Creating Rational Equations

Warm-Up

Dario, Elliott, and Ivan are in a basketball game. Dario has made 2 baskets in 8 attempts, Elliott has made 1 basket in 5 attempts, and Ivan has made 5 baskets in 14 attempts.

1. Dario's shooting percentage is now 25% because $\frac{2}{8}=\frac{1}{4}=25\%$. If he makes 4 baskets in his next 4 consecutive attempts, what will be his shooting percentage? Show your work.
2. If Elliott and Ivan also each make 4 baskets in their next 4 consecutive attempts, which player out of all three players will have the highest shooting percentage? Show your work.

1. Dario's shooting percentage is now 25% because $\frac{2}{8}=\frac{1}{4}=25\%$. If he makes 4 baskets in his next 4 consecutive attempts, what will be his shooting percentage? Show your work.

$$\frac{2}{8}$$

Original number of baskets made compared to baskets attempted

$$\frac{2+4}{8+4}$$

Additional 4 baskets attempted and made

$$\frac{6}{12} = 50\%$$ Simplify.

- Dario's shooting percentage will be 50%.

2. If Elliott and Ivan also each make 4 baskets in their next 4 consecutive attempts, which player out of all three players will have the highest shooting percentage? Show your work.

Dario's shooting percentage: $\frac{2+4}{8+4} = \frac{6}{12} = 50\%$

Elliott's shooting percentage: $\frac{1+4}{5+4} = \frac{5}{9} \approx 55.6\%$

Ivan's shooting percentage: $\frac{5+4}{14+4} = \frac{9}{18} = 50\%$

- Elliott will have the highest shooting percentage, at approximately 55.6%.

Instruction

Introduction

- There are many types of equations used in solving mathematics.
- The type of equation you work with depends largely on the problem situation, the information given, and what you are trying to find out.
- In this lesson, you will work with rational equations, equations that have a variable in a denominator.

Introduction, *continued*

- Creating rational equations to represent problem situations often involves ratios such as $\frac{20 \text{ customers}}{7 \text{ employees}}$, and rates such as $\frac{75 \text{ miles}}{2.5 \text{ gallons}}$.
- Therefore, solving rational equations frequently uses previously learned skills involving fractions.

Key Concepts

- A ratio is a relation between two quantities. A ratio can be expressed in words, fractions, decimals, or as a percentage.
- Examples of ratios include $\frac{2}{5}$, $\frac{15 \text{ successes}}{20 \text{ attempts}}$, and $\frac{66 \text{ students}}{3 \text{ teachers}}$.
- A rate is a ratio that compares measurements with different kinds of units.
- Examples of rates include $\frac{3 \text{ feet}}{2 \text{ inches}}$ and $\frac{75 \text{ miles}}{2.5 \text{ gallons}}$.

Key Concepts, *continued*

- The ratio $\frac{\text{successes}}{\text{attempts}}$ appears in various applications, sometimes using different words, but representing the same concept.
- For example, a shooting percentage in basketball can be written as $\frac{\text{7 successes}}{\text{10 attempts}}$.
- If you successfully make 7 shots out of 10 attempts, your shooting percentage is: $\frac{7}{10} = 0.7 = 70\%$.

Key Concepts, *continued*

- Another example is batting average in baseball or softball.
- If you get 3 hits in 8 at-bats, your batting average is $\frac{3\text{ successes}}{8\text{ attempts}} = \frac{3}{8} = 0.375.$
- Batting averages are typically left as a decimal rather than converted to a percent.

Key Concepts, *continued*

- A rational expression is an expression made of the ratio of two polynomials, in which a variable appears in the denominator of a polynomial.
- Examples of rational expressions include $\frac{2x}{3y^2}$, $\frac{12}{x}$, and $\frac{x^2+2x-15}{x-3}$.
- A rational expression is in simplest form when it is expressed as a quotient of polynomials whose greatest common factor is 1.

Key Concepts, *continued*

- To simplify a rational expression, factor either the numerator or the denominator or both so that common factors in the numerator and denominator can be identified, and then cancelled out.
- A rational equation is an equation that includes the ratio of two rational expressions, in which a variable appears in the denominator of at least one rational expression.
- Examples of rational equations include $\frac{5+x}{20+x}=0.90$, $\frac{12}{20}+\frac{15}{x}=1$, and $\frac{-4}{x^2-2x-3}+2=\frac{x-4}{x-3}$.

Key Concepts, *continued*

- The least common multiple (LCM) of two or more polynomials is the common multiple that has the least degree and the least positive constant factor.
- The least common denominator (LCD) of two or more fractions is the least common multiple of their denominators.
- Given the fractions $\frac{18}{20}$ and $\frac{5}{x}$, the least common multiple of 20 and x is $20x$, so the least common denominator is $20x$.

Key Concepts, *continued*

- Given the fractions $\frac{20}{x}$ and $\frac{4}{x+8}$, the least common multiple of x and $x + 8$ is $x(x + 8)$, so the least common denominator is $x(x + 8)$.

Key Concepts, *continued*

- A rational function is a function that can be written in the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.
- Examples of rational functions include $f(x) = \frac{5+x}{20+x}$, $f(x) = \frac{-4}{x^2 - 2x - 3}$, and $f(x) = \frac{12}{20} + \frac{15}{x}$.

Key Concepts, *continued*

- $f(x)=\frac{12}{20}+\frac{15}{x}$ can be written in the form $f(x)=\frac{p(x)}{q(x)}$:

$$f(x)=\frac{12}{20}+\frac{15}{x}=\frac{x}{x}\bullet\frac{12}{20}+\frac{20}{20}\bullet\frac{15}{x}=\frac{12x+300}{20x}$$

- Sometimes an equation representing a problem situation is naturally related to a rational function.

Key Concepts, *continued*

- For example, the function $f(x)=\frac{5+x}{20+x}$ might represent a shooting percentage in basketball.
- If you want to find the value of x that makes the shooting percentage 50%, then you could solve the equation $0.50=\frac{5+x}{20+x}$.

Key Concepts, *continued*

- Solutions to quadratic functions of the form $ax^2 + bx + c = 0$ can be determined using the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Key Concepts, *continued*

- An extraneous solution, or extraneous root**,** of an equation is a solution of an equation that arises during the solving process, but which is not a solution of the original equation.
- Extraneous solutions sometimes occur when solving rational equations, so it is important to check all apparent solutions by substituting them into the original equation and determining if they make it true.

Key Concepts, *continued*

- When solving rational equations, restrictions on the variable must be considered.
- Possible solutions that result in 0 as the denominator must be excluded from the solutions of the rational equation since denominators cannot equal 0.

Common Errors/Misconceptions

- mistakenly thinking the time to complete a task by two persons or objects working at different rates is the average of the individual times or half the average of the individual times
- neglecting to check apparent solutions to determine if they make the equation true
- using an equation other than the original equation when checking an apparent solution

Guided Practice

Example 1

Simplify the rational expression $\frac{x^2+2x-15}{x-3}$.

Guided Practice: Example 1, *continued*

1. **If possible, determine the factors of the numerator.**

- The numerator $x^2 + 2x - 15$ can be written as the product of the two factors $(x - 3)$ and $(x + 5)$.
- Rewrite the fraction with the numerator as a product of the factors.

$$\frac{x^2+2x-15}{x-3}=\frac{(x-3)(x+5)}{x-3}$$

Guided Practice: Example 1, *continued*

2. Divide by the common factor.

- The common factor of the fraction is $(x - 3)$.

$$\frac{x^2+2x-15}{x-3}=\frac{\cancel{(x-3)}(x+5)}{\cancel{x-3}}$$

Guided Practice: Example 1, *continued*

3. Simplify the fraction.

$$\frac{x^2 + 2x - 15}{x - 3} = \frac{x + 5}{1} = x + 5$$

Guided Practice: Example 1, *continued*

4. Determine if there are any restrictions on the variable.

- The denominator of the original expression is $x - 3$; therefore, x cannot be equal to 3 since this would mean the denominator would result in 0.
- The rational expression $\frac{x^2+2x-15}{x-3}$ simplifies to $x + 5$ where $x \neq 3$.

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Guided Practice

Example 2

Josie has made 12 free throws out of 26 attempts in her basketball games this season. If she can raise her free-throw percentage to 60%, she will tie her record from last season.

How many consecutive attempts will she have to make in order to reach a free-throw percentage of 60%? Write a function to complete the problem.

Guided Practice: Example 2, *continued*

1. Assign a variable to represent what you need to find.

- Let x represent the number of consecutive attempts Josie must make.

Guided Practice: Example 2, *continued*

2. Create a function to represent the situation.

- A free-throw percentage is the ratio $\frac{\text{successes}}{\text{attempts}}$, written in percent form, for shots attempted from the free-throw line.

- Josie has made 12 free throws out of 26 attempts so far.

Guided Practice: Example 2, *continued*

- If she is successful in x consecutive attempts, then her free-throw percentage will be represented by the following ratio:

$$\frac{12 \text{ successes} + x \text{ successes}}{26 \text{ attempts} + x \text{ attempts}} = \frac{12+x}{26+x}$$

- The function $f(x)=\frac{12+x}{26+x}$ represents the situation, where x is the number of consecutive successful attempts and the function value $f(x)$ is the equivalent of the free-throw percentage.

Guided Practice: Example 2, *continued*

3. Use the function to create an equation that can be solved to determine the answer to the problem.

- Find how many consecutive attempts Josie must make in order to achieve a free-throw percentage of 60%.
- To do this, you need to find the *x*-values that make $f(x) = 0.60$.
- Therefore, the equation to be solved is $\frac{12+x}{26+x} = 0.60$.

Guided Practice: Example 2, *continued*

4. Solve the equation.

$\frac{12+x}{26+x}=0.60$	Equation
$12 + x = 0.60(26 + x)$	Multiply both sides by $26 + x$.
$12 + x = 0.60(26) + 0.60(x)$	Apply Distributive Property.
$12 + x = 15.6 + 0.60x$	Simplify.
$12 + 0.40x = 15.60$	Subtract $0.60x$ from both sides.
$0.40x = 3.60$	Subtract 12 from both sides.
$x = 9$	Divide both sides by 0.40.

Guided Practice: Example 2, *continued*

5. Check the apparent solution and answer the question.

$$\frac{12+x}{26+x}=0.60$$ Equation

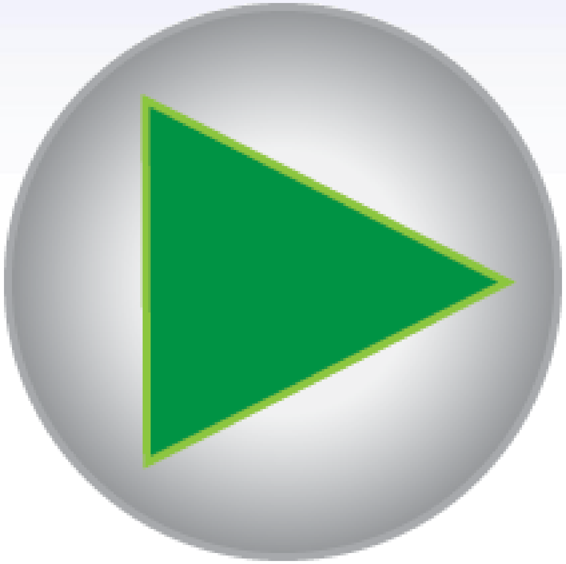
$$\frac{12+(9)}{26+(9)}=0.60$$ Substitute 9 for *x*.

$$\frac{21}{35}=0.60;\ 0.60=0.60$$

Guided Practice: Example 2, *continued*

- The answer checks out; therefore, $x = 9$ is the solution of $\frac{12+x}{26+x} = 0.60$.
- Josie will have to succeed at 9 consecutive attempts in order to raise her free-throw percentage to 60%.

Guided Practice: Example 2, *continued*



Guided Practice

Example 3

Moe and Marco deliver a regular order to a specialty foods store. Moe can unload the truck in 40 minutes if he works alone. Moe and Marco can unload the truck in 15 minutes if they work together.

How many minutes does it take Marco to unload the truck if he works alone? Assume both men work at steady rates and their rates are not affected by either working alone or working together.

Guided Practice: Example 3, *continued*

1. Assign a variable to represent what you need to find.

- Let x represent the number of minutes it takes Marco to unload the truck if he works alone.

Guided Practice: Example 3, *continued*

2. Create an equation that represents the situation.

- Moe, working alone, unloads at the rate of $\frac{1}{40}$ of the truck per minute.
- So, in 15 minutes Moe unloads $15 \bullet \frac{1}{40}$ of the truck.
- Marco, working alone, unloads at the rate of $\frac{1}{x}$ of the truck per minute.
- In 15 minutes, Marco unloads $15 \bullet \frac{1}{x}$ of the truck.

Guided Practice: Example 3, *continued*

- They complete the job working together in 15 minutes.
- Write these elements as an equation, then simplify.

(part of job done by Moe) + (part of job done by Marco) = (one whole job)	Equation
$\left(15\bullet\frac{1}{40}\right)+\left(15\bullet\frac{1}{x}\right)=(1)$	Substitute values for each part of the equation.
$\frac{15}{40}+\frac{15}{x}=1$	Simplify.

Guided Practice: Example 3, *continued*

3. Solve the equation.

$\frac{15}{40}+\frac{15}{x}=1$	Equation
$40x\left(\frac{15}{40}+\frac{15}{x}\right)=40x(1)$	Multiply both sides by $40x$, the least common denominator (LCD).
$40x\left(\frac{15}{40}\right)+40x\left(\frac{15}{x}\right)=40x(1)$	Distribute.
$15x+600=40x$	Simplify.
$600=25x$	Subtract $15x$ from both sides.
$24=x$	Divide both sides by 25.

Guided Practice: Example 3, *continued*

4. Check the apparent solution and answer the question.

$\frac{15}{40}+\frac{15}{x}=1$	Equation
$\frac{15}{40}+\frac{15}{(24)}=1$	Substitute 24 for x.
$\frac{3}{8}+\frac{5}{8}=1$	Simplify each fraction to obtain the LCD, 8.
$\frac{8}{8}=1;\ 1=1$	Simplify.

- It takes Marco 24 minutes to unload the truck if he works alone.

Guided Practice: Example 3, *continued*

Guided Practice

Example 4

A new street sweeping machine can sweep a town's streets in 20 fewer hours than it takes the old machine. Working together, the two machines can complete the job in 24 hours.

How long does it take each machine to complete the job, working alone? Assume the machines' rates are not affected by whether one machine is being used or both machines are being used.

Guided Practice: Example 4, *continued*

1. Assign a variable to represent what you need to find.

- Let x represent the number of hours it takes the new machine to complete the job, working alone.
- Therefore, $x + 20$ represents the number of hours it takes the old machine to complete the job, working alone.

Guided Practice: Example 4, *continued*

2. Create an equation that represents the situation.

- The new machine completes $\frac{1}{x}$ of the job per hour.
- So, in 24 hours the new machine completes $24 \bullet \frac{1}{x}$ of the job.
- The old machine completes $\frac{1}{x+20}$ of the job per hour.
- Thus, in 24 hours, the old machine completes $24 \bullet \frac{1}{x+20}$ of the job.

Guided Practice: Example 4, *continued*

- The machines complete the job working together in 24 hours.

- Write these elements as an equation, then simplify.

(part of job done by new machine) +
(part of job done by old machine) = (one whole job)

$$\left(24 \bullet \frac{1}{x}\right) + \left(24 \bullet \frac{1}{x+20}\right) = (1)$$

Substitute values for each part of the equation.

$$\frac{24}{x} + \frac{24}{x+20} = 1$$

Simplify.

Guided Practice: Example 4, *continued*

3. Solve the equation.

$$\frac{24}{x}+\frac{24}{x+20}=1$$

Equation

$$x(x+20)\left(\frac{24}{x}+\frac{24}{x+20}\right)=x(x+20)(1)$$

Multiply both sides by (x)(x + 20), the LCD.

$$x(x+20)\left(\frac{24}{x}\right)+x(x+20)\left(\frac{24}{x+20}\right)$$
$$=x(x+20)(1)$$

Distribute.

$$\frac{x(x+20)24}{x}+\frac{x(x+20)24}{x+20}=x(x+20)(1)$$

Simplify.

Guided Practice: Example 4, *continued*

$$\frac{\cancel{x}(x+20)24}{\cancel{x}}+\frac{x\cancel{(x+20)}24}{\cancel{x+20}}=x(x+20)(1)$$

Divide out common factors.

$(x + 20)24 + x(24) = x(x + 20)$ — Simplify.

$24x + 480 + 24x = x^2 + 20x$ — Distribute.

$48x + 480 = x^2 + 20x$ — Simplify.

$480 = x^2 - 28x$ — Subtract $48x$ from both sides.

$0 = x^2 - 28x - 480$ — Subtract 480 from both sides.

Guided Practice: Example 4, *continued*

- In this example, it is possible to factor the equation, but using the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, will work for all quadratic equations.

$$x = \frac{-(-28) \pm \sqrt{(-28)^2 - 4(1)(-480)}}{2(1)}$$

Substitute values for *a*, *b*, and *c* into the quadratic formula.

Guided Practice: Example 4, *continued*

$$x = \frac{28 \pm \sqrt{784 + 1920}}{2}$$

Simplify.

$$x = \frac{28 \pm \sqrt{2704}}{2} = \frac{28 \pm 52}{2}$$

$$x = 40 \text{ or } x = -12$$

- Since *x* represents a number of hours, $x = -12$ does not make sense in this situation because time cannot be negative.
- Therefore, $x = 40$.

Guided Practice: Example 4, *continued*

4. Check the apparent solution and answer the question.

$\frac{24}{x}+\frac{24}{x+20}=1$	Equation
$\frac{24}{(40)}+\frac{24}{(40)+20}=1$	Substitute 40 for x.
$\frac{24}{40}+\frac{24}{60}=1$	Simplify.
$\frac{3}{5}+\frac{2}{5}=1$	Simplify each fraction to obtain the LCD, 5.
$\frac{5}{5}=1;\ 1=1$	Simplify.

Guided Practice: Example 4, *continued*

- Therefore, $x = 40$ and $x + 20 = 60$.
- It takes the new machine 40 hours to complete the job, working alone.
- It takes the old machine 60 hours to complete the job, working alone.

Guided Practice

Example 5

Serena drove 54 miles in the city, averaging 24 miles per gallon. Then she drove 251 miles on highways. For the entire 305 miles, she averaged 30 miles per gallon.

What was her average fuel consumption rate on highways, in miles per gallon?

Guided Practice: Example 5, *continued*

1. Assign a variable to represent what you need to find.

- Let x represent Serena’s fuel consumption rate on highways, in miles per gallon.

Guided Practice: Example 5, *continued*

2. Organize the information.

- Recall that to divide by a fraction, you multiply by its reciprocal.

- So, if you divide miles by the rate of miles per gallon, you get gallons:

$$\frac{\text{miles}}{\frac{\text{miles}}{\text{gallons}}} = \text{miles} \bullet \frac{\text{gallons}}{\text{miles}} = \text{gallons}$$

- Use a table to organize the quantities in the problem.

Guided Practice: Example 5, *continued*

	Distance (mi)	Fuel consumption rate $\left(\frac{\text{mi}}{\text{gal}}\right)$	Fuel consumption (gal)
City	54	24	$\frac{54}{24}$
Highway	251	x	$\frac{251}{x}$
Overall (Total)	305	30	$\frac{305}{30}$

Guided Practice: Example 5, *continued*

3. Create an equation that represents the situation.

- The equation is:

(gallons used in city) + (gallons used on highway) = (gallons used overall)

$$\frac{54}{24}+\frac{251}{x}=\frac{305}{30}$$

Guided Practice: Example 5, *continued*

4. Solve the equation.

$$\frac{54}{24}+\frac{251}{x}=\frac{305}{30}$$ Equation

$$\frac{9}{4}+\frac{251}{x}=\frac{61}{6}$$ Simplify the fractions.

$$12x\left(\frac{9}{4}+\frac{251}{x}\right)=12x\left(\frac{61}{6}\right)$$ Multiply both sides by 12*x*, the LCD.

$$12x\left(\frac{9}{4}\right)+12x\left(\frac{251}{x}\right)=12x\left(\frac{61}{6}\right)$$ Distribute.

Guided Practice: Example 5, *continued*

$$\frac{12x(9)}{4}+\frac{12x(251)}{x}=\frac{12x(61)}{6}$$

Simplify.

$$\frac{\not{1}\not{2}x(9)}{\not{4}}+\frac{12\not{x}(251)}{\not{x}}=\frac{\not{1}\not{2}x(61)}{\not{6}}$$

Divide out common factors.

$$27x+3012=122x$$

Simplify.

$$3012=95x$$

Subtract 27*x* from both sides.

$$31.7\approx x$$

Divide both sides by 95.

Guided Practice: Example 5, *continued*

5. Check the apparent solution and answer the question.

$$\frac{54}{24}+\frac{251}{x}=\frac{305}{30}$$ Equation

$$\frac{54}{24}+\frac{251}{(31.7)}=\frac{305}{30}$$ Substitute 31.7 for *x*.

$$2.25+7.918\approx 10.167$$

$$10.168\approx 10.167$$

Use a calculator to write the fractions as decimals, approximating as needed.

Guided Practice: Example 5, *continued*

- Serena's average fuel consumption rate on highways was approximately 31.7 miles per gallon.

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