Rational and Irrational Numbers and Their Properties

Warm-Up

Ahmed started a savings account in 2000. The balance of the account is modeled by the equation $f(t) = (225)(1.04)^t$, where $t = 0$ represents the year 2012.

1. What was the balance in the account in 2010?

2. What was the balance in the account in 2012?

1. What was the balance in the account in 2010?

- For the year 2010, the value of *t* is –2, since 2010 is two years before 2012.

$$f(-2) = 225(1.04)^{-2}$$

$$f(-2) = 225\left(\frac{1}{1.04^2}\right)$$

$f(-2)$ » 208.03

- In 2010, the balance was approximately $208.03.

2. What was the balance in the account in 2012?

- For the year 2012, the value of t is 0.

$$f(0) = 225(1.04)^0$$

$$f(0) = 225(1)$$

$$f(0) = 225$$

- In 2012, the balance in the account was $225.00.

Instruction

Introduction

- The properties of exponents apply even when the result is an irrational number.
- In this section, we will see how the properties of exponents can be used to write equivalent exponential expressions involving irrational numbers.
- We will also learn how to tell if a sum or product will be rational or irrational, and we will solve equations involving irrational expressions.

Key Concepts

- An irrational number is a real number that cannot be expressed as the ratio of two integers. In other words, it cannot be written as a fraction that has integers for both the numerator and denominator.
- The decimal representations of irrational numbers neither end nor repeat.
- The root of a number for which there is no rational root is an irrational number, such as $\sqrt{2}$ or $\sqrt[3]{5}$.
- The set of integers is closed under addition and multiplication, meaning that the sum of any integers is an integer, and the product of any integers is also an integer.

Key Concepts, *continued*

- Equations involving a power can be solved by taking a root, and equations involving a root can be solved by raising to a power.
- In order to solve an equation involving a power or a root, isolate the variable.
- To solve an equation that has a rational exponent, raise both sides to the multiplicative inverse (reciprocal) of the exponent.

Key Concepts, *continued*

- If an expression is written using a power and a root, it can be rewritten using a rational exponent and then solved using the reciprocal of the rational exponent.
- The same operation must be performed on both sides of the equation.

Key Concepts, *continued*

- In the following equations, x is the variable being solved for and a, b, m, and n represent integers, with $m \neq 0$ and $n \neq 0$.

Power:

$$x^a = b$$

$$\sqrt[a]{x^a} = \sqrt[a]{b}$$

$$x = \sqrt[a]{b}$$

Root:

$$\sqrt[a]{x} = b$$

$$\left(\sqrt[a]{x}\right)^a = b^a$$

$$x = b^a$$

Key Concepts, *continued*

Rational exponent:

$$x^{\frac{m}{n}} = b$$

$$\left(x^{\frac{m}{n}}\right)^{\frac{n}{m}} = b^{\frac{n}{m}}$$

$$x^{\frac{m}{n} \cdot \frac{n}{m}} = b^{\frac{n}{m}}$$

$$x = b^{\frac{n}{m}}$$

Rational exponent:

$$\sqrt[n]{x^m} = b$$

$$\left(\sqrt[n]{x^m}\right)^n = b^n$$

$$x^m = b^n$$

$$\sqrt[m]{x^m} = \sqrt[m]{b^n}$$

$$x = \sqrt[m]{b^n}$$

Key Concepts, *continued*

- The set of rational numbers is closed under addition and multiplication.
- The sum of two rational numbers is rational and the product of two rational numbers is rational.
- On the other hand, the sum of a rational number and an irrational number is an irrational number, as is the product of a rational number and an irrational number.

Key Concepts, *continued*

- An irrational number has a decimal that never ends or repeats.
- If a rational number is added to or multiplied by an irrational number, the result will always be irrational because its decimal will never end or repeat. For example, $2+\sqrt{2} \approx 2 + 1.41421... \approx 3.41421...$, and $2 \cdot \sqrt{2} \approx 2 \cdot 1.41421... \approx 2.82842...$.
- The sum or product of two irrational numbers can be either rational or irrational. To determine which it is, simplify the expression.

Common Errors/Misconceptions

- incorrectly categorizing a root as irrational when an exact root exists
- incorrectly categorizing a repeating or terminating decimal as irrational
- incorrectly raising a power to a power when solving an exponential equation
- trying to solve an exponential equation without first isolating the variable
- confusing the rules for the Power of a Power Property, the Power of a Product Property, and the Product of Powers Property

Guided Practice

Example 1

Simplify the expression $a^{\frac{6}{5}} \cdot a^{\frac{3}{2}}$.

Guided Practice: Example 1, *continued*

1. Identify which property can be used to simplify the expression.

This is the product of two exponential expressions with the same base.

Use the Product of Powers Property to simplify.

Guided Practice: Example 1, *continued*

2. Apply the property to simplify the expression.

The Product of Powers Property states that if the bases are the same, the expression can be written as the single base raised to the sum of the powers.

$$a^{\frac{6}{5}} \bullet a^{\frac{3}{2}} = a^{\frac{6}{5}+\frac{3}{2}} = a^{\frac{12}{10}+\frac{15}{10}} = a^{\frac{27}{10}}$$

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Guided Practice: Example 1, *continued*

Guided Practice

Example 2

Simplify the expression $\dfrac{b^{\frac{7}{9}}}{b^{\frac{8}{3}}}$.

Guided Practice: Example 2, *continued*

1. **Identify which property can be used to simplify the expression.**

This is the quotient of two exponential expressions with the same base.

Use the Quotient of Powers Property to simplify.

Guided Practice: Example 2, *continued*

2. Apply the property to simplify the expression.

The Quotient of Powers Property states that since the bases are the same, the expression can be written as the base raised to the difference between the power of the numerator and the power of the denominator.

Guided Practice: Example 2, *continued*

If the difference is negative, the Negative Exponent Property can be used to rewrite the expression so that the exponent is positive.

$$\frac{b^{\frac{7}{9}}}{b^{\frac{8}{3}}} = b^{\frac{7}{9} - \frac{8}{3}} = b^{\frac{7}{9} - \frac{24}{9}} = b^{-\frac{17}{9}} \text{ or } \frac{1}{b^{\frac{17}{9}}}$$

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Guided Practice: Example 2, *continued*

Guided Practice

Example 3

Evaluate the exponential expression $\left(3^{\frac{1}{2}}\right)^{\frac{4}{3}}$. Round your answer to the nearest thousandth, if necessary. Then, determine whether the answer is rational or irrational.

Guided Practice: Example 3, *continued*

1. Simplify the expression using properties of exponents.

An expression with a power of a power can be rewritten using the product of the powers.

$$\left(3^{\frac{1}{2}}\right)^{\frac{4}{3}} = 3^{\frac{1}{2} \cdot \frac{4}{3}} = 3^{\frac{4}{6}}$$

Guided Practice: Example 3, *continued*

2. Write the rational exponent in simplest form. Be sure to include absolute value if the original expression involved finding an even root.

The exponent, $\frac{4}{6}$, can be reduced to $\frac{2}{3}$.

The original root is even, so the sixth root is positive.

$$3^{\frac{4}{6}} = 3^{\frac{2}{3}}$$

Guided Practice: Example 3, *continued*

3. Evaluate the power and root of the function, using a calculator if needed.

Note that the power of a power exponent property can be used to rewrite the expression $x^{\frac{a}{b}}$ as

$\left(x^{\frac{1}{b}}\right)^{a}$ or $\left(x^{a}\right)^{\frac{1}{b}}$, so either the root or power can be

evaluated first.

Guided Practice: Example 3, *continued*

3. Evaluate the power and root of the function, using a calculator if needed.

The cube root of 3 is not an integer, so a calculator will be needed to approximate the root.

The expression 3^2 can be evaluated first without using a calculator: $3^2 = 9$.

$$3^{\frac{2}{3}} = \left(3^2\right)^{\frac{1}{3}} = 9^{\frac{1}{3}} \approx 2.080$$

Guided Practice: Example 3, *continued*

4. **Determine whether the answer is rational or irrational.**

When evaluated, the expression $\left(3^{\frac{1}{2}}\right)^{\frac{4}{3}}$ is equal to $9^{\frac{1}{3}}$,

which cannot be expressed as a ratio of integers.

Therefore, this expression is irrational. ✔

Guided Practice

Example 4

Lochlan has a savings account. The total account balance, y, after any number of years, t, can be found using the equation $y = 5000(x)^t$, where x is equal to 1 plus the annual interest rate.

The total balance in the account is currently $6,203.74, and Lochlan has had the account for $5\frac{1}{2}$ years. What is the annual interest rate?

Guided Practice: Example 4, *continued*

1. **Replace any variables with known quantities.**

   The equation showing the total account balance y after t years is $y = 5000(x)^t$.

   If the total account balance is $6,203.74, then y = $6,203.74, $t = 5\frac{1}{2} = \frac{11}{2}$, and x is unknown.

$$6203.74 = 5000 \bullet x^{\frac{11}{2}}$$

Guided Practice: Example 4, *continued*

2. **Use inverse operations to find an equivalent equation in the form $x^a = b$, where x is the variable.**

Multiply both sides by the reciprocal of 5,000 to isolate the exponential expression, $x^{\frac{11}{2}}$.

$$\left(\frac{1}{5000}\right)\bullet 6203.74=\left(\frac{1}{5000}\right)\bullet 5000\bullet x^{\frac{11}{2}}$$

$$1.2407 = x^{\frac{11}{2}}$$

Guided Practice: Example 4, *continued*

3. **Solve by raising both sides to the reciprocal of the rational exponent.**

$$\left(1.2407\right)^{\frac{2}{11}} = \left(x^{\frac{11}{2}}\right)^{\frac{2}{11}}$$

$$1.040 = x$$

Since x is equal to 1 plus the interest rate, the interest rate must be 1.04 – 1 = 0.04, or 4%.

Guided Practice

Example 5

Solve the equation $\sqrt[4]{x^3} = 125$.

Guided Practice: Example 5, *continued*

1. Rewrite the equation using a rational exponent.

The variable x is being raised to the third power with a fourth root, so the rational exponent is $\frac{3}{4}$.

$$\sqrt[4]{x^3} = 125$$

$$x^{\frac{3}{4}} = 125$$

Guided Practice: Example 5, *continued*

2. Determine the reciprocal of the rational exponent.

The rational exponent is $\frac{3}{4}$.

The reciprocal is $\frac{4}{3}$.

Guided Practice: Example 5, *continued*

3. **Raise both sides to the reciprocal of the rational exponent.**

$$\left(x^{\frac{3}{4}}\right)^{\frac{4}{3}} = (125)^{\frac{4}{3}}$$

$$x = 125^{\frac{4}{3}}$$

Guided Practice: Example 5, *continued*

4. If possible, find the root of the quantity before raising to the power to find *x*.

$125 = 5^3$, so the third root of 125 is 5.

$$x = 125^{\frac{4}{3}} = \left(125^{\frac{1}{3}}\right)^4 = 5^4$$

$x = 625$

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