by 5,280 feet to convert from miles per minute to feet per minute.

$$\frac{1 \text{ mile}}{10 \text{ minutes}} \bullet \frac{5280 \text{ feet}}{1 \text{ mile}} = 528 \text{ feet per minute}$$

c. How many feet does the carousel horse travel in 2 minutes?

Multiply the number of feet traveled in 1 minute by 2 to find the number of feet traveled in 2 minutes.

$528 \bullet 2 = 1056$ feet

d. What is the radius of the merry-go-round?

Find the circumference.

$C = 160 \text{ feet} = 2\pi r$

Divide each side by 2π to find the radius.

$$r = \frac{160}{2\pi} = \frac{80}{\pi} \approx 25.46 \text{ feet}$$

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Instruction

e. Use the distance traveled and the radius of the merry-go-round to find the radian measure of the angle.

Divide the distance traveled (the arc length) by the radius to find the radian measure of the angle. Use $\frac{80}{\pi}$ for the radius since this value is more accurate.

$$\theta = \frac{s}{r} = \frac{1056}{\frac{80}{\pi}} = \frac{66\pi}{5} \text{ radians} \approx 41.47 \text{ radians}$$

f. How many revolutions does the merry-go-round make in this time? Does your answer seem reasonable?

To find the number of revolutions in 2 minutes, divide the radian measure by 2π, which is one revolution. Use $\frac{66\pi}{5}$ for the radian measure since this value is more accurate than 41.47.

$$\frac{\frac{66\pi}{5}}{2\pi} \approx 6$$

The merry-go-round goes through approximately 6.6 revolutions in 2 minutes. This seems reasonable.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 6.4.1: Defining Radians

Use your knowledge of radian measures to complete the following problems.

1. Convert 80° to radians. Leave the answer in terms of π.

2. Convert $\frac{5\pi}{6}$ radians to degrees.

3. A circle has a radius of 5 units. Find the radian measure of a central angle that intercepts an arc length of 15 units.

4. A circle has a radius of 18 units. Find the length of an arc intercepted by a central angle measuring 2.9 radians.

5. A circle has a radius of 11 units. Find the length of an arc intercepted by a central angle measuring 72°.

6. A central angle of $\frac{5\pi}{2}$ radians intercepts an arc length of 46 units. What is the radius of the circle, rounded to the nearest hundredth?

7. A standard dartboard has a radius of 170 mm and is split into 20 equal sections. What is the arc length of a single section on a dartboard rounded to the nearest millimeter?

8. How many radians does the hour hand on a clock travel through from 12 to 5?

9. A 26-inch diameter bicycle tire rotates 500 times. How many feet does the bicycle travel?

10. What is the difference between arc measure and arc length?

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Lesson 6.4.2: Deriving the Formula for the Area of a Sector

Warm-Up 6.4.2

A new, decorative stained-glass window is in the shape of a semicircle with a 2-foot diameter. The front of the window is to be covered with fabric and outlined with ribbon until an unveiling ceremony.

1. How many feet of ribbon will be needed if you need 2 extra feet of ribbon to tie a bow?

2. How many square feet of fabric will be needed to cover one side of the window?

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Instruction

Lesson 6.4.2: Deriving the Formula for the Area of a Sector

Common Core State Standard

G–C.5 Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Warm-Up 6.4.2 Debrief

1. How many feet of ribbon will be needed if you need 2 extra feet of ribbon to tie a bow?

   Find the perimeter of the window (the semicircular arc plus the flat side of the semicircle) and add 2 feet for the bow. First calculate the perimeter of the window.

   $$\frac{1}{2} \bullet C + d = \frac{1}{2} \bullet \pi d + d$$
   $$= \frac{1}{2} \bullet 2\pi + 2$$
   $$= \pi + 2 \approx 5.14$$

   The perimeter of the window is 5.14 feet. If you add 2 feet of ribbon for the bow, the total length of ribbon needed is 5.14 + 2 = 7.14 feet. You will need at least 7 feet 2 inches of ribbon to make sure there is enough ribbon for the bow.

2. How many square feet of fabric will be needed to cover one side of the window?

   Find the area of the window.

   $$\frac{1}{2} \bullet A = \frac{1}{2} \bullet \pi r^2 = \frac{1}{2} \bullet \pi \bullet 1^2 = \frac{1}{2} \bullet 1\pi = \frac{\pi}{2} \approx 1.57 \text{ square feet}$$

   Approximately 1.57 square feet of fabric will be needed.

Connection to the Lesson

- Students will draw parallels between arc length and the area of a sector.
- Students will extend their understanding of area by thinking about the area of a sector as a fraction of the area of a circle.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

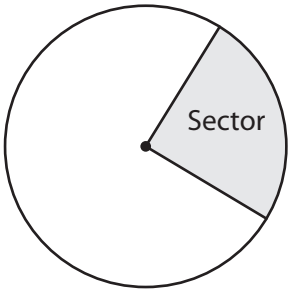
- finding the area of a circle (7.G.4)
- understanding scale factor in similar shapes (7.G.1)
- using ratios and proportions (7.RP.3)

Introduction

A **sector** is the portion of a circle bounded by two radii and their intercepted arc. Previously, we thought of arc length as a fraction of the circumference of the circle. In a similar way, we can think of a sector as a fraction of the area of the circle. In the same way that we found arc length, we can set up proportions to find the area of a sector.

Key Concepts

- A sector is the portion of a circle bounded by two radii and their intercepted arc.



- To find the area of a sector, A_{sector}, when the central angle θ is given in radians, we can set up a proportion using the area of a circle, $A = \pi r^2$.

$$\frac{A_{sector}}{\pi r^2} = \frac{\theta}{2\pi}$$

- We can solve this proportion for the area of the sector and simplify to get a formula for the area of a sector in terms of the radius of the circle and the radian measure of the central angle θ.

$$A_{sector} = \frac{r^2\theta}{2}$$

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Instruction

- To find the area of a sector when the central angle is given in degrees, we can set up a proportion using the area of a circle.

$$\frac{A_{\text{sector}}}{A_{\text{circle}}} = \frac{\text{degree measure}}{360^\circ}$$

Common Errors/Misconceptions

- inconsistently setting up ratios
- incorrectly simplifying ratios involving π

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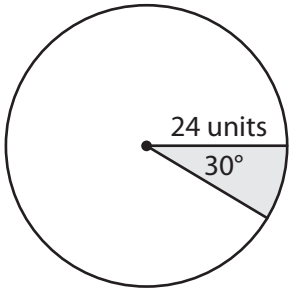
Instruction

Guided Practice 6.4.2

Example 1

A circle has a radius of 24 units. Find the area of a sector with a central angle of 30°.



1. Find the area of the circle.

   area = π • radius^2

   $A = \pi r^2 = \pi \cdot 24^2 = 576\pi$ square units

2. Set up a proportion.

   $$\frac{\text{degree measure of sector}}{360^\circ} = \frac{\text{area of sector}}{\text{area of circle}}$$

   $$\frac{30^\circ}{360^\circ} = \frac{A_{\text{sector}}}{576\pi \text{ square units}}$$

3. Multiply both sides by the area of the circle to find the area of the sector.

   $$A_{\text{sector}} = \frac{30^\circ}{360^\circ} \cdot 576\pi = 48\pi$$

   ≈ 150.80 square units

   The area of the sector is approximately 150.80 units^2.

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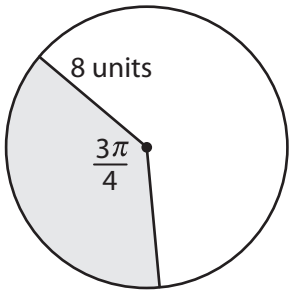
Instruction

Example 2

A circle has a radius of 8 units. Find the area of a sector with a central angle of $\frac{3\pi}{4}$ radians.



1. Substitute radius and radian measure into the formula for the area of a sector.

$$\text{area of sector} = \frac{\text{radius}^2 \bullet \text{radian measure of angle}}{2}$$

$$A_{\text{sector}} = \frac{r^2\theta}{2} = \frac{8^2 \bullet \frac{3\pi}{4}}{2}$$

2. Simplify.

$$A_{\text{sector}} = \frac{8^2 \bullet \frac{3\pi}{4}}{2} = \frac{64 \bullet \frac{3\pi}{4}}{2} = \frac{48\pi}{2} = 24\pi$$

≈ 75.40 square units

The area of the sector is approximately 75.40 units^2.

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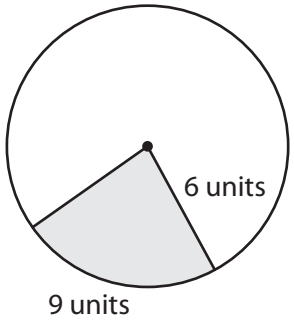
Instruction

Example 3

A circle has a radius of 6 units. Find the area of a sector with an arc length of 9 units.



1. Use the radian measure formula to find the measure of the central angle.

$$\text{radian measure} = \frac{\text{arc length}}{\text{radius}}$$

$$\theta = \frac{s}{r} = \frac{9}{6} = 1.5 \text{ radians}$$

2. Substitute radius and radian measure into the formula for the area of a sector and simplify.

$$A_{\text{sector}} = \frac{r^2\theta}{2} = \frac{6^2 \cdot 1.5}{2} = \frac{54}{2} = 27 \text{ square units}$$

The area of the sector is 27 units^2.

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Problem-Based Task 6.4.2: Pizza Specials

SMP	
1 ✓	2
3 ✓	4 ✓
5	6
7	8

A pizza parlor has a $5 lunch special this week for 3 pieces of a small cheese pizza and a soft drink. Next week, the $5 special will be for 2 pieces of a large cheese pizza and a soft drink. A small pizza measures 10 inches in diameter and is cut into 6 equal slices. A large pizza measures 14 inches in diameter and is cut into 8 equal slices. Which special is the better deal?

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Problem-Based Task 6.4.2: Pizza Specials

Coaching

a. What is the radius of a small pizza?

b. What is the area of a small pizza?

c. One slice of a small pizza makes up what fraction of a whole small pizza?

d. What is the area of a 1-piece sector of a small pizza?

e. What is the area of 3 pieces of a small pizza?

f. What is the radius of a large pizza?

g. What is the area of a large pizza?

h. One slice of a large pizza makes up what fraction of a whole large pizza?

i. What is the area of a 1-piece sector of a large pizza?

j. What is the area of 2 pieces of a large pizza?

k. How does the area of 3 pieces of a small pizza compare to the area of 2 pieces of a large pizza?

l. What can you conclude about the lunch specials?

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Instruction

Problem-Based Task 6.4.2: Pizza Specials

Coaching Sample Responses

a. What is the radius of a small pizza?

$$r = \frac{1}{2} \bullet d = \frac{1}{2} \bullet 10 = 5 \text{ inches}$$

b. What is the area of a small pizza?

$A_{small} = \pi r^2 = \pi \bullet 5^2 = 25\pi \approx 78.54$ square inches

c. One slice of a small pizza makes up what fraction of a whole small pizza?

A small pizza is cut into 6 equal slices, so 1 slice is $\frac{1}{6}$ of a whole pizza.

d. What is the area of a 1-piece sector of a small pizza?

$$\frac{1}{6} \bullet A_{small} = \frac{1}{6} \bullet 25\pi = \frac{25\pi}{6} \approx 13.09 \text{ square inches}$$

e. What is the area of 3 pieces of a small pizza?

$$3 \bullet \frac{25\pi}{6} = \frac{25\pi}{2} \approx 39.27 \text{ square inches}$$

f. What is the radius of a large pizza?

$$r = \frac{1}{2} \bullet d = \frac{1}{2} \bullet 14 = 7 \text{ inches}$$

g. What is the area of a large pizza?

$A_{large} = \pi r^2 = \pi \bullet 7^2 = 49\pi \approx 153.94$ square inches

h. One slice of a large pizza makes up what fraction of a whole large pizza?

A large pizza is cut into 8 equal slices, so 1 slice is $\frac{1}{8}$ of a whole pizza.

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Instruction

i. What is the area of a 1-piece sector of a large pizza?

$$\frac{1}{8} \bullet A_{\text{large}} = \frac{1}{8} \bullet 49\pi = \frac{49\pi}{8} \approx 19.24 \text{ square inches}$$

j. What is the area of 2 pieces of a large pizza?

$$2 \bullet \frac{49\pi}{8} = \frac{49\pi}{4} \approx 38.48 \text{ square inches}$$

k. How does the area of 3 pieces of a small pizza compare to the area of 2 pieces of a large pizza?

The area of 3 pieces of a small pizza is about 39.27 square inches. This is greater than the area of 2 pieces of a large pizza, which is about 38.48 square inches.

l. What can you conclude about the lunch specials?

The lunch special this week is a slightly better deal than the lunch special next week because the area of 3 pieces of a small pizza is slightly greater than the area of 2 pieces of a large pizza.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 6.4.2: Deriving the Formula for the Area of a Sector

Use your knowledge of the areas of sectors to complete the following problems.

1. Find the area of a sector with a central angle of 9.6 radians and a radius of 21.4 units.

2. Find the area of a sector with a central angle of $\frac{\pi}{3}$ radians and a radius of 12 units.

3. Find the area of a sector with a central angle of 44° and a radius of 56 units.

4. A circle has a radius of 8 units. Find the area of a sector with an arc length of 6 units.

5. A circle has a radius of 2 units. Find the arc length of a sector with an area of 12 square units.

6. A sector has a central angle of $\frac{\pi}{2}$ radians and an area of 13 square units. What is the area of the circle?

7. A personal pizza with a 6-inch diameter is cut into slices with a central angle of $\frac{\pi}{2}$ radians. What is the area of each slice?

8. A blueberry pie is made in a $10\frac{1}{4}$-inch diameter pie pan. The pie is cut into 8 equal slices. What is the area of 1 slice?

9. A rotating sprinkler sprays a stream of water 40 feet long. The sprinkler rotates 190°. What is the area of the portion of the yard that is watered by the sprinkler?

10. A lighthouse projects a beam of light that can be seen from up to 4 miles away and covers an angle of 35°. What is the area of the region from which a ship can see the light from the lighthouse?

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Progress Assessment

Circle the letter of the best answer.

1. Convert 52° to radians.

   a. $\frac{13\pi}{45}$ radians
   b. $\frac{13}{45}$ radians
   c. $\frac{45\pi}{13}$ radians
   d. $\frac{45}{13}$ radians

2. Convert $\frac{3\pi}{10}$ radians to degrees.

   a. 54°
   b. 60°
   c. 540°
   d. 600°

3. A circle has a radius of 2.4 units. Find the radian measure of a central angle that intercepts an arc length of 12 units.

   a. 0.2 radian
   b. 5 radians
   c. 5π radians
   d. 28.8 radians

4. A circle has a radius of 7 units. Find the length of the arc intercepted by a central angle measuring 4.5 radians.

   a. 0.64 unit
   b. 1.55 units
   c. 2.5 units
   d. 31.5 units

5. A circle has a radius of 76 units. Find the area of a sector with a central angle of 100°. Round your answer to the nearest hundredth.

   a. 907,291.96 square units
   b. 288,800 square units
   c. 5,040.51 square units
   d. 1,604.44 square units

6. A circle has a radius of 3.4 units. Find the area of a sector with a central angle of $\frac{9\pi}{7}$ radians.

   a. 0.41 square unit
   b. 23.35 square units
   c. 46.69 square units
   d. 93.39 square units

continued

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Assessment

7. A circle has a radius of 7 units. Find the area of a sector with an arc length of 24.5 units.

   a. 600.25 square units
   b. 171.5 square units
   c. 85.75 square units
   d. 12.25 square units

8. A sector has a central angle of $\frac{3\pi}{4}$ radians and an area of 12 square units. Find the area of the circle.

   a. 4.5 square units
   b. 9π square units
   c. 32 square units
   d. 96 square units

9. An extra-large pizza has a diameter of 16 inches and is cut into 10 equal pieces. Find the area of 2 pieces, rounded to the nearest hundredth.

   a. 20.11 square inches
   b. 40.21 square inches
   c. 80.42 square inches
   d. 160.85 square inches

10. A 24.5-inch wheel on a wheelchair rotates 50 times. How many feet does the wheelchair move in this time?

    a. 3,848.45 feet
    b. 1,225 feet
    c. 641.41 feet
    d. 320.70 feet

Use what you know about arcs and sectors to solve the following problem.

11. You have made a model Ferris wheel with 8 seats and a diameter of 1.5 feet. You want to decorate the wheel with fabric so that each seat is in a sector that has a different color of fabric. You then plan to line the perimeter of each sector with ribbon. All the sectors are congruent.

    a. What is the radian measure of each central angle?
    b. What is the degree measure of each central angle?
    c. How much of each color of fabric will you need?
    d. How much ribbon will you need for each sector?

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. What is the area of a circle that has a circumference of 50 meters?
   a. $\approx 198.94\ m^2$
   b. $\approx 625\ m^2$
   c. $\approx 1350\ m^2$
   d. $\approx 7853.982\ m^2$

2. What is the volume of a cylinder that has a radius of 20 cm and a height of 30 cm?
   a. $\approx 37{,}699.112\ cm^3$
   b. $\approx 12{,}000\ cm^3$
   c. $\approx 1884.956\ cm^3$
   d. $\approx 12{,}566.371\ cm^3$

3. What is the volume of a square pyramid that has a side length of 100 meters and a height of 50 meters?
   a. $\approx 5000\ m^3$
   b. $\approx 500{,}000\ m^3$
   c. $\approx 166{,}667\ m^3$
   d. $\approx 83{,}333\ m^3$

4. What is the volume of a cone that has a circumference of 50 meters and a height of 30 meters?
   a. $\approx 633.26\ m^3$
   b. $\approx 500\ m^3$
   c. $\approx 1989.44\ m^3$
   d. $\approx 25{,}000\ m^3$

5. A golf ball has a diameter of 1.68 in. What is the volume of the golf ball?
   a. $11.82\ in^3$
   b. $2.48\ in^3$
   c. $7.04\ in^3$
   d. $3.52\ in^3$

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Lesson 5: Explaining and Applying Area and Volume Formulas

Instruction

Common Core State Standards

SMP
1 ✓ 2
3 ✓ 4 ✓
5 6 ✓
7 ✓ 8

G–GMD.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*

G–GMD.3 Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.★

Essential Questions

1. How can you prove that the circumference of a circle equals $2\pi r$?
2. How can you prove that the area of a circle equals πr^2?
3. How can you use the formulas for a circle to solve real-world problems?
4. What is the formula for the volume of a cylinder, and how do you prove the formula?
5. What is the formula for the volume of a pyramid, and how do you prove the formula?
6. What is the formula for the volume of a cone, and how do you prove the formula?
7. What is the formula for the volume of a sphere, and how do you prove the formula?

WORDS TO KNOW

Archimedes	a Greek mathematician, physician, engineer, and inventor who lived from 287–212 B.C.; considered to be one of the greatest mathematicians of all time
bisect	to cut in half
Cavalieri's Principle	The volumes of two objects of equal height are equal if the areas of their corresponding cross sections are in all cases equal.
cone	a solid or hollow object that tapers from a circular or oval base to a point
cylinder	a solid or hollow object that has two parallel bases connected by a curved surface; the bases are usually circular
dissection	breaking a figure down into its components

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dodecagon	a 12-sided polygon
irrational number	a real number that cannot be written as $\frac{m}{n}$, where m and n are integers and $n \neq 0$; a non-terminating or non-repeating decimal
limit	the value that a sequence approaches as a calculation becomes more and more accurate
polyhedron	a three-dimensional object that has faces made of polygons
pyramid	a solid or hollow polyhedron object that has three or more triangular faces that converge at a single vertex at the top; the base may be any polygon
sphere	a three-dimensional surface that has all its points the same distance from its center

Recommended Resources

- CutOutFoldUp.com. "The Volume of a Pyramid is One-Third that of a Prism."

  http://www.walch.com/rr/00065

  This website has many three-dimensional nets and projects for classroom use. The link here is for making prisms to show how a pyramid can contain three prisms.

- Math Open Reference. "Cylinder."

  http://www.walch.com/rr/00066

  At the top, use the interactive diagram of a cylinder to watch the volume change. Transform the right cylinder into an oblique cylinder, and see that the volume stays the same. Farther down the page, use the applet to see that a cylinder is a prism with an infinite number of faces. Click on the "more" link to add more faces and watch the prism transform to a cylinder.

- Math Open Reference. "Volume of a cone."

  http://www.walch.com/rr/00067

  Use the interactive diagram of a cone to see how the volume changes based on changing the height and/or radius of the cone.

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- Math Open Reference. "Volume of a pyramid."

  http://www.walch.com/rr/00068

  Use the interactive diagram of a pyramid to see how the volume changes based on the height and base dimensions.

- Math Open Reference. "Volume of a sphere."

  http://www.walch.com/rr/00069

  Use the interactive diagram of a sphere to see how the volume changes based on the length of the sphere's radius.

- Roskes, Bonnie. "Dividing a Cube into Three Identical Pyramids using Google SketchUp."

  http://www.walch.com/rr/00070

  This video shows a cube being constructed with free geometry software. Three congruent pyramids are then created inside the cube.

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Lesson 6.5.1: Circumference and Area of a Circle

Warm-Up 6.5.1

An athletic field is in the shape of a rectangle with a semicircle on each end. The field is 180 meters long and 60 meters wide. The amount of work needed to maintain the field depends on its total area. What is the total area of the field?

1. Draw a diagram of the field.

2. How can you break the shape of the field into separate pieces?

3. What area formulas do you need to use to find the total area of the field?

4. What is the radius of the circle? What is the length of the rectangle?

5. What is the total area of the field?

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Instruction

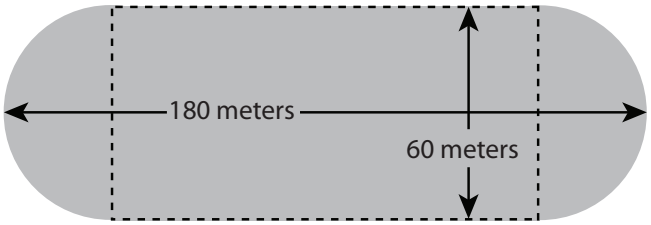
Lesson 6.5.1: Circumference and Area of a Circle

Common Core State Standard

G–GMD.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*

Warm-Up 6.5.1 Debrief

1. Draw a diagram of the field.



2. How can you break the shape of the field into separate pieces?

   There are two semicircles and one rectangle. The two semicircles make one whole circle.

3. What area formulas do you need to use to find the total area of the field?

   Formula for area of a rectangle: $A = l \bullet w$

   Formula for area of a circle: $A = \pi r^2$

4. What is the radius of the circle? What is the length of the rectangle?

   The radius of the circle is half the width of the field.

$$r = \frac{60}{2}$$
$$r = 30 \text{ m}$$

   To find the length of the rectangle, subtract the two radii of the semicircles from 180 (the full length of the field). Since there are two semicircles, and therefore two radii, subtract the radius twice.

   $180 - 30 - 30 = 120$ m

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5. What is the total area of the field?

   The total area of the field is the rectangular area plus the two semicircle areas. Together the two semicircles make a complete circle.

   $A = l \bullet w + \pi r^2$

   $A = (120 \bullet 60) + (\pi \bullet 30^2) \approx 10{,}027.4 \text{ m}^2$

Connection to the Lesson

- Students will learn the logic that shows why the formulas for circles work, including informal proofs of the formulas for the circumference and area of a circle.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

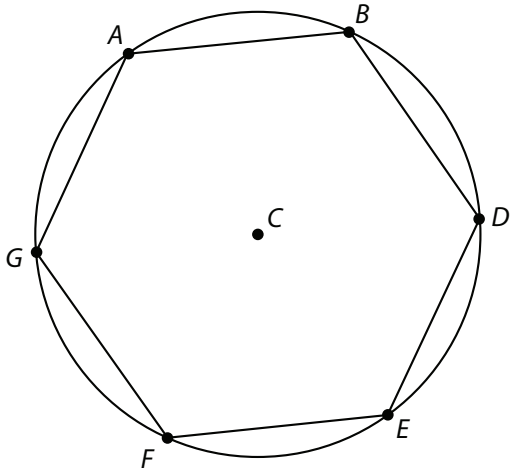
- using formulas for the areas of polygons and circles (7.G.6)
- performing calculations with the angles in circles (G–C.2)
- using the Pythagorean Theorem (8.G.7)
- using ratios of trigonometry (G–SRT.8)
- understanding how to bisect angles and side lengths (G–CO.12)

Introduction

You have used the formulas for finding the circumference and area of a circle. In this lesson, you will prove why the formulas for circumference and area work. You will also see how the value of π can be proven.

Key Concepts

- The circumference of a specific circle divided by its diameter is the ratio *pi*, written as π.
- *Pi* (π) is an **irrational number**, which is a number that cannot be written as a repeating decimal or as a fraction. It has an infinite number of non-repeating decimal places.
- We know that the circumference of a circle is equal to $\pi \bullet$ diameter or $2\pi \bullet$ radius.
- Therefore, $\pi = \dfrac{\text{circumference}}{\text{diameter}} = \dfrac{\text{circumference}}{2 \bullet \text{radius}}$.
- Long ago, mathematicians didn't yet know the value of *pi*. **Archimedes**, a great mathematician from ancient Greece, used inscribed polygons to determine the value of *pi*. He started by inscribing a regular hexagon in a circle.



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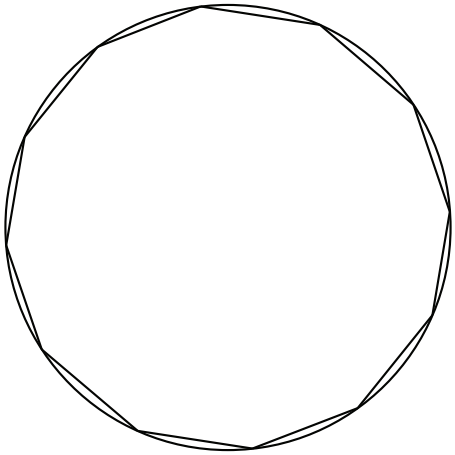
Instruction

- He determined that each side of the hexagon equals the radius of the circle.

  $AB = BD = DE = EF = FG = GA = AC$

- Archimedes realized that if the perimeter of the hexagon were equal to the circumference of the circle, then both would equal $6r$. This would mean that $\pi = 3$. However, the circumference is larger than the hexagon; therefore, Archimedes thought, π must be larger than 3.
- Next, Archimedes inscribed a regular **dodecagon**—a 12-sided polygon—in the circle. The perimeter of the dodecagon was much closer to the actual perimeter of the circle.



- He calculated the perimeter of the dodecagon to be approximately 6.21166. This means $\pi \approx 3.10583$. However, the circumference of the circle is still larger than the dodecagon, so π must be greater than 3.10583.
- Next, Archimedes inscribed a 24-sided regular polygon in the circle and calculated its perimeter. This polygon's perimeter is even closer to the circumference of a circle. Archimedes found that the ratio of the perimeter to the diameter is closer to the value of π.
- Archimedes kept going with this process until he had inscribed a 48-sided polygon in the circle. As the number of sides of a polygon increases, the polygon looks more and more like a circle. As he added more sides to the polygon, the value of the ratio of π became more and more accurate. The more sides an inscribed polygon has, the closer its perimeter is to the circumference of the circle.
- Therefore, Archimedes determined that the limit of the perimeter of a regular polygon inscribed in a circle divided by the radius of the circle, as the number of sides of the polygon approaches infinity, is equal to the value of π.
- A **limit** is the value that a sequence approaches as a calculation becomes more and more accurate. This limit cannot be reached, because the number of sides can always be increased.
- Theoretically, if the polygon had an infinite number of sides, π could be calculated. This is the basis for the formula for finding the circumference of a circle.

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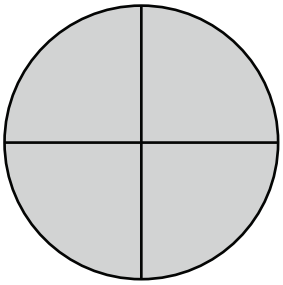
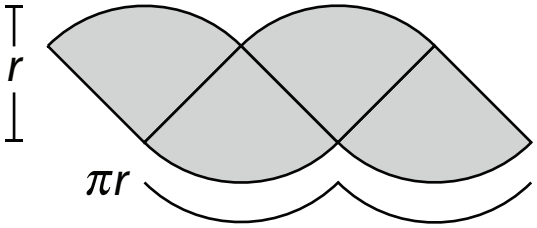
Instruction

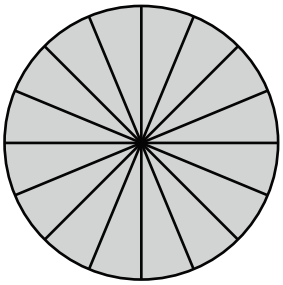
- Increasing the number of side lengths for the inscribed polygon causes the polygon's perimeter to get closer and closer to the circumference of the circle.
- The area of the circle can be derived similarly using dissection principles. **Dissection** involves breaking a figure down into its components.
- In the diagram that follows, a circle has been divided into four equal sections.



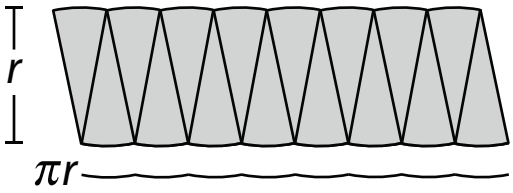
- If you cut the four sections from the circle apart, you can arrange them to resemble a parallelogram.



- The height of the "parallelogram" equals the radius, r, of the original circle. The base is equal to half of the circumference, or $\pi \bullet r$.
- The circle in the following diagram has been divided into 16 equal sections.



- You can arrange the 16 segments to form a new "parallelogram."



- This figure looks more like a parallelogram.

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Instruction

- As the number of sections increases, the rounded "bumps" along its base become less and less distinct. The more sections, the closer the figure is to a parallelogram. This is another example of a limit; as the number of sections approaches infinity, the shape of the assembled sections approaches a parallelogram.
- The formula for the area of a parallelogram is $A = b \bullet h = a$. The base of the parallelogram made out of the circle segments is $\pi \bullet r$. The height is r. Thus, the area of the circle is $a = r \bullet \pi r = \pi r^2$. This proof is based on the dissection of the circle.
- Remember that a sector is a part of a circle that is determined by a central angle.
- A central angle has its vertex on the center of the circle. A sector will have an angular measure greater than 0° and less than 360°.

Common Errors/Misconceptions

- not realizing that there is more than one way to prove a formula
- using the wrong formula for area, circumference, or the area of a sector
- using the diameter in a formula instead of the radius

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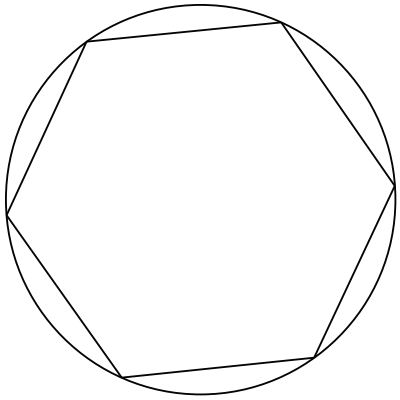
Instruction

Guided Practice 6.5.1

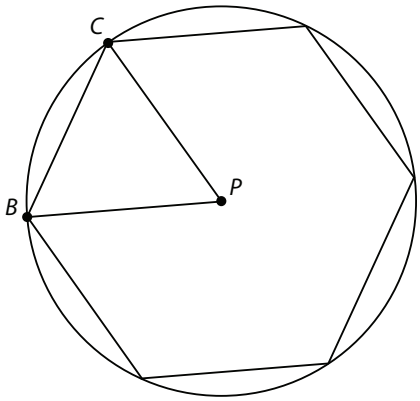
Example 1

Show how the perimeter of a hexagon can be used to find an estimate for the circumference of a circle that has a radius of 5 meters. Compare the estimate with the circle's perimeter found by using the formula $C = 2\pi r$.

1. Draw a circle and inscribe a regular hexagon in the circle.



2. Create a triangle with a vertex at the center of the circle. Draw two line segments from the center of the circle to consecutive vertices on the hexagon.



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Instruction

3. To find the length of $\overline{BC}$, first determine the known lengths of $\overline{PB}$ and $\overline{PC}$.

   Both lengths are equal to the radius of circle P, 5 meters.

4. Determine $m\angle CPB$.

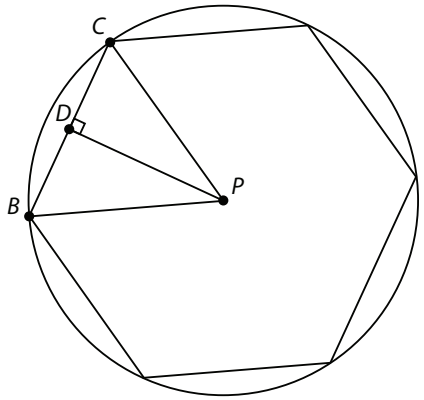
   The hexagon has 6 sides. A central angle drawn from P will be equal to one-sixth of the number of degrees in a circle.

   $$m\angle CPB = \frac{1}{6} \bullet 360 = 60$$

   The measure of $\angle CPB$ is 60°.

5. Use trigonometry to find the length of $\overline{BC}$.

   Make a right triangle inside of $\triangle PBC$ by drawing a perpendicular line, or altitude, from P to $\overline{BC}$.



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Instruction

6. Determine $m\angle BPD$.

   $\overline{DP}$ **bisects**, or cuts in half, $\angle CPB$. Since the measure of $\angle CPB$ was found to be 60°, divide 60 by 2 to determine $m\angle BPD$.

   $$\frac{60}{2} = 30$$

   The measure of $\angle BPD$ is 30°.

7. Use trigonometry to find the length of $\overline{BD}$ and multiply that value by 2 to find the length of $\overline{BC}$.

   $\overline{BD}$ is opposite $\angle BPD$. The length of the hypotenuse, $\overline{PB}$, is 5 meters, and the measure of $\angle BPD$ is 30°.

   The trigonometry ratio that uses the opposite and hypotenuse lengths is sine.

   $$\sin 30° = \frac{BD}{5}$$

   $$0.5 = \frac{BD}{5}$$ Evaluate the sine of 30°.

   $5 \cdot 0.5 = BD$ Multiply both sides of the equation by 5.

   $BD = 2.5$

   The length of $\overline{BD}$ is 2.5 meters.

   Since $\overline{BC}$ is twice the length of $\overline{BD}$, multiply 2.5 by 2.

   $BC = 2 \cdot 2.5 = 5$

   The length of $\overline{BC}$ is 5 meters.

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Instruction

8. Find the perimeter of the hexagon.

   Perimeter = $BC \cdot 6 = 5 \cdot 6 = 30$

   The perimeter of the hexagon is 30 meters.

9. Compare the estimate with the calculated circumference of the circle.

   Calculate the circumference.

   $C = 2\pi r$ Formula for circumference

   $C = 2\pi \cdot 5$ Substitute 5 for r.

   $C \approx 31.416$ meters

   Find the difference between the perimeter of the hexagon and the circumference of the circle.

   $31.416 - 30 = 1.416$ meters

   The formula for circumference gives a calculation that is 1.416 meters longer than the perimeter of the hexagon. You can show this as a percentage difference between the two values.

   $$\frac{1.416}{31.416} = 0.0451 = 4.51\%$$

   From a proportional perspective, the circumference calculation is approximately 4.51% larger than the estimate that came from using the perimeter of the hexagon.

   If you inscribed a regular polygon with more side lengths than a hexagon, the perimeter of the polygon would be closer in value to the circumference of the circle.

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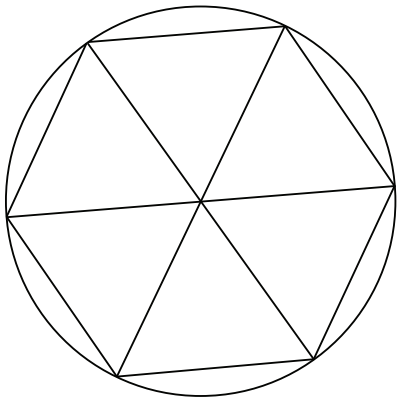
Instruction

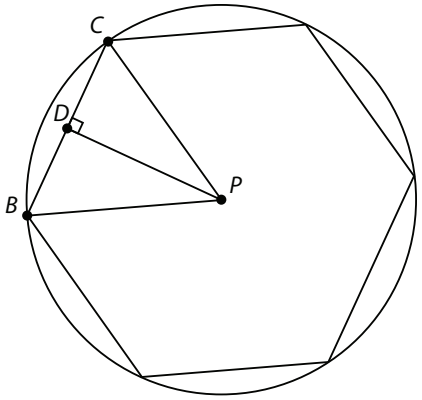
Example 2

Show how the area of a hexagon can be used to find an estimate for the area of a circle that has a radius of 5 meters. Compare the estimate with the circle's area found by using the formula $A = \pi r^2$.

1. Inscribe a hexagon into a circle and divide it into 6 equal triangles.



2. Use the measurements from Example 1 to find the area of one of the six triangles.



First, determine the formula to use.

$A = \frac{1}{2}bh$ — Area formula for a triangle

$A_{\triangle PBC} = \frac{1}{2}BC \bullet PD$ — Rewrite the formula with the base and height of the triangle whose area you are trying to determine.

(*continued*)

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From Example 1, the following information is known:

$BC = 5$ meters

$BD = 2.5$ meters

$m\angle BPD = 30°$

You need to find the height, $\overline{PD}$. In $\triangle BPD$, the height, $\overline{PD}$, is the adjacent side length.

Since the hypotenuse, $\overline{BP}$, is a radius of the circle, it is 5 meters. Since the measure of $\angle BPD$ and the hypotenuse are known, use the cosine of 30° to find $\overline{PD}$.

$$\cos 30° = \frac{PD}{5}$$

$$0.866054038 = \frac{PD}{5}$$ Evaluate the cosine of 30°.

$5 \bullet 0.866054038 = PD$ Multiply both sides by 5.

$PD \approx 4.33027019$ meters

Now that the length of $\overline{PD}$ is known, use that information to find the area of $\triangle PBC$ using the formula determined earlier.

$$A_{\triangle PBC} = \frac{1}{2} BC \bullet PD$$ Area formula for $\triangle PBC$

$$A_{\triangle PBC} = \frac{1}{2}(5) \bullet (4.33027019)$$ Substitute the values of BC and PD.

$$A_{\triangle PBC} \approx 10.82567548 \text{ m}^2$$

3. Find the area of the hexagon.

Multiply the area of one triangle times 6, the number of triangles in the hexagon.

$6 \bullet 10.82567548 = 64.95405288$ m²

The area of hexagon is about 64.95 m².

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4. Compare the area of the hexagon with the area of the circle.

Find the area of the circle.

$A_{\text{circle } P} = \pi r^2$ Formula for the area of a circle

$A_{\text{circle } P} = \pi \bullet 5^2$ Substitute the value for the radius.

$A_{\text{circle } P} \approx 78.53981634 \text{ m}^2$

The actual area of circle P is about 78.54 m^2.

Find the difference between the area of the hexagon and the area of the circle.

78.53981634 – 64.95405288 = 13.58576346

The actual area of the circle is approximately 13.59 m^2 greater than the hexagon's area.

Show the difference as a percentage.

$$\frac{13.59}{78.54} \approx 0.1730 \approx 17.30\%$$

The actual area of the circle is about 17.30% larger than the estimate found by using the area of the hexagon.

The estimate of a circle's area calculated by using an inscribed polygon can be made closer to the actual area of the circle by increasing the number of side lengths of the polygon.

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Instruction

Example 3

Find the area of a circle that has a circumference of 100 meters.

1. First, find the measure of the radius by using the formula for circumference.

$C = 2\pi r$

$100 = 2\pi r$

$r = \frac{100}{2\pi}$

$r \approx 15.9155$ m

2. Calculate the area by using the formula for the area of a circle.

$A = \pi r^2$

$A \approx \pi(15.9155)^2$

$A \approx 253.3031\pi$ m^2

$A \approx 795.775$ m^2

The area of a circle with a circumference of 100 meters is approximately 796 m^2.

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Instruction

Example 4

What is the circumference of a circle that has an area of 1,000 m^2?

1. First, find the radius by solving for r in the formula for the area of the circle.

$A = \pi r^2$	Formula for the area of a circle
$1000 = \pi r^2$	Substitute A into the equation.
$\frac{1000}{\pi} = r^2$	Divide both sides by π.
$r^2 \approx 318.3099$	Simplify.
$\sqrt{r^2} = \pm\sqrt{318.3099}$	Take the square root of both sides.
$r \approx 17.8412$ m	Use only the positive result, as distance is always positive.

The radius is approximately 17.8412 meters.

2. Find the circumference using the formula $C = 2\pi r$.

$C = 2\pi r$	Formula for circumference of a circle
$C = 2\pi(17.8412)$	Substitute 17.8412 for r.
$C \approx 112.1$ m	

The circumference of a circle with an area of 1,000 m^2 is approximately 112.1 meters.

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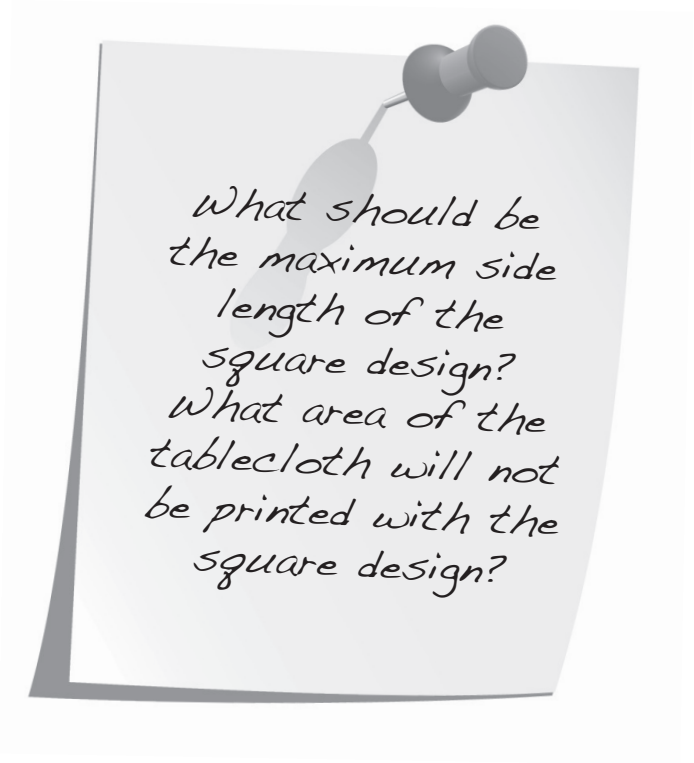
Problem-Based Task 6.5.1: Designing a Tablecloth

SMP	
1	2
3 ✓	4 ✓
5	6 ✓
7	8

A factory is printing a large square design on a circular tablecloth. The square design should be as large as possible. The tablecloth has an area of 30 square feet. What should be the maximum side length of the square design? What area of the tablecloth will not be printed with the square design?



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Problem-Based Task 6.5.1: Designing a Tablecloth

Coaching

a. Make a drawing of this problem.

b. What length of the circle corresponds to a length in the square?

c. How can you use the area of the circle to find the diagonal of the square?

d. What angle does the square's diagonal make at the vertex of the side length?

e. What ratio of trigonometry can you use to find the maximum length of the square's side?

f. What is the area of the square?

g. What is the area of the circle minus the area of the square?

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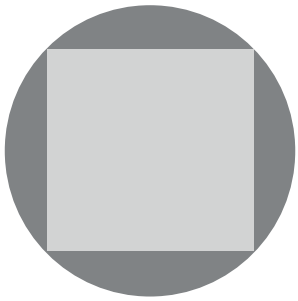
Instruction

Problem-Based Task 6.5.1: Designing a Tablecloth

Coaching Sample Responses

a. Make a drawing of this problem.



b. What length of the circle corresponds to a length in the square?

The diameter of the circle is equal to the diagonal of the square.

c. How can you use the area of the circle to find the diagonal of the square?

The diameter is twice the length of the radius of the tablecloth. The area of the tablecloth is given. Substitute the area of the tablecloth into the formula for the area of a circle.

$A = \pi r^2$

$30 = \pi r^2$

$\frac{30}{\pi} = r^2$

$r = \sqrt{9.549296586}$

$r = \pm 3.090193616$

$r \approx 3.0902$ feet

The diameter is twice the radius, so multiply the radius by 2.

$2 \cdot 3.0902 = 6.1804$ feet

The length of the diagonal of the square design is approximately 6.1804 feet.

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d. What angle does the square's diagonal make at the vertex of the side length?

The diagonal bisects the 90° angle in the square. Therefore, each angle measures 45°.

e. What ratio of trigonometry can you use to find the maximum length of the square's side?

The hypotenuse of the triangle created by the diagonal is 6.1804 feet.

You are trying to find either side length s, so use either the sine of 45° or the cosine of 45° to find the side length. Calculations with sine are shown below.

$$\sin 45^\circ = \frac{s}{6.1804}$$

$$6.1804 \bullet \sin 45^\circ = s$$

$$s \approx 4.37 \text{ feet}$$

The maximum side length of the square design should be approximately 4.37 feet.

f. What is the area of the square?

$$A_{\text{square}} = s^2$$

$$A_{\text{square}} = 19.0969 \text{ ft}^2$$

The area of the square is approximately 19.1 ft^2.

g. What is the area of the circle minus the area of the square?

$$30 - 19.1 = 10.9 \text{ ft}^2$$

The tablecloth will have about 10.9 square feet of fabric that is not printed with a square.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 6.5.1: Circumference and Area of a Circle

Use your knowledge of circumference and area to complete each problem.

1. A circle has a regular octagon inscribed in it. The circle has a radius of 4 meters. Find the perimeter of the octagon. Use the formula $C = 2\pi r$ to find the circumference of the circle. Why is the circumference found by using the formula a different length than the perimeter of the octagon?

2. A circle has a regular dodecagon (12-sided polygon) inscribed in it. The circle has a radius of 4 meters. Find the perimeter of the dodecagon. Then, find the circumference of the circle using $C = 2\pi r$. Why is the circumference found by using the formula a different length than the perimeter of the dodecagon?

3. Compare the results of problems 1 and 2. Which dissection is a better approximation of the circumference of the circle? Then, use a 15-sided regular polygon as the inscribed figure in a circle that has a radius of 4 meters. Calculate the polygon's perimeter and compare it with the circle's circumference.

4. A circle has a regular octagon inscribed in it. The circle has a radius of 4 meters. Find the area of the circle using the formula $A = \pi r^2$, then find the area of the octagon. Why is the area of the circle different from the area of the octagon?

5. A circle has a regular dodecagon inscribed in it. The circle has a radius of 4 meters. Find the area of the circle and then of the dodecagon. Why is the area of the circle different from the area of the dodecagon?

continued

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6. Compare the results of problems 4 and 5. Which inscribed area is a better approximation of the area of the circle? Then, use a 36-sided regular polygon as the inscribed figure in a circle that has a radius of 4 meters. Calculate the polygon's area and compare it with the circle's area.

7. A round dining room table has a wood top with a circumference of 32 feet. A woodworker is refinishing the top. He needs to find the area of the top to buy materials and know how long the job will take. What is the area of the tabletop?

8. A pizza has a circumference of 40 inches. What is the area of the pizza?

9. A carpenter is installing curved wood trim around a circular window. The window is a circle that has an area of 50 square feet. How many feet of wood trim are needed to surround the window? Determine the measurement of the trim based on the length of the wood next to the window.

10. An artist paints a large blue circle on a yellow wall. The area of the painted circle is 150 square feet. What is the circumference of the circle?

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Lesson 6.5.2: Volumes of Cylinders, Pyramids, Cones, and Spheres

Warm-Up 6.5.2

A gardener in Texas is designing a rainwater collection system. She will collect rainwater from her roof to water her garden. The collection barrels are cylinders that are 1.3 meters high and have a radius of 40 cm. A **cylinder** is a solid or hollow object that has two parallel bases connected by a curved surface. The gardener wants to collect at least 5 cubic meters of water.

1. In what way does a cylinder resemble the shape of a prism?

2. How are the formulas for the volumes of prisms and cylinders similar and different?

3. What are the formulas for the volumes of a prism and cylinder?

4. How many collection barrels does the gardener need?

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Lesson 6.5.2: Volumes of Cylinders, Pyramids, Cones, and Spheres

Common Core State Standards

G–GMD.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*

G–GMD.3 Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.★

Warm-Up 6.5.2 Debrief

1. In what way does a cylinder resemble the shape of a prism?

   Both are three-dimensional objects. Both have a height and two congruent parallel bases.

2. How are the formulas for the volumes of prisms and cylinders similar and different?

   Since a cylinder resembles a prism in some ways, the volume formulas are similar. For both figures, the volume is found by multiplying the area of the base times the height. The formulas are different because the areas of the bases are calculated differently. While the prism base is a polygon (for example, in a rectangular prism, the area of the base is A = length • width), the area of the base of a cylinder is $A = \pi r^2$.

3. What are the formulas for the volumes of a prism and cylinder?

   If you multiply the area of a prism's base by the prism's height, the formula for the volume of a prism is as follows:

   $V_{\text{prism}} = A_{\text{base}}$ • height

   The circle at the base of a cylinder has an area formula of $A = \pi r^2$.

   If you multiply the area of a cylinder's base by the cylinder's height, the formula for the volume of a cylinder is as follows:

   $V_{\text{cylinder}} = A_{\text{base}}$ • height

   $V_{\text{cylinder}} = \pi r^2 \bullet h$

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4. How many collection barrels does the gardener need?

   The radius of a barrel is 40 cm. Convert this to meters.

   $$40 \text{ cm} \bullet \frac{1 \text{ m}}{100 \text{ cm}} = 0.4 \text{ m}$$

   The radius of the barrel is 0.4 meter.

   The height of the barrel is 1.3 meters. Substitute these dimensions into the formula.

   $V_{\text{cylinder}} = \pi r^2 \bullet h$ Formula for volume of a cylinder

   $V = \pi \bullet 0.4^2 \bullet 1.3$ Substitute known dimensions of the water barrel.

   $V \approx 0.6534 \text{ m}^3$

   One barrel will hold approximately 0.6534 cubic meters of water. The gardener wants to collect at least 5 cubic meters of water. Therefore, divide 5 by 0.6534 to determine the number of needed barrels.

   $$\frac{5}{0.6534} \approx 7.65 \text{ barrels}$$

   Since the gardener can't have 0.65 of a barrel, round up to 8 barrels.

   The gardener needs 8 water barrels to collect at least 5 cubic meters of water.

Connection to the Lesson

- Students will learn why the volume formulas for a prism and a cylinder are the same (*volume = area of base • height*).
- Students will learn about the formulas for the volumes of pyramids, cones, and spheres.
- Students will explore proofs for why the formulas for cylinders, pyramids, cones, and spheres work.
- Students will solve real-world problems by calculating the volumes of cylinders, pyramids, cones, and spheres.

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Prerequisite Skills

This lesson requires the use of the following skills:

- understanding and using formulas for the volume of prisms, cylinders, pyramids, cones, and spheres (8.G.9)
- calculating with fractions and decimals (7.NS.1)
- understanding operations with exponents (8.EE.1)
- understanding and applying the formula for circumference (7.G.4)

Introduction

Think about the dissection arguments used to develop the area of a circle formula. These same arguments can be used to develop the volume formulas for cylinders, pyramids, cones, and spheres. You have already used volume formulas for prisms, cylinders, pyramids, cones, and spheres, but how are the formulas derived? In this lesson, you will learn how to find the volumes of cylinders, pyramids, cones, and spheres. As part of learning the formulas, you will see proofs of why the formulas work for those objects. The real world is filled with these objects. Using the formulas of volume for these objects expands your problem-solving skills.

Key Concepts

- The formula for finding the volume of a prism is $V = length \cdot width \cdot height$. This can also be shown as $V = area\ of\ base \cdot height$.
- Remember to use cubic units or volume measures when calculating volume. Some examples are cubic feet (ft^3), cubic meters (m^3), liters (L), and gallons (gal).
- If the dimensions of an object are all multiplied by a scale factor, k, then the volume will be multiplied by a scale factor of k^3. For example, if the dimensions are enlarged by a scale factor of 5, then the volume will be enlarged by a scale factor of 125.

Cylinders

- A cylinder has two bases that are parallel. This is also true of a prism.
- Bonaventura Cavalieri, an Italian mathematician, formulated **Cavalieri's Principle**. This principle states that the volumes of two objects of equal height are equal if the areas of their corresponding cross sections are in all cases equal.

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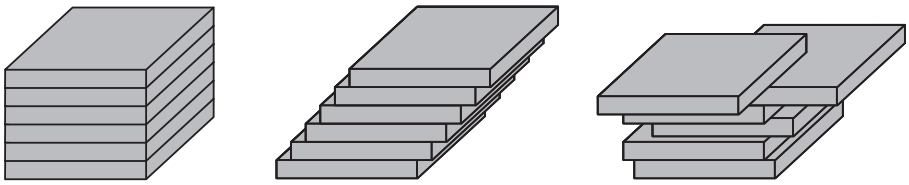
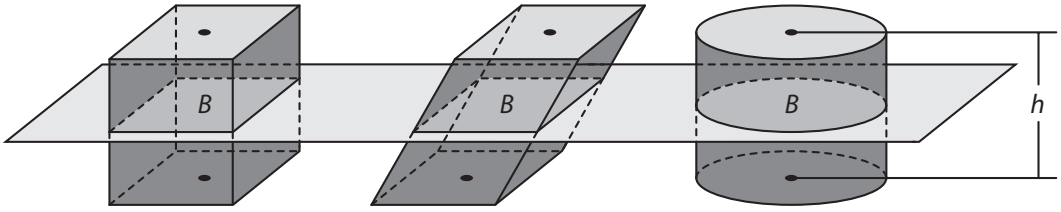
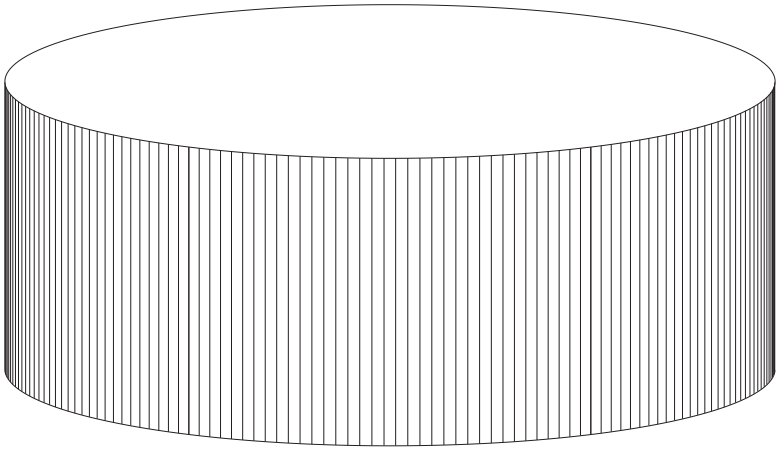
- This principle is illustrated by the following diagram. A rectangular prism has been sliced into six pieces and is shown in three different ways.



- The six pieces maintain their same volume regardless of how they are moved.
- Cavalieri's Principle describes how each piece is a thin slice in the plane of the prism.
- If each thin slice in each object has the same area, then the volumes of the objects are the same.
- The following diagram shows a prism, a prism at an oblique angle, and a cylinder.



- The three objects meet the two criteria of Cavalieri's Principle. First, the objects have the same height. Secondly, the areas of the objects are the same when a plane slices them at corresponding heights. Therefore, the three objects have the same volume.
- A square prism that has side lengths of $r\sqrt{\pi}$ will have a base area of $r\sqrt{\pi} \bullet r\sqrt{\pi} = \pi r^2$ on every plane that cuts through it. The same is true of a cylinder, which has a radius, *r*.
- The base area of the cylinder will be πr^2. This shows how a square prism and a cylinder can have the same areas at each plane.
- Another way of thinking about the relationship of a polygonal prism and the cylinder is to remember the earlier proof about the area of a circle.
- A cylinder can be thought of as a prism with an infinite number of sides.
- The following diagram shows a polygonal prism with 200 sides.



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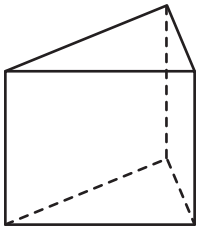
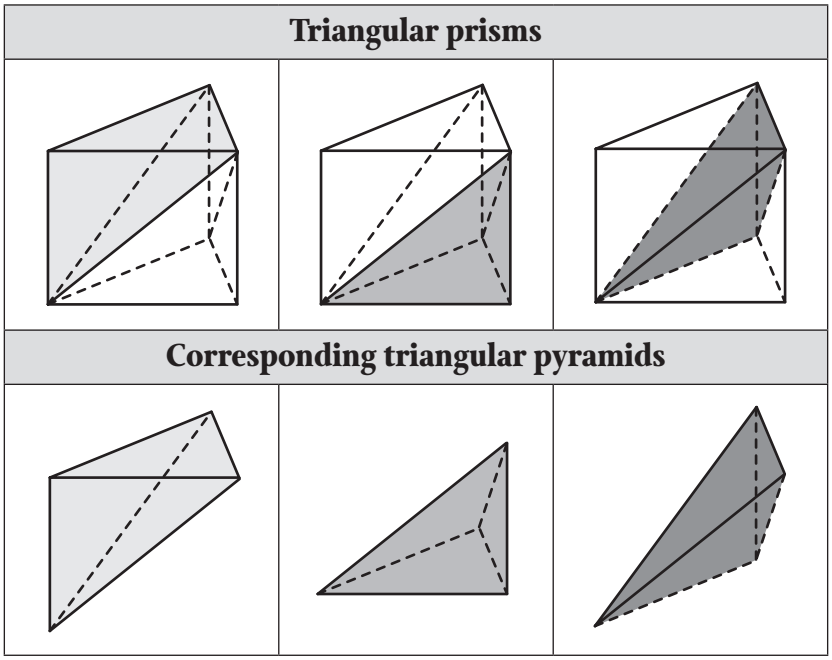
- The area of its base can be calculated by dividing it into quadrilaterals and triangles, but its base is approaching the limit of being a circle.
- If the base area and the height of a prism and a cylinder are the same, the prism and cylinder will have the same volume.
- The formula for finding the volume of a cylinder is $V = \pi r^2 \bullet h$.

Pyramids

- A **pyramid** is a solid or hollow polyhedron object that has three or more triangular faces that converge at a single vertex at the top; the base may be any polygon.
- A **polyhedron** is a three-dimensional object that has faces made of polygons.
- A triangular prism can be cut into three equal triangular pyramids.





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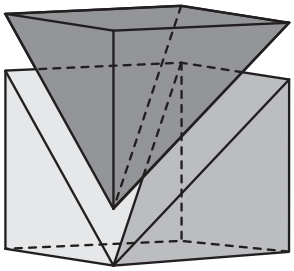
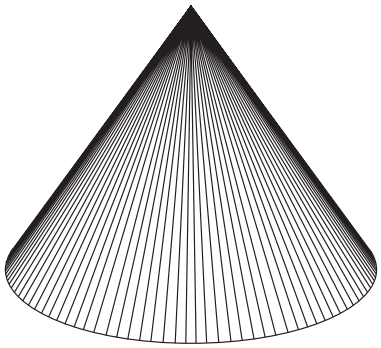
- A cube can be cut into three equal square pyramids.



- This dissection proves that the volume of a pyramid is one-third the volume of a prism: $V_{\text{pyramid}} = \frac{1}{3}B \bullet h$.

Cones

- A **cone** is a solid or hollow object that tapers from a circular base to a point.
- A cone and a pyramid use the same formula for finding volume.
- This can be seen by increasing the number of sides of a pyramid.
- The limit approaches that of being a cone.
- A pyramid with 100 sides follows. With such a large number of sides, it looks like a cone.



- The formula for the volume of a cone is $V_{\text{cone}} = \frac{1}{3}\pi r^2 \bullet h$.
- Cavalieri's Principle shows how pyramids and cones have the same volume.
- The diagram that follows shows cross sections of areas with the same planes.

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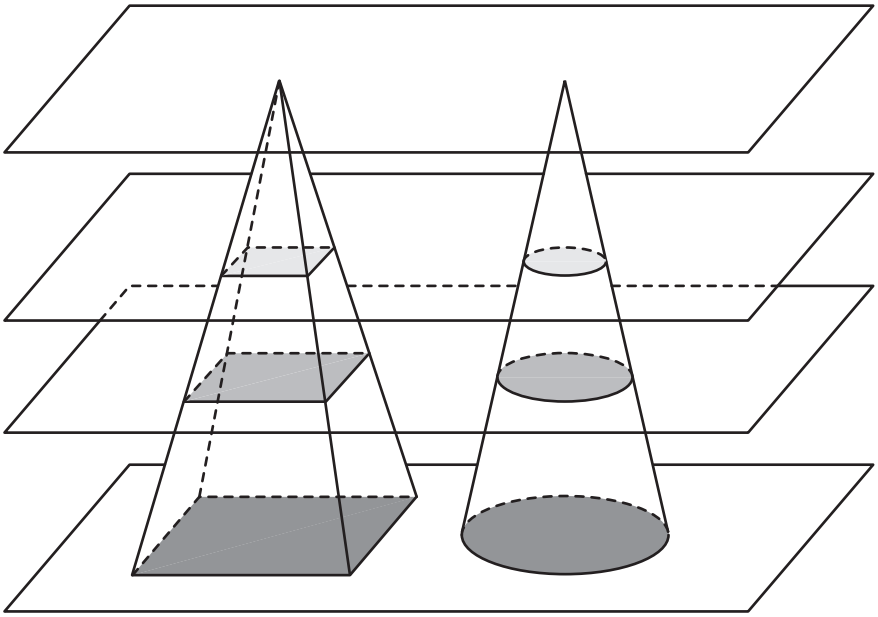
- Each object has the same area at each cross section.
- Therefore, the volumes of both objects are the same.



Spheres

- A **sphere** is a three-dimensional surface that has all its points the same distance from its center.
- The volume of a sphere can be derived in several ways, including by using Cavalieri's Principle and a limit process.
- The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

Common Errors/Misconceptions

- confusing formulas; for example, using the prism volume formula for a pyramid or a cone
- calculating the wrong base area for a pyramid due to not noticing the shape of the pyramid's base (e.g., triangle, square)
- performing miscalculations with the number π
- making arithmetic errors when dealing with the formulas, especially with fractions
- forgetting to express a calculation in the correct units for volume (e.g., m^3, liters)

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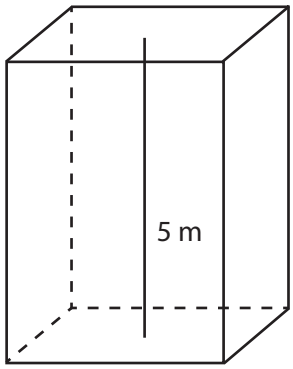
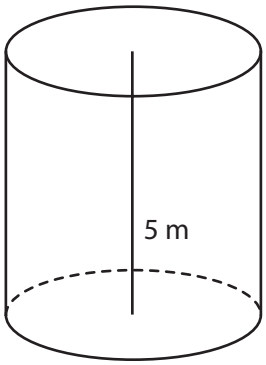
Instruction

Guided Practice 6.5.2

Example 1

Find the dimensions for a cylinder that has the same volume as a square prism with a base area of 9 square meters. The cylinder and the square prism both have heights of 5 meters.





1. Determine the relationship between two objects with the same volume.

   Cavalieri's Principle states that if two objects have the same area in every plane, or cross section, then their volumes are the same. The cylinder and prism have uniform width and length throughout their heights. Both need to have the same height. Therefore, in order for the cylinder and prism to have the same volume, they need to have the same areas for their bases.

2. Set up the formulas for the area of the base of the cylinder and the area of the base of the prism so that they are equal.

   The formula for the area of the circular base of the cylinder is $A = \pi r^2$, where r is the radius of the base of the cylinder.

   The area of the base of the prism is 9 square meters.

   $A_{\text{base of square prism}} = A_{\text{base of cylinder}}$ Set the areas of the bases equal.

   $9 = \pi r^2$ Substitute the known information.

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3. Solve the equation for r.

$9 = \pi r^2$

$\frac{9}{\pi} = r^2$ Divide both sides by π.

$2.864788976 \approx r^2$

$\pm\sqrt{2.864788976} \approx r$ Take the square root of both sides.

$r \approx 1.692568751$ The length of the radius must be a positive number.

For the cylinder to have the same volume as the prism, the cylinder's base must have a radius of 1.692568751.

Substitute the value of the radius into the equation to check your work.

$9 = \pi \bullet r^2$

$9 = \pi \bullet (1.692568751)^2$

$9 \approx 9.0000000001$

The difference is infinitesimal. The two objects essentially have bases that have the same area, 9 m^2.

4. Calculate the volume for each object.

The volume for a prism or a cylinder can be seen as many thin slices or cross sections stacked on top of one another. Thus, their volume formulas are the same: $V = B \bullet h$, where B is the area of the base of the object and h is the height.

Prism	Cylinder
$V = B \bullet h$	$V = B \bullet h$
$V = 9 \bullet 5$	$V \approx 9.0000000001 \bullet 5$
$V = 45\ m^3$	$V \approx 45\ m^3$

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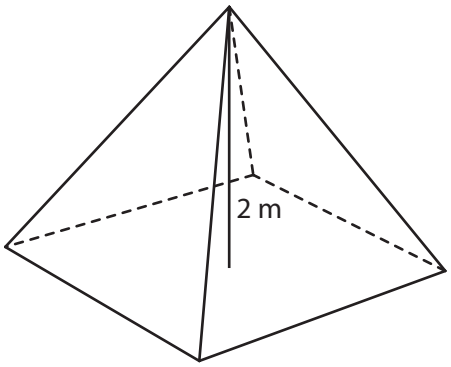
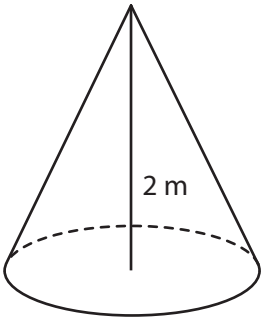
5. Verify that the two objects will have the same area at a height of 1 meter or any other height.

   Yes, this is true. At every height, both objects have uniform dimensions that are equal to the dimensions of their bases. Therefore, using Cavalieri's Principle, both objects, the prism and the cylinder, have the same volume.

Example 2

A cone and a pyramid both have a height of 2 meters. The volume of the pyramid is 3 cubic meters. A cone and a pyramid both taper to a point or vertex at the top. The "slant" of the taper is linear, meaning it is a straight line. The dimensions of both the cone and the pyramid change at a constant rate from base to tip. Find the radius of the cone that has the same volume as this pyramid.





1. Cavalieri's Principle states that the pyramid and cone will have the same volume if the area of each cross section of a plane is the same at every height of the two objects. This means that if the cone and pyramid have bases of equal area and equal heights, then their volumes will also be equal.

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2. Set up an equation to find the radius of the cone.

The volume of the pyramid is 3 m^3.

Both objects must have bases of equal area to have the same volume.

The area of the base of the pyramid can be found by solving for B in the formula for the volume of a pyramid.

$V_{\text{pyramid}} = \frac{1}{3}B \bullet h$

$3 = \frac{1}{3}B \bullet 2$ Substitute the volume and the height of the pyramid.

$3 = \frac{2}{3}B$ Simplify.

$\frac{3}{2} \bullet 3 = B$ Multiply both sides by $\frac{3}{2}$.

$\frac{9}{2} = B$ Simplify.

$B = 4.5$

The area of the base of the pyramid is 4.5 square meters.

$A_{\text{pyramid base}} = A_{\text{cone base}} = A_{\text{circle}} = \pi r^2$ Set up the equation.

$4.5 = \pi r^2$ Substitute the value found for the area of the base of the pyramid.

$\frac{4.5}{\pi} = r^2$ Divide both sides by π.

$\pm\sqrt{\frac{4.5}{\pi}} = r$ Take the square root of both sides.

$r \approx 1.19682684$

The radius of the cone is approximately 1.197 meters.

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Example 3

A new art museum is being built in the shape of a square pyramid. The height will be 50 meters. The art museum needs 86,400 cubic meters of space inside. What should be the side lengths of the base of the pyramid?

1. Use the formula for the volume of a pyramid to find the unknown side lengths of the base.

$$V = \frac{1}{3}B \bullet h$$

$86{,}400 = \frac{1}{3}B \bullet 50$ Substitute the known values.

$\frac{3 \bullet 86{,}400}{50} = B$ Multiply both sides by $\frac{3}{50}$.

$B = 5184$

The base is 5,184 square meters.

2. Find the side length of the base.

The base is a square. It has an area of *side* • *side*.

Find the length of one side by taking the square root of the area of the base.

$\pm\sqrt{5184} = 72$

Take the positive square root since length is a positive number.

The length of each side of the art museum's pyramid base should be 72 meters.

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Instruction

Example 4

Weston has two round balloons. One balloon has a radius that is 3 times the radius of the other balloon. How much more air will the larger balloon need than the smaller balloon?

1. Use the formula for the volume of a sphere for the smaller balloon.

$$V = \frac{4}{3}\pi r^3$$

2. For the larger balloon, since the radius is 3 times larger, use $3r$ instead of r in the volume formula.

$$V = \frac{4}{3}\pi (3r)^3$$

3. Simplify the formula for the volume of the larger balloon.

$V = \frac{4}{3}\pi (3r)^3$	Formula for the volume of the larger balloon
$V = \frac{4}{3}\pi (27r^3)$	Distribute the exponent.
$V = 36\pi r^3$	Multiply the coefficients.

4. Compare the coefficients of both formulas (for the smaller balloon and for the larger balloon).

   The coefficients are 36 and $\frac{4}{3}$.

   Divide to see how many times larger the volume of the larger balloon is.

$$36 \div \left(\frac{4}{3}\right) = 27$$

   The larger balloon will need 27 times as much air as the smaller balloon.

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Instruction

Example 5

A teenager buying some chewing gum is comparing packages of gum in order to get the most gum possible. Each package costs the same amount. Package 1 has 20 pieces of gum shaped like spheres. Each piece has a radius of 5 mm. Package 2 has 5 pieces of gum shaped like spheres. Each piece has a radius of 10 mm. Which package should the teenager buy?

1. Find the volume of a piece of gum in package 1 by using the volume formula for a sphere.

$V = \frac{4}{3}\pi r^3$ Formula for volume of a sphere

$V = \frac{4}{3}\pi(5)^3$ Substitute 5 for r.

$V \approx 524 \text{ mm}^3$

2. Multiply the volume of the single piece of gum in package 1 by 20 to get the total volume of the gum in the package.

$(524)(20) = 10{,}480 \text{ mm}^3$

3. The radius of each piece of gum in package 2 is 10 mm. Find the volume of a piece of gum in package 2 by using the volume formula for a sphere.

$V = \frac{4}{3}\pi r^3$ Formula for volume of a sphere

$V = \frac{4}{3}\pi(10)^3$ Substitute 10 for r.

$V \approx 4189 \text{ mm}^3$

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Instruction

4. Multiply the volume of a single piece of gum in package 2 by 5 to get the total volume of gum in the package.

   $(4189)(5) = 20{,}945 \text{ mm}^3$

5. Compare the volumes of the two packages to determine the best purchase.

   $10{,}480 < 20{,}945$

   The teenager should buy package 2.

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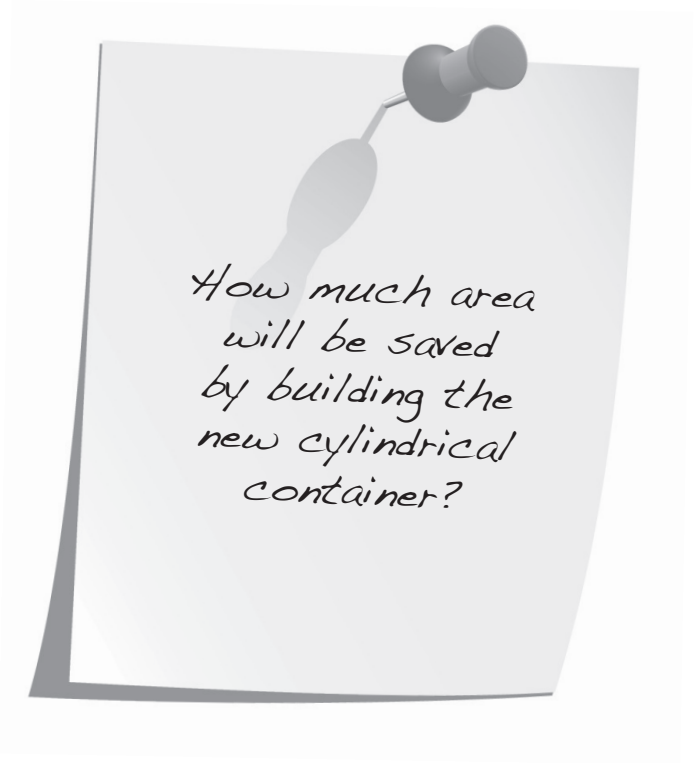
Problem-Based Task 6.5.2: Cylinders of Sand

SMP	
1 ✓	2
3	4 ✓
5	6
7 ✓	8

The manager of road maintenance for a large city wants to reduce the amount of space needed to store piles of sand. The sand is used on icy roads in the winter. There are 3 piles of sand that are cone-shaped. Each pile has a circumference of 157 feet and a height of 20 feet. The manager is thinking of building a storage area that is in the shape of a cylinder. The cylinder will have the same height as the cone-shaped sand piles, and the cylinder's circumference will be the same as that of the bases of the sand piles. How much area will be saved by building the new cylindrical container?



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Problem-Based Task 6.5.2: Cylinders of Sand

Coaching

a. How much area does 1 pile of sand take up?

b. How much area do 3 piles of sand take up?

c. What is the volume of 1 pile of sand?

d. What is the volume of 3 piles of sand?

e. What is the volume of the cylindrical container?

f. How much area does the cylindrical container take up?

g. How much sand is left over after the piles are put into the cylindrical container?

h. How much area will be saved by the new cylindrical container?

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Instruction

Problem-Based Task 6.5.2: Cylinders of Sand

Coaching Sample Responses

a. How much area does 1 pile of sand take up?

The area of the base of the pile of sand is $A = \pi r^2$.

Find the value of r by solving for r in the formula for circumference.

$C = 2\pi r$

$157 = 2\pi r$ Substitute the known value of the circumference.

$\frac{157}{2\pi} = r$ Divide both sides by 2π.

$r \approx 24.987$

The radius of the base of the sand pile is about 24.987 feet.

$A = \pi(24.987^2)$ Substitute the radius value into the area formula.

$A \approx 1961.51$

The area of the base of the pile of sand is 1,961.51 square feet.

b. How much area do 3 piles of sand take up?

$3 \bullet 1961.51 \approx 5884.36$

The area of 3 piles of sand is about 5,884 square feet.

c. What is the volume of 1 pile of sand?

Use the formula for the volume of a cone.

$V = \frac{1}{3}B \bullet h$

$V = \frac{1}{3}\pi r^2 \bullet h$ Substitute the expression for the area of a circle.

$V = \frac{1}{3}(1961.51) \bullet 20$ Substitute the above values for the area of the cone's base and the height of the cone.

$V \approx 13{,}076.67 \text{ ft}^3$

The volume of 1 pile of sand is 13,076.67 cubic feet.

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Instruction

d. What is the volume of 3 piles of sand?

$3 \bullet 13{,}076.67 = 39{,}230.01 \text{ ft}^3$

The volume of 3 piles of sand is about 39,230.01 cubic feet.

e. What is the volume of the cylindrical container?

Use the formula for the volume of a cylinder.

$V = B \bullet h$

$V = 1961.51 \bullet 20 = 39{,}230.20$ cubic feet — Substitute the known area of the base because it has the same dimensions as the base of the cone.

f. How much area does the cylindrical container take up?

It takes up the same area as 1 cone of sand, 1,961.51 square feet.

g. How much sand is left over after the piles are put into the cylindrical container?

There is no sand remaining. All 3 piles fit exactly into the 1 cylinder. The volume of the 3 cones of sand was the same as the volume of the cylinder.

h. How much area will be saved by the new cylindrical container?

The area saved equals the area of 2 piles of sand. The cylinder has the same base area as 1 pile of sand, but now all 3 piles of sand are in one location (the cylinder).

$2 \bullet 1961.51 = 3923.02$

The new cylinder saves approximately 3,923 square feet of space.

This answer makes sense because the formula for the volume of a cone is very similar to the formula for the volume of cylinder, except the cone is one-third the volume of the cylinder. Therefore, 3 piles of sand will fit in 1 cylinder of the same size base and height as the cones of sand.

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 6.5.2: Volumes of Cylinders, Pyramids, Cones, and Spheres

Use your knowledge of volume to complete each problem. Round each answer to the nearest whole number.

1. A gasoline fuel storage tank at an oil refinery is a cylinder with a radius of 20 meters and a height of 10 meters. How many gallons of gasoline will the tank hold? There are 264.172 gallons in 1 cubic meter.

2. A storage container is a cylinder with a height of 28.2 cm and a radius of 5 cm. How many liters of water will the storage container hold? 1 liter = 1000 cm^3.

3. A company makes candles in the shape of cones. Their best-selling candle has a height of 6 inches and a circumference of 12 inches. What volume of wax is needed to make 1 candle?

4. For problem 3 above, what would the base side lengths be for a square pyramid candle that has the same volume and height as the cone?

5. In July 2012, an ice cream company in England set a new world's record for the largest ice cream cone ever made. The total height was 13 feet including the ice cream on top. The cone itself was approximately 9 feet tall and had a diameter of about 3.5 feet. How many gallons of ice cream were needed to fill just the cone part (not including the ice cream on top)? 1 cubic foot = 7.40852 gallons.

continued

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6. A small city wants to build a cylindrical water tank that holds 10,000,000 liters. The city wants the height of the tank to be 10 meters. What will be the tank's diameter?
1 cubic meter = 1000 liters.

7. A history museum is building an outdoor model of a Mesoamerican pyramid located in South America. The outside of the pyramid will be made of small square stones. The inside will be concrete. The concrete part of the square pyramid will have side lengths of 5 feet and a height of 3 feet. What volume of concrete is needed to make the inside?

Use your knowledge of volume to complete each problem. Round each answer to the nearest hundredth.

8. The diameter of Mars is approximately 6,794 kilometers. What is the volume of Mars to the nearest cubic kilometer?

9. If the radius of a sphere is doubled, by how much does the volume of the sphere increase?

10. A manufacturer is shipping a spherical globe that fits exactly in a box shaped like a cube. The globe is touching all six sides of the box. If the volume of the box is 343 in^3, what is the volume of the globe?

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Progress Assessment

Circle the letter of the best answer.

1. In center-pivot irrigation, a long sprinkler on wheels rotates in a field of crops around a fixed pipe at the center of the circle made by the sprinkler. The sprinkler rotates 360° around the fixed pipe. If the sprinkler is 40 meters long, how much area of field is being watered by this sprinkler?

   a. $\approx 5026.55\ m^2$
   c. $\approx 20{,}106.19\ m^2$
   β. $\approx 251.33\ m^2$
   d. $\approx 3756.22\ m^2$

2. A large circular table has an area of 400 square feet. What is the table's circumference?

   a. 60.9 feet
   c. 11.3 feet
   b. 127.3 feet
   d. 70.9 feet

3. If the following regular polygons are inscribed in a circle, which one has a perimeter that is closest in length to the circumference of the circle?

   a. triangle
   c. hexagon
   b. octagon
   d. square

4. A storage tank has a circumference of 200 meters. What is the area of the base of the tank?

   a. $\approx 3183.1\ m^2$
   c. $\approx 1256.6\ m^2$
   b. $\approx 628.3\ m^2$
   d. $\approx 314.2\ m^2$

5. A new house has a conical pile of topsoil that needs to be spread on the ground. The pile of soil has a radius of 15 feet and a height of 12 feet. What is the volume of the pile of soil?

   a. $\approx 1413.72\ ft^3$
   c. $\approx 8482.3\ ft^2$
   b. $\approx 5654.87\ ft^3$
   d. $\approx 11{,}309.73\ ft^2$

continued

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6. A triangular pyramid has a base with equal lengths of 5 meters. The height of the pyramid is also 5 meters. What is the volume of the pyramid?

   a. $\approx 18.04\ m^3$
   b. $\approx 54.13\ m^3$
   c. $\approx 10.83\ m^3$
   d. $\approx 65.45\ m^3$

7. A dairy farmer stores milk in a stainless steel cylindrical tank. The tank is 6 feet tall and has a radius of 2 feet. How many gallons of milk does the tank hold? 1 cubic foot = 7.40852 gallons.

   a. ≈ 75.4 gallons
   b. ≈ 139.65 gallons
   c. ≈ 558.59 gallons
   d. ≈ 754.92 gallons

8. What is the volume of a square pyramid with a height of 6 meters and a base with a side length of 10 meters?

   a. $120\ m^3$
   b. $200\ m^3$
   c. $400\ m^3$
   d. $360\ m^3$

9. A spherical storage container at a factory has a radius of 5 meters. How many liters of liquid will it hold? 1 cubic meter = 1000 liters.

   a. ≈ 52,360 liters
   b. ≈ 104,720 liters
   c. ≈ 523,600 liters
   d. ≈ 1,570,796 liters

10. A worker at a bedding shop is stuffing a spherical pillow that has a radius of 7 inches. He is also stuffing a cylindrical pillow that has a radius of 7 inches and a length of 15 inches. To the nearest cubic inch, what is the volume of stuffing that the worker will need to completely fill both pillows?

    a. $3{,}744\ in^3$
    b. $2{,}308\ in^3$
    c. $3{,}743\ in^3$
    d. $1{,}436\ in^3$

continued

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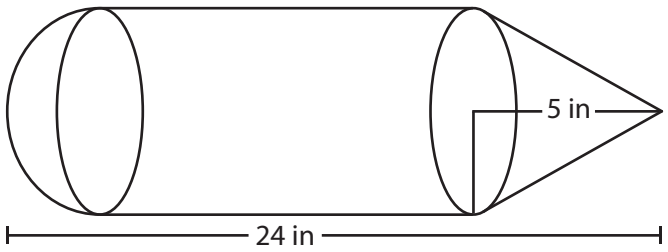
Assessment

Use what you know about the volume formulas of solids to answer the questions that follow.

11. A student has designed a model rocket as shown in the diagram below. The rocket is constructed of a hemisphere, a cylinder, and a cone. The student needs to check the following things below to make sure she can fly the rocket.



a. The volume of the cone is 47.1 in^3 and the height is 5 inches. In order for the rocket to fly, it must have a radius of at least 2 inches. What is the radius of the cone?

b. The length of the rocket from end to end is 24 inches. What is the volume of the cylindrical part of the rocket?

c. The hemispherical end of the rocket holds the rocket fuel. The rocket must be able to hold at least 50 in^3 of fuel. Is the hemispherical end big enough to hold the fuel?

d. The volume of the rocket cannot exceed 550 in^3. What is the volume of the rocket?

e. Can the student fly the rocket? Why or why not?

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Assessment

Pre-Assessment

Circle the letter of the best answer.

1. Which is an equation of the circle with center (0, 0) and radius 3?

   a. $x + y = 3$
   b. $x + y = 9$
   c. $x^2 + y^2 = 3$
   d. $x^2 + y^2 = 9$

2. Which is an equation of the circle with center (3, –2) and radius $2\sqrt{5}$?

   a. $(x-3)^2 + (y+2)^2 = 20$
   b. $(x+3)^2 + (y-2)^2 = 20$
   c. $(x-3)^2 + (y+2)^2 = 100$
   d. $(x+3)^2 + (y-2)^2 = 100$

3. Which is an equation of the parabola with focus (–1, –5) and directrix $y = 1$?

   a. $(x-1)^2 = y-5$
   b. $(x-1)^2 = -(y-5)$
   c. $(x-1)^2 = -12(y-2)$
   d. $(x+1)^2 = -12(y+2)$

4. What are the center and radius of the circle whose equation is $x^2 + y^2 - 4x - 6y + 8 = 0$?

   a. The center is (2, 3) and the radius is $\sqrt{2}$.
   b. The center is (2, 3) and the radius is $\sqrt{5}$.
   c. The center is (–2, –3) and the radius is $\sqrt{2}$.
   d. The center is (–2, –3) and the radius is $\sqrt{5}$.

5. Which option describes the parabola whose equation is $(y+1)^2 = x-4$?

   a. The focus is (4, –1) and the parabola opens up.
   b. The vertex is (4, –1) and the parabola opens to the right.
   c. The focus is (–4, 1) and the parabola opens up.
   d. The vertex is (–4, 1) and the parabola opens to the right.

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Lesson 6: Deriving Equations

Instruction

Common Core State Standards

G–GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

G–GPE.2 Derive the equation of a parabola given a focus and directrix.

SMP	
1 ✓	2 ✓
3	4 ✓
5	6
7 ✓	8

Essential Questions

1. How can a circle be described in different ways?
2. How can a parabola be described in different ways?
3. How are the definitions of a circle and a parabola similar?
4. How are the definitions of a circle and a parabola different?

WORDS TO KNOW

center of a circle	the point in the plane of the circle from which all points on the circle are equidistant. The center is not part of the circle; it is in the interior of the circle.
circle	the set of all points in a plane that are equidistant from a reference point in that plane, the center. The set of points forms a 2-dimensional curve that measures 360°.
directrix of a parabola	a line that is perpendicular to the axis of symmetry of a parabola and that is in the same plane as both the parabola and the focus of the parabola; the fixed line referenced in the definition of a parabola
distance formula	a formula that states the distance between points (x_1, y_1) and (x_2, y_2) is equal to $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
focus of a parabola	a fixed point on the interior of a parabola that is not on the directrix of the parabola but is on the same plane as both the parabola and the directrix; the fixed point referenced in the definition of a parabola
general form of an equation of a circle	$Ax^2 + By^2 + Cx + Dy + E = 0$, where $A = B$, $A \neq 0$, and $B \neq 0$

SRB p. U6-152

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Instruction

parabola	the set of all points that are equidistant from a fixed line, called the directrix, and a fixed point not on that line, called the focus. The parabola, directrix, and focus are all in the same plane. The vertex of the parabola is the point on the parabola that is closest to the directrix.
perfect square trinomial	a trinomial of the form $x^2 + bx + \left(\frac{b}{2}\right)^2$
Pythagorean Theorem	a theorem that relates the length of the hypotenuse of a right triangle (c) to the lengths of its legs (a and b). The theorem states that $a^2 + b^2 = c^2$.
quadratic function	a function that can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. The graph of a quadratic function is a parabola
radius	the distance from the center to a point on the circle
standard form of an equation of a circle	$(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the center and r is the radius
standard form of an equation of a parabola	$(x - h)^2 = 4p(y - k)$ for parabolas that open up or down; $(y - k)^2 = 4p(x - h)$ for parabolas that open right or left. For all parabolas, $p \neq 0$ and the vertex is (h, k).
vertex of a parabola	the point on a parabola that is closest to the directrix and lies on the axis of symmetry

Recommended Resources

- MathIsFun.com. "Circle Equations."

  http://www.walch.com/rr/00071

  This interactive site reviews the standard form of the equation of a circle and how it is derived. It reminds users that the equation is based on the Pythagorean Theorem, and reviews how to expand binomials and simplify the resulting polynomials to obtain the general form, given the standard form. The site also demonstrates how to use the technique of completing the square to obtain the standard form, given the general form. Questions are provided to assess understanding.

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Instruction

- MathIsFun.com. "Definition of Parabola."

  http://www.walch.com/rr/00072

  An interactive soccer player kicks a ball that follows a parabolic path. Users can change the values of a, b, and c to obtain equations of the form $y = ax^2 + bx + c$, and see how the graph changes to match the equation. This site might be useful as a review of what has already been learned about quadratic functions and their graphs, which are parabolas.

- MathIsFun.com. "Parabola."

  http://www.walch.com/rr/00073

  This site offers a hands-on activity showing how to use the definition of a parabola to draw one. The only required materials are paper, a pencil, and a ruler. The site explains that rays parallel to the axis of symmetry are reflected from the parabola to its focus, and it lists some devices that use that property, such as a satellite dish. Users can assess understanding by answering quiz questions.

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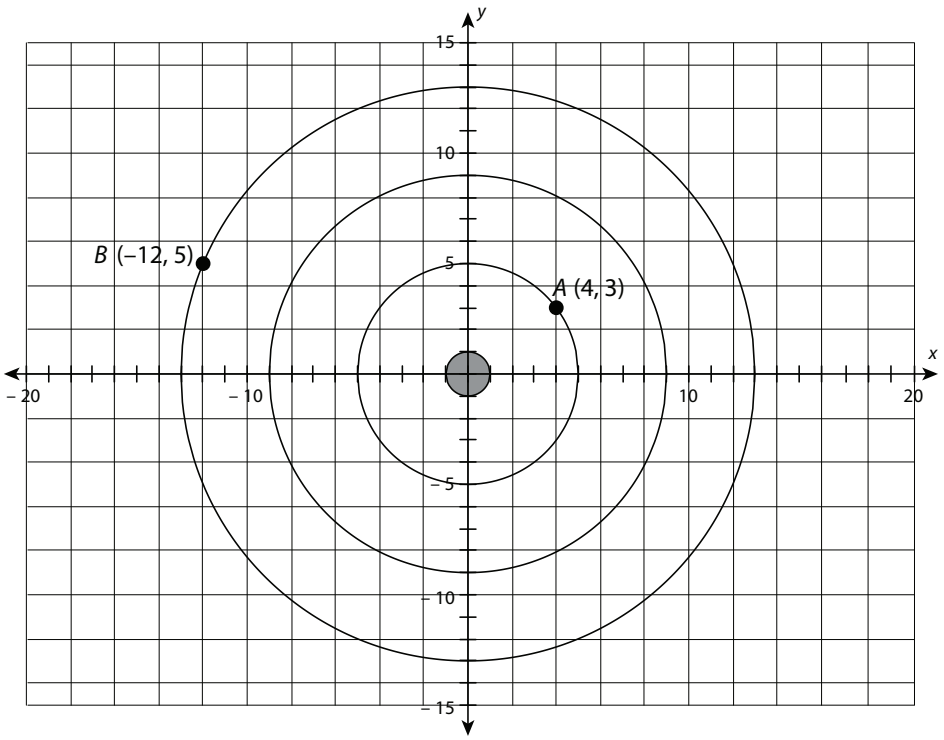
Lesson 6.6.1: Deriving the Equation of a Circle

Warm-Up 6.6.1

A video game designer created the following diagram of a target.



1. What is the radius of the circle that contains point *A*? Find this distance using the Pythagorean Theorem.

2. What is the radius of the circle that contains point *B*? Find this distance using the Pythagorean Theorem.

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Instruction

Lesson 6.6.1: Deriving the Equation of a Circle

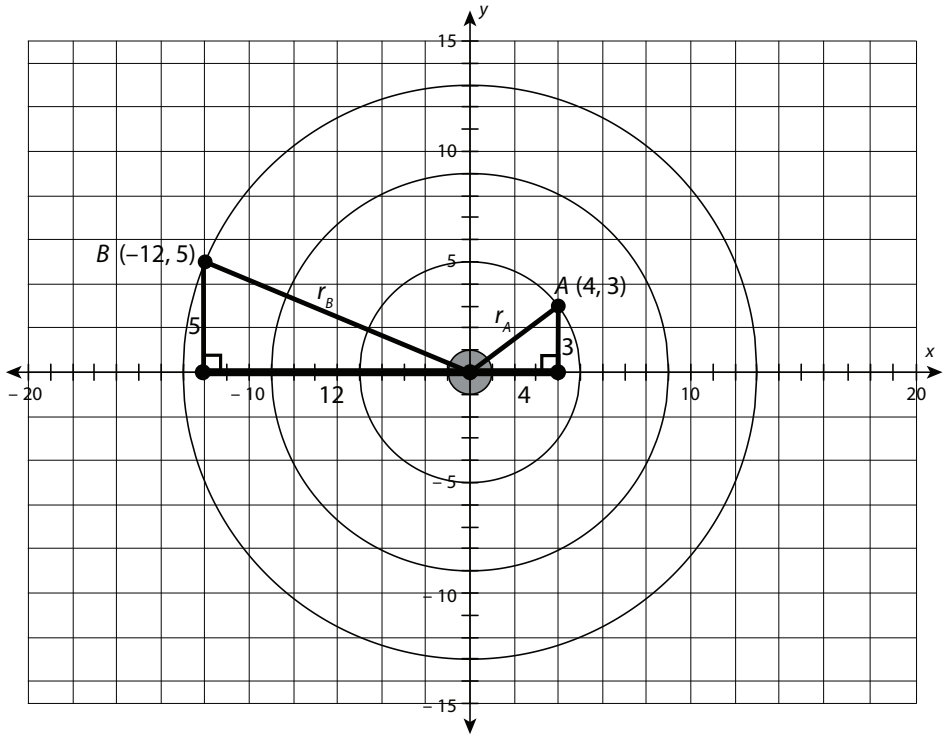
Common Core State Standard

G–GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Warm-Up 6.6.1 Debrief

1. What is the radius of the circle that contains point *A*? Find this distance using the Pythagorean Theorem.

   The following diagram depicts the triangles for both questions 1 and 2.



   $(r_A)^2 = 4^2 + 3^2$

   $(r_A)^2 = 16 + 9$

   $(r_A)^2 = 25$

   $r_A = 5$

   The radius of the circle that contains point *A* is 5.

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Instruction

2. What is the radius of the circle that contains point *B*? Find this distance using the Pythagorean Theorem.

   Refer to the diagram in the answer to the previous question and determine the radius.

   $(r_B)^2 = (-12)^2 + 5^2$

   $(r_B)^2 = 144 + 25$

   $(r_B)^2 = 169$

   $r_B = 13$

   The radius of the circle that contains point *B* is 13.

Connection to the Lesson

- Students will use right triangles and the Pythagorean Theorem to derive equations of circles.
- Students will use specific points on circles such as (4, 3) and the general point (x, y).

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- applying the Pythagorean Theorem (8.G.7)
- representing horizontal and vertical distances in a coordinate plane (6.NS.8)
- simplifying square roots (N–RN.2)
- writing equivalent forms of expressions involving squares and square roots (N–RN.2)
- writing equivalent forms of expressions involving perfect square trinomials and other polynomials (A–SSE.2)
- completing the square to form perfect square trinomials (A–REI.4a)

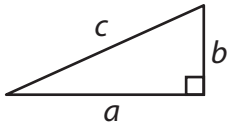
Introduction

The graph of an equation in x and y is the set of all points (x, y) in a coordinate plane that satisfy the equation. Some equations have graphs with precise geometric descriptions. For example, the graph of the equation $y = 2x + 3$ is the line with a slope of 2, passing through the point (0, 3). This geometric description uses the familiar concepts of line, slope, and point. The equation $y = 2x + 3$ is an algebraic description of the line.

In this lesson, we will investigate how to translate between geometric descriptions and algebraic descriptions of circles. We have already learned how to use the Pythagorean Theorem to find missing dimensions of right triangles. Now, we will see how the Pythagorean Theorem leads us to the distance formula, which leads us to the standard form of the equation of a circle.

Key Concepts

- The standard form of the equation of a circle is based on the distance formula.
- The distance formula, in turn, is based on the Pythagorean Theorem.
- The **Pythagorean Theorem** states that in any right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.



$$a^2 + b^2 = c^2$$

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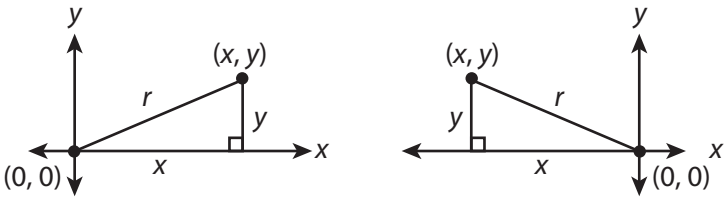
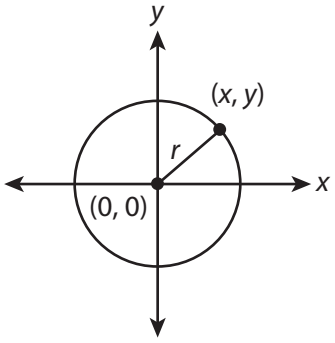
Instruction

- If r represents the distance between the origin and any point (x, y), then $x^2 + y^2 = r^2$.



  - x can be positive, negative, or zero because it is a coordinate.
  - y can be positive, negative, or zero because it is a coordinate.
  - r cannot be negative because it is a distance.
- A **circle** is the set of all points in a plane that are equidistant from a reference point in that plane, called the center. The set of points forms a 2-dimensional curve that has a central angle of 360°.
- The **center of a circle** is the point in the plane of the circle from which all points on the circle are equidistant. The center is in the interior of the circle.
- The **radius** of a circle is a line segment that extends from the center to a point on the circle. Its length is half of the diameter.
- For a circle with center (0, 0) and radius r, a point (x, y) is on the circle if and only if $x^2 + y^2 = r^2$.



- The distance formula is used to find the distance between any two points on a coordinate plane.
- The **distance formula** states that the distance d between $A(x_1, y_1)$ and $B(x_2, y_2)$ is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

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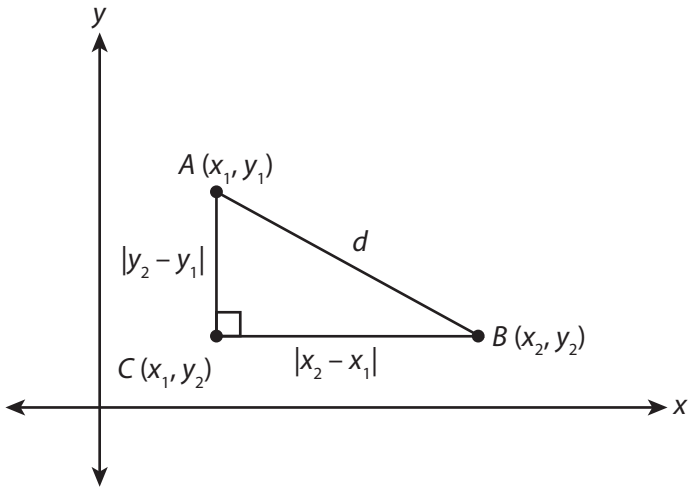
Instruction

- The distance formula is based on the Pythagorean Theorem.
- Look at this diagram of a right triangle with vertices *A*, *B,* and *C*. The distance *d* between points *A* and *B* is unknown.



- The worked example that follows shows how the distance formula is derived from the Pythagorean Theorem, using the points from the diagram to find *d*:

$AB^2 = BC^2 + AC^2$	Pythagorean Theorem
$d^2 = \lvert x_2 - x_1 \rvert^2 + \lvert y_2 - y_1 \rvert^2$	Substitute values for sides *AB*, *BC*, and *AC* of the triangle.
$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$	Simplify. All squares are nonnegative.
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Take the square root of each side of the equation to arrive at the distance formula.

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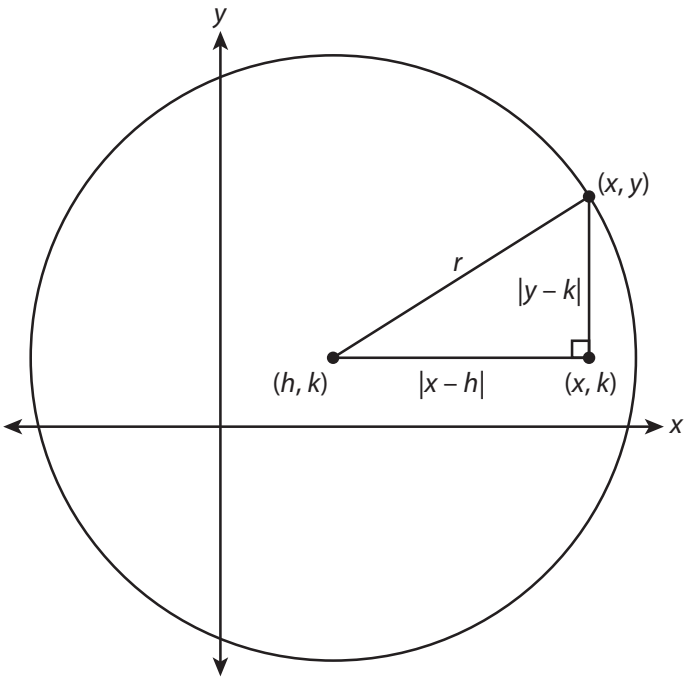
Instruction

- For a circle with center (h, k) and radius r, a point (x, y) is on the circle if and only if $\sqrt{(x-h)^2+(y-k)^2} = r$. Squaring both sides of this equation yields the **standard form of an equation of a circle** with center (h, k) and radius r: $(x - h)^2 + (y - k)^2 = r^2$.



- If a circle has center (0, 0), then its equation is $(x - 0)^2 + (y - 0)^2 = r^2$, or $x^2 + y^2 = r^2$.
- If the center and radius of a circle are known, then either of the following two methods can be used to write an equation for the circle:
  - Apply the Pythagorean Theorem to derive the equation.
  - Or, substitute the center coordinates and radius directly into the standard form.
- The **general form of an equation of a circle** is $Ax^2 + By^2 + Cx + Dy + E = 0$, where $A = B$, $A \neq 0$, and $B \neq 0$.
- If any one of the following three sets of facts about a circle is known, then the other two can be determined:
  - center (h, k) and radius r
  - standard equation: $(x - h)^2 + (y - k)^2 = r^2$
  - general equation: $Ax^2 + By^2 + Cx + Dy + E = 0$

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Instruction

- The general form of the equation of a circle comes from expanding the standard form of the equation of the circle.
- The standard form of the equation of a circle comes from completing the squares from the general form of the equation of a circle.
- Every **perfect square trinomial** has the form $x^2+bx+\left(\frac{b}{2}\right)^2$ because it is the square of a binomial: $x^2+bx+\left(\frac{b}{2}\right)^2=\left(x+\frac{b}{2}\right)^2$.
- Completing the square is the process of determining the value of $\left(\frac{b}{2}\right)^2$ and adding it to x^2+bx to form the perfect square trinomial, $x^2+bx+\left(\frac{b}{2}\right)^2$.

Common Errors/Misconceptions

- confusing the radius with the square of the radius
- forgetting to square half the coefficient of x when completing the square
- neglecting to square the denominator when squaring a fraction

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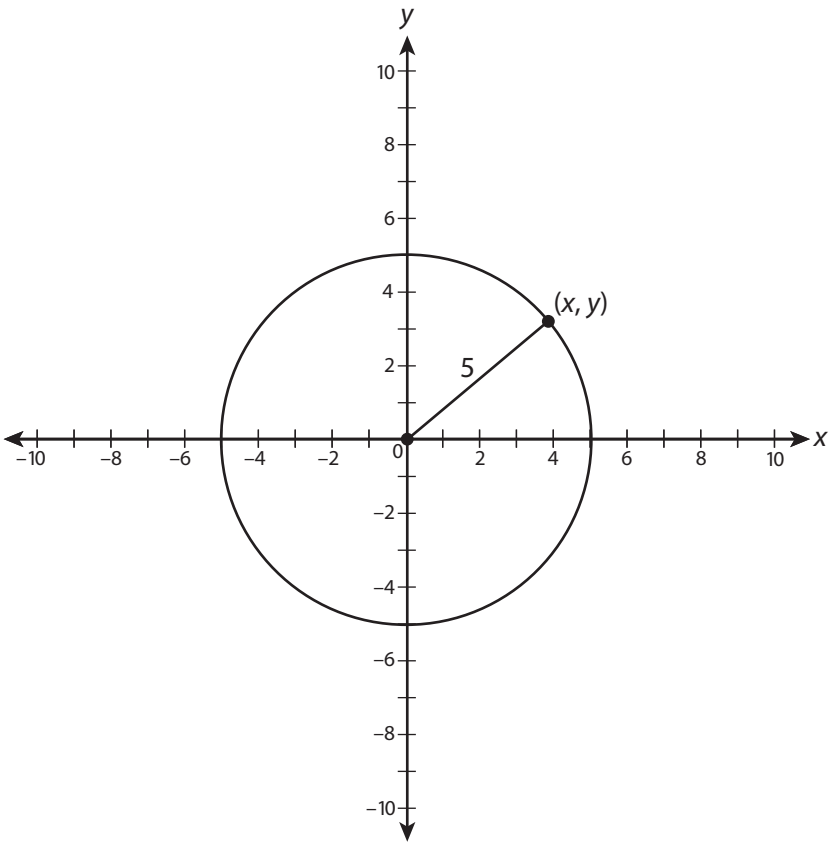
Instruction

Guided Practice 6.6.1

Example 1

Derive the standard equation of the circle with center (0, 0) and radius 5.

1. Sketch the circle and label a point (x, y) on the circle.



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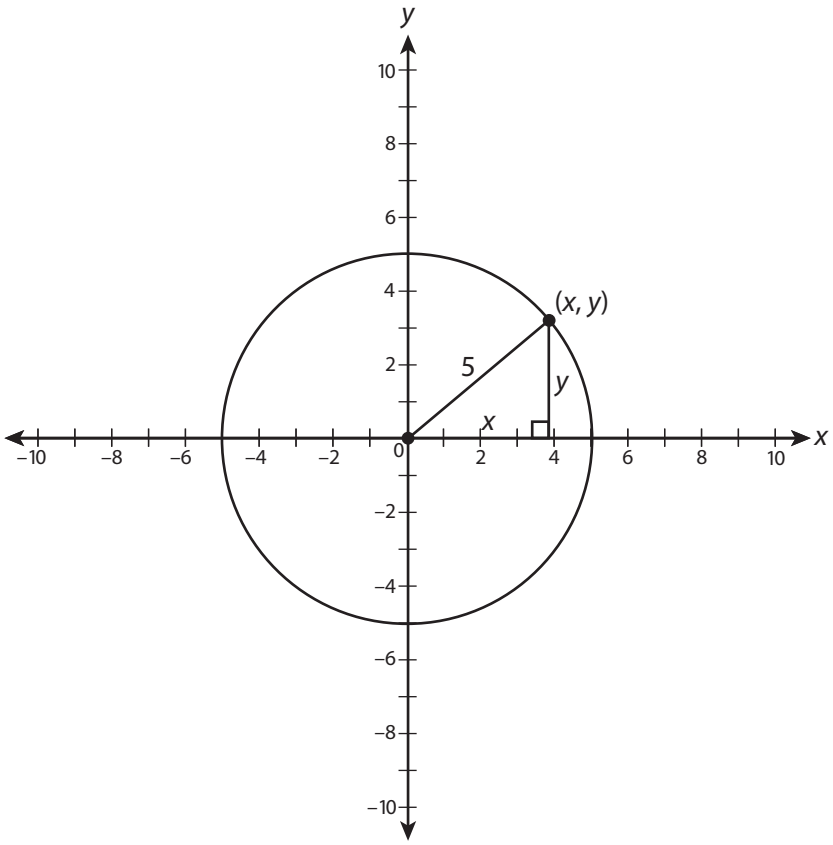
Instruction

2. Use the Pythagorean Theorem to derive the standard equation.

In order to use the Pythagorean Theorem, there must be a right triangle.

To create a right triangle, draw a line from point (x, y) that is perpendicular to the x-axis. Label the resulting sides of the triangle x and y.



Substitute the values for each side of the triangle into the formula for the Pythagorean Theorem, $a^2 + b^2 + c^2$.

$a^2 + b^2 + c^2$	Pythagorean Theorem
$x^2 + y^2 = 5^2$	Substitute values from the triangle.
$x^2 + y^2 = 25$	Simplify.

The standard equation of this circle is $x^2 + y^2 = 25$.

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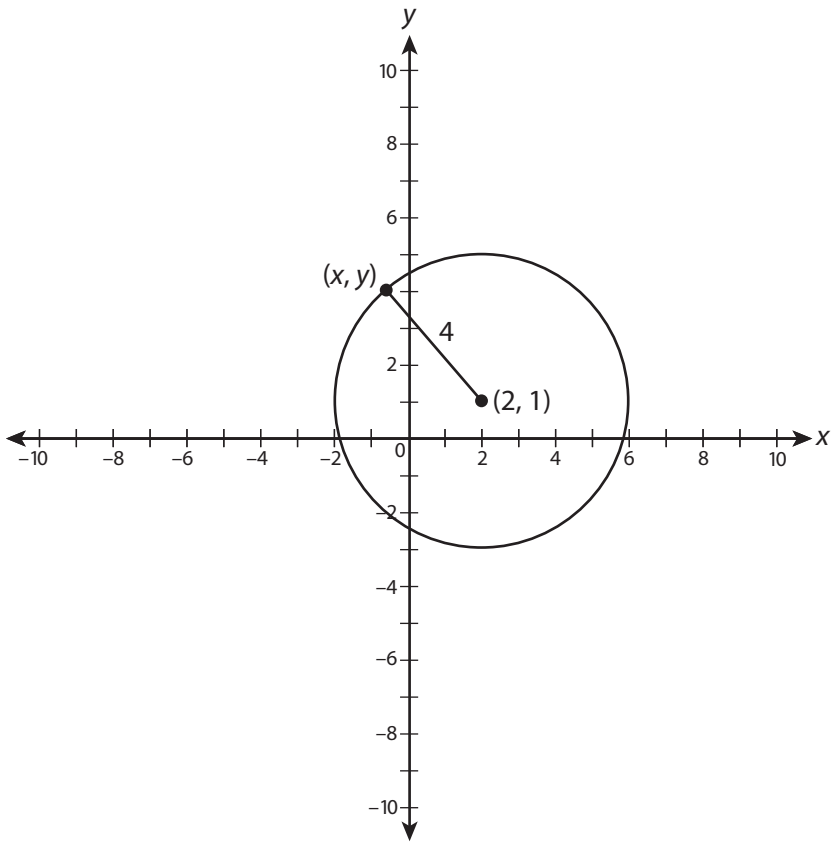
Instruction

Example 2

Derive the standard equation of the circle with center (2, 1) and radius 4. Then use a graphing calculator to graph your equation.

1. Sketch the circle and label a point on the circle (x, y).



2. Use the Pythagorean Theorem to derive the standard equation.

   Create a right triangle. Draw a vertical segment down from point (x, y) and a horizontal segment left from point (2, 1) to create the triangle.

   The length of the base of the triangle is equal to the absolute value of the difference of the x-coordinates of the endpoints.

   The height of the triangle is equal to the absolute value of the difference of the y-coordinates of the endpoints.

(*continued*)

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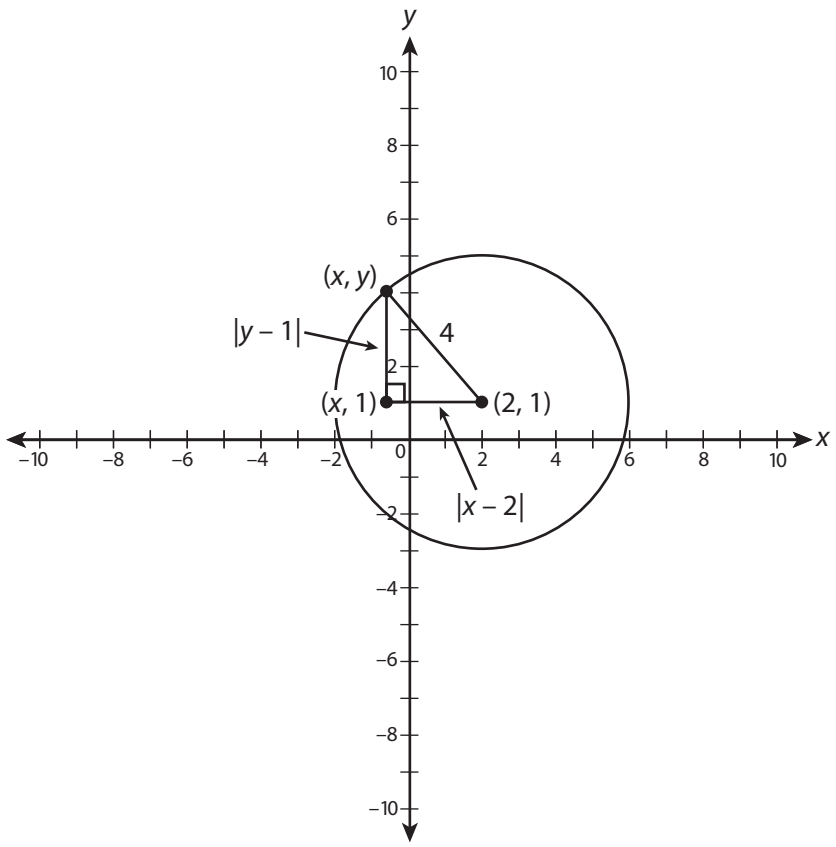
Instruction



Substitute the resulting values for the sides of the triangle into the Pythagorean Theorem.

$a^2 + b^2 + c^2$	Pythagorean Theorem
$\lvert x-2\rvert^2 + \lvert y-1\rvert^2 = 4^2$	Substitute values from the triangle.
$(x-2)^2 + (y-1)^2 = 4^2$	All squares are nonnegative, so replace the absolute value symbols with parentheses.
$(x-2)^2 + (y-1)^2 = 16$	Simplify.

The standard equation of this circle is $(x-2)^2 + (y-1)^2 = 16$.

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Instruction

3. Solve the standard equation for y to obtain functions that can be graphed.

$(x-2)^2+(y-1)^2=16$	Standard equation
$(y-1)^2=16-(x-2)^2$	Subtract $(x-2)^2$ from both sides.
$y-1=\pm\sqrt{16-(x-2)^2}$	If $a^2=b^2$, then $a=\pm\sqrt{b}$.
$y=1\pm\sqrt{16-(x-2)^2}$	Add 1 to both sides to solve for y.

4. Now graph the two functions, $y=1+\sqrt{16-(x-2)^2}$ and $y=1-\sqrt{16-(x-2)^2}$.

On a TI-83/84:

Step 1: Press [Y=].

Step 2: At Y_1, type in [1][+][$\sqrt{\ }$][16][–][(][X, T, θ, n][–][2][)][x^2][)].

Step 3: At Y_2, type in [1][–][$\sqrt{\ }$][16][–][(][X, T, θ, n][–][2][)][x^2][)].

Step 4: Press [WINDOW] to change the viewing window.

Step 5: At Xmin, enter [(–)][9].

Step 6: At Xmax, enter [9].

Step 7: At Xscl, enter [1].

Step 8: At Ymin, enter [(–)][6].

Step 9: At Ymax, enter [6].

Step 10: At Yscl, enter [1].

Step 11: Press [GRAPH].

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow to the graphing icon and press [enter].

Step 3: At the blinking cursor at the bottom of the screen, type [1][+][$\sqrt{\ }$][16][–][(][x][–][2][)][x^2][enter].

(*continued*)

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Instruction

Step 4: Move the cursor to the bottom left of the screen and click on the double right-facing arrows.

Step 5: At the blinking cursor, type [1][–][$\sqrt{\ }$][16][–][(][x][–][2][)][x^2][)][enter].

Step 6: Change the viewing window by pressing [menu], using the arrows to navigate down to number 4: Window/Zoom, and clicking the center button of the navigation pad.

Step 7: Choose 1: Window settings by pressing the center button.

Step 8: Enter in an appropriate XMin value, –9, by pressing [(–)] and [9], then press [tab].

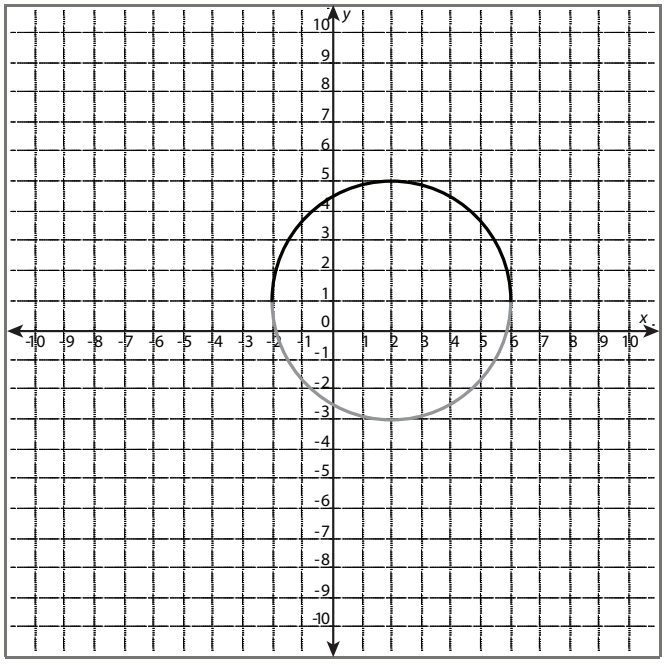
Step 9: Enter in an appropriate XMax value, [9], then press [tab].

Step 10: Leave the XScale set to "Auto." Press [tab] twice to navigate to YMin and enter an appropriate YMin value, –6, by pressing [(–)] and [6].

Step 11: Press [tab] to navigate to YMax. Enter [6]. Press [tab] twice to leave YScale set to "Auto" and to navigate to "OK."

Step 12: Press [enter].

Step 13: Press [menu] and select 2: View and 5: Show Grid.



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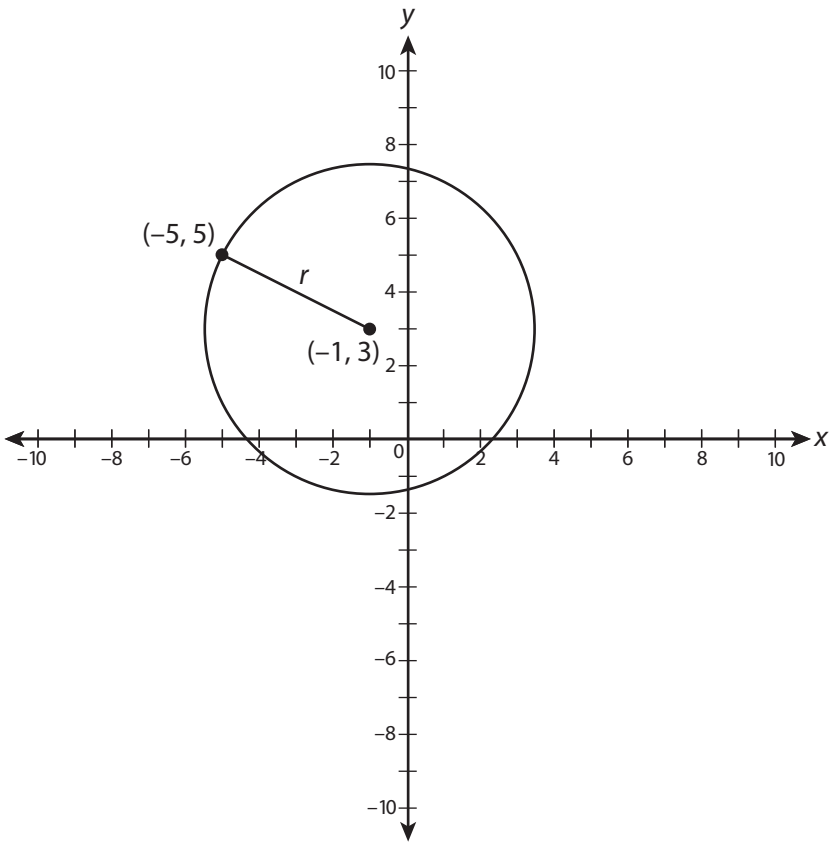
Instruction

Example 3

Write the standard equation and the general equation of the circle that has center (–1, 3) and passes through (–5, 5).

1. Sketch the circle.



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Instruction

2. Use the distance formula to find the radius, r.

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	Distance formula
$r=\sqrt{[(-5)-(-1)]^2+(5-3)^2}$	Substitute (–1, 3) and (–5, 5) for (x_1, y_1) and (x_2, y_2).
$r=\sqrt{(-4)^2+(2)^2}$	Simplify.
$r=\sqrt{16+4}$	
$r=\sqrt{20}$	
$r=\sqrt{4\bullet5}$	Write 20 as a product with a perfect square factor.
$r=\sqrt{4}\bullet\sqrt{5}$	Apply the property $\sqrt{ab}=\sqrt{a}\bullet\sqrt{b}$.
$r=2\sqrt{5}$	Simplify.

3. Substitute the center and radius directly into the standard form.

$(x-h)^2+(y-k)^2=r^2$	Standard form
$[x-(-1)]^2+(y-3)^2=(2\sqrt{5})^2$	Substitute values into the equation, using the center (–1, 3), and the radius $2\sqrt{5}$.
$(x+1)^2+(y-3)^2=20$	Simplify to obtain the standard equation.

The standard equation is $(x+1)^2+(y-3)^2=20$.

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Instruction

4. Square the binomials and rearrange terms to obtain the general form.

$(x + 1)^2 + (y - 3)^2 = 20$	Standard equation
$(x + 1)(x + 1) + (y - 3)(y - 3) = 20$	Expand the factors.
$x^2 + 2x + 1 + y^2 - 6y + 9 = 20$	Square the binomials to obtain trinomials.
$x^2 + 2x + y^2 - 6y + 10 = 20$	Combine the constant terms on the left side of the equation.
$x^2 + 2x + y^2 - 6y - 10 = 0$	Subtract 20 from both sides to get 0 on the right side.
$x^2 + y^2 + 2x - 6y - 10 = 0$	Rearrange terms in descending order to obtain the general equation.

The general equation is $x^2 + y^2 + 2x - 6y - 10 = 0$.

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Instruction

Example 4

Find the center and radius of the circle described by the equation $x^2 + y^2 - 8x + 2y + 2 = 0$.

1. Rewrite the equation in standard form.

$x^2 + y^2 - 8x + 2y + 2 = 0$	General form of the equation
$x^2 + y^2 - 8x + 2y = -2$	Subtract 2 from both sides to get the constant term on one side.
$x^2 - 8x + y^2 + 2y = -2$	Group same-variable terms.

Next, complete the square for both variables. Add the same values to both sides of the equation as shown:

$$x^2 - 8x + \left(\frac{-8}{2}\right)^2 + y^2 + 2y + \left(\frac{2}{2}\right)^2 = -2 + \left(\frac{-8}{2}\right)^2 + \left(\frac{2}{2}\right)^2$$

$x^2 - 8x + 16 + y^2 + 2y + 1 = -2 + 16 + 1$	Simplify the equation shown above.
$(x - 4)^2 + (y + 1)^2 = 15$	Write the perfect square trinomials as squares of binomials.

The standard equation is $(x - 4)^2 + (y + 1)^2 = 15$.

2. Determine the center and radius.

$(x - 4)^2 + (y + 1)^2 = 15$	Write the standard equation from step 1.
$(x-4)^2 + [y-(-1)]^2 = (\sqrt{15})^2$	Rewrite to match the form $(x - h)^2 + (y - k)^2 = r^2$.

For the equation $(x - h)^2 + (y - k)^2 = r^2$, the center is (h, k) and the radius is r, so for the equation $(x-4)^2 + [y-(-1)]^2 = (\sqrt{15})^2$, the center is $(4, -1)$ and the radius is $\sqrt{15}$.

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Instruction

Example 5

Find the center and radius of the circle described by the equation $4x^2 + 4y^2 + 20x - 40y + 116 = 0$.

1. Rewrite the equation in standard form.

$4x^2 + 4y^2 + 20x - 40y + 116 = 0$	General form of the equation
$x^2 + y^2 + 5x - 10y + 29 = 0$	Divide each term on both sides by 4 to make the leading coefficient 1.
$x^2 + y^2 + 5x - 10y = -29$	Subtract 29 from both sides to get the constant term on one side.
$x^2 + 5x + y^2 - 10y = -29$	Group same-variable terms.

Next, complete the square for both variables. Add the same values to both sides of the equation, as shown:

$$x^2 + 5x + \left(\frac{5}{2}\right)^2 + y^2 - 10y + \left(\frac{-10}{2}\right)^2 = -29 + \left(\frac{5}{2}\right)^2 + \left(\frac{-10}{2}\right)^2$$

$x^2 + 5x + \frac{25}{4} + y^2 - 10y + 25 = -29 + \frac{25}{4} + 25$	Simplify the equation from above.
$\left(x + \frac{5}{2}\right)^2 + (y - 5)^2 = \frac{9}{4}$	Write the perfect square trinomials as squares of binomials.

The standard equation is $\left(x + \frac{5}{2}\right)^2 + (y - 5)^2 = \frac{9}{4}$.

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Instruction

2. Determine the center and radius.

$\left(x+\frac{5}{2}\right)^2+(y-5)^2=\frac{9}{4}$ Write the standard equation from step 1.

$\left[x-\left(-\frac{5}{2}\right)\right]^2+(y-5)^2=\left(\frac{3}{2}\right)^2$ Rewrite to match the form $(x-h)^2+(y-k)^2=r^2$.

For the equation $(x-h)^2+(y-k)^2=r^2$, the center is (h, k) and the radius is r, so for the equation $\left[x-\left(-\frac{5}{2}\right)\right]^2+(y-5)^2=\left(\frac{3}{2}\right)^2$, the center is $\left(-\frac{5}{2}, 5\right)$ and the radius is $\frac{3}{2}$.

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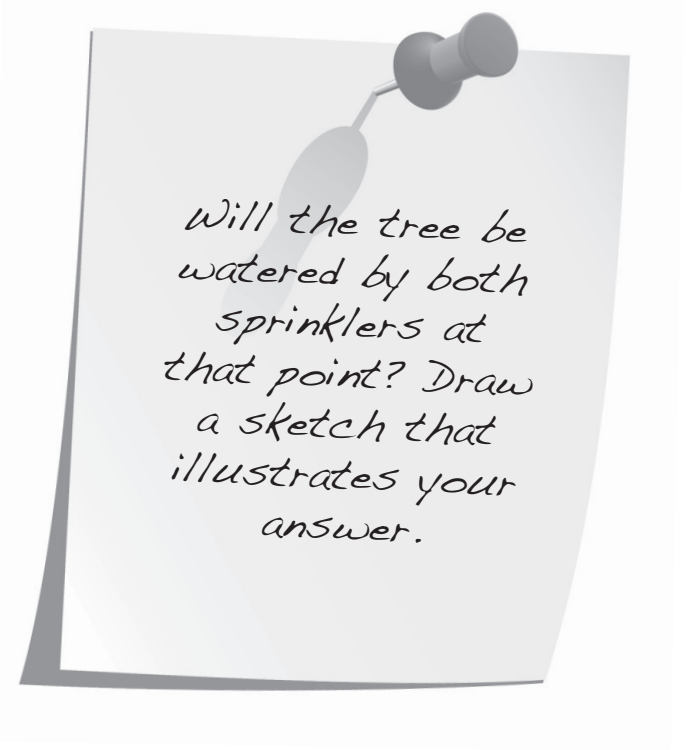
Problem-Based Task 6.6.1: Nurturing an Investment

SMP	
1	2 ✓
3	4 ✓
5	6
7 ✓	8

Anna's landscaping company has a contract to improve and maintain a municipal park. Anna made a scale drawing of the park on a coordinate system, using meters as the unit of distance. She has already installed two permanent sprinkler outlets. Sprinkler 1 waters inside the region whose boundary has the equation $x^2 + y^2 - 20x - 20y + 136 = 0$. Sprinkler 2 waters inside the region whose boundary has the equation $x^2 + y^2 - 50x - 24y + 669 = 0$. Anna bought an expensive tree and she wants to plant it at the point (17, 8), where she thinks it will be watered by both sprinklers. Will the tree be watered by both sprinklers at that point? Draw a sketch that illustrates your answer.



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Problem-Based Task 6.6.1: Nurturing an Investment

Coaching

a. What geometric figures are represented by the given equations? How do you know?

b. What facts do you need to know in order to determine the regions watered by the sprinklers?

c. What form of equation do you need to find the facts referenced in part b?

d. What are the main steps needed to obtain the form referenced in part c?

e. For sprinkler 1, apply the necessary steps to obtain the form referenced in parts c and d.

f. What is the geometric description of the region watered by sprinkler 1?

g. What is the geometric description of the region watered by sprinkler 2?

h. What is the distance between sprinkler 1 and the planned location of the tree?

i. Will the tree be watered by sprinkler 1? Explain.

j. Will the tree be watered by sprinkler 2? Explain.

k. Will the tree be watered by both sprinklers? Draw a sketch that illustrates your answer.

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Instruction

Problem-Based Task 6.6.1: Nurturing an Investment

Coaching Sample Responses

a. What geometric figures are represented by the given equations? How do you know?

The figures are circles. The equations are in the general form of an equation of a circle.

b. What facts do you need to know in order to determine the regions watered by the sprinklers?

You need to know the center and radius of each circle.

c. What form of equation do you need to find the facts referenced in part b?

You need the standard form of an equation of a circle.

d. What are the main steps needed to obtain the form referenced in part c?

For each given equation, add or subtract as needed from both sides to get the constant term on one side. Group same-variable terms. Complete the square for both variables, making sure to add the same values to both sides of the equation. Write the perfect square trinomials as squares of binomials.

e. For sprinkler 1, apply the necessary steps to obtain the form referenced in parts c and d.

$x^2 + y^2 - 20x - 20y + 136 = 0$

$x^2 + y^2 - 20x - 20y = -136$

$x^2 - 20x + y^2 - 20y = -136$

Next, complete the square for both variables. Add the same values to both sides of the equation, as shown below:

$$x^2 - 20x + \left(\frac{-20}{2}\right)^2 + y^2 - 20y + \left(\frac{-20}{2}\right)^2 = -136 + \left(\frac{-20}{2}\right)^2 + \left(\frac{-20}{2}\right)^2$$

$x^2 - 20x + 100 + y^2 - 20y + 100 = -136 + 100 + 100$

$(x - 10)^2 + (y - 10)^2 = 64$

The standard equation is $(x - 10)^2 + (y - 10)^2 = 64$.

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Instruction

f. What is the geometric description of the region watered by sprinkler 1?

Use the standard equation to determine the center and radius of the region watered by sprinkler 1. Sprinkler 1 waters the interior of the circle with center (10, 10) and radius 8 meters.

g. What is the geometric description of the region watered by sprinkler 2?

Convert the general equation to standard form, and then determine the center and radius of the region watered by sprinkler 2.

$x^2 + y^2 - 50x - 24y + 669 = 0$

$x^2 + y^2 - 50x - 24y = -669$

$x^2 - 50x + y^2 - 24y = -669$

Next, complete the square for both variables. Add the same values to both sides of the equation, as shown:

$$x^2 - 50x + \left(\frac{-50}{2}\right)^2 + y^2 - 24y + \left(\frac{-24}{2}\right)^2 = -669 + \left(\frac{-50}{2}\right)^2 + \left(\frac{-24}{2}\right)^2$$

$x^2 - 50x + 625 + y^2 - 24y + 144 = -669 + 625 + 144$

$(x - 25)^2 + (y - 12)^2 = 100$

Sprinkler 2 waters the interior of the circle with center (25, 12) and radius 10 meters.

h. What is the distance between sprinkler 1 and the planned location of the tree?

Use the distance formula, substituting points (10, 10) and (17, 8) for (x_1, y_1) and (x_2, y_2).

$d = \sqrt{(17-10)^2 + (8-10)^2}$

$d = \sqrt{49+4}$

$d = \sqrt{53}$ meters

i. Will the tree be watered by sprinkler 1? Explain.

Yes; the radius of the circular region watered by sprinkler 1 is 8 meters, or $\sqrt{64}$ meters. The tree will be $\sqrt{53}$ meters from the sprinkler, which is less than the radius.

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Instruction

j. Will the tree be watered by sprinkler 2? Explain.

Use the distance formula, substituting points (25, 12) and (17, 8) for (x_1, y_1) and (x_2, y_2).

$$d = \sqrt{(17-25)^2 + (8-12)^2}$$

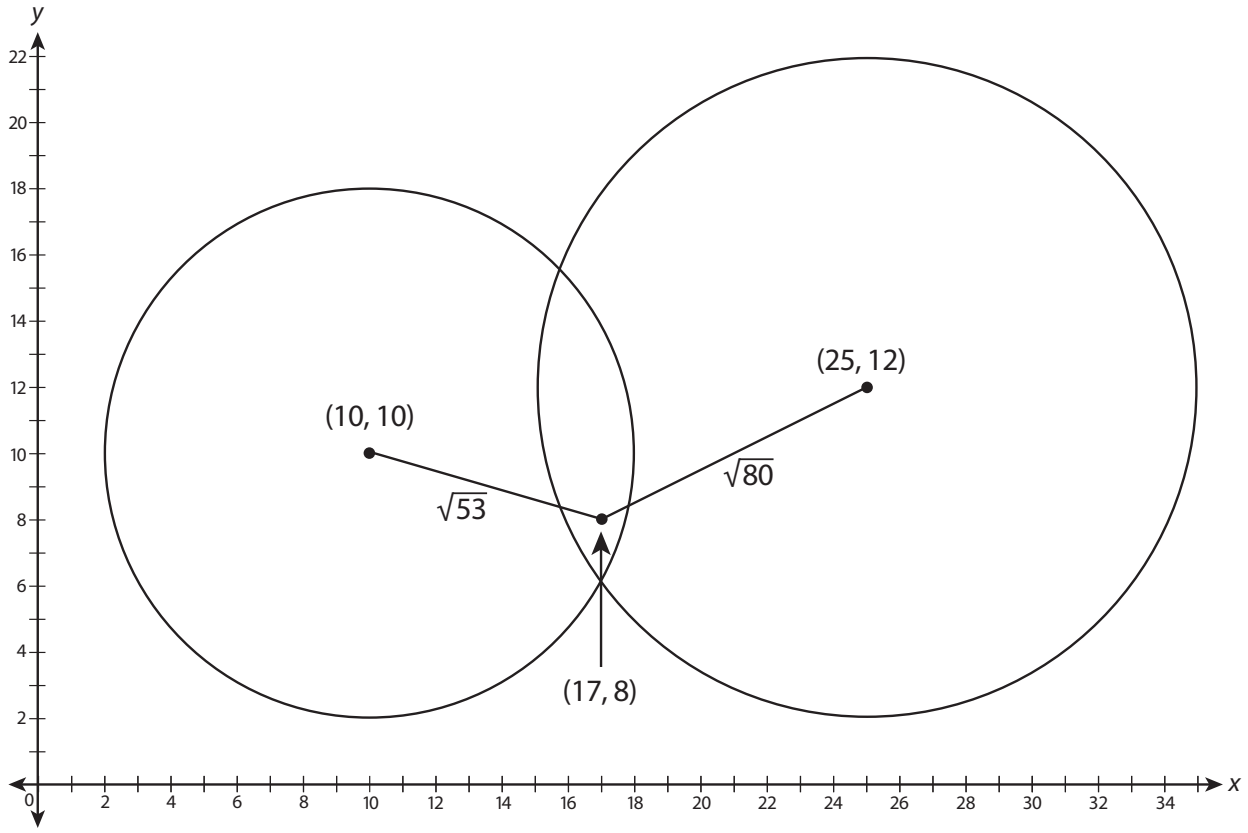
$$d = \sqrt{64+16}$$

$$d = \sqrt{80} \text{ meters}$$

Yes; the radius of the circular region watered by sprinkler 2 is 10 meters, or $\sqrt{100}$ meters. The tree will be $\sqrt{80}$ meters from the sprinkler, which is less than the radius.

k. Will the tree be watered by both sprinklers? Draw a sketch that illustrates your answer.

Yes, the tree will be watered by both sprinklers. Sample sketch:



Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 6.6.1: Deriving the Equation of a Circle

For problems 1–4, write the standard equation of the circle described.

1. The center is (0, 0) and the radius is 9.

2. The center is (–5, 1) and the radius is 4.

3. The center is (0, –3.2) and the radius is 2.2.

4. The center is (–2, 2) and the circle passes through (1, –3).

Use the provided information in each problem that follows to solve.

5. Write the general equation of the circle with center (5, 0) and radius $3\sqrt{2}$.

6. Find the center and radius of the circle described by the equation $x^2 + y^2 + 6x - 4y - 27 = 0$.

7. Find the center and radius of the circle described by the equation $2x^2 + 2y^2 + 2x - 6y + \frac{9}{2} = 0$.

8. A particular cell phone tower is designed to service a 12-mile radius. The tower is located at (–3, 5) on a coordinate plane whose units represent miles. What is the standard equation of the outer boundary of the region serviced by the tower? Is a cell phone user at (8, 0) within the service range? Explain.

9. A pizza restaurant will deliver up to 5 miles. The restaurant is located at the origin on a coordinate plane whose units represent miles. What is the standard equation of the outer boundary of the delivery region? Customers are located at A (4, 3), B (5, 0), and $C\left(2, \sqrt{21}\right)$. Which of these customers, if any, are on the outer boundary? Explain.

10. Marco is a park ranger stationed in a fire tower. The tower is on a coordinate plane whose units represent miles. Marco is responsible for monitoring a region whose boundary has the equation $x^2 + y^2 + 3x + y - 6.5 = 0$. What is the geometric description of the region for which Marco is responsible?

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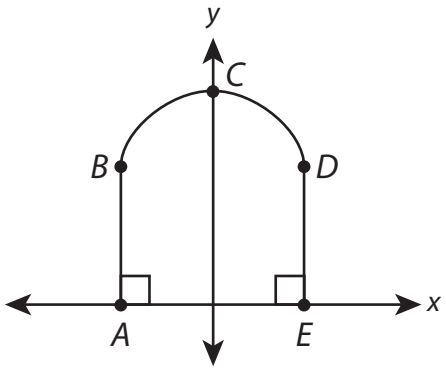
Lesson 6.6.2: Deriving the Equation of a Parabola

Warm-Up 6.6.2

The diagram below shows a window on a coordinate plane with feet as the unit of distance. The width of the window is 4 feet. Points *B*, *C*, and *D* follow the equation $y = -\frac{1}{4}x^2 + 6$.



1. What are the coordinates of point *C*? Explain.

2. What are the coordinates of points *A*, *B*, *D*, and *E*? Explain.

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Instruction

Lesson 6.6.2: Deriving the Equation of a Parabola

Common Core State Standard

G–GPE.2 Derive the equation of a parabola given a focus and directrix.

Warm-Up 6.6.2 Debrief

1. What are the coordinates of point *C*? Explain.

   Point *C* is at (0, 6). *C* is on the *y*-axis, so its *x*-coordinate has to be 0. To find the *y*-coordinate, substitute 0 for *x* in the equation of the parabola: $y=-\frac{1}{4}(0)^2+6=6$.

2. What are the coordinates of points *A*, *B*, *D*, and *E*? Explain.

   A is at (–2, 0), *B* is at (–2, 5), *D* is at (2, 5), and *E* is at (2, 0). Since the width of the window is given as 4 feet and the window is symmetric about the *y*-axis, the *x*-coordinates of *A* and *B* are –2 and the *x*-coordinates of *D* and *E* are 2. Points *A* and *E* are on the *x*-axis, so each has a *y*-coordinate of 0. To find the *y*-coordinate of *B*, substitute –2 into the equation of the parabola: $y=-\frac{1}{4}(-2)^2+6=-1+6=5$. To find the *y*-coordinate of *D*, substitute 2 into the equation of the parabola: $y=-\frac{1}{4}(2)^2+6=-1+6=5$.

Connection to the Lesson

- Students will derive equations of parabolas, but in a different form than that of $y=-\frac{1}{4}x^2+6$.
- Students will substitute known coordinates into equations to find unknown coordinates.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- graphing one-variable linear equations (A–CED.2★)
- applying the distance formula (8.G.8)
- simplifying expressions involving squares of binomials (A–APR.1)
- calculating horizontal and vertical distances in a coordinate plane (6.NS.8)
- solving quadratic equations by identifying square roots (A–REI.4b)

Introduction

Earlier we studied the circle, which is the set of all points in a plane that are equidistant from a given point in that plane. We have investigated how to translate between geometric descriptions and algebraic descriptions of circles. Now we will investigate how to translate between geometric descriptions and algebraic descriptions of parabolas.

Key Concepts

- A **quadratic function** is a function that can be written in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$.
- The graph of any quadratic function is a parabola that opens up or down.
- A **parabola** is the set of all points that are equidistant from a given fixed point and a given fixed line that are both in the same plane as the parabola.
- That given fixed line is called the **directrix** of the parabola.
- The fixed point is called the **focus**.
- The parabola, directrix, and focus are all in the same plane.

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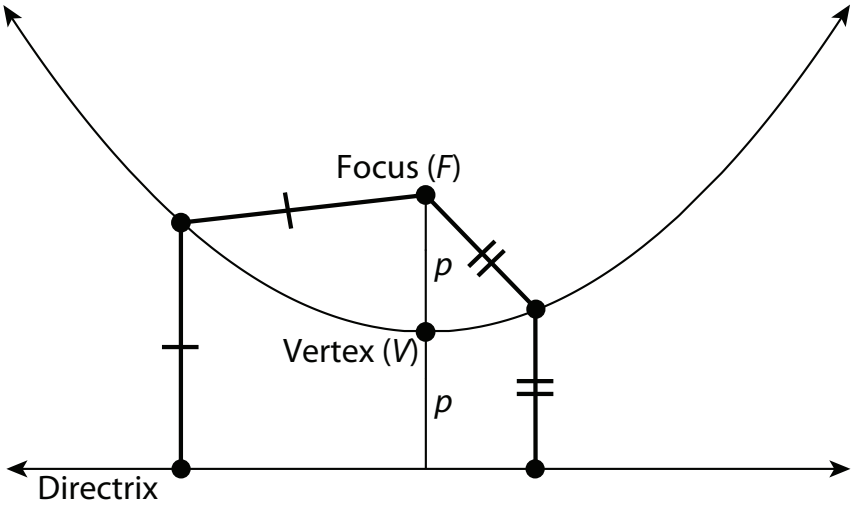
Instruction

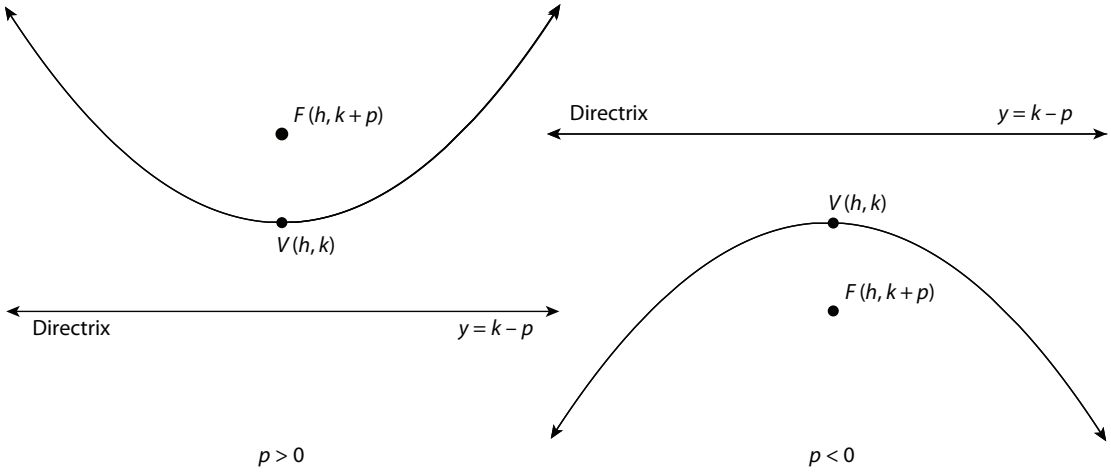
- The **vertex** of the parabola is the point on the parabola that is closest to the directrix.



- Parabolas can open in any direction. In this lesson, we will work with parabolas that open up, down, right, and left.
- As with circles, there is a **standard form for the equation of a parabola**; however, that equation differs depending on which direction the parabola opens (right/left or up/down).

Parabolas That Open Up or Down

- The standard form of an equation of a parabola that opens up or down and has vertex (h, k) is $(x - h)^2 = 4p(y - k)$, where $p \neq 0$.



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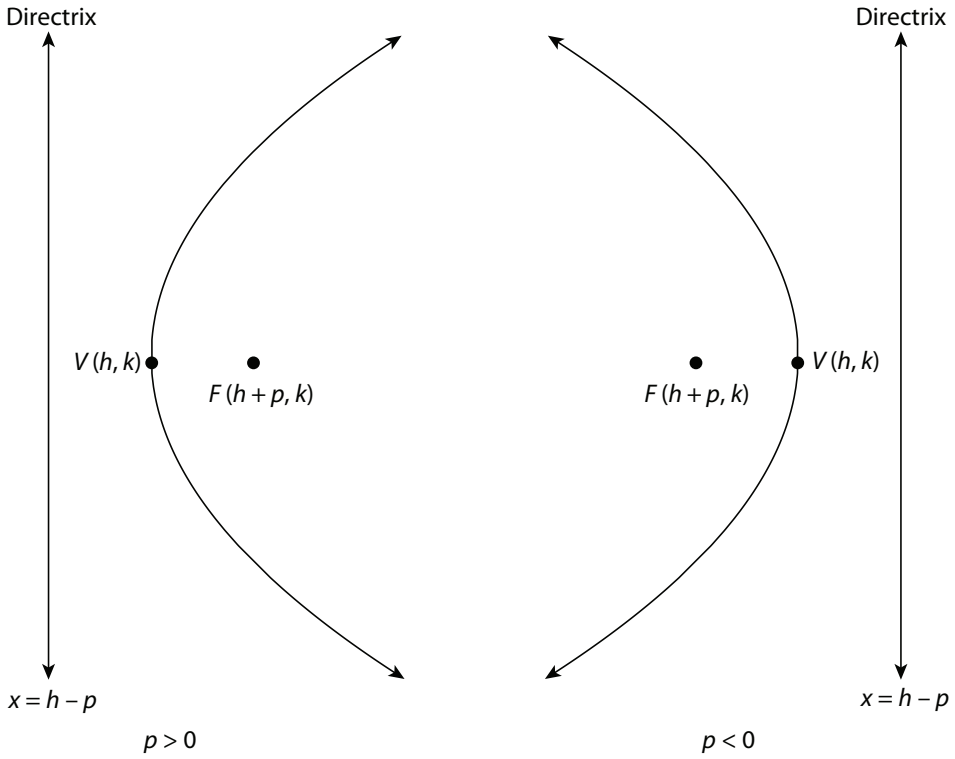
Parabolas That Open Right or Left

- The standard form of an equation of a parabola that opens right or left and has vertex (h, k) is $(y - k)^2 = 4p(x - h)$, where $p \neq 0$.



All Parabolas

- For any parabola, the focus and directrix are each $|p|$ units from the vertex.
- Also, the focus and directrix are $2|p|$ units from each other.
- If the vertex is (0, 0), then the standard equation of the parabola has a simple form.
  - $(x - 0)^2 = 4p(y - 0)$ is equivalent to the simpler form $x^2 = 4py$.
  - $(y - 0)^2 = 4p(x - 0)$ is equivalent to the simpler form $y^2 = 4px$.
- Either of the following two methods can be used to write an equation of a parabola:
  - Apply the geometric definition to derive the equation.
  - Or, substitute the vertex coordinates and the value of p directly into the standard form.

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Instruction

Common Errors/Misconceptions

- confusing vertical and horizontal when graphing one-variable linear equations
- subtracting incorrectly when negative numbers are involved
- using the wrong sign for the last term when writing the result of squaring a binomial
- omitting the middle term when writing the result of squaring a binomial
- using the incorrect standard form of the equation based on the opening direction of the parabola

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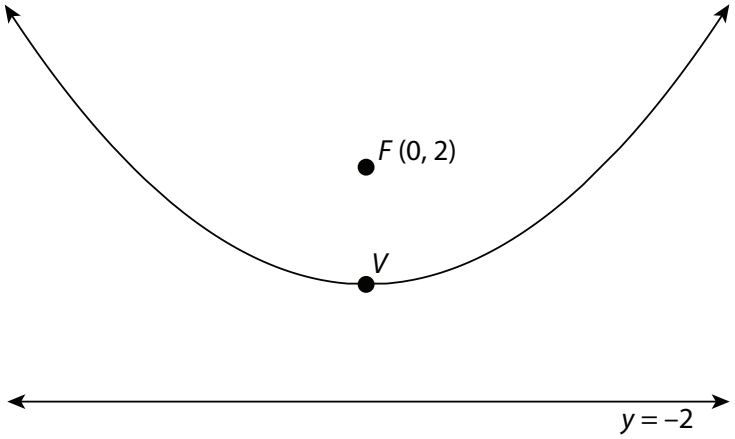
Guided Practice 6.6.2

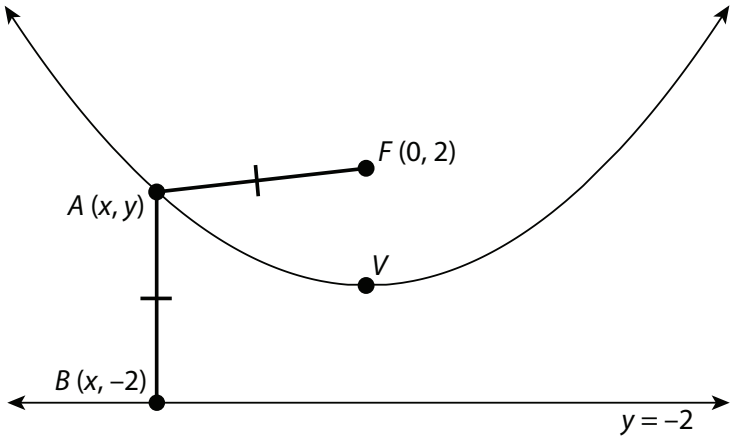
Example 1

Derive the standard equation of the parabola with focus (0, 2) and directrix $y = -2$ from the definition of a parabola. Then write the equation by substituting the vertex coordinates and the value of p directly into the standard form.

1. To derive the equation, begin by plotting the focus. Label it F (0, 2). Graph the directrix and label it $y = -2$. Sketch the parabola. Label the vertex V.



2. Let A (x, y) be any point on the parabola.



Point A is equidistant from the focus and the directrix. The distance from A to the directrix is the vertical distance AB, where B is on the directrix directly below A. Since the directrix is at $y = -2$, the y-coordinate of B is –2. Because B is directly below A, it has the same x-coordinate as A. So B has coordinates $(x, -2)$.

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Instruction

3. Apply the definition of a parabola to derive the standard equation using the distance formula.

Since the definition of a parabola tells us that $AF = AB$, use the graphed points for AF and AB to apply the distance formula to this equation.

$$\sqrt{(x-0)^2+(y-2)^2} = \sqrt{(x-x)^2+[y-(-2)]^2}$$

$\sqrt{x^2+(y-2)^2} = \sqrt{(y+2)^2}$	Simplify.
$x^2 + (y-2)^2 = (y+2)^2$	Square both sides to yield an equivalent equation. (See note.)
$x^2 = (y+2)^2 - (y-2)^2$	Subtract $(y-2)^2$ from both sides to get all x terms on one side and all y terms on the other side.
$x^2 = (y^2+4y+4) - (y^2-4y+4)$	Square the binomials.
$x^2 = y^2+4y+4-y^2+4y-4$	Distribute the negative sign.
$x^2 = 8y$	Simplify.

The standard equation is $x^2 = 8y$, or $(x-0)^2 = 8(y-0)$.

Note: Squaring both sides of an equation sometimes does not yield an equivalent equation. For example, the only solution to $2x = 6$ is 3, but squaring both sides of that equation yields $4x^2 = 36$, which has two solutions: 3 and –3. Note that if –3 is substituted for x in the equation $2x = 6$, the value of the left side is negative. However, every ordered pair (x, y) that satisfies $x^2 + (y-2)^2 = (y+2)^2$ also satisfies $\sqrt{x^2+(y-2)^2} = \sqrt{(y+2)^2}$ and vice versa because all terms in both equations are squared terms and therefore nonnegative. Note that if any negative value is substituted for x or y in these equations, both sides of the equation remain nonnegative.

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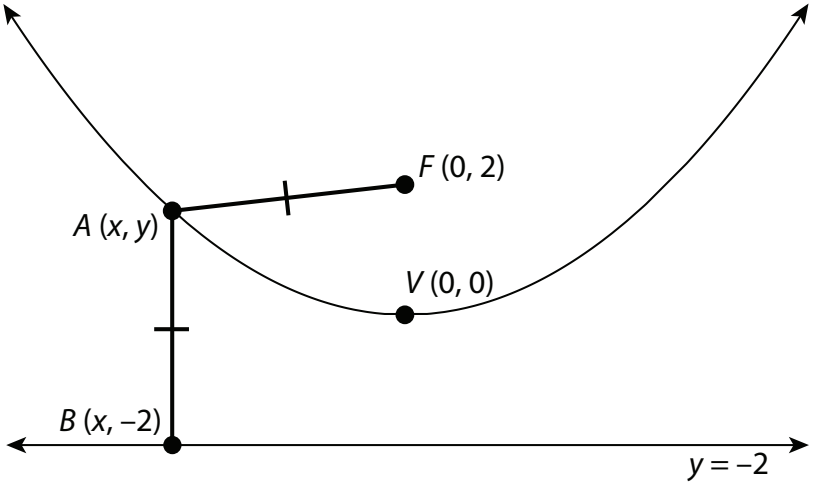
Instruction

4. To write the equation using the standard form, first determine the coordinates of the vertex and the value of p.



For any parabola, the distance between the focus and the directrix is $2|p|$. In this case, $2|p| = 2-(-2) = 4$, thus $|p| = 2$. The parabola opens up, so p is positive, and therefore $p = 2$. For any parabola, the distance between the focus and the vertex is $|p|$. Since $|p| = 2$, the vertex is 2 units below the focus. Therefore, the vertex coordinates are (0, 0).

5. Use the results found in step 4 to write the equation.

$(x-h)^2 = 4p(y-k)$	Standard form for a parabola that opens up or down
$(x-0)^2 = 4(2)(y-0)$	Substitute the values for h, p, and k.
$x^2 = 8y$	Simplify.

The standard equation is $x^2 = 8y$, or $(x-0)^2 = 8(y-0)$.

The results found in steps 3 and 5 match; so, either method of finding the equation (deriving it using the definition or writing the equation using the standard form) will yield the same equation.

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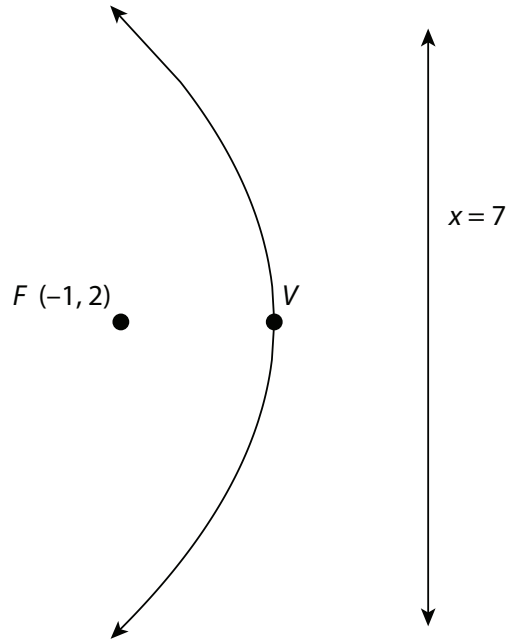
Instruction

Example 2

Derive the standard equation of the parabola with focus (–1, 2) and directrix $x = 7$ from the definition of a parabola. Then write the equation by substituting the vertex coordinates and the value of p directly into the standard form.

1. To derive the equation, begin by plotting the focus. Label it F (–1, 2). Graph the directrix and label it $x = 7$. Sketch the parabola. Label the vertex V.



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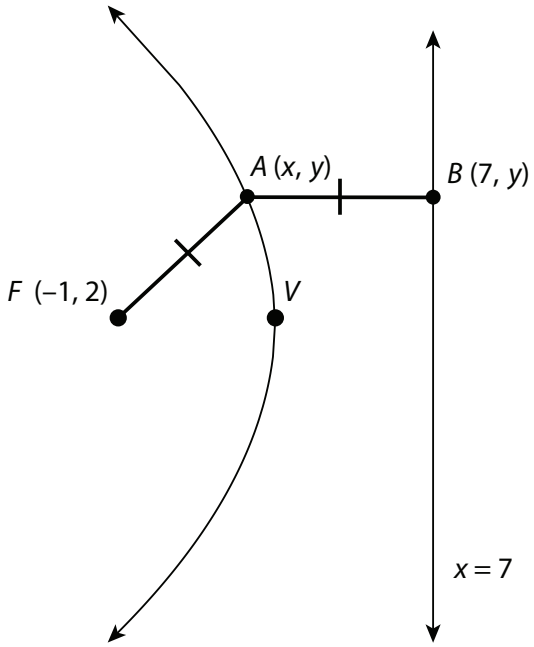
Instruction

2. Let $A\ (x, y)$ be any point on the parabola.



Point A is equidistant from the focus and the directrix. The distance from A to the directrix is the horizontal distance AB, where B is on the directrix directly to the right of A. The x-value of B is 7 because the directrix is at $x = 7$. Because B is directly to the right of A, it has the same y-coordinate as A. So, B has coordinates $(7, y)$.

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Instruction

3. Apply the definition of a parabola to derive the standard equation using the distance formula.

Since the definition of a parabola tells us that $AF = AB$, use the graphed points for AF and AB to apply the distance formula to this equation.

$$\sqrt{[x-(-1)]^2+(y-2)^2} = \sqrt{(x-7)^2+(y-y)^2}$$

$\sqrt{(x+1)^2+(y-2)^2} = \sqrt{(x-7)^2}$	Simplify.
$(x + 1)^2 + (y - 2)^2 = (x - 7)^2$	Square both sides.
$(y - 2)^2 = (x - 7)^2 - (x + 1)^2$	Subtract $(x + 1)^2$ from both sides to get all x terms on one side and all y terms on the other side.
$(y - 2)^2 = (x^2 - 14x + 49) - (x^2 + 2x + 1)$	Square the binomials on the right side.
$(y - 2)^2 = x^2 - 14x + 49 - x^2 - 2x - 1$	Distribute the negative sign.
$(y - 2)^2 = -16x + 48$	Simplify.
$(y - 2)^2 = -16(x - 3)$	Factor on the right side to obtain the standard form $(y - k)^2 = 4p(x - h)$.

The standard equation is $(y - 2)^2 = -16(x - 3)$.

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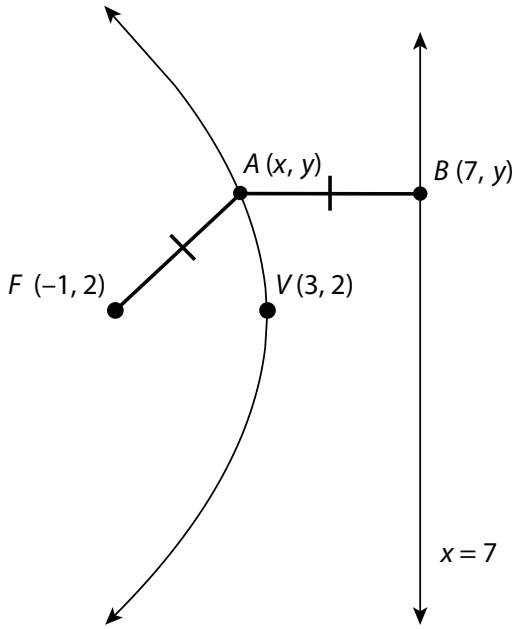
Instruction

4. Write the equation using standard form.

To write the equation using the standard form, first determine the coordinates of the vertex and the value of p.



For any parabola, the distance between the focus and the directrix is $2|p|$. In this case, $2|p| = 7-(-1) = 8$, thus $|p| = 4$. The parabola opens left, so p is negative, and therefore $p = -4$. For any parabola, the distance between the focus and the vertex is $|p|$. Since $|p| = 4$, the vertex is 4 units right of the focus. Therefore, the vertex coordinates are (3, 2).

5. Use the results found in step 4 to write the equation.

$(y-k)^2 = 4p(x-h)$	Standard form for a parabola that opens right or left
$(y-2)^2 = 4(-4)(x-3)$	Substitute the values for h, p, and k.
$(y-2)^2 = -16(x-3)$	Simplify.

The standard equation is $(y-2)^2 = -16(x-3)$.

The results shown in steps 3 and 5 match; so, either method of finding the equation (deriving it using the definition or writing the equation using the standard form) will yield the same equation.

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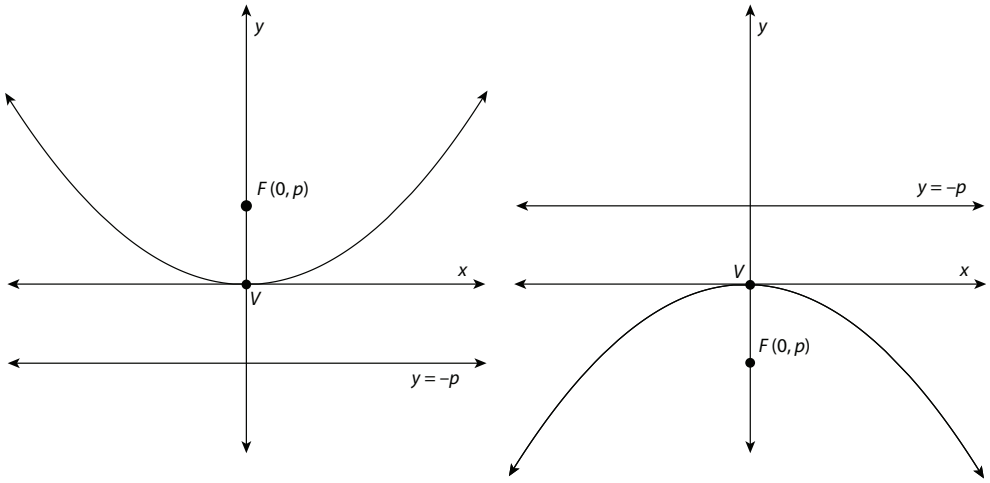
Instruction

Example 3

Derive the standard equation of the parabola with focus $(0, p)$ and directrix $y = -p$, where p is any real number other than 0.

1. Sketch diagrams showing the two possible orientations of the parabola. Include the focus, directrix, and vertex.



Because the focus is $(0, p)$ and the directrix is $y = -p$, the parabola can open either up or down.

The vertex is the origin because it is equidistant from the focus and directrix.

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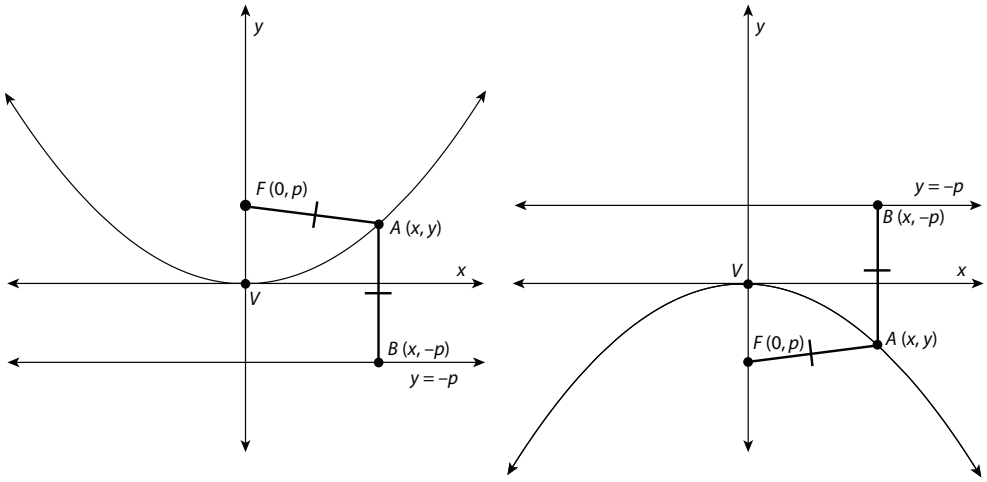
Instruction

2. Let $A\,(x, y)$ be any point on the parabola.



Point A is equidistant from the focus and the directrix. The distance from A to the directrix is the vertical distance AB, where B is on the directrix directly above or below A. Because B is directly above or below A, it has the same x-coordinate as A. So B has coordinates $(x, -p)$.

3. Apply the definition of a parabola to derive the standard equation using the distance formula.

Since the definition of a parabola tells us that $AF = AB$, use the graphed points for AF and AB to apply the distance formula to this equation.

$$\sqrt{(x-0)^2+(y-p)^2}=\sqrt{(x-x)^2+[y-(-p)]^2}$$

$\sqrt{x^2+(y-p)^2}=\sqrt{(y+p)^2}$	Simplify.
$x^2 + (y - p)^2 = (y + p)^2$	Square both sides.
$x^2 = (y + p)^2 - (y - p)^2$	Subtract $(y - p)^2$ from both sides to get all x terms on one side and all y terms on the other side.
$x^2 = (y^2 + 2py + p^2) - (y^2 - 2py + p^2)$	Square the binomials on the right side.
$x^2 = y^2 + 2py + p^2 - y^2 + 2py - p^2$	Distribute the negative sign.
$x^2 = 4py$	Simplify.

The standard equation is $x^2 = 4py$, or $(x - 0)^2 = 4p(y - 0)$.

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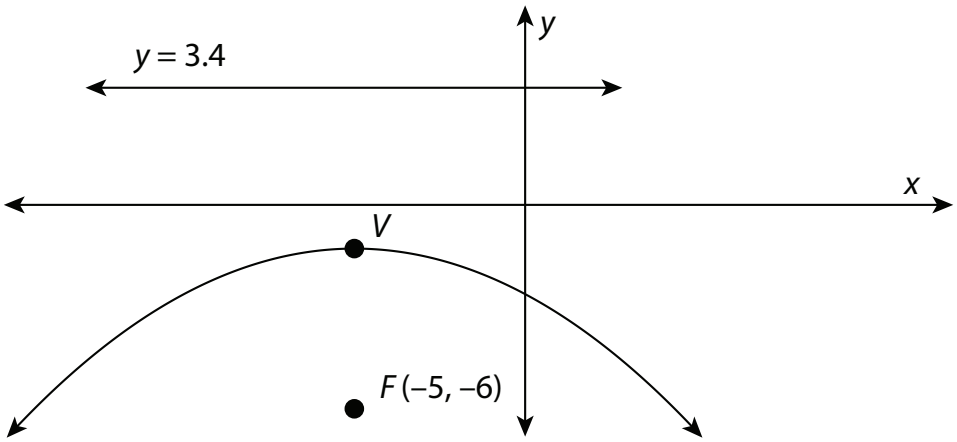
Instruction

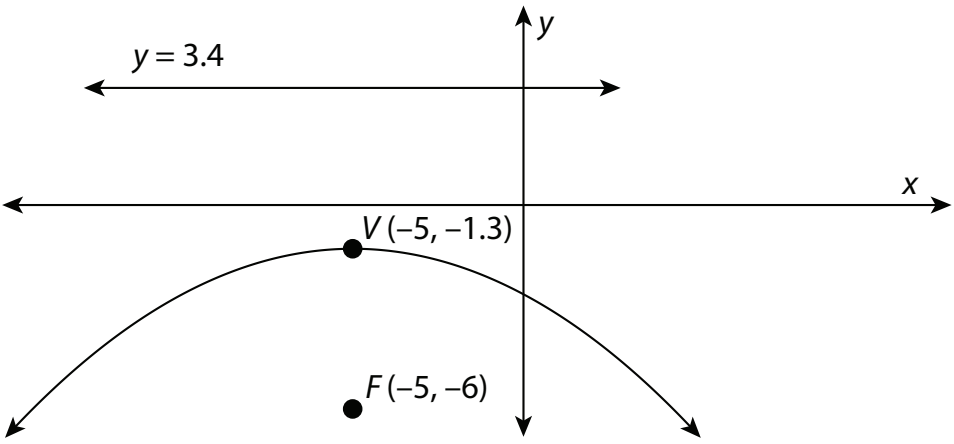
Example 4

Write the standard equation of the parabola with focus (–5, –6) and directrix $y = 3.4$. Then use a graphing calculator to graph your equation.

1. Plot the focus and graph the directrix. Sketch the parabola. Label the vertex V.



2. To write the equation, first determine the coordinates of the vertex and the value of p.



The distance between the focus and the directrix is $2|p|$. So $2|p| = 3.4 - (-6) = 9.4$, and $|p| = 4.7$. The parabola opens down, so p is negative, and therefore $p = -4.7$. The distance between the focus and the vertex is $|p|$, so the vertex is 4.7 units above the focus. Add to find the y-coordinate of the vertex: $-6 + 4.7 = -1.3$. The vertex coordinates are (–5, –1.3).

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Instruction

3. Use the results found in step 2 to write the equation.

$(x-h)^2 = 4p(y-k)$	Standard form for a parabola that opens down
$[x-(-5)]^2 = 4(-4.7)[y-(-1.3)]$	Substitute the values for h, p, and k.
$(x+5)^2 = -18.8(y+1.3)$	Simplify.

The standard equation is $(x+5)^2 = -18.8(y+1.3)$.

4. Solve the standard equation for y to obtain a function that can be graphed.

$(x+5)^2 = -18.8(y+1.3)$	Standard equation
$-\frac{1}{18.8}(x+5)^2 = y+1.3$	Multiply both sides by $-\frac{1}{18.8}$.
$-\frac{1}{18.8}(x+5)^2 - 1.3 = y$	Add –1.3 to both sides.

5. Graph the function using a graphing calculator.

On a TI-83/84:

Step 1: Press [Y=].

Step 2: At Y_1, type in [(][(–)][1][÷][18.8])][(][X, T, θ, n][+][5])][x^2] [–][1.3].

Step 3: Press [WINDOW] to change the viewing window.

Step 4: At Xmin, enter [(–)][12].

Step 5: At Xmax, enter [12].

Step 6: At Xscl, enter [1].

Step 7: At Ymin, enter [(–)][8].

Step 8: At Ymax, enter [8].

Step 9: At Yscl, enter [1].

Step 10: Press [GRAPH].

(*continued*)

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Instruction

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow to the graphing icon and press [enter].

Step 3: At the blinking cursor at the bottom of the screen, enter [(] [(–)][1][÷][18.8][)][(][x][+][5][)][x^2][–][1.3].

Step 4: Change the viewing window by pressing [menu], arrowing down to number 4: Window/Zoom, and clicking the center button of the navigation pad.

Step 5: Choose 1: Window settings by pressing the center button.

Step 6: Enter in an appropriate XMin value, –12, by pressing [(–)] and [12], then press [tab].

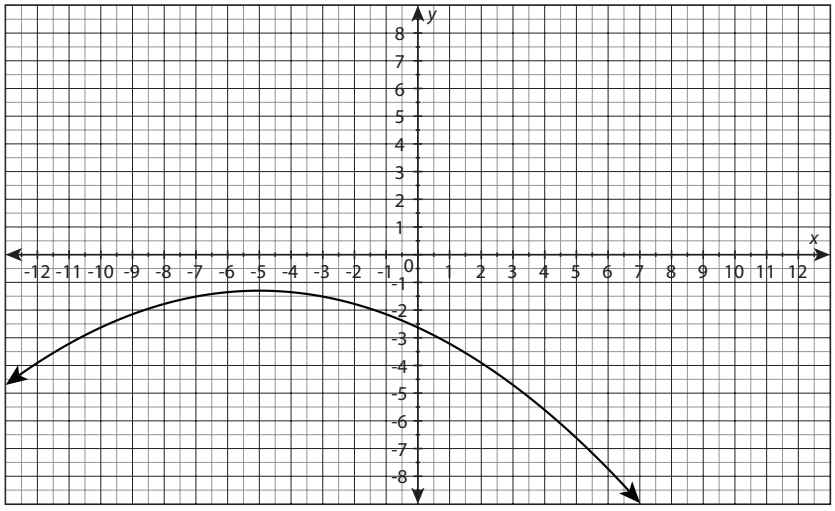
Step 7: Enter in an appropriate XMax value, [12], then press [tab].

Step 8: Leave the XScale set to "Auto." Press [tab] twice to navigate to YMin and enter an appropriate YMin value, –8, by pressing [(–)] and [8].

Step 9: Press [tab] to navigate to YMax. Enter [8]. Press [tab] twice to leave YScale set to "Auto" and to navigate to "OK."

Step 10: Press [enter].

Step 11: Press [menu] and select 2: View and 5: Show Grid.



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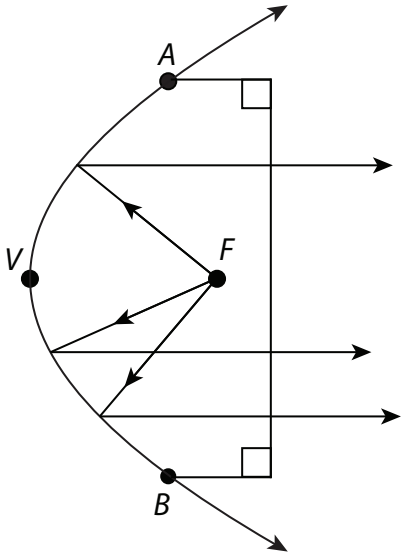
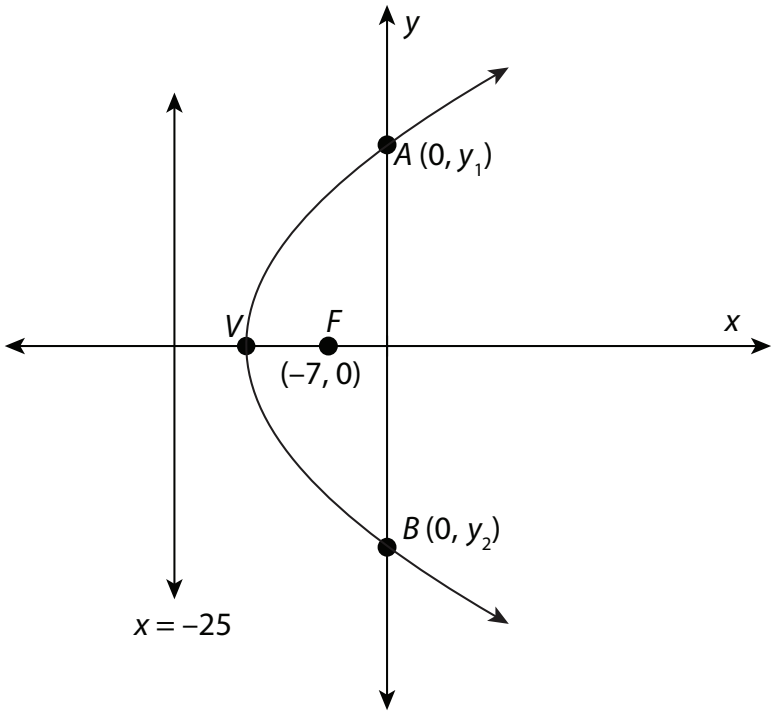
Instruction

Example 5

The following diagram shows a plan for a top view of a stage. The back wall is to be on a parabolic curve from *A* to *B* so that all sound waves coming from point *F* that hit the wall are redirected in parallel paths toward the audience. *F* is the focus of the parabola and *V* is the vertex.



An engineer draws the parabola on a coordinate plane, using feet as the unit of distance. The focus is (–7, 0), the directrix is $x = -25$, and points *A* and *B* are on the *y*-axis. What is the equation of the parabola? What is the width of the stage, *AB*?



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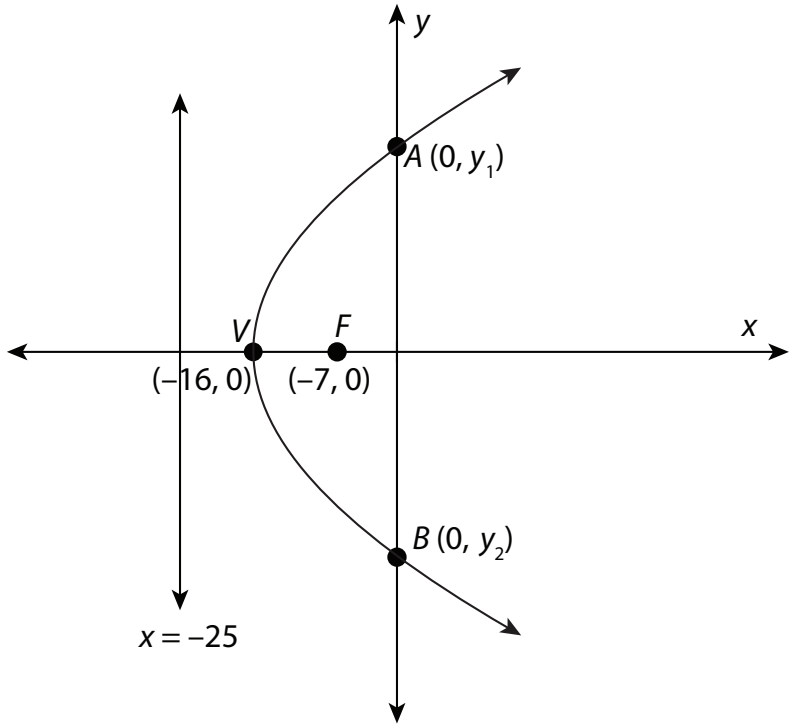
Instruction

1. To write the equation, first determine the coordinates of the vertex and the value of p.



The distance between the focus and the directrix is $2|p|$, so $2|p| = -7-(-25) = 18$ and $|p| = 9$. The parabola opens right, so p is positive, and therefore $p = 9$. For any parabola, the distance between the focus and the vertex is $|p|$. In this case $|p| = 9$, so the vertex is 9 units left of the focus and therefore the vertex coordinates are (–16, 0).

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Instruction

2. Use the results found in step 1 to write the equation.

$(y-k)^2 = 4p(x-h)$	Standard form for a parabola that opens right or left
$(y-0)^2 = 4(9)[x-(-16)]$	Substitute the values for h, p, and k.
$y^2 = 36(x+16)$	Simplify.

The standard equation is $y^2 = 36(x+16)$.

3. To find the width of the stage, AB, first find the y-intercepts y_1 and y_2.

$y^2 = 36(x+16)$	Equation of the parabola from step 2
$y^2 = 36(0+16)$	Substitute 0 for x.
$y^2 = 576$	Simplify.

$y = \pm\sqrt{576}$

$y = \pm 24$

So, $y_1 = 24$ and $y_2 = -24$.

$AB = y_1 - y_2 = 24 - (-24) = 48$

The width of the stage is 48 feet.

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Problem-Based Task 6.6.2: A Ball in Flight

SMP	
1 ✓	2 ✓
3	4 ✓
5	6
7 ✓	8

A softball player hits the ball when it is directly above home plate. The ball then follows a parabolic path until it hits the ground. The path of the ball is represented by a portion of a parabola on a coordinate plane, where the origin represents home plate and the unit of distance is feet. The focus of the parabola is (100, –18.5) and the directrix is $y = 106.5$.

Write the equation of the path of the ball and supply the following key facts: the maximum height of the ball, the height of the ball when it is hit by the bat, and how far the ball travels horizontally before it hits the ground.

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Problem-Based Task 6.6.2: A Ball in Flight

Coaching

a. Make a sketch using the given information.

b. What is the x-coordinate of the vertex? Explain.

c. In what direction does the parabola open? Explain.

d. What is the value of p for this parabola? Explain.

e. What is the y-coordinate of the vertex? Explain.

f. Summarize what you have found.

g. What is the equation of the parabola? Show your work.

h. What is the maximum height of the ball? Explain.

i. Where on the field is the ball when it is hit by the bat?

j. What point on the graph represents that location? Explain.

k. What is the height of the ball when it is hit by the bat? Explain.

l. What point on the graph represents the point where the ball hits the ground?

m. How can you find the coordinates of the points where the graph intersects the x-axis?

n. What are the approximate coordinates of those points? Show your work.

o. Sketch the graph of the parabola. Plot and label all the key points you have found. Identify the portion of the graph that represents the path of the ball.

p. How far does the ball travel horizontally before it hits the ground? Explain.

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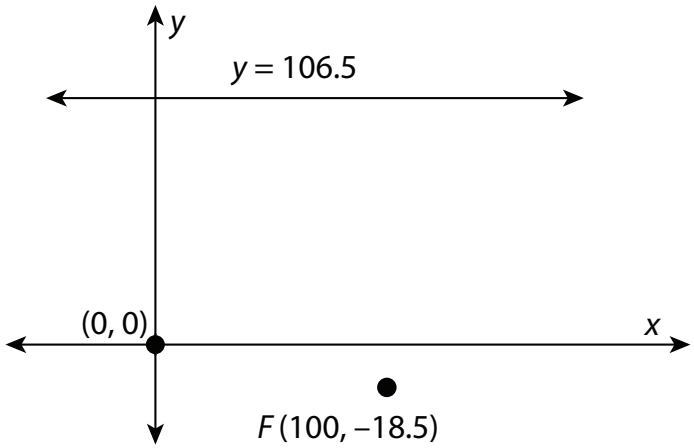
Instruction

Problem-Based Task 6.6.2: A Ball in Flight

Coaching Sample Responses

a. Make a sketch using the given information.



b. What is the *x*-coordinate of the vertex? Explain.

The *x*-coordinate is 100 because the vertex is directly above the focus.

c. In what direction does the parabola open? Explain.

The parabola opens down because the focus is below the directrix.

d. What is the value of *p* for this parabola? Explain.

$p = -62.5$. The distance between the focus and the directrix is $2|p|$, so $2|p| = 106.5 - (-18.5) = 125$, and $|p| = 62.5$. The parabola opens down, so p is negative.

e. What is the *y*-coordinate of the vertex? Explain.

The *y*-coordinate is 44. The distance between the focus and the vertex is $|p|$, which is 62.5 in this case. So the vertex is 62.5 units above the focus. Add to find the *y*-coordinate:

$-18.5 + 62.5 = 44$

f. Summarize what you have found.

The vertex is (100, 44) and $p = -62.5$.

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Instruction

g. What is the equation of the parabola? Show your work.

The equation is $(x - 100)^2 = -250(y - 44)$. This can be determined as follows:

$(x - h)^2 = 4p(y - k)$

$(x - 100)^2 = 4(-62.5)(y - 44)$

$(x - 100)^2 = -250(y - 44)$

h. What is the maximum height of the ball? Explain.

The maximum height is 44 feet because the vertex is (100, 44).

i. Where on the field is the ball when it is hit by the bat?

It is directly above home plate.

j. What point on the graph represents that location? Explain.

The point where the graph intersects the *y*-axis represents this location. The origin represents home plate and the ball is directly above home plate when it is hit, so the corresponding point on the graph is on the *y*-axis.

k. What is the height of the ball when it is hit by the bat? Explain.

The ball is 4 feet above the ground when it is hit. This is determined by finding the *y*-intercept, so substitute 0 for *x* in the equation.

$(x - 100)^2 = -250(y - 44)$

$(0 - 100)^2 = -250(y - 44)$

$10{,}000 = -250(y - 44)$

$$\frac{10{,}000}{-250} = \frac{-250}{-250}(y - 44)$$

$-40 = y - 44$

$4 = y$

l. What point on the graph represents the point where the ball hits the ground?

One of the points where the graph intersects the *x*-axis.

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m. How can you find the coordinates of the points where the graph intersects the *x*-axis?

Substitute 0 for *y* in the equation.

n. What are the approximate coordinates of those points? Show your work.

(–5, 0) and (205, 0). This can be determined as follows:

$(x-100)^2 = -250(y-44)$

$(x-100)^2 = -250(0-44)$

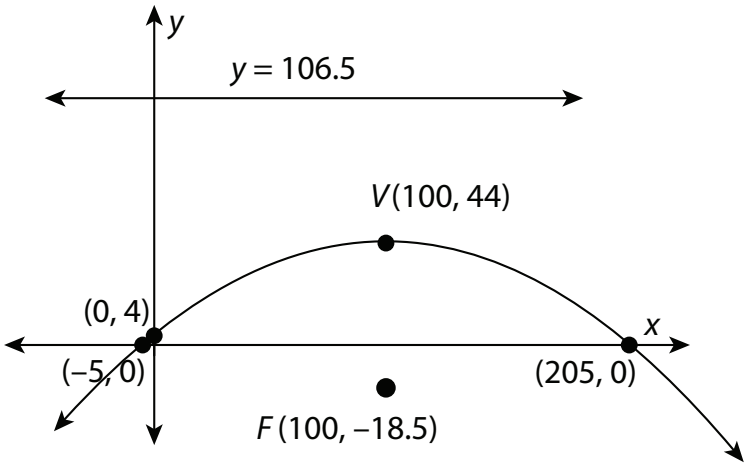
$(x-100)^2 = 11{,}000$

$x-100 = \pm\sqrt{11{,}000}$

$x = 100 \pm \sqrt{11{,}000}$

$x = 100+\sqrt{11{,}000} \approx 205$ or $x = 100-\sqrt{11{,}000} \approx -5$

o. Sketch the graph of the parabola. Plot and label all the key points you have found. Identify the portion of the graph that represents the path of the ball.



The path of the ball is represented by the portion of the parabola from (0, 4), where it is hit by the bat, to (205, 0), where it hits the ground.

p. How far does the ball travel horizontally before it hits the ground? Explain.

The ball travels approximately 205 feet horizontally because that is the horizontal distance between (0, 4) and (205, 0).

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 6.6.2: Deriving the Equation of a Parabola

For problems 1–4, derive the standard equation of the parabola with the given focus and directrix. Also, write the equation that shows how you applied the distance formula.

1. focus: (0, 5); directrix: $y = -5$

2. focus: (–2, 0); directrix: $x = 2$

3. focus: (–3, 5); directrix: $y = -3$

4. focus: (0, –3); directrix: $x = 8$

For problems 5 and 6, write the standard equation of the parabola with the given focus and directrix.

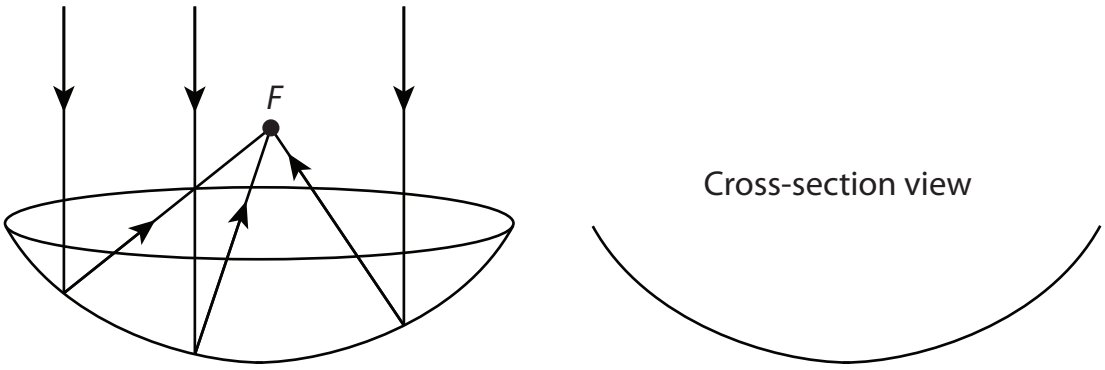
5. focus: (4.5, –5.5); directrix: $y = 1.5$

6. focus: (–10, 20); directrix: $x = 110$

Use what you know about parabolas to solve problems 7–10.

7. Identify the vertex, focus, and directrix of the parabola whose equation is $(x - 3)^2 = 10(y + 1)$.

8. The diagram below shows a parabolic satellite dish antenna. Incoming TV signals reflect off of the dish and toward the feed, or receiver, of the antenna, located at point *F*. A cross section of the dish is a section of a parabola.



The parabola is placed on a coordinate plane whose unit of distance is feet. The focus *F* is (0, 3.25) and the directrix is $y = -3.25$. What is the standard equation of the parabola?

continued

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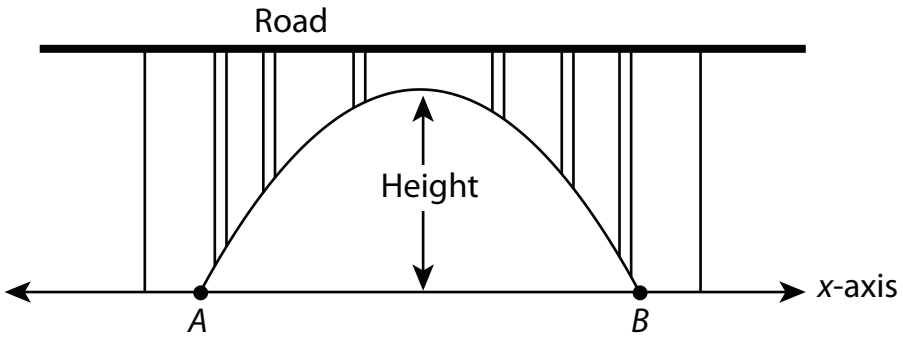
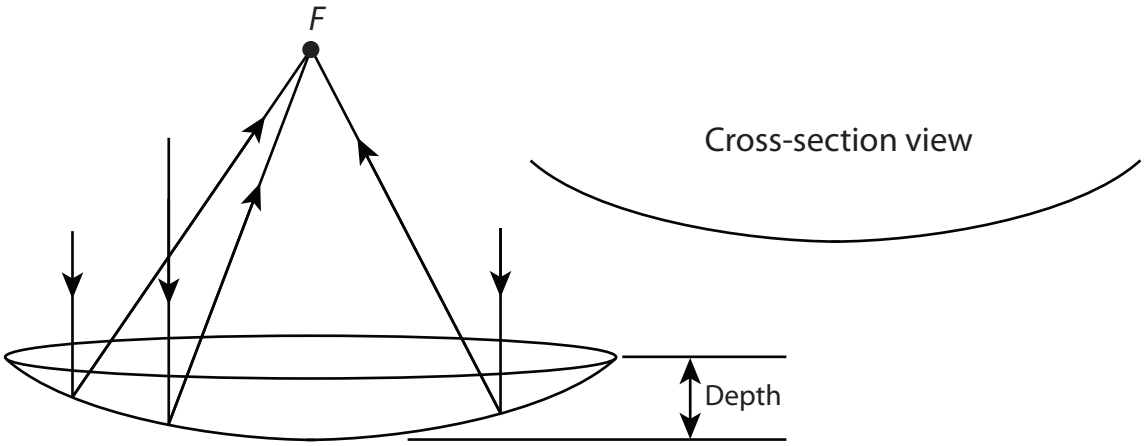
9. The following diagram shows a parabolic arch bridge.



An engineer draws the bridge on a coordinate plane so that points A and B are on the x-axis, the focus is (60, –5.25), and the directrix is $y = 55.25$. The unit of distance on the coordinate plane is feet. What is the standard equation of the parabola? What is the height of the bridge? What is the distance AB? Sketch the parabola, showing the coordinates of A, B, and the vertex.

10. The following diagram shows a radio telescope dish. Incoming light rays reflect off of the dish and toward the feed, located at point F. A cross section of the dish is a section of a parabola. The feed is 40 inches above the vertex. The diameter of the dish at the top is 8 feet.



A technician draws the parabola on a coordinate plane so that the vertex is at the origin. What is the equation of the parabola on the coordinate plane? What is the depth of the dish?

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Progress Assessment

Circle the letter of the best answer.

1. Which is an equation of the circle with center (–2, 5) and radius 4?
   a. $(x-2)^2+(y+5)^2=4$
   b. $(x+2)^2+(y-5)^2=4$
   c. $(x-2)^2+(y+5)^2=16$
   d. $(x+2)^2+(y-5)^2=16$

2. Which is an equation of the circle that has center (0, 2) and passes through (2, –3)?
   a. $x^2+(y-2)^2=7$
   b. $x^2+(y-2)^2=29$
   c. $(x-2)^2+(y+3)^2=7$
   d. $(x-2)^2+(y+3)^2=29$

3. Which is a general equation of the circle with center (3, 2) and radius 10?
   a. $x^2+y^2-87=0$
   b. $x^2+y^2-113=0$
   c. $x^2+y^2-6x-4y-87=0$
   d. $x^2+y^2-6x-4y-113=0$

4. What are the center and radius of the circle whose equation is $x^2+y^2-6x-21=0$?
   a. The center is (–3, 0) and the radius is $\sqrt{12}$.
   b. The center is (–3, 0) and the radius is 144.
   c. The center is (3, 0) and the radius is $\sqrt{30}$.
   d. The center is (3, 0) and the radius is 900.

5. What are the center and radius of the circle whose equation is $2x^2+2y^2+3x-16y+\frac{9}{8}=0$?
   a. The center is $\left(-\frac{3}{4},\ 4\right)$ and the radius is 4.
   b. The center is $\left(-\frac{3}{2},\ 8\right)$ and the radius is 4.
   c. The center is $\left(-\frac{3}{4},\ 4\right)$ and the radius is 16.
   d. The center is $\left(-\frac{3}{2},\ 8\right)$ and the radius is 16.

continued

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Assessment

6. Which is an equation of the parabola that has focus (0, 6) and directrix $y = -6$?

   a. $x^2 = -6y$
   b. $x^2 = 24y$
   c. $y^2 = -6x$
   d. $y^2 = 24x$

7. Which is an equation of the parabola that has focus (3, –4) and directrix $x = 7$?

   a. $(y+4)^2 = 8(x-3)$
   b. $(y+4)^2 = -8(x-3)$
   c. $(y+4)^2 = 8(x-5)$
   d. $(y+4)^2 = -8(x-5)$

8. Which is an equation of the parabola that has focus (4, 4.2) and directrix $y = -2$?

   a. $(x-4)^2 = 4.4(y-1.1)$
   b. $(x-4)^2 = 12.4(y-1.1)$
   c. $(x+4)^2 = 4.4(y+4.2)$
   d. $(x+4)^2 = 12.4(y+4.2)$

9. Which equation appears when you are deriving the equation of the parabola with focus (–2, –1) and directrix $y = 5$?

   a. $\sqrt{(x+2)^2+(y+1)^2} = \sqrt{(x-x)^2+(y-5)^2}$
   b. $\sqrt{(x-2)^2+(y-1)^2} = \sqrt{(x-x)^2+(y-5)^2}$
   c. $\sqrt{(x+2)^2+(y-2)^2} = \sqrt{(x-x)^2+(y-5)^2}$
   d. $\sqrt{(x-2)^2+(y+2)^2} = \sqrt{(x-x)^2+(y-5)^2}$

10. What are the focus and directrix of the parabola whose equation is $(x+6)^2 = -12(y+4)$?

    a. The focus is (–6, –7) and the directrix is $y = -1$.
    b. The focus is (–6, –4) and the directrix is $y = -1$.
    c. The focus is (–6, –7) and the directrix is $y = 1$.
    d. The focus is (–6, –4) and the directrix is $y = 1$.

continued

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Assessment

Use the given information to answer the questions that follow.

11. Lift-Ease Towing Company and Car Care Towing Company both charge the same flat fee to tow any vehicle inside a defined region. Each company has their region drawn on a coordinate plane, using miles as the distance unit. The boundary of the region for Lift-Ease has the equation $x^2 + y^2 - 24x + 2y + 45 = 0$. The boundary of the region for Car Care has the equation $x^2 + y^2 - 60x + 16y + 739 = 0$.

    a. What is the specific geometric description of the region for Lift-Ease?

    b. What is the specific geometric description of the region for Car Care?

    c. A motorist is stranded at (24, 5) on the coordinate plane. Is the motorist within either company's region? Explain.

    d. Make a sketch that illustrates your answers to the questions in parts a, b, and c.

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Pre-Assessment

Circle the letter of the best answer.

1. Which point is on the circle centered at (1, 3) and passing through the point (3, 4)?

   a. (–2, 0)
   b. (–4, 3)
   c. $\left(1, 3+\sqrt{3}\right)$
   d. $\left(1, 3+\sqrt{5}\right)$

2. Which is an equation of the parabola with vertex (0, 0) and focus (0, p)?

   a. $y = 4px^2$
   b. $x = 4py^2$
   c. $y^2 = 4px$
   d. $x^2 = 4py$

3. What is the x-coordinate of the vertex of the graph of $y=\frac{1}{4}x^2+4x+3$?

   a. –8
   b. –2
   c. 3
   d. 4

4. What are the coordinates of the focus of the parabola whose equation is $y=\frac{1}{8}x^2-\frac{1}{2}x+\frac{3}{2}$?

   a. (2, –1)
   b. (2, 1)
   c. (2, 3)
   d. (2, 4)

5. Which point is on the circle that has a center at (2, –3) and a radius of $\sqrt{10}$?

   a. (–1, –3)
   b. (2, –7)
   c. (–1, –2)
   d. (2, 0)

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Lesson 7: Using Coordinates to Prove Geometric Theorems About Circles and Parabolas

Instruction

Common Core State Standard

G–GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0, 2).*

SMP	
1	2
3 ✓	4
5	6 ✓
7 ✓	8

Essential Questions

1. How can you determine whether a point is on a graph?
2. What are some formulas that help describe how points and figures in a coordinate plane are situated with respect to each other?
3. How are equations of quadratic functions related to certain types of parabolas and their equations?

WORDS TO KNOW

axis of symmetry of a parabola	the line through the vertex of a parabola about which the parabola is symmetric. For a quadratic function in the form $f(x) = ax^2 + bx + c$, the equation of the axis of symmetry is $x = \frac{-b}{2a}$.
circle	the set of all points in a plane that are equidistant from a reference point in that plane, the center. The set of points forms a 2-dimensional curve that measures 360°.
directrix of a parabola	a line that is perpendicular to the axis of symmetry of a parabola and that is in the same plane as both the parabola and the focus of the parabola; the fixed line referenced in the definition of a parabola
distance formula	a formula that states the distance between points (x_1, y_1) and (x_2, y_2) is equal to $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

SRB p. U6-194

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focus of a parabola	a fixed point on the interior of a parabola that is not on the directrix of the parabola but is on the same plane as both the parabola and the directrix; the fixed point referenced in the definition of a parabola
midpoint formula	a formula that states the midpoint of the line segment created by connecting (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
parabola	the set of all points that are equidistant from a fixed line, called the directrix, and a fixed point not on that line, called the focus. The parabola, directrix, and focus are all in the same plane. The vertex of the parabola is the point on the parabola that is closest to the directrix.
perfect square trinomial	a trinomial of the form $x^2 + bx + \left(\frac{b}{2}\right)^2$
quadratic function	a function that can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. The graph of a quadratic function is a parabola.
slope formula	a formula that states the slope of the line through (or the line segment connecting) $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$ is $\frac{y_2 - y_1}{x_2 - x_1}$
standard form of an equation of a circle	$(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the center and r is the radius
standard form of an equation of a parabola	$(x - h)^2 = 4p(y - k)$ for parabolas that open up or down; $(y - k)^2 = 4p(x - h)$ for parabolas that open right or left. For all parabolas, $p \neq 0$ and the vertex is (h, k).
theorem	a statement that is shown to be true
vertex of a parabola	the point on a parabola that is closest to the directrix and lies on the axis of symmetry

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Recommended Resources

- IXL Learning. "Lines in the coordinate plane: Slopes of lines."

  http://www.walch.com/rr/00074

  This interactive website gives a series of problems involving slopes of lines and scores them immediately. If the user submits a wrong answer, the correct answer is given and then a thorough explanation is provided, including the slope formula and how it is applied to the problem.

- IXL Learning. "Lines in the coordinate plane: Slopes of parallel and perpendicular lines."

  http://www.walch.com/rr/00075

  This interactive website gives a series of problems involving slopes of lines and whether the lines are parallel, perpendicular, or neither, and scores the problems immediately. If the user submits a wrong answer, the correct answer is given and then a thorough explanation is provided, including the slope conditions required for parallel and perpendicular lines, and how those conditions are applied to the problem.

- IXL Learning. "Points, lines, and segments: Midpoint formula."

  http://www.walch.com/rr/00076

  This interactive website gives a series of problems asking for midpoints of line segments and scores them immediately. If the user submits a wrong answer, the correct answer is given and then a thorough explanation is provided, including the midpoint formula and how it is applied to the problem.

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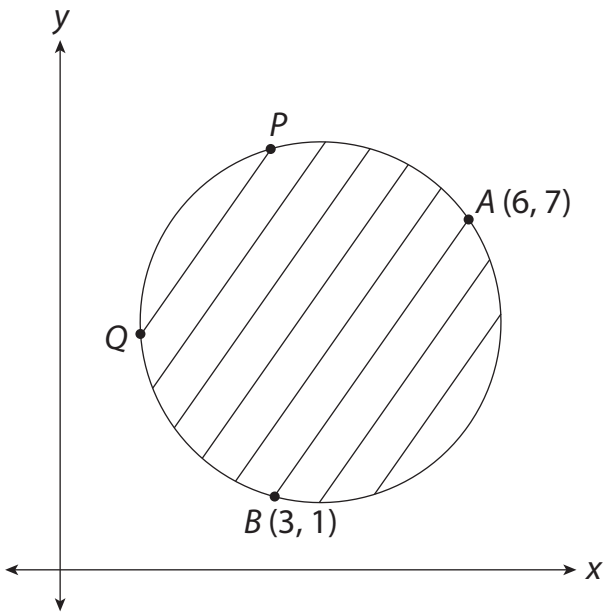
Lesson 6.7.1: Using Coordinates to Prove Geometric Theorems About Circles and Parabolas

Warm-Up 6.7.1

Juan is planning to make a circular tabletop by joining wooden planks. He draws the following diagram of the tabletop on a coordinate system in which distance is represented in feet. The line segments represent the edges of the planks, which are all parallel.



1. What is the length of line segment *AB*? Show your work.

2. What is the slope of line segment *PQ*? Explain.

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Instruction

Lesson 6.7.1: Using Coordinates to Prove Geometric Theorems About Circles and Parabolas

Common Core State Standard

G–GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0, 2).*

Warm-Up 6.7.1 Debrief

1. What is the length of line segment *AB*? Show your work.

   Use the distance formula.

   $$AB = \sqrt{(6-3)^2 + (7-1)^2} = \sqrt{9+36} = \sqrt{45} = \sqrt{9} \bullet \sqrt{5} = 3\sqrt{5}$$

2. What is the slope of line segment *PQ*? Explain.

   The slope of $\overline{PQ}$ is 2. Since the slope of $AB = \frac{7-1}{6-3} = \frac{6}{3} = 2$, and $\overline{PQ}$ and $\overline{AB}$ are parallel, they have the same slope.

Connection to the Lesson

- Students will work with circles, slopes, and distances.
- Students will use slope and distance to prove or disprove statements involving parallelism, perpendicularity, and special types of polygons.

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Instruction

Prerequisite Skills

This lesson requires the use of the following skills:

- applying the distance, midpoint, and slope formulas (G–GPE.5)
- using slope to determine whether lines are parallel, perpendicular, or neither (G–GPE.5)
- identifying congruent angle pairs formed by parallel lines and a transversal (8.G.5)
- simplifying expressions involving square roots (N–RN.2)
- applying the standard form of an equation of a circle (G–GPE.1)
- applying the standard forms of equations of parabolas (G–GPE.2)
- converting between different forms of equations of parabolas that represent functions (F–IF.8a)
- using an equation of a parabola that represents a function to identify the vertex (F–IF.8a)
- applying properties involving circles, tangent lines, and inscribed polygons (G–C.3)
- understanding function notation (F–IF.2)

Introduction

A **theorem** is statement that is shown to be true. Some important theorems have names, such as the Pythagorean Theorem, but many theorems do not have names. In this lesson, we will apply various geometric and algebraic concepts to prove and disprove statements involving circles and parabolas in a coordinate plane. If a statement is proven, it is a theorem. If a statement is disproved, it is not a theorem.

The directions for most problems will have the form "Prove or disprove...," meaning we will work through those problems to discover whether each statement is true or false. Then, at the end of the work, we will state whether we have proved or disproved the statement.

Key Concepts

- A theorem is any statement that is proven or can be proved to be true.
- The standard form of an equation of a circle with center (h, k) and radius r is $(x-h)^2+(y-k)^2=r^2$. This is based on the fact that any point (x, y) is on the circle if and only if $\sqrt{(x-h)^2+(y-k)^2}=r$.

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Instruction

- Completing the square is the process of determining the value of $\left(\frac{b}{2}\right)^2$ and adding it to $x^2 + bx$ to form the perfect square trinomial $x^2 + bx + \left(\frac{b}{2}\right)^2$.
- A quadratic function can be represented by an equation of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$.
- The graph of any quadratic function is a parabola that opens up or down.
- A parabola is the set of all points that are equidistant from a fixed line, called the directrix, and a fixed point not on that line, called the focus.
- The parabola, directrix, and focus are all in the same plane.
- The distance between the focus and a point on the parabola is the same as the distance from that point to the directrix.
- The vertex of the parabola is the point on the parabola that is closest to the directrix.
- Every parabola is symmetric about a line called the **axis of symmetry**.
- The axis of symmetry intersects the parabola at the vertex.
- The x-coordinate of the vertex is $-\frac{b}{2a}$.
- The y-coordinate of the vertex is $f\left(-\frac{b}{2a}\right)$.
- The standard form of an equation of a parabola that opens up or down and has vertex (h, k) is $(x - h)^2 = 4p(y - k)$, where $p \neq 0$ and p is the distance between the vertex and the focus and between the vertex and the directrix.
- Parabolas that open up or down represent functions, and their equations can be written in either of the following forms: $y = ax^2 + bx + c$ or $(x - h)^2 = 4p(y - k)$. If one form is known, the other can be found.
- The standard form of an equation of a parabola that opens right or left and has vertex (h, k) is $(y - k)^2 = 4p(x - h)$, where $p \neq 0$ and p is the distance between the vertex and the focus and between the vertex and the directrix.

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- In any parabola:
  - The focus and directrix are each $|p|$ units from the vertex.
  - The focus and directrix are $2|p|$ units from each other.

Common Errors/Misconceptions

- neglecting to square the radius for the equation of the circle
- using the equation of a parabola that opens up or down for a parabola that opens left or right, and vice versa

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Instruction

Guided Practice 6.7.1

Example 1

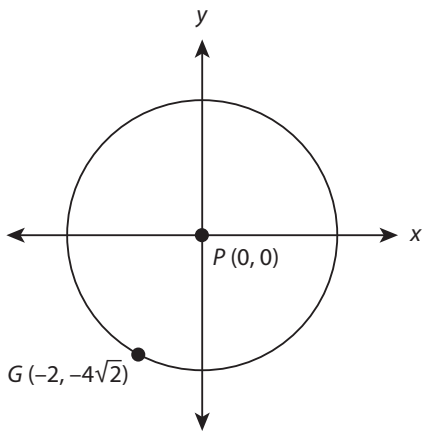
Given the point A (–6, 0), prove or disprove that point A is on the circle centered at the origin and passing through $\left(-2,-4\sqrt{2}\right)$.

1. Draw a circle on a coordinate plane using the given information.

   You do not yet know if point A lies on the circle, so don't include it in your diagram.

   In the diagram that follows, the name P is assigned to the origin and G is assigned to the known point on the circle.

   To help in plotting points, you can use a calculator to find decimal approximations.



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Instruction

2. Find the radius of the circle using the distance formula.

Use the known points, P and G, to determine the radius of the circle.

$r=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$ Distance formula

$r=\sqrt{\left[(-2)-(0)\right]^2+\left[\left(-4\sqrt{2}\right)-(0)\right]^2}$ Substitute (0, 0) and $\left(-2,-4\sqrt{2}\right)$ for (x_1, y_1) and (x_2, y_2).

$r=\sqrt{(-2)^2+\left(-4\sqrt{2}\right)^2}$ Simplify, then solve.

$r=\sqrt{4+32}$

$r=\sqrt{36}$

$r=6$

The radius of the circle is 6 units.

For point A to be on the circle, it must be precisely 6 units away from the center of the circle.

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3. Find the distance of point A from the center P to determine whether it is on the circle.

   The coordinates of point P are (0, 0).

   The coordinates of point A are (–6, 0).

   $AP = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance formula

   $AP = \sqrt{[(-6) - (0)]^2 + [(0) - (0)]^2}$ Substitute (0, 0) and (–6, 0) for (x_1, y_1) and (x_2, y_2).

   $AP = \sqrt{(-6)^2 + (0)^2}$ Simplify, then solve.

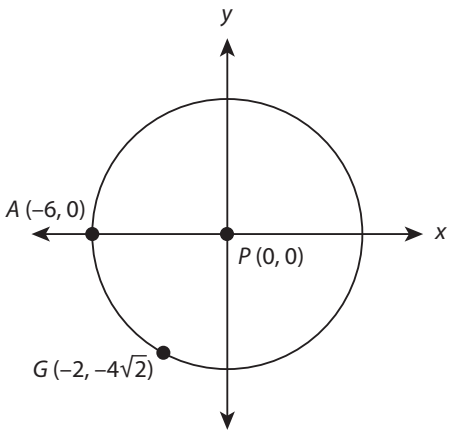
   $AP = \sqrt{(-6)^2}$

   $AP = \sqrt{36}$

   $AP = 6$

   Point A is 6 units from the center, and since the radius of the circle is 6 units, point A is on the circle.

   The original statement has been proved, so it is a theorem.



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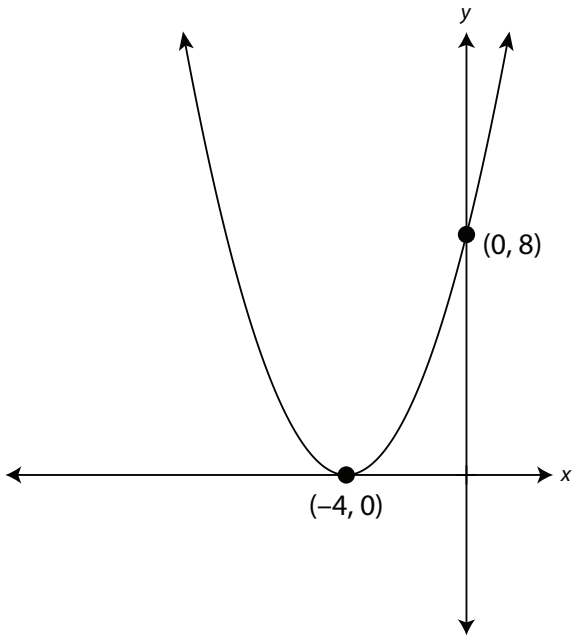
Example 2

Prove or disprove that the quadratic function graph with vertex (–4, 0) and passing through (0, 8) has its focus at (–4, 1).

1. Sketch the graph using the given information.



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Instruction

2. Derive an equation of the parabola from its graph.

The parabola opens up, so it represents a function. Therefore, its equation can be written in either of these forms: $y = ax^2 + bx + c$ or $(x - h)^2 = 4p(y - k)$. We were given a vertex and a point on the parabola; therefore, we'll use the form $(x - h)^2 = 4p(y - k)$ and the vertex to begin deriving the equation.

The vertex is (–4, 0), so $h = -4$ and $k = 0$.

$(x - h)^2 = 4p(y - k)$	Standard form of an equation for a parabola that opens up or down
$[x - (-4)]^2 = 4p(y - 0)$	Substitute the vertex (–4, 0) into the equation.
$(x + 4)^2 = 4py$	Simplify, but do not expand the binomial.

The equation of the parabola is $(x + 4)^2 = 4py$.

3. Substitute the given point on the parabola into the standard form of the equation to solve for p.

The point given is (0, 8).

$(x + 4)^2 = 4py$	Simplified equation from step 2
$(0 + 4)^2 = 4p(8)$	Substitute the point (0, 8) into the equation.
$16 = 32p$	Simplify and solve for p.
$p = \frac{1}{2}$	

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4. Use the value of p to determine the focus.

   p is positive, so the focus is directly above the vertex.

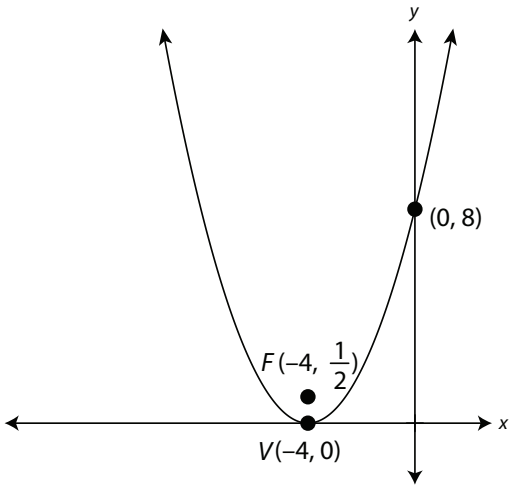
   $p = \frac{1}{2}$, so the focus is $\frac{1}{2}$ unit above the vertex.

   The vertex is (–4, 0), so the focus is $\left(-4, \frac{1}{2}\right)$.

   This result disproves the statement that the quadratic function graph with vertex (–4, 0) and passing through (0, 8) has its focus at (–4, 1).

   The statement has been disproved, so it is not a theorem.

   Instead, the following statement has been proved: The quadratic function graph with vertex (–4, 0) and passing through (0, 8) has its focus at $\left(-4, \frac{1}{2}\right)$.



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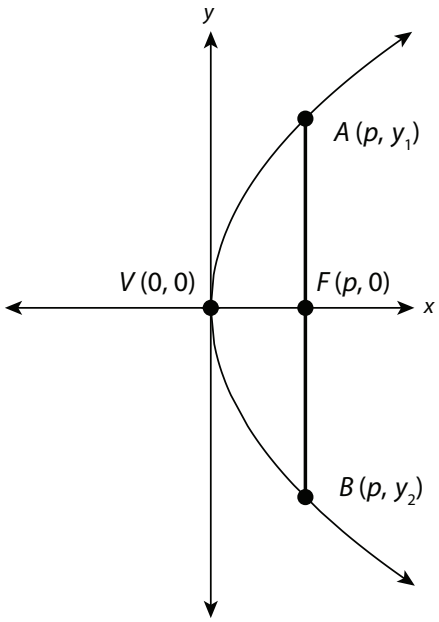
Example 3

The following information is given about a parabola:

- The vertex V is at (0, 0).
- The focus F is at $(p, 0)$, with $p > 0$.
- The line segment through F is perpendicular to the axis of symmetry and connects two points of the parabola.

Prove that the line segment through F has length $4p$.

1. Make a sketch using the given information.



The vertex is (0, 0) and the focus is $F(p, 0)$ with $p > 0$, so F is on the positive x-axis. The axis of symmetry is the x-axis.

The line segment through F, perpendicular to the axis of symmetry, and connecting two points of the parabola is the vertical line segment, $\overline{AB}$. Because the segment is vertical, both A and B have p as their x-coordinate, matching the x-coordinate of F. The y-coordinates of A and B are unknown; they are named y_1 and y_2 in the diagram.

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2. Because the parabola opens to the right, its equation has the form $(y - k)^2 = 4p(x - h)$. Use this equation to solve for y_1 and y_2 in terms of p.

$(y - k)^2 = 4p(x - h)$	Standard form of the equation for a parabola that opens right or left
$(y - 0)^2 = 4p(x - 0)$	Substitute (0, 0) for (h, k).
$y^2 = 4px$	Simplify.
$(y_1)^2 = 4p \bullet p$	Substitute the coordinates of point A.
$(y_1)^2 = 4p^2$	Simplify.
$y_1 = \sqrt{4p^2}$	y_1 is positive, so take the positive square root of $4p^2$.
$y_1 = 2p$	Solve for y_1.
$(y_2)^2 = 4p \bullet p$	Substitute the coordinates of point B into the equation $y^2 = 4px$.
$(y_2)^2 = 4p^2$	Simplify.
$y_2 = -\sqrt{4p^2}$	y_2 is negative, so take the negative square root of $4p^2$.
$y_2 = -2p$	Solve for y_2.

3. Use the results from step 2 to find the length of $\overline{AB}$.

$AB = y_1 - y_2 = 2p - (-2p) = 2p + 2p = 4p$

For the parabola with vertex V (0, 0) and focus F $(p, 0)$ with $p > 0$, the line segment through F has length $4p$. The original statement has been proved; therefore, it is a theorem.

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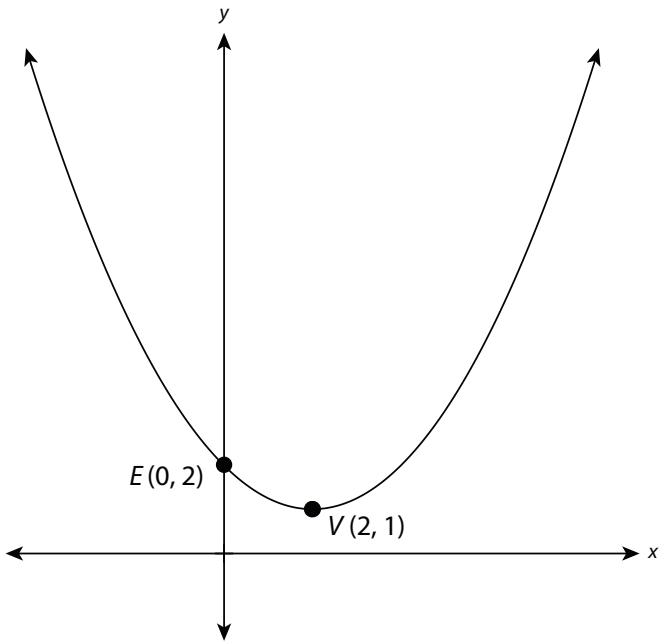
Example 4

Prove or disprove that the points A (4, 2), B (–2, 5), C (6, 5), and D (–4, 10) are all on the quadratic function graph with vertex V (2, 1) that passes through E (0, 2).

1. Make a sketch with the given information.

   A quadratic function graph is a parabola. You do not yet know if any of the points A, B, C, or D lie on the parabola, so do not show them in your sketch.



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2. Derive an equation of the parabola from its graph.

The parabola opens up, so it represents a function. Therefore, its equation can be written in either of these forms: $y = ax^2 + bx + c$ or $(x - h)^2 = 4p(y - k)$. We were given a vertex and a point on the parabola, so we'll use the form $(x - h)^2 = 4p(y - k)$ and the vertex to begin deriving the equation.

The vertex is (2, 1), so $h = 2$ and $k = 1$.

$(x - h)^2 = 4p(y - k)$	Standard form of the equation for a parabola that opens up or down
$(x - 2)^2 = 4p(y - 1)$	Substitute (2, 1) for (h, k).

3. Continue to derive the equation of the parabola by finding p.

Use the given point (0, 2) and the equation derived from step 2.

$(x - 2)^2 = 4p(y - 1)$	Derived equation
$[(0) - 2]^2 = 4p[(2) - 1]$	Substitute point E (0, 2) for (x, y).
$(-2)^2 = 4p(1)$	Simplify, then solve for p.
$4 = 4p$	
$p = 1$	

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4. Convert the standard form of the equation into the general form of the equation.

   Use the value of $p = 1$ and the standard form of the equation derived in step 2.

$(x-2)^2 = 4p(y-1)$	Standard form of the equation with the vertex substituted
$(x-2)^2 = 4(1)(y-1)$	Standard form of the equation with the value of p substituted
$x^2 - 4x + 4 = 4(y-1)$	Expand the binomial.
$\frac{1}{4}x^2 - x + 1 = y - 1$	Divide both sides of the equation by 4.
$\frac{1}{4}x^2 - x + 2 = y$	Add 1 to both sides.

   The equation of the parabola is $y = \frac{1}{4}x^2 - x + 2$.

5. Determine whether the points A, B, C, and D are on the parabola by substituting their coordinates into the equation.

A (4, 2): $2 = \frac{1}{4}(4)^2 - (4) + 2$	Yes, the equation is true, so A is on the parabola.
B (–2, 5): $5 = \frac{1}{4}(-2)^2 - (-2) + 2$	Yes, the equation is true, so B is on the parabola.
C (6, 5): $5 = \frac{1}{4}(6)^2 - (6) + 2$	Yes, the equation is true, so C is on the parabola.
D (–4, 10): $10 = \frac{1}{4}(-4)^2 - (-4) + 2$	Yes, the equation is true, so D is on the parabola.

   A, B, C, and D are all on the parabola.

   The statement has been proved, so it is a theorem.

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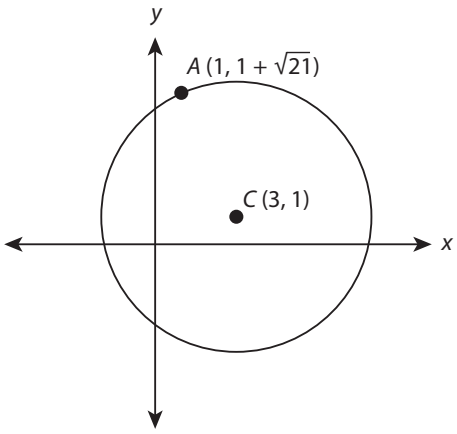
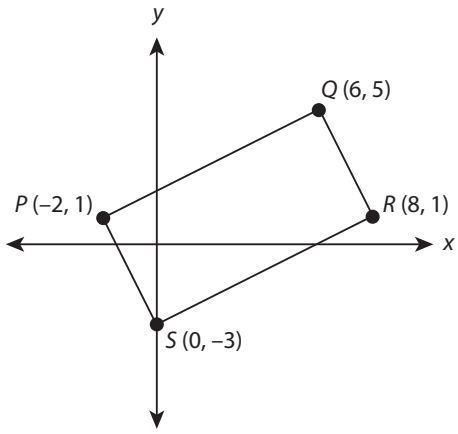
Example 5

Prove or disprove that P (–2, 1), Q (6, 5), R (8, 1), and S (0, –3) are vertices of a rectangle that is inscribed in the circle centered at C (3, 1) and passing through $A\left(1, 1+\sqrt{21}\right)$.

1. Make sketches using the given information.

   You do not yet know if any of the points P, Q, R, or S lie on the circle, so show the polygon on a separate coordinate system. To help in plotting point A, you can use a calculator to find a decimal approximation.





2. Find the radius of the circle.

$$\text{radius} = AC = \sqrt{(1-3)^2 + \left(1+\sqrt{21}-1\right)^2} = \sqrt{(-2)^2 + \left(\sqrt{21}\right)^2}$$

$$\text{radius} = AC = \sqrt{4+21} = \sqrt{25} = 5$$

The radius of the circle is 5 units.

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3. Find the distance of points *P*, *Q*, *R*, and *S* from the center *C* to determine whether they are on the circle.

$$PC=\sqrt{\left[3-(-2)\right]^2+(1-1)^2}=\sqrt{(5)^2+(0)^2}=\sqrt{25}=5$$

$$QC=\sqrt{(3-6)^2+(1-5)^2}=\sqrt{(-3)^2+(-4)^2}=\sqrt{9+16}=\sqrt{25}=5$$

$$RC=\sqrt{(3-8)^2+(1-1)^2}=\sqrt{(-5)^2+(0)^2}=\sqrt{25}=5$$

$$SC=\sqrt{(3-0)^2+\left[1-(-3)\right]^2}=\sqrt{(3)^2+(4)^2}=\sqrt{9+16}=\sqrt{25}=5$$

P, *Q*, *R*, and *S* are each 5 units from the center, and since the radius is also 5 units, the points are all on the circle.

4. Determine whether *PQRS* is a rectangle. Use slopes to identify parallel and perpendicular segments.

slope of $\overline{PQ}=\dfrac{5-1}{6-(-2)}=\dfrac{4}{8}=\dfrac{1}{2}$ slope of $\overline{QR}=\dfrac{1-5}{8-6}=\dfrac{-4}{2}=-2$

slope of $\overline{RS}=\dfrac{-3-1}{0-8}=\dfrac{-4}{-8}=\dfrac{1}{2}$ slope of $\overline{SP}=\dfrac{1-(-3)}{-2-0}=\dfrac{4}{-2}=-2$

$\overline{PQ}$ is parallel to $\overline{RS}$ and $\overline{QR}$ is parallel to $\overline{SP}$ because they have equal slopes.

Therefore, *PQRS* is a parallelogram because it has both pairs of opposite sides parallel.

$\overline{PQ}$ is perpendicular to $\overline{QR}$ because the product of their slopes is –1: $\left(\dfrac{1}{2}\right)(-2)=-1$.

Thus, *PQRS* is a rectangle because it is a parallelogram with a right angle.

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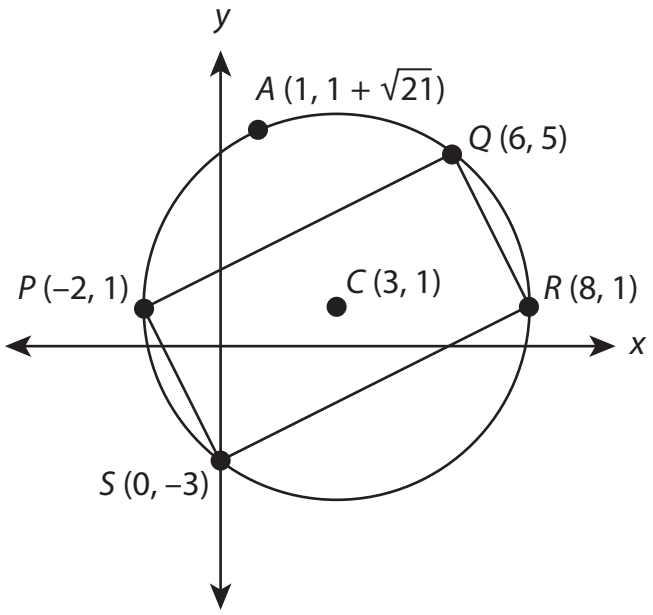
5. Steps 2 and 3 allowed us to determine that P, Q, R, and S are all on the circle. Step 4 shows that $PQRS$ is a rectangle. Therefore, P (–2, 1), Q (6, 5), R (8, 1), and S (0, –3) are vertices of a rectangle that is inscribed in the circle centered at C (3, 1) and passing through $A\left(1, 1+\sqrt{21}\right)$.

   The statement has been proved, so it is a theorem.



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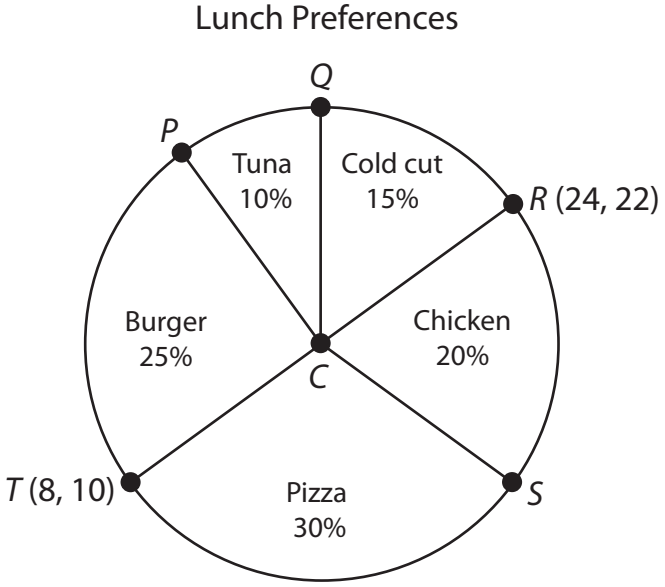
Problem-Based Task 6.7.1: A Circle Graph for Lunch

SMP	
1	2
3 ✓	4
5	6 ✓
7 ✓	8

Alena conducted a survey of her classmates' lunch preferences. She drew a circle graph on a transparency sheet to show the results. She placed the circle graph on a coordinate plane and positioned it so that points R and T have the coordinates R (24, 22) and T (8, 10). Alena claims that the coordinates of point P are (8, 22). Is she correct? Prove or disprove her claim.



Alena claims that the coordinates of point P are (8, 22). Is she correct? Prove or disprove her claim.

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Problem-Based Task 6.7.1: A Circle Graph for Lunch

Coaching

a. What are the two key facts needed for a complete geometric description of this circle on a coordinate plane?

b. What are the only coordinates you are given?

c. What fact about line segment RT will be useful in determining the coordinates of C, the center of the circle?

d. How do you know this fact about $\overline{RT}$?

e. How can you find the coordinates of C?

f. What are the coordinates of C? Explain.

g. What is the radius of the circle? Explain.

h. Now you know the radius and center of the circle. What condition must the point (8, 22) satisfy to be on the circle?

i. Does the point (8, 22) satisfy that condition? Explain.

j. Have you proved that P is at (8, 22)? Explain.

k. How can you prove that (8, 22) is the endpoint of $\overline{PC}$, the radius segment that separates the "burger" sector and the "tuna" sector? Explain.

l. Is Alena's claim correct? Explain.

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Problem-Based Task 6.7.1: A Circle Graph for Lunch

Coaching Sample Responses

a. What are the two key facts needed for a complete geometric description of this circle on a coordinate plane?

The radius and the coordinates of the center are needed.

b. What are the only coordinates you are given?

You are given R (24, 22) and T (8, 10).

c. What fact about line segment RT will be useful in determining the coordinates of C, the center of the circle?

$\overline{RT}$ is the diameter of the circle.

d. How do you know this fact about $\overline{RT}$?

The "chicken" and "pizza" sectors of the circle amount to 50%.

e. How can you find the coordinates of C?

Find the coordinates of the midpoint of $\overline{RT}$.

f. What are the coordinates of C? Explain.

C is the midpoint of $\overline{RT}$, so use the midpoint formula to find its coordinates.

R is at (24, 22) and T is at (8, 10).

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

$$\left(\frac{8+24}{2},\frac{10+22}{2}\right)=(16,16)$$

C is at (16, 16).

g. What is the radius of the circle? Explain.

The radius is the distance from T to the center C, which is found using the distance formula:

$$CT=\sqrt{(16-8)^2+(16-10)^2}=\sqrt{64+36}=\sqrt{100}=10$$

The radius is 10.

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h. Now you know the radius and center of the circle. What condition must the point (8, 22) satisfy to be on the circle?

It must be 10 units from the center, (16, 16).

i. Does the point (8, 22) satisfy that condition? Explain.

Yes; using the distance formula, we can determine that (8, 22) is 10 units from (16, 16).

$$\sqrt{(16-8)^2+(16-22)^2}=\sqrt{64+36}=\sqrt{100}=10$$

j. Have you proved that *P* is at (8, 22)? Explain.

No; we have proved that (8, 22) is on the circle, but not that *P* is at (8, 22). It is necessary to prove that (8, 22) is the endpoint of the radius segment that separates the "burger" sector and the "tuna" sector.

k. How can you prove that (8, 22) is the endpoint *P* of $\overline{PC}$, the radius segment that separates the "burger" sector and the "tuna" sector? Explain.

The "burger" sector is 25%, so $\overline{PC}$ is perpendicular to $\overline{CT}$. It is necessary to show that the segment connecting the point (8, 22) and the point *C* (16, 16) is perpendicular to $\overline{CT}$.

l. Is Alena's claim correct? Explain.

No, Alena's claim is not correct.

The slope of the segment connecting (8, 22) and *C* (16, 16) is $\frac{16-22}{16-8}=\frac{-6}{8}=\frac{-3}{4}$.

The slope of the radius segment $\overline{CT}$ is $\frac{16-10}{16-8}=\frac{6}{8}=\frac{3}{4}$.

The segment connecting (8, 22) and *C* (16, 16) is not perpendicular to the radius segment $\overline{CT}$ because $\left(\frac{-3}{4}\right)\left(\frac{3}{4}\right)\neq -1$.

Therefore, point *P* is not at (8, 22).

Recommended Closure Activity

Select one or more of the essential questions for a class discussion or as a journal entry prompt.

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Practice 6.7.1: Using Coordinates to Prove Geometric Theorems About Circles and Parabolas

Use the given information to prove or disprove each statement. Justify your reasoning.

1. Prove or disprove that point $P\left(2, 2\sqrt{3}\right)$ lies on the circle centered at the origin R and passing through the point Q (4, 0).

2. Prove or disprove that point $Q\left(6, 2\sqrt{5}\right)$ is on the circle centered at the origin A and passing through the point $B\left(-3, \sqrt{55}\right)$.

3. Given the points A (–2, 4), B (1, 1), and C (3, 9), prove or disprove that these points are on the parabola with focus $F\left(0, \frac{1}{4}\right)$ and directrix $y = -\frac{1}{4}$.

4. Given the points A (–2, 4), B (–1, 1), C (2, 4), and D (3, 9), prove or disprove that points are on the quadratic function graph with vertex V (0, 0) and passing through E (5, 25).

5. Prove or disprove that the points P (8, 18), Q (–4, 18), R (6, 8), and S (–2, 8) are all on the quadratic function graph with vertex V (2, 0) that passes through T (0, 2).

continued

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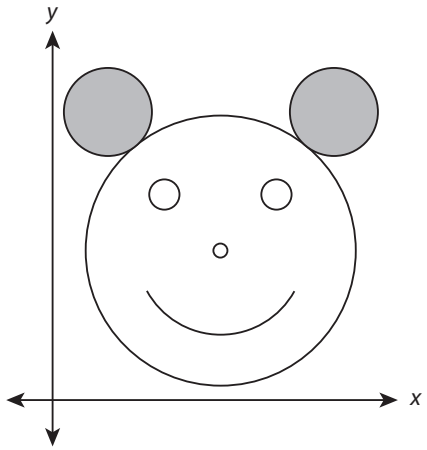
6. Prove or disprove that the point P (–5, 4) lies on the circle centered at C (–1, 2) and passing through the point D (1, –2).

7. Prove or disprove that the points A (8, 6), B (8, –6), and C (–10, 0) are the vertices of an isosceles triangle inscribed in the circle centered at the origin Q and passing through the point $P\left(-3, \sqrt{91}\right)$.

8. The diagram below shows a target at a carnival dart game. The diagram is on a coordinate system. A player wins a prize by hitting a shaded ear. The shaded ears are circles as follows:

   - a circle centered at C_1 (10, 50) and passing through P_1 (13, 54)
   - a circle centered at C_2 (30, 50) and passing through P_2 (26, 47)

   Bradley throws two darts, hitting the points A (14, 46) and B (26, 55). Does he win a prize? Justify your answer.



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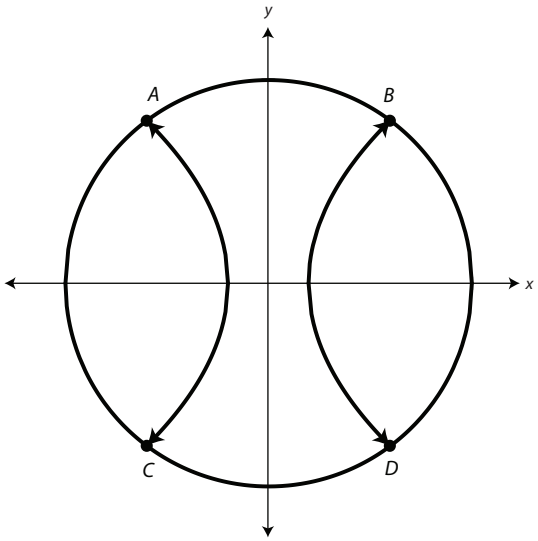
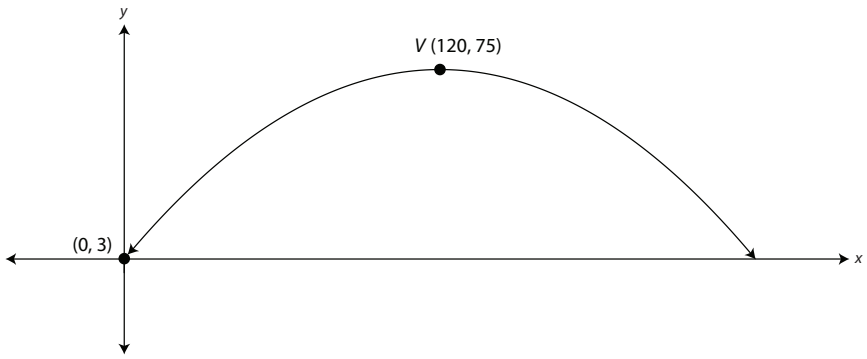
9. A graphic artist created the sketch of a baseball shown below, using the following graphs on a coordinate system:

   - a parabola with vertex $V_1(1, 0)$ and focus $F_1(3, 0)$
   - a parabola with vertex $V_2(-1, 0)$ and focus $F_2(-3, 0)$
   - a circle with center $R(0, 0)$ and passing through the point $P\left(2, \sqrt{21}\right)$

   Prove or disprove that the coordinates of the four labeled points are as follows: $A(-3, 4)$, $B(3, 4)$, $C(-3, -4)$, and $D(3, -4)$.



10. The following graph represents the parabolic path of a baseball. A batter hit the ball when the ball was 3 feet above the ground. After being hit, the ball reached a maximum height of 75 feet, which occurred when it had traveled 120 feet horizontally. Each unit on the graph represents 1 foot. Prove or disprove that the ball was 43 feet above the ground when it had traveled 200 feet horizontally.



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Progress Assessment

Circle the letter of the best answer.

1. Which point is on the circle centered at (2, –1) and passing through the point (–3, 3)?

   a. (–1, –1)
   b. (5, –1)
   c. $\left(-3, 3+\sqrt{41}\right)$
   d. $\left(2+\sqrt{41}, -1\right)$

2. Which point is on the parabola with vertex (0, 0) and focus (0, 2)?

   a. $\left(2, \frac{1}{2}\right)$
   b. (2, 2)
   c. (2, 4)
   d. (2, 32)

3. What is the focus of the parabola whose equation is $y=-\frac{1}{4}x^2+x-2$?

   a. (–2, 2)
   b. (–2, 3)
   c. (2, –2)
   d. (2, –1)

4. Points *P* and *Q* have coordinates *P* (0, 4) and *Q* (2, 1). Line segments *PQ* and *QR* are perpendicular. Which of the following coordinates could be point *R*?

   a. (–2, –5)
   b. (–5, –2)
   c. (–3, –4)
   d. (–4, –3)

5. Points *A* (0, 6), *B* (6, 3), and *C* (–2, 11) are all on the parabola with focus (4, 3) and vertex (4, 2). If point *D* is on the parabola and line segments *AB* and *CD* are parallel, what are the coordinates of point *D*?

   a. (–4, 18)
   b. (0, 10)
   c. (8, 6)
   d. (10, 11)

continued

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Assessment

6. Polygon $ABCD$ is a trapezoid inscribed in the circle whose equation is $x^2 + y^2 = 100$. Points A, B, and C have the coordinates A (6, 8), B (10, 0), and C (0, –10). Which of the following coordinates could be point D?

   a. (–10, 0)
   b. (–8, –6)
   c. (–6, –8)
   d. (–2, 0)

7. Points A, B, C, and D have the coordinates A (–4, 2), B (4, 2), C (4, –2), and D (–4, –2). Which description of polygon $ABCD$ is accurate?

   a. It is a trapezoid inscribed in a circle whose equation is $x^2 + y^2 = 6$.
   b. It is a trapezoid inscribed in a circle whose equation is $x^2 + y^2 = 20$.
   c. It is a rectangle inscribed in a circle whose equation is $x^2 + y^2 = 6$.
   d. It is a rectangle inscribed in a circle whose equation is $x^2 + y^2 = 20$.

8. Points A, B, and C are on the parabola whose equation is $y = x^2 + 2$. Points A and B have the coordinates A (–1, 3) and B (2, 6). Angle ABC is a right angle. Which of the following coordinates could be point C?

   a. (–3, 11)
   b. (–2, 6)
   c. (–2, 10)
   d. (3, 11)

9. Which point is on the parabola with focus (1, 2) and directrix $x = –3$?

   a. (–1, –2)
   b. (1, –2)
   c. (–3, 4)
   d. (3, 4)

10. Points A, B, and C have coordinates A (1, 2), B (5, 10), and C (4, 2). Line segments AB and CD are perpendicular. Which of the following coordinates could be point D?

    a. (0, 4)
    b. (1, –4)
    c. (7, –4)
    d. (10, 5)

continued

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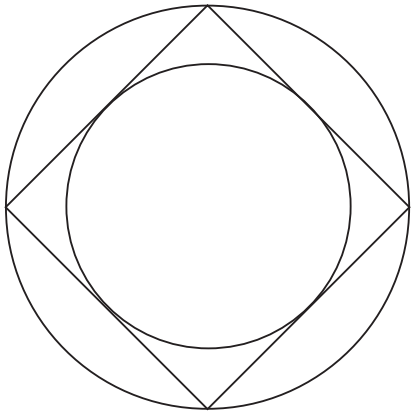
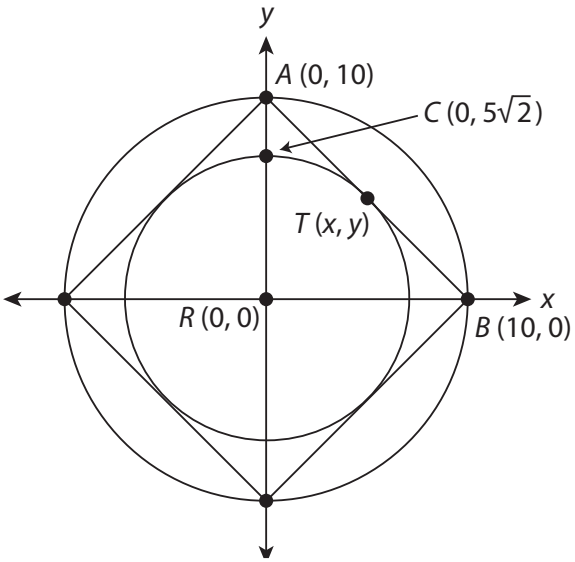
Assessment

Use the given information and diagrams to prove or disprove the statement.

11. Ella is experimenting with designs for a tile mosaic. She draws the design shown on the left and then places it on a coordinate system as shown on the right. She claims that the coordinates of T, the point at which side AB of the square is tangent to the circle, are $T(5, 5)$. Prove or disprove her claim.





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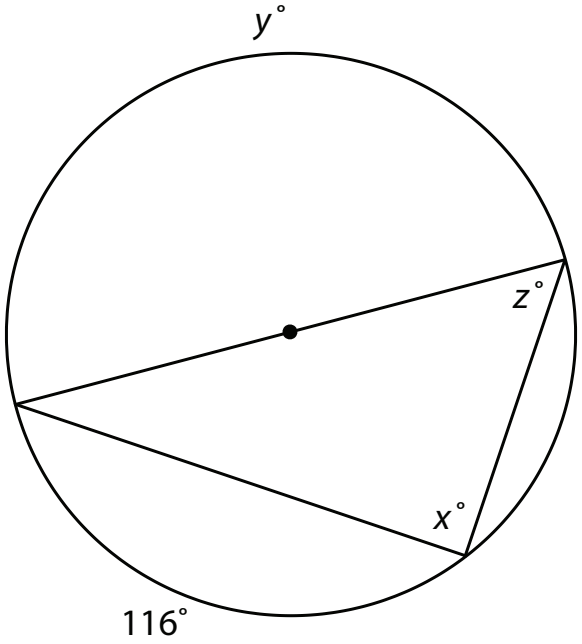
Unit 6 Assessment

Circle the letter of the best answer.

1. What are the values of x, y, and z?



a. $x = 180$, $y = 90$, and $z = 58$

b. $x = 90$, $y = 180$, and $z = 116$

c. $x = 90$, $y = 180$, and $z = 58$

d. $x = 180$, $y = 90$, and $z = 116$

continued

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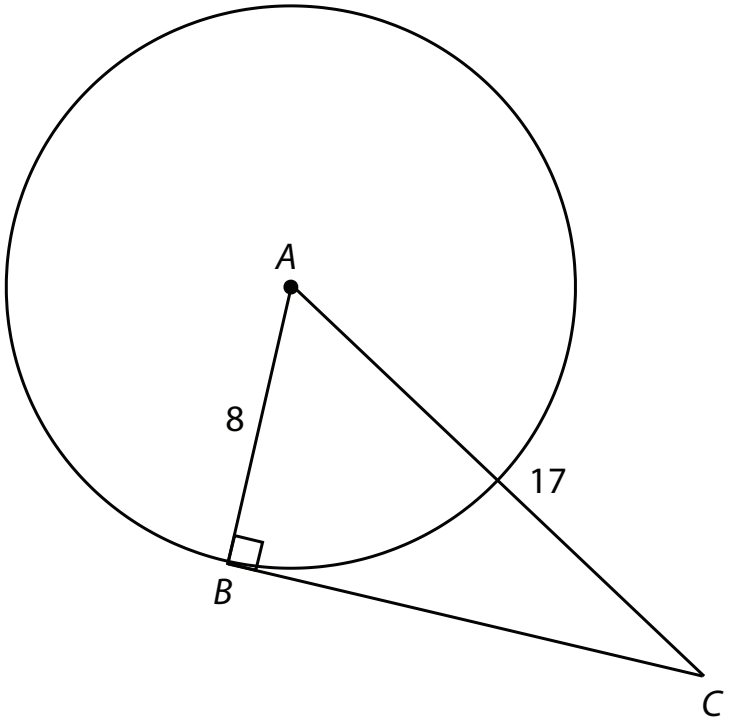
2. $\overline{BC}$ is tangent to circle A at point B. What is the length of $\overline{BC}$?



a. 18.8 units

b. 15 units

c. 16 units

d. 8.5 units

3. The center point of a circle inscribed in a triangle is which point of concurrency?

a. incenter

b. circumcenter

c. centroid

d. orthocenter

continued

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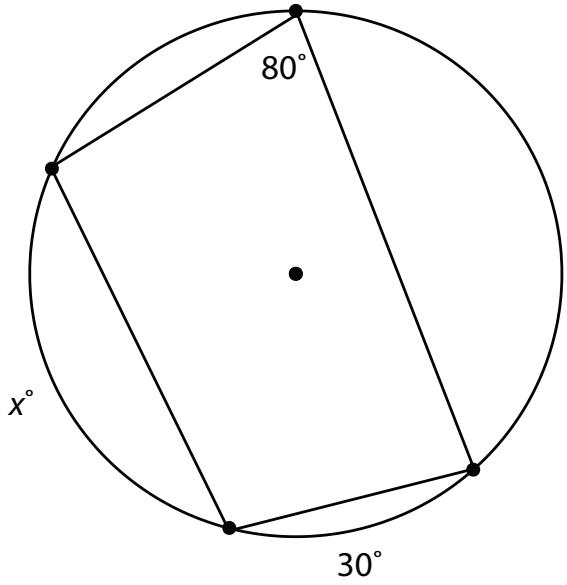
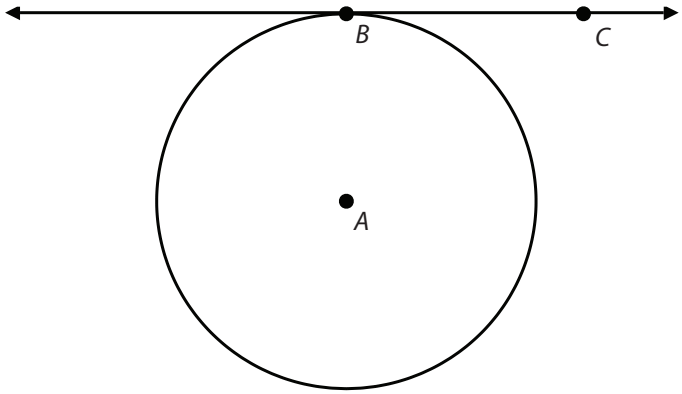
4. What is the value of x?



a. 150
b. 100
c. 160
d. 130

5. $\overleftrightarrow{BC}$ is tangent to circle A, as shown. Which is NOT a step in the process for constructing the tangent?



a. Use a protractor to measure a 90° angle at point A.
b. Draw a straight line from center A through point B and beyond B.
c. Draw an arc on either side of point B.
d. Put the compass on point B and set the compass to a width less than the distance of $\overline{AB}$.

continued

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6. A circle has a circumference of 113 units. Find the length of the arc intercepted by a central angle of 35°.

   a. 10.98 units

   b. 197.48 units

   c. 21.38 units

   d. 3,955 units

7. A circle has a diameter of 12 units. Find the area of a sector with a central angle of 135°.

   a. 14.14 units^2

   b. 169.65 units^2

   c. 810 units^2

   d. 42.41 units^2

8. If the following regular polygons were inscribed in a circle, which one would have a perimeter that was closest in length to the circumference of the circle?

   a. octagon

   b. pentagon

   c. dodecagon

   d. hexagon

9. If the radius of a sphere is cut in half, how will the volumes of the original sphere and the new sphere compare?

   a. The volume of the new sphere will be $\frac{1}{8}$ of the volume of the original sphere.

   b. The volume of the new sphere will be $\frac{1}{4}$ of the volume of the original sphere.

   c. The volume of the new sphere will be $\frac{1}{2}$ of the volume of the original sphere.

   d. The volume of the new sphere will be 2 times the volume of the original sphere.

continued

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10. Which is an equation of the circle that has center (–3, 1) and radius 3?

    a. $(x-3)^2+(y+1)^2=3$

    b. $(x+3)^2+(y-1)^2=3$

    c. $(x-3)^2+(y+1)^2=9$

    d. $(x+3)^2+(y-1)^2=9$

11. Which is an equation of the parabola that has focus (–4, 1) and directrix $y = -5$?

    a. $(x+4)^2=8(y+1)$

    b. $(x+4)^2=12(y+1)$

    c. $(x+4)^2=8(y+2)$

    d. $(x+4)^2=12(y+2)$

12. *ABCD* is a rectangle inscribed in a circle centered at (5, 2). *A* is at (0, 2), *B* is at (8, 6), and *C* is at (10, 2). What are the coordinates of *D*?

    a. (1, –1)

    b. (1, 0)

    c. (2, –2)

    d. (5, –3)

continued

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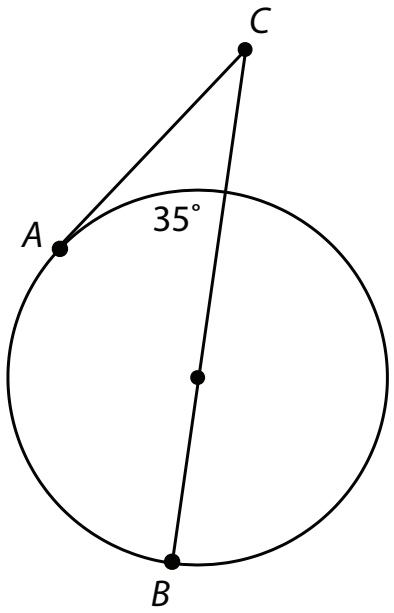
Use the diagrams and given information to complete problems 13–15.

13. A weather satellite is represented by point C in the diagram below. The satellite covers the area in between the signals, represented by the segments extending from point C. One signal hits the exterior of the coverage area in exactly one point. The other signal hits the exterior of the coverage area in exactly 2 points.



a. What is the measure of $\angle ACB$?

b. What is the measure of minor $\overset{\frown}{AB}$?

c. What is the measure of $\angle ABC$?

continued

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14. A spherical scoop of ice cream is 2.5 inches in diameter and sits atop a waffle cone that has the same diameter. The height of the waffle cone is 6 inches.

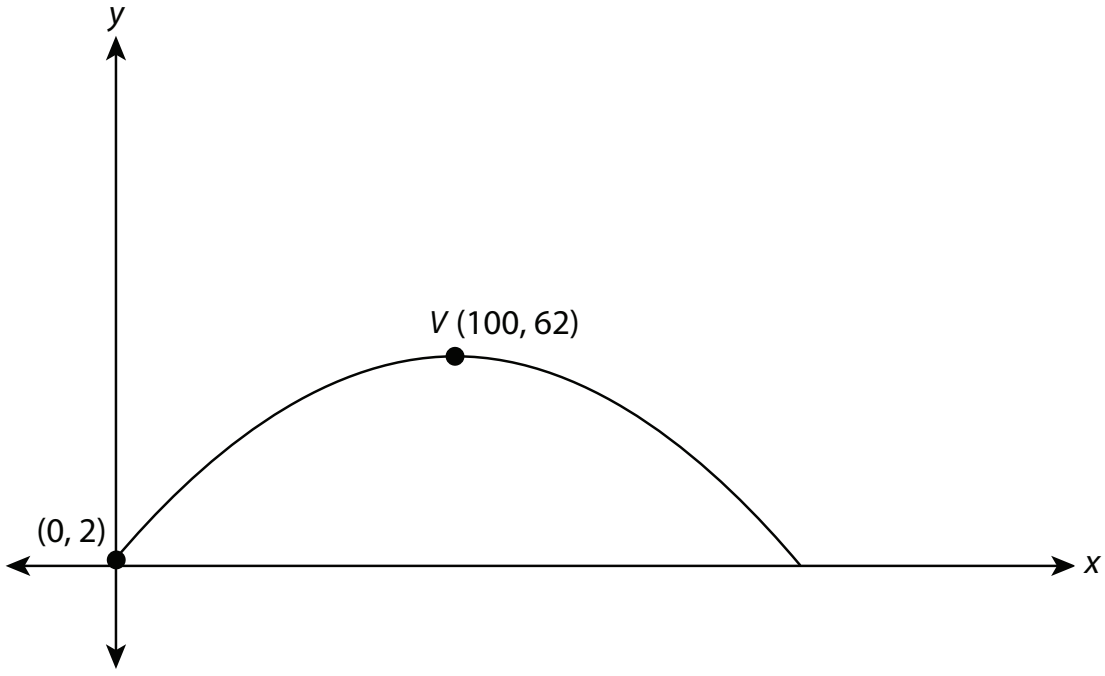
    a. Assuming the scoop is solid, what is the volume of the ice cream scoop?

    b. What is the volume of the waffle cone?

    c. If the ice cream melts and drips directly into the cone, will the cone be able to hold all of the melted ice cream? Explain.

    d. If there were two identical ice cream scoops and they both melted directly into the cone, would the cone be able to hold all of the melted ice cream? Explain.

15. The graph below represents the parabolic path of a softball, which a batter hit when the ball was 2 feet above the ground. The *x*-axis represents the ground and the units are in feet. The ball reached a maximum height of 62 feet after traveling 100 feet horizontally. Prove or disprove that the ball was 47 feet above the ground after it traveled 150 feet horizontally.



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Answer Key

Lesson 1: Introducing Circles

Pre-Assessment, p. U6-1

1. a
2. c
3. b
4. d
5. b

Warm-Up 6.1.1, p. U6-5

1. The scale factor used to create the smaller plate is 3/5.
2. $FD = 4.5$ inches; $ED = 4.2$ inches

Practice 6.1.1: Similar Circles and Central and Inscribed Angles, p. U6-24

1. 3/2
2. 15/11
3. 4/3 ft
4. 18.8 in
5. $x = 75$; $y = 73$
6. $x = 18$; $m\overset{\frown}{AB} = 148$
7. $x = 90$; $y = 180$; $z = 31.5$
8. $m\angle C = m\angle D = 20$
9. $m\angle B = 55$; $m\angle C = 35$
10. $m\overset{\frown}{BC} = 96$; $m\overset{\frown}{CA} = 154$

Warm-Up 6.1.2, p. U6-27

1. $\angle D$ is a central angle and $\angle B$ is an inscribed angle. The measure of the central angle is twice that of the inscribed angle. Therefore, $m\angle D$ is twice that of $m\angle B$.
2. Equation: $x + 12 = 2x$; $x = 12$
3. $m\angle D = 24$; $m\angle B = 12$

Practice 6.1.2: Chord Central Angles Conjecture, p. U6-38

1. $m\overset{\frown}{BDC} = 296$
2. $m\angle BAC = 122$
3. $t = 2$
4. $\overset{\frown}{HI} \cong \overset{\frown}{KJ}$
5. No; no information about chord lengths or arc measures is given. There is no way to determine the measure of the central angles without this information.
6. $y = 35$
7. $t = 3$
8. $t = 23$
9. 8
10. 210

Warm-Up 6.1.3, p. U6-41

1. 25
2. It can be concluded that both lines form right angles with the radius of the circle, thus they are perpendicular with the radius. It can also be concluded, by the Pythagorean Theorem, that both segments form right triangles with the radius and an additional line drawn from the center of the circle to a point on the segment exterior to the circle.
3. $TW = TF = \sqrt{65}$
4. Lennon can conclude that the two segments are congruent or equal in length.

Practice 6.1.3: Properties of Tangents of a Circle, p. U6-57

1. $x = 17$
2. yes
3. $QR = 35$
4. 24 cm
5. $BC = 21$
6. If $\triangle ABC$ is a right triangle, then $\angle CAB$ is a right angle, so radius $\overline{CA}$ is perpendicular to $\overline{BA}$, which must be true for a line to be tangent to a circle.
7. $QR = 11$
8. Yes, the logo will fit. The tangent is 45 feet and the radius of the circle is 5 feet. Added together, they do not exceed 60 feet.
9. 28°
10. $x = 11.5$ feet; each side measures 92 feet.

Progress Assessment, p. U6-61

1. a
2. b
3. a
4. d
5. c
6. b
7. c
8. c
9. c
10. c
11. a. –2
    b. 74°
    c. approximately 7.8 units
    d. approximately 7.8 units
    e. Subtract the length of $\overline{EB}$ from the length of $\overline{EF}$. Draw a line from point O to point F and use the Pythagorean Theorem to find $\overline{OF}$.

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Lesson 2: Inscribed Polygons and Circumscribed Triangles

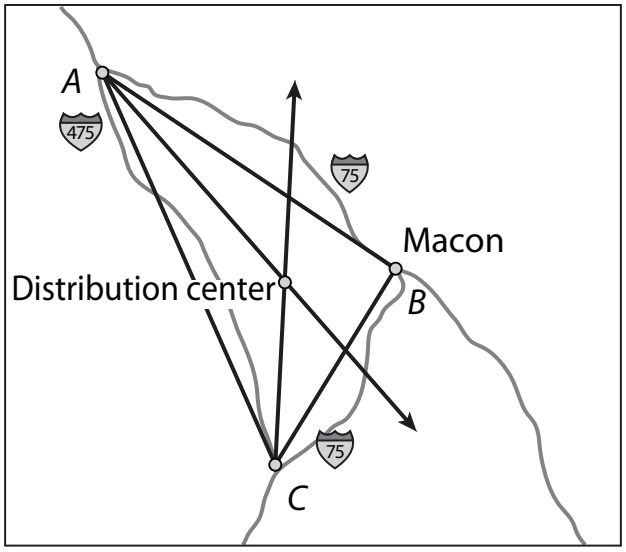
Pre-Assessment, p. U6-66

1. b
2. b
3. d
4. c
5. b

Warm-Up 6.2.1, p. U6-69

1. To stay the same distance from either path, you must walk along the angle bisector of the two paths.
2. The ray with endpoint *C* creates two equal angles; therefore, it is the angle bisector of the two paths.

Practice 6.2.1: Constructing Inscribed Circles, p. U6-90

1. Students' constructions may vary. Check for accuracy.
2. Students' constructions may vary. Check for accuracy.
3. Students' constructions may vary. Check for accuracy.
4. Yes, the incenter will always be found within the interior of the triangle. In order to be equidistant from all three sides, that point cannot occur outside or on the triangle. For the incenter to be the center of the inscribed circle, it must occur within the interior.
5. 

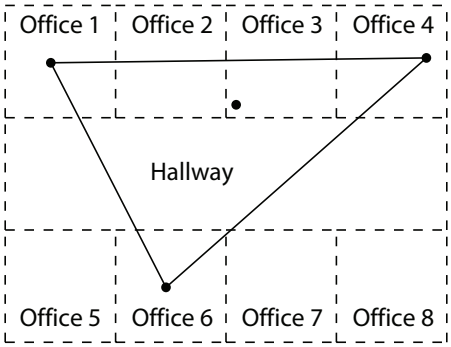
6. $BI \approx 6.71$ units
7. 145°
8. 112.5°
9. $\triangle AIB \cong \triangle AIC$ by ASA. Therefore, $\overline{BI} \cong \overline{CI}$.
10. Quadrilateral *AFDG* is a kite. Since tangents drawn from the same external point are congruent, $AG = AF$. Since they are radii from the same circle, $DG = DF$. The same argument applies to quadrilaterals *BGDE* and *CEDF*.

Warm-Up 6.2.2, p. U6-94

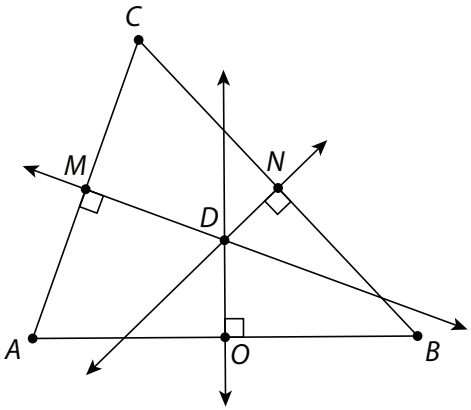
1. Figure A is a perpendicular bisector; therefore, it represents the relationship between the fold and the triangle when the paper is folded correctly.
2. Choice b is correct.
3. The sides of the paper along the hypotenuse of the original triangle must be equal. The fold must lie on the perpendicular bisector of the segment formed by the ends of the longest side of the triangular paper.

Practice 6.2.2: Constructing Circumscribed Circles, p. U6-108

1. Students' constructions may vary. Check for accuracy.
2. Students' constructions may vary. Check for accuracy.
3. Students' constructions may vary. Check for accuracy.
4. circumcenter
5. $AB = 10$
6. No; the circumcenter is located outside of the obtuse triangle.
7. The control box should be placed in the corner of Office 3 that is closest to Office 2 and the hallway.



8. Since every point on the perpendicular bisector is equidistant from the endpoints of the segment, $DA = DC$ and $DB = DC$. Through the Transitive Property, $DA = DB$; therefore, the circumcenter is equidistant from the vertices of the triangle.



9. It is possible for the incenter and the circumcenter to be located at the same point when the triangle is equilateral. This point of concurrency is equidistant from the sides of the triangle and equidistant from the vertices.

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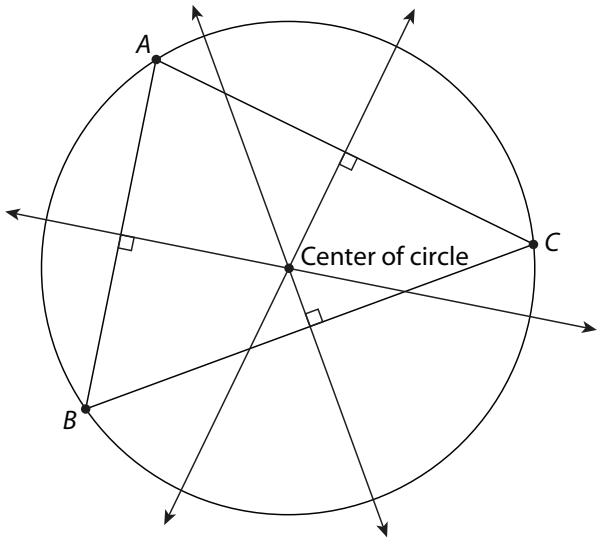
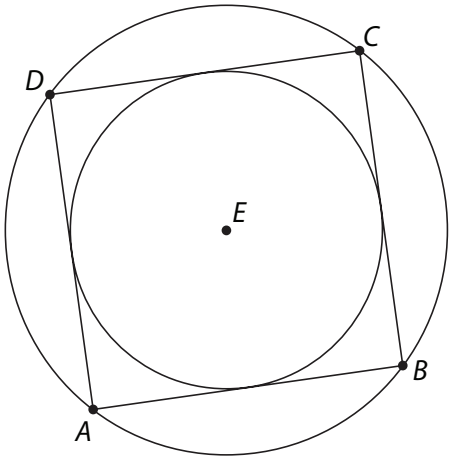
10. Methods may vary. Sample method: Choose any three points on the circle and construct the inscribed triangle determined by those three points. Then, find the perpendicular bisector of at least two of the sides.



Warm-Up 6.2.3, p. U6-111

1. Jordan needs to turn another 80°.
2. Jordan is turning $\frac{7}{9}$ of a full 360° turn.

Practice 6.2.3: Proving Properties of Inscribed Quadrilaterals, p. U6-123

1. $x = 65; y = 90$
2. $x = 44$
3. Yes, $\overline{EF} \cong \overline{FD}$. $m\overset{\frown}{EFD} = 140$ because it is twice the measure of $\angle ECD$. If $m\overset{\frown}{FD} = 70$, then $m\overset{\frown}{EF} = 70$, since congruent arcs determine congruent chords.
4. 

5. $x = 124; y = 112; z = 42$
6. $m\overset{\frown}{BC} = 78$
7. $m\angle B = 105$
8. $m\angle BCD = 85$
9. Yes, if it is an isosceles trapezoid. The opposite angles and also the consecutive angles must be supplementary.
10. The diagonals are also diameters.

Progress Assessment, p. U6-126

1. d
2. c
3. b
4. b
5. b
6. c
7. b
8. c
9. c
10. c
11. Opposite angles of an inscribed quadrilateral are supplementary. Create two equations: $26y + 2x = 180$ and $21y + 3x = 180$. Solve the first equation for x and substitute into the second equation: $21y + 3(90 - 13y) = 170$. Solve the equation for y; $y = 5$. Substitute the value of y into either of the original equations and solve for x; $x = 25$.

Lesson 3: Constructing Tangent Lines

Pre-Assessment, p. U6-130

1. b
2. c
3. a
4. c
5. b

Warm-Up 6.3.1, p. U6-134

1. The slope of the radius is needed to determine the slope of the line (representing the trail) that is tangent to the circular playground. The slope is necessary since the radius and the tangent line are perpendicular to each other, and, therefore, their slopes are negative reciprocals.
2. The slope of the radius is $\frac{4}{3}$. The slope of the perpendicular line representing the trail is $-\frac{3}{4}$. Use the slope $-\frac{3}{4}$ and the point (6, 3) to find another point on the trail.
3. $y - 3 = -\frac{3}{4}(x - 6)$ or $y = -\frac{3}{4}x + \frac{15}{2}$

Practice 6.3.1: Constructing Tangent Lines, p. U6-158

1. Answers will vary; check students' work for accuracy.
2. Answers will vary; check students' work for accuracy.
3. Answers will vary; check students' work for accuracy.
4. Draw a straight line connecting the center of circle *D* with point *A*.
5. It was placed between points *R* and *S*.
6. Draw a line from *A* through *B* and extending beyond *B*. Put the compass on point *B* and set it for a width less than the distance of $\overline{AB}$. Draw an arc on the ray on either side of point *B*. Label the intersection points *D* and *E*. Set the compass greater than the distance from *B* to *D*. Put the sharp edge of the compass on point *D* and draw an arc on $\overline{AB}$. Without changing the compass, put the sharp edge on point *E* and draw an arc intersecting the first one. The intersection is point *C*. Draw the tangent $\overleftrightarrow{CB}$.
7. Usually, two lines that are tangent to the same circle intersect. The exception is if the points of tangency are directly across from each other, creating parallel lines.
8. Answers will vary; check students' work for accuracy.

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9. They are the same length because when the construction was done, the student placed the sharp edge of the compass on the perpendicular bisector of $\overline{AL}$, and then marked the points of tangency on the circle correctly without changing the compass setting.
10. You can conclude that a line perpendicular to $\overline{BC}$ was constructed at point *B*. Point *A* is a point on this line, which is tangent to circle *C*.

Progress Assessment, p. U6-160

1. d
2. a
3. b
4. c
5. b
6. c
7. c
8. b
9. b
10. a
11. a. two walkways
    b. Constructions will vary; check students' work for accuracy.

Lesson 4: Finding Arc Lengths and Areas of Sectors

Pre-Assessment, p. U6-164

1. b
2. c
3. d
4. b
5. c

Warm-Up 6.4.1, p. U6-167

1. 60π feet $\approx$ 188.50 feet
2. 30π feet $\approx$ 94.25 feet

Practice 6.4.1: Defining Radians, p. U6-179

1. $4\pi/9$ radians
2. 150°
3. 3 radians
4. 52.2 units
5. $22\pi/5$ units $\approx$ 13.82 units
6. 5.86 units
7. 53 mm
8. $5\pi/6$ radians
9. $\approx$ 3,403.39 feet
10. Arc measure is the measure of an arc in terms of the central angle that it intercepts, generally measured in degrees or radians, whereas arc length is a measure of length or a fraction of the circumference.

Warm-Up 6.4.2, p. U6-180

1. $\pi + 2$ feet $\approx$ 5.14 feet
2. $\pi/2$ square feet $\approx$ 1.57 square feet

Practice 6.4.2: Deriving the Formula for the Area of a Sector, p. U6-191

1. 2,198.208 square units
2. $24\pi \approx$ 75.40 square units
3. $17{,}248\pi/45 \approx$ 1204.14 square units
4. 24 square units
5. 12 units
6. 52 square units
7. $9\pi/4 \approx$ 7.07 square inches
8. $\approx$ 10.31 square inches
9. $\approx$ 2,652.90 square feet
10. $\approx$ 4.89 square miles

Progress Assessment, p. U6-192

1. a
2. a
3. b
4. d
5. c
6. b
7. c
8. c
9. b
10. d
11. a. $\pi/4$ radians
    b. 45°
    c. $\approx$ 0.22 square foot
    d. $\approx$ 2.09 feet

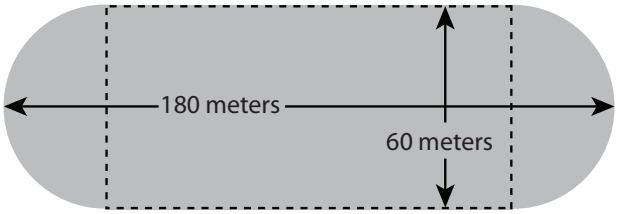
Lesson 5: Explaining and Applying Area and Volume Formulas

Pre-Assessment, p. U6-194

1. a
2. a
3. c
4. c
5. b

Warm-Up 6.5.1, p. U6-198

1. Check students' diagrams for accuracy. Sample diagram:



2. There are two semicircles (which make one full circle) and one rectangle.
3. Formula for area of a rectangle: $l \bullet w$; formula for area of a circle: πr^2
4. The radius of the circle is half the width of the field: $r = 30$ m. The length of the rectangle is 180 – 30 – 30 = 120 m.
5. $(120 \bullet 60) + (\pi \bullet 30^2) \approx 10{,}027.4$ m^2

Practice 6.5.1: Circumference and Area of a Circle, p. U6-218

1. $C \approx$ 25.133 m; perimeter $\approx$ 24.492 m. The circle is larger than the inscribed octagon.
2. $C \approx$ 25.133 m; perimeter $\approx$ 24.8464 m. The circle is larger than the inscribed dodecagon.
3. The dodecagon is the better approximation. The perimeter of a 15-sided polygon is approximately 24.949 m. This is even closer to the circumference.
4. Area of circle $\approx$ 50.26 m^2; octagon area $\approx$ 45.25 m^2. Some of the area of the circle is not included in the octagon.

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5. Area of circle ≈ 50.26 m^2; dodecagon area ≈ 48 m^2. Some of the area of the circle is not included in the dodecagon.
6. The dodecagon is closer; 36-sided area: 50.01 m^2. This is very close to the actual area of the circle.
7. ≈ 81.5 square feet
8. ≈ 127.3 square inches
9. ≈ 25.07 square feet
10. ≈ 43.42 square feet

Warm-Up 6.5.2, p. U6-220

1. Both are three-dimensional objects. Both have a height and two parallel bases.
2. Since a cylinder resembles a prism in some ways, the volume formulas are similar. They are similar in that the basic premise of the formula is to multiply the area of the base times the height. They are different because the areas of the bases are calculated differently. While the prism base is *length* • *width*, the base of the cylinder is πr^2.
3. $V_{prism} = A_{base}$ • height; $V_{cylinder} = A_{base}$ • height; $V_{cylinder} = \pi r^2 \bullet h$
4. The gardener needs 8 water barrels to collect at least 5 cubic meters of water.

Practice 6.5.2: Volumes of Cylinders, Pyramids, Cones, and Spheres, p. U6-240

1. ≈ 3,319,683 gallons
2. ≈ 2.215 liters
3. ≈ 22.92 cubic inches
4. ≈ 3.38 inches
5. ≈ 213.835 gallons
6. ≈ 35.68 meters
7. 25 cubic feet
8. 164,200,792,894 km^3
9. The volume increases eightfold.
10. 179.50 in^3

Progress Assessment, p. U6-242

1. a
2. d
3. b
4. a
5. b
6. a
7. c
8. b
9. c
10. a
11. a. 3 in
    b. 452.16 in^3
    c. Yes, it is big enough. It holds 56.52 in^3 of rocket fuel.
    d. 555.78 in^3
    e. No, it cannot fly. The volume of the rocket is too large.

Lesson 6: Deriving Equations

Pre-Assessment, p. U6-245

1. d
2. a
3. d
4. b
5. b

Warm-Up 6.6.1, p. U6-249

1. 5
2. 13

Practice 6.6.1: Deriving the Equation of a Circle, p. U6-274

1. $x^2 + y^2 = 81$
2. $(x + 5)^2 + (y - 1)^2 = 16$
3. $x^2 + (y + 3.2)^2 = 4.84$
4. $(x + 2)^2 + (y - 2)^2 = 34$
5. $x^2 + y^2 - 10x + 7 = 0$
6. center: (–3, 2); radius = $\sqrt{40} = 2\sqrt{10}$
7. center: (–1/2, 3/2); radius = 1/2
8. Standard equation: $(x + 3)^2 + (y - 5)^2 = 144$. No, a cell phone user at (8, 0) is not within the service range because the distance from (8, 0) to (–3, 5) is $\sqrt{146}$ or approximately 12.08 miles, which is greater than 12 miles.
9. Standard equation: $x^2 + y^2 = 25$. All three customers are on the outer boundary of the delivery region because the distance from each customer to the origin is 5 miles.
10. The region is a circle with center (–1.5, –0.5) and radius 3 miles.

Warm-Up 6.6.2, p. U6-275

1. Point *C* is at (0, 6). Explanation: It is on the *y*-axis, so its *x*-coordinate is 0. To find the *y*-coordinate, substitute 0 for *x* in the equation of the parabola: $y = -\frac{1}{4}(0)^2 + 6 = 6$.
2. *A* is at (–2, 0), *B* is at (–2, 5), *D* is at (2, 5), and *E* is at (2, 0). Since the width of the window is 4 feet and the window is symmetric about the *y*-axis, the *x*-coordinates of *A* and *B* are –2 and the *x*-coordinates of *D* and *E* are 2. Points *A* and *E* are on the *x*-axis, so each has a *y*-coordinate of 0. To find the *y*-coordinate of *B*, substitute –2 into the equation of the parabola: $y = -\frac{1}{4}(-2)^2 + 6 = -1 + 6 = 5$. To find the *y*-coordinate of *D*, substitute 2 into the equation of the parabola: $y = -\frac{1}{4}(2)^2 + 6 = -1 + 6 = 5$.

Practice 6.6.2: Deriving the Equation of a Parabola, p. U6-301

1. $x^2 = 20y$; $\sqrt{(x-0)^2 + (y-5)^2} = \sqrt{(x-x)^2 + (y+5)^2}$
2. $y^2 = -8x$; $\sqrt{(x+2)^2 + (y-0)^2} = \sqrt{(x-2)^2 + (y-y)^2}$
3. $(x + 3)^2 = 16(y - 1)$; $\sqrt{(x+3)^2 + (y-5)^2} = \sqrt{(x-x)^2 + (y+3)^2}$
4. $(y + 3)^2 = -16(x - 4)$; $\sqrt{(x-0)^2 + (y+3)^2} = \sqrt{(x-8)^2 + (y-y)^2}$
5. $(x - 4.5)^2 = -14(y + 2)$
6. $(y - 20)^2 = -240(x - 50)$
7. focus: (3, 1.5); vertex: (3, –1); directrix: $y = -3.5$
8. $x^2 = 13y$

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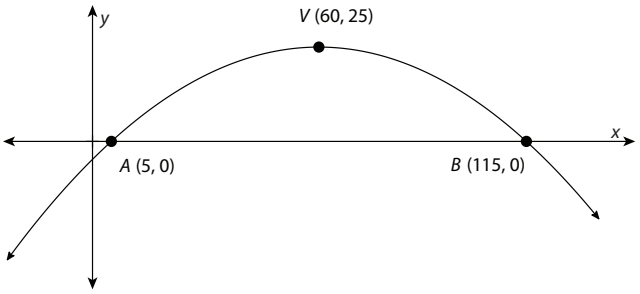
9. equation: $(x-60)^2 = -121(y-25)$; height = 25 ft; AB = 110 ft

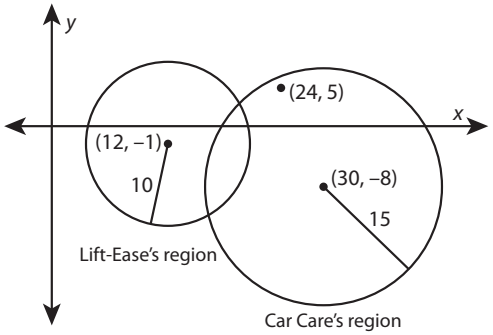


10. equation: $x^2 = 160y$; depth = 14.4 inches

Progress Assessment, p. U6-303

1. d
2. b
3. c
4. c
5. a
6. b
7. d
8. b
9. a
10. a

11. a. The region for Lift-Ease is the interior of the circle with center (12, –1) and radius 10.
    b. The region for Car Care is the interior of the circle with center (30, –8) and radius 15.
    c. The motorist is not within Lift-Ease's region because the distance between (24, 5) and (12, –1) is $\sqrt{180}$ or approximately 13.42 miles, which is greater than 10 miles, the radius for Lift-Ease. The motorist is within Car Care's region because the distance between (24, 5) and (30, –8) is $\sqrt{205}$ or approximately 14.32 miles, which is less than 15 miles, the radius for Car Care.
    d.



Lesson 7: Using Coordinates to Prove Geometric Theorems About Circles and Parabolas

Pre-Assessment, p. U6-306

1. d
2. d
3. a
4. c
5. c

Warm-Up 6.7.1, p. U6-310

1. $AB = \sqrt{(6-3)^2 + (7-1)^2} = \sqrt{9+36} = \sqrt{45} = \sqrt{9} \bullet \sqrt{5} = 3\sqrt{5}$
2. The slope of $\overline{PQ}$ is 2. Since the slope of $AB = \frac{7-1}{6-3} = \frac{6}{3} = 2$, and $\overline{PQ}$ and $\overline{AB}$ are parallel, they have the same slope.

Practice 6.7.1: Using Coordinates to Prove Geometric Theorems About Circles and Parabolas, p. U6-333

1. The statement has been proved. Justification: The radius = $QR = 4$, and $PR = \sqrt{(2-0)^2 + (2\sqrt{3}-0)^2} = \sqrt{4+12} = \sqrt{16} = 4$; therefore, P lies on the circle.
2. The statement has been disproved. Justification: The radius $= AB = \sqrt{(-3-0)^2 + (\sqrt{55}-0)^2} = \sqrt{9+55} = \sqrt{64} = 8$ and $AQ = \sqrt{(6-0)^2 + (2\sqrt{5}-0)^2} = \sqrt{36+20} = \sqrt{56} \neq 8$; therefore, Q does not lie on the circle.
3. The statement has been proved. Justification: $p = 1/4$, the vertex is (0, 0), and the equation is $x^2 = y$. A, B, and C are all on the parabola.
4. The statement has been proved. Justification: Use the form $y = ax^2 + bx + c$. The point (0, 0) is on the graph, so $c = 0$ and $y = ax^2 + bx$. The vertex is (0, 0), so $-b/2a = 0$, $b = 0$ and $y = ax^2$. The point (5, 25) is on the graph, so $25 = a(5)^2$, $a = 1$, and the equation is $y = x^2$. A, B, C, and D are all on the parabola.
5. The statement has been proved. Justification: Use the form $y = ax^2 + bx + c$. The point (0, 2) is on the graph, so $c = 2$ and $y = ax^2 + bx + 2$. The vertex is (2, 0), so $-b/2a = 2$, $b = -4a$, and $y = ax^2 - 4ax + 2$. The point (2, 0) is on the graph, so $0 = a(2)^2 - 4a(2) + 2$, $a = 1/2$, and the equation is $y = (1/2)x^2 - 2x + 2$. P, Q, R, and S are all on the parabola.
6. The statement has been proved. Justification: The radius = $CD = \sqrt{[1-(-1)]^2 + (-2-2)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$ and $CP = \sqrt{[-5-(-1)]^2 + (4-2)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$; therefore, P lies on the circle.
7. The statement has been proved. Justification: The radius = $PQ = \sqrt{(-3-0)^2 + (\sqrt{91}-0)^2} = \sqrt{9+91} = \sqrt{100} = 10$; A, B, and C are all on the circle.
   $AC = \sqrt{[8-(-10)]^2 + (6-0)^2} = \sqrt{324+36} = \sqrt{360} = 6\sqrt{10}$ and
   $BC = \sqrt{[8-(-10)]^2 + (-6-0)^2} = \sqrt{324+36} = \sqrt{360} = 6\sqrt{10}$;
   therefore, ABC is an isosceles triangle.

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8. Bradley does not get a prize. The ears have radii $r_1 = \sqrt{(13-10)^2 + (54-50)^2} = 5$ and $r_2 = \sqrt{(26-30)^2 + (47-50)^2} = 5$. Point A (14, 46) is closer to the left ear with center C_1, and $AC_1 = \sqrt{(14-10)^2 + (46-50)^2} = \sqrt{16+16} = \sqrt{32}$. Since $\sqrt{32} > 5$, point A is outside the shaded ear, so hitting point A does not result in winning. Point B (26, 55) is closer to the right ear with center C_2, and $BC_2 = \sqrt{(26-30)^2 + (55-50)^2} = \sqrt{16+25} = \sqrt{41}$. Since $\sqrt{41} > 5$, point B is outside the shaded ear, and hitting point B does not result in winning.

9. The statement has been proved. Justification: The left parabola, through A and C, has an equation of the form $(y-k)^2 = 4p(x-h)$ with $p = -2$ and vertex $(h, k) = (-1, 0)$, so its equation is $(y-0)^2 = 4(-2)[x-(-1)]$, or $y^2 = -8(x+1)$. The right parabola, through B and D, has an equation of the form $(y-k)^2 = 4p(x-h)$ with $p = 2$ and vertex $(h, k) = (1, 0)$, so its equation is $(y-0)^2 = 4(2)(x-1)$, or $y^2 = 8(x-1)$. The radius of the circle is $RP = \sqrt{(2-0)^2 + (\sqrt{21}-0)^2} = \sqrt{4+21} = \sqrt{25} = 5$, so the equation of the circle is $x^2 + y^2 = 25$. Points A (–3, 4) and C (–3, –4) are on the left parabola and the circle. Points B (3, 4) and D (3, –4) are on the right parabola and the circle.

10. The statement has been proved. Justification: Use the form $y = ax^2 + bx + c$. The point (0, 3) is on the graph, so $c = 3$ and $y = ax^2 + bx + 3$. The vertex is (120, 75), so $-b/2a = 120$, $b = -240a$, and $y = ax^2 - 240ax + 3$. The point (120, 75) is on the graph, so $75 = a(120)^2 - 240a(120) + 3$ and $a = -0.005$. Therefore, the equation is $y = -0.005x^2 - 240(-0.005)x + 3$, or $y = -0.005x^2 + 1.2x + 3$. When $x = 200$, $y = -0.005(200)^2 + 1.2(200) + 3 = 43$.

Progress Assessment, p. U6-336

1. d	4. d	7. d	10. a
2. a	5. c	8. a	
3. c	6. b	9. b	

11. The statement has been proved. Justification: The radius of the inner circle is $5\sqrt{2}$, so the equation of the inner circle is $x^2 + y^2 = (5\sqrt{2})^2$, or $x^2 + y^2 = 50$. The point (5, 5) is on the inner circle because $5^2 + 5^2 = 50$. The slope of the line through points A and B is $\frac{10-0}{0-10} = -1$ and its y-intercept is 10, so its equation is $y = -x + 10$. The point (5, 5) is on that line because $5 = -5 + 10$. So (5, 5) is on the circle and also on the line containing side AB of the square. Furthermore, because a line is tangent to a circle at only one point, the point (5, 5) and the point T are the same point. Therefore, the coordinates of T are (5, 5).

Unit Assessment p. U6-339

1. c	4. d	7. d	10. d
2. b	5. a	8. c	11. d
3. a	6. a	9. a	12. c

13. a. $m\angle ACB = 55$
    b. $m\overset{\frown}{AB} = 145$
    c. $m\angle ABC = 17.5$
14. a. approximately 8.2 in^3
    b. approximately 9.8 in^3
    c. Yes, the cone will be able to hold all of the ice cream because the volume of the cone is greater than the volume of the scoop of ice cream. In fact, there are about 1.6 in^3 of space left over.
    d. No; the ice cream would overflow because the volume of two scoops of ice cream is approximately 16.4 in^3. This is more than the volume of the cone, which is approximately 9.8 in^3. The overflow volume is about 6.6 in^3.
15. The statement has been proved. Justification: Use the form $y = ax^2 + bx + c$. The point (0, 2) is on the graph, so $c = 2$ and $y = ax^2 + bx + 2$. The vertex is (100, 62), so $-\frac{b}{2a} = 100$, $b = -200a$, and $y = ax^2 - 200ax + 2$. The point (100, 62) is on the graph, so $62 = a(100)^2 - 200a(100) + 2$ and $a = -0.006$. So the equation is $y = -0.006x^2 - 200(-0.006)x + 2$, or $y = -0.006x^2 + 1.2x + 2$. When $x = 150$, $y = -0.006(150)^2 + 1.2(150) + 2 = 47$.

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Student Book Answer Key

Lesson 1: Introducing Circles

Practice 6.1.1: Similar Circles and Central and Inscribed Angles, p. U6-17

1. 1/48
2. 6/25
3. 8/7
4. 47 in
5. $x = 100$; $y = 37.5$; $z = 85$
6. $x = 6$; $m\overset{\frown}{AB} = 26$
7. $x = 90$; $y = 30$
8. $m\angle C = m\angle D = 10$
9. $m\angle B = 50$; $m\angle C = 40$
10. $m\overset{\frown}{AB} = 140$; $m\overset{\frown}{CA} = 124$

Practice 6.1.2: Chord Central Angles Conjecture, p. U6-27

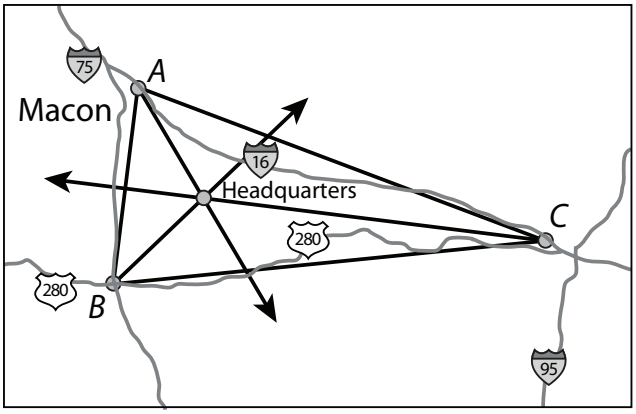
1. $m\overset{\frown}{BDC} = 288$
2. $m\angle BAC = 45$
3. $w = 8.5$
4. The circles are not congruent.
5. $a = 8$
6. $b = 2.4$
7. $b = 5.5$
8. $b = 4$
9. 27.3 mm
10. 2.8 ft

Practice 6.1.3: Properties of Tangents of a Circle, p. U6-38

1. $x = 8$
2. You must prove that radius $\overline{SP}$ is perpendicular to $\overline{PR}$ at point P.
3. $m\angle ACB = 79$
4. 12°
5. Yes. Sample explanation: Since the converse of the Pythagorean Theorem is true, $\triangle XYZ$ is a right triangle; therefore, $\angle ZXY$ is a right angle. This makes $\overline{XY}$ perpendicular to $\overline{XZ}$; therefore, $\overline{XY}$ is tangent to $\odot Z$.
6. There is an infinite number because the slopes can vary through one point.
7. $\frac{3}{2}$
8. 144 yards
9. 54 feet
10. A tangent only touches a circle at one point. Tangents can be lines or segments. A chord is a segment only. It touches a circle twice because both endpoints must be on the circle.

Lesson 2: Inscribed Polygons and Circumscribed Triangles

Practice 6.2.1: Constructing Inscribed Circles, p. U6-58

1. Students' constructions may vary. Check for accuracy.
2. Students' constructions may vary. Check for accuracy.
3. Students' constructions may vary. Check for accuracy.
4. No, the incenter will never be located outside of the triangle. In order to be equidistant from all three sides, that point cannot occur outside of or on the triangle. For the incenter to be the center of the inscribed circle, it must occur within the interior.
5. 

6. $BI \approx 21.54$ units
7. 135°
8. 107.5°
9. $\triangle ABI$ and $\triangle ACI$ are congruent by ASA.
10. Students' constructions may vary. Check that tangents were constructed at points B, C, and D.

Practice 6.2.2: Constructing Circumscribed Circles, p. U6-70

1. Students' constructions may vary. Check for accuracy.
2. Students' constructions may vary. Check for accuracy.
3. Students' constructions may vary. Check for accuracy.
4. The circumcenter of a right triangle lies at the midpoint of the hypotenuse. This is true of all right triangles.

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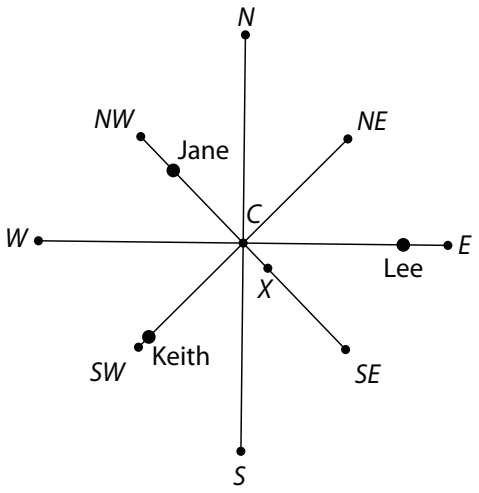
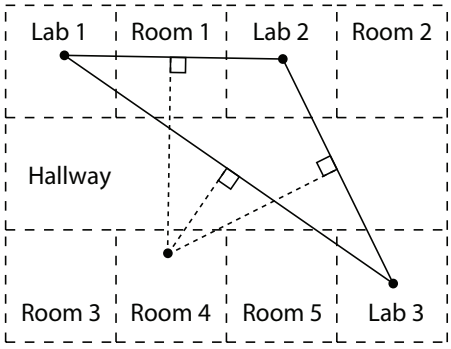
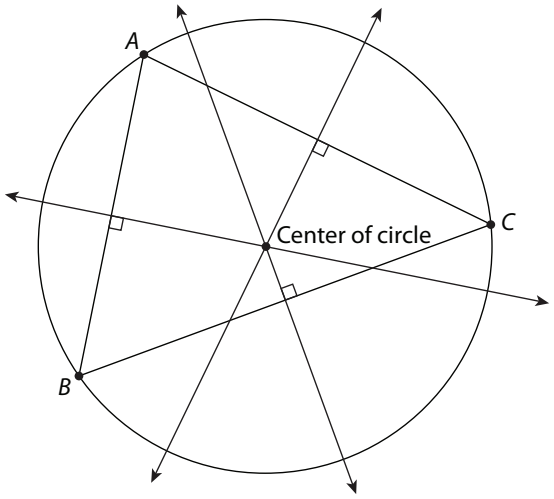
5. Construct the circumcenter to find the location where the friends should meet, point *X*.



6. $AB = 16$
7. The circumcenter is in Room 4; the server should be placed there.

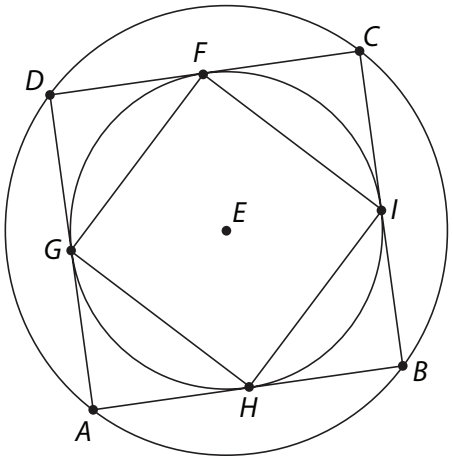


8. The circumcenter will be located outside of the triangle when the triangle is obtuse.
9. Yes, it is possible if the triangle is equilateral. In an equilateral triangle, the angle bisector is the same as the perpendicular bisector.
10. Methods may vary. Sample method: Choose any three points on the circle and construct the inscribed triangle determined by those three points. Then, find the perpendicular bisector of at least two of the sides.



Practice 6.2.3: Proving Properties of Inscribed Quadrilaterals, p. U6-80

1. $x = 104$; $y = 116$
2. $k = 80$; $n = 78$
3. Yes, $\overline{BC} \parallel \overline{DE}$ because $m\overset{\frown}{CD} = m\overset{\frown}{BE}$; these equal arcs have equal chords.
4. Yes, the two squares are similar. All squares are similar.



5. No, the consecutive angles are not supplementary.
6. $m\overset{\frown}{AE} = 30$
7. $m\angle A = 95$
8. $m\angle D = 75$
9. Yes, but only if one pair of opposite angles are right angles.
10. Each diagonal is also a diameter of the circle.

Lesson 3: Constructing Tangent Lines

Practice 6.3.1: Constructing Tangent Lines, p. U6-102

1. Answers will vary; check students' work for accuracy.
2. Answers will vary; check students' work for accuracy.
3. Answers will vary; check students' work for accuracy.
4. Draw a straight line connecting the center of circle *G* with point *A*.
5. It was constructed at the midpoint of point *F* and point *H*.
6. Draw a line from the center of the circle to each of the vertices, then construct the perpendicular bisector of that line.
7. Check that point *M* was properly constructed using the steps to construct a tangent at point *L* on the diagram.
8. The compass could have been placed between points *J* and *K*.
9. Answers will vary; check students' work for accuracy.
10. You can conclude that a line perpendicular to $\overline{AB}$ was constructed at point *B*, and $\overline{BC}$ is tangent to circle *A*.

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Lesson 4: Finding Arc Lengths and Areas of Sectors

Practice 6.4.1: Defining Radians, p. U6-113

1. $13\pi/9$ radians
2. 280.7°
3. 2.9 radians
4. 28 units
5. 138°
6. 83.17 units
7. 11.78 feet
8. $4\pi/3$ radians
9. ≈ 1,570.80 feet
10. Answers will vary. Sample answer: the measure of the central angle that intercepts an arc equal in length to the radius of the circle; π radians = 180°

Practice 6.4.2: Deriving the Formula for the Area of a Sector, p. U6-118

1. 705.6 square units
2. $12\pi \approx 37.70$ square units
3. $289\pi/45 \approx 20.18$ square units
4. 84.27 square units
5. 4 units
6. 376 square units
7. $25\pi/6 \approx 13.09$ square inches
8. ≈ 4.91 square inches
9. ≈ 1,965.94 square feet
10. ≈ 6,414.09 square miles

Lesson 5: Explaining and Applying Area and Volume Formulas

Practice 6.5.1: Circumference and Area of a Circle, p. U6-135

1. Circumference ≈ 6.28318 m; perimeter ≈ 6.18 m. The circumference is larger than the perimeter of the polygon because the circle is larger than the inscribed polygon.
2. Perimeter ≈ 6.26 m. The circumference is larger than the perimeter of the polygon because the circle is larger than the inscribed polygon.
3. The 20-sided polygon gave the closest approximation. Perimeter ≈ 6.2803 m; the perimeter of the 60-sided polygon is very close to the actual circumference.
4. Area of circle ≈ 3.1416 m^2; area of polygon ≈ 2.94 m^2. Some of the area of the circle is not included in the polygon's area.
5. Area of circle ≈ 3.1416 m^2; area of polygon ≈ 3.09 m^2. Some of the area of the circle is not included in the polygon's area.
6. The 20-sided polygon gave the closest approximation to the area; 60-sided area ≈ 3.1358 m^2. This is very close to the actual area.
7. ≈ 97.5 square feet
8. ≈ 127.3 square inches
9. ≈ 29.66 feet
10. ≈ 61.4 feet

Practice 6.5.2: Volumes of Cylinders, Pyramids, Cones, and Spheres, p. U6-150

1. ≈ 2.09 cubic feet
2. ≈ 47.75 cubic centimeters
3. The cone must have a radius that is $\sqrt{3}$ times larger than the cylinder.
4. ≈ 6216.99 ft^3; ≈ 155.02 hours
5. ≈ 2,583,283.33 m^3
6. ≈ 0.25 liters
7. ≈ 105.22 cubic centimeters
8. 21,990,642,870 km^3
9. It will overflow because there is more ice cream than space in the cone. The volume of the cone is 45.79 cm^3, and the volume of the ice cream is 65.42 cm^3.
10. 64 times more air

Lesson 6: Deriving Equations

Practice 6.6.1: Deriving the Equation of a Circle, p. U6-171

1. $x^2 + y^2 = 5$
2. $(x - 3)^2 + (y - 3)^2 = 9$
3. $(x - 1.1)^2 + (y + 2)^2 = 1$
4. $(x - 5)^2 + (y + 2)^2 = 41$
5. $x^2 + y^2 + 7x - 2y + 7.49 = 0$
6. center: (4, –5); radius = $\sqrt{108} = 6\sqrt{3}$
7. center: (5/4, –3); radius = 3/2
8. Standard equation: $(x - 10)^2 + (y + 12)^2 = 1024$. Yes, Marcy receives a strong signal because the distance from (15, 18) to (10, –12) is $\sqrt{925}$ or approximately 30.41 miles, which is less than 32 miles.
9. Standard equation: $x^2 + y^2 = 1600$. Yes, a customer at $\left(-10\sqrt{7}, 30\right)$ will get free delivery because the distance from $\left(-10\sqrt{7}, 30\right)$ to (0, 0) is 40 miles.
10. The center is (0, 80) and the radius is 200 feet.

Practice 6.6.2: Deriving the Equation of a Parabola, p. U6-191

1. $x^2 = -12y$; $\sqrt{(x-0)^2+(y+3)^2} = \sqrt{(x-x)^2+(y-3)^2}$
2. $y^2 = 16x$; $\sqrt{(x-4)^2+(y-0)^2} = \sqrt{(x+4)^2+(y-y)^2}$
3. $y^2 = 4(x - 4)$; $\sqrt{(x-5)^2+(y-0)^2} = \sqrt{(x-3)^2+(y-y)^2}$
4. $(x - 4)^2 = -24(y + 2)$; $\sqrt{(x-4)^2+(y+8)^2} = \sqrt{(x-x)^2+(y-4)^2}$
5. $(y - 2)^2 = -4(x + 2)$
6. $(y + 1.8)^2 = 12.8(x - 3)$
7. focus: (–4 1/8, –2); vertex: (–4, –2); directrix: $x = -3\ 7/8$

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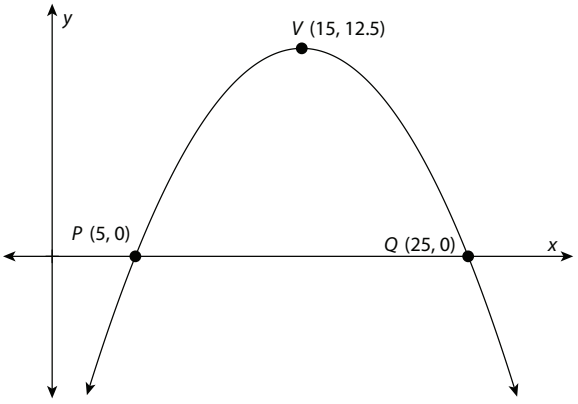
8. $y^2 = 2x$
9. equation: $(x - 15)^2 = -8(y - 12.5)$; height = 12.5 ft; width of PQ = 20 ft



10. equation: $x^2 = 192y$; depth = 18.75 inches

Lesson 7: Using Coordinates to Prove Geometric Theorems About Circles and Parabolas

Practice 6.7.1: Using Coordinates to Prove Geometric Theorems About Circles and Parabolas, p. U6-213

1. The statement has been disproved. Justification: The radius $= AR = 3$ and $QR = \sqrt{(-1-0)^2 + (-\sqrt{2}-0)^2} = \sqrt{1+2} = \sqrt{3}$. Since $\sqrt{3} \neq 3$, Q does not lie on the circle.
2. The statement has been proved. Justification: The radius $= PR = \sqrt{(5-0)^2 + (2\sqrt{6}-0)^2} = \sqrt{25+24} = \sqrt{49} = 7$, and $AP = \sqrt{(0-0)^2 + (0-7)^2} = \sqrt{49} = 7$. Therefore, point A lies on the circle.
3. The statement has been proved. Justification: $p = 1/2$, the vertex is R (0, 0), and the equation is $x^2 = 2y$. P, Q, and R are all on the parabola.
4. The statement has been proved. Justification: $p = -1/8$, the vertex is (0, 0), and the equation is $x^2 = (-1/2)y$. A, B, and C are all on the parabola.
5. The statement has been proved. Justification: Use the form $y = ax^2 + bx + c$. The point (0, –14) is on the graph, so $c = -14$ and $y = ax^2 + bx - 14$. The vertex is (4, 2), so $-b/2a = 4$, $b = -8a$, and $y = ax^2 - 8ax - 14$. The point (4, 2) is on the graph, so $2 = a(4)^2 - 8a(4) - 14$, $a = -1$, and the equation is $y = -x^2 + 8x - 14$. A, B, C, and D are all on the parabola.
6. The statement has been disproved. Justification: The parabola has an equation of the form $(y - k)^2 = 4p(x - h)$ with $p = 3$ and vertex $(h, k) = (2, 1)$, so its equation is $(y - 1)^2 = 4(3)(x - 2)$, or $(y - 1)^2 = 12(x - 2)$. The point A (5, 6) does not lie on the parabola because $(6 - 1)^2 \neq 12(5 - 2)$.
7. The statement has been proved. Justification: The radius $= PQ = \sqrt{(10-0)^2 + (\sqrt{69}-0)^2} = \sqrt{100+69} = \sqrt{169} = 13$; A, B, and C are all on the circle.
   $AC = \sqrt{[0-(-5)]^2 + (-13-12)^2} = \sqrt{25+625} = \sqrt{650} = 5\sqrt{26}$
   and
   $BC = \sqrt{(0-5)^2 + (-13-12)^2} = \sqrt{25+625} = \sqrt{650} = 5\sqrt{26}$;
   therefore, ABC is an isosceles triangle.
8. Natasha gets a prize. The circles have radii $r_1 = \sqrt{(12-6)^2 + (12-20)^2} = 10$ and $r_2 = \sqrt{(12-12)^2 + (12-23)^2} = 11$. The distance from point Q (19, 4) to center C (12, 12) is $QC = \sqrt{(12-19)^2 + (12-4)^2} = \sqrt{49+64} = \sqrt{113}$. Since $10 < \sqrt{113} < 11$, point Q is in the shaded ring.
9. The statement has been proved. Justification: The left parabola has an equation of the form $(x - h)^2 = 4p(y - k)$ with $p = -3$ and vertex $(h, k) = (6, 8)$, so its equation is $(x - 6)^2 = -12(y - 8)$. The right parabola has an equation of the form $(x - h)^2 = 4p(y - k)$ with $p = -3$ and vertex $(h, k) = (18, 8)$, so its equation is $(x - 18)^2 = -12(y - 8)$. The point A (12, 5) is on both parabolas because it satisfies both equations.
10. The statement has been proved. Justification: Use the form $y = ax^2 + bx + c$. The point (0, 10) is on the graph, so $c = 10$ and $y = ax^2 + bx + 10$. The vertex is (0, 10), so $-b/2a = 0$, $b = 0$, and $y = ax^2 + 10$. The point (20, 10.4) is on the graph, so $10.4 = a(20)^2 + 10$ and $a = 0.001$. Therefore, the equation is $y = 0.001x^2 + 10$. This can be rewritten in the form $4p(y - k) = (x - h)^2$ as follows: $y = 0.001x^2 + 10$ is equivalent to $y - 10 = 0.001x^2$, which is equivalent to $1000(y - 10) = (x - 0)^2$. Therefore, $p = 250$, the focus is (0, 260), and the directrix is $y = -240$. By definition, all points on the parabola are equidistant from the focus and the directrix. Therefore, all points on the parabola are equidistant from the point (0, 260), and the line $y = -240$.

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Station Activities Set 1: Circumference, Angles, Arcs, Chords, and Inscribed Angles

Instruction

Goal: To provide opportunities for students to develop concepts and skills related to circumference, arc length, central angles, chords, and inscribed angles

Common Core State Standards

G–C.2 Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*

G–C.5 Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Student Activities Overview and Answer Key

Station 1

Students will be given a plastic coffee can lid, a tape measure, a black marker, a compass, a ruler, and white paper. Students will measure the radius, diameter, and circumference of the coffee can lid. They will derive the relationships among these three measurements. Then they will solve two real-world problems using circumference.

Answers

1. Circumference; answers will vary.
2. Verify the circle on students' papers.
3. Answers will vary; answers will vary; $d = 2r$
4. Students' results should be close to 3.14; the circumference divided by the diameter is approximately equal to π.
5. Answers will vary; yes, this calculation finds the circumference.
6. Answers will vary; yes, this calculation finds the circumference.
7. Yes, both calculations give the circumference.
8. $C = 2\pi r$; $C = \pi d$
9. $C = \pi d = \pi(20) \approx 62.8$ ft
10. $C = 2\pi r$; $6.28 = 2\pi r$; $6.28 \approx 6.28r$; $r \approx 1$ inch

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Station 2

Students will be given a compass, a protractor, a red marker, a calculator, and a ruler. They will construct a circle, radii, a central angle, and an arc. Students will derive the relationship between the arc length and the circumference. Then they will find the length of the arc.

Answers

1. Check students' constructions; 6.28 inches
2. Check students' constructions; $\frac{1}{4}$; $\frac{90^\circ}{360^\circ} = \frac{1}{4}$
3. The arc length is $\frac{1}{4}$ of the circle's circumference; $\frac{1}{4}(6.28) = 1.57$ inches
4. length of arc $= \left(\frac{\text{central angle}}{360^\circ}\right)(\text{circumference})$
5. 2.62 cm
6. 8.85 in
7. 2.36 m
8. 38.47 ft

Station 3

Students will be given white computer paper, a compass, a ruler, a red marker, a blue marker, and a calculator. Students will construct circles, radii, and chords. They will derive the relationship between chords, arcs, and the triangles created by chords and radii. Then they will solve a real-world problem.

Answers

1. 9.42 in
2. 1.03 in; 1.05 in
3. 1.03 in; 1.05 in
4. The arc length is slightly longer than the chord length, and chord and arc lengths that have equal central angles are equal.
5. $\triangle PAB$ and $\triangle PCD$ are congruent because of the SAS theorem. Both triangles have congruent sides (the radii) and the included angles are congruent since they each measure 40°.

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6. $m\overset{\frown}{AB}$ and $m\overset{\frown}{CD}$ are congruent because the central angles are congruent.
7. $C = \pi d = \pi(10) = 31.4$ in. Cut the pie into 6 slices by dividing 31.4 by 6 to find the arc length of one slice: 31.4/6 = 5.23 in.

Station 4

Students will be given white computer paper, a compass, a protractor, a ruler, and a calculator. They will construct circles, inscribed angles, and central angles. Students will find the degree measure of the inscribed angle and its intercepted arc. Then they will derive the relationship between the inscribed angle and its intercepted arc.

Answers

1. Answers will vary.
2. Answers will vary. (Should be twice the answer in problem 1.)
3. Answers will vary.
4. Answers will vary. (Should be twice the answer in problem 3.)
5. The measure of an inscribed angle is equal to one-half the degree measure of its intercepted arc.

Materials List/Setup

Station 1 plastic coffee can lid; tape measure; black marker; compass; ruler; white paper

Station 2 compass; protractor; red marker; calculator; ruler

Station 3 white paper; compass; ruler; red marker; blue marker; calculator

Station 4 white paper; compass; protractor; ruler; calculator

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. How do you find the circumference of a circle given the radius? Given the diameter?
2. In a circle, how do you find the length of an arc given its central angle?
3. In a circle, what is the relationship between arcs and chords?
4. In a circle, what is the relationship between an inscribed angle and its intercepted arc?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in the same station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. $C = 2\pi r$; $C = \pi d$
2. length of arc $= \left(\dfrac{\text{central angle}}{360^\circ} \right)(\text{circumference})$
3. The arc length is greater than the length of the chord that has the same endpoints.
4. The measure of an inscribed angle is equal to one-half the degree measure of its intercepted arc.

Possible Misunderstandings/Mistakes

- Not understanding the difference between a radius and a chord
- Not finding the measure of the correct arc
- Not having the vertex of an inscribed angle on the circle

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Station Activities Set 1: Circumference, Angles, Arcs, Chords, and Inscribed Angles

Station 1

You will be given a plastic coffee can lid, a tape measure, a black marker, a compass, a ruler, and white paper.

1. As a group, use the black marker to mark a starting point on your coffee lid.

   Roll the coffee can lid along the tape measure so you can measure the distance around the edge of the coffee can lid.

   What is the mathematical name for this distance?

   What is the distance around the edge of the lid in inches?

   Repeat this measurement to verify your answer.

2. Trace the coffee can lid on the white paper.

   Use the ruler and compass to find the center of the circle.

3. What is the radius of the circle?

   What is the diameter of the circle?

   How does the radius relate to the diameter?

4. Divide your measurement from problem 1 by the diameter of the circle. Compare your result to the value of π, which is approximately 3.14. What do you notice?

continued

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5. What is π times twice the radius of your circle?

   Does this match your answer in problem 1? Why or why not?

6. What is π times the diameter of your circle?

   Does this match your answer in problem 1? Why or why not?

7. Do your answers for problems 5 and 6 match? Why or why not?

8. Based on your observations in problems 1–7, what is the formula for the circumference of a circle written in terms of the radius?

   What is the formula for the circumference of a circle written in terms of the diameter?

9. Larry installed a circular pool in his backyard. The pool has a diameter of 20 feet. What is the circumference of the pool? Show your work in the space below.

10. Lisa is running for class president and passed out buttons that each have a circumference of 6.28 inches. What is the radius of each button? Show your work in the space below.

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Station 2

You will be given a compass, a protractor, a red marker, a calculator, and a ruler. Work as a group to construct the circles and answer the questions.

1. In the space below, construct a circle with a diameter of 2 inches. Label the center of the circle as point *C*.

What is the circumference of the circle?

continued

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2. On the circle, construct a horizontal radius. Use the protractor to create a vertical radius that creates a 90° angle with your horizontal radius.

   Use the red marker to highlight the arc of the circle between the endpoints of these radii.

   What fraction of the circle's circumference is this arc?

   What is the ratio of the central angle you created between the two radii and the total angle measure of the circle? Justify your answer.

3. How can you use the circumference of the circle and the ratio of the central angle to the total angle measure of the circle to find the length of the arc? Explain your answer and find the arc length described in problem 2.

4. In general, what method can you use to find the length of an arc given the central angle?

continued

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For problems 5–8, find the length of the arc using the given information. Use 3.14 as an approximation of π.

5. Circle with radius 5 cm and central angle of 30°

6. Circle with diameter 7 inches and central angle of 145°

7. Circle with radius 0.5 meters and central angle of 270°

8. Circle with diameter 14 feet and central angle of 315°

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Station 3

You will be given white paper, a compass, a ruler, a red marker, a blue marker, and a calculator. Work as a group to construct the circles and answer the questions.

On the paper, construct a circle that has a diameter of 3 inches. Mark the center of the circle as point *P*.

1. What is the circumference of the circle?

2. Construct two radii, $\overline{PA}$ and $\overline{PB}$, that create a 40° angle. Draw chord $\overline{AB}$.

   What is the length of chord $\overline{AB}$?

   What is the length of $\overset{\frown}{AB}$? (*Hint:* Use $\left(\frac{\text{central angle}}{360°}\right)(\text{circumference})$.)

3. Construct two radii, $\overline{PC}$ and $\overline{PD}$, that also create a 40° angle. Draw chord $\overline{CD}$.

   What is the length of chord $\overline{CD}$?

   What is the length of $\overset{\frown}{CD}$?

continued

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4. Based on your observations in problems 1–3, what is the relationship between chords and arcs?

5. What is the relationship between $\triangle PAB$ and $\triangle PCD$? Explain your answer.

6. What is the relationship between $m\overset{\frown}{AB}$ and $m\overset{\frown}{CD}$? Explain your answer.

7. An apple pie 10 inches in diameter is cut into 6 equal size slices. What is the arc length for each slice of pie? Show your work and answer in the space below.

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Station 4

You will be given white paper, a compass, a protractor, a ruler, and a calculator. Work as a group to construct the circles and answer the questions.

On the white paper, construct a circle with a radius of 0.75 inches.

1. Plot a point on the circle. Label this point P.

   Construct two chords, $\overline{PA}$ and $\overline{PB}$, to create an inscribed angle.

   What is the measure of the inscribed angle, $\angle APB$?

2. Label the center of the circle as point C.

   Construct two radii, $\overline{CA}$ and $\overline{CB}$.

   Find $m\overset{\frown}{AB}$. Show your work below.

continued

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On the white paper, construct a new circle with a radius of 2 inches.

3. Plot a point on the circle. Label this point P.
   Construct two chords, $\overline{PA}$ and $\overline{PB}$, to create an inscribed angle.

   What is the measure of the inscribed angle, $\angle APB$?

4. Label the center of the circle as point C.
   Construct two radii, $\overline{CA}$ and $\overline{CB}$.

   Find $m\overset{\frown}{AB}$. Show your work below.

5. Based on your observations in problems 1–4, what is the relationship between the measure of an inscribed angle and its intercepted arc?

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Instruction

Goal: To provide opportunities for students to develop concepts and skills related to special segments, secants, tangents, angle measurements, and equations of circles

Common Core State Standards

G–C.1 Prove that all circles are similar.

G–C.2 Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*

G–C.4 (+) Construct a tangent line from a point outside a given circle to the circle.

G–C.5 Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Student Activities Overview and Answer Key

Station 1

Students will be given a ruler and calculator. They will work together to construct secants that intersect outside a circle. They will derive the relationship between the secants using the secant segment and external portions.

Answers

1. The secants intersect at a point outside the circle.
2. $FA \approx 1.2$ inches
3. $FB \approx 2.9$ inches
4. $FC \approx 1.3$ inches
5. $FD \approx 2.7$ inches
6. $FA \times FB \approx 3.5$ square inches
7. $FC \times FD \approx 3.5$ square inches
8. If two secants intersect outside a circle, then the product of the secant segment with its external portion equals the product of the other secant segment with its external portion.

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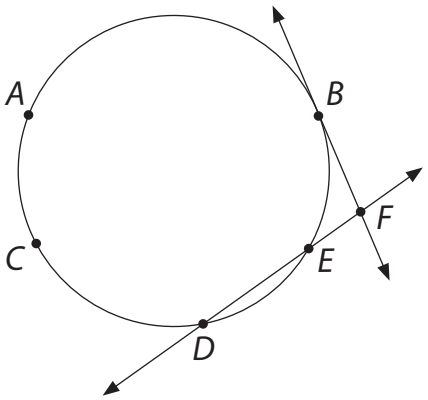
Instruction

Station 2

Students will be given a ruler, a compass, a protractor, and a calculator. Students will construct a secant and tangent on a circle. They will measure the angle created by the secant and tangent from a point drawn outside the circle. They will find the measure of the intercepted arcs. Then they will compare the angle and measure of intercepted arcs.

Answers

1. 

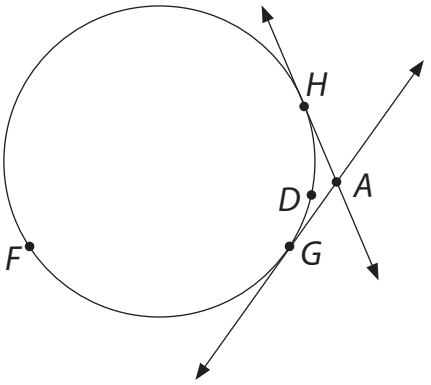
2. 105°
3. 260°
4. 50°
5. The angle formed equals half the difference of the intercepted arcs.

Station 3

Students will be given white paper, a ruler, a compass, a protractor, and a calculator. Students will construct a circle and tangents that intersect at a point outside the circle. They will derive the relationship between the angle formed by the tangents and the intercepted arcs.

Answers

1. Answers will vary. Sample answer: outside the circle



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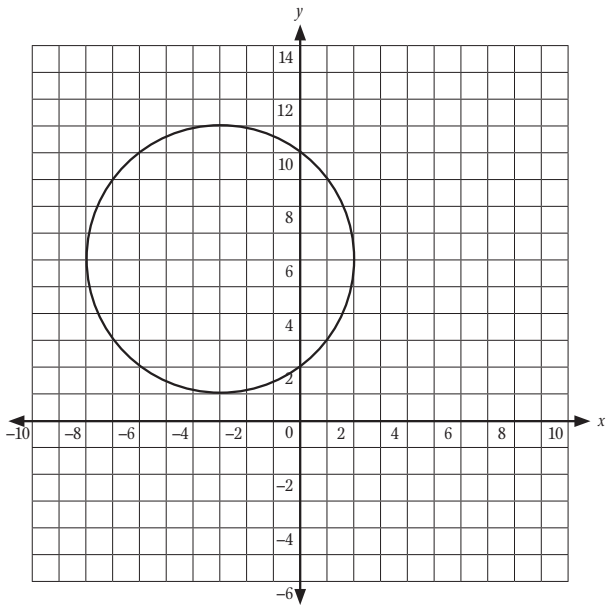
2. Answers will vary. Sample answer: 115°
3. Answers will vary. Possible answer based on answer to question 2: 295°
4. Answers will vary. Possible answer based on answer to question 2: 65°
5. The measure of an angle formed by two tangents drawn from a point outside the circle is half the difference of the intercepted arcs.

Station 4

Students will be given graph paper, a ruler, and a compass. Students will construct circles in the coordinate plane. They will find the equation of a circle given the center and radius, and state the center and radius of the circle when the equation of the circle is given in center-radius form.

Answers

1. $h = 0; k = 0; x^2 + y^2 = 25$
2. $h = 10; k = -4; (x - 10)^2 + (y + 4)^2 = 25$
3. Answers will vary. Possible answer: The equation of the circle, $x^2 + y^2 = r^2$, is based on a circle with a center at the origin (0, 0). Both h and k give the center of the circle in relationship to the center of the circle at $x^2 + y^2 = r^2$.
4. (–3, 6); $r = 5$; sample graph:



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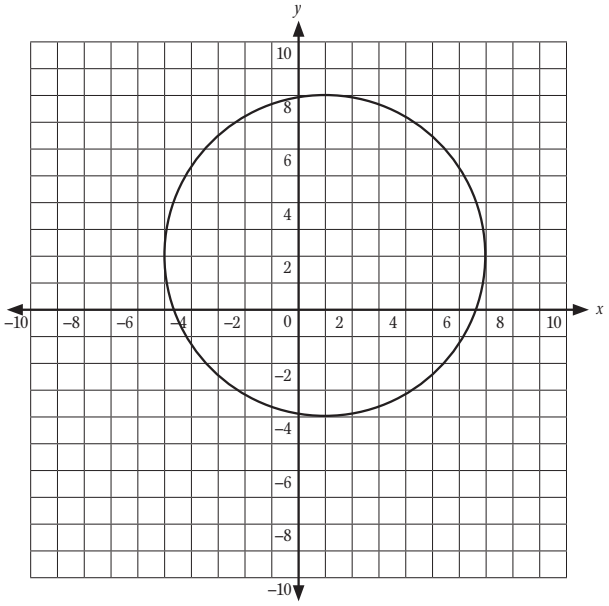
5. (1, 2); $r = 6$; sample graph:



Materials List/Setup

Station 1 ruler; calculator

Station 2 ruler; compass; protractor; calculator

Station 3 white paper; ruler; compass; protractor; calculator

Station 4 graph paper; ruler; compass

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. What is the relationship between two secants that intersect outside of a circle?
2. What is the relationship between the measure of the angle created by the secant and tangent drawn from a point outside the circle and the intercepted arcs?
3. What is the relationship between the measure of the angle created by two tangents drawn from a point outside the circle and the intercepted arcs?
4. How do you write the equation for a circle given the coordinates of the center point and radius?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in the same station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. If two secants intersect outside a circle, then the product of the secant segment with its external portion equals the product of the other secant segment with its external portion.
2. The angle formed equals half the difference of the intercepted arcs.
3. The measure of an angle formed by two tangents drawn from a point outside the circle is half the difference of the intercepted arcs.
4. Use $(x - h)^2 + (y - k)^2 = r^2$.

Possible Misunderstandings/Mistakes

- Not measuring the correct segments when comparing secants that intersect at a point outside the circle
- Not calculating the measure of intercepted arcs correctly
- Not identifying the intercepted arcs correctly
- Not changing the subtraction sign to a plus sign when a coordinate of the center of the circle is negative

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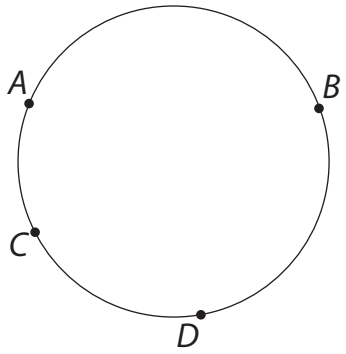
Station 1

You will be given a ruler and calculator. Work as a group to answer the questions.



On the circle above, construct a secant through points C and D. Construct a secant through points A and B.

1. Where do the secants intersect?

   Label this intersection as point F.

2. What is the length of $\overline{FA}$?

3. What is the length of $\overline{FB}$?

4. What is the length of $\overline{FC}$?

5. What is the length of $\overline{FD}$?

6. What is $\overline{FA} \times \overline{FB}$?

7. What is $\overline{FC} \times \overline{FD}$?

8. Based on your observations in problems 1–7, what is the relationship between two secants that intersect outside of a circle?

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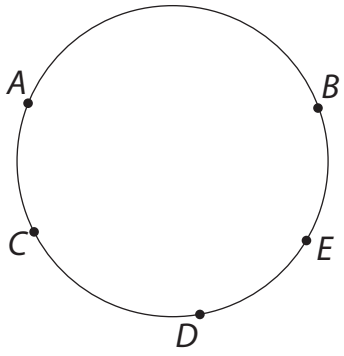
Station 2

You will be given a ruler, a compass, a protractor, and a calculator. Work as a group to answer the questions.



On the circle above, construct a secant through points D and E. Construct a tangent through point B.

1. Do the secant and tangent intersect inside or outside the circle?

2. Label the angle created by the secant and tangent as $\angle F$.

   What is the measure of $\angle F$?

3. What is $m\overset{\frown}{DCB}$? Show your work in the space below.

continued

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4. What is $m\overset{\frown}{BE}$? Show your work in the space below.

5. Based on your observations in problems 1–4, what is the relationship between the measure of the angle created by the secant and tangent drawn from a point outside the circle and the intercepted arcs?

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Station 3

You will be given white paper, a ruler, a compass, a protractor, and a calculator. Work as a group to construct the circles and answer the questions.

- Construct a circle on the white paper.
- Construct two points on the right edge of the circle. Label the points as H and G.
- Construct a point on the right edge of the circle between points H and G. Label this point D.
- Construct a point on the left edge of the circle between points H and G. Label this point F.
- Construct tangent lines through points H and G.

1. Where do the tangent lines intersect?

   Label this as point A.

2. What is the measure of $\angle A$?

3. What is $m\overset{\frown}{HFG}$? Show your work and answer in the space below.

continued

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4. What is $m\overset{\frown}{GDH}$? Show your work and answer in the space below.

5. What is the relationship between the measure of the angle formed by the point of intersection of the two tangent lines and the intercepted arcs? Show your work and answer in the space below.

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Station 4

You will be given graph paper, a ruler, and a compass. Work as a group to construct the circles and answer the questions.

The equation of a circle is $(x - h)^2 + (y - k)^2 = r^2$.

1. On your graph paper, construct a circle with its center at (0, 0) and a radius of 5 units.

   What is the value of h for this circle?

   What is the value of k for this circle?

   What is the equation of this circle?

2. On your graph paper, shift the circle in problem 1 down 4 units and right 10 units.

   What is the value of h for this circle?

   What is the value of k for this circle?

   What is the equation of this circle?

continued

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3. Why do you think h and k are used in the equation of a circle?

4. A circle has the equation $(x + 3)^2 + (y - 6)^2 = 25$.

   What is the center of the circle? Explain your answer.

   What is the radius of the circle? Explain your answer.

   Graph the circle on your graph paper to justify your answers.

5. A circle has the equation $(x - 1)^2 + (y - 2)^2 = 36$.

   What is the center of the circle? Explain your answer.

   What is the radius of the circle? Explain your answer.

   Graph the circle on your graph paper to justify your answers.

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Instruction

Goal: To provide opportunities for students to develop concepts and skills related to the circumcenter, incenter, orthocenter, and centroid of a circle

Common Core State Standards

G–CO.12 Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*

G–CO.13 Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

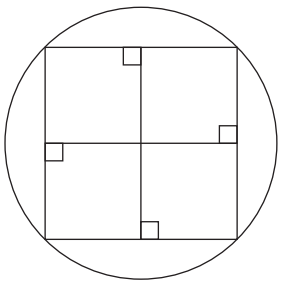
Student Activities Overview and Answer Key

Station 1

Students will be given a ruler, a protractor, and a compass. They will construct a square and circumscribe a circle around the square. Then they will construct the perpendicular bisectors of the square. They will realize that the circumcenter is the intersection of the perpendicular bisectors, which is also the center of the circle. They will repeat this process for an equilateral triangle.

Answers

1–2.



The circle circumscribes the square with the intersection of the perpendicular bisectors as the center of the circle. It passes through all four vertices of the square.

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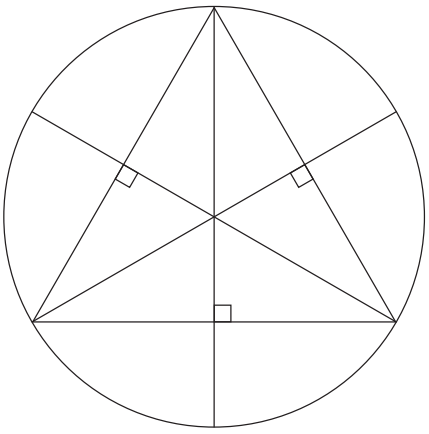
3. 

4. circumcenter
5. The circle circumscribes the triangle with each vertex being on the edge of the circle and the center being the intersection of the perpendicular bisectors.
6. The circumcenter is the center of the circle that passes through all of the vertices of the polygon, and the circumcenter is at the intersection of the polygon's perpendicular bisectors.

Station 2

Students will be given a ruler, a compass, and a protractor. They will construct the angle bisectors of an equilateral triangle. Then they will inscribe a circle in the triangle. They will realize that the incenter is the intersection of the angle bisectors, which is also the center of the circle.

Answers

1. Answers will vary.
2. incenter
3. Answers will vary.
4. Answers will vary.
5. The circle is inscribed in the triangle. The three sides of the triangle are all tangent to the circle.
6. The incenter is the center of a circle inscribed in a triangle, and the point where the angle bisectors of the triangle intersect.

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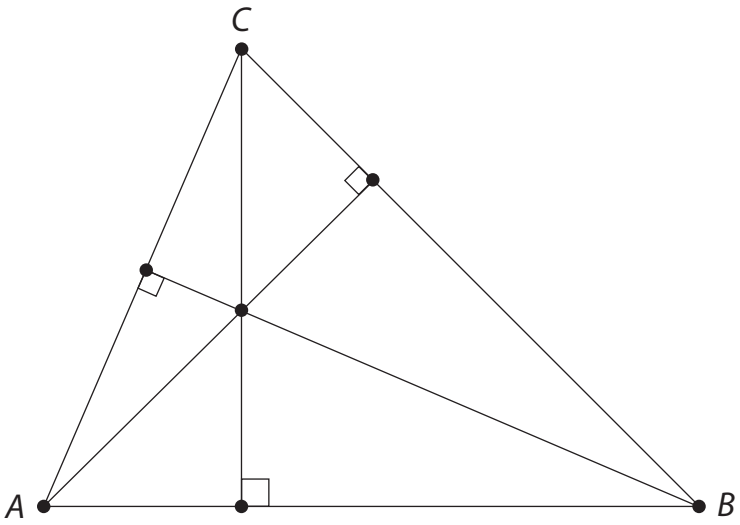
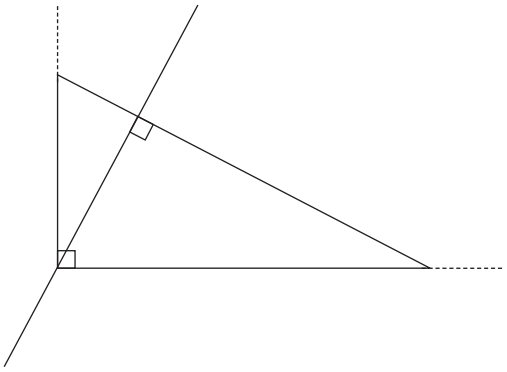
Instruction

Station 3

Students will be given a ruler, a protractor, and a compass. Students will construct the altitudes of an acute triangle to find the orthocenter. Then they will construct the altitudes of a right triangle and an obtuse triangle and discover that the orthocenter can lie inside, on, or outside the triangle.

Answers

1. 

2. orthocenter
3. acute
4. 
5. on the right angle of the triangle
6. right

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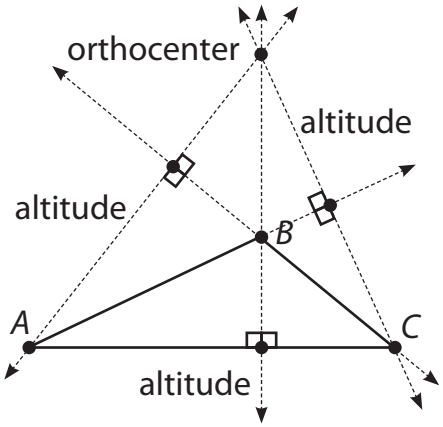
7. 

8. outside of the triangle

9. obtuse

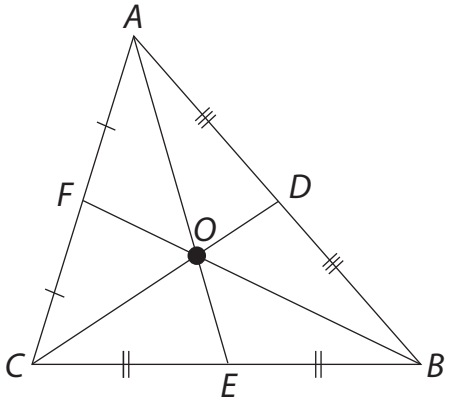
10. If the triangle is acute, the orthocenter lies inside of the triangle. If the triangle is right, the orthocenter lies at the right angle of the triangle. If the triangle is obtuse, the orthocenter lies outside of the triangle.

Station 4

Students will be given large index cards, scissors, a compass, a ruler, and a protractor. They will construct the centroid of three triangles. They will relate the centroid to the center of gravity or the balancing point.

Answers

1. Triangles may vary. Be sure all angles are acute in the triangle. Sample triangle with medians:



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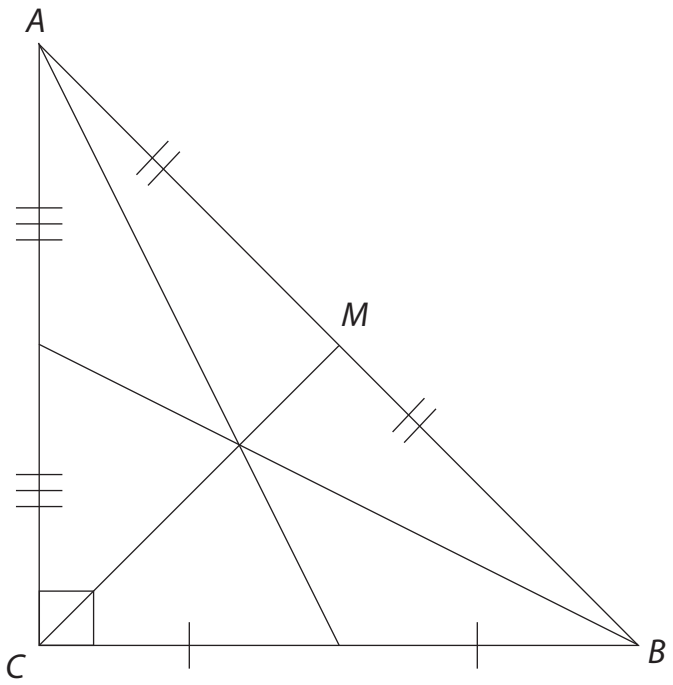
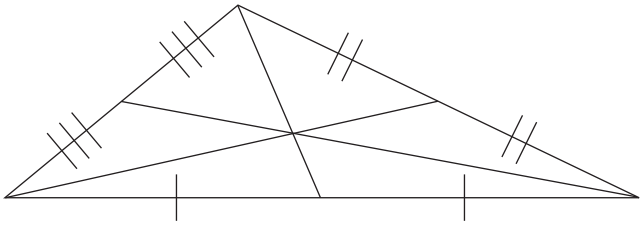
Instruction

2. Triangles may vary. Be sure there is a right angle in the triangle. Sample triangle with medians:



3. Triangles may vary. Be sure there is an obtuse angle in the triangle. Sample triangle with medians:



4. Students should place a point at the intersection of the medians for all three triangles.
5. The balancing point is at the intersection of the medians.
6. centroid
7. The centroid is the center of gravity and is formed by the intersection of the medians.

Materials List/Setup

Station 1 ruler; compass; protractor

Station 2 ruler; compass; protractor

Station 3 ruler; compass; protractor

Station 4 large index cards; scissors; ruler; compass; protractor

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Instruction

Discussion Guide

To support students in reflecting on the activities and to gather some formative information about student learning, use the following prompts to facilitate a class discussion to "debrief" the station activities.

Prompts/Questions

1. What is the circumcenter?
2. What is the incenter?
3. What is the orthocenter?
4. What is the centroid of a circle?

Think, Pair, Share

Have students jot down their own responses to questions, then discuss with a partner (who was not in their station group), and then discuss as a whole class.

Suggested Appropriate Responses

1. It is the intersection of the perpendicular bisectors of a triangle (or other polygon) and the center of the circumscribed circle.
2. It is the intersection of the angle bisectors of a triangle and the center of the inscribed circle.
3. It is the intersection of the altitudes of a triangle.
4. It is the intersection of the medians of a triangle and the center of gravity for the triangle.

Possible Misunderstandings/Mistakes

- Not realizing that the circumcenter is found in circumscribed circles
- Not realizing that the incenter is found in inscribed circles
- Incorrectly constructing perpendicular bisectors, angle bisectors, and altitudes

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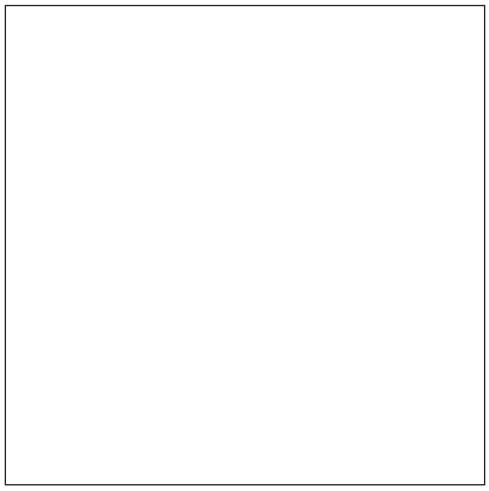
Station 1

At this station, you will find a ruler, a compass, and a protractor. Work as a group to answer the questions.

1. In the space below, construct the perpendicular bisectors of the sides of the square.



2. Draw a circle using the intersection of the perpendicular bisectors as the center of the circle. Let the radius be the distance from the center to one of the vertices of the square. What do you notice about the circle in relation to the square?

continued

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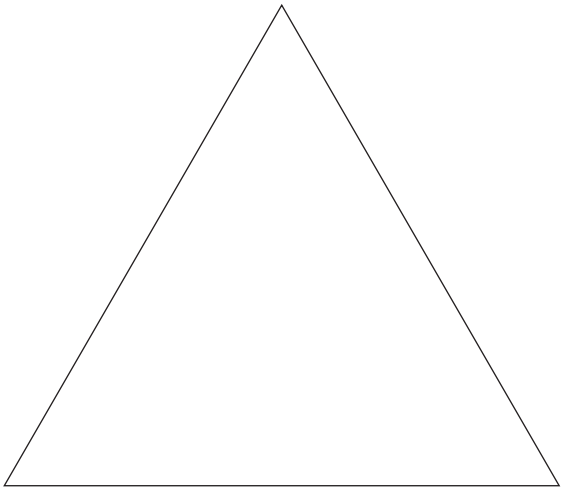
3. In the space below, construct the perpendicular bisectors of the sides of the triangle.



4. What is the name of the point of intersection of the perpendicular bisectors of a triangle?

5. Draw a circle using the intersection of the perpendicular bisectors as the center of the circle. Let the radius be the distance from the center to one of the vertices of the triangle. What do you notice about the circle in relationship to the triangle?

6. Based on your observations in problems 1–4, what is the definition of the circumcenter of a circumscribed polygon?

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Station 2

At this station, you will find a ruler, a compass, and a protractor. Work as a group to answer the questions.

1. Use a ruler to draw an acute triangle in the space below, and then bisect each angle of the triangle.

2. What is the name of the intersection point of the angle bisectors of a triangle?

3. Construct a perpendicular line segment from the intersection of the angle bisectors to one of the sides of the triangle.

4. Draw a circle using the intersection of the angle bisectors of the triangle as the center and the segment constructed in problem 3 as the radius.

continued

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5. What do you notice about the circle in relation to the triangle?

6. Based on your observations in problems 1–5, what is the definition of the incenter of an inscribed polygon?

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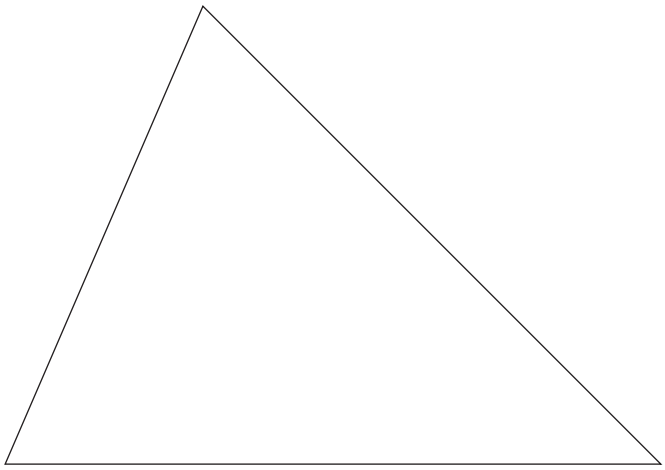
Station 3

At this station, you will find a ruler, a compass, and a protractor. Work as a group to answer the questions.

1. On the triangle below, construct the altitudes.



2. What is the name of the point of intersection of the altitudes of a triangle?

3. Based on the angles, what type of triangle is this?

continued

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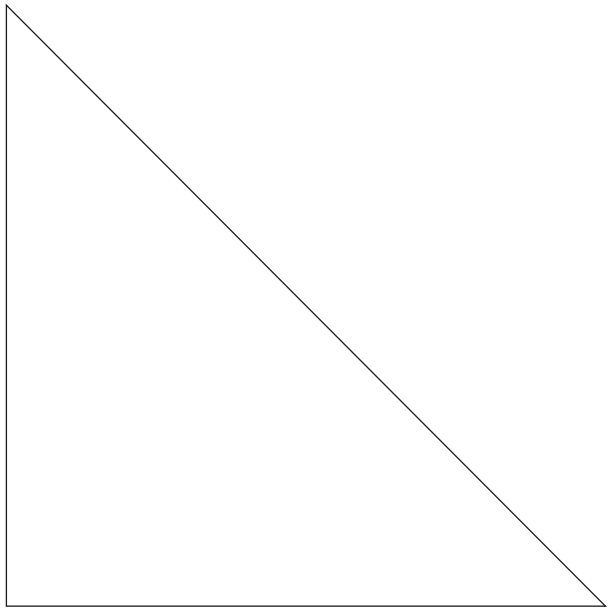
4. On the triangle below, construct the altitudes.



5. Where does the intersection of the altitudes occur?

6. Based on the angles, what type of triangle is this?

continued

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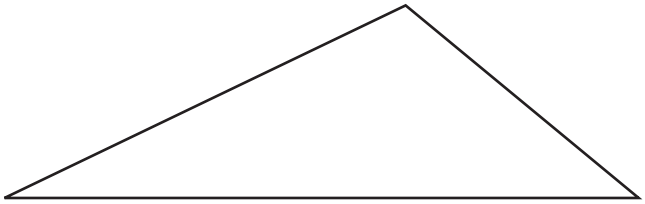
7. On the triangle below, construct the altitudes. Remember that you might need to extend the leg(s) of the triangle to create the altitude(s).



8. Where does the intersection of the altitudes occur?

9. Based on the angles, what type of triangle is this?

10. Based on your answers to problems 1–9, what conclusion can you draw about the location of the intersection of the altitudes of a triangle?

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Station 4

At this station, you will find large index cards, scissors, a compass, a ruler, and a protractor. Work together to construct the medians of the triangle and answer the questions.

1. Draw an acute triangle on one of the index cards and then construct the medians of the triangle.

2. Draw a right triangle on one of the index cards and then construct the medians of the triangle.

3. Draw an obtuse triangle on one of the index cards and then construct the medians of the triangle.

4. For all three triangles, place a point at the intersection of the medians.

5. Cut out each of the triangles and try balancing each one of them on the tip of your finger. Where does the balancing point occur?

6. What is the intersection point of the medians called?

7. What conclusion can you draw about the intersection point of the medians of a triangle?

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APPENDIX

Problem-Based Task Implementation Guides

Introduction

This addendum contains a collection of Implementation Guides (IGs) designed to support the use of the Problem-Based Tasks found throughout this unit.

Each Implementation Guide includes an overview and suggestions for introducing and facilitating the task. These contain specific recommendations for selected Standards for Mathematical Practice (SMPs) to highlight, guidance on addressing common errors and misconceptions associated with the task, and ideas for monitoring and coaching students as they work through the task. Finally, there is guidance for debriefing the task, with special attention to the SMPs and relevant Key Concepts from the instruction, as well as information on suitable extensions and alternate strategies.

Throughout the Implementation Guides, suggestions are provided in a question/prompt format to support your efforts to elicit student thinking and facilitate classroom discourse.

We developed these Implementation Guides for our Problem-Based Tasks at the request of math educators, and with advice and feedback from mathematics supervisors and teachers. Please let us know if they are helpful. Visit us at www.walch.com, follow us on Twitter (@WalchEd), or e-mail suggestions to customerservice@walch.com.

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Implementation Guide

Problem-Based Task 6.1.1: Following in Archimedes' Footsteps

Common Core State Standards

G–C.1 Prove that all circles are similar.

G–C.2 Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*

Task Overview

Focus

How can it be proven that all circles are similar? What experiments can be performed to verify the approximate value of *pi*? In this task, students will identify circular objects in their classroom and measure the circumference and diameter of these objects to verify the approximation of *pi*.

This activity will provide practice with:

- identifying circular objects in a classroom setting
- measuring circumference and diameter of circular objects with a string and a ruler
- approximating the center point of circles
- recording measurement data in a table
- calculating a ratio of the circumference of a circle to its diameter
- analyzing results of an experiment

Introduction

This task should be used to explore or to apply the skill of proving that all circles are similar and applying relationships between parts of a circle. Students should already be familiar with setting up and solving ratios and recording measurement data in a table.

Begin by reading the problem and clarifying the meaning of *circumference*, *diameter*, *pi*, and *revolution*.

circumference	the distance around a circle; $C = 2\pi r$ or $C = \pi d$, for which C represents circumference, r represents the circle's radius, and d represents the circle's diameter
diameter	a straight line passing through the center of a circle connecting two points on the circle; equal to twice the radius
***pi* (π)**	the ratio of circumference of a circle to the diameter; equal to approximately 3.14
revolution	a complete turn around the circumference of a circle

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 5:** Use appropriate tools strategically.

  Students will use strings and rulers for this task. They will use a piece of string to wrap around circular objects to measure their circumferences. They will use rulers to determine the numerical measurement of the circumferences, as they lay the string out on the ruler. They will also use rulers to measure the diameters of the circular objects.

- **SMP 6:** Attend to precision.

  Students will apply the definitions of circumference and diameter of a circle to measure these values for several circular objects in their classroom. They will carefully find the measurements of each circumference with a string, and they will use a ruler to measure the length of the string, as well as using a ruler to measure the length of the diameter, estimating the measurements as precisely as possible. Encourage students to use the correct terminology of the terms "circumference", "diameter", and "*pi*" throughout the task.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  Students will express regularity in repeated reasoning as they follow the process of measuring several circular objects with a string, laying the string out on a ruler and recording the numerical values in centimeters, measuring the lengths of the diameter of the objects, and recording this data in a table. They will also use repeated reasoning as they calculate the ratio of the circumference to the diameter for each object, to see how close their values are to the approximation of *pi*.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly measuring an object with the string

  Remind students that the string should be wrapped around the object so that it completes one revolution. As a way to double check, ask students to compare their value for the circumference of an object with that of another student's value for the same object.

- not choosing a clearly defined center for an object

  Remind students that it may be difficult to find an exact center for certain objects. Suggest that they try measuring the diameter from different endpoints on the object and see how close each measure is.

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- misreading the measurement from a ruler

  Remind students to hold one end of their string on the 0 marker on the ruler, and not the end of the ruler. Also remind them to pull the string tightly. Ask them to measure several times to make sure they are reading the numerical values correctly from the ruler.

- miscalculating the pi ratio from the data in the table

  Have students write $\pi = \frac{C}{d}$ at the top of their papers, and remind them that the value for the circumference is the numerator of the ratio, and the diameter is the denominator in the ratio.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "Why are all circles similar? How does this relate to all squares being similar, for example?" (**Answer**: In general, shapes of the same type are similar to each other because they have the same characteristics, although they may be scaled larger or smaller. Squares are similar because they all have 4 congruent sides and 4 right angles. All circles are similar because they all have the same shape, and they all have the same ratio of the circumference to the diameter, which is pi (π).)

- Before students begin measuring objects, discuss measurement errors with them. Ask them, "What possible measurement errors can occur during this task?" **Answer:** Some possible measurement errors are not marking the string off at the exact place when wrapping it around the circular item, incorrectly placing the string on the ruler, and reading the ruler incorrectly.)

- Before students begin measuring objects, ask them, "How could it be beneficial to have several people measure the same object, in terms of accuracy of results?" (**Answer:** If several people measure the same item, then all the measurements could be averaged to get a more accurate representation of the results.)

- Before students begin measuring objects, ask them, "When you use a ruler and measure the string, how many place values should be used when recording the data in the table, given that the measurements will be in centimeters? Should everyone have the same amount of decimal places for their data?" (**Answer:** Since the measurements will be in units of centimeters, using measurements estimated to the nearest tenth of a centimeter would probably be best. To get more consistent measurements throughout the class, everyone in the class should use the same number of decimal places when recording their data.)

- As students begin measuring objects, ask them, "Instead of a string, what else could be used to wrap around a circular object to measure the circumference?" (**Answer**: A piece of notebook paper could be wrapped around the object if a piece of string was not available.)

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- As students are working on the task, discuss the possible difficulty of choosing an exact center of an object to measure the diameter. Ask them, "Once you choose and mark a point off on an object that you think is the center, what is one way you could use a ruler to check your accuracy?" (**Answer**: If the surface of the object is flat, then a ruler could be used to measure the radii at various points. If several radii have measurements that are close, then the chosen center is reasonably accurate.)
- As students are measuring the diameters of their objects, ask them, "If it was only possible to measure the radius of an object, how could you use the radius in the *pi* ratio? What is the relationship between the radius and the diameter of a circle?" (**Answer:** The radius of a circle is exactly half the length of the diameter, so if the radius was measured, then that value would be doubled to get the value of the circumference.)
- As students are working on the task, ask them, "What is another tool that could be used to draw a circle and get an exact center for the circle?" (**Answer:** A compass can be used to accurately and precisely draw a circle. The center of the circle is the sharp point of the compass.)
- Discuss the importance of having a large sample size to draw conclusions about an experiment. Ask students, "Why is it better to find an average of 30 individual results of an experiment as opposed to finding the average of only 3 results?" (**Answer:** The more data samples will lead to a more accurate conclusion about an experiment. With only three samples, it is possible that all three results are not close to the desired result. With 30 samples, there is a greater chance that the average of the samples will be closer to the desired result.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to explain why all circles are similar. Focus specifically on the meaning of *pi*. Discuss ways in which students attempted to minimize measurement errors as they worked through the task and measured the circumference and diameter of the various objects. Ask for volunteers to discuss their methods of trying to determine the centers of the objects. Discuss how trying to measure several diameters could help provide a more accurate estimation of the center. Ask for volunteers to discuss the concepts of "estimation" and "approximation". Also discuss how a compass could be used to draw a circle representing some of the objects, if the objects were the appropriate size to trace on paper. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and procedures for their measurements, recording their data, and their calculations for the ratio of the circumference to the diameter for the objects. Ask them how

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they applied their knowledge of the characteristics and components of circles, as well as the fact that all circles are similar. Ask them why it is important to perform many trials in order to obtain more accurate results. Focus on the use of precise mathematical language and clarity, specifically when discussing circumference, diameter, and the *pi* ratio.

Connecting to Key Concepts

Make explicit connections to key concepts:

- A **circle** is the set of all points that are equidistant from a fixed point (the center) and form a 2-dimensional curve .

  In this task, students will explore circles as they identify circular objects in their classroom and measure the specific characteristics of circumference and diameter on these circles.

- The **circumference** is the distance around a circle.

  Students will measure the circumference of circular objects by taking a string and wrapping it around the object for one revolution.

- The **diameter** is a segment with endpoints on the circle that passes through the center of the circle.

  Students will measure the diameter of circular objects by measuring the segment that passes through the center of the circle and has endpoints on the circle.

- ***Pi***, π, is the ratio of the circumference to the diameter of a circle.

  Students will use their values for the circumference and diameter of each circular object and divide the circumference by the diameter to determine a value that should be close to the approximation of *pi*.

Extending the Task

- To extend the task, ask students to take as many circular objects as they can that they used in the task that can be placed on plain paper. Ask them to place an object on the paper and use their pencils to trace the object. Once they have the circle drawn, ask them to use a compass and work with a partner to try and find the exact center of the circle, or as close to the exact center as possible. Ask them to create methods of using a compass and a ruler to determine the center of their circles. Ask for volunteers to share their circles and explain their processes for attempting to find an exact center with the tools they were given. Brainstorm with the entire class about how to determine the center of a circle using a compass and a ruler.

- Another option is to ask students to construct circles given specified measurements for the circumference. For example, ask students to construct a circle that has a circumference of 20.41 centimeters. Instruct them to use 3.14 for the value of *pi*. Discuss how they can use the

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ratio of *pi* to solve for the unknown value of the diameter, so that they can construct the circle after they determine the value of the diameter. Also, give them values for the radius of a circle, and ask them to construct a circle with the given radius, and also calculate the circumference of that circle. Ask for volunteers to discuss how to manipulate the *pi* ratio in terms of solving for unknown values, as well as discussing any difficulties they experienced while constructing their circles given only one value.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** I first wrapped the string around several circular objects to measure their circumferences. Then I laid the string out on a ruler to get a numerical measurement for the circumference. Next, I chose a center for each circle, and then I used a ruler to measure the length of the diameters for each object.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision by applying the definitions of circumference and diameter of a circle to measure these values for several circular objects in my classroom. I carefully measured each circumference with a string, and then I used a ruler to measure the length of the string, and I measured the length of the diameter with a ruler.)
- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** I looked for and expressed regularity in repeated reasoning as I followed the process of measuring a circular object with a string, laying the string out on a ruler and recording the numerical value in centimeters, measuring the length of the diameter of the object, and recording this data in a table. I also used repeated reasoning when I calculated the ratio of the circumference to the diameter for each object, to see how close my values were to *pi*.)

Alternate Strategies or Solutions

Students may choose to use a piece of paper to wrap around various circular objects, instead of using a string. They may decide that measuring the length of the circumference from a paper will yield more accurate results. Encourage students to also try measuring with the string, so that they can compare the results of both methods. Students may also choose to place a compass on the circular objects to try and determine the center of the objects, as opposed to just estimating the center. Encourage students to use as many different tools and methods as possible to obtain the most accurate results possible.

Technology

Students can use scientific calculators.

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Implementation Guide

Problem-Based Task 6.1.3: The Circus Is in Town! Is It Safe?

Common Core State Standard

G–C.2 Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*

Task Overview

Focus

How can the relationship between tangent lines and circles be applied in a real-world scenario? What properties of tangent lines can be used to determine missing side lengths of right triangles? In this task, students will calculate side lengths of right triangles that are formed by tangent lines of circles, and they will decide if the sum of these lengths is enough to meet the requirements of a fire code in a scenario involving a circus setting.

This activity will provide practice with:

- examining a diagram representing a real-world scenario
- identifying properties of tangents of circles
- applying the Pythagorean Theorem to determine missing side lengths
- solving algebraic equations
- substituting values into an equation
- analyzing calculations to draw conclusions

Introduction

This task should be used to explore or to apply the skill of identifying and describing relationships among tangents of a circle, including properties of tangents, and the perpendicular relationship between the radius of a circle and the tangent where the radius intersects the circle. Students should already be familiar with performing calculations using the Pythagorean Theorem and solving algebraic equations for unknown values.

Begin by reading the problem and clarifying the meaning of *Pythagorean Theorem*, *radius*, and *tangent line*.

Pythagorean Theorem	a theorem that relates the length of the hypotenuse of a right triangle (c) to the lengths of its legs (a and b). The theorem states that $a^2 + b^2 = c^2$
radius	the distance from the center to a point on the circle; equal to one-half the diameter
tangent line	a line that intersects a circle at exactly one point and is perpendicular to the radius of the circle at the point of tangency

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine if the walkways in between components of a circus environment, represented by circles, are long enough to meet the requirements of the fire code. They will accomplish this goal by calculating the lengths of tangent lines which represent the distances of walkways between the various components of the circus. They will apply their knowledge about properties of tangent lines, as well as their knowledge about right triangles and using the Pythagorean Theorem to calculate the missing side lengths.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for concluding that the circus will pass inspection because the exterior walkways are long enough to meet the requirements of the code. They will explain their calculations of the lengths of the walkways, as well as how they applied the properties of tangent lines to determine the missing lengths. They will also justify their results by explaining their calculations when given algebraic expressions representing missing lengths. When presenting their arguments, they will explain their analysis of the results, and explain how these results are interpreted in the context of the scenario.

- **SMP 6:** Attend to precision.

  Students will apply the definitions of tangent line and point of tangency to determine how to solve for the missing lengths of the walkways. They will perform calculations using the Pythagorean Theorem, and they will also solve for an unknown value in an algebraic expression. Encourage students to use the correct terminology throughout the task, specifically when referring to the sides of a right triangle in the context of the circle (i.e. a leg of the right triangle is the radius). Also encourage them to refer to the congruency of the tangent lines when determining missing lengths.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- assuming that a radius and a line form right angles at the possible point of tangency simply by relying on observation

  Remind students of the relationship between a radius of a circle and a tangent line, in that there is a radius that forms a right angle with the tangent line at a point, and this point is called the point of tangency. Either the right angle must be proven, by using the Pythagorean Theorem, or else it must be stated as a given right angle in the problem. In this task, students will draw a radius to the point of tangency, thus creating a right angle.

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- assuming that two tangent lines are congruent by observation

  Remind students that two tangent lines are only congruent if they are tangent to the same circle and both lines originate from the same exterior point. Point out the congruent tangent lines on the diagram with students.

- making incorrect calculations when using the Pythagorean Theorem formula

  Have students write the Pythagorean Theorem down on their paper. Remind them to double check their calculations by substituting the value for the side lengths back into the formula to ensure that the sum of the square of the legs is equal to the sum of the square of the hypotenuse.

- assuming the value of x is a length of a tangent line

  Remind students that the variable x is part of an expression which represents the length of a tangent line. Remind them that once the value for x is determined, that value is substituted back into the expression to find the actual distance.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "After examining the diagram, what information can be drawn on it to visualize the components of the scenario?" (**Answer**: It might be helpful to use a ruler to draw the distances given in the verbal information in the problem. Specifically, the distance from the ticket booth at Exit A to the center of the concession stand can be drawn and labeled as 25 feet. Also, the distance of 25 feet can be drawn from the center of the concession stand to the petting zoo at Exit B.)
- Before students begin the task, review the properties of tangents that they will need to apply in this task. Ask them, "What do you know about tangent lines of a circle, and how does a radius of a circle relate to a tangent line?" (**Answer**: Tangent lines of a circle intersect the circle at a point on the exterior of the circle. A radius can be drawn from the center of the circle that intersects the point of tangency at a right angle.)
- As students begin the task, ask them, "Why is the information given in the problem, the distances of 25 feet, helpful in solving the problem? What do these distances represent in terms of right triangles?" (**Answer**: The distances of 25 feet represent the hypotenuse of right triangles. Since a distance of 7 feet is given in the problem, the Pythagorean Theorem can be used to find the missing leg of the triangles, which represents the missing distances in the problem.)
- As students are working on the task, ask them, "Which property of tangent lines can you apply to determine the unknown length of the walkway tangent to the big top, which begins on the edge of the petting zoo and extends from the same point as the walkway that is 13 feet long?" (**Answer:** The property of tangents that states that two lines extending from the same exterior point, tangent to the same circle, are congruent. So it can be concluded that this unknown distance is also 13 feet.)

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- As students are working on the task, ask them, "Why do you set the two given algebraic expressions equal to each other to solve for the value of x? What allows you to do this?" (**Answer:** Since the two expressions represent lengths that are congruent to each other they are set equal to each other. The property of tangents that states that two lines extending from the same exterior point, tangent to the same circle, are congruent, allows the expressions to be set equal to each other.)
- Once students have found the value of x to be 30, ask them, "How can you verify that you have solved the equation correctly, and that 30 is the value for x?" (**Answer:** To make sure that $x = 30$, substitute 30 back into both algebraic expressions to verify that they result in the same value.)
- Ask students, "What is an alternate format for representing the equation for the sum of the lengths of the walkways, $24 + 24 + 13 + 13 + 53 + 53 = 180$?" (**Answer**: An alternate form is: $2(24) + 2(13) + 2(53) = 180$.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their thought processes after initially reading the problem and examining the diagram. Discuss the properties of circles and tangent lines that were used in determining the missing lengths in the diagram. Also discuss which tangent lines were congruent, and which ones were not congruent in the diagram, focusing on the properties of tangent lines which state congruency of the tangent lines. Ask for volunteers to discuss how the Pythagorean Theorem was applied, specifically which sides of the right triangles were known and which were unknown. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning and procedures for their calculations for each of the missing lengths. Ask them how they applied their knowledge of the characteristics and components of circles, right triangles, and the theorems and properties of circles and right triangles. Also discuss their justification that the circus will pass inspection. Focus on the use of precise mathematical language and clarity when discussing tangent lines, right triangles, and congruency properties.

Connecting to Key Concepts

Make explicit connections to key concepts:

- **A tangent line** is a line that intersects a circle at exactly one point.

  In this task, students will recognize that the lines drawn in the diagram are tangent lines, and they will apply the properties of tangent lines throughout the task.

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- Tangent lines are perpendicular to the radius of the circle at the point of tangency.

  Students will apply the relationship of a tangent line and the radius of a circle to create right triangles that can be used to calculate missing side lengths which represent lengths in the given scenario.
- If two segments are tangent to the same circle, and originate from the same exterior point, then the segments are congruent.

  Students will apply this property of tangents by calculating the length of one tangent of a circle, and then determining other tangent lengths that are congruent on the same circle. They will use these lengths to represent distances in the given scenario.

Extending the Task

- To extend the task, ask students to use a compass and identify the centers of each of the circles. Ask them to work with a partner and discuss methods of finding the exact centers with a compass. Then, ask them to use a ruler to draw central angles to the two tangent points on each circle. Ask them to create a table and record their data of the measure of each central angle. Also ask them to measure the two chords that form the central angles for each circle (in millimeters), and record that data as well. Ask for volunteers to share their results, and discuss the relationships between central angles and the chords that form them.
- Another option for extending the task is to ask students to work in small groups of 3-4 students and create their own real-world scenario using different-sized circles. Ask them to provide some known values of lengths of tangent lines, but to leave some values unknown. Ask them to write down specific steps that can be taken to find the missing lengths on their diagrams. Ask for volunteers to share their scenarios, and brainstorm as a class on how to modify each scenario if necessary.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the task is to determine if the walkways in between components of a circus environment, represented by circles, are long enough to meet the requirements of the fire code. To accomplish this goal, I calculated the lengths of tangent lines which represented the distances of walkways between the various components of the circus. I applied my knowledge about properties of tangent lines, as well as my knowledge about right triangles and the Pythagorean Theorem, to calculate the missing side lengths.)

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- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining my choices for concluding that the circus will pass inspection because the exterior walkways are long enough to meet the requirements of the fire code. I explained my calculations of the lengths of the walkways, and I explained how I applied the properties of tangent lines to determine the missing lengths. I justified my results by explaining my calculations when given algebraic expressions representing missing lengths. Then, I explained my interpretation of the results in the context of the scenario.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision by applying the definitions of tangent line and point of tangency in order to determine how to solve for the missing lengths of the walkways. I performed calculations using the Pythagorean Theorem, and I solved for an unknown value in an algebraic expression, and used this value to find a tangent length.)

Alternate Strategies or Solutions

Students may choose to draw the given verbal information onto the diagram. They may also choose to draw dotted lines, or some type of dashed lines, to represent the missing walkway lengths. They may choose to label these missing lengths with variables. Encourage students to carefully draw and label these parts, making sure that the correct lengths are placed on the correct tangent lines, as well as making sure the correct lines are drawn between each circle.

Technology

Students can use scientific calculators.

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Implementation Guide

Problem-Based Task 6.2.1: First Aid Station

Common Core State Standard

G–C.3 Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

Task Overview

Focus

How can a circle be inscribed in a triangle to model a real-world scenario? Which geometric constructions are necessary to inscribe a circle inside a triangle? In this task, students will perform steps necessary to inscribe a circle inside a triangle, as the center of the triangle will represent the location of a first aid station at a cross-country race. The construction of perpendicular segments will represent the shortest paths for the runners on three different routes to get to the first aid station.

This activity will provide practice with:

- analyzing given verbal information
- determining the number of geometric constructions necessary to sufficiently accomplish the goal of the task
- examining a diagram and applying definitions of circle terminology
- constructing angle bisectors and perpendicular segments in a triangle
- interpreting the results of geometric constructions and drawing conclusions based on the context of the scenario

Introduction

This task should be used to explore or to apply the skill of constructing the inscribed circles of a triangle. Students should already be familiar with constructing angle bisectors, as well as constructing a perpendicular line from an external point to a line.

Begin by reading the problem and clarifying the meaning of *angle bisector*, *equidistant*, *incenter*, and *inscribed circle*.

angle bisector	a ray that divides an angle into two congruent angles
equidistant	the same distance from a reference point
incenter	the intersection of the angle bisectors of a triangle
inscribed circle	a circle that contains one point from each side of a triangle or other polygon

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled with geometric constructions, specifically by the constructions of angle bisectors and perpendicular segments. They will recognize that the sides of a given triangle represent three different running routes at a cross-country race. They will also recognize that by constructing angle bisectors, they will be able to construct a circle inside the triangle formed by the three routes. The center of this circle will represent the location of the first aid station, and the perpendicular segments that are constructed from the center of the circle represent roped paths to the first aid station that are equidistant from each route.

- **SMP 5:** Use appropriate tools strategically.

  Students will use a compass and a ruler for this task. They will use these tools to carefully construct angle bisectors and perpendicular segments. They will use the compass to create arcs from a vertex of a triangle, and they will also construct a circle with the compass from the center of the circles, which is the point of intersection of the bisectors. The compass will also be used to construct perpendicular segments. The rulers will be used to draw the segments representing the angle bisectors, as well as the perpendicular segments which represent the roped paths in the scenario.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure and procedures involved in constructing angle bisectors of a triangle, as well as the procedures for constructing perpendicular segments. They will recognize that these constructions are repeated three times, since there are three angles to bisect, and three sides of the triangle, which will have a perpendicular segment from a point on each side to the incenter.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly constructing an angle bisector or a perpendicular segment

  Review the procedures for bisecting an angle and creating a perpendicular segment with the students before they begin the task. Have them practice these procedures on sample angles and triangles independent of the task.

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- identifying the incenter after only constructing one angle bisector

  Remind students that although the incenter will lie on the ray of the angle bisector, another bisector is needed to identify the exact location of the incenter.

- failing to construct the perpendicular from the incenter to any side of the triangle

  Have students write down the definition of "incenter" on their papers. Remind them that the incenter is found by the intersection of the angle bisectors, and that the perpendicular segments must be constructed originating from the incenter.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "Given three different routes for runners at a cross country race, what types of paths to the first aid station should be created for the runners, and why?" (**Answer**: If a runner is hurt, he/she will want to find the shortest path to the first aid station. The shortest path between two points is a straight line, so the paths should be segments that are the shortest distance from each route.)
- Before students begin working on the task, ask them, "Without measuring any angles or sides of the triangle given in the diagram, what type of triangle do you think it is?" (**Answer:** Answers will vary. Some students may think it is a right triangle, because the angle at the top of the diagram appears to be close to a right angle. Or, some students may think that all three angles appear to be less than 90°, so they may classify the triangle as acute.)
- Before students begin the task, ask them, "Does the given triangle appear to be an equilateral triangle? How could you check?" (**Answer:** It appears that the sides of the triangle are not all the same length. A ruler could be used to measure the sides to determine if they were congruent or not.)
- As students begin working on the task and are constructing angle bisectors and perpendicular segments, ask them, "What are some possible errors that you could make when working with a compass for the constructions? (**Answer**: Some possible errors are moving the compass width when making arcs, or not making the compass width large enough so that the arcs intersect.)
- As students are working on the task, remind them of the terminology used when constructing angle bisectors of a triangle. Ask the, "What is the term for the intersection of the angle bisectors of the triangle?" (**Answer**: The incenter is the intersection of the angle bisectors in a triangle.)
- As students are working on the task, ask them, "Why do you think it is unnecessary to construct all three angle bisectors of the triangle?" (**Answer:** Since all three bisectors are concurrent, which means they intersect at a single point, it is only necessary to construct two bisectors to find their point of intersection. The third bisector will also intersect at this point.)

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- As students are working on constructing a perpendicular segment from the incenter to a side of the triangle, ask them, "Why is it only necessary to construct one perpendicular segment from the incenter to a side on the triangle? What does this segment represent in terms of a circle?" (**Answer:** The perpendicular segment from the incenter to a side on a triangle represents the radius of a circle. Since all radii are equidistant from the center of a circle to any point on a circle, only one perpendicular segment needs to be constructed to establish the radius.)
- Once students have drawn the perpendicular segments, ask them, "Do the points on the triangle where the perpendicular segments intersect appear to represent the midpoints of the sides of the triangle? If not, what type of triangle do you think would have segments drawn that were perpendicular bisectors of the sides?" (**Answer:** No, the perpendicular segments do not appear to represent the midpoints of the triangle, because it can be seen that the segments created by the points are not equal. An equilateral triangle would have perpendicular bisectors.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts as to where they thought the first aid station should be placed in the triangle, as well as how the roped paths should be drawn. Discuss their thoughts on how to create the shortest distance from the first aid station to each side of the triangle, as well as how each path is equidistant from the first aid station. Ask them how they related the goal of the task to choosing the appropriate geometric constructions that they used to accomplish the task. Discuss the procedures used for the construction of the angle bisectors, as well as the procedures used to construct the perpendicular segments, and then the construction of the circle. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process regarding their choices for the types of geometric constructions used to solve the task. Ask them how they applied their knowledge of the procedure for bisecting angles, and the procedure for constructing perpendicular lines. Ask them to explain why the incenter represented the location of the first aid station. Focus on the use of precise mathematical language and clarity, specifically when discussing the terminology used in the construction procedures, as well at the terms "equidistant" and "incenter".

Connecting to Key Concepts

Make explicit connections to key concepts:

- A point that is **equidistant** lies the same distance away from a reference point.

  In this task, students will apply the term "equidistant" as they construct an inscribed circle in a triangle and recognize that the radii of the circle are segments which are equidistant from the three points on the triangle.

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- An **angle bisector** is the ray that divides an angle into two congruent angles.

  Students will construct angle bisectors to find the point of intersection of the bisectors, which will be the center of the inscribed circle.

- The point of concurrency of the angle bisectors of a triangle is called the **incenter.**

  Students will use the incenter to create perpendicular segments which represent roped paths in the context of the scenario.

- A circle that contains one point from each side of a triangle or other polygon is referred to as an **inscribed circle.**

  Students will construct an inscribed circle and recognize that each side of the triangle contains one point of the circle.

- To construct an inscribed circle, determine the shortest distance from the incenter to each of the sides of the triangle. Remember that the shortest distance from a point to a line or line segment is a perpendicular line.

  Students will apply their prior knowledge about constructing perpendicular lines, and they will create three perpendicular segments from the incenter to a point on each side of the triangle. They will recognize that the perpendicular segments represent the shortest distances from the center of the circle, which represents the first aid station, to the points on each side of the triangle, which represent the different routes for the runners.

Extending the Task

- To extend the task, ask students to use a protractor to measure the angles of the triangle in the diagram. Based on their measurements, ask them to classify the triangle as acute, right, or obtuse. Then ask them to redraw the triangle on their own papers with the angle measurements. Have them measure the lengths of the three sides of the triangle, to the nearest tenth of a centimeter. If they measured any angle of the triangle as 90°, or relatively close to this value, ask them to use the Pythagorean Theorem to determine if the triangle is indeed a right triangle. Ask for volunteers to discuss their results and conclusions, as well as how measurement errors could have affected the results.

- Another option for extending the task is to ask students to draw an equilateral triangle on their paper. Ask them to use their rulers and protractors and carefully construct three 60° angles to form the triangle, and then have them measure the three sides to see how accurate their triangles are. Then, ask them to inscribe a circle in the triangle by first constructing the three angle bisectors, and using the incenter to then construct three perpendicular segments that will allow them to use their compass to inscribe the circle. Discuss the relationship of the segments to the sides, such that students recognize that in an equilateral triangle, the perpendicular segments are the bisectors of each side, so the points of intersection are the midpoints of the sides. Ask them if they can draw any other conclusions about equilateral triangles, based on their constructions.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by recognizing that it can be modeled with geometric constructions, specifically by the constructions of angle bisectors and perpendicular segments. I recognized that the given triangle represented three different running routes at a cross-country race. I also recognized that by constructing angle bisectors, I was able to construct a circle inside the triangle formed by the three routes. The center of this circle represented the location of the first aid station. Then I constructed perpendicular segments from the center of the circle, which represented roped paths to the first aid station that were equidistant from each running route.)
- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** I used a compass and a ruler strategically by following the steps necessary for constructing angle bisectors and perpendicular segments. I used my compass to create arcs from a vertex of a triangle, and I also used the compass to construct a circle from the point of intersection of the angle bisectors. My compass was also used to construct the perpendicular segments. I used my ruler to draw the segment representing the angle bisectors, as well as the perpendicular segments from the incenter.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** When solving this problem, I made use of structure by applying the structure and procedures involved in constructing angle bisectors of a triangle, as well as the procedures for constructing perpendicular segments. I recognized that these constructions were repeated three times, since there were three angles to bisect, and three sides of the triangle which had a perpendicular segment from a point on each side to the incenter.)

Alternate Strategies or Solutions

Students may choose to construct all three angle bisectors of the triangle, as opposed to only two. Remind them that constructing all three bisectors is a good way to verify the accuracy of their constructions, as they can verify that all three bisectors intersect at the same point. Some students may choose to inscribe the circle prior to constructing any perpendicular segments. They may use trial and error to set the compass width such that the circle intersects the circle at the three points, one on each side. Remind them that constructing at least one perpendicular segment will allow for greater accuracy and precision, as well as saving time, as opposed to the amount of time that might be necessary to find the exact correct length necessary to draw the circle accurately.

Technology

No technology is needed for this task.

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Implementation Guide

Problem-Based Task 6.2.3: King Arthur and His Round Table

Common Core State Standard

G–C.3 Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

Task Overview

Focus

How can a quadrilateral be inscribed in a circle? Which types of quadrilateral can be inscribed in a circle? What properties of inscribed quadrilaterals can be applied in order to determine missing angle measurements? In this task, students will apply properties of the angles of an inscribed quadrilateral, and they will also identify the types of quadrilaterals that can be inscribed in a circle, given that the four vertices of the quadrilateral represent the seating locations of four of King Arthur's Knights of the Round Table in the scenario.

This activity will provide practice with:

- analyzing a diagram
- defining supplementary angles
- applying properties of interior angle relationships of quadrilaterals
- calculating missing angle measures
- classifying quadrilaterals based on properties of sides and angles
- identifying possible quadrilaterals that can be inscribed in a circle

Introduction

This task should be used to explore or to apply the skill of proving properties of angles for a quadrilateral inscribed in a circle. Students should already be familiar with understanding the properties of special quadrilaterals, calculating the supplement of an angle, and finding measures of inscribed angles in a circle.

Begin by reading the problem and clarifying the meaning of *inscribed quadrilateral* and *supplementary*.

inscribed quadrilateral	a quadrilateral whose vertices are on a circle
supplementary	two angles whose sum is 180°

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goals of the task are to determine if there is enough information given in the problem to identify a missing opposite angle measure, and to identify which types of quadrilaterals can be inscribed in a circle. They will accomplish this goal by applying their knowledge about supplementary angles, properties of inscribed quadrilaterals, and properties of quadrilaterals such as squares, rectangles, and isosceles trapezoids.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for the types of quadrilaterals that can be formed using the vertices of the quadrilaterals, representing four Knights of the Round Table. They will justify their results by discussing properties of the different types of quadrilaterals, and inscribed quadrilaterals, focusing on the relationship between the pairs of opposite angles. When presenting their arguments to others, they will explain their analysis of the results, and explain how these results are interpreted in the context of the scenario.

- **SMP 8:** Look for and express regularity in repeated reasoning.

  Students will express regularity in repeated reasoning as they follow the same procedure of identifying the relationship between pairs of opposite angles of each type of quadrilateral given in the task. They will reason that a square, a rectangle, and an isosceles trapezoid all have pairs of opposite angles that are supplementary, based on the properties of the quadrilaterals.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- assuming that the angle opposite of angle A in the task, angle B, is congruent to angle A

  Review the properties of inscribed quadrilaterals with students. Remind them that the pairs of opposite angles of an inscribed quadrilateral are supplementary angles.

- assuming that the other pair of angles, angles G and L, are either right angles, or they are congruent to each other

  Remind students that the only information that is known about angles G and L is that they are supplementary angles. Not enough information is given in the diagram to determine their individual measures.

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- incorrectly identifying a non-isosceles triangle as a quadrilateral that can be inscribed in a circle

  Remind students that opposite pairs of angles must be supplementary in order for a quadrilateral to be inscribed in a circle. Have them draw non-isosceles triangles and measure the pairs of opposite angles to verify that they are not supplementary angles.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "After reading the information given in the task, how do you know which type of polygon will be inscribed in the circle, in relation to the context of the Knights of the Round Table?" (**Answer:** In the problem, there are four knights that are referenced, and there are four vertices shown on the diagram that each represent the location of a knight seated at the table. Since there are four knights, a quadrilateral will be inscribed in the circle.)
- Before students begin the task, review the term "inscribed quadrilateral". Ask them, "In regards to geometry, what does the word "inscribed" mean? What does the word "quadrilateral" mean?" (**Answer**: The term "inscribed" means to draw a figure inside another figure, touching as many points as possible on the outside figure. A quadrilateral is a four-sided polygon. So, an "inscribed quadrilateral" is a quadrilateral drawn inside of a circle whose four vertices are on the circle.)
- Before students begin the task, ask them, "What types of quadrilaterals do you think can possibly be inscribed in a circle?" (**Answer**: Answers will vary. Some students may list all possible quadrilaterals, not recognizing that not all of them can be inscribed in a circle. Some students may list a square, rectangle, and an isosceles trapezoid as quadrilaterals that can be inscribed.)
- As a follow-up to the previous question, discuss with students their thoughts on why certain quadrilaterals cannot be inscribed in a circle. Lead students to consider the relationships of opposite angles in different types of quadrilaterals. Ask them, "Consider a four-sided kite, and a rhombus. What do you know about the pairs of opposite angles of these types of quadrilaterals? Are they supplementary?" (**Answer**: In a rhombus the opposite pairs of angles are congruent, but they are not always supplementary. In a kite, the opposite angles where the adjacent pairs of congruent sides meet are congruent, but the other pair of opposite angles is not congruent or supplementary.)
- Ask students, "What can you conclude about the types of quadrilaterals that can be inscribed in a circle, in terms of their pairs of opposite angles?" (**Answer:** If the pairs of opposite angles of a quadrilateral are supplementary, the quadrilateral can be inscribed in a circle.)

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- As students are working on the task, ask them, "What is the measure of angle *B* once segments *BG* and *BL* are drawn? How do you know this value?" (**Answer:** The measure of angle *B* is 68°. This value is known because the angle opposite of *B*, which is angle *A*, has a measure of 112°. The opposite angles of an inscribed quadrilateral are supplementary.)
- As students are working on the task, ask them, "What is the sum of the interior angles of a quadrilateral? Since you know that the sum of angles *A* and *B* is 180°, what do you know about the sum of angles *G* and *L*?" (**Answer:** The sum of the interior angles of a quadrilateral is 360°. Since the opposite angles of an inscribed quadrilateral are supplementary, then angles *G* and *L* must also sum to 180°.)
- Ask students, "Although you know the sum of angles *G* and *L* is 180°, do you know enough information from the problem to determine each angle's individual measure? If not, what information could be given that would be helpful?" (**Answer:** There is not enough information given to determine the measure of angle *G* and angle *L*. If the measure of the intercepted arc of angle *G* or angle *L* were given, then the angle measurements could be found, since the measure of an inscribed angle is equal to half of the measure of its intercepted arc.)
- As students are working on determining which quadrilaterals can be made using the seating locations of the four knights at the round table, ask them, "Why is it possible for the knights to form a square, rectangle, and an isosceles trapezoid?" (**Answer:** In both a square and a rectangle, each angle measures 90°, so all pairs of opposite angles are supplementary, since 90 + 90 = 180. One property of an isosceles trapezoid is that the pairs of opposite angles are supplementary.)
- Ask students, "What other property of an isosceles trapezoid can be used to verify that it is an isosceles trapezoid?"(**Answer:** In addition to the property that pairs of opposite angles are supplementary, if each angle of the trapezoid is supplementary to its adjacent angle, then the trapezoid is isosceles.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts and predictions about which types of quadrilaterals can be inscribed inside a circle. Specifically discuss the relationships of the pairs of opposite angles in the different types of quadrilaterals, and how those relationships relate to quadrilaterals which are inscribed in a circle. Ask them to discuss what information about other angles can be obtained by the given angle of 112° in the diagram. Also discuss what information would be needed to determine the other missing angles in the diagram. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.

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- Compare students' strategies and ways of justifying responses. Ask students to explain and defend their reasons for stating which quadrilaterals can and cannot be inscribed in a circle. Encourage them to explain their choices by applying prior knowledge about properties of quadrilaterals and the connection those properties have with inscribed quadrilaterals. Focus on the use of precise mathematical language and clarity, specifically when discussing definitions and properties of quadrilaterals, properties of circles, and properties of inscribed quadrilaterals.

Connecting to Key Concepts

Make explicit connections to key concepts:

- An **inscribed quadrilateral** is a quadrilateral whose vertices are on a circle.

  In this task, students will translate the given verbal information and diagram and recognize that the four points on the circle represent the vertices of a quadrilateral.

- The opposite angles of an inscribed quadrilateral are supplementary.

  Students will calculate the missing measure of an opposite angle of the quadrilateral, and they will verify that the two angles sum to 180°, which is the definition of supplementary.

- Rectangles and squares can always be inscribed within a circle.

  Students will apply their knowledge about the properties of quadrilaterals and apply the fact that opposite angles of squares and rectangles are supplementary. They will conclude that this meets the criteria of inscribed quadrilaterals, in that the opposite angles of an inscribed quadrilateral are supplementary.

Extending the Task

To extend the task, ask students to redraw the given diagram using a compass, ruler, and a protractor. Ask them to then connect the vertices so that they can see the inscribed quadrilateral. Next, ask them to use their protractor and measure the missing angles, G and L. Discuss how they can obtain measurements as precise as possible, reminding them that since the angles are opposite angles, their measures must sum to 180°. Based on the angle measurements of G and L, ask them to discuss which type of quadrilateral has been inscribed. Ask them to also measure each side of the quadrilateral, to the nearest tenth of a centimeter. Discuss the properties of the quadrilateral, in terms of measures of sides and angles.

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Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of this problem by first recognizing that the goals of the task are to determine if there is enough information given in the problem to identify a missing opposite angle measure, and to identify which types of quadrilaterals can be inscribed in a circle. I accomplished this goal by applying my knowledge about supplementary angles, properties of inscribed quadrilaterals, and properties of quadrilaterals such as squares, rectangles, and isosceles trapezoids. I identified that these three types of quadrilaterals had pairs of opposite angles that were supplementary, and I concluded that they could be inscribed in a circle.)
- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by justifying my choices for the types of quadrilaterals that can be formed using the vertices of the quadrilaterals, representing four Knights of the Round Table. I justified my results by discussing properties of the different types of quadrilaterals, and inscribed quadrilaterals, focusing on the relationship between the pairs of opposite angles. I explained the analysis of my results, and I explained how these results are interpreted in the context of the scenario, such that the conclusion was made that the four knights could form a square, rectangle, and an isosceles trapezoid.)
- **For SMP 8, ASK:** "How did you look for and express regularity in repeated reasoning?" (**Answer:** I expressed regularity in repeated reasoning as I followed the same procedure of identifying the relationship between pairs of opposite angles of each type of quadrilateral given in the task. I reasoned that a square, a rectangle, and an isosceles trapezoid all have pairs of opposite angles that are supplementary, based on the properties of the quadrilaterals.)

Alternate Strategies or Solutions

Students may choose to draw the diagram on their papers, using a protractor and ruler to recreate the diagram. They may draw the segments *GB* and *LB*, and then measure angle *B*. Remind them about the relationship of pairs of opposite angles of inscribed quadrilaterals, such that they should find a measure of 68° for angle *B*, since opposite angle *A* is given as 112°. Students may also try to draw circles and attempt to inscribe squares and rectangles in the circles. Encourage them to also attempt to inscribe other quadrilaterals, such as kites, rhombuses, and certain types of trapezoids. Discuss why these quadrilaterals cannot be inscribed inside a circle.

Technology

No technology is needed for this task.

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Problem-Based Task 6.3.1: Designing a T-Shirt Logo

Common Core State Standard

G–C.4 (+) Construct a tangent line from a point outside a given circle to the circle.

Task Overview

Focus

How can the properties of circles and tangent lines be applied in a real-world scenario? How can the tangent lines be constructed so that a circle is inscribed in a triangle? In this task, students will apply their knowledge about circles and tangent lines, and they will also construct tangent lines, to calculate the perimeter of a triangle with an inscribed circle that represents a company logo on a t-shirt.

This activity will provide practice with:

- examining a diagram representing a real-world scenario
- constructing perpendicular bisectors of segments
- drawing tangent lines on a circle
- applying the Pythagorean Theorem to determine missing side lengths
- identifying properties of tangent lines to determine missing segment lengths
- calculating the perimeter of a triangle

Introduction

This task should be used to explore or to apply the skill of constructing a tangent line from a point outside a given circle to the circle. Students should already be familiar with using a compass and a straightedge, constructing a perpendicular bisector of a line segment, and applying the Pythagorean Theorem to determine a missing side length of a right triangle.

Begin by reading the problem and clarifying the meaning of *perimeter*, *perpendicular bisector*, *point of tangency*, *Pythagorean Theorem*, and *tangent line*.

perimeter	distance around a figure
perpendicular bisector	a line that divides a segment into two equal parts and forms a right angle at the line segment
point of tangency	the only point at which a tangent line and a circle intersect
Pythagorean Theorem	a theorem that relates the length of the hypotenuse of a right triangle (c) to the lengths of its legs (a and b). The theorem states that $a^2 + b^2 = c^2$
tangent line	a line that intersects a circle at exactly one point and is perpendicular to the radius of the circle

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to calculate the perimeter of a triangle which represents a company logo on a t-shirt. They will accomplish this task by applying their knowledge about properties of circles and properties of tangent lines. They will construct tangent lines to complete the given sketch, and they will identify missing values of the triangle using the Pythagorean Theorem, as well as by applying properties about congruent segments of tangent lines formed from the same external point. Once they have found the missing segment parts, they will calculate the sum of all the parts to find the perimeter of the logo on the shirt.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for the segment lengths that they find to find the total lengths of all three sides of the triangle. They will justify these choices by explaining how they constructed the tangent lines to complete the sides of the triangle. They will also explain which properties of tangent lines were applied to determine missing lengths, as well as how they applied the Pythagorean Theorem to determine missing lengths. When presenting their arguments to others, they will explain their analysis of the results, specifically how the segment pieces were added together to get the perimeter, and explain how these results are interpreted in the context of the scenario.

- **SMP 5:** Use appropriate tools strategically.

  Students will use a compass and a straightedge for this task. They will use these tools to carefully construct perpendicular bisectors and tangent lines. They will use the compass to create a perpendicular bisector of a segment, and they will then use a straightedge to draw the tangent lines resulting from the segment bisector.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- assuming that a radius and a line are perpendicular at the possible point of intersection simply by observation

  Remind students that the definition of a tangent line is a line that intersects a circle at exactly one point and is perpendicular to the radius of the circle. Once students have constructed the tangent lines, the radius is perpendicular to the tangent line by definition, and not observation.

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- assuming two tangent segments are congruent by observation

  Have students make a list of the properties of tangent lines. Remind them that the two segments from the external point to each point of tangency are congruent, so as long as the segments are formed from two tangent lines from the same external point, then the segments are congruent.

- incorrectly changing the compass settings

  Remind students to be careful when using the compass to draw arcs so that the length does not change when constructing bisectors. Have them check their compass and ensure that it is tight enough so it will not move at all during a construction.

- not making large enough arcs to find the points of intersection

  Review the construction process for perpendicular bisectors of segments. Remind students that it is alright to make arcs that are too large, but making arcs that are too small will result in an incorrect construction.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, review perimeter with them. Ask them, "What is the difference between the area of a figure and the perimeter of a figure?" (**Answer**: The area of a figure is the space inside the figure, and the perimeter is the distance around the figure.)
- Ask students, "How is the perimeter of any figure calculated?" (**Answer**: The perimeter of any figure is found by finding the sum of all the side lengths of the figure.)
- As students begin working on the task, ask them, "What further information can you obtain from knowing that the diameter of the circle is 8 cm? In other words, which segment lengths on the diagram can you determine, and why?" (**Answer:** Since the radius is half the value of the diameter, the segments which represent radii of the circle - *DG*, *IG*, *FG*, *HG*, *EG*, and *JG* - are all 4 cm in length.)
- As students begin working on the task and are constructing perpendicular segment bisectors and tangent lines, ask them, "What are some possible errors that you could make when working with a compass for the constructions? (**Answer**: Some possible errors are moving the compass width when making arcs, or not making the compass width large enough so that the arcs intersect.)
- After students draw the two tangent lines, ask them, "How do you know that triangle *CFG* is a right triangle?" (**Answer**: Triangle *CFG* is a right triangle because the tangent line that was drawn from point *C* to point *B* intersects the circle at point *F*, creating a right angle by definition of a tangent line.)

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- After students have applied the Pythagorean Theorem to find the length of segment *CF*, ask them, "Which triangle is congruent to triangle *CFG* that shares a vertex of *C*? How are these triangles congruent? Which Triangle Congruency Theorem can be used?" (**Answer:** Triangle *CDG* is congruent to triangle *CFG*. The corresponding sides are all congruent. Specifically, one leg of each triangle is the radius of the circle, and all radii of a circle are congruent. Both triangles share one side. Also, the hypotenuses of both triangles are the same because of the property that states that the two segments from the external point to each point of tangency are congruent. The SSS Triangle Congruency Theorem can be used.)
- Once students have established that triangles *CDG* and *CFG* are congruent, ask them, "Which other two triangles are congruent to triangles *CDG* and *CFG*?" (**Answer:** Triangles *AEG* and *BEG* are also congruent to triangles *CDG* and *CFG*.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts about how to accomplish the goal of the task after reading through the problem. Ask them to explain their processes for constructing perpendicular bisectors, and then drawing tangent lines to draw the complete sketch of the logo. Discuss the properties of tangent lines which were applied to find missing segment lengths. Ask them why they were able to use the given diameter of the circle, as well as the Pythagorean Theorem, to also find missing side lengths. Ask them to list various pairs of congruent segments and explain how they are congruent. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and procedures for the constructions of the tangent lines, as well as their calculations of the perimeter. Encourage them to discuss the specific properties of tangent lines and how they relate to the segments of the logo design. Also discuss specifically how each of the segment pieces of the triangle sides was determined. Focus on the use of precise mathematical language and clarity, specifically when discussing perpendicular bisector and tangent line constructions, properties of circles, and properties of tangent lines.

Connecting to Key Concepts

Make explicit connections to key concepts:

- If a line is tangent to a circle, it is perpendicular to the radius drawn to the **point of tangency,** the only point at which a tangent line and a circle intersect.

  In this task, students will construct two tangent lines and recognize that the tangent lines intersect the circle at only one point.

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- Exactly one tangent line can be constructed by using construction tools to create a line perpendicular to the radius at a point on the circle.

  Students will use a compass and a straightedge to first construct a perpendicular bisector of a segment, and they will use this bisector to construct two tangent lines to the circle.

- Exactly two lines can be constructed that are tangent to the circle through an exterior point not on the circle.

  Students will construct the two tangent lines that originate from the same exterior point that is not on the circle.

- The two segments from the external point to each point of tangency are congruent.

  Students will use this property of tangents to identify the lengths of segments of the triangle. They will use the known value of one segment and recognize that the other segment has the same value.

Extending the Task

To extend the task, ask students to use a work with a partner and use their compasses and straightedges to design a new company logo that will be placed on a t-shirt. Ask them to begin with a circle, and then ask them to try and inscribe an equilateral triangle in the circle. Once they have these constructions completed, ask them to be creative with drawing different radii of the circle, using the vertices of the triangle as endpoints, for example. Encourage them to draw other segments, and possibly inscribe a circle within the triangle. Ask them to write down any observations regarding congruent segments, congruent triangles, tangent lines, or anything else they encounter. Ask for volunteers to share their designs and discuss them using circle, triangle, and tangent line terminology.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the task is to calculate the perimeter of a triangle which represents a company logo on a t-shirt. I accomplished this task by applying my knowledge about properties of circles and properties of tangent lines. I constructed tangent lines to complete the given sketch, and I identified missing values of the triangle using the Pythagorean Theorem, as well as by using properties about congruent segments of tangent lines formed from the same external point. When I found the missing segment parts, I calculated the sum of all the parts to find the perimeter of the logo on the shirt.)

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- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining my choices for the segment lengths that I found to find the total lengths of all three sides of the triangle. I justified these choices by explaining how I constructed the tangent lines to complete the sides of the triangle. I also explained which properties of tangent lines were applied to determine missing lengths, as well as how I used the Pythagorean Theorem to determine missing lengths. Finally, I explained how the combined segment lengths were added together to find the perimeter of the triangle.)
- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** I used a compass and a straightedge strategically by following the steps necessary for constructing perpendicular bisectors and tangent lines. I used my compass to create a perpendicular bisector of a segment, and then I used a ruler to draw the tangent lines resulting from the segment bisector.)

Alternate Strategies or Solutions

When they draw the diagram on their papers, some students may choose to label all known side lengths, especially the segments representing the radii of the circle. They may use congruency marking notation to have a visual picture of the congruent segment pieces. Encourage students to also make chart, or table, to display the congruent segment pieces in an organized manner. Remind students to continue to fill in the diagram, as well as the chart or table, each time they calculate or determine a missing segment length.

Technology

Students can use scientific calculators.

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Implementation Guide

Problem-Based Task 6.4.1: Around the Merry-Go-Round

Common Core State Standard

G–C.5 Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Task Overview

Focus

How can a central angle of a circle be measured in radians? How does the radius of a circle relate to the length of the intercepted arc of its central angle? How can radian measures and the circumference of a circle apply to a real-world scenario? In this task, students will apply concepts of circumference, arc lengths, and radian measures to determine the distance a carousel horse on a merry-go-round travels, as well as how many radians and revolutions the horse travels in a specified amount of time.

This activity will provide practice with:

- calculating conversion factors given different units
- identifying common conversion factors
- applying definitions and formulas for circumference and radian measure
- determining missing values in a formula
- approximating and rounding decimals
- analyzing results and assessing reasonableness

Introduction

This task should be used to explore the skill of defining the radian measure of an angle as the constant of proportionality, and applying radians to a circle as another method of measuring angles. Students should already be familiar with finding the circumference of a circle, and performing unit conversions.

Begin by reading the problem and clarifying the meaning of the terms *circumference*, *radian*, and *radian measure*.

circumference	the distance around a circle; $C = 2\pi r$ or $C = \pi d$, for which C represents circumference, r represents the circle's radius, and d represents the circle's diameter
radian	a unit for measuring angles; one radian is equal to the measure of the central angle that intercepts an arc equal in length to the radius of the circle; π radians = 180°
radian measure	the measure of an angle in radians; the ratio of the arc intercepted by the central angle to the radius of the circle

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Students will recognize that the goal of the task is to determine the number of feet a carousel horse on the outer edge of a merry-go-round travels in a 2-minute ride, and also the number of radians the horse travels in a 2-minute ride. They will accomplish this goal by applying their knowledge about the characteristics of circles, specifically the circumference of a circle and radian measures of a circle. They will use conversion factors to determine the distance the horse travels, and they will calculate the radian measure by first calculating the radius of the merry-go-round by using the formula for the circumference of a circle. They will also determine the number of revolutions the merry-go-round makes in 2 minutes, and they will assess the reasonableness of these results in the context of the scenario.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled by applying components of a circle, specifically circumference, radius, radian measure, and the ratio *pi*. They will recognize that a circle represents the shape of the merry-go-round, and that one revolution of the merry-go-round represents the circumference of the circle. They will apply the formula for the circumference of a circle to determine the radius, which will then be used to find the radian measure of a central angle. These values will be used to answer the questions presented in the task.

- **SMP 6:** Attend to precision.

  Students will apply the definitions of circumference, radius, and radian measure to determine the distance a carousel horse on the edge of a merry-go-round travels in 2 minutes, as well as the number of radians the horse travels, and how many revolutions the merry-go-round makes in 2 minutes. They will first use appropriate conversion factors and make necessary conversions to obtain the desired unit rates for calculating the distance and number of revolutions. Remind students to use appropriate units when labeling their results. Also remind them to use care when rounding their answers. Encourage them to use the correct terminology throughout the task, specifically when referring to the various components of a circle.

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Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- using the wrong unit conversion when converting from miles per hour to miles per minute, or from miles per minute to feet per minute

  Have students write down the conversion factors for hours and minutes and feet and miles. Remind students that there are 60 minutes in one hour, and there are 5,280 feet in one mile.

- obtaining values for calculations which involve π that are different than the results given in the task

  Remind students that measurements will be approximate and may differ slightly, depending on if the value for *pi* (π) is used as 3.14, or if the π button on the calculator is used.

- incorrectly applying the formulas for circumference and radian measure

  Have students write down the formula for the circumference, $C = 2\pi r$, and remind them that the radian measure of an angle is found by dividing the arc length by the radius of the circle, $\theta = \frac{s}{r}$.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "What other terms can you use to describe the circumference of a circle?" (**Answer:** Another term for circumference is the distance around a circle. Additionally, a revolution is one full movement around a circle.)
- Before students begin the task, ask them, "What is generally true about the speed of a merry-go-round?" (**Answer:** A merry-go-round is typically slow, but the speed is constant. Although some merry-go-rounds may be faster than others, they will all usually go around at constant speeds.)
- After students have read through the task and are beginning to work through the steps, ask them, "Based on the information given in the problem in miles per hour, what conversions will you need to use for the units to be represented in feet per minute?" (**Answer:** To convert from miles to feet, the conversion of 1 mile = 5,280 feet will be used. To convert from hours to minutes, the conversion of 1 hour = 60 minutes will be used.)
- As students begin working on the task and the conversion factors, ask them, "What is important to remember when converting units in terms of the set-up of the ratios so that the necessary units cancel out?" (**Answer:** It is important that the ratios are set up such that the units that need to be canceled out are opposite each other, meaning one unit of one factor is in the numerator, and the other same unit is in the denominator of the other factor, so that the units cancel out, like cross products.)

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- Once students have determined that a carousel horse travels one-tenth of a mile each minute, ask them, "How do you convert one-tenth of a mile per minute to one mile per ten minutes? Why is it necessary to do this to use the next conversion factor involving feet and miles?" (**Answer:** Since the conversion factor that is needed to determine the number of feet the carousel horse travels in a minute is in terms of one mile, it is necessary to use the conversion factor 1 mile = 5,280 feet. Since the previous ratio is one-tenth of a mile per minute, and the desired value is 1 whole mile, the ratio of one-tenth mile per minute is multiplied by 10 to get the ratio of 1 mile per ten minutes.)
- Discuss the discrepancies that occur when using the value of π as 3.14 compared to using the π button on a calculator. Ask students, "Why will you get a more precise and accurate result when you use the *pi* button on your calculator, as opposed to using the value of *pi* as 3.14?" (**Answer:** The value of 3.14 is commonly used to represent the approximation of *pi*, and this can be used if the *pi* button is not available on a calculator. The *pi* button on a calculator uses a more accurate value of *pi* as 3.141592654, which will allow for more precise results from calculations involving *pi*.)
- As students are finding the radius of the merry-go-round, ask them to find the value using both the π button on a calculator, and then by using 3.14 as the value of π. Ask them, "When finding the value of the radius by dividing 80 by *pi*, find the values by first dividing 80 by 3.14, and then by dividing 80 by the value of the *pi* button on your calculator. What place value do you notice the difference in the results?" (**Answer:** When 80 is divided by 3.14, the result is 25.477770701. When 80 is divided by the value of π on the calculator, the result is 25.46479089. The difference in the values can be noted in the hundredths place value.)
- Once students have determined that the merry-go-round makes approximately 6.6 revolutions in 2 minutes, ask them, "Why is this result reasonable to you? Why do you think a merry-go-round does not move faster than the value given in the problem of 6 miles per hour? What are some examples of results that would not be reasonable?" (**Answer:** Since a merry-go-round often appeals to younger children, the speed at which it moves will not be very fast, as a faster rate of movement could possibly frighten small children. If the results had been less than 1 revolution in 2 minutes, or possibly greater than maybe 20 revolutions in 2 minutes, then the results would not have been reasonable, because a merry-go-round typically will not move that slow or that fast.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

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Debriefing the Task

- Ask for volunteers to discuss their thought processes after initially reading the problem. Discuss ideas about how to accomplish the goals of determining how many feet and how many revolutions a carousel horse will go on the outer edge of a merry-go-round. Ask for students to explain the relationships of the different parts of a circle and how these parts will be represented in the context of the merry-go-round scenario. Discuss how the conversion factors were used to find the necessary unit rates. Also discuss the differences in the calculations when the value of 3.14 is used for π as opposed to using the π button on the calculator. Focus on the concept of radian measure and how this was calculated in the task. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and procedures for their calculations for each of the components in the problems. Encourage them to share alternate methods of determining the missing information. Ask them how they applied their knowledge of the characteristics and components of circles and radian measures. Also discuss their justification that the final result of approximately 6.6 revolutions in 2 minutes is a reasonable answer. Focus on the use of precise mathematical language and clarity, specifically when discussing circumference, radius, and radian measure.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The **radian measure** of a central angle can be found by calculating the ratio of the length of the intercepted arc to the radius of the circle.

  In this task, students will find the distance traveled by a carousel horse, which represents the length of an arc, and they will find the radius of the circle, which represents the distance from the center of the merry-go-round to the outer edge, and they will use these values to determine the radian measure.

- This ratio leads us to a formula for the radian measure of a central angle θ in terms of the intercepted arc length, s, and the radius of the circle, r: $\theta = \frac{s}{r}$.

  Students will apply the formula for the radian measure of a central angle by substituting the values for the arc length, which is 1,056 feet in the task, and the value for the radius, which is 25.46 feet, and calculate the radian measure of the central angle of the merry-go-round.

- Recall that the **circumference**, or the distance around a circle, is given by $C = 2\pi r$ or $C = \pi d$, where C represents circumference, r represents radius, and d represents the circle's diameter.

  Students will recognize that the distance around the outside of the merry-go-round represents the circumference of a circle. They will use the values given for the circumference of the merry-go-round and determine the radius using the formula $C = 2\pi r$.

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Extending the Task

- To extend the task, ask students to work with a partner to create another task, using the same scenario as the one given, but with different values for the circumference and the revolution speed per minute. Ask them to work through the same steps as presented in the task, and ask them to determine how many feet their carousel horse will travel in 5 minutes, and also how many revolutions the horse will make in 5 minutes. Ask students to share their scenarios and discuss their calculations. Specifically discuss if their revolution speed per minute is reasonable and practical in the context of a merry-go-round. As an extension to this, ask students to create a scenario in which the initial given information includes the number of revolutions the horse will make in 5 minutes. Then, the goal of the task will be to determine the speed of the merry-go-round in miles per hour.
- Another option for extending the task is to provide a diagram of a circle representing the merry-go-round. Ask students to use a compass and a straightedge and sketch a drawing of a merry-go-round. Ask the students to work together to determine where they would place 6 carousel horses around the outer edge of the merry-go-round such that they are equally spaced apart. Ask students to determine the measure of each central angle from the center of the merry-go-round to each pair of adjacent horses. Discuss why the measure of each central angle is 60°. Also ask students to create a table listing the different numbers of horses that could be placed on the outer edge of the merry-go-round such that the central angles are congruent (for example; 12 horses could be placed on the outer edge, and each central angle would measure 30°). Ask for volunteers to share their results.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Answer:** I made sense of the problem by first recognizing that the goal of the task is to determine the number of feet a carousel horse on the outer edge of a merry-go-round travels in a 2-minute ride, the number of radians the horse travels in a 2-minute ride, and the number of revolutions the merry-go-round makes in 2 minutes. I accomplished this goal by applying my knowledge about the characteristics of circles, specifically the circumference of a circle and radian measures of a circle. I used conversion factors to determine the distance the horse travels, and then I calculated the radian measure by first calculating the radius of the merry-go-round by using the formula for the circumference of a circle. Finally, I assessed the reasonableness of my results in the context of the scenario and concluded that the merry-go-round making approximately 6.6 revolutions in 2 minutes was reasonable.)

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- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by recognizing that this scenario can be modeled by applying components involving a circle, specifically circumference, radius, radian measure, and the ratio *pi*. I recognized that a circle represents the shape of the merry-go-round, and that one revolution of the merry-go-round represents the circumference of the circle. I applied the formula for the circumference of a circle in order to determine the radius, which I then used to find the radian measure of a central angle.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision by applying the definitions of circumference, radius, and radian measure in order to determine the distance a carousel horse on the edge of a merry-go-round travels in 2 minutes, as well as how many revolutions the merry-go-round makes in 2 minutes. I used conversion factors and made necessary conversions to obtain the desired unit rates for calculating the distance and number of revolutions. I used appropriate units when labeling my results, and I was careful when rounding my answers.)

Alternate Strategies or Solutions

- Although the task follows the process of dividing the radian measure by 2π in order to determine the number of revolutions the carousel makes in 2 minutes, some students may observe that there is another way to determine the number of revolutions made in 2 minutes. They might recognize that since the total number of feet the carousel travels in 2 minutes is 1,056, and the circumference of the carousel is 160 feet, then the number of revolutions can be found by dividing the total number of feet traveled in 2 minutes by the given circumference of the carousel, which is 160 feet: $1{,}056 \div 160 = 6.6$. Encourage students to also follow the procedure given in the task to verify their results.
- Some students may choose to draw a diagram representing the scenario. This may help them visualize the various parts of the circle, which represent the parts of the merry-go-round. Remind students to carefully and clearly label the values, such as the circumference and the radius. They may also draw a central angle in order to understand the radian measure. Also encourage students to use arrows to indicate the revolutions of the merry-go-round.

Technology

Students can use scientific calculators.

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Problem-Based Task 6.5.1: Designing a Tablecloth

Common Core State Standard

G–GMD.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*

Task Overview

Focus

How can the area formula for a circle be used to determine the length of the radius of a circle? How can trigonometric ratios be applied in order to find missing side lengths of a square inscribed in a circle? How can areas of squares and circles be applied to solve real-world problems? In this task, students will apply area formulas for both circles and squares, and they will also use trigonometric ratios in order to calculate measures of side lengths that represent the material used for a circular-shaped tablecloth with a square-shaped design inside.

This activity will provide practice with:

- drawing a diagram to represent a scenario described with verbal information
- identifying parts of a circle and a square
- applying area formulas for circles and squares
- calculating side lengths of right triangles using trigonometric ratios
- analyzing results of calculations

Introduction

This task should be used to explore the skill of applying formulas for the areas of circles and squares in order to solve real-world problems. Students should already be familiar with area formulas for circles and polygons, applying trigonometric ratios, and understanding how to bisect angles and side lengths.

Begin by reading the problem and reviewing the meaning of diagonal of a square, diameter of a circle, radius of a circle, and trigonometric ratio.

diagonal of a square	line segment drawn from opposite corners of the square which bisect the corners into 45° angles
diameter of a circle	straight-line segment that passes through the center of the circle whose endpoints are on the circle
radius of a circle	segment which is half the length of the diameter
trigonometric ratio	ratio of the length of two sides of a right triangle and a related angle

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Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for the formulas used to determine the missing values of the circle and square necessary for accomplishing the goals of the task. They will justify these choices by explaining their calculations based on their knowledge of area formulas for a circle and a square. They will also explain their choice of applying a specific trigonometric ratio used to determine the side length of the inscribed square. When presenting their arguments to others, they will explain their analysis of the results, specifically how the area of the square design is subtracted from the area of the circular-shaped tablecloth in order to determine the area of the tablecloth that will not be covered by the square design.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled by applying area formulas of a circle and a square, as well as by applying trigonometric ratios. They will recognize that the difference of the areas of the circular-shaped tablecloth and the square-shaped design inscribed in the circle will represent the area of the tablecloth that does not contain the printed design. They will also recognize that the area of the square-shaped design of the tablecloth can be calculated by first using a trigonometric ratio to determine the side length of the square.

- **SMP 6:** Attend to precision.

  Students will apply the formulas for the area of a circle and a square, as well as the definitions of a trigonometric ratio, in order to determine the side length of the square inscribed in a circle, which represent a circular-shaped tablecloth and its square-shaped design. They will also apply the results of these formulas to determine the area of tablecloth which is not printed with the square design. They will determine a specified place value for rounding their results, and they will carefully perform the necessary calculations using their calculators. Remind students to use appropriate units when labeling their results. Also remind them to use care when rounding their answers. Encourage them to use the correct terminology throughout the task, specifically when referring to the various components of a circle and square, as well as the trigonometric ratio used when determining the side length of the square.

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Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly applying the formulas for the areas of a circle and a square

  Have students write down the area formulas on the tops of their papers. Remind them to isolate the radius when calculating its value, given the value of the area of the circle. Also remind them to take the square root of the results, since r^2 is in the formula, and not r.

- using the value for the radius when finding the length of the side of the square

  Remind students that the radius is only half the length of the diameter, and that the diameter represents the hypotenuse of the right triangle in the inscribed square. Remind students to double the length of the value of the radius to obtain the correct value used for the calculation of the square's side length.

- incorrectly applying a trigonometric ratio when finding the length of the side of the square

  Review the basic trigonometric ratios with students. Discuss why either the sine or cosine ratio can be used in this problem. Remind them that the hypotenuse in the right triangle is the value of the diameter of the circle.

- inconsistently rounding values throughout the task

  Discuss the types of rounding errors that can occur when faced with values that have a large number of decimal places. As a class, agree upon a specified place value to round answers to each time to achieve consistency.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, review the concept of inscribing a square inside of a circle. Ask them, "How do we know that a square can be inscribed in a circle? What property of inscribed quadrilaterals is applied?" (**Answer**: A square can be inscribed in a circle because of the property of inscribed quadrilaterals that states the opposite angles of an inscribed quadrilateral are supplementary. Since each angle of a square is 90°, then opposite angles are supplementary, because 90° + 90° = 180°. The definition of supplementary is two angles whose sum is 180°.)

- After students have drawn a diagram of the inscribed square in the circle, ask them. "How do you know that the diameter of the circle and the diagonal of a square are the same segments? Explain." (**Answer**: The diameter of a circle is the segment that goes through the center of the circle and has endpoints on the circle. The diagonal of a square has endpoints that are the opposite corners of the square, and these corners (vertices) of the square are on the circle. The diagonal of a square goes through the center of the circle. Therefore, the diameter of the circle and the diagonal of the inscribed square are the same segment.)

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- As students begin working on the task, encourage them to use their diagram to help them visualize the parts of the circle and the square that are used in the calculations. Ask them, "How can you use your sketch of the tablecloth to help you throughout the task? What can you draw on the diagram?" (**Answer:** It might be helpful to label the values on the diagram as they are calculated. It also might be helpful to sketch the right triangle that will be used to determine the side length of the square.)
- As students begin working on the task, discuss precision in calculations and rounding. Specifically, discuss the discrepancies that occur when using the value of π as 3.14 compared to using the π button on a calculator. Ask students, "Why will you get a more precise and accurate result when you use the *pi* button on your calculator, as opposed to using the value of *pi* as 3.14?" (**Answer:** The value of 3.14 is commonly used to represent the approximation of *pi*, and this can be used if the *pi* button is not available on a calculator. The *pi* button on a calculator uses a more accurate value of *pi* as 3.141592654, which will allow for more precise results from calculations involving *pi*.)
- As students are working through the task, point out the differences in place values that are used for rounding in the steps of the task. Ask them, "Based on the context of the scenario, what do you think is the most realistic and practical place value that the answers should be rounded to?" (**Answer:** Answers will vary. Since the material for the tablecloth is measured in units of feet, it might be the most practical to have answers rounded to the nearest hundredth of a foot. It would not be practical to round up to the nearest whole foot, because a half of a foot difference, for example, would be significant in terms of measuring fabric.)
- As students are working through the task and determining the side length of the square, ask them, "How do you know that you can create a right triangle which has two 45°angles? (**Answer**: Since the diagonals of a square bisect each corner, and each corner measures 90°, then each corner has two 45° angles created by the diagonals.)
- As students are using a trigonometric ratio to determine the side length of the square, ask them, "What other trigonometric ratio can be used besides the sine ratio? Why can't you use the tangent ratio?" (**Answer:** The cosine ratio could also be used to solve for the side length of the square. The tangent ratio cannot be used because it is the ratio of the opposite side and the adjacent side. Neither the length of the leg opposite or adjacent to the 45° angle is known, so a ratio using the hypotenuse length of the triangle must be used, since the hypotenuse length is already known because it is the same segment as the diameter of the circle, which was calculated as 6.18 feet. Both the sine and cosine ratios use the hypotenuse.)
- As students are working through the final steps of the task, ask them, "As you are subtracting the area of the square from the area of the circle, what is one way you could tell if you had made a mistake in the calculation for the area of the square?" (**Answer:** Since the area of the square is less than the area of the circle, if the area of the square was calculated at a value greater than 30 ft^2, then an error was made. The area of the square must be less than 30 ft^2, since the square is inscribed inside of the circle.)

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- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts about how to accomplish the goals of the task after reading through the problem. Specifically, ask them to explain their ideas for finding the value of the side length of the inscribed square, as well as how to determine the amount of the tablecloth that did not have the printed design on it. Discuss the properties of circles and squares that were applied in the task, as well as how the area formulas were applied. Also discuss how right triangles and trigonometric ratios were used to determine missing lengths. Ask them about any issues that arose with rounding, or with the values used in their calculations. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and procedures for each sequential part of the task. Ask them to explain their choices for the formulas used, as well as their choices for the specific values that were used in their calculations. Ask them to explain their results, and how they determined the amount of area of the tablecloth that did not have the printed design on it. Focus on the use of precise mathematical language and clarity, specifically when discussing components of circles and squares, as well as how the side lengths of the square related to side lengths of a right triangle.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The formula for the area of a circle is $A = \pi r^2$, where r represents the radius of the circle, and the approximate value of π can be used as 3.14.

  In this task, students will use the given area of a circle, which represents the circular-shaped tablecloth, and they will solve the area formula for the radius of the circle. They will then use this value to determine the diameter of the circle, which is also the diagonal of the inscribed square.

- Recall that the **hypotenuse** is the side opposite the vertex of the 90° angle in a right triangle. Every right triangle has one 90° angle. The hypotenuse is the longest side of a right triangle.

  Students will calculate the diameter of the circle, and they will recognize that the diameter is the same segment representing the diagonal of the inscribed square. They will use the value of the diagonal as the same value of the hypotenuse of a right triangle, which is formed by the two adjacent side lengths of the square and a corner of the square.

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- The **sine** of an acute angle in a right triangle is the ratio of the length of the opposite side to the length of the hypotenuse; $\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$.

  Students will apply the sine ratio to determine the length of the side opposite of the 45° angle in the right triangle. They will substitute in the value for the hypotenuse, which is the same as the diameter of the circle.

Extending the Task

- To extend the task, ask students to use a straightedge to first draw the other diagonal of the square, and then ask them to draw other segments which divide the square into equal parts. Ask them to be creative and make their own design for the square part of the tablecloth. Encourage them to use colored pencils or markers on their designs. Ask for volunteers to share their designs, and discuss what types of triangles and quadrilaterals were formed when they constructed additional segments on their designs. Ask them how these polygons relate to the parts of the circle. Encourage students to discuss their designs using proper terminology.
- Another option for extending the task it to ask students to work with a partner to create their own square-shaped tablecloth, with a circle inscribed in the square this time. Ask them to provide the value for the area of the square, and then create the problem such that it is necessary to find the area and the circumference of the inscribed circle. Have them model their problem based on the original task, such that they write down specific questions and steps to solve the problem. Ask for volunteers to share their problems and discuss the steps and calculations performed to solve the problem.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining my choices for the formulas used to determine the missing values of the circle and square necessary for accomplishing the goals of the task. I explained my calculations based on my knowledge of area formulas for a circle and a square. I also explained my choice of the trigonometric ratio used to determine the side length of the inscribed square. Once my calculations were complete, I explained how I analyzed my results, specifically how the area of the square design was subtracted from the area of the circular-shaped tablecloth to determine the area of the tablecloth that will not be covered by the square design.)

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- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by recognizing that this scenario can be modeled by applying area formulas of a circle and a square, as well as by applying trigonometric ratios. I recognized that the difference of the areas of the circular-shaped tablecloth and the square-shaped design inscribed in the circle represented the area of the tablecloth that does not contain the printed design. I also recognized that the area of the square-shaped design of the tablecloth could be calculated by first determining the side length of the square using a trigonometric ratio.)
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision by correctly applying the formulas for the area of a circle and a square, as well as the definitions of a trigonometric ratio, to determine the side length of the square inscribed in a circle, which represented a circular-shaped tablecloth and its square-shaped design. I also applied these formulas to determine the area of tablecloth which is not printed with the square design. I determined a specified place value for rounding my results, and I carefully performed the necessary calculations with my calculator. I made sure the appropriate units were used when labeling my results. I also made sure I was careful when rounding my answers each time.)

Alternate Strategies or Solutions

When they draw the diagram of a circle and an inscribed square, some students may choose to use a compass and a straightedge to have a more realistic picture of the scenario. They may also draw the diagonals of the square, as well as labeling the corners of the square with right angle markings. Encourage students to neatly label all known values of the scenario, such as the radius and diameter as they become known. Some students may also choose to use a consistent place value for rounding for each calculation. Remind students to choose a place value such as the hundredths or thousandths place, so that they can maintain a reasonable level of accuracy, and so that their results are close in approximation to those values given in the task.

Technology

Students can use scientific calculators, or else graphing calculators, if their scientific calculators do not have the trigonometric functions on them.

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Implementation Guide

Problem-Based Task 6.6.1: Nurturing an Investment

Common Core State Standard

G–GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Task Overview

Focus

How can the center and radius of a circle be determined from the standard form of the equation of the circle? How can the method of completing the square be used to convert an equation of a circle in general form to standard form? How can the distance formula be applied in conjunction with circles in order to solve real-world problems? In this task, students will determine the center and radius of circles which represent sprinkler outlets installed in a municipal park, and they will also apply the distance formula to determine if a tree planted at a certain location will be watered by both sprinklers.

This activity will provide practice with:

- identifying geometric figures based on a verbal description
- drawing a diagram to represent a scenario described verbally
- completing the square to get the equation of a circle in general form
- converting an equation of a circle given in general form to standard from
- applying the distance formula
- analyzing results of calculations and applying them in the context of the scenario

Introduction

This task should be used to explore or to apply the skill of completing the square to find the center and radius of a circle given by an equation and converting from the general form of an equation of a circle to standard form. Students should already be familiar with completing the square to form perfect square trinomials and using the distance formula to determine distances between points.

Begin by reading the problem and clarifying the meaning of the terms *distance formula*, *general form of an equation of a circle*, *perfect square trinomial,* and *standard form of an equation of a circle*.

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distance formula	formula that states the distance between points (x_1, y_1) and (x_2, y_2) is equal to $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
general form of an equation of a circle	$Ax^2 + By^2 + Cx + Dy + E = 0$, where $A = B$, $A \neq 0$, and $B \neq 0$
perfect square trinomial	trinomial of the form $x^2 + bx + \left(\frac{b}{2}\right)^2$
standard form of an equation of a circle	$(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the center and r is the radius

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they make sense of the goal of the task by recognizing that they will need to identify the centers and the radii of two sprinkler outlets to determine if a tree planted in a specified location will be watered by both sprinklers. They will examine the given information and conclude that the equations of the circles given will need to be converted into standard form to identify the center and the radius of each sprinkler outlet. They will apply their prior knowledge about completing the square to accomplish the task of converting the equations to standard from. They will reason quantitatively as they substitute values into the distance formula to calculate the distance from the desired location of the tree to each sprinkler outlet. They will also compare values given in radical format.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled with circles that will be drawn on a coordinate plane. The circles will represent two sprinkler outlets installed in a municipal park. The heads of these sprinklers will represent the center of a circle, and the streams of water that come from the sprinklers will represent the radius of a circle. They will recognize that converting the given forms of the equations for these circles into equations for the standard form of a circle will allow them to identify the center and radius of each circle. They will use this information to draw the circles on the coordinate plane. They will also draw the desired location of a tree on the coordinate plane, which is represented with coordinates. Once the sprinklers and tree have been drawn, students will determine the distances between objects and analyze the results.

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- **SMP 7:** Look for and make use of structure.

  Students will apply the structure and procedures involved in converting equations of a circle from general form to standard form to identify the center and the radius. They will also apply the process for completing the square when converting the general form to the standard form. They will recognize that these procedures will be repeated twice, since there are two circles representing two sprinklers in the problem. They will also follow the steps necessary to apply the distance formula, as they substitute values in the formula to determine distances between the sprinklers and the tree.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- confusing the radius with the square of the radius

  Have students write down both the standard form and the general form of the equation for a circle. Remind them that in these formulas, the radius is squared, so they will need to take the square root of the value of the radius.

- forgetting to square half the coefficient of x when completing the square

  Before starting the task, review the process of completing the square with students. Practice with an example or two, using values that are not used in the task. Encourage students to ask questions and discuss any parts of the process that are confusing to them.

- incorrectly sketching the circles on the coordinate plane

  Have students draw each circle separately. For the circle representing Sprinkler 1, have them plot the center of the circle first. Then, since the radius is 8, have them count either vertically or horizontally 8 units on the grid. Then once that point is drawn, have them use a compass and set its length as the radius of the circle, and then with the sharp point of the compass on the center of the circle, draw the full circle. Have them repeat this process for the circle representing Sprinkler 2.

- misinterpreting the results of the sketch of the circles and tree location and thinking that the tree will not get watered by both sprinklers

  Discuss with students that the tree will get watered by both sprinklers as long as it is located within the boundaries, or in the interior, of both circles. Remind them to carefully plot the centers of both sprinklers, as well as plotting points representing the circle, based on the radius of each circle, on the coordinate plane. Remind them to look for the area where the circles overlap.

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Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "How can a sprinkler that is installed in the ground be represented by a circle? Which part of a circle is the actual sprinkler? What does the stream of water represent in relation to a circle?" (**Answer:** The sprinkler that is installed in the ground in this task is represented by a circle because the equation given that represents its boundary is in the general form of the equation of a circle. A sprinkler that is installed in the ground represents a circle because it rotates and shoots out water in a uniform circular motion. The sprinkler itself represents the center of a circle, and the stream of water that it creates represents the radius of the circle.)
- After reading through the problem, ask students, "In general, how will it be possible for a tree to be watered by both sprinklers? Explain what this means in the context of where the tree must be located in relation to the sprinklers." (**Answer:** If a tree receives water from both sprinklers, then that means it must be located somewhere between both sprinklers so that the streams of water from each sprinkler reach the tree. It must be located in the interior of the circles, or the area where the circles overlap.)
- As an extension to the previous question, ask students, "Is it possible for the tree to get water from both sprinklers if the streams of water from the sprinklers do not overlap? Why or why not?" (**Answer:** No, the tree will not get water from both sprinklers if their streams of water do not overlap. If the tree is located in the interior of one sprinkler, but not the other, then it will only receive water from one sprinkler, but not both.)
- Before students begin working on through the steps in the task, discuss their initial predictions regarding if the tree will be watered by both sprinklers. Ask them, "Is there enough information given in the problem to make a valid prediction about whether the desired location of the tree will allow for it to be watered by both sprinklers? If not, what information do you think you will need to find to determine if the tree's location is an appropriate one or not?" (**Answer:** Since no other points are given for the location of the two sprinklers, and neither the center nor the radius is given for the sprinklers, there is not enough information to determine if the tree's location will allow it to be watered by both sprinklers. The center and the radius for each sprinkler will need to be identified.)
- As students begin working, discuss the two forms of equations of circles. Ask them, "Why is it necessary to convert the given equations into another form to find the center and radius for each circle? What is the equation for this form?" (**Answer:** The equations given for the circles representing the two sprinklers are given in general form. The general form does not provide the center and radius of a circle. The form of an equation of a circle that allows for the center and radius of a circle to be identified is the standard form, which is $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the center and r is the radius.)

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- If students are having difficulty with the process of completing the square, review additional examples of the process with them using sample problems. Ask them, "When you are finding the value of half of the "*bx*" term, what must you remember to do with this value?" (**Answer:** The value of half of the "*bx*" term must be added to both sides of the equation so that it is balanced.)
- Ask students, "Why are the radii of the circles changed to radical format when comparing their values to the distances from the tree to each sprinkler? How else could the values have been compared?" (**Answer:** The values of the radii are changed to radical format in order to more easily compare them to the values of the distances between the tree and each sprinkler, since the distances are given as radicals; $\sqrt{53}$ meters and $\sqrt{80}$ meters. These values could have been simplified to 7.28 meters and 8.94 meters and then compared to the radii values of 8 and 10, respectively.)
- As students are sketching the drawing of the sprinklers and the tree, ask them, "How can you use a compass to draw the circles? How do you know the location of the radius on the coordinate plane for each circle? What possible points represent the length of the radius for each circle?" (**Answer:** The center of each circle is plotted as a point on the coordinate plane. For Sprinkler 1, the radius is 8, so count either horizontally or vertically, left or right, 8 units from the center. Points (10, 18), (2, 10), (10, 2), and (18, 10) are points which represent the radius. For Sprinkler 2, points (25, 22), (15, 12), (25, 2), and (35, 12) are points which represent the radius.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts about how to accomplish the goal of the task after reading through the problem. Ask them to explain their ideas for how to determine if the desired location of the tree will allow it to be watered by both sprinklers. Discuss why the equations given for the circles representing the sprinklers do not give enough information. Also discuss the steps involved in using the process of completing the square to convert the given general form of the circle equations to standard form. Ask students to discuss how they applied the distance formula to find the distances between each sprinkler and the tree, focusing on how they chose the appropriate points to substitute into the formula. Discuss how they sketched the drawing of the sprinklers and the tree on the coordinate plane. Ask them about any issues that arose with rounding, or with the values used in their calculations. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.

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- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and procedures for each sequential part of the task. Ask them to explain their analysis of the results that led them to conclude that the desired location of the tree would allow it to be watered by both sprinklers. Ask them to justify these results by explaining the representation of the scenario on the coordinate plane. Focus on the use of precise mathematical language and clarity, specifically when discussing the general form and the standard form of the equations of a circle, as well as the components of the process of completing the square and the components of the distance formula.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The **distance formula** states that the distance d between $A\left(x_1, y_1\right)$ and $B\left(x_2, y_2\right)$ is $d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$.

  In this task, students will apply the distance formula to determine how far away the tree is from each of the sprinkler outlets, which are represented by the centers of two circles with different radii.

- For a circle with center (h, k) and radius r, a point (x, y) is on the circle if and only if $\sqrt{\left(x - h\right)^2 + \left(y - k\right)^2} = r$. Squaring both sides of this equation yields the **standard form of an equation of a circle** with center (h, k) and radius r: $(x - h)^2 + (y - k)^2 = r^2$.

  Students will convert the given equations of the two circles, which represent the two sprinkler outlets, from general form to standard form, which will allow them to identify the radius and center of each circle.

- The **general form of an equation of a circle** is $Ax^2 + By^2 + Cx + Dy + E = 0$, where $A = B$, $A \neq 0$, and $B \neq 0$.

  Students will identify that the given equations for the two circles representing the sprinkler outlets are in general form of an equation of a circle. They will convert this general form to standard form for each equation.

- The standard form of the equation of a circle comes from completing the squares from the general form of the equation of a circle.

  Students will perform the necessary steps in the process of completing the square with the general form of the equations of the circles representing the sprinkler outlets.

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- Completing the square is the process of determining the value of $\left(\frac{b}{2}\right)^2$ and adding it to $x^2 + bx$ to form the perfect square trinomial, $x^2 + bx + \left(\frac{b}{2}\right)^2$.

  Students will apply the process of completing the square to convert the equations of the circles given in general form to standard form. They will determine the value of the "*bx*" term for each equation, find half the value, and add the value to each side of the equation, for both circle equations.

Extending the Task

- To extend the task, ask students to work with a partner and determine several other possible locations where a tree could be planted and get watered by both sprinklers. Have them create a table of possible coordinates which represent the tree's location. Ask them to apply the distance formula to mathematically verify that their chosen location will be in the interior of both circles representing the two sprinklers. Ask for volunteers to share their list of points, and ask them to explain their calculations using the distance formula. Then, ask students to consider the question, "What if the circles representing the sprinklers intersected in only one point?" Ask the pairs of students to determine how to change the center of one, or both, of the sprinkler locations in order for the circles to intersect at one point. Encourage students to use terminology such as "point of tangency", and ask them what type of tangent line is drawn at this point of intersection.
- Another option for extending the task it to ask students to work with a partner to create their own scenario similar to the one presented in the original task, but this time have them provide the center and radius for each sprinkler, and then have them create the equations for the circles in both general and standard form. Ask them to draw a diagram of their circles, using a compass and a straightedge. Ask for volunteers to share their problems and discuss the steps and calculations performed to solve the problem.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I made sense of the goal of the task by recognizing that I needed to identify the centers and the radii of two sprinkler outlets to determine if a tree planted in a specified location would be watered by both sprinklers. I examined the information and concluded that the equations of the circles given would need to be converted into standard form to identify the center and the radius of each sprinkler outlet. I applied my prior knowledge about completing the square to convert the equations to standard form). "Which of your strategies represent quantitative reasoning?" (**Answer:** I used quantitative reasoning as I used the distance formula to calculate the distance from the desired location of the tree to each sprinkler outlet. I also compared values that were given in radical format.)

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- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I used mathematics to model this particular scenario by recognizing that it can be modeled with circles drawn on a coordinate plane. The circles represented two sprinkler outlets installed in a municipal park. The heads of these sprinklers represented the center of a circle, and the streams of water from the sprinklers represented the radius of a circle. I recognized that converting the given forms of the equations for these circles into equations for the standard form of a circle allowed me to identify the center and radius of each circle. I used this information to draw the circles on the coordinate plane. I also drew the desired location of a tree on the coordinate plane, which was represented with coordinates. Then I determined the distances between the objects and analyzed the results.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** When solving this problem, I made use of structure by applying the structure and procedures involved in converting equations of a circle from general form to standard form in order to identify the center and the radius. I also applied the process for completing the square when I converted the general form of the equations to the standard form. I performed all of these procedures twice, since there were two circles representing two sprinklers in the problem. I also followed the steps for the distance formula by substituting values in the formula in order to determine distances between the sprinklers and the tree.)

Alternate Strategies or Solutions

Some students may choose to express the results of their distance formula calculations in decimal format instead of radical format, as is presented in the task. Encourage students to also verify their conclusions when comparing the distances by also comparing the values in radical format.

Technology

Students can use scientific calculators if they choose to simplify the radical values found when using the distance formula.

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Problem-Based Task 6.7.1: A Circle Graph for Lunch

Common Core State Standard

G–GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point* $\left(1, \sqrt{3}\right)$ *lies on the circle centered at the origin and containing the point (0, 2).*

Task Overview

Focus

How can the location of a point on a circle be proven? How can the distance formula, the midpoint formula, and the slope formula be applied to prove the precise location of a point? In this task, students will first determine the radius and center of a circle which represents a graph of students' lunch preferences, and then they will apply various formulas to prove or disprove the location of a given point on the circle.

This activity will provide practice with:

- interpreting components of a circle based on a circle graph of a real-world scenario
- applying the midpoint and distance formulas
- determining the radius and center of a circle
- identifying the slope of segments
- classifying two slopes as perpendicular, parallel, or neither
- proving or disproving a claim using evidence from calculations

Introduction

This task should be used to explore or to apply the skill of proving or disproving that a point lies on a circle. Students should already be familiar with applying the midpoint formula, the distance formula, and the slope formula. They should also be familiar with using slope to determine if lines are parallel, perpendicular, or neither.

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Begin by reading the problem and reviewing the meaning of the terms *distance formula*, *midpoint formula*, and *slope formula*.

distance formula formula that states the distance between points (x_1, y_1) and (x_2, y_2) is equal to $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

midpoint formula formula that states the midpoint of the line segment created by connecting (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

slope formula formula that states the slope of the line through (or the line segment connecting) $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\frac{y_2 - y_1}{x_2 - x_1}$

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students will construct viable arguments and justify their choices for proving or disproving the claim that a given point on a circle has the specified coordinates given in the problem. They will apply the midpoint and distance formulas to first prove that the point is on the circle, by determining the center and radius of the circle. Then they will apply the slope formula to compare the slopes of the segment containing the center of the circle and the point in question, and the segment containing the center and another point on the circle. When presenting their arguments to others, they will explain their conclusion of whether or not the point has the specified coordinates by citing evidence from their calculations performed in the formulas. They will also explain their reasoning and analysis in the context of the scenario.

- **SMP 6:** Attend to precision.

  Students will apply the midpoint, distance, and slope formulas to accomplish the goal of this task. They will use these formulas to determine the center and radius of a circle, and they will use these values to prove whether or not a point on the circle is located at the given coordinates. They will carefully substitute values for the coordinates into each formula and perform the algebraic operations to simplify the results. Remind students to apply order of operations as they simplify the expressions in the formulas. Encourage them to use the correct terminology throughout the task, specifically when referring to the center and radius of a circle as they pertain to the location of a point on the circle, as well as when explaining why the slopes of the segments are not perpendicular.

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- **SMP 7:** Look for and make use of structure.

  Students will apply the structure and procedures involved in performing the calculations of the midpoint, distance, and slope formulas. They will repeat the process for the distance formula twice, as they determine the distance between the center of the circle and two points on the circle. They will also apply the slope formula twice to find the slopes of each segment and determine whether or not they are perpendicular to each other. For all formulas, they will substitute the necessary coordinates in each formula to determine the desired midpoint, distance, or slope.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- substituting the wrong coordinates into the formulas

  Have students write down the formulas for midpoint, distance, and slope on their papers. Remind them to carefully label each pair of coordinates they use in each formula as x_1, y_1, x_2, and y_2.

- incorrectly classifying the slopes of segments *PC* and *CT*

  Review the different types of slope with students. Have them write down the three types of slope (parallel, perpendicular, or neither), and remind them that two lines are parallel if their slopes are identical, two lines are perpendicular if their slopes are negative reciprocals of each other, and two lines are neither parallel nor perpendicular if the slopes have any other relationship to each other.

- mistakenly concluding that point *P* is located at (8, 22) only because it was proven to be located on the circle

  Remind students that the only thing that is certain when point P is proven to be on the circle is that it is a radius of the circle. Remind them that there are many radii of a circle, as well as many possible locations on the circle, so further proof is needed.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them, "What do you know about a circle graph in terms of the total percentage of its parts?" (**Answer:** A circle graph must total 100%.)
- Before students begin the task, ask them to examine the circle graph. Ask them, "What component of a circle does the segment *RT* represent? How do you know this based on the context of the graph in terms of the sectors?" (**Answer:** Segment RT represents the diameter of a circle. It is a segment that goes through the center of the circle and has its endpoints on the circle. On each side of the diameter, the sectors of food preferences each total 50%. Specifically, on one side the "chicken" and "pizza" sectors total to 50%, and on the other side the "burger", "tuna", and "cold cut" total to 50%.)

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- Before students begin the task, ask them, "What are your initial predictions about point *P* being located at the given coordinates of (8, 22)? Explain your thoughts." (**Answer:** Answers will vary. Some students may assume that the point is located at (8, 22) because the other point given, point *R*, also has a *y*-coordinate of 22. Some students may examine the diagram and determine that point *P* cannot have a *y*-coordinate of 22 because it is not "level" with point *R*, which does have a *y*-coordinate of 22.)
- As students are working on finding the radius of the circle, ask them, "In the task, the coordinates used to find the radius are the coordinates for the center of the circle, (16, 16), and the coordinates for point *T*, (8, 10). Which other point on the circle could be used to find the radius instead of point *T*, and why?" (**Answer:** Point *R* could also be used in conjunction with the center to find the radius. The coordinates for point *R* are given in the problem, and the segment from the center to point *R* is also a radius.)
- After students have determined that point *P* is a point on the circle, ask them, "Why can you not conclude that point *P* is located at the given coordinates of (8, 22), even though you have proved that point *P* is on the circle? Explain." (**Answer:** Although point *P* is on the circle, there are many possibilities for the exact location of *P*. As long as it was proven that the point is located 10 units away from the center, then it is a radius of the circle, but the endpoint of the radius has many possible locations.)
- Review the possible types of slope when comparing two lines. Ask students, "What are the three possibilities for the types of slope when comparing two lines?" (**Answer:** If two lines have the same slope, then they are parallel lines. If two lines have slopes that are negative reciprocals of each other, then they are perpendicular lines. If neither of these two cases occurs, then the lines are neither parallel nor perpendicular to each other.)
- Ask students, "Based on the context of the graph, why do segments *PC* and *CT* have to be perpendicular to each other?" (**Answer:** On the circle graph, the sector that is defined by segments *PC* and *CT* comprise 25% of the graph, which is exactly one quarter of the graph. Every circle is 360°, so exactly one quarter of 360° is 90°. So the angle formed by the center of the circle and these two segments is a right angle.)
- After students have found the slopes of segments *PC* and *CT*, ensure their understanding of the results. Ask them, "What was the slope of each segment, and what did you conclude from these results, and why?" (**Answer:** The slope of segment *PC* was $-\frac{3}{4}$, and the slope of segment *CT* was $\frac{3}{4}$. These slopes are not identical, so they are not parallel. The slopes are not negative reciprocals of each other, so they are not perpendicular. Since they are not perpendicular, then the point *P* cannot be located at the coordinates (8, 22).)

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- As an extension of the previous question, ask students, "What did the slope of segment *PC* need to be for it to be perpendicular to segment *CT*? Explain mathematically." (**Answer:** The slope of segment *PC* needed to be $-\frac{4}{3}$ for it to be perpendicular with segment *CT*. Since the slope of segment *CT* is $\frac{3}{4}$, these values would be negative reciprocals of each other, because when multiplied, their product is –1: $-\frac{4}{3} \bullet \frac{3}{4} = -1$.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their initial thoughts about point *P* being located at the coordinates (8, 22). Discuss their thoughts on how to mathematically prove this claim, and discuss how it relates to the context of the circle graph. Ask students to discuss how they found the midpoint of segment *RT*, as well as how they knew this segment was the diameter of the circle. Discuss how the distance formula was applied to determine the radius of the circle, and then how the value of the radius was used to prove point *P* was on the circle. Also discuss the necessity of proving if segments *PC* and *CT* are perpendicular, specifically based on the sectors on the circle graph. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and procedures for the use of the midpoint, distance, and slope formulas. Ask them to explain the results of the calculations for each formula and how these results related to each sequential part of the task. Ask them to explain their analysis of the results that led them to conclude that point *P* was not located at (8, 22), specifically focusing on the results of comparing the slopes of the segments *PC* and *CT*. Focus on the use of precise mathematical language and clarity, specifically when discussing the segments of the circle in terms of diameter and radius, and the terminology used when comparing the slopes of the two segments.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The standard form of an equation of a circle with center (h, k) and radius r is $(x-h)^2+(y-k)^2=r^2$. This is based on the fact that any point (x, y) is on the circle if and only if $\sqrt{(x-h)^2+(y-k)^2}=r$.

  In this task, students will prove or disprove that a point is on the circle by substituting in the coordinates of the point along with the coordinates of the radius, and they will determine if the calculations on the left side of the equation are equal to the value of the radius.

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- Two lines are parallel if their slopes are identical. Two lines are perpendicular if their slopes are negative reciprocals of each other.

  Students will use the slope formula and determine the slopes of two line segments and then classify the slopes as parallel, perpendicular, or neither.

Extending the Task

- To extend the task, ask students to work with a partner and calculate the number of degrees for each of the central angles of the circle graph. Ask them to create a table of values, such that they can relate the percentage of each sector of the graph to the corresponding degrees of the central angle. For example, $\angle PCQ$ represents 10% of the circle graph, which is one-tenth of the circle. Since a circle contains a total of 360°, then one-tenth of 360° is 36°. Therefore, the measure of $\angle PCQ$ is 36°. As them to follow this procedure for the remaining central angles, making sure to record the percent of each sector, as well as the number of degrees of the circle the sector represents. Ask for volunteers to share their tables and discuss their calculations. Remind students that the sum of the central angles must be 360°.
- Another option for extending the task it to ask students to work with a partner to create a survey that they can take of the students in their classroom. An example might be to survey students about their favorite color, or their favorite sport. Ask students to take their survey results and create a circle graph displaying the results. Ask them to sketch their circle graphs on a coordinate plane, creating whole number coordinates for the center and the radius. Ask them to analyze their graphs to determine if any radius segments are perpendicular. Ask them to find the distance between chords created by adjacent segments, whose endpoints are on the circle. Ask for volunteers to share their surveys and graphs, as well as how they applied prior knowledge about types of segments and segments lengths on their circles.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 3, ASK:** "How did you construct viable arguments and critique the reasoning of others?" (**Answer:** I constructed viable arguments by explaining how I disproved the claim that a given point on a circle has the specified coordinates given in the problem. I applied the midpoint and distance formulas to first prove that the point is on the circle, by determining the center and radius of the circle. Then I applied the slope formula to compare the slopes of the segment containing the center of the circle and the point in question and another point on the circle. Based on these results, I concluded that the point in question was not located at the given coordinates, because the line segment containing the point and the center of the circle was not perpendicular to the segment containing a right angle. I explained that the segments would have to be perpendicular because in the context of the scenario, this section of the graph represented one-quarter of the graph, and therefore the angle formed by the center of the circle and the segments is a right angle.)

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- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** I attended to precision by carefully performing the steps involved in the midpoint, distance, and slope formulas. I used these formulas to determine the center and radius of a circle, and then I used these values to prove whether or not a point on the circle was located at the coordinates given in the problem. I carefully substituted values for the coordinates into each formula and performed the algebraic operations in order to simplify the results. I applied order of operations when I simplified the expressions in the formulas.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** When solving this problem, I made use of structure by following the structure and procedures involved in performing the calculations of the midpoint, distance, and slope formulas. I repeated the process for the distance formula twice, as I determined the distance between the center of the circle and two points on the circle. I also applied the slope formula twice in order to find the slopes of each segment so I could determine whether or not they are perpendicular to each other.)

Alternate Strategies or Solutions

When calculating the radius of the circle, some students may choose to use the coordinates of point *R* instead of point *T*, as presented in the task, since the distance from the center of the circle, *C*, to point *T* also represents a radius. Encourage students to calculate the radius of both segments *CT* and *CR* in order to verify their calculations are correct. Some students may also choose to find and compare the slopes of segments *PC* and *CR*, instead of segments *PC* and *CT*, since the sectors between *PC* and *CT* also add up to 25%, which is one quarter of the graph. Also encourage students to find and compare the slopes for both pairs of segments so that they can verify their results.

Technology

No technology is needed for this task.

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