Interpreting Logarithmic Models

Warm-Up

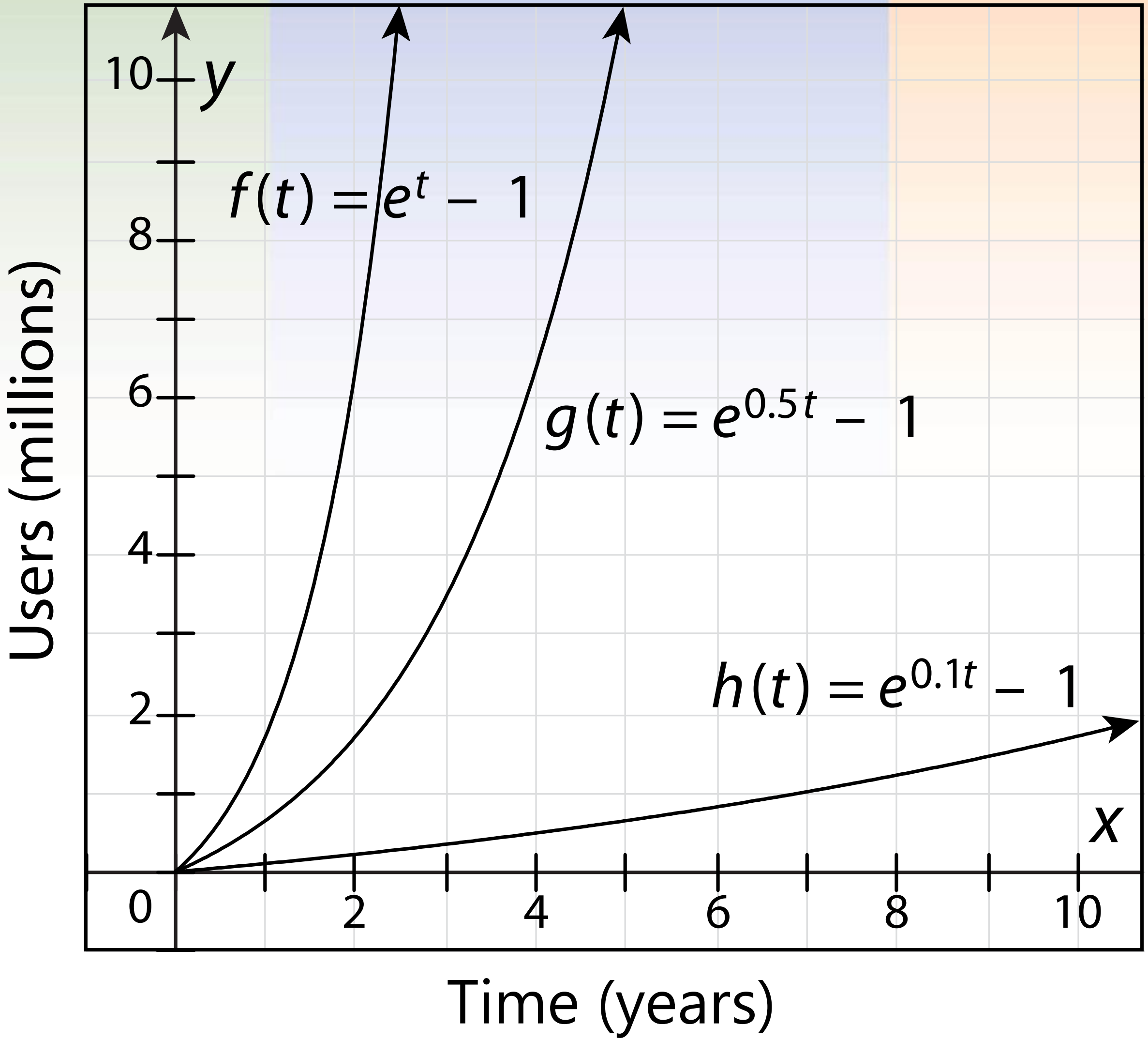
The number of people using social media, smartphones, and tablet computers has grown exponentially over the past 5 years.

The following exponential functions model the growth in the number of users, in millions, for three examples of these technologies, where t represents time in years:

- a particular social media website: $f(t) = e^{t} - 1$
- a popular smartphone device: $g(t) = e^{0.5t} - 1$
- a certain model of tablet computer: $h(t) = e^{0.1t} - 1$

Use the following graph of the exponential functions to complete the problems.

Assume that each technology was launched at the same time and thus the numbers of users are equal when $t = 0$.



y
10
8
6
4
2
0
2
4
6
8
10
x
f(t) = e^t – 1
g(t) = e^0.5t – 1
h(t) = e^0.1t – 1
Users (millions)
Time (years)

1. How does the growth in the number of users for each technology correlate with the coefficient of the power in the exponential term of each function?
2. Write the inverse logarithmic function of each given exponential function.
3. How do the terms in the logarithmic functions compare in relation to the exponential growth of each technology?
4. How can the logarithmic functions describe the growth of each technology?

1. How does the growth in the number of users for each technology correlate with the coefficient of the power in the exponential term of each function?

- As the coefficient of the power of *e* decreases, the growth of the number of users decreases for any given year *t*.

For example, the social media website users outnumber the smartphone users after 3 years, because

$f(3) = e^{3} - 1$

and

$g(3) = e^{0.5(3)} - 1$

$= e^{1.5} - 1.$

$f(3) > g(3)$ because $e^{3} > e^{1.5}$

2. **Write the inverse logarithmic function of each given exponential function.**

- For $f(t)$,

  $f(t) = e^t - 1$, so $f(t) + 1 = e^t$, which means that $\ln[f(t) + 1] = t$.

- Switch the variables and change f to f^{-1} and t to n, where n is the number of users measured in millions:

  $$\ln(n + 1) = f^{-1}(n)$$

- For $g(t)$, $g(t) = e^{0.5t} – 1$, so $g(t) + 1 = e^{0.5t}$, which means that $\ln [g(t) + 1] = 0.5 \bullet t$.

- Switch the variables and change g to g^{-1} and t to n.

$$\ln (n + 1) = 0.5 \bullet g^{-1}(n)$$

Then, simplify.

$2 \cdot \ln (n + 1) = g^{-1}(n)$	Multiply both sides by 2.
$\ln (n + 1)^2 = g^{-1}(n)$	Apply the power rule.

For $h(t)$, $h(t) = e^{0.1t} - 1$, so $h(t) + 1 = e^{0.1t}$, which means that $\ln [h(t) + 1] = 0.1 \cdot t$.

Switch the variables and change h to h^{-1} and t to n.

$$\ln (n + 1) = 0.1 \cdot h^{-1}(n)$$

Then, simplify.

$10 \cdot \ln (n + 1) = h^{-1}(n)$	Multiply both sides by 10.
$\ln (n + 1)^{10} = h^{-1}(n)$	Apply the power rule.

3. How do the terms in the logarithmic functions compare in relation to the exponential growth of each technology?

- The slower the growth of users of a technology, the larger the argument of the logarithmic function becomes.

- For example, for the social media website, the argument of the natural logarithm is $t + 1$, but for the tablet, the argument is $(t + 1)^{10}$.

4. How can the logarithmic functions describe the growth of each technology?

- The graphs depict the changing number of users per year for each technology.
- The inverse function is the number of years needed to see an increase in growth of a fixed number of users.
- Therefore, the logarithmic model implies that it will take more years as indicated by the inverse functions to see the same amount of user growth for a technology.

- For example, if the number of years elapsed is $n = 2$, then $\ln(3) = f^{-1}(2)$ for the social media website and $\ln(3)^2 = g^{-1}(2)$ for the smartphone.
- $g^{-1}(2) > f^{-1}(2)$ because $\ln 9 > \ln 3$, which implies that it will take the smartphone about twice as many years as the social media website to realize the same increase in the number of users.

Instruction

Introduction

Expressing or solving logarithmic functions in terms of exponential function models is one technique for solving real-world problems.

One factor in determining which type of function is best in a given situation is how the solution to a problem affects a particular audience.

For example, environmental scientists may need to present a study of the acidity or alkalinity of a freshwater pond to citizens at a town hall meeting.

Introduction, *continued*

The citizens might best understand the results if the logarithm-based pH factor is used to describe the chemical condition of the pond rather than the actual concentration of hydronium or hydroxide ions in a sample of the pond water.

Introduction, *continued*

Such initial conditions as the upper and lower bound of a domain are essential to the viability of such models.

For example, time is nearly always considered to be a positive quantity that moves in an ever-increasing direction. (There are exceptions to this in some of the leading-edge fields of physics, such as cosmology, but such discussions are generally beyond the scope of a mathematics course at this level.)

Introduction, *continued*

The ability to move accurately between a function and its inverse is often important in solving real-world problems that employ logarithms.

Also, a thorough mastery of the basic rules of exponents and logarithms is essential for such problems.

Introduction, *continued*

Finally, be aware of the potential for a graphical or tabular presentation of a problem to aid in the application of logarithmic functions.

In fact, visual models based on real data often provide a more accurate picture of a problem than an algebraic model that does not reveal the restricted domain or range of the problem in the way that a graph or table does.

Key Concepts

- Logarithmic functions have a wide range of applications in real-world problems, such as in the fields of biology, chemistry, ecology, and engineering.
- Logarithmic functions often provide an alternative approach to the use of exponential functions, which might help to increase the understanding of a problem or its solution.

Key Concepts, *continued*

- The logarithmic function is the inverse of an exponential function and vice versa.
- Recall two basic steps in writing one as the other:
  - A function's inverse switches the values of the domain and range values.
  - For example, if an ordered pair of a function is (3, 5), then the ordered pair (5, 3) solves the inverse of the function.
  - To find the inverse of a function, replace the domain variable (often x) with the range variable ($f(x)$), and change the range variable to $f^{-1}(x)$.

Key Concepts, *continued*

- Apply the basic rules of exponentials and logarithms, described earlier in this lesson.
- The ability to identify the restricted domain and/or range over which a logarithmic function is defined can mean the difference between finding a solution to a problem and misinterpreting the application of that function to the particulars of the problem setting.
- This is especially true if a function and its inverse have roles to play in formulating and/or solving the problem.

Key Concepts, *continued*

- A graphing calculator can help with the identification of the restraints on real-world problems.
- The tables that are created when functions are graphed offer approximations of solutions that cannot be easily found using algebraic methods.
- Adjust the table and window settings as necessary to reflect the particulars of the problem.

Common Errors/Misconceptions

- neglecting to ensure that the real-world domain and/or range of a problem is compatible with the logarithmic function that models the problem
- incorrectly interpreting the representation of a logarithmic function in a problem
- when working with two or more functions, choosing the less-appropriate function to model a real-world problem and its solution
- misinterpreting a graph or table of a logarithmic function model of a real-world problem

Guided Practice

Example 1

The number of electric vehicles (E) sold in the United States within 3 years of their introduction to the market can be modeled by the logarithmic function $E(y) = 1.67 + 5.74 \cdot \ln y$, where $E(y)$ is the number of vehicles sold in thousands and y is the number of years after introduction to the market.

The number of hybrid vehicles (H) sold within 3 years of their introduction to the market can be modeled by the logarithmic function $H(y) = 0.78 + 2.4 \cdot \ln y$.

Guided Practice, *continued*

Example 1, continued

At what value of *y* was an equal number of each vehicle sold, and which type of vehicle had greater sales over the 3 introductory years? Explain your reasoning with references to the terms in the function and how they compare.

Guided Practice: Example 1, *continued*

1. Set $E(y) = H(y)$.

The two functions are equal at a value of y that satisfies the equation $E(y) = H(y)$.

We are given that $E(y) = 1.67 + 5.74 \cdot \ln y$ and $H(y) = 0.78 + 2.4 \cdot \ln y$.

Therefore, $1.67 + 5.74 \cdot \ln y = 0.78 + 2.4 \cdot \ln y$.

Guided Practice: Example 1, *continued*

2. Solve the resulting equation for ln *y*.

Isolate the logarithmic terms on one side of the equation and the constants on the other.

Guided Practice: Example 1, *continued*

$1.67 + 5.74 \bullet \ln y = 0.78 + 2.4 \bullet \ln y$	Set $E(y)$ equal to $H(y)$.
$5.74 \bullet \ln y = 0.78 + 2.4 \bullet \ln y - 1.67$	Subtract 1.67 from both sides.
$5.74 \bullet \ln y - 2.4 \bullet \ln y = 0.78 - 1.67$	Subtract $2.4 \bullet \ln y$ from both sides.
$\ln y\,(5.74 - 2.4) = -0.89$	Factor out $\ln y$ and simplify.
$3.34 \bullet \ln y = -0.89$	Continue to simplify.
$\ln y \approx -0.2665$	Calculate the result.

Guided Practice: Example 1, *continued*

3. Rewrite the resulting equation using an inverse function.

The exponential function with base e is the inverse of the natural logarithm function in step 2.

If $\ln y \approx -0.2665$, then $y \approx e^{-0.2665}$, which simplifies to $y \approx 0.766$.

Guided Practice: Example 1, *continued*

4. Interpret the resulting value of *y* in terms of the conditions of the original problem.

The result also has to be checked for reasonableness as a member of the domain of the real-world problem.

Since *y* represents time in years, 0.766 would be the equivalent of approximately three-quarters of a year.

This is the amount of time after vehicle introduction that the same number of electric vehicles and hybrid vehicles were sold.

Guided Practice: Example 1, *continued*

However, the sales data in the problem are based on 3 full years of sales, which implies a domain of [1, 3].

Parts of a year (e.g., 1.5 years, or a year and a half) are defined within this domain, but not for a period of time that falls outside of the domain, such as 0.75 year.

Guided Practice: Example 1, *continued*

5. **Check the value of *y* found in step 3 by substituting it back into the functions *E*(*y*) and *H*(*y*).**

Values should always be checked to verify that the result is accurate and satisfies the condition(s) of the problem.

Recall that the functions represent the number of vehicles sold in thousands.

Guided Practice: Example 1, *continued*

$$E(0.766) \approx 1.67 + 5.74 \cdot \ln 0.766$$

$$\approx 0.140, \text{ or about 140 vehicles}$$

$$H(0.766) \approx 0.78 + 2.4 \cdot \ln 0.766$$

$$\approx 0.140, \text{ or about 140 vehicles}$$

The results for each function are equal, so the calculated value of *y* checks out.

Guided Practice: Example 1, *continued*

6. **Show which vehicle type had greater sales over the 3-year period by comparing the characteristics of each function.**

   Look at the terms in each function and use the values of the domain to support your claim(s).

Guided Practice: Example 1, *continued*

The constant term of $H(t)$ is **less than** the constant term of $E(t)$, which means that $E(t)_{\text{constant term}}$ **>** $H(t)_{\text{constant term}}$.

The coefficient of the logarithmic term of $E(t)$ is also **greater than** the coefficient of the logarithmic term of $H(t)$, which means that $E(t)_{\text{logarithmic term}}$ **>** $H(t)_{\text{logarithmic term}}$.

Furthermore, the logarithmic terms themselves are equal and are only defined for values of y in the range $1 \le y \le 3$, which also indicates they are positive.

Guided Practice: Example 1, *continued*

In summary:

$$E(t)_{\text{constant term}} + E(t)_{\text{logarithmic term}} \cdot \ln y >$$
$$H(t)_{\text{constant term}} + H(t)_{\text{logarithmic term}} \cdot \ln y$$

As the result of step 5 shows, the two vehicle types had the same number of sales during a period of time of about 0.766 of a year that was not included in the functions' domain, namely, before the first sales year was over.

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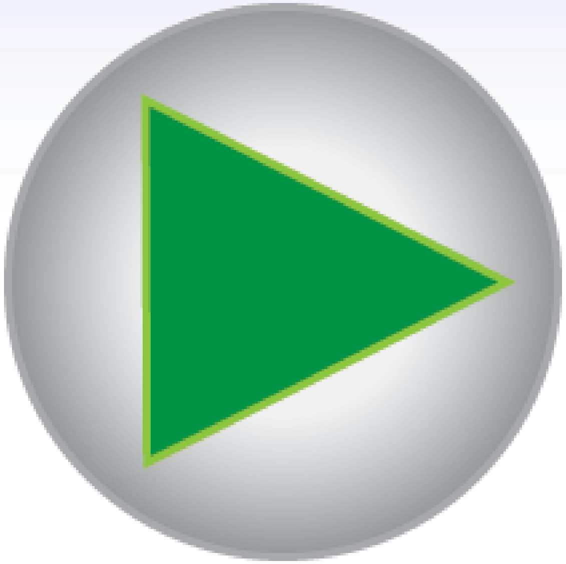
Guided Practice: Example 1, *continued*



Guided Practice

Example 2

A pyramid-shaped token for a board game consists of 4 congruent equilateral faces.

Each face is a different color: blue, green, red, or white. Each face is equally likely to end up on the bottom if the token is rolled on the game board.

What is the probability of the green face landing on the bottom?

Guided Practice, *continued*

Example 2

The probability of the same event *a* occurring *n* times is given by the function $f(n) = a^n$.

Write a function for the green face landing on the bottom *n* times in a row.

Then, write the inverse of the function, and explain what the inverse function describes in the context of this problem.

Guided Practice: Example 2, *continued*

1. State the number of faces on the token.

This is the first step in defining the probability event.

There are 4 faces on the token.

Guided Practice: Example 2, *continued*

2. State the probability of any of the 4 faces landing on the bottom.

Each face has a 1 in 4, or 0.25, chance of landing on the bottom.

Thus, the probability of the green face landing on the bottom is 0.25.

Guided Practice: Example 2, *continued*

3. Check your answer to step 2 by calculating the chance that *any* of the 4 faces lands on the bottom.

This answer should add up to 1.

The probability of a blue, green, red, or white face landing on the bottom is given by 0.25 + 0.25 + 0.25 + 0.25, which equals 1.

Guided Practice: Example 2, *continued*

4. **Write the function for the green face landing on the bottom *n* times in a row.**

Let $a = \frac{1}{4}$ in the function $f(n) = a^n$ and simplify.

$$f(n) = a^n$$

$$f(n) = \left(\frac{1}{4}\right)^n$$

$$f(n) = (4^{-1})^n$$

$$f(n) = 4^{-n}$$

Guided Practice: Example 2, *continued*

5. Write the inverse of the function determined in step 4.

This will give the logarithm requested in the problem.

The inverse of $f(n) = 4^{-n}$ is $\log_4 f(n) = -n$, which simplifies to $n = -\log_4 f(n)$.

Switch the variables and change $f(n)$ to $f^{-1}(n)$:

$$f^{-1}(n) = -\log_4 n = \log_4 n^{-1}$$

Therefore, the inverse function is $f^{-1}(n) = \log_4 n^{-1}$.

Guided Practice: Example 2, *continued*

6. Explain what the inverse function describes in the context of this problem.

Suppose $n = 4$.

The value of the function at $n = 4$ is given by the ordered pair $(4, 4^{-4})$, so the corresponding point for the inverse is $(4^{-4}, 4)$.

The inverse implies that when a probability of 4^{-4} occurs, the token has been rolled 4 times with the same outcome (4^{-1}) each time.

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Guided Practice: Example 2, *continued*

Guided Practice

Example 3

The owners of a West Virginia pine forest and grasslands preserve introduced a breeding pair of quail to a specific part of the property.

The population of quail can be modeled by an exponential function, but a statistics teacher at the local college came up with the logarithmic function $M(n) = 8.33 - \log{(50 - n)^4}$, which models how the number of quail n relates to the number of months M since the introduction of the first pair to the preserve.

Guided Practice

Example 3

What is the maximum number of quail (n_{max}) that can be estimated using this model?

Explain your answer, and state the domain for the function.

What does the constant 8.33 mean in terms of this function model? Write the inverse of the logarithmic function.

Guided Practice: Example 3, *continued*

1. **Name the type of number that has to be used for the variable *n*.**

The answer to this will also help in describing the domain.

The variable *n* refers to individual quail, so *n* has to be a positive whole number that is equal to at least 2, since the problem specifies a breeding pair.

Guided Practice: Example 3, *continued*

2. **State the value of *n* for which the logarithmic term of the function is undefined.**

This has to be considered since a logarithm is involved.

The argument of the logarithm, $50 - n$, cannot be less than or equal to 0.

Guided Practice: Example 3, *continued*

3. Use the results of steps 1 and 2 to list the domain.

The variable *n* is a positive whole number that is at least 2 but less than 50, so the domain is [2, 49].

Guided Practice: Example 3, *continued*

4. Name the maximum number of quail that can be estimated using the logarithmic model.

The maximum number of quail is 49, since that is the largest number for n in the domain.

Guided Practice: Example 3, *continued*

5. Determine the range of the function *M*(*n*).

M is the number of months, so *M*(*n*) should be computed using the end points of the domain.

Evaluate *M*(*n*) at the beginning of the domain values, 2.

$$M(2) = 8.33 - \log (50 - 2)^4$$

$$M(2) = 8.33 - \log(48)^4$$

$$M(2) \approx 1.6 \text{ months}$$

Guided Practice: Example 3, *continued*

Evaluate $M(n)$ at the end of the domain values, 49.

$$M(49) = 8.33 - \log(50 - 49)$$

$$M(49) = 8.33 - \log 1$$

$$M(49) = 8.33 \text{ months}$$

The time of 8.33 is the maximum time it will take for the quail population to "max out" at 49 birds.

Guided Practice: Example 3, *continued*

5. Write the inverse of the logarithmic function.

Simplify the logarithmic function using algebraic methods and the power rule of logarithms in order to write the inverse.

$$M(n) = 8.33 - \log (50 - n)^4$$

$$M(n) - 8.33 = -4 \bullet \log (50 - n)$$

$$8.33 - M(n) = 4 \bullet \log (50 - n)$$

$$\frac{8.33 - M(n)}{4} = \log(50 - n)$$

Guided Practice: Example 3, *continued*

Switch the variables, and change n to m and $M(n)$ to $N(m)$ to indicate that the inverse of the logarithmic function will use the number of months m as the independent variable and the variable $N(m)$ for the number of quail.

The inverse is $\frac{8.33 - n}{4} = \log\left[50 - M(n)\right]$ which can be written $50 - M(n) = 10^{\left(\frac{8.33 - m}{4}\right)}$ or $N(m) = 50 - 10^{2.0825 - 0.25m}$.