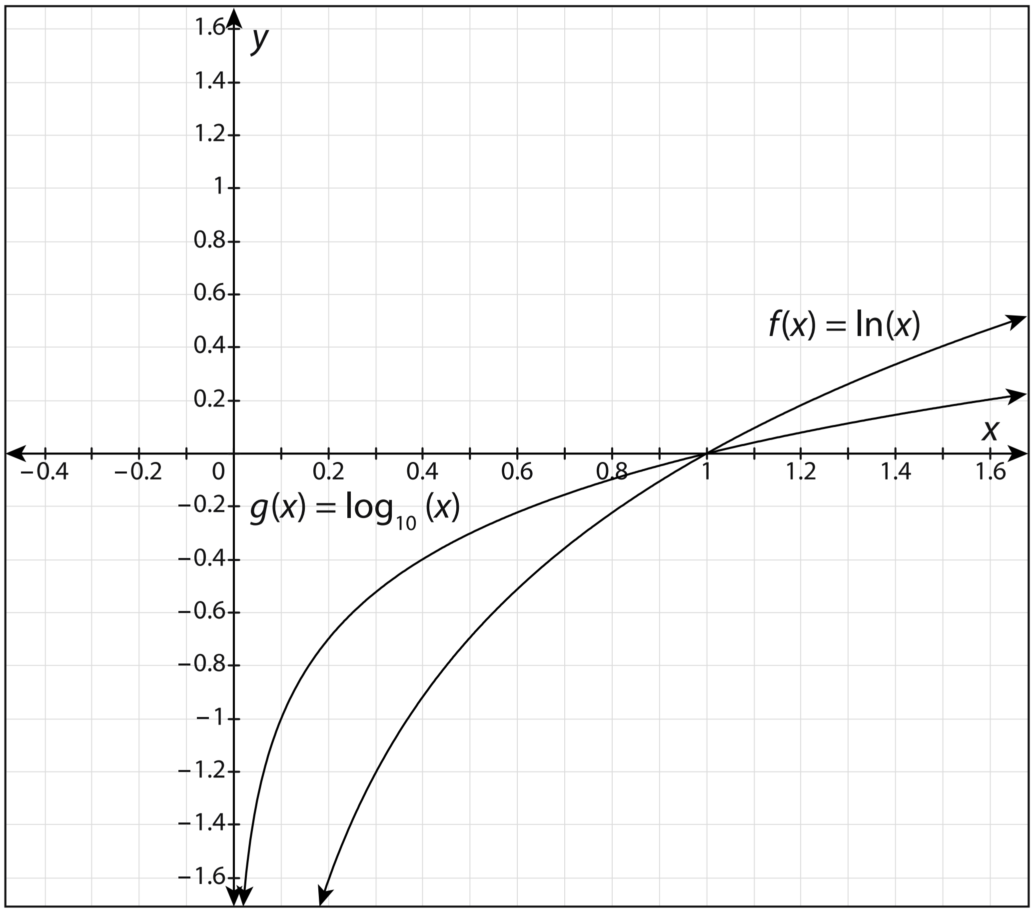
Graphing Logarithmic Functions

Warm-Up

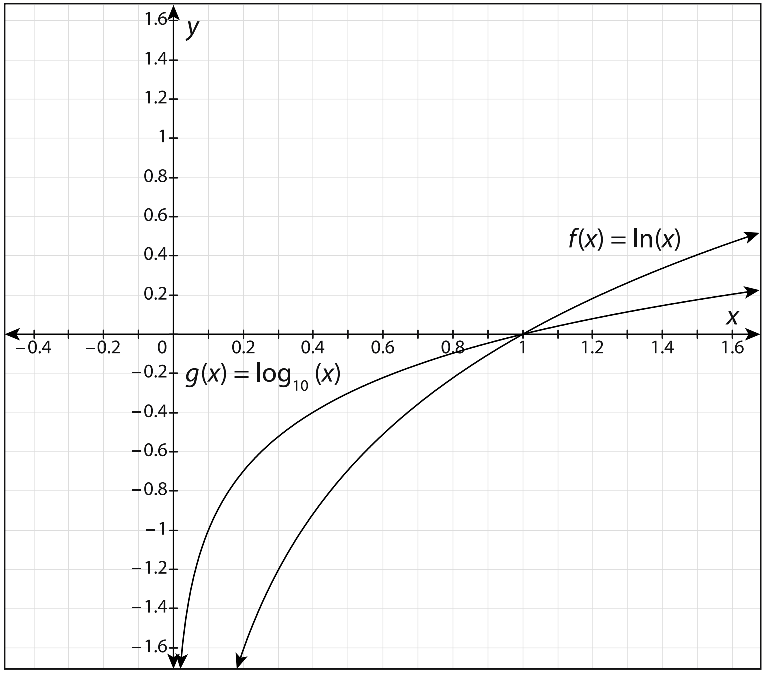


In preparation for the upcoming Mathlete Tournament, Mrs. Blake graphed two functions as shown. She asked her students to use this graph to compare the domain and range of each function.

1. Over which restricted domain are the values of $f(x) < g(x)$?
2. Over which restricted domain are the values of $f(x) > g(x)$?
3. At what domain value is $f(x) = g(x)$?
4. What are the ranges of $f(x)$ and $g(x)$?

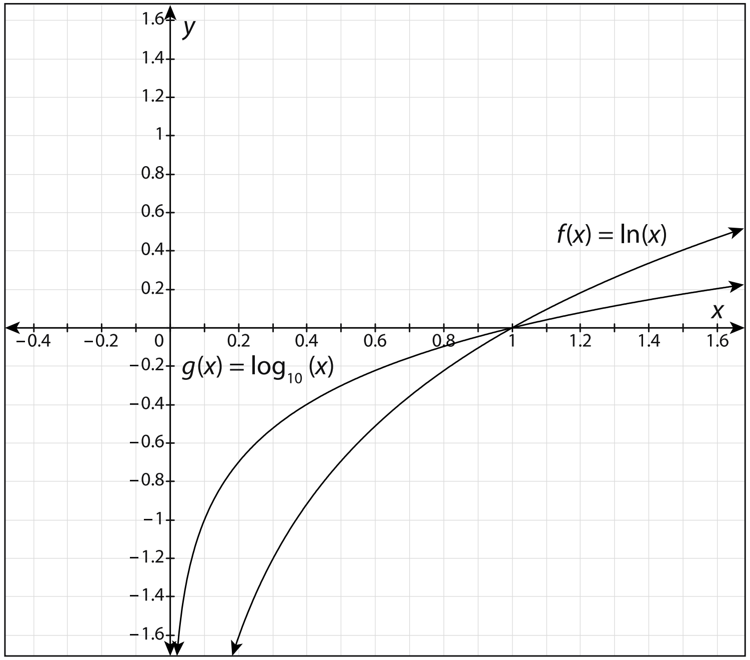
1. **Over which restricted domain are the values of *f*(*x*) < *g*(*x*)?**

As the graph shows, the restricted domain over which *f*(*x*) < *g*(*x*) is (0, 1).



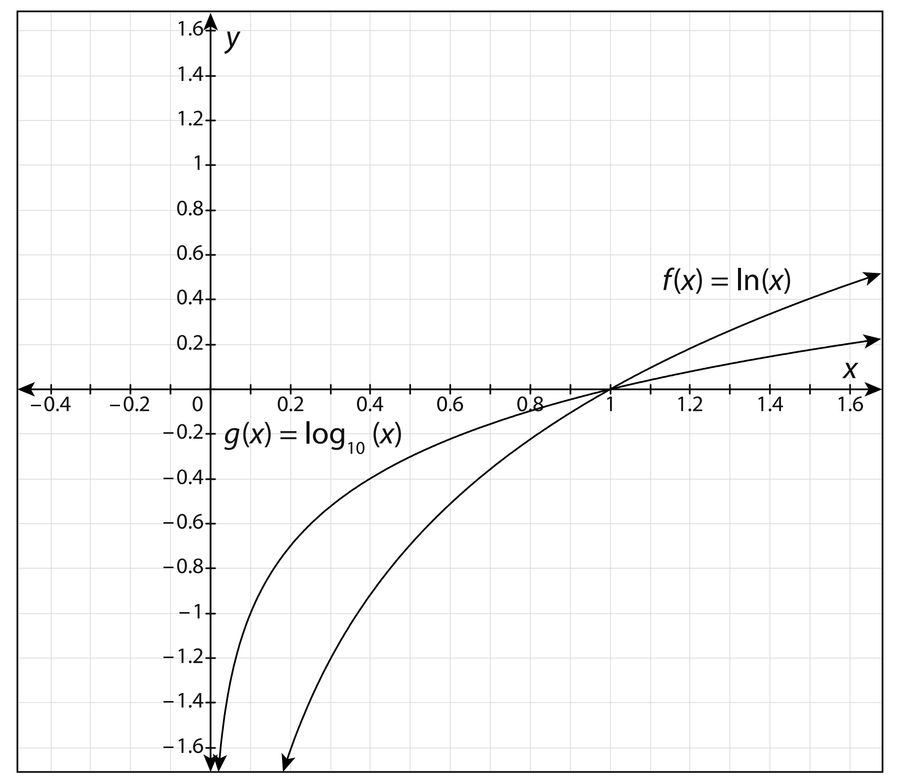
2. Over which restricted domain are the values of *f*(*x*) > *g*(*x*)?

As the graph shows, the restricted domain over which $f(x) > g(x)$ is $(1, \infty)$.



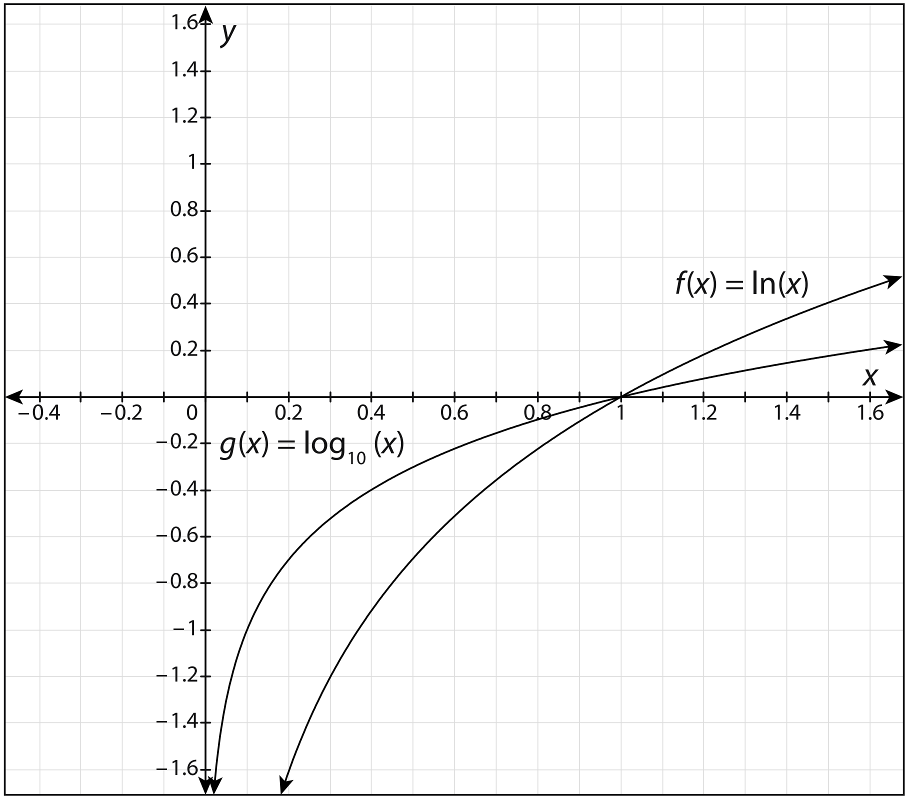
3. At what domain value is *f*(*x*) = *g*(*x*)?

As the graph shows, $f(x) = g(x)$ at $x = 1$.



4. What are the ranges of *f*(*x*) and *g*(*x*)?

Both functions have a range of $(-\infty, \infty)$.



Instruction

Introduction

The functions $f(x) = a \bullet \log_b c$, $g(x) = a + b \bullet \log_c d$, and $h(x) = a \bullet \log_d c$ are “families” of logarithmic functions.

The graphs of logarithmic functions exhibit patterns that can help in identifying these functions when the algebraic function is not present.

Key Concepts

- Logarithmic functions or the logarithmic terms within logarithmic functions are equal to 0 when their arguments equal 1.
- Recall that the argument is the result of raising the base of a logarithm to the power of the logarithm, so that b is the argument of the logarithm $\log_a b = c$.

Key Concepts, *continued*

- For example, let's look at $f(x) = 30 \cdot \ln x$ at $x = 1$:

$$f(1) = 30 \cdot \ln 1$$

$$= 30 \cdot 0$$

$$= 0$$

Key Concepts, *continued*

For another example, let's look at the function $g(x) = –250 + 6 \cdot \log_4 (x + 4)$.

Solve for $x = –3$:

$$g(–3) = –250 + 6 \cdot \log_4 [(–3) + 4]$$

$$g(–3) = –250 + 6 \cdot \log_4 (1)$$

$$g(–3) = –250 + 0 = –250$$

Notice that the argument for each example was equal to 1.

Key Concepts, *continued*

- The values of logarithmic functions or their logarithmic terms approach positive or negative infinity as the argument of the logarithmic term approaches 0.
- For example, in the function $f(x) = -\ln x$, $f(0.1)$ is about 2.3, but $f(0.001)$ is almost 7.
- If $x = 10^{-12}$, $f(x)$ is almost 28. The function value increases continuously as the value of x decreases.

Key Concepts, *continued*

- A graphing calculator will also show this function behavior.
- By looking at a variety of function values of interest (such as very small values of x), it is possible to see trends in the changing values of the function.
- Examine very small values of x by adjusting the axis scales or using the calculator's trace feature.
- Similarly, the value of logarithmic functions or their logarithmic terms as x becomes very large, positively or negatively, can be seen by substituting values for x or by using a calculator to calculate the values.

Key Concepts, *continued*

- For example, for the function $g(x) = \log x$, the domain is $(0, \infty)$.
- As x becomes very large, the value of $g(x)$ increases, too, but at a much slower rate.
- The value of $g(10^3) = 3$, but the value of $g(10^{200}) = 200$.
- If you suspect the function value of $g(x)$ has an upper bound, try to find the highest value of the function.

Key Concepts, *continued*

- Follow these basic rules to compare logarithmic functions.
- However, use caution when defining domains: positive, negative, or zero domain values can result in an undefined function, or change the ordering in a comparison of function values.

Key Concepts, *continued*

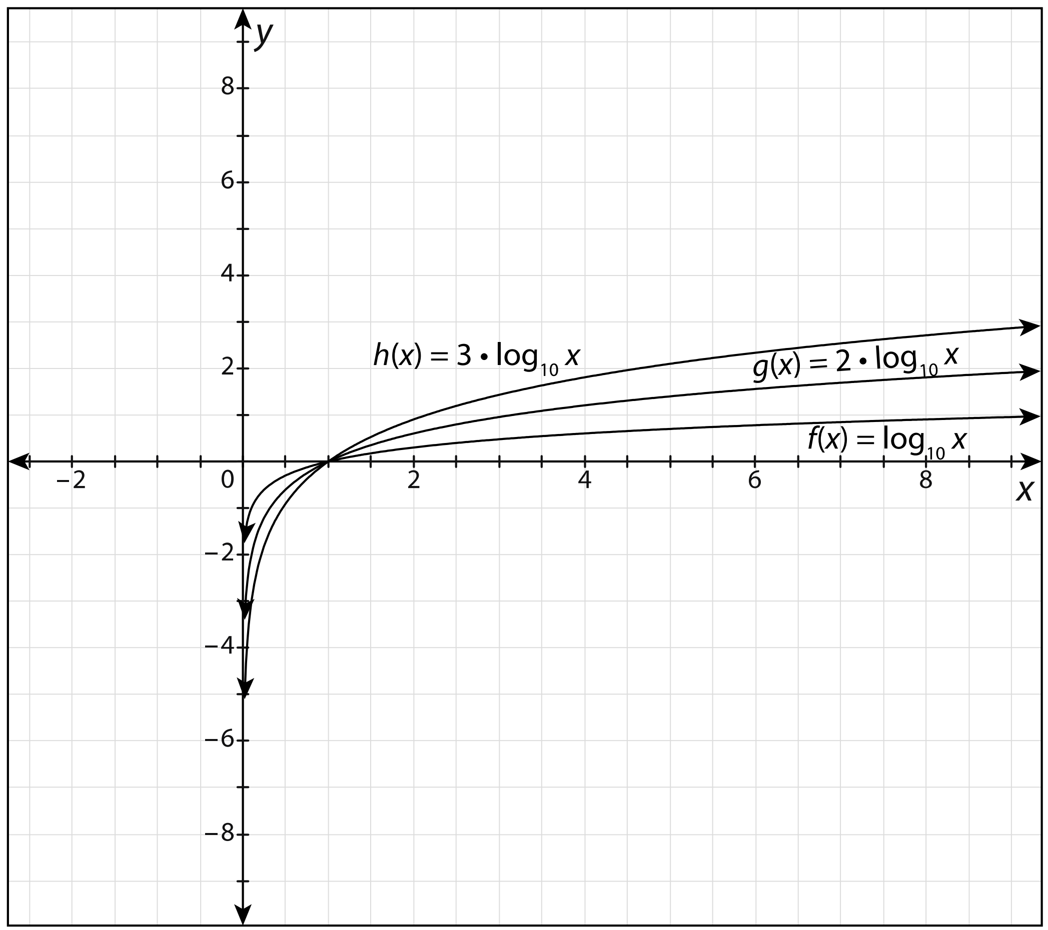
Powers, Products, Quotients, Roots, and Sums of Logarithmic Functions

- Families of logarithmic functions are grouped according to the operations shown in the equations of the functions.
- In real-world problems, you can calculate such combined operations with logarithmic functions best by approximation techniques or with calculators.
- Each example that follows shows how different operations affect the graph of a logarithmic function.

Key Concepts, *continued*

f*(*x*) = *a* • log$_b$ *c

- Compare the graphs of three functions of the form $f(x) = a \cdot \log_b c$.



Key Concepts, *continued*

- All three graphs pass through the point (1, 0) because any number raised to the 0 power (the *y*-value) is equal to 1 (the *x*-value).
- The coefficient in front of each logarithm multiplies the logarithmic value at that value of *x* by the magnitude of the coefficient.

Key Concepts, *continued*

This means that $h(x) = 3 \bullet f(x) = 3 \bullet \log x$, and $g(x)$ can be determined as follows.

$$g(x) = \frac{2}{3} \bullet h(x)$$

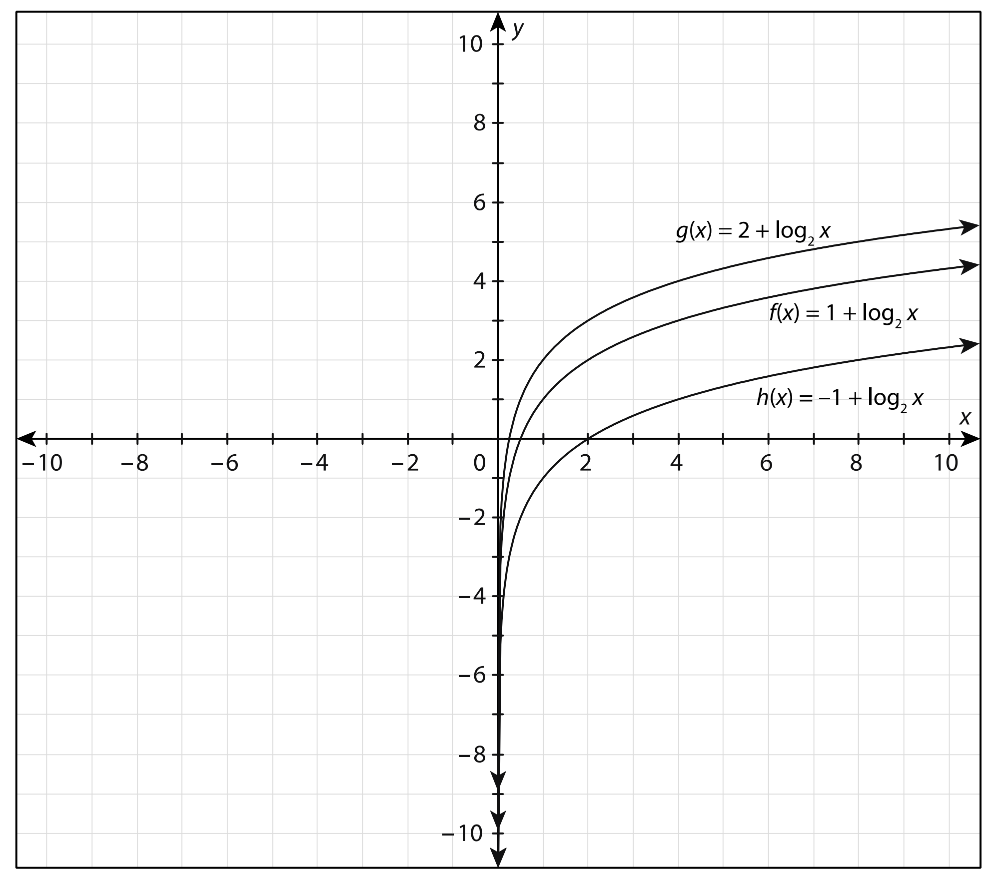
$$g(x) = 2 \bullet \log x$$

$$g(x) = \frac{2}{3} \bullet (3 \bullet \log x)$$

Key Concepts, *continued*

f*(*x*) = *a* + *b* • log$_c$ *d

- Observe the graphs of three functions of the form $f(x) = a + b \cdot \log_c d$.



Key Concepts, *continued*

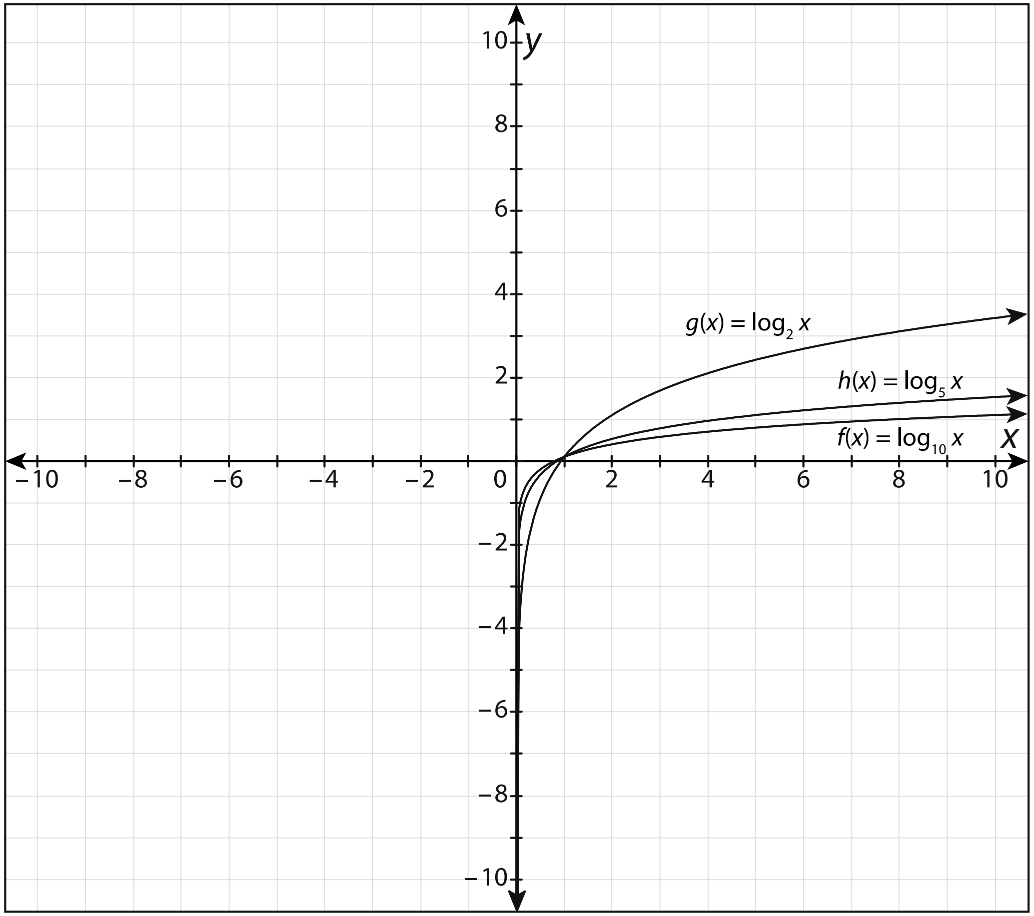
- All three graphs are continuously increasing across the domain $(0, \infty)$.
- For any value of x in the domain, the y-values are related by the inequality $g(x) > f(x) > h(x)$.
- The x-intercepts are determined by the constant added to $\log_2 x$.

Key Concepts, *continued*

$f(x) = a \cdot \log_b c$ and

$g(x) = a \cdot \log_d c$

- Compare the graphs of three functions with different bases.



Key Concepts, *continued*

- All three functions contain the point (1, 0), since the logarithm of any base to the power of 0 is equal to 1.
- As the graphs show, the functions are related by the inequality $g(x) > h(x) > f(x)$ for a specific positive value of x.
- Comparing the bases of the three functions reveals that they are ordered in the opposite "direction" from the functions: $2 < 5 < 10$ for $g(x) > h(x) > f(x)$.

Key Concepts, *continued*

- Alternately, as x increases, the powers of the smaller bases grow larger in order to remain proportional with larger bases raised to the same power.
- For example, for $x = 8$,

$$g(8) = 3$$

$$h(8) \approx 1.29$$

$$f(8) \approx 0.90.$$

Common Errors/Misconceptions

- confusing the domain and range in a logarithmic function problem
- using an argument in a logarithmic term that is less than 0
- failing to check all of the domain and extreme-value options in graphing one or more logarithmic functions; e.g., intercepts and function values for upper and lower bounds of domains

Guided Practice

Example 1

Sketch the graphs of the functions $f(x) = 2 \cdot \log_2 x$ and $g(x) = 2 - \log_2 x$ on a coordinate plane. Describe the end behavior of each function.

Guided Practice: Example 1, *continued*

1. **Determine function values for *f*(1) and *g*(1) and write the results as ordered pairs.**

Substitute $x = 1$ into each function and solve.

$$f(1) = 2 \cdot \log_2 (1) = 2 \cdot 0 = 0$$

$$g(1) = 2 - \log_2 (1) = 2 - 0 = 2$$

Write each point as an ordered pair.

For $f(x)$: (1, 0)

For $g(x)$: (1, 2)

Guided Practice: Example 1, *continued*

2. Find the value(s) of *x* at which $f(x) = g(x)$.

This will reveal any point(s) of intersection, or solutions, of the system of equations.

$2 \cdot \log_2 x = 2 - \log_2 x$	Set the functions equal to each other.
$3 \cdot \log_2 x = 2$	Add $\log_2 x$ to both sides.
$\log_2 x = \frac{2}{3}$	Divide both sides by 3.
$2^{\frac{2}{3}} = x$	Simplify.

Using a calculator, the approximate value of *x* is 1.6.

Guided Practice: Example 1, *continued*

3. Find the function value at which the functions are equal.

Substitute the approximate value of x from step 2 into each function.

$$f(1.6)\text{ » }g(1.6)\text{ » }\frac{4}{3}$$

Guided Practice: Example 1, *continued*

4. Write the approximate point at which the functions intersect.

Since the value of x is rounded, the functions intersect at approximately $\left(1.6, \frac{4}{3}\right)$ or $(1.6, 1.\overline{3})$.

Guided Practice: Example 1, *continued*

5. Determine additional points for sketching the graph of each function.

Choose values of x that can be evaluated easily.

Let $x = 0.25, 0.5, 2,$ and 4.

Guided Practice: Example 1, *continued*

Solve $f(x)$ for each given value.

$$f(0.25) = 2 \cdot \log_2(0.25) = 2 \cdot -2 = -4$$

$$f(2) = 2 \cdot \log_2(2) = 2 \cdot 1 = 2$$

$$f(0.5) = 2 \cdot \log_2(0.5) = 2 \cdot -1 = -2$$

$$f(4) = 2 \cdot \log_2(4) = 2 \cdot 2 = 4$$

Guided Practice: Example 1, *continued*

Solve $g(x)$ for each given value.

$$g(0.25) = 2 - \log_2 (0.25)$$
$$= 2 - (-2)$$
$$= 4$$

$$g(2) = 2 - \log_2 (2)$$
$$= 2 - 1$$
$$= 1$$

$$g(0.5) = 2 - \log_2 (0.5)$$
$$= 2 - (-1)$$
$$= 3$$

$$g(4) = 2 - \log_2 (4)$$
$$= 2 - 2$$
$$= 0$$

Guided Practice: Example 1, *continued*

6. Write the additional points.

Make sure to keep the points with the correct function.

For $f(x)$: (0.25, –4), (0.5, –2), (2, 2), and (4, 4)

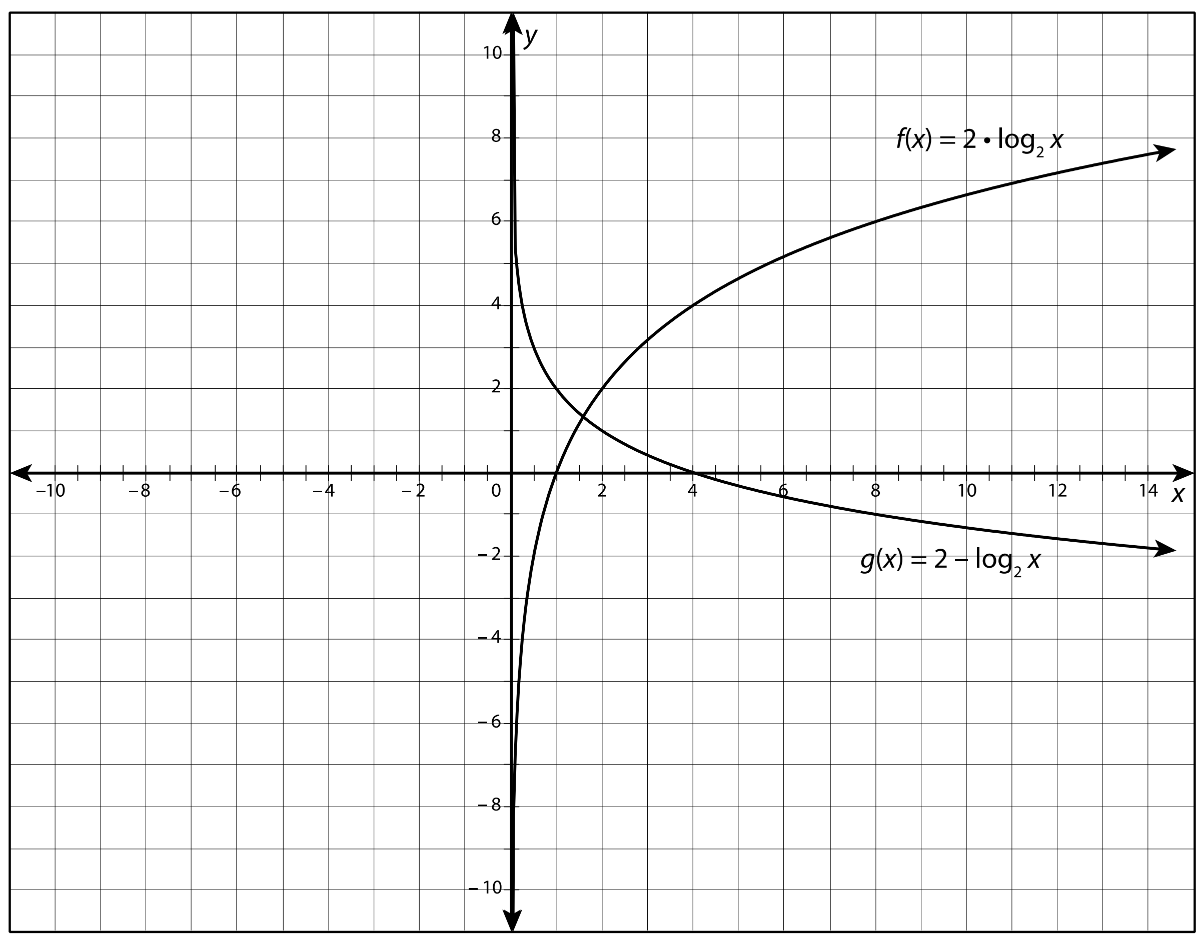
For $g(x)$: (0.25, 4), (0.5, 3), (2, 1), and (4, 0)

Guided Practice: Example 1, *continued*

7. Plot and sketch a curve to connect the points for each function.

The completed curves might suggest that additional points should be plotted to better determine the actual shape of each function’s graph.

Guided Practice: Example 1, *continued*



Guided Practice: Example 1, *continued*

8. Describe what happens to the function values as *x* becomes very large and as *x* approaches 0.

It is useful to know the limits of function values as the values go beyond the finite values used for a sketch of a graph.

For $f(x)$:

- As x approaches positive infinity, $f(x)$ approaches positive infinity.
- As x approaches 0, $f(x)$ approaches negative infinity.

Guided Practice: Example 1, *continued*

For $g(x)$:

- As x approaches positive infinity, $g(x)$ approaches negative infinity.
- As x approaches 0, $g(x)$ approaches positive infinity.

Guided Practice: Example 1, *continued*

Guided Practice

Example 2

Sketch the graphs of $f(x) = x + \log x$ and $g(x) = \log x$ on a coordinate plane, using a graphing calculator if needed.

Write $f(x)$ in terms of $g(x)$ to simplify sketching.

Then, describe the end behavior of the functions.

Guided Practice: Example 2, *continued*

1. **Determine function values for *f*(1) and *g*(1) and write the results as ordered pairs.**

Substitute $x = 1$ into each function and solve.

$$f(1) = 1 + \log(1) \\ = 1 + 0 \\ = 0$$

$$g(1) = \log(1) \\ = 0$$

Rewrite the points as ordered pairs.

For $f(x)$: (1, 1)

For $g(x)$: (1, 0)

Guided Practice: Example 2, *continued*

2. Find the value(s) of x at which $f(x) = g(x)$.

This will reveal any point(s) of intersection, or solutions, of the system of equations.

$$f(x) = g(x)$$

$$x + \log x = \log x$$

$$x = 0$$

The value $x = 0$ is not in the domain of these functions. Therefore, there is no point of intersection or solution for $f(x)$ and $g(x)$.

Guided Practice: Example 2, *continued*

3. Determine additional points for sketching the graph of each function.

Choose values of x for each function, and then solve the resulting equations.

Let $x = 0.1, 1, 2, 3,$ and 4.

The x-values of 2, 3, and 4 will result in common logarithms that cannot be found easily without a calculator that has the capability of computing common logarithms.

Guided Practice: Example 2, *continued*

Solve $f(x)$ for each given value.

$$f(0.1) = (0.1) + \log(0.1) = 0.1 - 1 = -0.9$$

$$f(1) = (1) + \log(1) = 1$$

$$f(2) = (2) + \log(2) \approx 2.3$$

$$f(3) = (3) + \log(3) \approx 3.5$$

$$f(4) = (4) + \log(4) \approx 4.6$$

Guided Practice: Example 2, *continued*

Solve $g(x)$ for each given value.

$g(0.1) = \log(0.1)$
$= -1$

$g(1) = \log(1)$
$= 0$

$g(2) = \log(2)$
≈ 0.3

$g(3) = \log(3)$
≈ 0.5

$g(4) = \log(4)$
≈ 0.6

Guided Practice: Example 2, *continued*

4. Write and plot the additional points.

Make sure to keep the points with the correct function.

For $f(x)$: (0.1, –0.9), (1, 1), approximately (2, 2.3), approximately (3, 3.5), and approximately (4, 4.6)

For $g(x)$: (0.1, –1), (1, 0), approximately (2, 0.3), approximately (3, 0.5), and approximately (4, 0.6)

Guided Practice: Example 2, *continued*

5. Write $f(x)$ in terms of $g(x)$.

- In some problems involving two or more functions graphed on the same coordinate plane, the functions are related, which can simplify graph sketching.

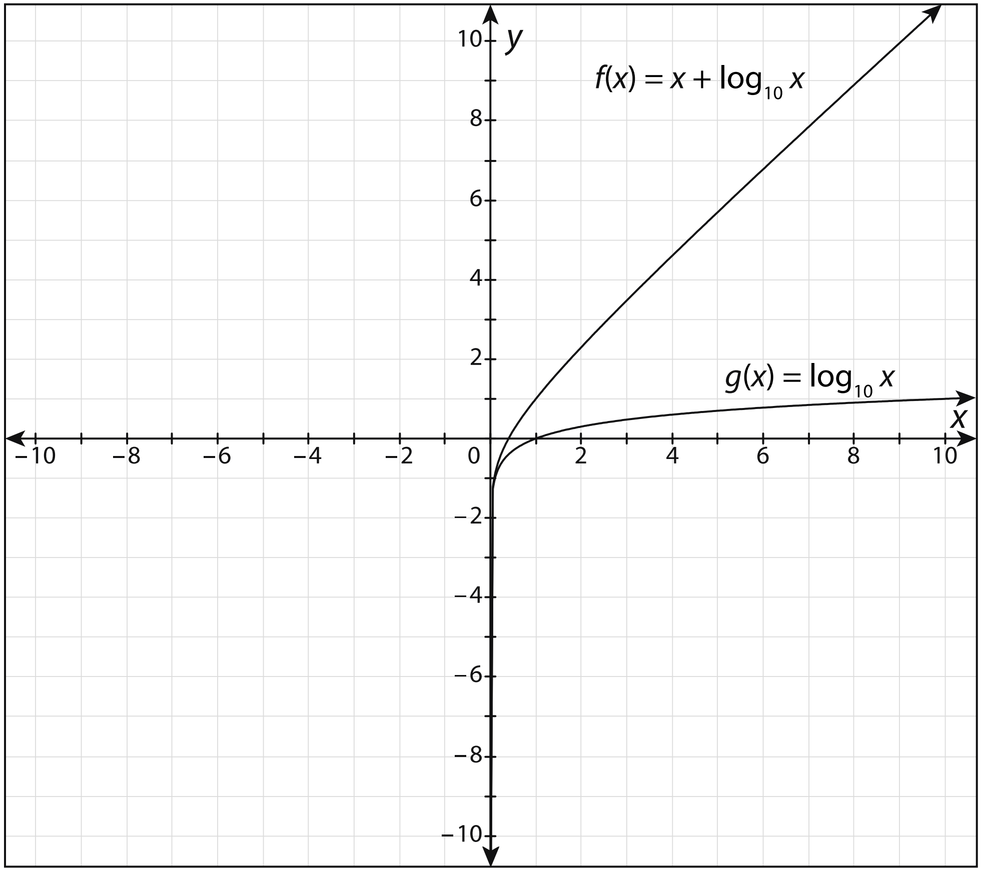
- $f(x) = x + \log x = x + g(x)$ or $f(x) - g(x) = x$, which means the difference between the function values at any value of x is x.

Guided Practice: Example 2, *continued*

6. **Plot and sketch a curve to connect the points for each function.**

The completed curves might suggest that additional points should be plotted to better determine the actual shape of each function’s graph.

Guided Practice: Example 2, *continued*



Guided Practice: Example 2, *continued*

7. Describe what happens to the function values for domain values less than 1.

The graph indicates that the function values diverge.

For $f(x)$:

- The function values approach negative infinity as x approaches 0.
- As x approaches 1, the function values converge on 1.

Guided Practice: Example 2, *continued*

For $g(x)$:

- The function values approach negative infinity as x approaches 0.
- As x approaches 1, the function values converge on 0.

Guided Practice: Example 2, *continued*

8. Describe what happens to the function values for domain values greater than 1.

The graph indicates that the function values increase.

For $f(x)$:

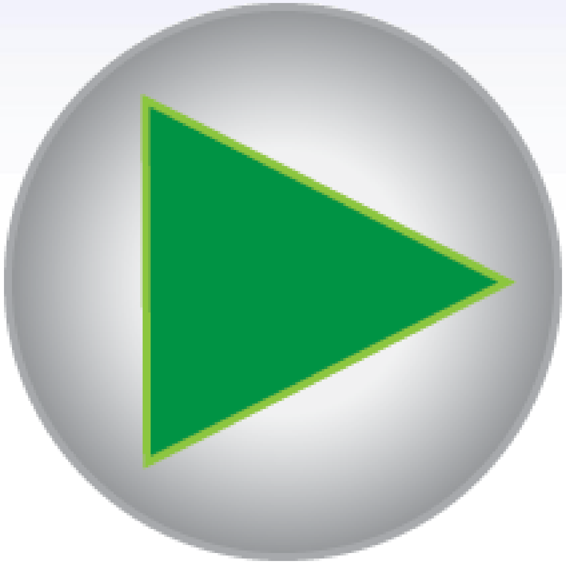
- The function values approach positive infinity as x increases from $x = 1$ to positive infinity.
- As x approaches 1 from the right, the function values converge on 1.

Guided Practice: Example 2, *continued*

For $g(x)$:

- The function values approach positive infinity as x increases from $x = 1$ to positive infinity.
- As x approaches 1 from the right, the function values converge on 0.

Guided Practice: Example 2, *continued*



Guided Practice

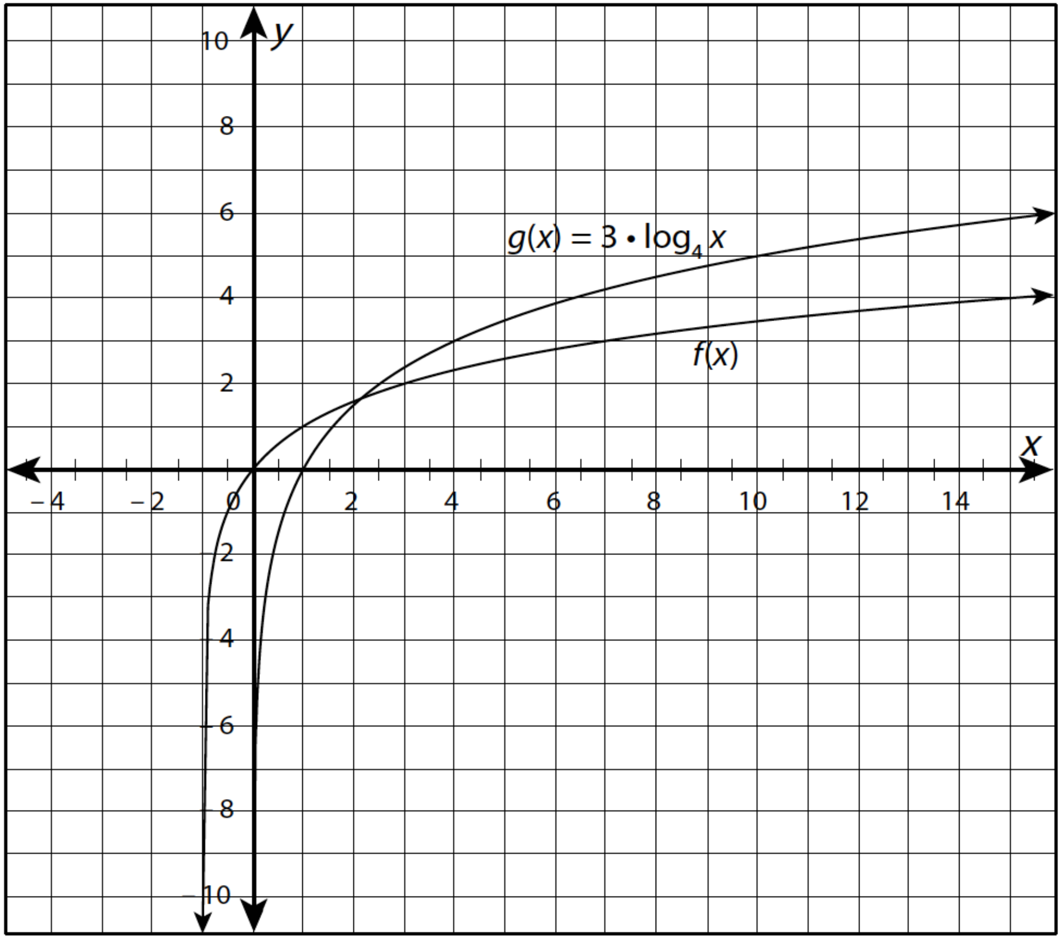
Example 3

Use the graph of the function $f(x)$ to write the algebraic form of $f(x)$.

Assume that $f(x)$ is of the form $a \cdot \log_4 (x + b)$ and includes the point $(15, 4)$.

Then, describe how to find the common solution of $f(x)$ and the function $g(x) = 3 \cdot \log_4 x$.

Guided Practice



Guided Practice: Example 3, *continued*

1. **Determine the *x*-intercepts and their coordinates for each function.**

Inspect the graph to determine this information.

The *x*-intercept of *f*(*x*) is 0, so the point is (0, 0).

The x-intercept of g(x) is 1, so the point is (1, 0).

Guided Practice: Example 3, *continued*

2. **Substitute the coordinates of the x-intercept of $f(x)$ into the form $f(x) = a \cdot \log_4 (x + b)$.**

This is the first step in writing $f(x)$.

For $x = 0$, $f(x) = 0$. Substitute and solve.

$$f(x) = a \cdot \log_4 (x + b)$$

$$f(0) = a \cdot \log_4 [(0) + b]$$
$$= 0$$

Guided Practice: Example 3, *continued*

Divide the logarithmic term by a.

$$\log_4 b = 0$$

$b = 1$ since $4^0 = 1$.

Substitute $b = 1$ in $f(x) = a \bullet \log_4 (x + b)$.

$$f(x) = a \bullet \log_4 (x + 1)$$

Guided Practice: Example 3, *continued*

3. **Use the other given points on the graph of $f(x)$ to find a.**

Substitute the given point (15, 14) into the $f(x)$ function and solve.

$f(x) = a \cdot \log_4 (x + 1)$	Function for $f(x)$ found in the previous step
$(4) = a \cdot \log_4 [(15) + 1]$	Substitute 15 for x and 4 for $f(x)$.
$4 = a \cdot \log_4 (16)$	Add.

Guided Practice: Example 3, *continued*

$4 = a \cdot 2$	Simplify.
$2 = a$	Divide both sides by 2.

Guided Practice: Example 3, *continued*

4. Substitute the value of *a* into the equation for *f*(*x*) found in step 2.

This is the last step in writing $f(x)$.

$$f(x) = 2 \cdot \log_4 (x + 1)$$

Guided Practice: Example 3, *continued*

5. Set $f(x) = g(x)$.

This is the first step in finding the common solution of the two functions.

If $f(x) = 2 \cdot \log_4 (x + 1)$ and $g(x) = 3 \cdot \log_4 x$, then $2 \cdot \log_4 (x + 1) = 3 \cdot \log_4 x$ for the value of x at which a common solution exists.

Guided Practice: Example 3, *continued*

6. Simplify the resulting equation.

Use the rules of logarithms.

$2 \cdot \log_4 (x + 1) = 3 \cdot \log_4 x$ can be rewritten as

$\log_4 (x + 1)^2 = \log_4 x^3$.

Guided Practice: Example 3, *continued*

7. Rewrite the equation without logarithms.

When logarithms of the same base are equal, the arguments of those logarithms are equal.

Each logarithm has a base of 4; therefore, set the arguments equal to each other.

$$(x + 1)^2 = x^3$$

Guided Practice: Example 3, *continued*

8. Explain how to find *x* in the resulting equation.

Since this is a cubic equation, expand the binomial and collect all terms on one side of the equation.

$$(x + 1)^2 = x^3$$

$$x^2 + 2x + 1 = x^3$$

$$x^3 – x^2 – 2x – 1 = 0$$

Guided Practice: Example 3, *continued*

There are no small integer solutions to this cubic equation, so *x* can be found by determining the root of this cubic equation on a graphing calculator.

Alternatively, the point of intersection can be found by calculating it with the "calculate intersect" feature on a graphing calculator.

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Guided Practice

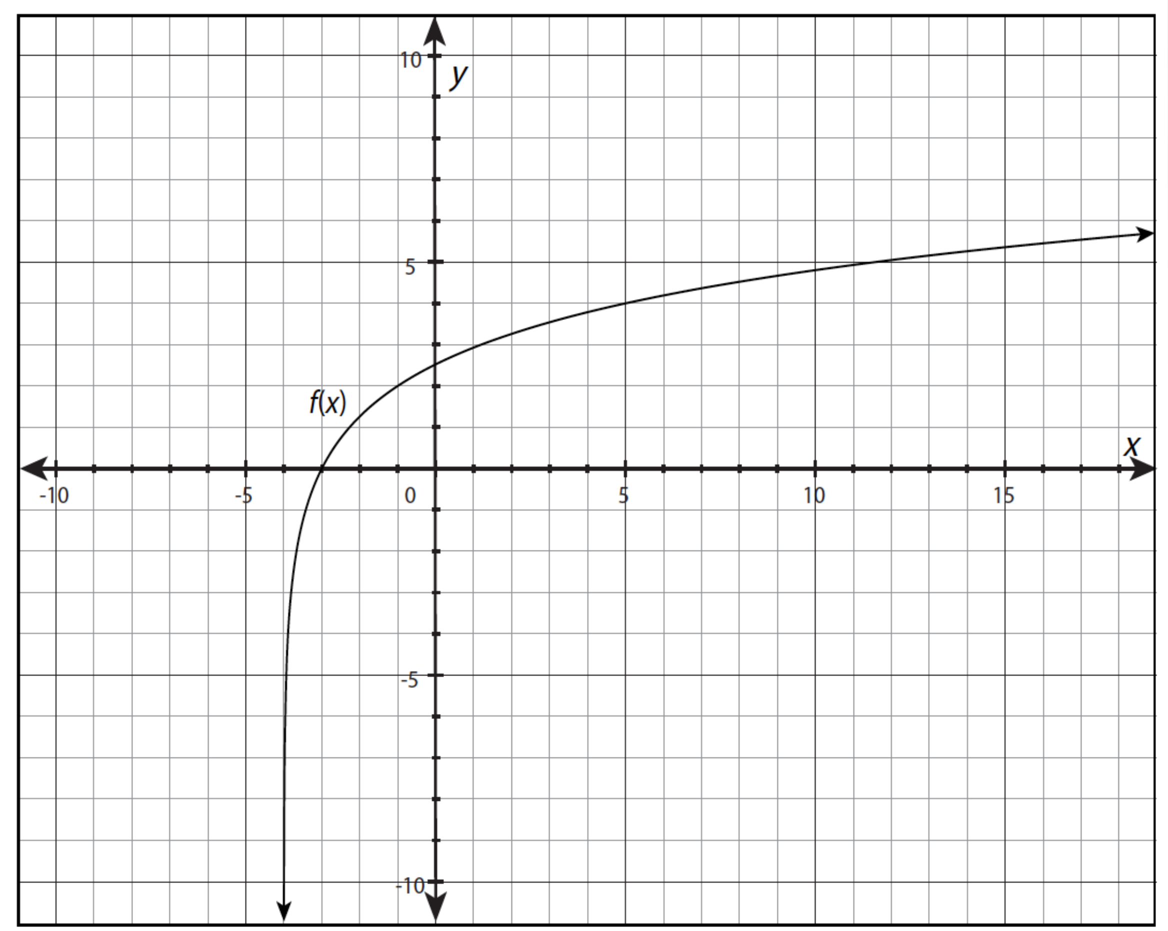
Example 4

The graph shows a logarithmic function of the form $f(x) = a \bullet \log_b (x + c)$, where (–1, 2) is a point on the graph.

The exact *x*-intercept can be determined directly from the graph. Write the *x*-intercept and estimate the value of the *y*-intercept from the graph.

Then, calculate the actual *y*-intercept and compare this value with the estimate.

Guided Practice



Guided Practice: Example 4, *continued*

1. Identify the *x*-intercept from the graph and write its coordinates.

The *x*-intercept can be used to help identify the constants in the general form of the function and find the *y*-intercept.

The *x*-intercept is located at the point where the graph crosses the *x*-axis, or where $y = 0$.

The *x*-intercept is –3, so the point containing it is (–3, 0).

Guided Practice: Example 4, *continued*

2. Estimate the *y*-intercept from the graph.

The *x*-intercept can be used to help identify the constants in the general form of the function and find the *y*-intercept.

The *x*-intercept is located at the point where the graph crosses the *x*-axis, or where $y = 0$.

The *x*-intercept is –3, so the point containing it is (–3, 0).

Guided Practice: Example 4, *continued*

3. Determine the equation for the *y*-intercept.

The *y*-intercept can be calculated by evaluating the function at $x = 0$.

To do this, first determine the equation for the *y*-intercept.

The function is in the form $f(x) = a \cdot \log_b (x + c)$, so substitute 0 for x and simplify.

Guided Practice: Example 4, *continued*

$$f(0) = a \cdot \log_b (0 + c)$$

$$= a \cdot \log_b c$$

The equation for the *y*-intercept can be written as $f(0) = a \cdot \log_b c$.

Guided Practice: Example 4, *continued*

4. Use the *x*-intercept and the given point (–1, 2) to find the constants in the function.

To find one or more of the constants, substitute the coordinates of the *x*-intercept found in step 1 into the form $f(x) = a \cdot \log_b (x + c)$.

The *x*-intercept is located at (–3, 0).

Guided Practice: Example 4, *continued*

$f(x) = a \cdot \log_b (x + c)$	Given function form
$(0) = a \cdot \log_b [(-3) + c]$	Substitute –3 for *x* and 0 for *f*(*x*).
$0 = \log_b (c - 3)$	Divide both sides by *a* and simplify.
$b^0 = c - 3$	Apply the power rule of logarithms.
$1 = c - 3$	A quantity raised to a 0 power is equal to 1.
$4 = c$	Add 3 to both sides.

Guided Practice: Example 4, *continued*

Therefore, if $c = 4$, then $f(x) = a \cdot \log_b (x + 4)$.

Now substitute (–1, 2) into $f(x) = a \cdot \log_b (x + 4)$ to find a.

$f(x) = a \cdot \log_b (x + 4)$	Function from the previous step
$2 = a \cdot \log_b [(-1) + 4]$	Substitute –1 for x and 2 for $f(x)$.
2 = a • logb 3	Add.
$\frac{2}{\log_b 3} = a$	Divide both sides by $\log_b 3$.

The known constants are $\frac{2}{\log_b 3} = a$ and $c = 4$.

Guided Practice: Example 4, *continued*

5. Substitute the value of *a* into the equation for the *y*-intercept.

Use the equation for the *y*-intercept determined in step 3.

$f(0) = a \bullet \log_b c$	Equation for the y-intercept
$f(0)=\left(\dfrac{2}{\log_b 3}\right)\bullet\log_b(4)$	Substitute $\dfrac{2}{\log_b 3}$ for *a* and 4 for *c*.
$f(0)=2\bullet\left(\dfrac{\log_b 4}{\log_b 3}\right)$	Rewrite the multiplication.
$f(0) = (\log_3 4)^2$	Simplify.

Guided Practice: Example 4, *continued*

6. Use a calculator to evaluate $(\log_3 4)^2$.

Rewrite $(\log_3 4)^2$ as $2 \bullet \left(\frac{\log_b 4}{\log_b 3}\right)$ using the power and quotient rules.

$$f(0) = 2 \bullet \left(\frac{\log_b 4}{\log_b 3}\right) \approx 2.52$$

The *y*-intercept is approximately 2.52.

Guided Practice: Example 4, *continued*

7. Compared the estimated *y*-intercept from the graph with the calculated *y*-intercept.

Though the graph suggests that the *y*-intercept appears to be at (0, 2.5), the *y*-intercept is actually closer to approximately (0, 2.52), a difference of approximately 0.02.

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