Problem-Based Task Implementation Guide: Golden Rectangles and the Golden Ratio

Task Overview

Focus

How can a rational equation be created to represent the ratio of corresponding sides of two rectangles? What properties of quadrilaterals can be applied when determining missing side lengths of a quadrilateral? In this task, students will create and solve a rational equation that represents the ratio of lengths of sides of two rectangles, and they will use the solution of this equation to determine the golden ratio.

This activity will provide practice with:

- creating ratios and proportions
- identifying corresponding sides of rectangles
- substituting values representing side lengths to create a proportion
- simplifying rational expressions
- solving quadratic equations using the quadratic formula
- analyzing solutions of a quadratic equation in the context of the problem
- converting an expression containing a radical to a decimal format

Introduction

This task should be used to explore or to apply the skill of creating rational equations in one variable and using them to solve problems. Students should already be familiar with representing comparisons of quantities as ratios. They should also be familiar with simplifying expressions involving radicals and applying the quadratic formula to solve quadratic equations.

Begin by reading the problem and reviewing and clarifying the meaning of the terms *equivalent ratios*, *golden ratio*, *golden rectangle*, *proportion*, and *similar polygons*.

equivalent ratios	two ratios that are equal
golden ratio	the ratio of the length to width in a golden rectangle
golden rectangle	a rectangle that can be divided into a square and a smaller rectangle that is similar to the original rectangle
proportion	the relationship of one thing to another in terms of size, number, or amount; the ratio
similar polygons	two polygons that have all corresponding angles congruent and all corresponding sides proportional

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 2:** Reason abstractly and quantitatively.

  Students will reason abstractly as they make sense of the goal of the task by recalling prior knowledge of ratios, proportions, properties of quadrilaterals, and solving quadratic equations using the quadratic formula. They will represent the side lengths of two rectangles algebraically, and they will use these lengths to create a proportion. After simplifying the proportion and creating a quadratic equation, they will analyze the solutions of the equation and determine which solution is correct in the context of the problems. They will reason quantitatively as they perform the algebraic steps necessary to simplify the proportion, as well as when they work through the steps of the quadratic formula, as they determine the solutions to the equation.

- **SMP 4:** Model with mathematics.

  Students will recognize that this scenario can be modeled by using proportional reasoning involving two similar rectangles. They will sketch the drawings of the two rectangles and use algebraic expressions to represent the lengths of the sides. They will draw conclusions about the ratio of the length to the width of the golden rectangle after solving a quadratic equation created as a result of simplifying the proportion representing the side lengths of both rectangles.

- **SMP 7:** Look for and make use of structure.

  Students will apply the structure and procedures involved in solving an equation using the quadratic formula. They will also apply the structure of a proportion by multiplying both sides of the proportion by the LCD. Students will recognize the structure of the side lengths of rectangle as they determine the algebraic expressions representing the lengths.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly matching up the corresponding sides of the newly created rectangle to the original rectangle

  Remind students that the points representing the sides of a rectangle must correspond with the "matching" side of the other rectangle. Have students draw both rectangles with the correct orientation on their papers. Discuss which vertices match on both rectangles.

- choosing the wrong values for sides *BC* and *AP* of the rectangles

  Review with students the properties of squares and rectangles. Remind them that all four sides of a square have the same length, and that opposite sides of a rectangle have the same length.

- incorrectly identifying the negative solution of the equation as the correct solution for the length of the rectangle

  Remind students to consider the context of the problem and recall that side lengths cannot be negative values.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, review the concept of ratios with them. Ask them, "What are some common ratios that are used in everyday life?" (**Answer**: Answers will vary, but examples may include miles per gallon, miles per hour, words per minute in typing, earned run average, etc.)
- Before students begin the task, ask them, "What is one of the most well-known ratios involving circles? Explain the ratio." (**Answer:** The ratio π (*pi*) is one of the most well-known ratios involving circles. Pi has a value of 3.14 (rounded to the nearest hundredth), and it is the ratio of the length of the circumference of any circle to the length of the diameter of that circle. In other words, the circumference of any circle is always 3.14 (rounded) times larger than the diameter of the circle.)
- As students begin the task, ask them, "What is a way to visually examine the proportional sides of the newly created rectangle to the original rectangle?" (**Answer**: A diagram showing the original rectangle separate from the newly created rectangle can be drawn. The new rectangle will have to be rotated 90 degrees clockwise in order for the corresponding sides to match.)
- If students are having difficulty determining which sides of the new rectangle are proportional to the sides of the original rectangle, remind students that the names of the points representing each corner (vertex) of the sides in the original triangle must match the corresponding points representing the new triangle. Ask them, "List all of the matching corresponding sides of the original rectangle to the new rectangle." (**Answer:** Side *AB* of the original matches with side *BC* of the new, *BC* of the original matches with *CQ* of the new, *CD* of the original matches with *QP* of the new, and *AD* of the original matches with *BP* of the new: $\overline{AB}$ is proportional to $\overline{BC}$, $\overline{BC}$ is proportional to $\overline{CQ}$, $\overline{CD}$ is proportional to $\overline{QP}$, and $\overline{AD}$ is proportional to $\overline{BP}$.)

- As students are working through the task, ask them, "What properties of squares and rectangles will be applied in this task?" (**Answer:** The properties of the relationship between the side lengths of a square (four congruent sides) and the side lengths of a rectangle (opposite sides are congruent) will be applied in this task.)
- As students are working to find the value of x once they substitute the known values for the side lengths ($\frac{AB}{BC}=\frac{AD}{BP}$; $\frac{(x)}{(1)}=\frac{(1)}{(x-1)}$), ask them, "What is another method that can be used to simplify the proportion, instead of multiplying by the LCD to achieve the same resulting quadratic equation of $x^2 - x - 1 = 0$?" (**Answer:** Cross multiplication can be used to simplify the proportion: $(x)(x - 1) = (1)(1)$; $x^2 - x = 1$; $x^2 - x - 1 = 0$.)
- Once students have solved the quadratic equation using the quadratic formula, ask them, "There are two solutions to the equation, so how can you determine which solution is the correct one in the context of this task?" (**Answer:** Solving the equation results in both a positive and a negative value. Because the context of the task involves the length of a side of a rectangle, length cannot be negative. Therefore, the negative solution is discarded, and the positive one is applied instead.)
- Once students have found the value of the length of the rectangle, which is the golden ratio, ask them, "How can you test this value to ensure that it is indeed the golden ratio for any rectangle?" (**Answer:** One way to test the value is to create other rectangles and measure the sides, such that a square is created inside of the original rectangle. Once all the sides are drawn and measured as precisely as possible, then the ratio of the length to the width can be calculated, which should be as close to 1.618 as possible.)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask for volunteers to discuss their thoughts and processes for setting up the proportion for the similar rectangles, as well as how they determined how to represent the side lengths of the rectangles algebraically. Also ask for volunteers to explain how they analyzed the two solutions of the quadratic equation and determined which one was applicable in the context of the problem. Encourage students to discuss any difficulties or confusion they experienced when working through the various parts of the task.
- Compare students' strategies and ways of justifying responses. Ask students to share their reasoning process and how they were able to create the proportion and then create a resulting quadratic equation. Ask them to explain their strategies for choosing the side lengths, and ultimately how they calculated the golden ratio. Focus on the use of precise mathematical language and clarity, specifically when referring to ratios, proportions, and similarity.

Connecting to Key Concepts

Make explicit connections to key concepts:

- A **ratio** is a relation between two quantities. A ratio can be expressed in words, as fractions, as decimals, or as a percentage.

  In this task, students will determine the ratio between the length and width of a golden rectangle. They will express this ratio in decimal format.

- A **rational equation** is an equation that includes the ratio of two rational expressions, in which a variable appears in the denominator of at least one rational expression.

  Students will create a rational equation that represents the ratio of the sides of the original rectangle and the new rectangle. The variable x appears in the denominator of the second rational expression: $\frac{(x)}{(1)} = \frac{(1)}{(x-1)}$.

- Solutions to quadratic functions of the form $ax^2 + bx + c = 0$ can be determined using the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

  Students will solve the quadratic equation, $x^2 - x - 1 = 0$, using the quadratic formula. They will also analyze the two solutions to determine which one is appropriate for the given context of the task.

- When solving rational equations, restrictions on the variable must be considered.

  Students will analyze the two given solutions of the quadratic equation and determine that only the positive solution is correct, because the variable, x, cannot be negative in the context of this task since it represents the length of the rectangle.

Extending the Task

- To extend the task, ask students to draw their own rectangle. Ask them to carefully measure the side lengths, record the values, and draw a segment within the rectangle that will create a square and an additional rectangle. Next, have students create proportions for the corresponding sides of the two rectangles, using the exact values for each side length. Ask them to calculate the golden ratio, then ask for volunteers to share their rectangles and calculations. Discuss how measurement errors can lead to incorrect values for the golden ratio.

- Another option for extending the task is to ask students to work with a partner and have each person draw a rectangle. Ask them to have their partners follow the same steps as given in the task, and ask them to write down all calculations necessary to find the golden ratio for their rectangle. Ask for volunteers to share their rectangles and explain their thought processes when calculating the golden ratio.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 2, ASK:** "How did you reason abstractly and quantitatively? Which of your strategies represent abstract reasoning?" (**Answer:** I used abstract reasoning as I recalled and applied my prior knowledge of ratios, proportions, properties of quadrilaterals, and solving quadratic equations using the quadratic formula. I also reasoned that the side lengths of the two similar rectangles could be represented with algebraic expressions that were used in a proportion. Then I solved the proportion by creating a quadratic equation.) "Which of your strategies represents quantitative reasoning?" (**Answer:** I used quantitative reasoning as I worked through the steps of the quadratic formula in order to determine the solutions of the equation.)
- **For SMP 4, ASK:** "How did you use mathematics to model this particular scenario?" (**Answer:** I recognized that this scenario could be modeled using proportional reasoning involving two similar rectangles. I sketched the drawings of the two rectangles and used algebraic expressions to represent the lengths of the sides. I also drew conclusions about the ratio of the length to the width of the golden rectangle after solving a quadratic equation created as a result of simplifying the proportion representing the side lengths of both rectangles.)
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** When solving this problem, I made use of structure by applying the procedures and steps involved in solving an equation using the quadratic formula. I also applied the structure of a proportion by multiplying both sides of the proportion by the LCD in order to simplify it.)

Alternate Strategies or Solutions

- Students may choose to draw the original rectangle on a coordinate plane, in order to visually recognize the side lengths with whole number units. Remind them to ensure that the line segment drawn in the rectangle creates a square, and remind them to check the four sides of the square to verify that all four sides are equal in length.
- Students may choose to cross multiply instead of multiplying by the LCD in order to simplify the proportion representing the side lengths. Encourage students to apply both methods and verify that their results are the same with each method. Students may also choose to solve the quadratic equation using another method, such as completing the square. Encourage students to ensure their results are the same regardless of the method chosen.

Technology

Students can use scientific calculators in order to determine the decimal value of the golden ratio.