Logarithmic Functions as Inverses

Warm-Up

In the metric system, sound intensity is measured in watts per square meter. For example, the average sound intensity at a rock concert is 0.1 watt per square meter. The threshold of human hearing is about 10^{-12} watt per square meter.

1. What is 0.1 watt per square meter written as an expression using a base of 10 and an exponent?
2. How much greater is the sound intensity at an average rock concert than the threshold of human hearing?
3. Write an exponential function with a base of 10 and a power of x to represent sound intensity, $I(x)$.

1. **What is 0.1 watt per square meter written as an expression using a base of 10 and an exponent?**

- Recall that in order to rewrite 0.1 using a base of 10 and an exponent, first move the decimal place to the right to create the whole number 1.
- Then determine the exponent by counting how many places the decimal was moved.
- To move from 0.1 to 1.0, the decimal was moved 1 unit to the right, creating an exponent of –1.
- Therefore, 0.1 written using base 10 with an exponent is 10^{-1}.

2. How much greater is the sound intensity at an average rock concert than the threshold of human hearing?

- To determine this, divide the intensity at the rock concert by the threshold of human hearing.

$$\frac{10^{-1}}{10^{-12}} = 10^{(-1)-(-12)} = 10^{-1+12} = 10^{11}$$

- The sound intensity of the average rock concert is 100 billion times greater than the threshold of human hearing.

3. Write an exponential function with a base of 10 and a power of *x* to represent sound intensity, *I*(*x*).

The simplest exponential function is given by $I(x) = 10^x$.

Instruction

Introduction

In this course, you have studied a variety of functions, such as trigonometric functions, quadratic functions, and the inverses of functions.

You have worked with exponents in the past and probably realize that exponents are not always whole numbers.

You may also recall that sometimes exponents contain variables.

Introduction, *continued*

An exponential function is a function that has a variable in the exponent, such as $f(x) = 5^x$.

The power is the result of raising a base to an exponent; 32 is a power of 2 since $2^5 = 32$.

The power is also the value of the function's logarithm, such as x in the logarithmic function $x = \log_5 f(x)$ and its exponential function, $f(x) = 5^x$.

Introduction, *continued*

Like other functions, exponential functions have inverses, which are called logarithmic functions.

A logarithmic function is the inverse of an exponential function.

For example, for the exponential function $f(x) = 5^x$, the inverse logarithmic function is $x = \log_5 f(x)$.

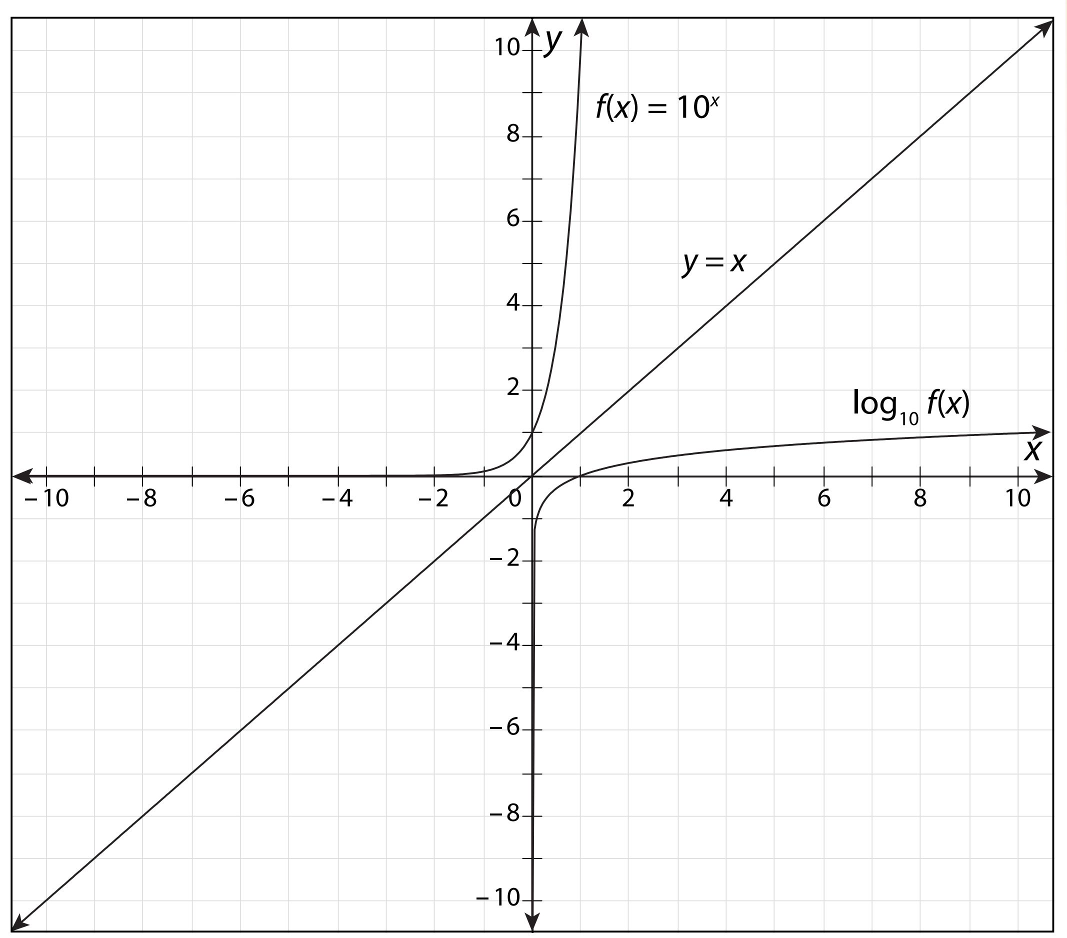
Introduction, *continued*

If the exponential function is of the form $f(x) = a^x$, then the logarithmic function is of the form $\log_a f(x) = x$.

This confirms the relationship between a function $f(x) = y$ and its inverse, $g(y) = x$.

This relationship can also be seen from the following graph of an exponential function, $f(x) = 10^x$, and its inverse logarithmic function, $\log_{10} f(x)$.

Introduction, *continued*



Introduction, *continued*

Notice that the exponential function and its inverse logarithmic function are reflected across the line $f(x) = x$ (often written as $y = x$).

For example, this means that for the value $x = 3$, the exponential function is given by $f(3) = 10^3$ and its inverse logarithmic function is $\log_{10} f(3) = \log_{10} (10^3) = 3$.

In real-world problems, such as the sound-intensity example in the Warm-Up, there will be situations in which the inverse function is more effectively used than the function from which the inverse is derived.

Introduction, *continued*

Knowledge of the real-world domain of the function can help make the decision about whether the function or its inverse has more meaning.

Another factor in deciding which function to work with is how simplified the expressions and numbers are for each function.

Key Concepts

- As the graph in the Introduction shows, the exponential function and its inverse are one-to-one over their domains. The domain of the exponential function $(-\infty, \infty)$. However, the domain of the logarithmic function is $(0, \infty)$.

- The range of the exponential function is $(0, \infty)$. The range of the logarithmic function is $(-\infty, \infty)$. This information provides more evidence that the logarithmic function is the inverse of the exponential function.

Key Concepts, *continued*

- In the graphed example, the value of the exponential function is 1 at $x = 0$ because $f(0) = 10^0 = 1$.
- Correspondingly, the value of the inverse logarithmic function is 0 at $x = 1$ because $\log_{10}(1) = 0$.
- Exponential functions with more constants can be explored using the properties of exponents or by looking at data tables generated by a graphing calculator.

Key Concepts, *continued*

- Use a graphing calculator to explore the domain, range, and other key points of the function $4 \cdot 3^{2x}$ and its inverse logarithmic function by looking at data tables of domain and function values.

- Follow the directions appropriate to your calculator model.

Key Concepts, *continued*

On a TI-83/84:

Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

Step 2: At Y1, use your keypad to enter values for the function. Use [X, T, θ, n] for x and [x^2] for any exponents.

Step 3: Press [GRAPH]. Press [WINDOW] to adjust the graph’s axes.

Step 4: Press [2ND][GRAPH] to display a table of values. Look at the domain values around $x = 0$.

Key Concepts, *continued*

On a TI-Nspire:

Step 1: Press [home] to display the Home screen.

Step 2: Arrow down to the graphing icon, the second icon from the left, and press [enter].

Step 3: Enter the function to the right of "$f1(x) =$" and press [enter].

Step 4: To adjust the x- and y-axis scales on the window, press [menu] and select 4: Window and then 1: Window Settings. Enter each setting as needed. Tab to "OK" and press [enter].

Step 5: To see a table of values, press [menu] and scroll down to 2: View, then 5: Show Table.

Key Concepts, *continued*

- Either calculator will show exponential function values that approach 0 as *x* becomes negative and that increase as *x* becomes positive.
- To show the corresponding function values for the inverse logarithmic function, switch the *x*- and *y*-values, as shown in the following table.

Exponential function	***X***	–2	–1	0	1	2
	Y	0.05	0.44	4	36	324
Logarithmic function	***X***	—	—	4	36	324
	Y	–2	–1	0	1	2

Key Concepts, *continued*

- Notice that the logarithmic function does not exist for negative domain values.
- The logarithmic function values can be verified with the data table.
- For example, $f(x) = 4 \cdot 3^{2x}$, so $x = \frac{1}{2}\log_3\left(\frac{f(x)}{4}\right)$.

For $x = 0$, $\log_3\left(\frac{f(x)}{4}\right) = 0 \rightarrow \log_3\left(\frac{4}{4}\right) = \log_3(1) = 0$

or $3^0 = 1$.

Key Concepts, *continued*

- Notice that the coefficient of 4 in the function changes the value of the function to 4 at $x = 0$, and it changes the value of x to 4 when the inverse function is 0.
- Finally, the basic definitions and rules of exponents and logarithms will be needed in order to manipulate and calculate exponential and logarithmic functions, summarized as follows.

Key Concepts, *continued*

Terms and Rules for Logarithms

- In a logarithmic equation, $\log_a b = c$, a is the base, b is the argument, and c is the logarithm of b to the base a.
- The base is the quantity that is being raised to an exponent in an exponential expression, such as a in the expression a^x, or the quantity that is raised to an exponent which is the value of the logarithm, such as 2 in the function $\log_2 g(x) = 3 - x$.

Key Concepts, *continued*

- The argument is the result of raising the base of a logarithm to the power that is the value of the logarithm, so that b is the argument of the logarithm $\log_a b = c$.
- You may recall the rules for working with exponents; for example, according to the Product of Powers Property, when multiplying two exponents with the same base, keep the base and add the powers: $a^x \bullet a^y = a^{x+y}$.

Key Concepts, *continued*

- The rules for various operations with logarithms are derived from the rules for exponents.
- The following table lists some exponent rules, followed by the equation and name of the related logarithmic rule.

Key Concepts, *continued*

Exponent rule	Related logarithm rule	Logarithm rule name
$a^x \cdot a^y = a^{x+y}$	$\log_a (x \cdot y) = \log_a x + \log_a y$	Product rule
$\frac{a^x}{a^y} = a^{x-y}$	$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$	Quotient rule
$(a^x)^y = a^{x \cdot y}$	$\log_a x^y = y \cdot \log_a x$	Power rule

- Another rule, the base change rule, allows for computing with logarithms other than base 10; one form of the equation for this rule is $\log_b a = \frac{\log_{10} a}{\log_{10} b}$. (Other forms will be discussed later.)

Key Concepts, *continued*

- This rule is particularly useful when working with calculators that only calculate with logarithms with bases of *e* (natural logarithms) and 10 (common logarithms).
- The irrational number *e* has a value of approximately 2.71828.
- A natural logarithm is a logarithm with a base of *e*. Natural logarithms are usually written in the form "ln," which means "$\log_e$." For example, $f(x) = \ln(1 - x)$ is understood to be the inverse of the function for the exponential function $g(x) = 1 - e^x$.

Key Concepts, *continued*

- A common logarithm, on the other hand, is a logarithm with a base of 10.
- When writing a common logarithm, the 10 is usually omitted, such that $\log x = \log_{10} x$.
- For example, the logarithmic function $f(x) = \log (2x - 1)$ is understood to be the inverse function for the exponential function $g(x) = 10^{2x - 1}$.

Common Errors/Misconceptions

- incorrectly identifying the domain and range variables in an exponential function and in its inverse logarithmic function
- confusing the base with the power in expressing an exponential function as a logarithmic function, or vice versa
- misidentifying the domains of exponential functions and their inverse logarithmic functions

Common Errors/Misconceptions, *continued*

- misinterpreting the coefficients of a base and of a variable in a power in an exponential function when writing the inverse logarithmic function
- misapplying the rules of exponents and logarithms in rewriting exponential and logarithmic functions

Guided Practice

Example 1

Write the inverse of the exponential function

$f(x) = 0.1 \cdot 2^{0.3x}$.

Guided Practice: Example 1, *continued*

1. Isolate the exponential term.

- This is necessary in order to use the logarithmic function definition $\log_a f(x) = x$.
- Divide both sides of the equation by 0.1; this is equal to multiplication by 10.
- $f(x) = 0.1 \cdot 2^{0.3x}$ becomes $10 \cdot f(x) = 2^{0.3x}$.

Guided Practice: Example 1, *continued*

2. Rewrite the result as a logarithm.

$10 \cdot f(x) = 2^{0.3x}$ becomes $\log_2 [10 \cdot f(x)] = 0.3x$.

Guided Practice: Example 1, *continued*

3. Isolate the exponent variable, *x*.

$\log_2 [10 \bullet f(x)] = 0.3x$ becomes $x = \frac{10}{3} \bullet \log_2 \left[10 \bullet f(x) \right]$

Guided Practice: Example 1, *continued*

4. **Use the rules of logarithms to rewrite the result so that the simplest expression possible can be used to evaluate the function numerically.**

Use the product rule for logarithms to rewrite the expression as the sum of two logarithms:

$$\frac{10}{3}\bullet\log_2\left[10\bullet f\left(x\right)\right]=\frac{10}{3}\left[\log_2 10\right]+\frac{10}{3}\left[\log_2 f\left(x\right)\right]=x$$

Guided Practice: Example 1, *continued*

Use the power rule to rewrite the separate logarithms as exponentials:

$$\frac{10}{3}\left[\log_2 10\right]+\frac{10}{3}\left[\log_2 f\left(x\right)\right]=\log_2 10^{\frac{10}{3}}+\log_2 f\left(x\right)^{\frac{10}{3}}=x$$

Guided Practice: Example 1, *continued*

5. **Switch the domain and function variables to write the logarithmic inverse as a logarithmic function.**

This step is necessary if the logarithmic function is considered independent of the exponential function from which it was derived.

$$\log_2 10^{\frac{10}{3}} + \log_2 x^{\frac{10}{3}} = f^{-1}(x)$$

Guided Practice: Example 1, *continued*

The inverse of the exponential function

$10 \cdot f(x) = 2^{0.3x}$ is $f^{-1}(x) = \log_2 10^{\frac{10}{3}} + \log_2 x^{\frac{10}{3}}$

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Guided Practice

Example 2

Find the exponential function on which the logarithmic function $g(x) = \log_6 x^2 - \log_6 25$ is based.

Guided Practice: Example 2, *continued*

1. **Switch the domain and function variables to write the logarithmic function as an exponential function.**

$g(x) = \log_6 x^2 - \log_6 25$ becomes $x = \log_6 [g(x)]^2 - \log_6 25$.

Guided Practice: Example 2, *continued*

2. **Use the rules of logarithms to rewrite the result so that the simplest expression possible can be found for the logarithmic function before it is converted to exponential form.**

$x = \log_6 \left[g(x)\right]^2 - \log_6 25$	Exponential function from step 1
$x = \log_6 \left[g(x)\right]^2 - \log_6 5^2$	Rewrite 25 as 5^2.

Guided Practice: Example 2, *continued*

$$x = 2 \cdot \log_6 g(x) - 2 \cdot \log_6 5$$

Apply the power rule to both logarithmic terms.

$$x = 2 \cdot \left[\log_6 g(x) - \log_6 5\right]$$

Factor 2 out from both terms.

$$x = 2 \cdot \log_6 \left[\frac{g(x)}{5}\right]$$

Use the quotient rule to rewrite the logarithmic terms.

Guided Practice: Example 2, *continued*

3. Solve for the logarithmic term and rewrite the logarithmic function as an exponential function.

The equation may be easier to work with by applying the Symmetric Property of Equality so that x is on the right side of the equation.

Guided Practice: Example 2, *continued*

$$2 \cdot \log_6\left[\frac{g(x)}{5}\right] = x$$

$$\log_6\left[\frac{g(x)}{5}\right] = \frac{x}{2}$$

$$\log_6\left[0.2 \cdot g(x)\right] = 0.5x$$

Guided Practice: Example 2, *continued*

4. **Write the exponential function from the simplified logarithmic function by using the definition of an exponential function and its inverse.**

$\log_6\left[0.2\cdot g(x)\right]=0.5x$	Logarithmic function
$0.2\cdot g(x)=6^{0.5x}$	Rewrite as the inverse.
$g(x)=5\cdot 6^{0.5x}$	Divide both sides by 0.2.

Guided Practice: Example 2, *continued*

The logarithmic function $g(x) = \log_6 x^2 - \log_6 25$ is based on the exponential function $g(x) = 5 \cdot 6^{0.5x}$.

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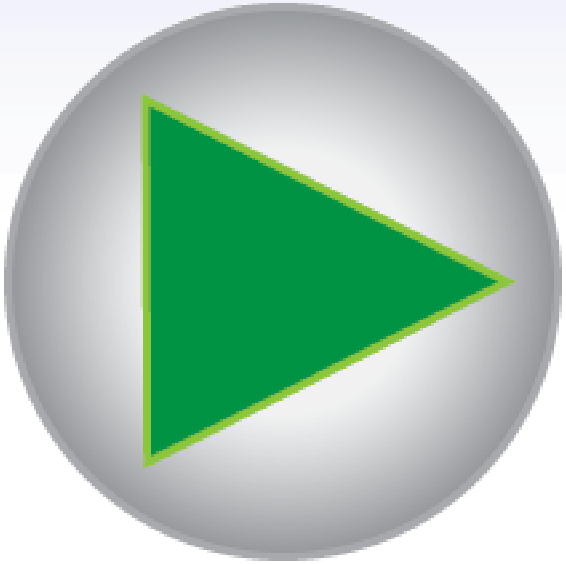
Guided Practice: Example 2, *continued*



Guided Practice

Example 3

Use a logarithmic function to solve the exponential equation $4^{\left(\frac{x-3}{x}\right)}=5.$

Guided Practice: Example 3, *continued*

1. Rewrite the exponential function as its inverse logarithmic function.

This is an alternative method to using the properties of exponents to solve the equation.

$4^{\left(\frac{x-3}{x}\right)} = 5$ becomes $\log_4 5 = \frac{x-3}{x}$.

Guided Practice: Example 3, *continued*

2. Simplify the result algebraically.

$\log_4 5 = \frac{x - 3}{x}$	Inverse logarithmic function
$x \cdot \log_4 5 = x - 3$	Multiply both sides by x.
$3 = x - x \cdot \log_4 5$	Add and subtract from both sides.
$3 = x(1 - \log_4 5)$	Factor out x.
$x = \frac{3}{1 - \log_4 5}$	Divide both sides by $1 - \log_4 5$.

Guided Practice: Example 3, *continued*

3. Solve the original exponential equation using the rules of exponents.

This will serve as a check on the logarithmic approach used in steps 1 and 2.

$4^{\left(\frac{x-3}{x}\right)}=5$	Original equation
$4^{x-3}=5^x$	Simplify the fractional exponent.
$\frac{4^x}{4^3}=5^x$	Rewrite subtracted exponents as a fraction.

Guided Practice: Example 3, *continued*

$$\left(\frac{4}{5}\right)^x = 4^3$$

Cross multiply and simplify.

$$\frac{4}{5} = 4^{\frac{3}{x}}$$

Simplify.

Simplify using the definition of logarithm and the quotient rule.

$$\frac{4}{5} = 4^{\frac{3}{x}}$$

Simplified exponential equation

Guided Practice: Example 3, *continued*

$$\log_4\left(\frac{4}{5}\right)=\frac{3}{x}$$

Rewrite as a logarithm.

$$\log_4 4-\log_4 5=\frac{3}{x}$$

Apply the quotient rule.

The resulting expression can be rearranged to equal x, yielding the result found with the logarithmic function in step 2:

$\log_4 4-\log_4 5=\frac{3}{x}$ becomes $x=\frac{3}{1-\log_4 5}$.

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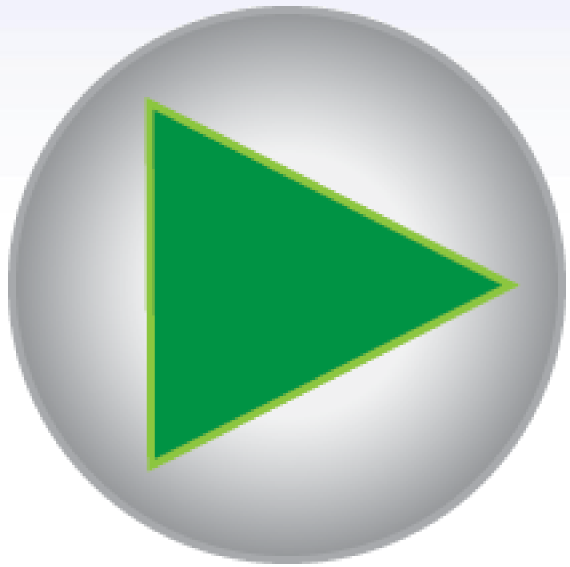
Guided Practice: Example 3, *continued*



Guided Practice

Example 4

Write the domain and function value of the exponential function $f(x) = 1.23 \bullet 2^{0.7x}$ and its inverse at a domain value of $x = 1.05$.

Use a graphing or scientific calculator and the rules of exponents and logarithms to verify your results.

Guided Practice: Example 4, *continued*

1. **Substitute $x = 1.05$ into the function $f(x) = 1.23 \bullet 2^{0.7x}$.**

Evaluate the function using either a graphing calculator or a scientific calculator with exponentiation functionality.

$$f(1.05) = 1.23 \bullet 2^{0.7(1.05)} \approx 2.05$$

Guided Practice: Example 4, *continued*

2. Write the domain and function value of the inverse of the exponential function at *x* = 1.05.

Switch the domain and function values of the exponential function to determine the domain and function values of the inverse logarithmic function at $x = 1.05$.

The corresponding values for the inverse logarithmic function are given by the ordered pair (2.05, 1.05).

Guided Practice: Example 4, *continued*

3. Verify that the ordered pair determined in step 2 satisfies the inverse logarithmic function.

To verify that (2.05, 1.05) satisfies the inverse logarithmic function, first determine the inverse logarithmic function from the original function using the definition of logarithm and the quotient and power rules.

Guided Practice: Example 4, *continued*

$f(x) = 1.23 \bullet 2^{0.7x}$	Original function
$0.7x = \log_2\left[\dfrac{f(x)}{1.23}\right]$	Divide by 1.23 and rewrite as a logarithm.
$x = \dfrac{10}{7}\log_2 f(x) - \dfrac{10}{7}\log_2 1.23$	Divide by 0.7 and apply the quotient rule.
$x = \log_2 f(x)^{\frac{10}{7}} - \log_2 1.23^{\frac{10}{7}}$	Apply the power rule.

Guided Practice: Example 4, *continued*

Switch x and $f(x)$ to write the inverse logarithmic function.

$$f(x) = \log_2 x^{\frac{10}{7}} - \log_2 1.23^{\frac{10}{7}}$$

Guided Practice: Example 4, *continued*

Substitute the value of x found in step 2.

$$f(2.05) = \log_2 (2.05)^{\frac{10}{7}} - \log_2 1.23^{\frac{10}{7}}$$

Substitute 2.05 for x.

$$f(2.05) = \log_2 \left(\frac{2.05}{1.23}\right)^{\frac{10}{7}}$$

Apply the quotient rule.

$$f(2.05) = \frac{10}{7} \log_2 1.67$$

Apply the power rule.

Guided Practice: Example 4, *continued*

4. Use a graphing calculator to estimate the value of the inverse logarithmic function at $x = 2.05$.

The process for estimating the function value will depend on the calculator used. Follow the directions specific to your calculator model.

Guided Practice: Example 4, *continued*

On a TI-83/84:

Note: The TI-83/84 can only calculate natural (base-e) and common (base-10) logarithms, so the base change rule $\log_b a = \dfrac{\log_{10} a}{\log_{10} b}$ must be used to rewrite the expression found in step 3 before proceeding:

$$\frac{10}{7}\log_2 1.67 = \frac{10}{7}\left(\frac{\log_{10} 1.67}{\log_{10} 2}\right)$$

Guided Practice: Example 4, *continued*

Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

Step 2: Use the keypad and [LOG] key to enter the values for the expression. Press [ENTER].

Guided Practice: Example 4, *continued*

On a TI-Nspire:

Note: The TI-Nspire can calculate the logarithm directly without first converting it to a base-10 logarithm.

Step 1: Press [home].

Step 2: Arrow down to the calculator icon, the first icon from the left, and press [enter]. Press [ctrl][clear] to create a new document or to clear any previous calculations on the current document.

Guided Practice: Example 4, *continued*

Step 3: Press [ctrl][10x] to bring up the $\log_{\square}\square$ field.

Step 4: Enter the argument of the logarithm into the blank subscript field. Tab to the next blank field to enter the base. Press [enter].

Step 5: To multiply the result by $\frac{10}{7}$, press [ctrl][(–)]. Then, enter the numbers and the operations using your keypad. Press [enter].

Either calculator will return a result of approximately 1.057.

Guided Practice: Example 4, *continued*

5. Compare the results of step 2 and 4.

Both procedures result in an ordered pair of approximately (2.05, 1.06) for the inverse logarithmic function.

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