The Remainder Theorem

Prerequisite Skills

This lesson requires the use of the following skills:

- multiplying and dividing monomials
- evaluating functions for a given value of *x*

Introduction

In mathematics, the word "remainder" is often used in relation to the process of long division. You are probably familiar with dividing whole numbers. For example, the result (or quotient) of 8 divided by 4 is 2; that is, 8 ÷ 4 = 2. In other words, if the dividend of 8 units were divided by a divisor of 4, each group would have 2 units.

The process of division does not always result in a whole number. For instance, the result of 14 divided by 5, or 14 ÷ 5, is 2 with a remainder of 4. Recall that this means that if a dividend of 14 units were divided by a divisor of 5, then each group would have 2 units and there would be 4 units left over. 4 is referred to as the remainder. In earlier grades, you often wrote the quotient and its remainder as 2R4 or $2\frac{4}{5}$. The idea of having remainders also extends to dividing polynomials. Sometimes when polynomials are divided, the result is a polynomial; other times there is a remainder.

Key Concepts

- Long division, used to divide whole numbers, can also be used to divide polynomials. Recall that the dividend is the quantity being divided, and the divisor is the quantity by which it is being divided: dividend ÷ divisor = quotient.
- Long division with polynomials can sometimes be long and tedious; fortunately, there is another way to divide polynomials.
- The process of **synthetic division**, a shorthand way of dividing a polynomial by a linear binomial by using only the coefficients, is often used instead.
- In order to use synthetic division, the dividend must be a polynomial written in standard form, ordered by the power of the variables, with the largest power listed first. If a term is missing, 0 must be used in its place.
- For example, compare $4x^2 + 16$ to the standard form of a polynomial, $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x^1 + a_0$. Notice that $4x^2 + 16$ does not have a value for $a_{n-1}x^{n-1}$, or in this case, ax. To prepare this polynomial for synthetic division, substitute 0 as the coefficient (a) in the ax term: $4x^2 + 0x + 16$. Now you can use the coefficients 4, 0, and 16 to perform the synthetic division.

- For synthetic division, the divisor must be of the form $(x - a)$, where a is a real number.
- Use the following steps to divide polynomials using synthetic division. An example has been provided for clarity.

Synthetic Division of Polynomials	
Example: $(3x^2 - 20x + 12) \div (x - 3)$	
1. Write the coefficients of the dividend, $3x^2 - 20x + 12$: 3, –20, and 12.	$\begin{array}{rrr} 3 & -20 & 12 \end{array}$
2. Identify the value of a in the divisor, $(x - a)$. Write this value, 3, in the upper left corner.	$\begin{array}{r\rfloor rrr} 3 & 3 & -20 & 12 \end{array}$
3. Create a horizontal line below the coefficients, allowing for space to write above and below the line.	$\begin{array}{r\rfloor rrr} 3 & 3 & -20 & 12 \\ & & & \\ \cline{2-4} & & & \end{array}$
4. Write the first coefficient in the dividend below the horizontal line.	$\begin{array}{r\rfloor rrr} 3 & 3 & -20 & 12 \\ & & & \\ \cline{2-4} & 3 & & \end{array}$
5. Multiply the number below the horizontal line by the value of a and write the product under the next coefficient and above the horizontal line.	$\begin{array}{r\rfloor rrr} 3 & 3 & -20 & 12 \\ & & 9 & \\ \cline{2-4} & 3 & & \end{array}$
6. Add the numbers in the new column. Write the result below the horizontal line in that column.	$\begin{array}{r\rfloor rrr} 3 & 3 & -20 & 12 \\ & & 9 & \\ \cline{2-4} & 3 & -11 & \end{array}$
7. Repeat steps 5 and 6 until addition has been completed for all columns.	$\begin{array}{r\rfloor rrr} 3 & 3 & -20 & 12 \\ & & 9 & -33 \\ \cline{2-4} & 3 & -11 & -21 \end{array}$
8. Draw a box around the far right sum.	$\begin{array}{r\rfloor rrr} 3 & 3 & -20 & 12 \\ & & 9 & -33 \\ \cline{2-4} & 3 & -11 & \boxed{-21} \end{array}$
9. The numbers below the horizontal line represent the quotient. These numbers are the coefficients of the polynomial quotient in the order of decreasing degree. The boxed number is the remainder. Place the remainder over the divisor to express the final term of the polynomial.	$= 3x - 11 - \dfrac{21}{x - 3}$

- If you are dividing by a linear function, $(x - a)$, the order of the quotient is 1 less than the dividend. The remainder, if any, is a constant.
- If the remainder is 0, then the divisor is a factor of the polynomial.
- Synthetic division can also be used to find the value of a function. This is known as **synthetic substitution**.
- To evaluate a polynomial using synthetic substitution, follow the same process described for synthetic division. For example, given the function $3x^2 - 20x + 12$, if you must determine the value of the function at $x = 3$, use 3 as the a value in the divisor of the synthetic division. The resulting remainder gives the value of the polynomial when evaluated at $x = 3$.
- If a polynomial $p(x)$ is divided by $(x - a)$, then the remainder, r, is equal to $p(a)$.
- This process leads to the **Remainder Theorem**.

Remainder Theorem
For a polynomial $p(x)$ and a number a, dividing $p(x)$ by $x - a$ results in a remainder of $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

- To take a closer look at the Remainder Theorem, let's work with the same polynomial we used to demonstrate synthetic division of polynomials: $3x^2 - 20x + 12$.
- Let $p(x) = 3x^2 - 20x + 12$. We can use synthetic substitution (by following the process of synthetic division), to evaluate this function for $x = 3$. The result is the remainder, or –21. Because this result is a number other than 0, the Remainder Theorem allows us to conclude that $(x - 3)$ is not a factor of $p(x)$. Only when the remainder is 0 will $(x - a)$ be a factor of the polynomial. A remainder of any number other than 0 indicates that $(x - a)$ is not a factor of the given polynomial.
- If we evaluate the same function, $p(x) = 3x^2 - 20x + 12$, for $x = 6$, the remainder is 0. By the Remainder Theorem, $(x - 6)$ is a factor of the given polynomial.
- Dividing a polynomial by one of its binomial factors results in a **depressed polynomial**. When $3x^2 - 20x + 12$ is divided by 6, the depressed polynomial is $3x - 2$.
- It is sometimes easier to evaluate a function for a given value by using direct substitution. For instance, when evaluating the expression $3x^2 + 2$ when $x = 4$, directly substitute 4 into the polynomial and follow the order of operations: $3(4)^2 + 2 = 3(16) + 2 = 48 + 2 = 50$. Other times, when the polynomials are more complicated, synthetic substitution is more efficient. Both methods are helpful when working with polynomials.
- As you will see in this lesson, synthetic division and substitution can also be applied to real-world problems.

Common Errors/Misconceptions

- not using the standard form of a polynomial before attempting to use synthetic division
- using the incorrect sign when substituting or dividing
- forgetting to use 0 as a placeholder for coefficients that are not present in the standard form of the dividend