Describing End Behavior and Turns

Prerequisite Skills

This lesson requires the use of the following skills:

- determining the degree and leading coefficient of a polynomial
- writing a polynomial in standard form

Introduction

By this point in your mathematics experience, you have worked extensively with functions: determining the slopes of linear functions, identifying the vertices of quadratic functions as maxima or minima, determining the end behavior of exponential functions, and identifying intercepts of many types of functions. You can extend the skills you have used to analyze functions you have previously studied in order to understand the graphs of other polynomial functions.

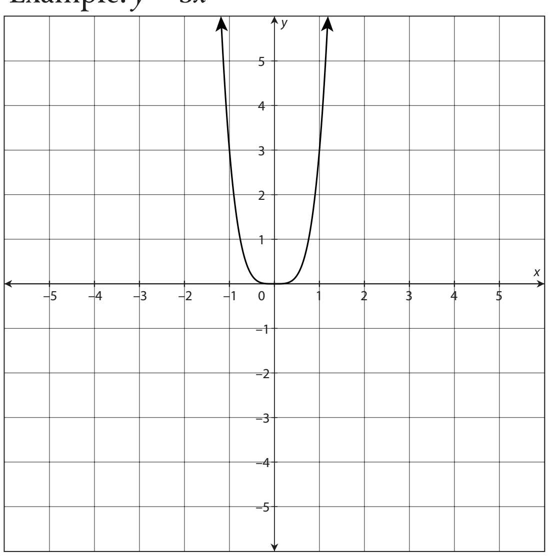
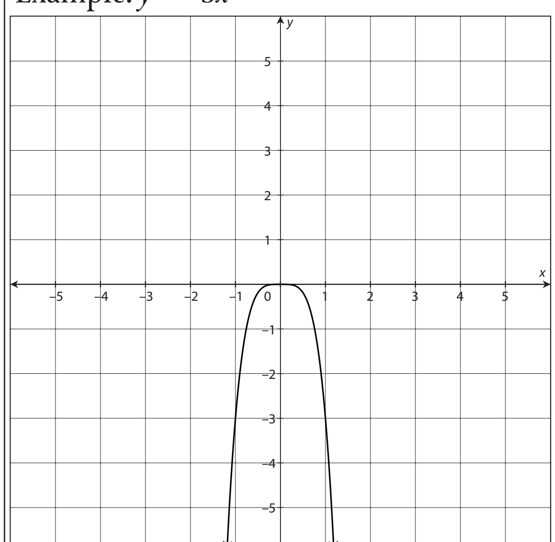
Key Concepts

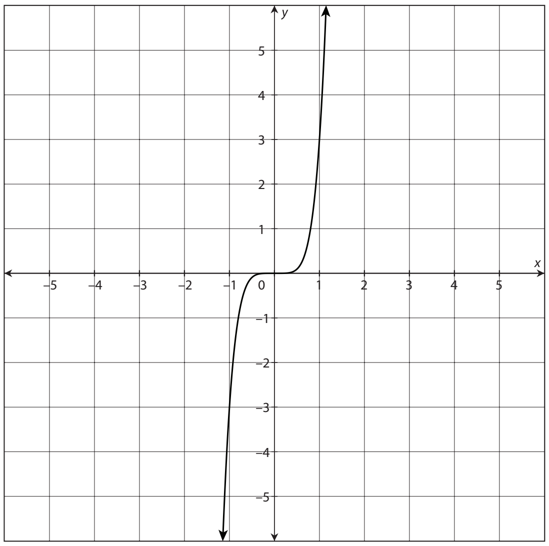
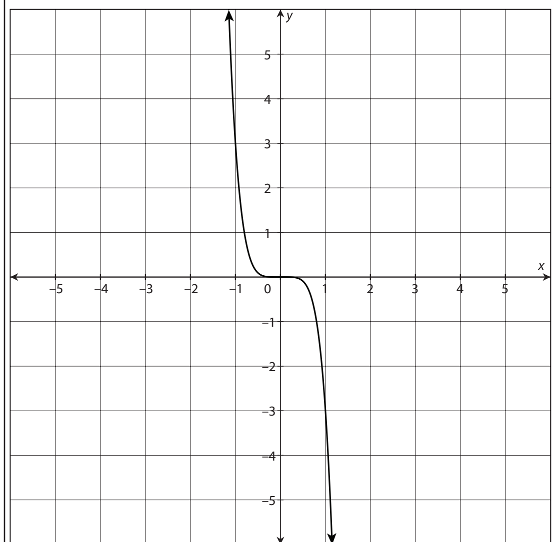
- Recall that a polynomial function is a function with a general form of $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x^1 + a_0$, where a_1 is a rational number, $a_n \neq 0$, and n is the highest degree of the polynomial.
- Polynomial functions are defined for any function that contains positive integer exponents.
- Recall that **integers** are numbers that are not fractions or decimals.
- The degree of a polynomial function is the highest exponent to which the dependent variable is raised.
- For example, the equation $y = 3x^7 + 9x^3 - x + 4$ is a seventh-degree polynomial function because its highest exponent is 7 and all other exponents are positive whole numbers.
- $y = 4x^{\frac{3}{5}} + 6x - 3$ is not a polynomial function because the exponent is not a whole number. $y = -3\sqrt{x} + 4$ is also not a polynomial function since the square root of a number can be written as a power of $\frac{1}{2}$, which is not a whole number either. And $y = 5x^3 + 2x^{-4} + 6$ is not a polynomial function because not all exponents are non-negative integers.

End Behavior

- To determine the **end behavior** of a polynomial function, or the behavior of the graph as x approaches positive or negative infinity, consider the highest degree of the polynomial and its coefficient, ax^n.
- If n is even, the polynomial function is considered an **even-degree polynomial function**.

- When n is even and a is positive, then both ends of the function will extend upward. That is, the value of $f(x)$ approaches positive infinity as x approaches negative infinity, and also when x approaches positive infinity. Symbolically, this can be written $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.
- When n is even and a is negative, then both ends of the function will extend downward. That is, the value of $f(x)$ approaches negative infinity as x approaches negative infinity, and also when x approaches positive infinity. Symbolically, this can be written $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$.
- If n is odd, the polynomial function is considered an **odd-degree polynomial function**.
- When n is odd and a is positive, then one end of the function will extend down to the left and the other end will extend up to the right. That is, the value of $f(x)$ approaches positive infinity as x approaches positive infinity, and the value of $f(x)$ approaches negative infinity as x approaches negative infinity. Symbolically, this can be written $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.
- When n is odd and a is negative, then one end of the function will extend up to the left and the other end will extend down to the right. That is, the value of $f(x)$ approaches positive infinity as x approaches negative infinity, and the value of $f(x)$ approaches negative infinity as x approaches positive infinity. Symbolically, this can be written $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$.

Even-degree polynomials	
Positive leading coefficient Example: $y = 3x^4$	**Negative leading coefficient** Example: $y = -3x^4$
	
$f(x) \to +\infty$ as $x \to -\infty$ $f(x) \to +\infty$ as $x \to +\infty$	$f(x) \to -\infty$ as $x \to -\infty$ $f(x) \to -\infty$ as $x \to +\infty$

Odd-degree polynomials	
Positive leading coefficient	**Negative leading coefficient**
Example: $y = 3x^5$	Example: $y = -3x^5$
	
$f(x) \to -\infty$ as $x \to -\infty$	$f(x) \to +\infty$ as $x \to -\infty$
$f(x) \to +\infty$ as $x \to +\infty$	$f(x) \to -\infty$ as $x \to +\infty$

Turning Points

- A **turning point** of a function is a point where the graph of the function changes from sloping upward to sloping downward or, alternatively, from sloping downward to sloping upward.
- To determine the maximum number of turning points of a function, subtract 1 from the highest degree of the polynomial. In other words, find $n - 1$.
- For instance, the polynomial function $y = 3x^7 + 9x^3 - x + 4$ can have no more than 7 – 1, or 6, turning points.
- The maximum number of turning points does not necessarily indicate the actual number of turning points of a function, just that it can have no more than that number. Some functions may have fewer turning points than the number calculated.
- A turning point corresponds to a **local maximum**, the greatest value of a function for a particular interval of the function, or a **local minimum**, the least value of a function for a particular interval of the function. A local maximum may also be referred to as a **relative maximum** and a local minimum may also be referred to as a **relative minimum**.

Roots of a Polynomial Function

- The highest degree of the polynomial determines the maximum number of **roots**, or *x*-intercepts of a function.
- A polynomial function with a degree of 10 could have up to 10 roots, but could also have 0 to 9 roots, depending on the specific equation.
- Recall that real numbers include all rational and irrational numbers, but do not include imaginary and complex numbers.

Sketching a Polynomial Function

- Being able to identify the general end behavior, the possible number of turning points, and the maximum number of roots of a polynomial function can be helpful in creating a rough sketch of the function.
- Start by choosing at least six *x*-values that are both positive and negative. It is also useful to choose the value of 0.
- As you've done in previous courses, substitute each chosen *x*-value into the given function and evaluate to determine the corresponding *y*-value. Then, plot the points on a graph.
- Be sure to smoothly connect all chosen points to illustrate the graph of the function.
- Graphing calculators are especially helpful when sketching a complicated function.

Common Errors/Misconceptions

- assuming the maximum number of calculated turning points is the actual number of turning points of a function
- assuming the maximum number of roots is the actual number of roots of a function
- not using the highest degree term to identify the end behavior of the polynomial function
- not using the highest degree of the polynomial function to determine the maximum number of turns
- confusing even-degree and odd-degree functions with even and odd functions