## Conceptual Task: Rational Decisions

## Exploration Questions Sample Responses

a. Recall that rational numbers can be written as a ratio of integers. Will the numerator and denominator of a product of rational numbers also be integers? How do you know?

Students should recognize that multiplying integers by integers yields integer results. Since the product of two rational numbers can be written as a product of two ratios of integers, the numerator and denominator of the product will also be integers. Therefore, the product of two rational numbers is rational, and the second statement provided in the stem is true.

b. Will the numerator and denominator of a sum of rational numbers also be integers? How do you know?

Students should recognize that adding integers to integers yields integer results. Since the sum of two rational numbers can be written as a sum of two ratios of integers, the numerator and denominator of the sum will also be integers. Therefore, the sum of two rational numbers is rational, and the first statement provided in the stem is true.

*Note*: since adding fractions may require multiplication in order to obtain matching denominators, both addition and multiplication of integers may be involved; remind students of this.

c. Recall that if you write a rational number as a decimal, the decimal either ends or repeats. The decimal 0.101001000100001 ... is an irrational number. Why?

Students should recognize that although there is a distinct pattern, the decimal itself never repeats because the strings of zeros are getting longer and longer.

d. Add 1 to 0.101001000100001 ... . Is the result still irrational? Why?

Students should recognize that the result (1.101001000100001 ...) is irrational because the decimal expansion still never repeats.

e. Multiply 0.101001000100001 ... by 2. Is the result still irrational? Why?

Students should recognize that the result (0.202002000200002 ...) is irrational because the decimal expansion still never repeats.

f. Consider the sum $a + b$, where $a$ is rational and $b$ is irrational. What happens if you subtract $a$? Is the result rational or irrational?

Students should recognize that $a + b - a = b$, which is irrational by definition.

g. Joe is convinced that the sum of a rational and irrational number can be rational. How could you prove or disprove his claim?

Students can use the argument presented in the previous question to disprove Joe's claim:

> Suppose $a$ is rational and $b$ is irrational. Further suppose that $a + b$ is rational. Then $a + b - a = b$ is rational because the sum of two rational numbers is rational. But $b$ is irrational, so this is a contradiction.

If students struggle to find the connection, ask them what happens if you assume the sum $a + b$ is rational.

h. Can you use a similar argument to show that the product of a rational number and an irrational number is irrational? Explain. (You need not write a proof.)

A similar argument can be used for the product:

> Suppose $a$ is rational and $b$ is irrational. Further suppose that $a \bullet b$ is rational. Then $a \bullet b \bullet \frac{1}{a} = b$ is rational because $\frac{1}{a}$ is rational and the product of two rational numbers is rational. But $b$ is irrational, so this is a contradiction.

Students need not write out this proof, but should recognize that the logic used to prove the addition case can be adapted for the multiplication case.

i. The expressions $\sqrt{2} + \sqrt{2}$ and $\sqrt{2} \bullet \sqrt{3}$ are both irrational. Will the sum or product of two irrational numbers always be irrational? Or can it be rational? Provide specific examples.

Answers may vary. Students should realize that the sum can indeed be rational. For example, $0.1010010001... + 0.0101101110... = 0.1111... = \frac{1}{9}$. Another example is that $\sqrt{2} + \left(-\sqrt{2}\right) = 0$.

Similarly, the product of two irrational numbers can be rational. For example, $\sqrt{2} \bullet \sqrt{2} = 2$.

j. Formulate additional statements about the sums and products of irrational numbers to add to the list from the problem scenario.

Answers may vary. Sample statements:

- The sum of two irrational numbers can be rational or irrational.
- The product of two irrational numbers can be rational or irrational.