

# Introduction

A **rational function** is a function that is the quotient of two polynomials; it can be written in the form  $f(x) = \frac{p(x)}{q(x)}$ , where  $p(x)$  and  $q(x)$  are polynomials and  $q(x) \neq 0$ . For example,  $f(x) = \frac{x + 6}{3x - 1}$  is a rational function. In order to graph a rational function, first identify its key features. This information, along with additional points found using an input/output table, can be used to determine the basic shape of the graph.

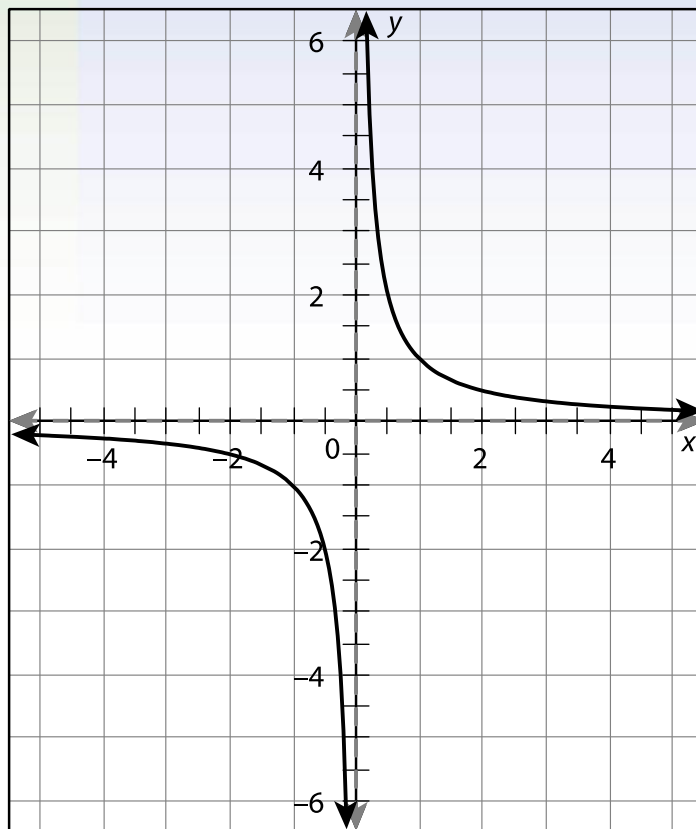


# Key Concepts

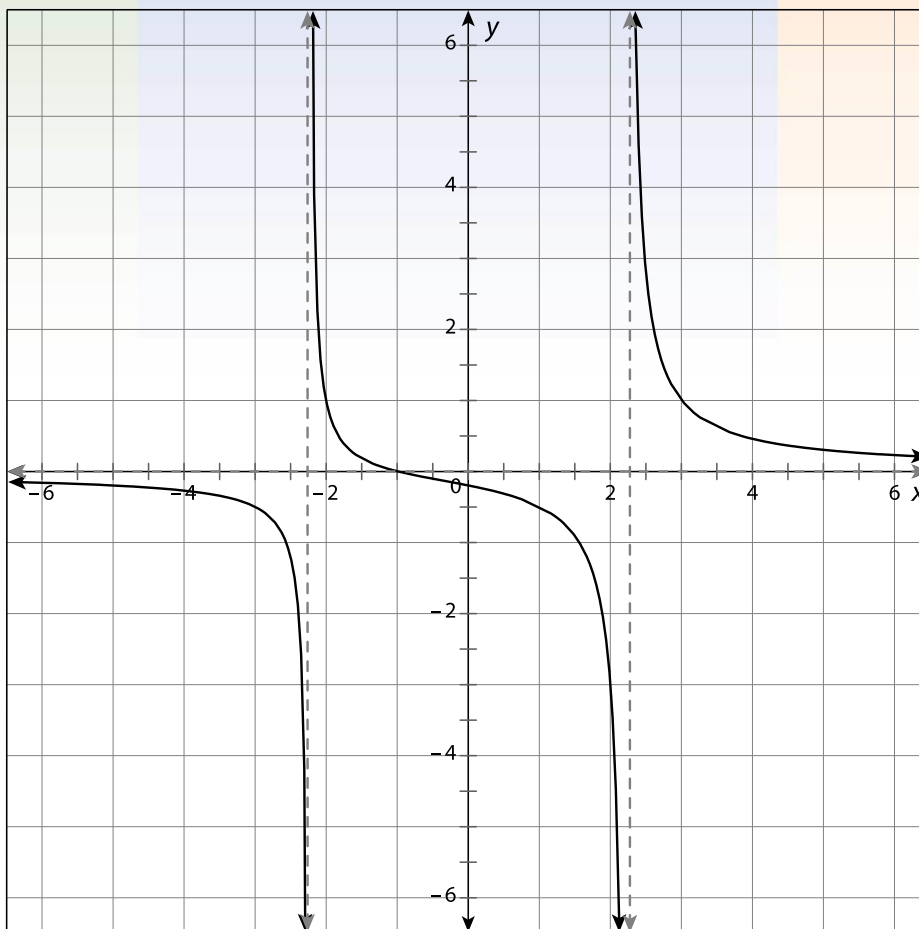
- The shape of a rational function will depend on the highest exponent, or degree, of the polynomials in the numerator and denominator. Consider the following examples on the next slides. These are all graphs of rational functions. Notice that they are in two or more branches and not one smooth curve like the graphs of most functions.



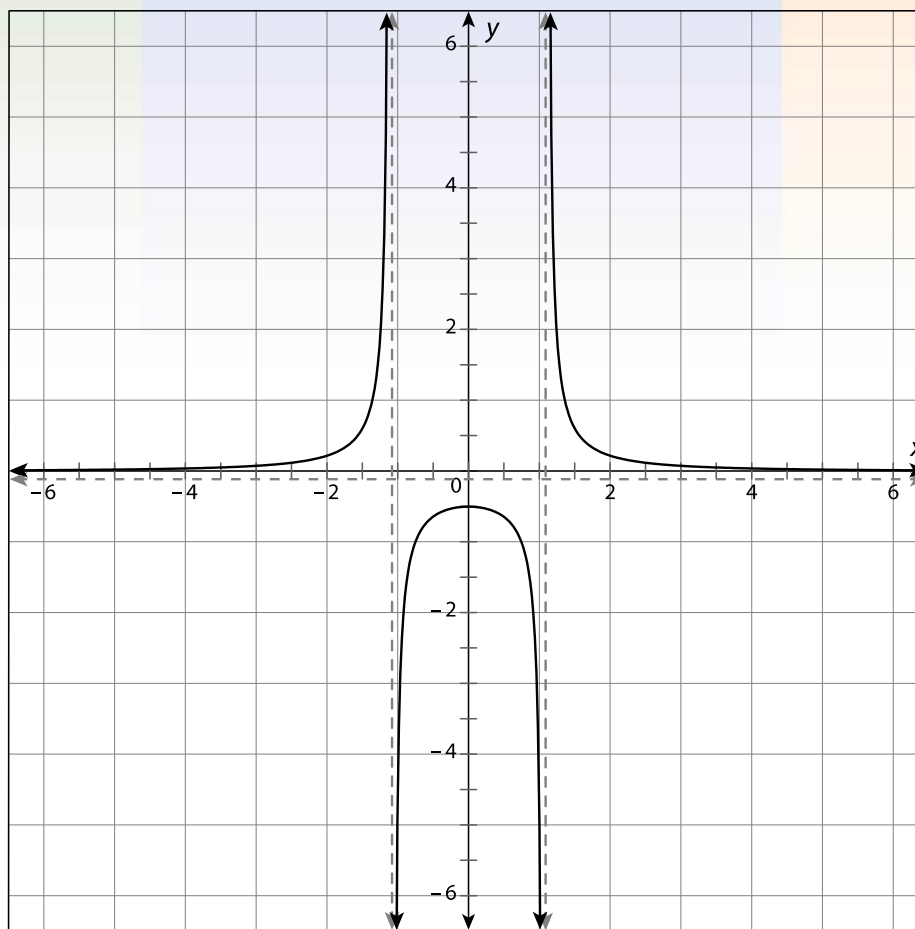
# Key Concepts, *continued*



# Key Concepts, *continued*



# Key Concepts, *continued*



## Key Concepts, *continued*

### Determining the Asymptotes of a Rational Function

- Notice that in each of these examples, the rational function has **asymptotes**, or lines that the function gets closer and closer to as one of the variables increases or decreases without bound.
- A **vertical asymptote** of the function  $f$  is the line  $x = a$  where  $f(x)$  either increases or decreases without bound as  $x$  gets closer to  $a$ . To find the vertical asymptote(s) of a rational function, set the denominator equal to 0 and solve for  $x$ .

## Key Concepts, *continued*

- A **horizontal asymptote** of the function  $f$  is the line  $y = b$  of the function  $f$  where  $f(x)$  gets closer to  $b$  as  $x$  either increases or decreases without bound.
- The method for finding the horizontal asymptote of a rational function differs depending on how the degrees of the polynomials in the numerator and the denominator compare.
- If the numerator has a lower degree than the denominator, then the  $x$ -axis is the horizontal asymptote.

## Key Concepts, *continued*

- If the numerator and denominator have the same degree, the horizontal asymptote can be found by dividing the coefficient of the highest-degree variable in the numerator by the coefficient of the highest-degree variable in the denominator. For example, in the function  $f(x) = \frac{6x^2 - 1}{2x^2 - 9x + 5}$ , 6 divided by 2 is 3, so the asymptote is at  $y = 3$ .

## Key Concepts, *continued*

### Determining the Zeros and $y$ -Intercepts of a Rational Function

- The **zeros** of a function are the  $x$ -values at which a function equals 0. Therefore, they are also the points at which the function crosses the  $x$ -axis and are also known as the  **$x$ -intercepts**.
- To find the zeros (or  $x$ -intercepts) of a function, replace  $f(x)$  or  $y$  with 0 and solve for  $x$ . Then write the intercept as the coordinate  $(x, 0)$ .

## Key Concepts, *continued*

- The **y-intercept** of a function is the point at which the graph crosses the  $y$ -axis. To find the  $y$ -intercept, replace  $x$  with 0 and solve for  $f(x)$  or  $y$ . Then write the intercept as the coordinate  $(0, y)$ .
- The table that follows summarizes how to use these key features to graph a rational function.

### Graphing a Rational Function

1. Identify and sketch all the asymptotes.
2. Identify and sketch the zero(s) and  $y$ -intercept.
3. Plot additional points as needed using an input/output table.
4. Draw the curve of the function.

# Key Concepts, *continued*

## Identifying the End Behavior of the Function

- The end behavior of a function describes the behavior of the graph as  $x$  approaches positive infinity and as  $x$  approaches negative infinity.
- Once a function has been graphed, the graph can be examined to determine the end behavior.
- If the graph appears to be nearly horizontal as  $x$  approaches negative or positive infinity, then the end behavior is the fact that  $y$  approaches this  $y$ -value.

## Key Concepts, *continued*

- If instead of approaching a specific  $y$ -value the graph is constantly increasing or decreasing as  $x$  approaches negative or positive infinity, then the end behavior is that the function approaches negative or positive infinity.
- The left and right sides of a graph may have different end behaviors; therefore, consider both cases separately.



# Common Errors/Misconceptions

- incorrectly applying the process for finding the vertical and/or horizontal asymptote(s)
- incorrectly applying the process for finding the  $x$ - and/or  $y$ -intercept(s)
- misidentifying which process to use to find the horizontal asymptote (depending on how the degrees of the numerator and denominator compare)
- not plotting enough points to accurately identify the curve of the function

## Guided Practice

### Example 1

Find all asymptotes of the function  $f(x) = \frac{3x^2 - 1}{x^2 - 2x - 3}$ .

## Guided Practice: Example 1, *continued*

### 1. Find the vertical asymptote(s).

To find the vertical asymptote, set the denominator equal to 0 and solve for  $x$ .

$$x^2 - 2x - 3 = 0$$

Set the denominator equal to 0.

$$(x - 3)(x + 1) = 0$$

Factor the trinomial.

$$x - 3 = 0 \text{ and } x + 1 = 0$$

Set each factor equal to 0.

$$x = 3 \text{ and } x = -1$$

Solve each equation for  $x$ .

The vertical asymptotes of  $f(x) = \frac{3x^2 - 1}{x^2 - 2x - 3}$  are the lines  $x = 3$  and  $x = -1$ .

## Guided Practice: Example 1, *continued*

### 2. Determine how to find the horizontal asymptote.

The method for finding the horizontal asymptote of a rational function differs depending on how the degree of the numerator compares to the degree of the denominator.



## Guided Practice: Example 1, *continued*

For the function  $f(x) = \frac{3x^2 - 1}{x^2 - 2x - 3}$ , both the numerator and denominator are second-degree polynomials.

Since the numerator and denominator are of the same degree, the horizontal asymptote can be found by dividing the coefficients of the highest-degree variables.

## Guided Practice: Example 1, *continued*

### 3. Find the horizontal asymptote.

To find the horizontal asymptote, divide the coefficient of the highest-degree variable in the numerator by the coefficient of the highest-degree variable in the denominator.

The highest-degree variable in both the numerator and the denominator is  $x^2$ . The coefficient of  $x^2$  in the numerator is 3 and the coefficient of  $x^2$  in the denominator is an understood 1.

## Guided Practice: Example 1, *continued*

Divide 3 by 1 to find the horizontal asymptote:  $\frac{3}{1} = 3$ .

The horizontal asymptote of  $f(x) = \frac{3x^2 - 1}{x^2 - 2x - 3}$  is the

line  $y = 3$ .

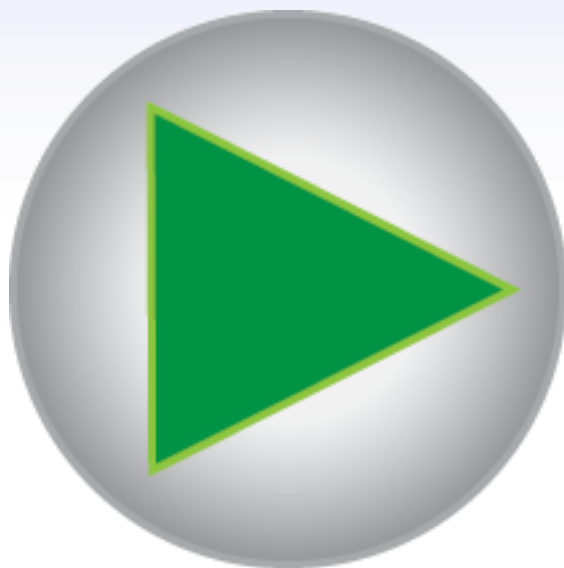
## Guided Practice: Example 1, *continued*

### 4. Determine whether the function has an oblique asymptote and, if so, find it.

There is only an oblique asymptote if the polynomial in the numerator is 1 degree higher than the polynomial in the denominator. Therefore, this function has no oblique asymptote.



## Guided Practice: **Example 1, *continued***



## Guided Practice

### Example 4

Graph  $f(x) = \frac{x^2 - 4}{2x + 3}$  by identifying and using its key

features, then describe the function's end behavior.

## Guided Practice: Example 4, *continued*

### 1. Identify any vertical and horizontal asymptotes.

To find the vertical asymptote, set the denominator equal to 0 and solve for  $x$ .

$$2x + 3 = 0$$

Set the denominator equal to 0.

$$2x = -3$$

Subtract 3 from both sides.

$$x = -1.5$$

Divide both sides by 2.

The vertical asymptote is  $x = -1.5$ .

## Guided Practice: Example 4, *continued*

The method for finding the horizontal asymptote of a rational function differs depending on how the degrees of the polynomials in the numerator and the denominator compare.

For the function  $f(x) = \frac{x^2 - 4}{2x + 3}$ , the polynomial in the numerator is a higher degree than the polynomial in the denominator. Therefore, there is no horizontal asymptote.

## Guided Practice: **Example 4, *continued***

Because the degree of the numerator is exactly 1 more than the degree of the denominator, there is an oblique asymptote.

To find an oblique asymptote, divide the numerator of the function by the denominator using long division, as shown on the next slide.

## Guided Practice: Example 4, continued

$$\begin{array}{r} \frac{1}{2}x - \frac{3}{4} \\ 2x + 3 \overline{) x^2 + 0x - 4} \\ \underline{- \left( x^2 + \frac{3}{2}x \right)} \\ \phantom{2x + 3 \overline{) x^2 + 0x - 4}} \frac{3}{2}x - 4 \\ \underline{- \left( \frac{3}{2}x - \frac{9}{4} \right)} \\ \phantom{2x + 3 \overline{) x^2 + 0x - 4}} \phantom{\frac{3}{2}x} - \frac{7}{4} \end{array}$$

## Guided Practice: **Example 4, continued**

Disregard the remainder  $-\frac{7}{4}$ . The rest of the quotient is the equation of the oblique asymptote.

Thus, the oblique asymptote is  $y = \frac{1}{2}x - \frac{3}{4}$ .

## Guided Practice: Example 4, *continued*

### 2. Sketch the asymptotes.

Sketch the asymptotes  $x = -1.5$  and  $y = \frac{1}{2}x - \frac{3}{4}$  on a

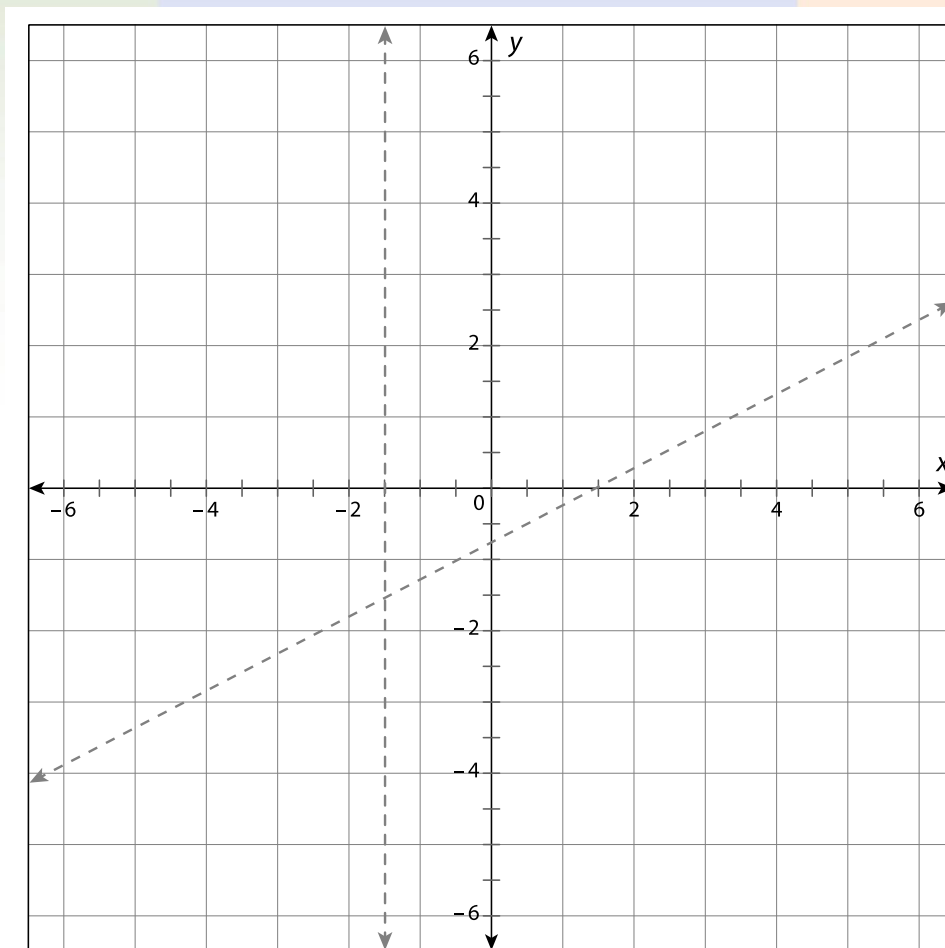
coordinate plane. Note that the equation of this

oblique asymptote is in slope-intercept form. It can be

graphed using  $\left(0, -\frac{3}{4}\right)$  as the  $y$ -intercept and  $\frac{1}{2}$  as

the slope.

## Guided Practice: Example 4, continued



## Guided Practice: Example 4, *continued*

### 3. Identify the zero(s) and y-intercept.

To find the zero or x-intercept, substitute 0 for  $f(x)$  and solve for  $x$ :

$$f(x) = \frac{x^2 - 4}{2x + 3}$$

Original function,  $f(x)$

$$f(0) = \frac{x^2 - 4}{2x + 3}$$

Substitute 0 for  $f(x)$ .

$$0 = x^2 - 4$$

Multiply both sides by  $2x + 3$ .

## Guided Practice: **Example 4, *continued***

$$4 = x^2$$

Add 4 to both sides.

$$\pm 2 = x$$

Take the square root of both sides.

$$x = \pm 2$$

Apply the Symmetric Property of Equality.

The zeros or x-intercepts are  $(2, 0)$  and  $(-2, 0)$ .

## Guided Practice: Example 4, *continued*

To find the  $y$ -intercept, substitute 0 for  $x$  and solve for  $f(x)$ .

$$f(x) = \frac{x^2 - 4}{2x + 3}$$

Original function,  $f(x)$

$$f(0) = \frac{(0)^2 - 4}{2(0) + 3}$$

Substitute 0 for  $x$ .

$$f(0) = \frac{-4}{3}$$

Simplify.

$$f(0) = -1.\bar{3}$$

Divide.

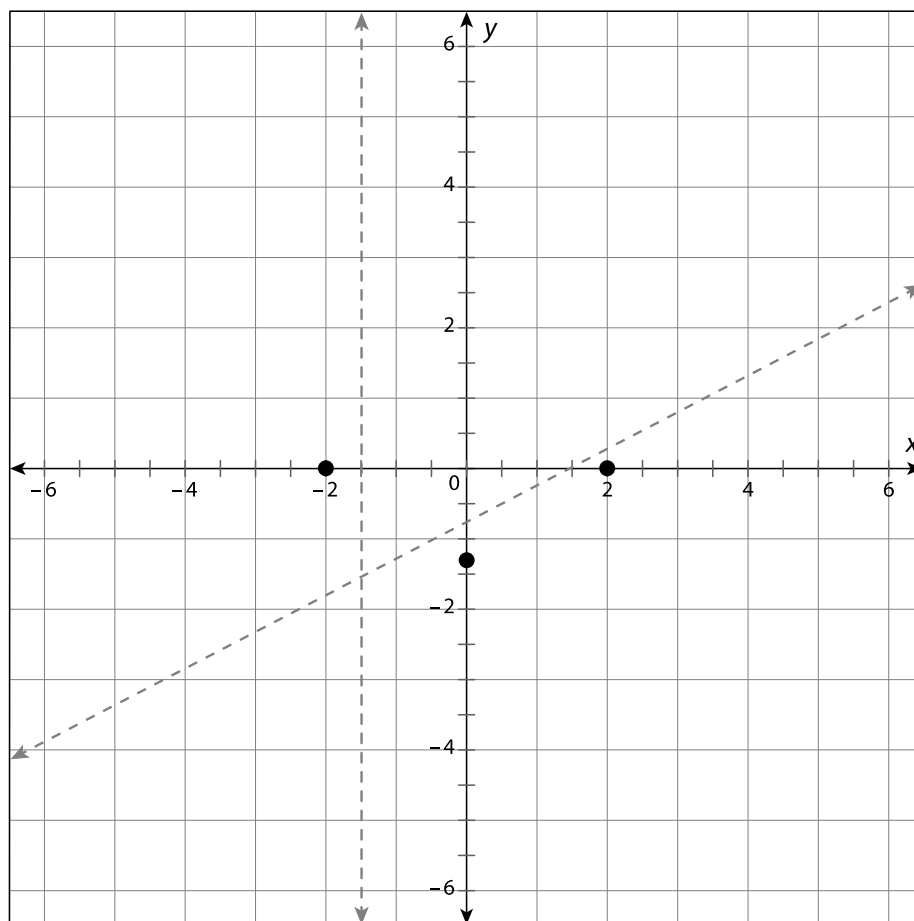
The  $y$ -intercept is  $(0, -1.\bar{3})$ .

## Guided Practice: Example 4, *continued*

### 4. Plot the zeros and $y$ -intercept.

The zeros ( $x$ -intercepts) are  $(2, 0)$  and  $(-2, 0)$  and the  $y$ -intercept is  $(0, -1.\bar{3})$ .

## Guided Practice: Example 4, continued



## Guided Practice: Example 4, *continued*

### 5. Determine and plot additional points as needed using an input/output table.

The general shape of the curve can be determined based on the location of the asymptotes and the  $x$ - and  $y$ -intercepts. However, additional points are needed to more accurately graph the curve.

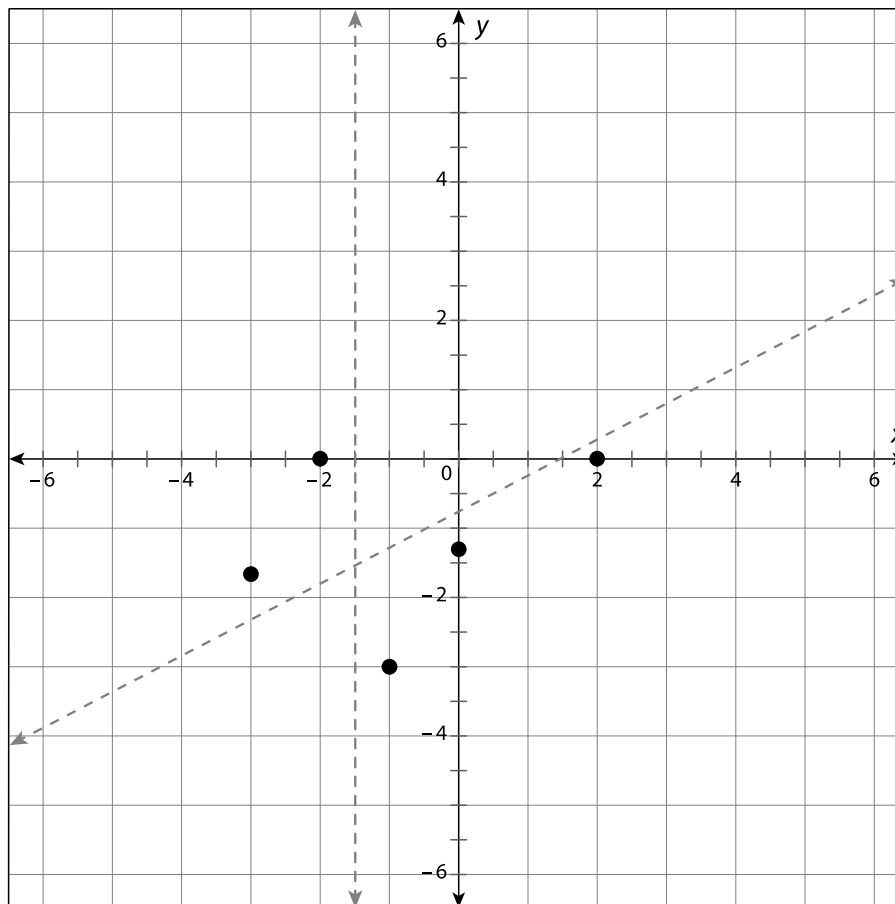
Substitute values for  $x$  and solve for  $f(x)$ . The input/output table that follows shows the results for some sample values of  $x$ .

## Guided Practice: Example 4, *continued*

$x$	$f(x)$
$-1$	$-3$
$-3$	$-1.\bar{6}$

Plot the resulting points.

## Guided Practice: Example 4, continued



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### Instruction

Graphing Rational Functions and Identifying Key Features

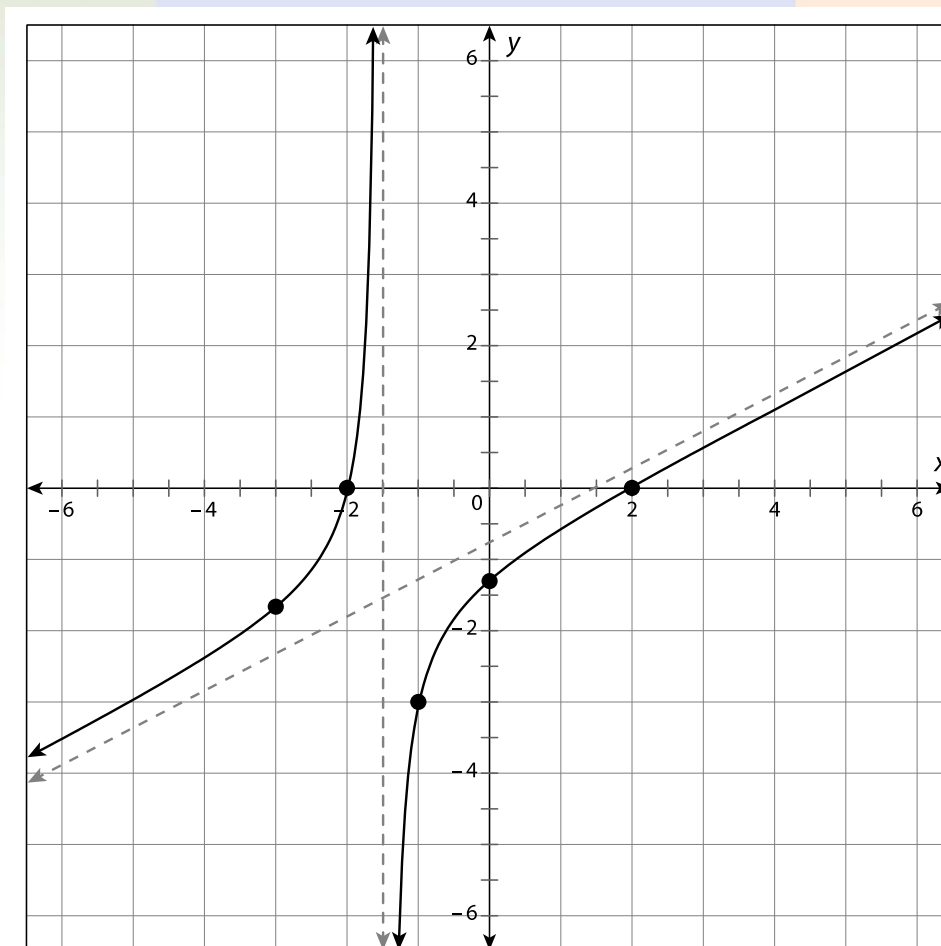
## Guided Practice: Example 4, *continued*

### 6. Draw the curve of the function.

Draw the curve of the function, remembering that the curve gets closer and closer to the asymptotes as  $x$  and  $y$  increase.

You can plot more points as needed to better approximate the curves.

## Guided Practice: Example 4, continued



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### Instruction

Graphing Rational Functions and Identifying Key Features

## Guided Practice: Example 4, *continued*

### 7. Describe the function's end behavior.

As  $x$  approaches positive infinity, the function's value (the  $y$ -value) approaches positive infinity. As  $x$  approaches negative infinity, the function's value approaches negative infinity.



## Guided Practice: **Example 4, *continued***

