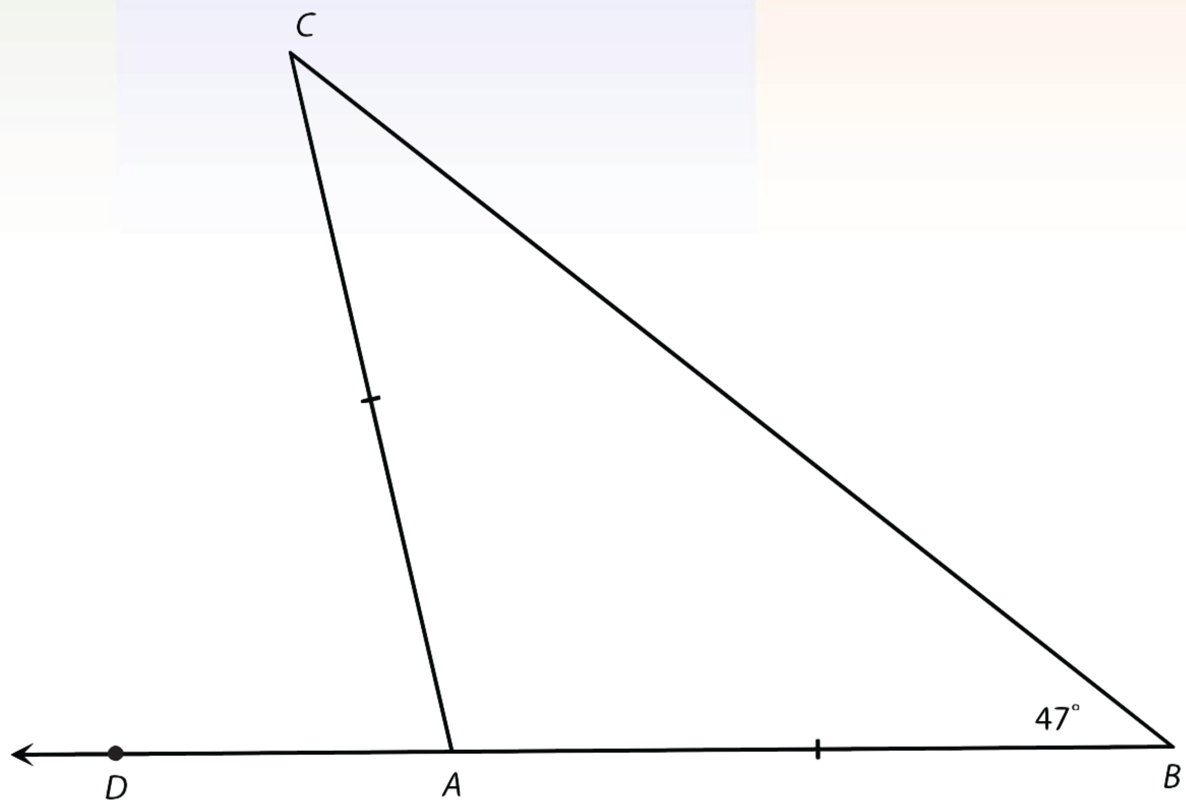
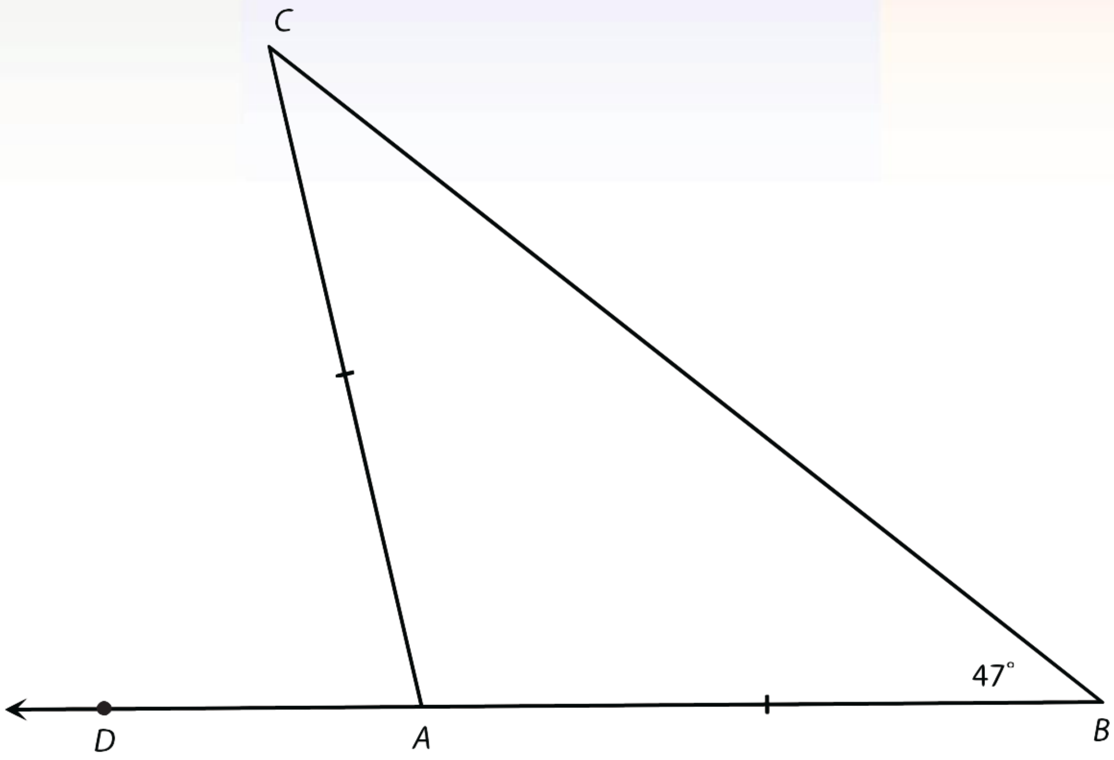
Proving Theorems About Isosceles Triangles

Warm-Up

In the diagram, a captain is aboard a ship at point *B*, a lighthouse is located at point *C*, and the ship is sailing in the direction of $\overrightarrow{BD}$. There is a buoy floating in the water at point *A*.



The captain measured $\angle CBD$ to be $47°$. The ship is as far from the buoy as the buoy is from the lighthouse, so the measures of $\overline{AB}$ and $\overline{AC}$ are equal.



1. What kind of triangle is $\triangle ABC$? Explain your reasoning.

2. What is the measure of Ð*ACB*?

3. What is the measure of Ð*CAD*?

1. What kind of triangle is $\triangle ABC$? Explain your reasoning

$\triangle ABC$ has two congruent sides, so by definition,

$\triangle ABC$ is an isosceles triangle.

2. What is the measure of Ð*ACB* ?

- By definition, an isosceles triangle has <u>two</u> congruent sides and <u>two</u> congruent angles.
- If *AB* is congruent to *AC*, then the measure of Ð*C* is congruent to the measure of Ð*B* .
- Ð*B* measures 47º; therefore, Ð*C* also measures 47º.

3. What is the measure of ∠*CAD*?

- ∠*C* and ∠*B* are remote interior angles to ∠*CAD*, an exterior angle.
- The measure of an exterior angle is equal to the sum of the measure of its remote interior angles.
- The sum of the interior angles is found by adding *m*∠*C* and *m*∠*B*.

  $47 + 47 = 94$
- The measure of ∠*CAD* is 94º.

Instruction

Introduction

Isosceles triangles can be seen throughout our daily lives in structures, supports, architectural details, and even bicycle frames.

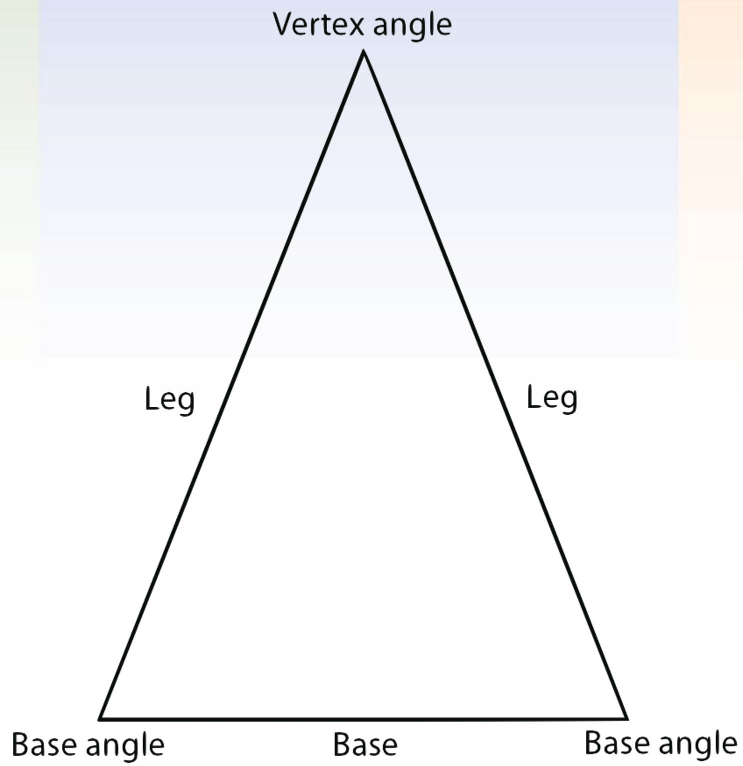
Isosceles triangles are a distinct classification of triangles with unique characteristics and parts that have specific names.

In this lesson, we will explore the qualities of isosceles triangles.

Key Concepts

- Isosceles triangles have at least two congruent sides, called legs.
- The angle created by the intersection of the legs is called the vertex angle.
- The side that is opposite the vertex angle is the base of the isosceles triangle.
- Each of the remaining angles is referred to as a base angle. The intersection of one leg and the base of the isosceles triangle creates a base angle.

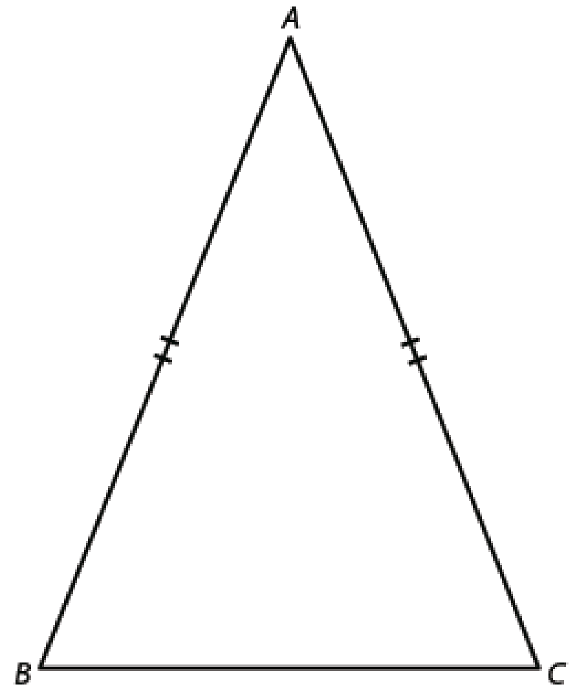
Key Concepts, *continued*



The following theorem is true of every isosceles triangle.

Key Concepts, *continued*

Theorem
Isosceles Triangle Theorem If two sides of a triangle are congruent, then the angles opposite the congruent sides are congruent. $m\angle B \cong m\angle C$

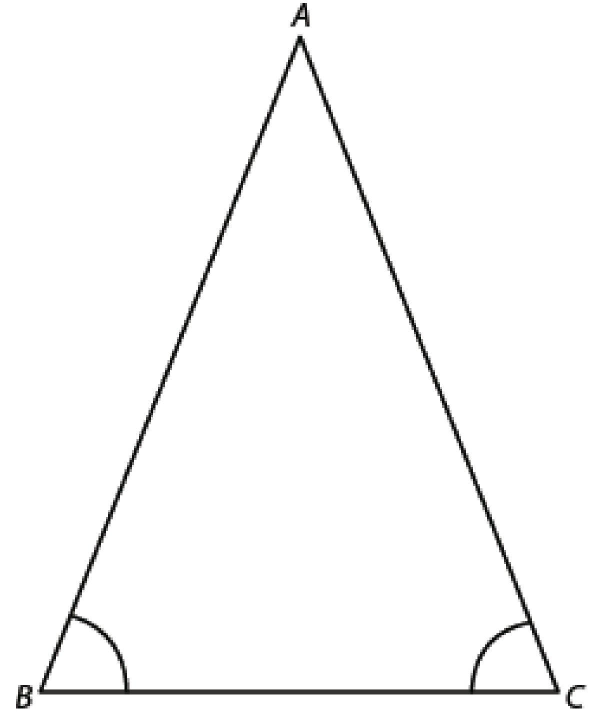


Key Concepts, *continued*

- If the Isosceles Triangle Theorem is reversed, then that statement is also true.
- This is known as the converse of the Isosceles Triangle Theorem.

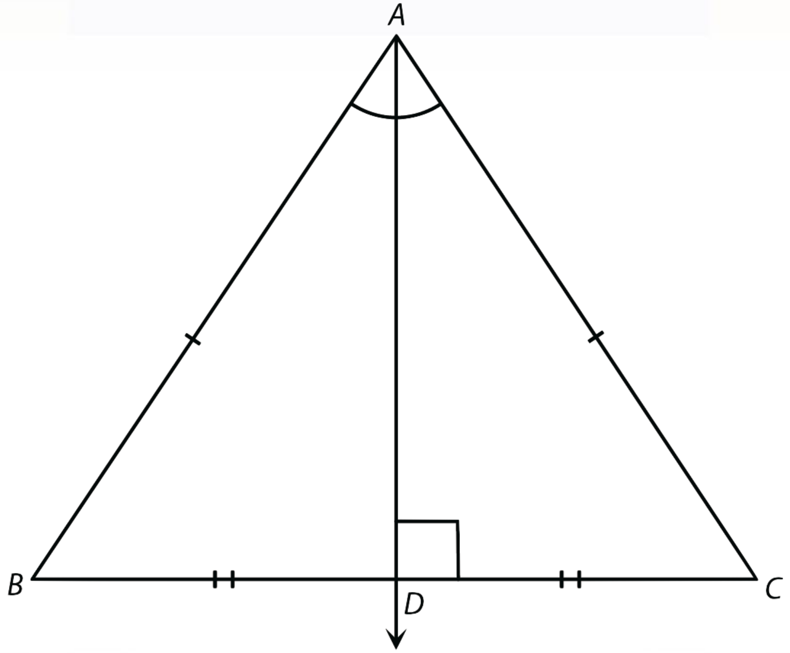
Key Concepts, *continued*

Theorem
Converse of the Isosceles Triangle Theorem   If two angles of a triangle are congruent, then the sides opposite those angles are congruent. $\overline{AB} \cong \overline{AC}$



Key Concepts, *continued*

- If the vertex angle of an isosceles triangle is bisected, the bisector is perpendicular to the base, creating two right triangles.
- In the diagram that follows, *D* is the midpoint of $\overline{BC}$.



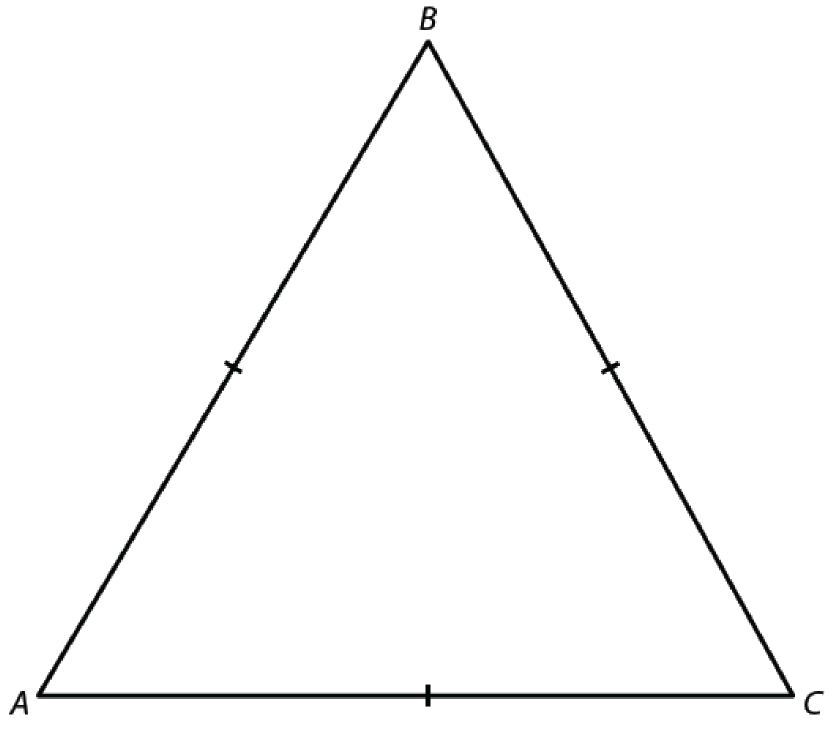
Key Concepts, *continued*

- Equilateral triangles are a special type of isosceles triangle, for which each side of the triangle is congruent.

- If all sides of a triangle are congruent, then all angles have the same measure.

Key Concepts, *continued*

Theorem
If a triangle is equilateral then it is equiangular, or has equal angles.



$\angle A \cong \angle B \cong \angle C$

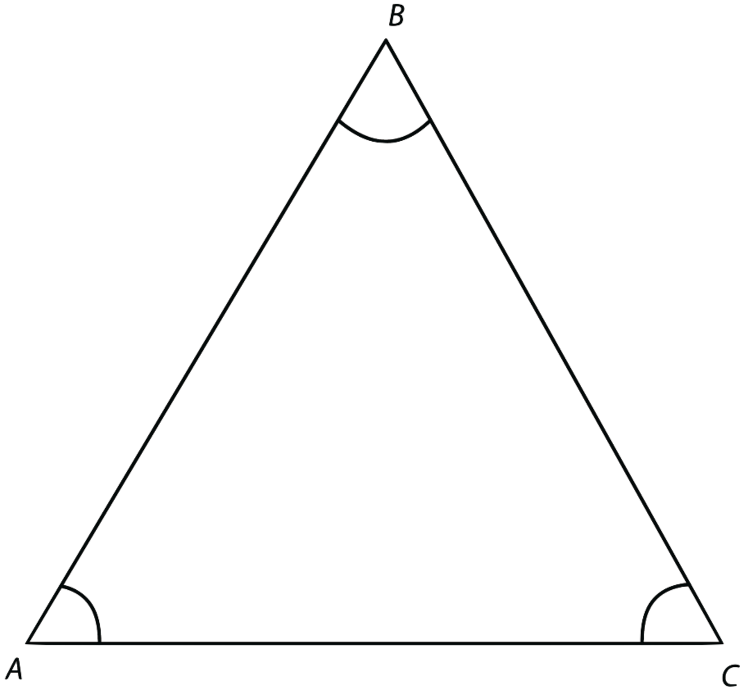
Key Concepts, *continued*

- Each angle of an equilateral triangle measures 60° $\left(180 \div 3 = 60\right)$.

- Conversely, if a triangle has equal angles, it is equilateral.

Key Concepts, *continued*

Theorem
If a triangle is equiangular, then it is equilateral.



$\overline{AB}$ @ $\overline{BC}$ @ $\overline{AC}$

Key Concepts, *continued*

- These theorems and properties can be used to solve many triangle problems.

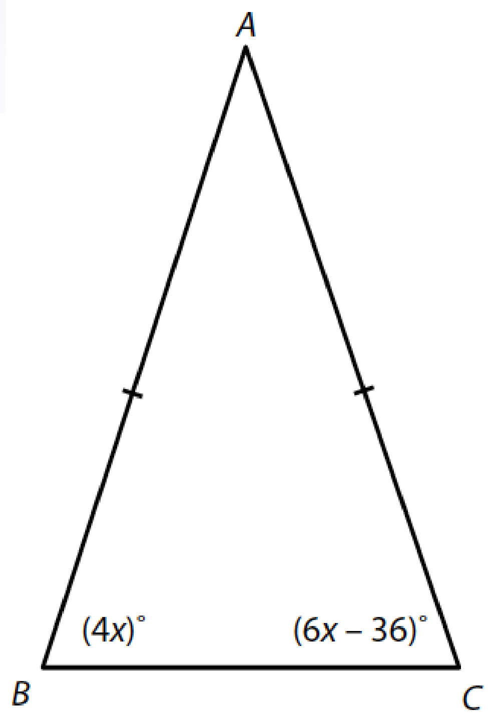
Common Errors/Misconceptions

- incorrectly identifying parts of isosceles triangles
- not identifying equilateral triangles as having the same properties of isosceles triangles
- incorrectly setting up and solving equations to find unknown measures of triangles
- misidentifying or leaving out theorems, postulates, or definitions when writing proofs

Guided Practice

Example 1

Find the measure of each angle of $\triangle ABC$.



Guided Practice: Example 1, *continued*

1. Identify the congruent sides and angles.

The legs of an isosceles triangle are congruent; therefore, $\overline{AB} \cong \overline{AC}$.

The base of $\triangle ABC$ is $\overline{BC}$.

$\angle B$ and $\angle C$ are base angles and are congruent.

Guided Practice: Example 1, *continued*

2. Calculate the value of *x*.

Congruent angles have the same measure.

Create an equation.

$m\angle B = m\angle C$ — The measures of base angles of isosceles triangles are equal.

$(4x) = (6x - 36)$ — Substitute values for $m\angle B$ and $m\angle C$.

$-2x = -36$ — Subtract $6x$ from both sides.

$x = 18$ — Divide both sides by –2.

The value of *x* is 18.

Guided Practice: Example 1, *continued*

3. Calculate each angle measure.

$m\angle B = 4x$	Given
$m\angle B = 4(18)$	Substitute the known value of *x* into the expression for *m*∠*B*.
$m\angle B = 72$	Solve for *m*∠*B*.

Guided Practice: Example 1, *continued*

3. Calculate each angle measure.

$m\angle C = 6x - 36$	Given
$m\angle C = 6(18) - 36$	Substitute the known value of *x* into the expression for *m∠C*.
$m\angle C = 72$	Solve for *m∠C*.

Guided Practice: Example 1, *continued*

3. Calculate each angle measure.

$m\angle A + m\angle B + m\angle C = 180$ — The sum of the angles of a triangle is 180º.

$m\angle A + (72) + (72) = 180$ — Substitute the known values.

$m\angle A = 36$ — Solve for *m∠A*.

Guided Practice: Example 1, *continued*

4. **Summarize your findings.**

$$m\angle A = 36°$$

$$m\angle B = 72°$$

$$m\angle C = 72°$$

Guided Practice

Example 2

Determine whether $\triangle ABC$ with vertices A (–4, 5), B (–1, –4), and C (5, 2) is an isosceles triangle.

If it is isosceles, name a pair of congruent angles.

Guided Practice: Example 2, *continued*

1. **Use the distance formula to calculate the length of each side.**

   Calculate the length of $\overline{AB}$.

Guided Practice: Example 2, *continued*

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$AB = \sqrt{[(-1) - (-4)]^2 + [(-4) - (5)]^2}$$

Substitute (–4, 5) and (–1, –4) for (x_1, y_1) and (x_2, y_2).

$$AB = \sqrt{(3)^2 + (-9)^2}$$

Simplify.

$$AB = \sqrt{9 + 81}$$

$$AB = \sqrt{90} = 3\sqrt{10}$$

Guided Practice: Example 2, *continued*

Calculate the length of $\overline{BC}$.

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$BC=\sqrt{[(5)-(-1)]^2+[(2)-(-4)]^2}$$

Substitute (–1, –4) and (5, 2) for (x_1, y_1) and (x_2, y_2).

$$BC=\sqrt{(6)^2+(6)^2}$$

Simplify.

$$BC=\sqrt{36+36}$$

$$BC=\sqrt{72}=6\sqrt{2}$$

Guided Practice: Example 2, *continued*

Calculate the length of $\overline{AC}$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$AC = \sqrt{[(5) - (-4)]^2 + [(2) - (5)]^2}$$

Substitute (–4, 5) and (5, 2) for (x_1, y_1) and (x_2, y_2).

$$AC = \sqrt{(9)^2 + (-3)^2}$$

Simplify.

$$AC = \sqrt{81 + 9}$$

$$AC = \sqrt{90} = 3\sqrt{10}$$

Guided Practice: Example 2, *continued*

2. **Determine if the triangle is isosceles.**

   A triangle with at least two congruent sides is an isosceles triangle.

   $\overline{AB} \cong \overline{AC}$, so $\triangle ABC$ is isosceles.

Guided Practice: Example 2, *continued*

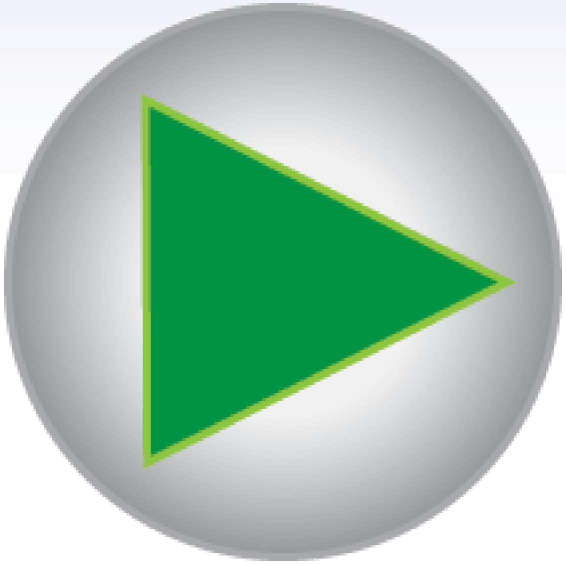
3. Identify congruent angles.

If two sides of a triangle are congruent, then the angles opposite the sides are congruent.

$\angle B \cong \angle C$

✔

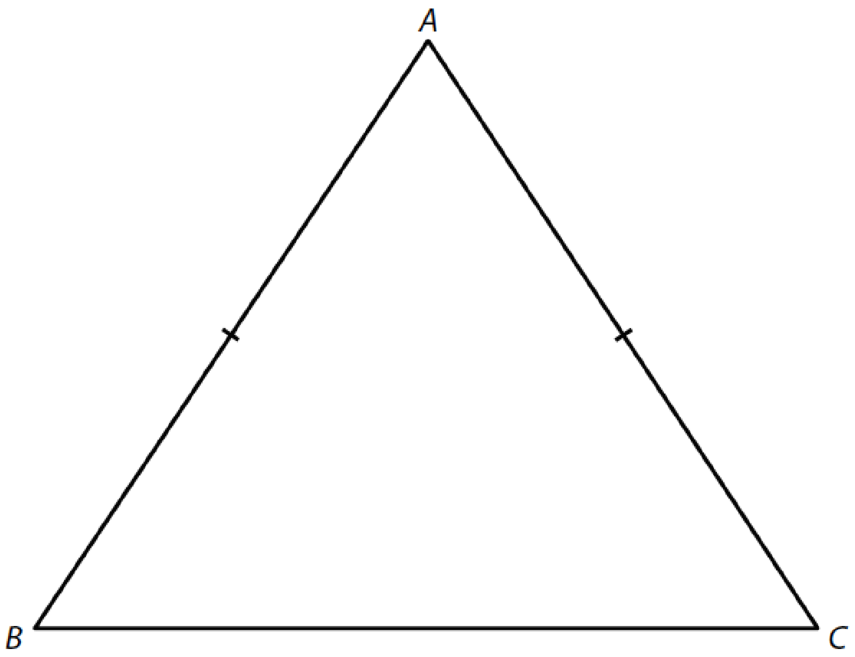
Guided Practice: Example 2, *continued*



Guided Practice

Example 3

Given $\overline{AB} \cong \overline{AC}$, prove that $\angle B \cong \angle C$.



Guided Practice: Example 3, *continued*

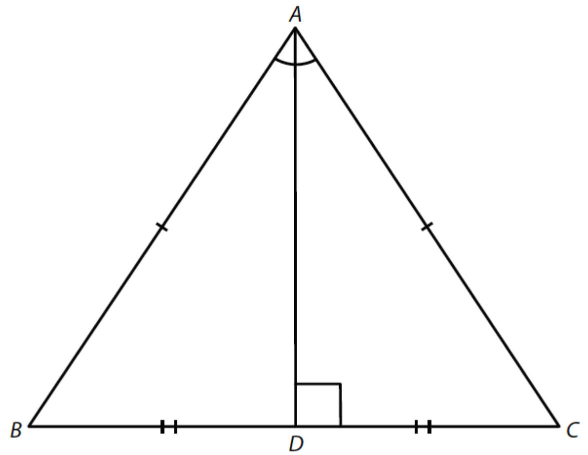
1. **State the given information.**

$$\overline{AB} \cong \overline{AC}$$

Guided Practice: Example 3, *continued*

2. **Draw the angle bisector of $\angle A$ and extend it to $\overline{BC}$, creating the perpendicular bisector of $\overline{BC}$. Label the point of intersection *D*.**

Indicate congruent sides.



$\angle B$ and $\angle C$ are congruent corresponding parts.

Guided Practice: Example 3, *continued*

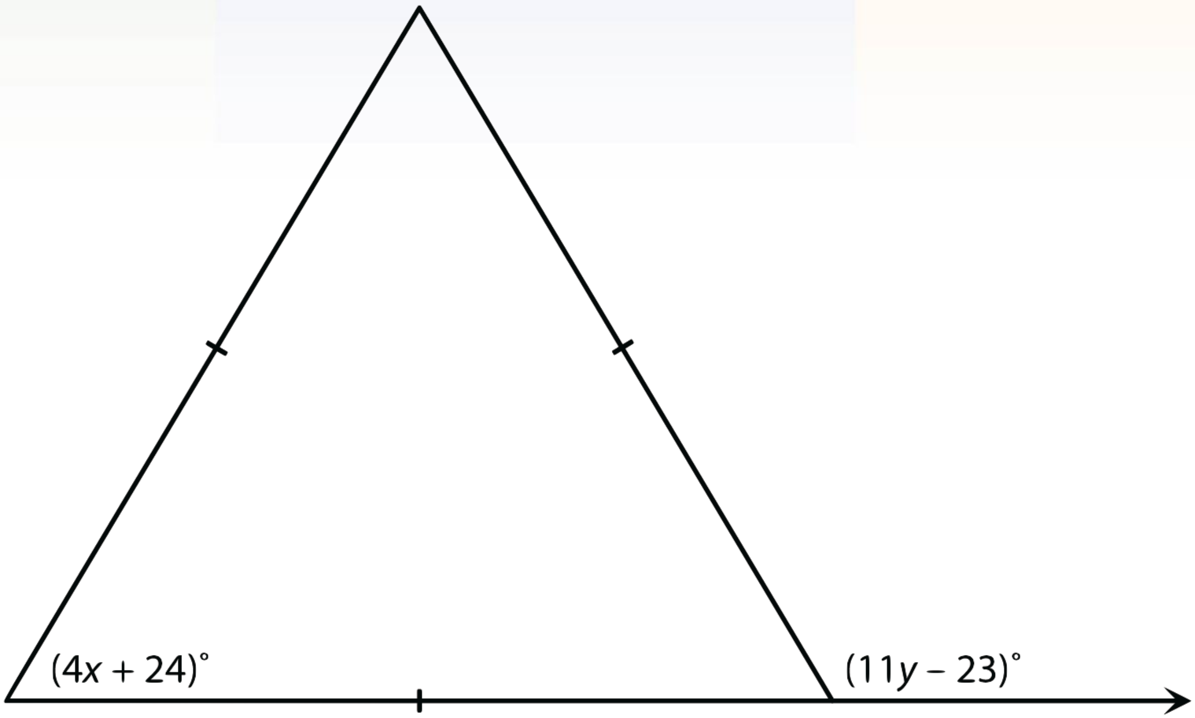
2. Write the information in a two-column proof.

Statements	Reasons
1. $\overline{AB} \cong \overline{AC}$	1. Given
2. Draw the angle bisector of $\angle A$ and extend it to $\overline{BC}$, creating a perpendicular bisector of $\overline{BC}$ and the midpoint of $\overline{BC}$.	2. There is exactly one line through two points.
3. $\overline{BD} \cong \overline{BC}$	3. Definition of midpoint
4. $\overline{AD} \cong \overline{AD}$	4. Reflexive Property
5. $\triangle ABD \cong \triangle ACD$	5. SSS Congruence Statement
6. $\angle B \cong \angle C$	6. Corresponding Parts of Congruent Triangles are Congruent

Guided Practice

Example 4

Find the values of *x* and *y*.



Guided Practice: Example 4, *continued*

1. Make observations about the figure.

- The triangle in the diagram has three congruent sides.
- A triangle with three congruent sides is equilateral.
- Equilateral triangles are also equiangular.

Guided Practice: Example 4, *continued*

1. Make observations about the figure.

- The measure of each angle of an equilateral triangle is 60º.
- An exterior angle is also included in the diagram.
- The measure of an exterior angle is the supplement of the adjacent interior angle.

Guided Practice: Example 4, *continued*

2. Determine the value of *x*.

The measure of each angle of an equilateral triangle is 60°.

Create and solve an equation for x using this information.

Guided Practice: Example 4, *continued*

$4x + 24 = 60$	Equation
$4x = 36$	Subtract 24 from both sides.
$x = 9$	Divide both sides by 4.

The value of x is 9.

Guided Practice: Example 4, *continued*

3. Determine the value of *y*.

- The exterior angle is the supplement to the interior angle.
- The interior angle is 60º by the properties of equilateral triangles.
- The sum of the measures of an exterior angle and interior angle pair equals 180.
- Create and solve an equation for *y* using this information.

Guided Practice: Example 4, *continued*

$11y - 23 + 60 = 180$	Equation
$11y + 37 = 180$	Simplify.
$11y = 143$	Subtract 37 from both sides.
$y = 13$	Divide both sides by 11.

The value of *y* is 13.

✔

Guided Practice: Example 4, *continued*

Guided Practice

Example 5

$\triangle ABC$ is equilateral. Prove that it is equiangular.

Guided Practice: Example 5, *continued*

1. State the given information.

$\triangle ABC$ is an equilateral triangle.

Guided Practice: Example 5, *continued*

2. Plan the proof.

Equilateral triangles are also isosceles triangles.

Isosceles triangles have at least two congruent sides.

$\overline{AB} \cong \overline{BC}$

$\angle A$ and $\angle C$ are <u>base angles</u> in relation to $\overline{AB}$ and $\overline{BC}$.

$\angle A \cong \angle C$ because of the Isosceles Triangle Theorem.

Guided Practice: Example 5, *continued*

2. Plan the proof.

$\overline{BC} \cong \overline{AC}$

$\angle A$ and $\angle B$ are <u>base angles</u> in relation to $\overline{BC}$ and $\overline{AC}$.

$\angle A \cong \angle B$ because of the Isosceles Triangle Theorem.

By the Transitive Property, $\angle A \cong \angle B \cong \angle C$; so $\triangle ABC$ is equiangular.

Guided Practice: Example 5, *continued*

2. Write the information in a paragraph proof.

Since $\triangle ABC$ is equilateral, $\overline{AB} \cong \overline{BC}$ and $\overline{BC} \cong \overline{AC}$.

By the Isosceles Triangle Theorem, $\angle A \cong \angle C$ and $\angle A \cong \angle B$.

By the Transitive Property, $\angle A \cong \angle B \cong \angle C$; therefore $\triangle ABC$ is equiangular.