Key Features of Trigonometric Functions

Warm-Up

If you stick a pinwheel into the ground and then step back to watch it, the direction the wind is blowing will determine whether the pinwheel spins clockwise or counterclockwise from your perspective. Nathan wants to test this using a pinwheel that has a radius of 10 cm. He painted a white dot at the very tip of one of the pinwheel blades, and positioned the blade with this dot as far to the right as possible when he stuck the pinwheel into the ground. If the pinwheel moves counterclockwise, the dot will move upward first. If the pinwheel moves clockwise, the dot will move downward first. Use this information to solve the problems that follow.

1. Write a function $f(t)$ that describes the horizontal position of the dot relative to the center of the pinwheel as a function of time t (in seconds) if the wheel starts spinning counterclockwise at 1 revolution per second.

2. Write a function $g(t)$ that describes the horizontal position of the dot relative to the center of the pinwheel as a function of time t (in seconds) if the wheel starts spinning clockwise at 1 revolution per second.

(*continued*)

3. Create a table of values with columns for t, $f(t)$, and $g(t)$. Use the values $\frac{1}{12}$, $\frac{1}{8}$, $\frac{1}{6}$, and $\frac{1}{4}$ for t. Fill in the values for $f(t)$ and $g(t)$ using your functions from problems 1 and 2. Describe the pattern you see in your table of values. Why does it occur?

1. **Write a function *f*(*t*) that describes the horizontal position of the dot relative to the center of the pinwheel as a function of time *t* (in seconds) if the wheel starts spinning counterclockwise at 1 revolution per second.**
   - Think of the pinwheel as a circle on a coordinate plane, with the circle's center at the origin. The white dot is positioned "as far to the right as possible," so the dot can be thought of as resting on the *x*-axis. Thus, the radius of 10 cm (from the center to the white dot) represents $x = 10$.
   - The dot begins at position (10, 0) at time $t = 0$, and moves up from there.

- The cosine function provides us with the horizontal position, and its amplitude is 10, which is given by the radius of the pinwheel.
- To obtain an initial value of 10, the cosine function's initial input can be 0 when $t = 0$, so a potential function to model this could be $f(t) = 10 \cos t$.
- Next, adjust the period so that it models 1 revolution (2π radians) per second. Multiply t by 2π. Our final model is $f(t) = 10 \cos (2\pi t)$.

2. Write a function *g*(*t*) that describes the horizontal position of the dot relative to the center of the pinwheel as a function of time *t* (in seconds) if the wheel starts spinning clockwise at 1 revolution per second.

- The answer to problem 1 provides most of the analysis needed for this answer. The only change required is to the direction of travel, so the t term must be negative instead of positive, causing the angle to rotate clockwise: $g(t) = 10\cos(-2\pi t)$.

3. Create a table of values with columns for *t*, *f*(*t*), and *g*(*t*). Use the values $\frac{1}{12}$, $\frac{1}{8}$, $\frac{1}{6}$, and $\frac{1}{4}$ for *t*. Fill in the values for *f*(*t*) and *g*(*t*) using your functions from problems 1 and 2. Describe the pattern you see in your table of values. Why does it occur?

- Substitute each of the given *t*-values into the functions for *f*(*t*) and *g*(*t*) and solve.

- The completed table is shown:

t	$f(t)$	$g(t)$
$\frac{1}{12}$	$\frac{10\sqrt{3}}{2}$	$\frac{10\sqrt{3}}{2}$
$\frac{1}{8}$	$5\sqrt{2}$	$5\sqrt{2}$
$\frac{1}{6}$	5	5
$\frac{1}{4}$	0	0

- The values of $f(t)$ and $g(t)$ are equal for all values of t. This is caused by the symmetry of the pinwheel. It will not matter whether the pinwheel is rotating clockwise or counterclockwise, since for each value of t the horizontal position of the white dot relative to the center of the pinwheel will be the same.

Instruction

Introduction

- Trigonometric functions are used to model many repetitive phenomena in our universe.
- Examples include the location of the moon relative to Earth, the position of the hands on a clock, and the height of a weight bouncing at the end of a spring.
- The input values for such models are not always angle measures. They are often a function of time or distance, neither of which is measured in degrees or radians.

Introduction, *continued*

- Yet, by adjusting for period, amplitude, phase shift, and vertical displacement, trigonometric functions model the desired situations well.
- In addition, we must be able to evaluate trigonometric functions of any real number, positive or negative, and not just of the **special angles** whose trigonometric values can be computed exactly.
- This work becomes easier when considering the symmetries that exist in the unit circle and in the graphs of the trigonometric functions.

Key Concepts

- Trigonometric functions of an angle in the unit circle are based on the coordinates of the point where the angle’s terminal side intersects the unit circle, no matter how many times the angle measure may have caused the terminal side to rotate around the unit circle. The reference angle and quadrant of the angle’s terminal side help to evaluate the result.
- Recall that a **reference angle** is the acute angle between the *x*-axis and the terminal side of an angle in standard position.

Key Concepts, *continued*

- Angles that have the same terminal side when in standard position are called **coterminal angles**. For example, $\frac{p}{4}$ is coterminal with $\frac{9p}{4}$.
- Remember that positive angle measures cause the terminal side of the angle to rotate counterclockwise around the unit circle, and negative angles cause the terminal side to rotate clockwise.

Key Concepts, *continued*

- Also recall that the cosine of an angle is the *x*-coordinate of the point where the angle's terminal side intersects the unit circle, and the sine of an angle is the *y*-coordinate of this point.
- As the terminal side of an angle completes a full circle and begins to pass through points it has previously passed through, trigonometric functions of that angle will produce the same results they produced the previous time around. A function that exhibits such repetitive, periodic behavior is called a **periodic function**. A periodic function will repeat itself identically over and over.

Key Concepts, *continued*

- The **argument** of a trigonometric function is the value at which the trigonometric function is being evaluated. It is the independent variable in a function.
- A **cycle** is the smallest representation of a cosine or sine function graph as defined over a restricted domain; it is equal to one repetition of the period of a function.
- The **period** of a trigonometric function is the horizontal distance from the beginning to the end of a cycle on the graph. It is the interval required in a function's argument to get from the start of one repetition of output values to the start of the next repetition.

Key Concepts, *continued*

- The graphs of all trigonometric functions are periodic and, therefore, a particular result will repeat every period. For some trigonometric functions, most values will occur twice per period.
- Trigonometric functions having arguments greater than the size of the period can be simplified by subtracting an integer multiple of the period from the argument to reduce it to less than one period.

Key Concepts, *continued*

- The **phase shift** is the horizontal distance by which the curve of a parent function is shifted by the addition of a constant or other expression in the argument of the function. If $f(x) = \sin(ax + b)$, the phase shift is found by setting the argument of the function equal to 0 and solving for x, resulting in a phase shift of $-\frac{b}{a}$.

Key Concepts, *continued*

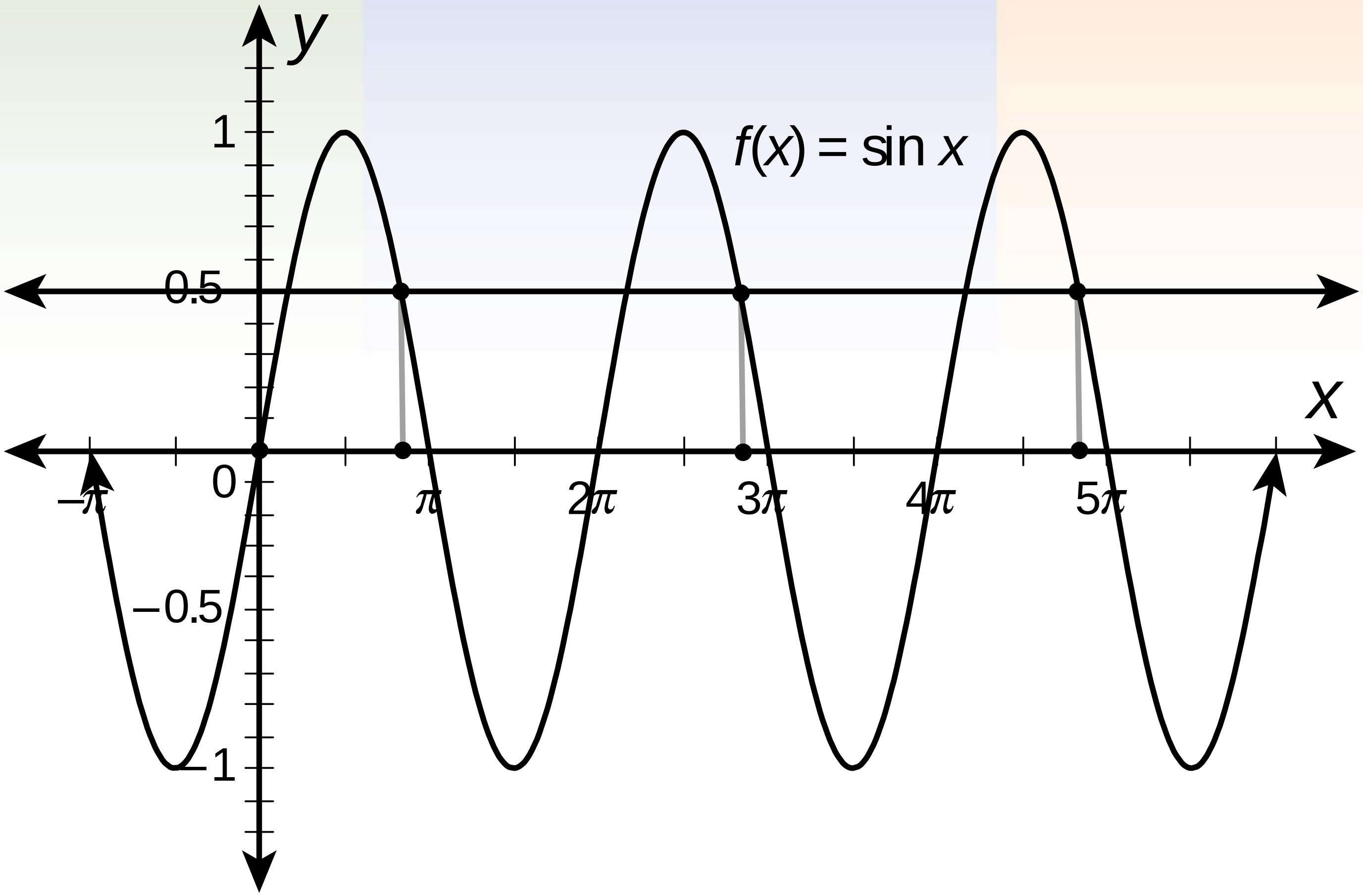
- By adjusting the period and the phase shift of a trigonometric function, you customize the period and "starting point" to match your model's input requirements.
- By adjusting the amplitude and vertical displacement of a trigonometric function, you customize its range to match your model's output requirements.

Key Concepts, *continued*

Periodicity of Trigonometric Functions

- Trigonometric function arguments that differ by an integer multiple of the period will have identical reference angles, and therefore produce identical results.
- The following diagram shows a graph of the sine function, $f(x) = \sin x$, which has a period of 2π. If a horizontal line is drawn anywhere between $y = 1$ and $y = -1$ (for example, the horizontal line $y = 0.5$ in the diagram), the line will intersect the sine curve at multiple locations.

Key Concepts, *continued*



Key Concepts, *continued*

- If you choose a point where the horizontal line intersects the sine curve (for example, at the leftmost vertical gray line), you may add any integer multiple of the period to its *x*-coordinate and end up at another point of intersection. In the diagram, all of the vertical lines are 2π apart.
- This repetitive, or periodic, process works for any trigonometric function when you add an integer multiple of its period to the argument.

Key Concepts, *continued*

- The horizontal line in the graph of $f(x) = \sin x$ also intersects the curve at x-values that are not noted with vertical gray lines. All of the intersections of the sine curve and horizontal line have x-coordinates that share equivalent reference angles on the unit circle. The intersection points in the graph that are not marked by gray vertical lines represent reference angles that lie in Quadrant I. Those *with* vertical lines represent reference angles that lie in Quadrant II. The quadrant is determined by the angle measurement of the first positive occurrence of an intersection.

Key Concepts, *continued*

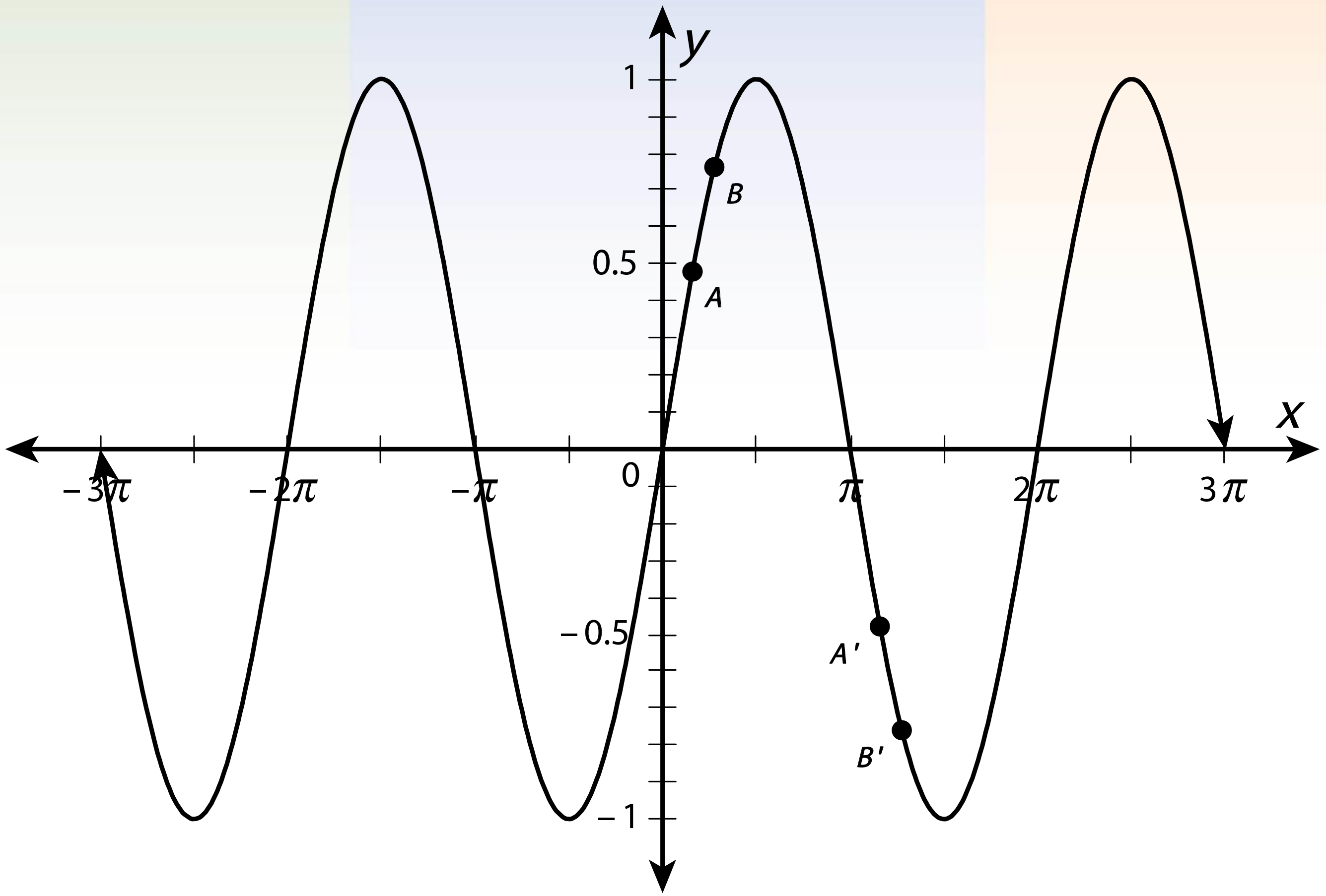
- Unit circle reference angles that are equivalent to those determined by the vertical lines in the diagram, but are located in quadrants III and IV, will have negative *y*-coordinates, so points corresponding to these angles all appear below the *x*-axis in the graph. For example, if another horizontal line was drawn at $y = –0.5$, the intersections with the sine curve would correspond with the points in quadrants III and IV.

Key Concepts, *continued*

Symmetries of Trigonometric Functions

- Equivalent reference angles are most easily recognized by their symmetry on a unit circle, but they can also be recognized from symmetries on a graph of a trigonometric function. Consider the following graph of $f(x) = \sin x$, labeled this time with four points: A, A', B, and B'.

Key Concepts, *continued*



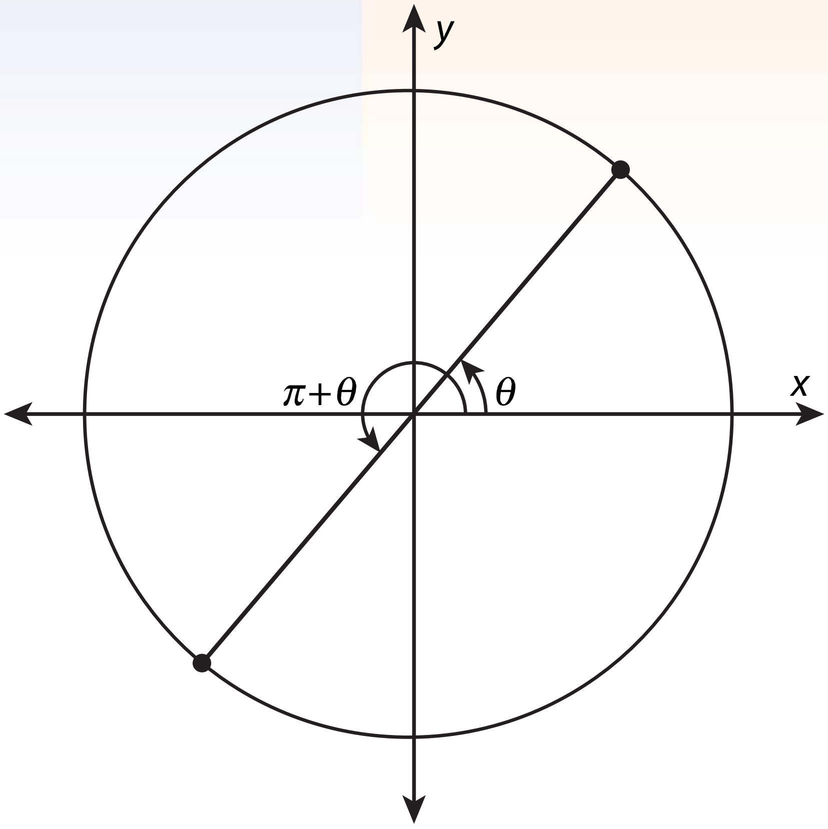
Key Concepts, *continued*

- The sines of angles with equivalent reference angles, but that lie in opposite quadrants (I and III, or II and IV) will have the same magnitude, but opposite signs. Such reference angles are symmetric about the origin on a unit circle.
- Two examples of pairs of points with reference angles that are symmetric about the origin on the unit circle are labeled in the previous graph using the same letter: A and A', and B and B'.
- The y-coordinates of each pair of points, for example A and A', on the graph have the same magnitudes, but different signs, as predicted from our understanding of the unit circle.

Key Concepts, *continued*

- Angles that are symmetric about the origin on the unit circle have reference angles that are congruent, as shown in the diagram.

Angles symmetric about the origin on the unit circle



Key Concepts, *continued*

- This diagram can be used to illustrate the following:
  - The sines of angles that are symmetric about the origin will have the same magnitudes, but opposite signs. For example, sin ($\pi + \theta$) = –sin θ.
  - The cosines of angles that are symmetric about the origin will have the same magnitudes, but opposite signs. For example, cos ($\pi + \theta$) = –cos θ.
  - The tangents of angles that are symmetric about the origin will be equal. This can be derived from the previous two identities:

$$\tan(p+q)=\frac{\sin(p+q)}{\cos(p+q)}=\frac{-\sin q}{-\cos q}=\tan q.$$

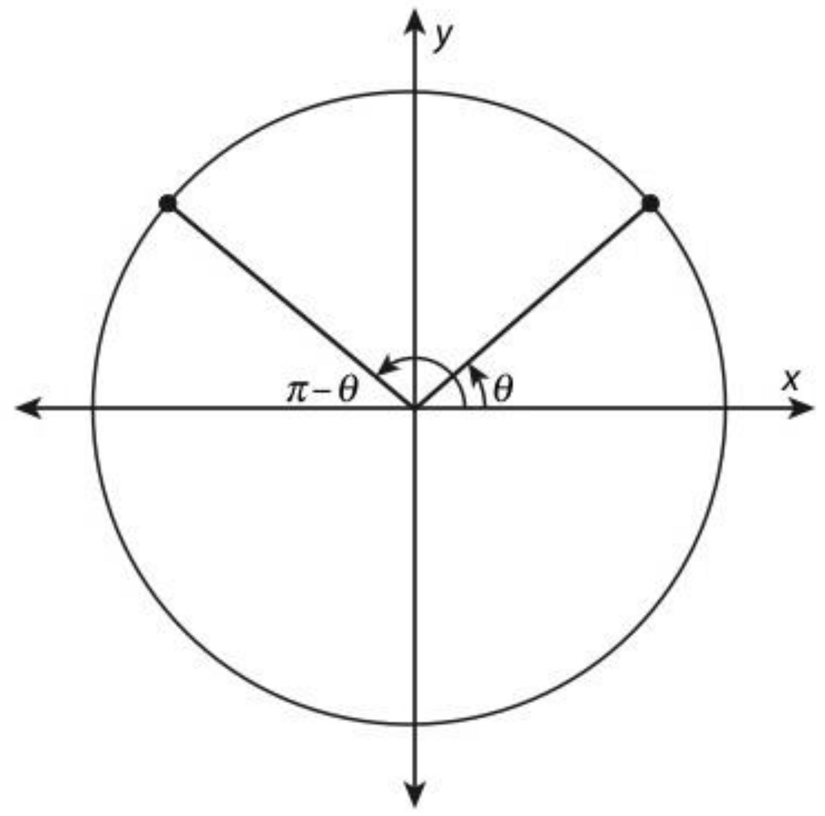
Key Concepts, *continued*

- A **point of symmetry** is a central point such that if a graph is rotated 180° about the point, the resulting graph will look exactly like the original graph. The sine function is an example of a function that exhibits point symmetry about the origin.
- Symmetry about the origin is a property solely of the argument. The angles θ and $\pi + \theta$ will always be symmetric about the origin.

Key Concepts, *continued*

- **Supplementary angles** are two angles that add up to 180° or π radians. Supplementary angles in standard position on a unit circle (as shown in the diagram) exhibit vertical **line symmetry** about the *y*-axis, and have equivalent reference angles.

Supplementary angles on the unit circle



Key Concepts, *continued*

- This diagram can be used to verify the following:
  - The sines (y-coordinates) of supplementary angles on the unit circle are equal. For example, $\sin(\pi - \theta) = \sin\theta$.
  - The cosines (x-coordinates) of supplementary angles on the unit circle will have the same magnitudes, but opposite signs. For example, $\cos(\pi - \theta) = -\cos\theta$.

Key Concepts, *continued*

- The tangents of supplementary angles on the unit circle will have the same magnitudes, but opposite signs. This can be derived from the previous two identities:

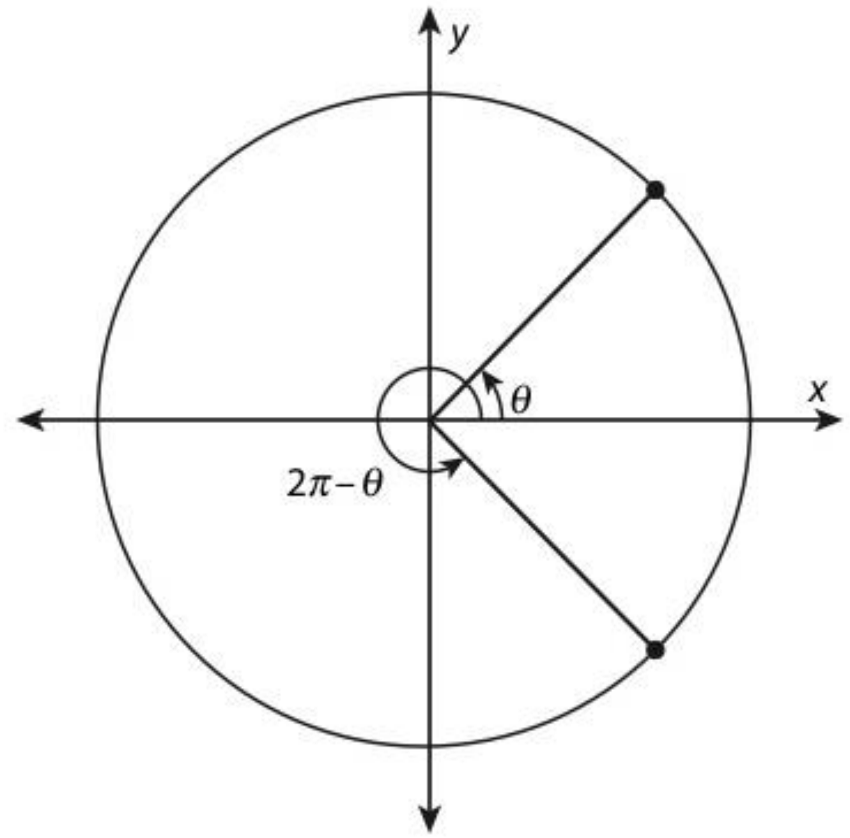
$$\tan(p - q) = \frac{\sin(p - q)}{\cos(p - q)} = \frac{\sin q}{-\cos q} = -\tan q.$$

- The angles θ and $\pi - \theta$ will always be supplementary angles.

Key Concepts, *continued*

- Angles exhibiting horizontal **line symmetry** about the *x*-axis when plotted in standard position on a unit circle, as shown in the diagram, have equivalent reference angles.

Angles with horizontal line symmetry



Key Concepts, *continued*

- This diagram can be used to verify the following:
  - The sines of horizontally symmetric angles on the unit circle will have the same magnitudes, but opposite signs. For example, $\sin (2\pi - \theta) = -\sin \theta$.
  - The cosines of horizontally symmetric angles on the unit circle will be equal. For example, $\cos (2\pi - \theta) = \cos \theta$.

Key Concepts, *continued*

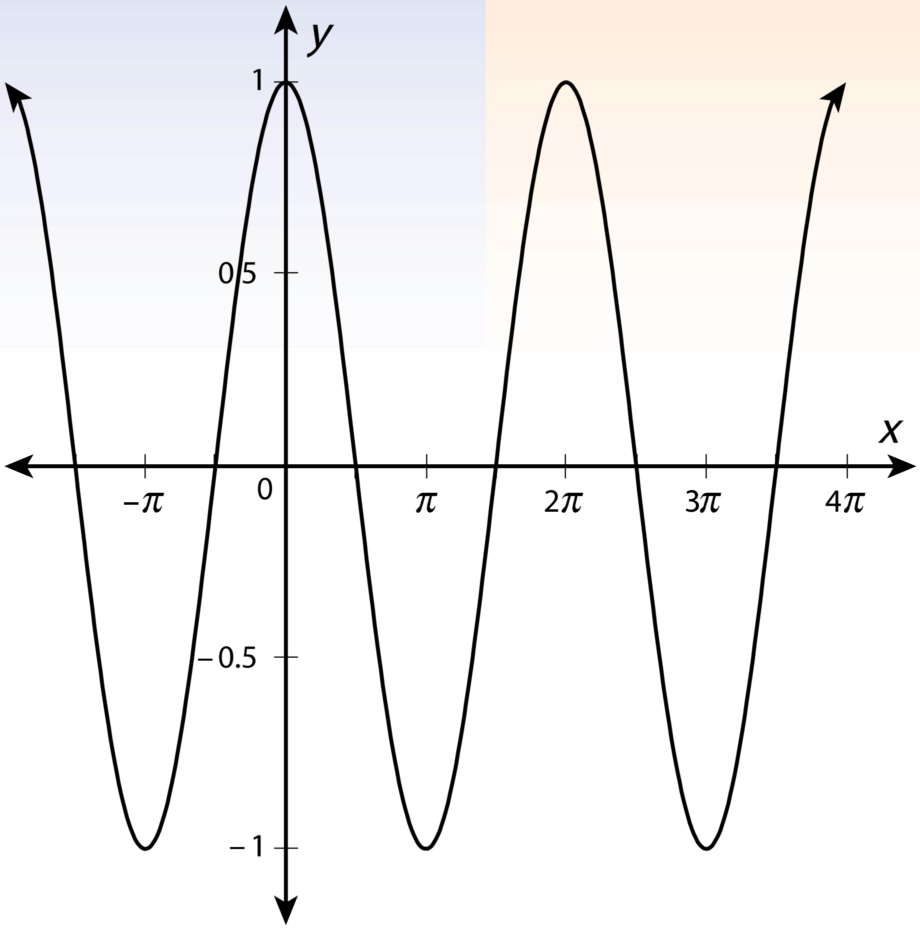
- The tangents of horizontally symmetric angles on the unit circle will have the same magnitudes, but opposite signs. This can be derived from the previous two identities:

$$\tan(2p - q) = \frac{\sin(2p - q)}{\cos(2p - q)} = \frac{-\sin q}{\cos q} = -\tan q.$$

- The angles θ and $2\pi - \theta$ will always have horizontal line symmetry.

Key Concepts, *continued*

- Functions that exhibit vertical line symmetry about the *y*-axis, such as the graph of the cosine function $f(x) = \cos x$ at right, are called **even functions**.



Key Concepts, *continued*

- To determine if a function is even, test whether $f(-x) = f(x)$. If this equality is true, the function is even.
- Functions whose graphs exhibit symmetry about the origin, such as the sine functions shown previously, are known as **odd functions**.
- To determine if a function is odd, test whether $f(-x) = -f(x)$. If this equality is true, the function is odd.
- Some functions are even, some are odd, and many are neither even nor odd: they do not exhibit symmetry about either the *y*-axis or the origin.

Common Errors/Misconceptions

- using the incorrect sign due to not considering the quadrant in which a reference angle lies
- incorrectly recalling a rule or fact that could be double-checked using a sketch of a unit circle
- confusing *x*- and *y*-coordinates, or cosine and sine values, with one another
- not checking that the calculator is in the correct mode (degrees vs. radians) for a given problem
- neglecting to consider one or more complete periods of the function, and focusing only on where a condition is first satisfied
- incorrectly recalling memorized identities, which can be avoided by referring to a unit circle

Guided Practice

Example 1

Without using a calculator, evaluate $\sin\left(\frac{17p}{3}\right)$.

Guided Practice: Example 1, *continued*

1. **Determine the quadrant that contains the terminal side of the angle.**

   Subtract 2π (which is one complete revolution) from the given angle until the angle is less than 2π. Because the given angle, $\frac{17p}{3}$, has a denominator of 3, write 2π in the form $\frac{6p}{3}$ so that the fractions can be subtracted conveniently.

Guided Practice: Example 1, *continued*

Subtract one complete revolution from the given angle.

$$\frac{17p}{3} - \frac{6p}{3} = \frac{11p}{3}$$

Because $\frac{11p}{3}$ is still greater than 2π, subtract another complete revolution.

$$\frac{11p}{3} - \frac{6p}{3} = \frac{5p}{3}$$

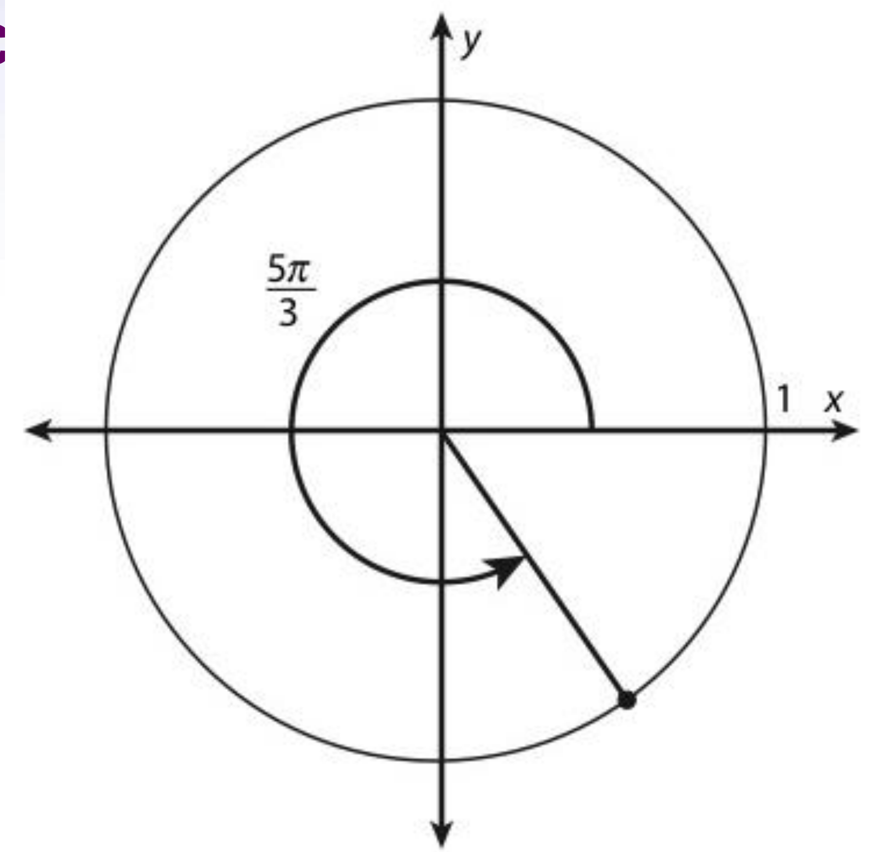
Guided Practice: Example 1, *continued*

The result, $\frac{5p}{3}$ radians, is coterminal with the given angle; that is, it is in the exact same position. Therefore, all trigonometric values of $\frac{17p}{3}$ and $\frac{5p}{3}$ will be the same.

Guided Practice: Example 1, *continued*

2. **Sketch the location of the coterminal angle on a unit circ**

The unit circle shows us that $\frac{5p}{3}$ radians is in Quadrant IV. Mark this position on a unit circle.



Guided Practice: Example 1, *continued*

3. Using your sketch from step 2, determine the reference angle that applies to this situation.

The smallest angle between the terminal side and the *x*-axis for this angle is found by continuing around the unit circle counterclockwise until you return to the point (1, 0). The measure of this reference angle can be found by subtracting.

$$2p - \frac{5p}{3} = \frac{6p}{3} - \frac{5p}{3} = \frac{p}{3}$$

The reference angle is $\frac{p}{3}$ radians.

Guided Practice: Example 1, *continued*

4. Evaluate the sine of the reference angle, using your knowledge of special angles.

The angle $\frac{p}{3}$ is a special angle; that is, it is a known value on the unit circle, so we can evaluate its sine value without a calculator.

Find the sine of $\frac{p}{3}$ radians.

$$\sin\left(\frac{p}{3}\right) = \frac{\sqrt{3}}{2}$$

Guided Practice: Example 1, *continued*

The sine of $\frac{p}{3}$, the reference angle, is $\frac{\sqrt{3}}{2}$. This means that $\frac{\sqrt{3}}{2}$ is the sine of both $\frac{5p}{3}$ and the given angle $\frac{17p}{3}$, which is in the same position. However, the sign of $\frac{\sqrt{3}}{2}$ may be positive or negative depending on the quadrant.

Guided Practice: Example 1, *continued*

5. Use the quadrant information from step 2 to modify the sign of your result as necessary.

In Quadrant IV, all *y*-coordinates (sine values) are negative. Therefore, we must modify the sign of our result from the previous step—it should be negative instead of positive.

$$\sin\left(\frac{5p}{3}\right) = -\frac{\sqrt{3}}{2}$$

Guided Practice: Example 1, *continued*

The sine of $\frac{5p}{3}$ is $-\frac{\sqrt{3}}{2}$. Therefore, the sine of $\frac{17p}{3}$ is also $-\frac{\sqrt{3}}{2}$, since these two angles are coterminal.

It is a good habit to visualize the quadrant of an angle, and then double-check the sign of the result based on the quadrant in which it lies. This applies to results obtained from your knowledge of special angles, as well as to results provided by calculators.

Guided Practice: Example 1, *continued*

Guided Practice

Example 5

A weight is attached to a rod by a spring. The weight is pulled down, and then released. From the moment of release, the weight's height above the floor (in feet) can be modeled approximately by $h(t) = -0.75 \cdot \cos(2\pi t) + 5$ from time $t = 0$ until $t = 4$ seconds. Use a calculator to determine, to the nearest tenth of a second, all the times during this time interval when the weight will be at a height of 5.6 feet, rounded to the nearest tenth.

Guided Practice: Example 5, *continued*

1. Analyze the function to determine how many answers there are.

From the function $h(t) = -0.75 \cdot \cos(2\pi t) + 5$, we know the following information:

Amplitude: 0.75

Period: 1 second

Phase shift: 0

Vertical shift: 5

Guided Practice: Example 5, *continued*

Four seconds will elapse between $t = 0$ and $t = 4$, which corresponds to 4 periods of the function. Given that the function's maximum will be 5 + 0.75 = 5.75, the weight should pass through the value of 5.6 twice during each period (once on the way up, and a second time on the way back down). Therefore, there will be a total of 2 • 4 = 8 times that the weight is at that height during the specified time interval.

Guided Practice: Example 5, *continued*

2. Use your calculator to determine the times that will produce a height of 5.6 feet.

Follow the steps appropriate to your calculator model.

On a TI-83/84:

Step 1: Verify that your calculator is in radian mode.

Step 2: Press [Y =], then enter the given function, $-0.75 \cdot \cos(2\pi x) + 5$, at Y1.

Step 3: Press [2ND][WINDOW]. Set TblStart to 0 and ΔTbl to 0.1 (the precision specified in the problem).

(*continued*)

Guided Practice: Example 5, *continued*

Step 4: Press [2ND][GRAPH] to view the table of the function values.

Step 5: Use the up and down arrow buttons to scroll through the table's "X" values from 0 to 4 to find values of interest in the Y1 column. For this problem, the value of interest will be 5.6068, which is the only value that rounds to 5.6. Record the times (the "X" values) at which the height of the weight will be approximately 5.6 feet (the height above the floor).

Guided Practice: Example 5, *continued*

On a TI-Nspire:

Step 1: Verify that your calculator is in radian mode, then click on the graph icon at the bottom of the home screen.

Step 2: Enter the given function, $-0.75 \cdot \cos(2\pi x) + 5$, at "$f(x)=$". Press [enter] to view the graph.

Step 3: Press [menu], then choose option 7: Table, then 1: Splitscreen, and then press [enter].

(*continued*)

Guided Practice: Example 5, *continued*

Step 4: To set up the table, press [menu] again and choose 2: Table, then 5: Edit Table Settings. Set Table Start to 0 and set Table Step to 0.1 (the precision specified in the problem). Make sure Independent and Dependent are set to “Auto.” Press [enter].

Step 5: Use the up and down arrow buttons to scroll through the table’s “X” values from 0 to 4 to find values of interest in the Y1 column. For this problem, the value of interest will be 5.60676, which is the only value that rounds to 5.6. Record the times (the “X” values) at which the height of the weight will be approximately 5.6 feet (the height above the floor).

Guided Practice: Example 5, *continued*

3. State the answer in the context of the original problem.

According to the results of either calculator, the weight will pass through a height of 5.6 feet (rounded to the nearest tenth) at times of 0.4, 0.6, 1.4, 1.6, 2.4, 2.6, 3.4, and 3.6 seconds, rounded to the nearest tenth of a second.

Guided Practice: Example 5, *continued*

Note that if the domain were not restricted to values between 0 and 4, the answer could be expressed as $0.4 + n$ and $0.6 + n$, where n is an integer greater than or equal to 0. Adding n, in this example, corresponds to adding as many full periods of the function as we wish (in this case, a full period equals 1 second). Note that you will often see the answers to trigonometry problems given as "a value plus n times the period," which gives you a formula to find any of the infinite number of possible answers.

Guided Practice: Example 5, *continued*