Key Features of Trigonometric Functions

Prerequisite Skills

This lesson requires the use of the following skills:

- measuring angles in degrees and radians
- being familiar with the unit circle, quadrant names, and reference angles
- evaluating trigonometric functions of special angles
- determining the period, amplitude, phase shift, and vertical shift of a trigonometric function
- knowing how to use the table functionality of a graphing calculator
- being familiar with supplementary and vertical angles

Introduction

Trigonometric functions are used to model many repetitive phenomena in our universe. Examples include the location of the moon relative to Earth, the position of the hands on a clock, and the height of a weight bouncing at the end of a spring.

The input values for such models are not always angle measures. They are often a function of time or distance, neither of which is measured in degrees or radians. Yet, by adjusting for period, amplitude, phase shift, and vertical displacement, trigonometric functions model the desired situations well.

In addition, we must be able to evaluate trigonometric functions of any real number, positive or negative, and not just of the **special angles** whose trigonometric values can be computed exactly. This work becomes easier when considering the symmetries that exist in the unit circle and in the graphs of the trigonometric functions.

Key Concepts

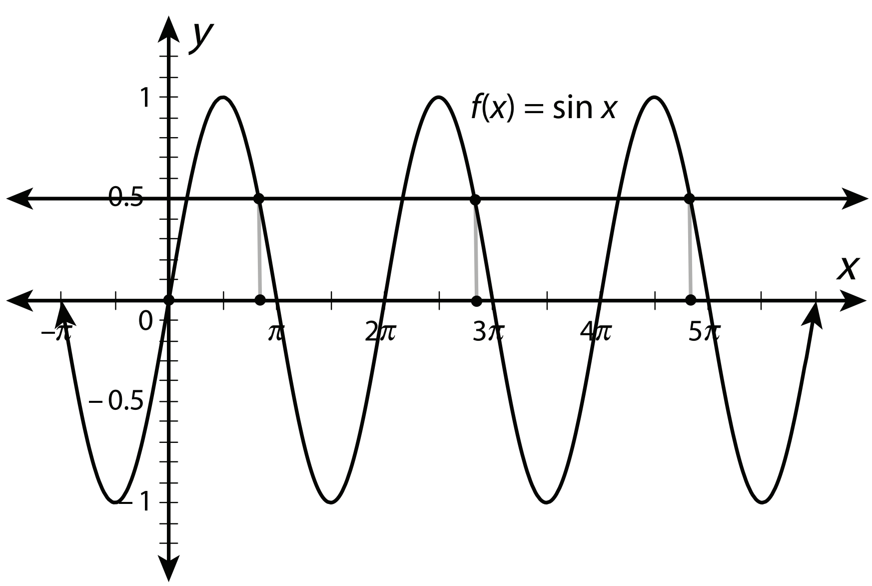
- Trigonometric functions of an angle in the unit circle are based on the coordinates of the point where the angle's terminal side intersects the unit circle, no matter how many times the angle measure may have caused the terminal side to rotate around the unit circle. The reference angle and quadrant of the angle's terminal side help to evaluate the result.
- Recall that a **reference angle** is the acute angle between the *x*-axis and the terminal side of an angle in standard position.
- Angles that have the same terminal side when in standard position are called **coterminal angles**. For example, $\frac{\pi}{4}$ is coterminal with $\frac{9\pi}{4}$.

- Remember that positive angle measures cause the terminal side of the angle to rotate counterclockwise around the unit circle, and negative angles cause the terminal side to rotate clockwise.
- Also recall that the cosine of an angle is the x-coordinate of the point where the angle's terminal side intersects the unit circle, and the sine of an angle is the y-coordinate of this point.
- As the terminal side of an angle completes a full circle and begins to pass through points it has previously passed through, trigonometric functions of that angle will produce the same results they produced the previous time around. A function that exhibits such repetitive, periodic behavior is called a **periodic function**. A periodic function will repeat itself identically over and over.
- The **argument** of a trigonometric function is the value at which the trigonometric function is being evaluated. It is the independent variable in a function.
- A **cycle** is the smallest representation of a cosine or sine function graph as defined over a restricted domain; it is equal to one repetition of the period of a function.
- The **period** of a trigonometric function is the horizontal distance from the beginning to the end of a cycle on the graph. It is the interval required in a function's argument to get from the start of one repetition of output values to the start of the next repetition.
- The graphs of all trigonometric functions are periodic and, therefore, a particular result will repeat every period. For some trigonometric functions, most values will occur twice per period.
- Trigonometric functions having arguments greater than the size of the period can be simplified by subtracting an integer multiple of the period from the argument to reduce it to less than one period. Thus, while the domain of a trigonometric function is infinite, its behavior can be deduced by considering a restricted domain.
- The **phase shift** is the horizontal distance by which the curve of a parent function is shifted by the addition of a constant or other expression in the argument of the function. If $f(x) = \sin(ax + b)$, the phase shift is found by setting the argument of the function equal to 0 and solving for x, resulting in a phase shift of $-\frac{b}{a}$.
- By adjusting the period and the phase shift of a trigonometric function, you customize the period and "starting point" to match your model's input requirements.
- By adjusting the amplitude and vertical displacement of a trigonometric function, you customize its range to match your model's output requirements.

- Recall that the **amplitude** of a sine or cosine function is the difference between the minimum and the maximum of the function. If $f(x) = c\sin(ax + b) + d$, the amplitude is c.
- Recall that the **vertical displacement** of a trigonometric function is the amount by which its graph is moved up or down. If $f(x) = c\sin(ax + b) + d$, the vertical displacement is d.
- The line $y = d$ is called the **midline**. The midline bisects the distance between the minimum and the maximum values of the function.
- The **range** of a periodic function is related to its amplitude and vertical displacement. If $f(x) = c\sin(ax + b) + d$, the range of the function will be $[d - |c|, d + |c|]$. This is because the midline bisects the distance between the maximums and minimums of the function, so the function rises $|c|$ above the midline and falls $|c|$ below the midline.

Periodicity of Trigonometric Functions

- Trigonometric function arguments that differ by an integer multiple of the period will have identical reference angles, and therefore produce identical results.
- The following diagram shows a graph of the sine function, $f(x) = \sin x$, which has a period of 2π. If a horizontal line is drawn anywhere between $y = 1$ and $y = -1$ (for example, the horizontal line $y = 0.5$ in the diagram), the line will intersect the sine curve at multiple locations.

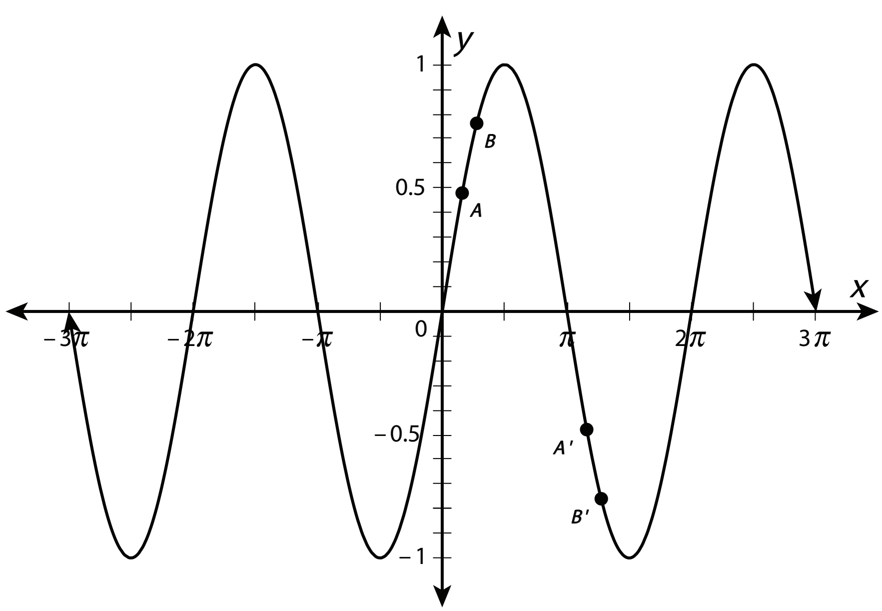


- If you choose a point where the horizontal line intersects the sine curve (for example, at the leftmost vertical gray line), you may add any integer multiple of the period to its x-coordinate and end up at another point of intersection. In the diagram, all of the vertical lines are 2π apart.

- This repetitive, or periodic, process works for any trigonometric function when you add an integer multiple of its period to the argument.
- The horizontal line in the graph of $f(x) = \sin x$ also intersects the curve at x-values that are not noted with vertical gray lines. All of the intersections of the sine curve and horizontal line have x-coordinates that share equivalent reference angles on the unit circle. The intersection points in the graph that are not marked by gray vertical lines represent reference angles that lie in Quadrant I. Those *with* vertical lines represent reference angles that lie in Quadrant II. The quadrant is determined by the angle measurement of the first positive occurrence of an intersection.
- Unit circle reference angles that are equivalent to those determined by the vertical lines in the diagram, but are located in quadrants III and IV, will have negative y-coordinates, so points corresponding to these angles all appear below the x-axis in the graph. For example, if another horizontal line was drawn at $y = -0.5$, the intersections with the sine curve would correspond with the points in quadrants III and IV.

Symmetries of Trigonometric Functions

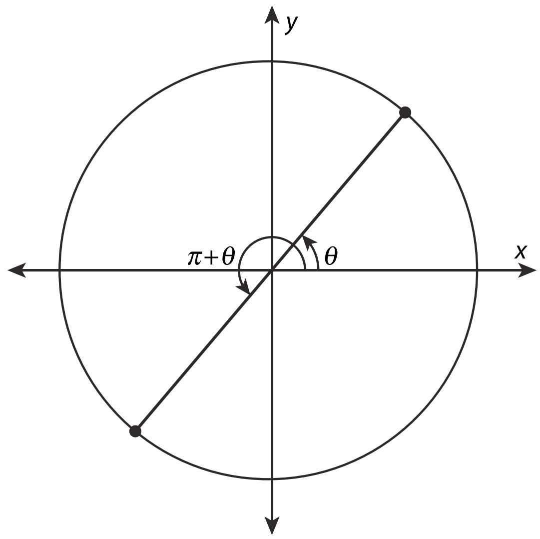
- Equivalent reference angles are most easily recognized by their symmetry on a unit circle, but they can also be recognized from symmetries on a graph of a trigonometric function. Consider the following graph of $f(x) = \sin x$, labeled this time with four points: A, A', B, and B'.



- The sines of angles with equivalent reference angles, but that lie in opposite quadrants (I and III, or II and IV) will have the same magnitude, but opposite signs. Such reference angles are symmetric about the origin on a unit circle.

- Two examples of pairs of points with reference angles that are symmetric about the origin on the unit circle are labeled in the previous graph using the same letter: A and A', and B and B'.
- The y-coordinates of each pair of points, for example A and A', on the graph have the same magnitudes, but different signs, as predicted from our understanding of the unit circle.
- Angles that are symmetric about the origin on the unit circle have reference angles that are congruent, as shown in the following diagram.

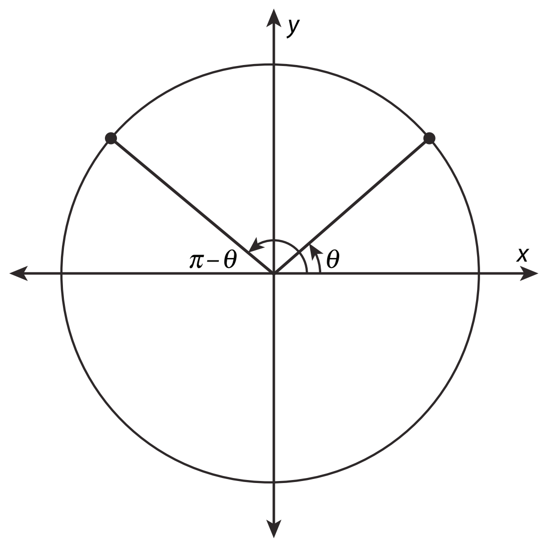
Angles symmetric about the origin on the unit circle



- This diagram can be used to illustrate the following:
  - The sines of angles that are symmetric about the origin will have the same magnitudes, but opposite signs. For example, $\sin(\pi + \theta) = -\sin\theta$.
  - The cosines of angles that are symmetric about the origin will have the same magnitudes, but opposite signs. For example, $\cos(\pi + \theta) = -\cos\theta$.
  - The tangents of angles that are symmetric about the origin will be equal. This can be derived from the previous two identities: $\tan(\pi+\theta)=\dfrac{\sin(\pi+\theta)}{\cos(\pi+\theta)}=\dfrac{-\sin\theta}{-\cos\theta}=\tan\theta$.

- A **point of symmetry** is a central point such that if a graph is rotated 180° about the point, the resulting graph will look exactly like the original graph. The sine function is an example of a function that exhibits point symmetry about the origin.
- Symmetry about the origin is a property solely of the argument. The angles θ and $\pi + \theta$ will always be symmetric about the origin.
- **Supplementary angles** are two angles that add up to 180° or π radians. Supplementary angles in standard position on a unit circle (as shown in the following diagram) exhibit vertical **line symmetry** about the *y*-axis, and have equivalent reference angles.

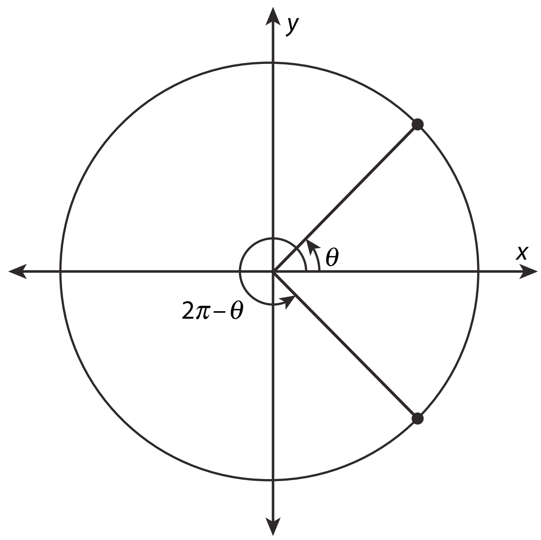
Supplementary angles on the unit circle



- This diagram can be used to verify the following:
  - The sines (*y*-coordinates) of supplementary angles on the unit circle are equal. For example, $\sin(\pi - \theta) = \sin\theta$.
  - The cosines (*x*-coordinates) of supplementary angles on the unit circle will have the same magnitudes, but opposite signs. For example, $\cos(\pi - \theta) = -\cos\theta$.
  - The tangents of supplementary angles on the unit circle will have the same magnitudes, but opposite signs. This can be derived from the previous two identities: $\tan(\pi-\theta)=\dfrac{\sin(\pi-\theta)}{\cos(\pi-\theta)}=\dfrac{\sin\theta}{-\cos\theta}=-\tan\theta$.
- The angles θ and $\pi - \theta$ will always be supplementary angles.

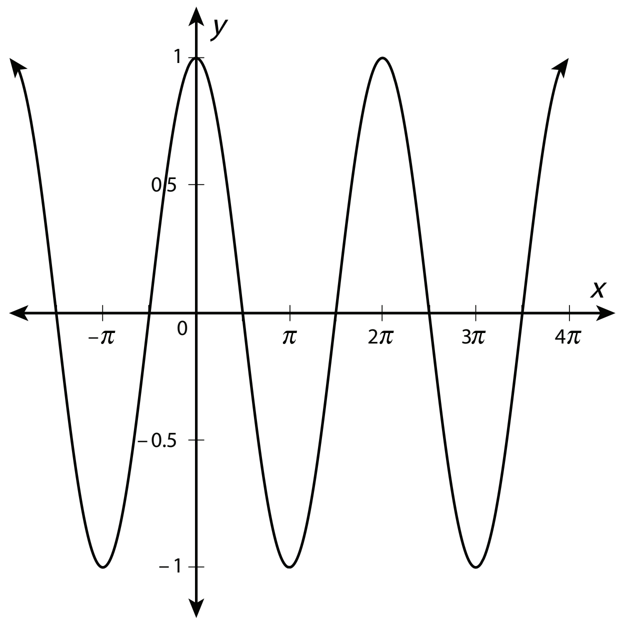
- Angles exhibiting horizontal line symmetry about the *x*-axis when plotted in standard position on a unit circle, as shown in the following diagram, have equivalent reference angles.

Angles with horizontal line symmetry



- This diagram can be used to verify the following:
  - The sines of horizontally symmetric angles on the unit circle will have the same magnitudes, but opposite signs. For example, $\sin(2\pi - \theta) = -\sin\theta$.
  - The cosines of horizontally symmetric angles on the unit circle will be equal. For example, $\cos(2\pi - \theta) = \cos\theta$.
  - The tangents of horizontally symmetric angles on the unit circle will have the same magnitudes, but opposite signs. This can be derived from the previous two identities: $\tan(2\pi - \theta) = \frac{\sin(2\pi - \theta)}{\cos(2\pi - \theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$.
- The angles θ and $2\pi - \theta$ will always have horizontal line symmetry.

- Functions that exhibit vertical line symmetry about the y-axis, such as the graph of the cosine function $f(x) = \cos x$ that follows, are called **even functions**.



- To determine if a function is even, test whether $f(-x) = f(x)$. If this equality is true, the function is even.
- Functions whose graphs exhibit symmetry about the origin, such as the sine functions shown previously, are known as **odd functions**.
- To determine if a function is odd, test whether $f(-x) = -f(x)$. If this equality is true, the function is odd.
- Some functions are even, some are odd, and many are neither even nor odd: they do not exhibit symmetry about either the y-axis or the origin.

Common Errors/Misconceptions

- using the incorrect sign due to not considering the quadrant in which a reference angle lies
- incorrectly recalling a rule or fact that could be double-checked using a sketch of a unit circle
- confusing x- and y-coordinates, or cosine and sine values, with one another
- not checking that the calculator is in the correct mode (degrees vs. radians) for a given problem
- neglecting to consider one or more complete periods of the function, and focusing only on where a condition is first satisfied
- incorrectly recalling memorized identities, which can be avoided by referring to a unit circle