Logarithmic Functions as Inverses

Prerequisite Skills

This lesson requires the use of the following skills:

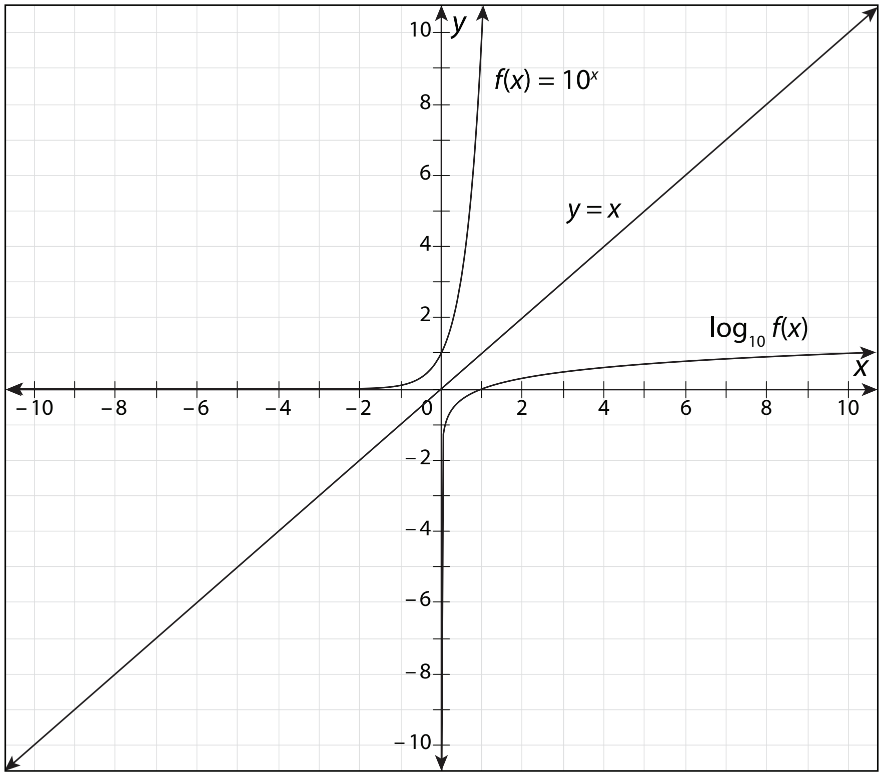
- determining the dependent and independent variables in an exponential function based on data from a graph or in a problem statement
- recognizing that a logarithmic function is the inverse of the related exponential function by comparing their graphs on the same axes

Introduction

In this course, you have studied a variety of functions, such as trigonometric functions, quadratic functions, and exponential functions. You have worked with exponents in the past and probably realize that exponents are not always whole numbers. You may also recall that sometimes exponents contain variables.

An **exponential function** is a function that has a variable in the exponent, such as $f(x) = 5^x$. The **power** is the result of raising a base to an exponent; 32 is a power of 2 since $2^5 = 32$. The exponent is the value of the function's corresponding logarithm, such as x in the logarithmic function $x = \log_5 f(x)$ and its corresponding exponential function, $f(x) = 5^x$.

Like other functions, exponential functions have inverses, which are called logarithmic functions. A **logarithmic function** is the inverse of an exponential function. For example, for the exponential function $f(x) = 5^x$, the inverse logarithmic function is $x = \log_5 f(x)$. If the exponential function is of the form $f(x) = a^x$, then the logarithmic function is of the form $\log_a f(x) = x$. This confirms the relationship between a function $f(x) = y$ and its inverse, $g(y) = x$. This relationship can also be seen from the following graph of an exponential function, $f(x) = 10^x$, and its inverse logarithmic function, $\log_{10} f(x)$.



Notice that the exponential function and its inverse logarithmic function are reflected across the line $f(x) = x$ (often written as $y = x$). For example, this means that for the value $x = 3$, the exponential function is given by $f(3) = 10^3$ and its inverse logarithmic function is $\log_{10} f(3) = \log_{10}(10^3) = 3$. In real-world problems, such as the sound-intensity example in the Warm-Up, there will be situations in which the inverse function is more effectively used than the function from which the inverse is derived. Knowledge of the real-world domain of the function can help make the decision about whether the function or its inverse has more meaning. Another factor in deciding which function to work with is how simplified the expressions and numbers are for each function.

Key Concepts

- As the graph in the Introduction shows, the exponential function and its inverse are one-to-one over their domains. The domain of the exponential function is $(-\infty, +\infty)$. However, the domain of the logarithmic function is $(0, +\infty)$.
- The range of the exponential function is $(0, +\infty)$. The range of the logarithmic function is $(-\infty, +\infty)$. This information provides more evidence that the logarithmic function is the inverse of the exponential function.

- In the graphed example, the value of the exponential function is 1 at $x = 0$ because $f(0) = 10^0 = 1$. Correspondingly, the value of the inverse logarithmic function is 0 at $x = 1$ because $\log_{10}(1) = 0$.
- Exponential functions with more constants can be explored using the properties of exponents or by looking at data tables generated by a graphing calculator.
- Use a graphing calculator to explore the domain, range, and other key points of the function $4 \bullet 3^{2x}$ and its inverse logarithmic function by looking at data tables of domain and function values. Follow the directions appropriate to your calculator model.

On a TI-83/84:

Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

Step 2: At Y1, use your keypad to enter values for the function. Use [X, T, θ, n] for *x* and [x^2] for any exponents.

Step 3: Press [GRAPH]. Press [WINDOW] to adjust the graph's axes.

Step 4: Press [2ND][GRAPH] to display a table of values. Look at the domain values around $x = 0$.

On a TI-Nspire:

Step 1: Press [home] to display the Home screen.

Step 2: Arrow down to the graphing icon, the second icon from the left, and press [enter].

Step 3: Enter the function to the right of "$f1(x)$ =" and press [enter].

Step 4: To adjust the *x*- and *y*-axis scales on the window, press [menu] and select 4: Window and then 1: Window Settings. Enter each setting as needed. Tab to "OK" and press [enter].

Step 5: To see a table of values, press [menu] and scroll down to 2: View, then 5: Show Table.

- Either calculator will show exponential function values that approach 0 as *x* becomes negative and that increase as *x* becomes positive.

- To show the corresponding function values for the inverse logarithmic function, switch the *x*- and *y*-values, as shown in the following table.

Exponential function	***x***	–2	–1	0	1	2
	y	0.05	0.44	4	36	324
Logarithmic function	***x***	0.05	0.44	4	36	324
	y	–2	–1	0	1	2

- Notice that the logarithmic function does not exist for negative domain values.
- The logarithmic function values can be verified with the data table.
- For example, $f(x) = 4 \cdot 3^{2x}$, so $x = \frac{1}{2}\log_3\left(\frac{f(x)}{4}\right)$. For $x = 0$, $\log_3\left(\frac{f(0)}{4}\right) = 0 \rightarrow \log_3\left(\frac{4}{4}\right) =$ $\log_3(1) = 0$.
- Notice that the coefficient of 4 in the function changes the value of the function to 4 at $x = 0$, and it changes the value of x to 4 when the value of the inverse function is 0.
- Finally, the basic definitions and rules of exponents and logarithms will be needed in order to manipulate and calculate exponential and logarithmic functions, summarized as follows.

Terms and Rules for Logarithms

- In a logarithmic equation, $\log_a b = c$, a is the base, b is the argument, and c is the logarithm of b to the base a.
- The **base** is the quantity that is being raised to an exponent in an exponential expression, such as a in the expression a^x, or the quantity that is raised to an exponent which is the value of the logarithm, such as 2 in the function $\log_2 g(x) = 3 - x$.
- The **argument** is the result of raising the base of a logarithm to the power that is the value of the logarithm, so that b is the argument of the logarithm $\log_a b = c$.
- You may recall the rules for working with exponents; for example, according to the Product of Powers Property, when multiplying two exponents with the same base, keep the base and add the powers: $a^x \cdot a^y = a^{x+y}$. The rules for various operations with logarithms are derived from the rules for exponents. The following table lists some exponent rules, followed by the equation and name of the related logarithmic rule.

Exponent rule	Related logarithm rule	Logarithm rule name
$a^x \bullet a^y = a^{x+y}$	$\log_a (x \bullet y) = \log_a x + \log_a y$	Product rule
$\frac{a^x}{a^y} = a^{x-y}$	$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$	Quotient rule
$(a^x)^y = a^{x \bullet y}$	$\log_a x^y = y \bullet \log_a x$	Power rule

- Another rule, the base change rule, allows for computing with logarithms other than base 10; one form of the equation for this rule is $\log_b a = \frac{\log_{10} a}{\log_{10} b}$. (Other forms will be discussed later.)
- This rule is particularly useful when working with calculators that only calculate with logarithms with bases of e (natural logarithms) and 10 (common logarithms).
- The irrational number ***e*** has a value of approximately 2.71828. A **natural logarithm** is a logarithm with a base of e. Natural logarithms are usually written in the form "ln," which means "$\log_e$." For example, $f(x) = \ln (1 - x)$ is understood to be the inverse of the function for the exponential function $g(x) = 1 - e^x$.
- A **common logarithm**, on the other hand, is a logarithm with a base of 10. When writing a common logarithm, the 10 is usually omitted, such that $\log x = \log_{10} x$. For example, the logarithmic function $f(x) = \log x$ is understood to be the inverse function for the exponential function $g(x) = 10^x$.

Common Errors/Misconceptions

- incorrectly identifying the domain and range variables in an exponential function and in its inverse logarithmic function
- confusing the base with the power in expressing an exponential function as a logarithmic function, or vice versa
- misidentifying the domains of exponential functions and their inverse logarithmic functions
- misinterpreting the coefficients of a base and of a variable in a power in an exponential function when writing the inverse logarithmic function
- misapplying the rules of exponents and logarithms in rewriting exponential and logarithmic functions