Rational and Irrational Numbers and Their Properties

Prerequisite Skills

This lesson requires the use of the following skills:

- applying the properties of integer exponents
- understanding the real number system, including rational numbers
- differentiating between roots and powers

Introduction

The properties of exponents apply even when the result is an irrational number. In this section, we will see how the properties of exponents can be used to write equivalent exponential expressions involving irrational numbers. We will also learn how to tell if a sum or product will be rational or irrational, and we will solve equations involving irrational expressions.

Key Concepts

- An **irrational number** is a real number that cannot be expressed as the ratio of two integers. In other words, it cannot be written as a fraction that has integers for both the numerator and denominator.
- The decimal representations of irrational numbers neither end nor repeat.
- The root of a number for which there is no rational root is an irrational number, such as $\sqrt{2}$ or $\sqrt[3]{5}$.
- The set of integers is closed under addition and multiplication, meaning that the sum of any integers is an integer, and the product of any integers is also an integer.
- Equations involving a power can be solved by taking a root, and equations involving a root can be solved by raising to a power.
- In order to solve an equation involving a power or a root, isolate the variable.
- To solve an equation that has a rational exponent, raise both sides to the multiplicative inverse (reciprocal) of the exponent.
- If an expression is written using a power and a root, it can be rewritten using a rational exponent and then solved using the reciprocal of the rational exponent. Or, each operation, the root and the power, can be "undone" in separate steps.
- The same operation must be performed on both sides of the equation.

- In the following equations, x is the variable being solved for and a, b, m, and n represent integers, with $m \neq 0$ and $n \neq 0$.

Power:

$$x^a = b$$
$$\sqrt[a]{x^a} = \sqrt[a]{b}$$
$$x = \sqrt[a]{b}$$

Root:

$$\sqrt[a]{x} = b$$
$$\left(\sqrt[a]{x}\right)^a = b^a$$
$$x = b^a$$

Rational exponent:

$$x^{\frac{m}{n}} = b$$
$$\left(x^{\frac{m}{n}}\right)^{\frac{n}{m}} = b^{\frac{n}{m}}$$
$$x^{\frac{m}{n} \bullet \frac{n}{m}} = b^{\frac{n}{m}}$$
$$x = b^{\frac{n}{m}}$$

Rational root:

$$\sqrt[n]{x^m} = b$$
$$\left(\sqrt[n]{x^m}\right)^n = b^n$$
$$x^m = b^n$$
$$\sqrt[m]{x^m} = \sqrt[m]{b^n}$$
$$x = \sqrt[m]{b^n}$$

- The set of rational numbers is closed under addition and multiplication, so the sum of two rational numbers is rational and the product of two rational numbers is rational.
- On the other hand, the sum of a rational number and an irrational number is an irrational number, as is the product of a rational number and an irrational number. An irrational number has a decimal that never ends or repeats. So, if a rational number is added to or multiplied by an irrational number, the result will always be irrational because its decimal will never end or repeat. For example, $2+\sqrt{2} \approx 2 + 1.41421... \approx 3.41421...$, and $2 \bullet \sqrt{2} \approx 2 \bullet 1.41421... \approx 2.82842...$.
- The sum or product of two irrational numbers can be either rational or irrational. To determine which it is, simplify the expression.

Common Errors/Misconceptions

- incorrectly categorizing a root as irrational when an exact root exists
- incorrectly categorizing a repeating or terminating decimal as irrational
- incorrectly raising a power to a power when solving an exponential equation
- trying to solve an exponential equation without first isolating the variable
- confusing the rules for the Power of a Power Property, the Power of a Product Property, and the Product of Powers Property