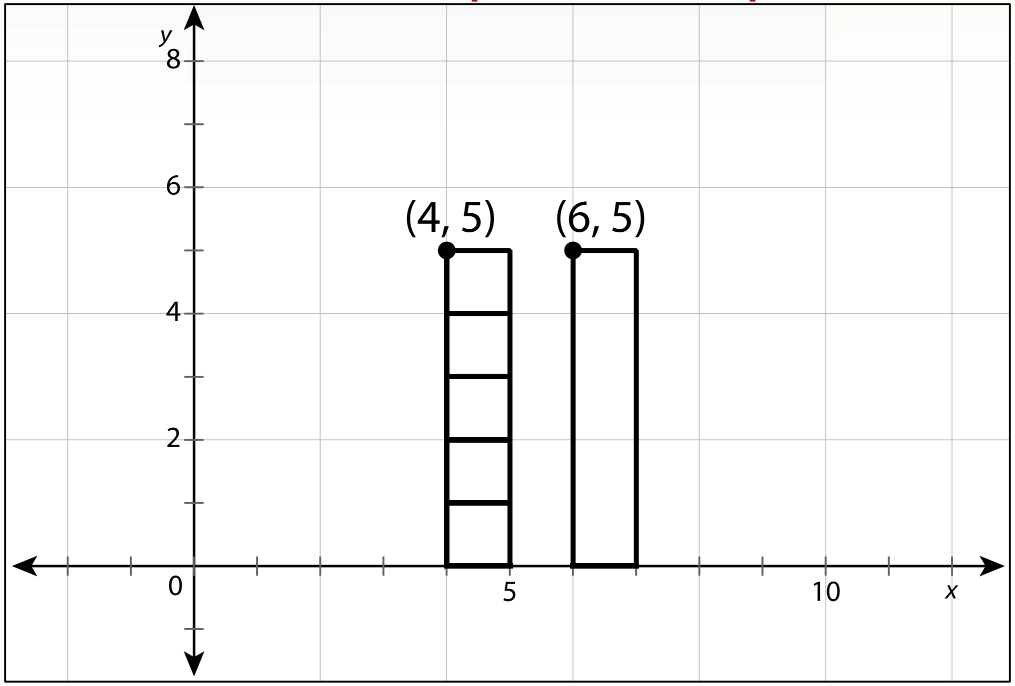
Interpreting Various Forms of Quadratic Functions

Warm-Up

Joe is trying to use a hose to spray water over a stack of moving boxes and into a bird bath. The graph represents a stack of boxes on the left that the water must clear and a birdbath on the right that the water must fill. The water will follow a parabolic path.



1. One possible path the water could travel is given by $y = -\frac{1}{5}x^2 + \frac{8}{5}x + \frac{9}{5}$, where *y* represents the height in feet and *x* represents the horizontal distance traveled in feet. What is the vertex of this parabola?
2. Determine the second *x*-intercept if one *x*-intercept of the path of the water is –1.
3. What is the maximum value of the quadratic function?
4. Sketch the graph of the path of the water.
5. Based on the graph, will the water clear the boxes? If it clears the boxes, will the water fill the birdbath?

1. **One possible path the water could travel is given by $y = -\frac{1}{5}x^2 + \frac{8}{5}x + \frac{9}{5}$, where *y* represents the height in feet and *x* represents the horizontal distance traveled in feet. What is the vertex of this parabola?**

- The vertex is of the form $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.
- Use the original equation to determine the values of *a* and *b* in order to find the *x*-coordinate of the vertex.

1. What is the vertex of this parabola?

$$x = \frac{-b}{2a}$$

Formula to find the *x*-coordinate of the vertex of a parabola

$$x = \frac{-\left(\frac{8}{5}\right)}{2\left(-\frac{1}{5}\right)}$$

Substitute $-\frac{1}{5}$ for *a* and $\frac{8}{5}$ for *b*.

$$x = 4$$

Simplify.

The *x*-coordinate of the vertex is 4.

1. What is the vertex of this parabola?

- Substitute 4 into the original equation to find the *y*-coordinate of the vertex.

$y=-\frac{1}{5}x^2+\frac{8}{5}x+\frac{9}{5}$	Original equation
$y=-\frac{1}{5}(4)^2+\frac{8}{5}(4)+\frac{9}{5}$	Substitute 4 for *x*.
$y=5$	Simplify.

- The *y*-coordinate of the vertex is 5.
- The vertex is located at (4, 5).

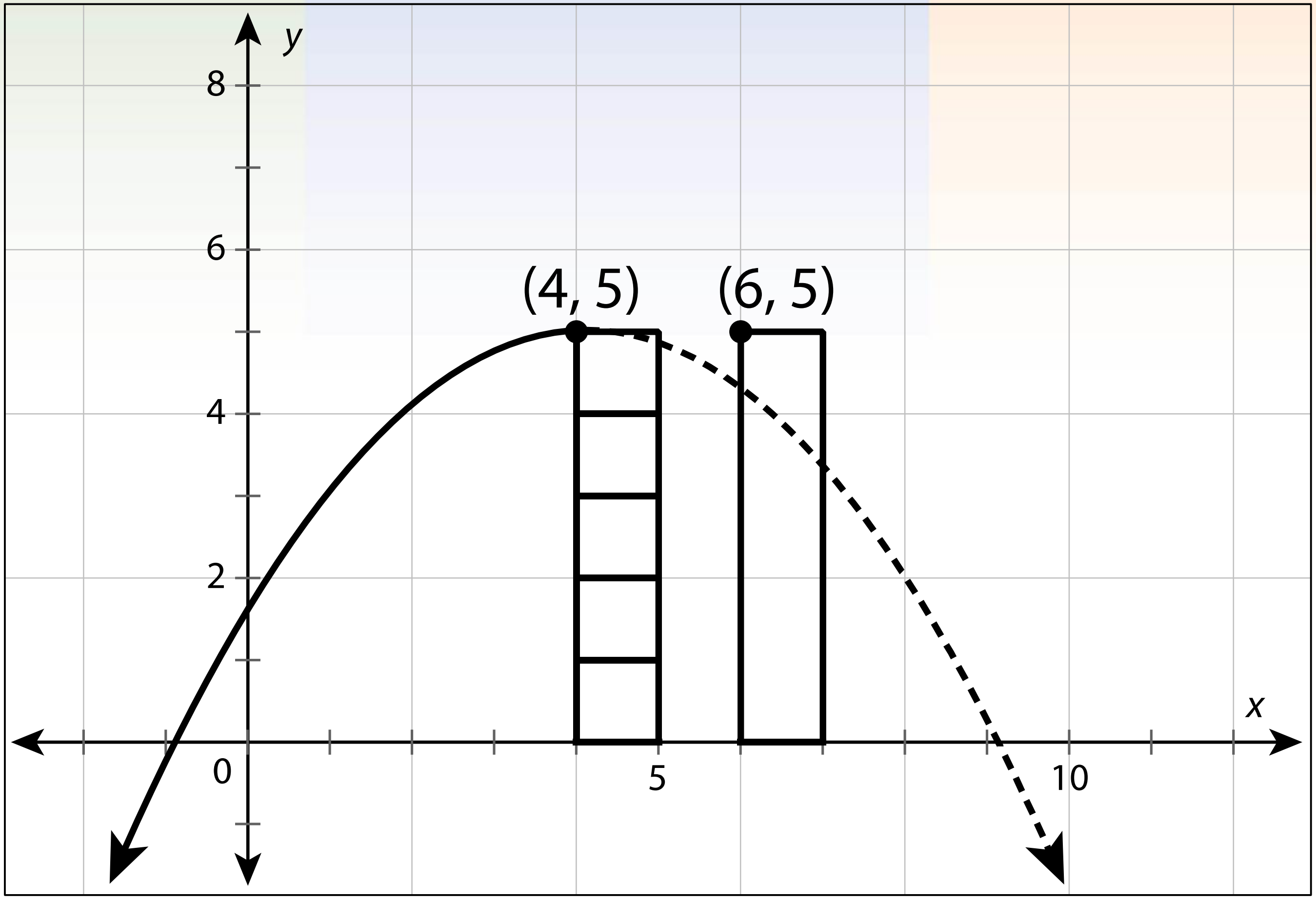
2. Determine the second *x*-intercept if one *x*-intercept of the path of the water is –1.

- Use symmetry to identify the second *x*-intercept.
- The axis of symmetry goes through the vertex, so the axis of symmetry is $x = 4$.
- For each point to the left of the axis of symmetry, there is another point the same distance on the right side of the axis.
- The point (–1, 0) is on the graph; $x = -1$ is 5 units to the left of the axis of symmetry.
- The point that is 5 units to the right of the axis is 9, so the point (9, 0) is also on the graph. The other *x*-intercept is 9.

3. **What is the maximum value of the quadratic function?**

- The maximum value is the *y*-coordinate of the vertex, 5.

4. Sketch the graph of the path of the water.



5. Based on the graph, will the water clear the boxes? If it clears the boxes, will the water fill the birdbath?

- No, the water will not clear the boxes.
- If the boxes were not in the way, the water would still not fill the birdbath because the birdbath is too tall.

Instruction

Introduction

- Quadratic equations can be written in several forms, including standard form, vertex form, and factored form.
- While each form is equivalent, certain forms easily reveal different features of the graph of the quadratic function.
- In this lesson, you will learn to use the various forms of quadratic functions to show the key features of the graph and determine how these key features relate to the characteristics of a real-world situation.

Key Concepts

Standard Form

- Recall that the standard form, or general form, of a quadratic function is written as $f(x) = ax^2 + bx + c$, where a is the coefficient of the quadratic term, b is the coefficient of the linear term, and c is the constant term.
- When a function is written in standard form, the y-intercept is the value of c.

Key Concepts, *continued*

- The vertex of the function can be found by first determining the value of x, $x = \frac{-b}{2a}$, and then finding the corresponding y-value, $y = f\left(\frac{-b}{2a}\right)$.
- The vertex is often written as $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

Key Concepts, *continued*

- If $a > 0$, the function has a minimum. The minimum is the value of k, the y-coordinate of the vertex.
- If $a < 0$, the function has a maximum. The maximum is the value of k, the y-coordinate of the vertex.

Key Concepts, *continued*

Vertex Form

- The vertex form of a quadratic function is written as $f(x) = a(x - h)^2 + k$.
- In vertex form, the maximum or minimum of the function is identified using the vertex of the parabola, the point (h, k).
- If $a > 0$, the function has a minimum, where k is the y-coordinate of the minimum and h is the x-coordinate of the minimum.

Key Concepts, *continued*

- If $a < 0$, the function has a maximum, where k is the y-coordinate of the maximum and h is the x-coordinate of the maximum.
- Because the axis of symmetry goes through the vertex, the axis of symmetry can be identified from vertex form as $x = h$.
- The graph of a quadratic function is symmetric about the axis of symmetry.

Key Concepts, *continued*

Factored Form

- The factored form, or intercept form, of a quadratic function is written as $f(x) = a(x - p)(x - q)$.
- Recall that the x-intercepts of a function are the x-values when $y = 0$.
- In factored form, the x-intercepts of the function are identified as p and q.
- Recall that the y-intercept of a function is the point at which the function intersects the y-axis.

Key Concepts, *continued*

- To determine the *y*-intercept, substitute 0 for *x* and simplify.
- The axis of symmetry can be identified from the factored form since it passes through the midpoint between the *x*-intercepts. Therefore, the axis of symmetry is $x = \frac{p+q}{2}$.
- To determine the vertex of the parabola, calculate the *y*-value that corresponds to the *x*-value of the axis of symmetry.

Key Concepts, *continued*

- If $a > 0$, the function has a minimum and the graph opens up.
- If $a < 0$, the function has a maximum and the graph opens down.

Common Errors/Misconceptions

- confusing the attributes of different forms
- incorrectly identifying x-intercepts of the factored form
- incorrectly identifying the vertex as a maximum or minimum

Guided Practice

Example 1

Suppose that the flight of a launched bottle rocket can be modeled by the function $f(x) = -(x - 1)(x - 6)$, where $f(x)$ measures the height above the ground in meters and x represents the horizontal distance in meters from the launching spot at the point (1, 0).

- How far does the bottle rocket travel in the horizontal direction from launch to landing?
- What is the maximum height the bottle rocket reaches?
- How far has the bottle rocket traveled horizontally when it reaches its maximum height? Graph the function.

Guided Practice: Example 1, *continued*

1. Identify the *x*-intercepts of the function.

- In the function, $f(x)$ represents the height of the bottle rocket. At launch and landing, the height of the bottle rocket is 0.

- The function $f(x) = -(x - 1)(x - 6)$ is of the form $f(x) = a(x - p)(x - q)$, where p and q are the *x*-intercepts.

- The *x*-intercepts of the function are 1 and 6.

Guided Practice: Example 1, *continued*

1. Identify the *x*-intercepts of the function.

- Find the distance between the two points to determine how far the bottle rocket traveled in the horizontal direction.

  $6 - 1 = 5$

- The bottle rocket traveled 5 meters in the horizontal direction from launch to landing.

Guided Practice: Example 1, *continued*

2. **Determine the maximum height of the bottle rocket.**

- The maximum height occurs at the vertex.
- Find the *x*-coordinate of the vertex using the formula $x = \frac{p+q}{2}$.

Guided Practice: Example 1, *continued*

2. Determine the maximum height of the bottle rocket.

$x = \dfrac{p+q}{2}$	Formula to determine the x-coordinate of the vertex
$x = \dfrac{6+1}{2}$	Substitute 6 for p and 1 for q.
$x = 3.5$	Simplify.

The x-coordinate of the vertex is $x = 3.5$.

Guided Practice: Example 1, *continued*

2. Determine the maximum height of the bottle rocket.

- Use this value to determine the vertex.

$f(x) = -(x - 1)(x - 6)$	Original function
$f(3.5) = -[(3.5) - 1][(3.5) - 6]$	Substitute 3.5 for x.
$f(3.5) = -(2.5)(-2.5)$	Simplify.
$f(3.5) = 6.25$	Multiply.

- The y-coordinate of the vertex is 6.25.
- The maximum height reached by the bottle rocket is 6.25 meters.

Guided Practice: Example 1, *continued*

3. Determine the horizontal distance from the launch point to the maximum height of the bottle rocket.

- We know that the bottle rocket is launched from the point (1, 0) and reaches a maximum height at (3.5, 6.25).

- Subtract the *x*-value of the two points to find the distance traveled horizontally.

$$3.5 – 1 = 2.5$$

Guided Practice: Example 1, *continued*

3. Determine the horizontal distance from the launch point to the maximum height of the bottle rocket.

Another method is to take the total distance traveled horizontally from launch to landing and divide by 2. The maximum value occurs halfway between the *x*-intercepts of the function:

$$\frac{5}{2} = 2.5.$$

The bottle rocket travels 2.5 meters horizontally when it reaches its maximum.

Guided Practice: Example 1, *continued*

4. Graph the function.

Use a graphing calculator or complete a table of values. Use the *x*-intercepts and vertex as three of the known points. Choose *x*-values on either side of the vertex for two additional *x*-values.

x	*y*
1	0
2	
3.5	6.25
5	
6	0

To determine the *y*-coordinates of the additional points, substitute each *x*-value into the original function and evaluate.

Guided Practice: Example 1, *continued*

4. Graph the function.

$f(x) = –(x – 1)(x – 6)$	Original function
$f(2) = –[(2) – 1][(2) – 6]$	Substitute 2 for x.
$f(2) = –(1)(–4)$	Simplify.
$f(2) = 4$	Multiply.

$f(x) = –(x – 1)(x – 6)$	Original function
$f(5) = –[(5) – 1][(5) – 6]$	Substitute 5 for x.
$f(5) = –(4)(–1)$	Simplify.
$f(5) = 4$	Multiply.

Guided Practice: Example 1, *continued*

4. Graph the function.

Fill in the missing table values.

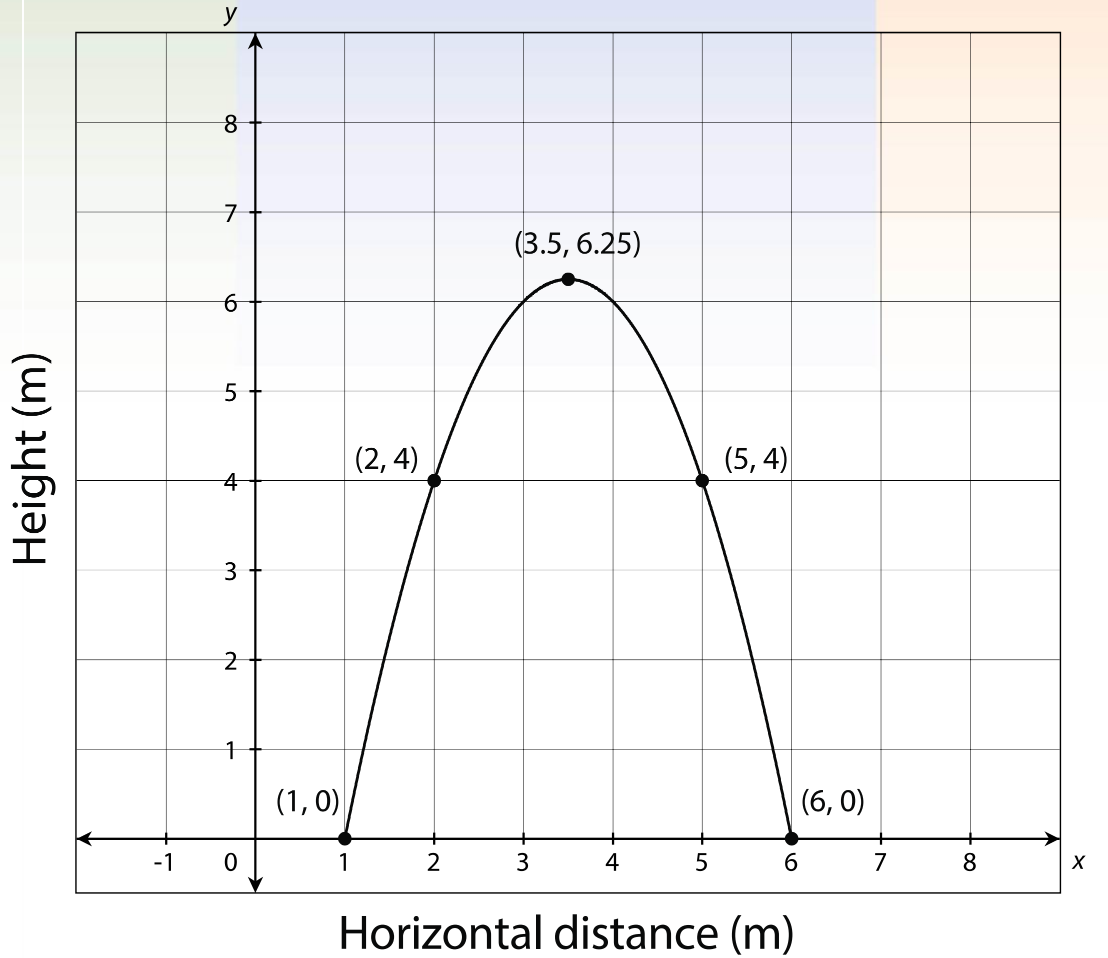
x	*y*
1	0
2	**4**
3.5	6.25
5	**4**
6	0

Guided Practice: Example 1, *continued*

4. Graph the function.

- Plot the points on a coordinate plane and connect them using a smooth curve.
- Since the function models the flight of a bottle rocket, it is important to only show the portion of the graph where both horizontal distance and height are positive.

Guided Practice: Example 1, *continued*



Guided Practice: Example 1, *continued*

Guided Practice

Example 2

Reducing the cost of an item can result in a greater number of sales. The revenue function that predicts the revenue in dollars, $R(x)$, for each \$1 change in price, x, for a particular item is $R(x) = -100(x - 7)^2 + 28{,}900$.

- What is the maximum value of the function?
- What does the maximum value mean in the context of the problem?
- What price increase maximizes the revenue and what does it mean in the context of the problem? Graph the function.

Guided Practice: Example 2, *continued*

1. Determine the maximum value of the function.

- The function $R(x) = –100(x – 7)^2 + 28,900$ is written in vertex form, $f(x) = a(x – h)^2 + k$, where (h, k) is the vertex.
- The vertex of the function is $(7, 28,900)$; therefore, the maximum value is 28,900.

Guided Practice: Example 2, *continued*

2. Determine what the maximum value means in the context of the problem.

- The maximum value of 28,900 means that the maximum revenue resulting from increasing the price by *x* dollars is $28,900.

Guided Practice: Example 2, *continued*

3. Determine the price increase that will maximize the revenue and what it means in the context of the problem.

- The maximum value occurs at the vertex, (7, 28,900).

- This means an increase in price of $7 will result in the maximum revenue.

Guided Practice: Example 2, *continued*

4. Graph the function.

Use a graphing calculator or complete a table of coordinates. Use the vertex as one known point. Choose *x*-values on either side of the vertex to have four additional *x*-values.

x	*y*
0	
5	
7	28,900
9	
14	

To determine the *y*-coordinates of the additional points, substitute each *x*-value into the original function and solve.

Guided Practice: Example 2, *continued*

4. Graph the function.

$R(x) = –100(x – 7)^2 + 28,900$	Original function
$R(0) = –100[(0) – 7]^2 + 28,900$	Substitute 0 for *x*.
$R(0) = –4,900 + 28,900$	Simplify.
$R(0) = 24,000$	Add.

$R(x) = –100(x – 7)^2 + 28,900$	Original function
$R(5) = –100[(5) – 7]^2 + 28,900$	Substitute 5 for *x*.
$R(5) = –400 + 28,900$	Simplify.
$R(5) = 28,500$	Add.

Guided Practice: Example 2, *continued*

4. Graph the function.

$R(x) = -100(x - 7)^2 + 28{,}900$	Original function
$R(9) = -100[(9) - 7]^2 + 28{,}900$	Substitute 9 for x.
$R(9) = -400 + 28{,}900$	Simplify.
$R(9) = 28{,}500$	Add.

$R(x) = -100(x - 7)^2 + 28{,}900$	Original function
$R(14) = -100[(14) - 7]^2 + 28{,}900$	Substitute 14 for x.
$R(14) = -4{,}900 + 28{,}900$	Simplify.
$R(14) = 24{,}000$	Add.

Guided Practice: Example 2, *continued*

4. Graph the function.

Fill in the missing table values.

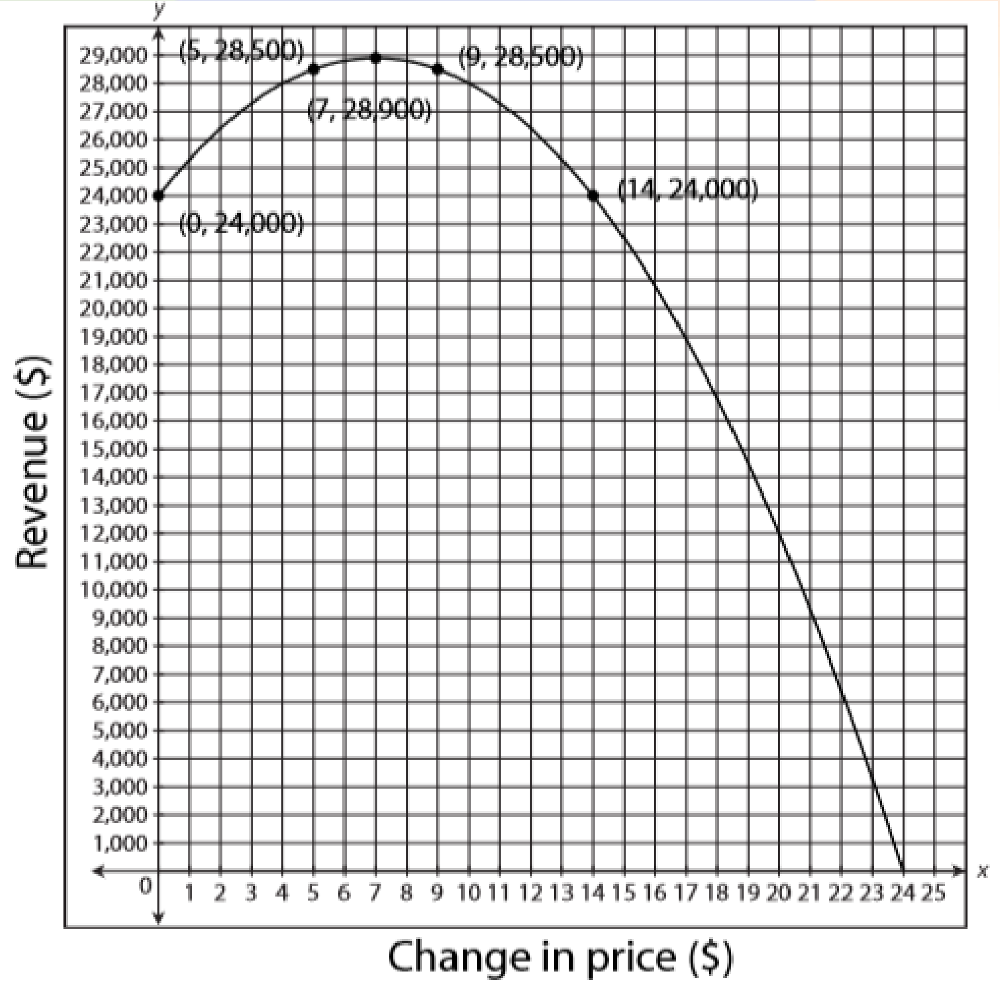
x	*y*
0	**24,000**
5	**28,500**
7	28,900
9	**28,500**
14	**24,000**

Guided Practice: Example 2, *continued*

4. Graph the function.

- Notice that the points (0, 24,000) and (14, 24,000) are the same horizontal distance from the vertex on either side. The same is true for (5, 28,500) and (9, 28,500).
- Plot the points on a coordinate plane and connect using a smooth curve.

Guided Practice: Example 2, *continued*



Guided Practice

Example 3

A football is kicked and follows a path given by $f(x) = -0.03x^2 + 1.8x$, where $f(x)$ represents the height of the ball in feet and x represents the horizontal distance in feet.

- What is the maximum height the ball reaches?
- What horizontal distance maximizes the height?
- Graph the function.

Guided Practice: Example 3, *continued*

1. Determine the maximum height of the ball.

- The function $f(x) = -0.03x^2 + 1.8x$ is written in standard form, $f(x) = ax^2 + bx + c$, where $a = -0.03$, $b = 1.8$, and $c = 0$.

- The maximum occurs at the vertex, $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

Guided Practice: Example 3, *continued*

1. Determine the maximum height of the ball.

- Determine the *x*-coordinate of the vertex.

$x = \frac{-b}{2a}$	Formula to find the *x*-coordinate for the vertex of a parabola
$x = \frac{-(1.8)}{2(-0.03)}$	Substitute values for *a* and *b*.
$x = 30$	Simplify.

Guided Practice: Example 3, *continued*

1. Determine the maximum height of the ball.

- Determine the *y*-coordinate of the vertex.

$f(x) = –0.03x^2 + 1.8x$	Original function
$f(30) = –0.03(30)^2 + 1.8(30)$	Substitute 30 for *x*.
$f(30) = 27$	Simplify.

- The vertex is (30, 27), so the maximum value is 27 feet.
- The maximum height the ball reaches is 27 feet.

Guided Practice: Example 3, *continued*

2. Determine the horizontal distance of the ball when it reaches its maximum height.

- This horizontal distance is determined by the *x*-coordinate of the vertex.
- The vertex is (30, 27).
- The ball will have traveled 30 feet in the horizontal direction when it reaches its maximum height.

Guided Practice: Example 3, *continued*

3. Graph the function.

Use a graphing calculator or complete a table of coordinates. Use the vertex as one known point. Choose *x*-values on either side of the vertex to have four additional *x*-values.

x	*y*
5	
20	
30	27
40	
55	

To determine the *y*-coordinates of the additional points, substitute each *x*-value into the original function and solve.

Guided Practice: Example 3, *continued*

3. Graph the function.

$f(x) = –0.03x^2 + 1.8x$	Original function
$f(5) = –0.03(5)^2 + 1.8(5)$	Substitute 5 for x.
$f(5) = 8.25$	Simplify.

$f(x) = –0.03x^2 + 1.8x$	Original function
$f(20) = –0.03(20)^2 + 1.8(20)$	Substitute 20 for x.
$f(20) = 24$	Simplify.

Guided Practice: Example 3, *continued*

3. Graph the function.

$f(x) = -0.03x^2 + 1.8x$	Original function
$f(40) = -0.03(40)^2 + 1.8(40)$	Substitute 40 for x.
$f(40) = 24$	Simplify.

$f(x) = -0.03x^2 + 1.8x$	Original function
$f(55) = -0.03(55)^2 + 1.8(55)$	Substitute 55 for x.
$f(55) = 8.25$	Simplify.

Guided Practice: Example 3, *continued*

- Fill in the missing table values.

x	*y*
5	**8.25**
20	**24**
30	27
40	**24**
55	**8.25**

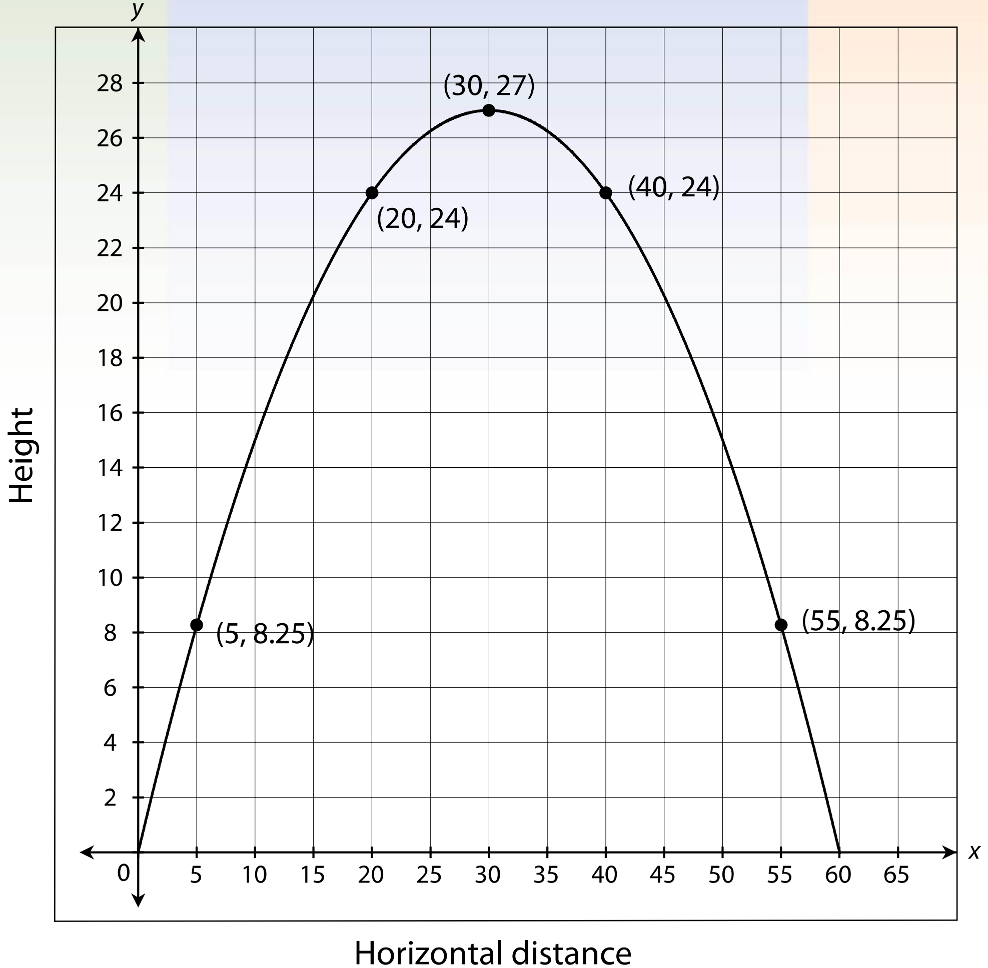
- Notice that the points (5, 8.25) and (55, 8.25) are the same horizontal distance from the vertex on either side. The same is true for (20, 24) and (40, 24).

Guided Practice: Example 3, *continued*

3. Graph the function.

- Notice that the points (5, 8.25) and (55, 8.25) are the same horizontal distance from the vertex on either side. The same is true for (20, 24) and (40, 24).
- Plot the points on a coordinate plane and connect them using a smooth curve.
- Since the function models the path of a kicked football, it is important to only show the portion of the graph where both height and horizontal distance are positive.

Guided Practice: Example 3, *continued*



Guided Practice: Example 3, *continued*