Problem-Based Task: Estimating Depreciation

Task Overview

Focus

Assuming that a car depreciates in value at a constant rate each year, how can you determine that rate if you know how much you paid for the car when it was new, how much it's worth now, and how much time has passed since you bought it new? Do you have to wait for the start of the next year so that you can use integer exponents to calculate the value? In this task, students will use what they know about exponents and rates of growth and decay to solve for a decay rate using a non-integer, rational exponent.

This activity will provide practice with:

- substituting for variables
- determining the independent and dependent variables
- isolating a variable that has a rational exponent
- raising a number to a rational exponent
- applying the properties of exponents and extending them to rational exponents
- applying the order of operations
- analyzing a solution for reasonableness
- interpreting a solution in terms of a context

Introduction

This task should be used to explore or apply the skill of developing an exponential model for a real-life situation. Students should already be familiar with exponential models from previous lessons.

Begin by reading the problem and clarifying the meaning of *depreciate*:

depreciate to decrease in value

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 5:** Use appropriate tools strategically.

  Students will most likely be using a scientific or graphing calculator that has the ability to store the latest calculation for use in future calculations. Be prepared to show students how to efficiently use this feature, especially if calculating the value of the car for a specific number of years. Students will often write intermediate calculations on a piece of paper and retype them into the calculator, even though this is unnecessary. Also, students often struggle with raising a quantity to a rational power on a calculator since there are no buttons for this. Encourage students to think about how they raise a quantity to a power of 4 and to follow the same procedure. Stress the importance of placing the rational exponent in parentheses.

- **SMP 6:** Attend to precision.

  Students might want to write down the final number exactly as it appears on their calculator as their solution. Ask students to think about what level of precision makes sense for the context of the problem. (**Answer:** "Because the problem involves estimating future values for a car, it would make sense to round to the nearest 100 or 1,000.") Also, students might want to round before finishing their calculations. Encourage students to store intermediate results on their calculators to maintain precision as much as possible until the last step, when rounding should occur.

- **SMP 7:** Look for and make use of structure.

  Students might have difficulty seeing that the rational exponent 2.5 can be converted into a fraction. Ask students to think about how they can change the form of the exponent so that it is easier to simplify the variable with that exponent (**Answer:** "I can rewrite the exponent as a fraction.")

  Ask students how to use the structure of the fraction to find the power to which you can raise both sides of the equation in order to isolate the variable. (**Answer:** "You find the multiplicative inverse, or the reciprocal, of the fraction by 'flipping' the fraction.")

  Ask students about the structure of the equation and what type of model this equation follows. (**Answer:** "This is an exponential decay model.")

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- forgetting to isolate the variable with the exponent before solving for the variable

  When solving $16{,}905 = 22{,}000 \bullet d^{2.5}$ for d, the first step is to isolate $d^{2.5}$ by dividing both sides by 22,000.

- attempting to eliminate the rational exponent by treating it as a factor rather than an exponent

  Students may try to eliminate the exponent by dividing by it or multiplying by the reciprocal.

  Point out that to eliminate the rational exponent and isolate the variable d, you must write the exponent 2.5 as a fraction, $\frac{5}{2}$, and then raise both sides of the equation to this power.

- converting the decimal to a fraction incorrectly

  Explain that 2.5 can be read "two and a half." Another way to write "two and a half" is $2\frac{1}{2}$. Written as an improper fraction, this becomes $\frac{5}{2}$.

- making division errors when isolating the variable

  Make sure students isolate the variable by dividing 16,905 by 22,000 and not the other way around.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.
- If students have difficulty starting the task:
  - Ask them to identify which variable needs to be determined in the equation $c = 22{,}000 \bullet d^t$. (**Answer:** "We are trying to find the value of the variable *d*.")
  - Ask students how they know this information. (**Answer:** "The equation has three variables: *c*, *d*, and *t*. The only value not given in the problem statement is *d*.")
- If students have difficulty determining which numbers to substitute for which variables, guide them to think about the structure of the equation and what we are solving for. Ask:
  - "What type of model does this equation follow?" (**Answer:** an exponential model)

- "What is the general form of an exponential equation?" (**Answer:** $y = a \bullet b^x$)
- "What does each variable in the general form represent in terms of this context?" (**Answer:** "The variable y represents the car's value after x years, a represents the initial value, and b represents the rate of growth or decay.")
- "From the general form, which variable are we trying to determine?" (**Answer:** "We are trying to determine b, or the rate of growth/decay.")
- Ask students whether this situation reflects exponential growth or decay, and how they know. (**Answer:** "This situation represents exponential decay because the value of the car is decreasing.")
- From this information, ask students to substitute in the known values for the corresponding variables in the given equation, $c = 22{,}000 \bullet d^t$. (**Answer:** "Comparing the given equation $c = 22{,}000 \bullet d^t$ to the general form $y = a \bullet b^x$, c corresponds to y, 22,000 corresponds to a, d corresponds to b, and t corresponds to x. I can substitute known values for c and t: $c = 16{,}905$ (the current value of the car) and $t = 2.5$ years. The resulting equation is $16{,}905 = 22{,}000 \bullet d^{2.5}$.")

- If students are uncertain about how to work with a variable that has an exponent:
  - Encourage students to isolate the variable as much as possible by using inverse operations. (**Answer:** Divide both sides of the equation by 22,000 to get $0.768 \approx d^{2.5}$.)
  - Ask, "How would you handle solving for a variable if the exponent was 2?" (**Answer:** "I would take the square root of both sides.") "What if the exponent was 3?" (**Answer:** "I would take the cube root of both sides.")
  - "What's another way to write the square root and the cube root?" (**Answer:** "By using the exponents $\frac{1}{2}$ and $\frac{1}{3}$.")
  - "What is the relationship between an exponent of 2 on a squared variable and an exponent of $\frac{1}{2}$ when taking the square root?" (**Answer:** "The numbers 2 and $\frac{1}{2}$ are multiplicative inverses.")
  - "How can you use this structure here to solve for the variable?" (**Answer:** "I can rewrite 2.5 as a fraction and then find its multiplicative inverse.")
  - "What is 2.5 written as a fraction?" (**Answer:** $\frac{5}{2}$)
  - "What is the multiplicative inverse of $\frac{5}{2}$?" (**Answer:** $\frac{2}{5}$)

- "How can you use the multiplicative inverse to solve for the variable?" (**Answer:** "I can raise both sides of the equation to that power; when raising a power to a power, multiply the exponents. When I multiply multiplicative inverses, the result is 1. This means that the exponent on the variable is now 1.")
- Ask students to show how they raise both sides of the equation to the power of $\frac{2}{5}$ symbolically. (**Answer:** $(0.768)^{\frac{2}{5}} \approx \left(d^{\frac{5}{2}}\right)^{\frac{2}{5}}$)

• Once students have solved the resulting equation for *d*, ask:

- "What does the answer to this equation mean?" (**Answer:** "This is the approximated depreciation rate of Yasmina's car each year. Because the value for *d* is 0.90 or, as a percentage, 90%, this means that each year, her car retains 90% of its previous value.")
- "How can this information be used to create a model for the value of Yasmina's car for the value of her car at any time?" (**Answer:** "Substitute the value of *d* into the equation given in the task: $c = 22{,}000 \bullet (0.90)^t$.")

• For students who complete the Coaching questions:

- Once students have calculated the value of Yasmina's car after 6 years, ask them to write the solution in a sentence to interpret its meaning. (**Answer:** "After 6 years, the car will be worth approximately $11,692, or about half of its original value.")
- Ask, "Do you think that it's reasonable for the car to lose nearly half of its original value after 6 years?" (**Sample answer:** "I was surprised to find out how quickly cars can depreciate.")

Debriefing the Task

Compare students' strategies for solving the problem. Encourage students to explain their thinking as they were solving the problem, particularly the connections they were able to recognize between this model and previous exponential decay models they have seen. Ask students to draw similarities between solving this equation and solving other exponential decay equations. (**Answer:** "This problem and other exponential decay problems I've seen had the same general structure, $y = a \bullet b^x$, where $b < 0$ because it represented decay.")

Connecting to Key Concepts

Make explicit connections to key concepts:

- The properties of exponents apply even when the result is an irrational number.

  When raising a power to a power, you multiply exponents.

- To solve an equation that has a rational exponent, raise both sides to the multiplicative inverse (reciprocal) of the exponent.

  In this task, the exponent on the variable is $\frac{5}{2}$, so raise both sides of the equation to the power of $\frac{2}{5}$.

Extending the Task

- To extend the task, have students use the model that they developed to calculate the estimated value of Yasmina's car after some particular number of years. Then, have students graph the model and analyze the graph of the function. Ask students to describe what they notice about the graph. (**Sample answer:** "The graph shows that the value of the car depreciates less each year as the years go by, so that the car never reaches a value of $0.")

- Another way to extend the task is to have students look up various cars' prices when new, and then find the cars' values depending on the condition of the car after a given number of years. Students could determine the depreciation rate of a car that they are familiar with, and see if the depreciation rate is similar to that of Yasmina's car. The following website can be used to find current values, though researching an older car's original price will require additional investigation.

  Kelley Blue Book. "Get Your Car Value."

  http://www.walch.com/rr/07002

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Answer:** "On my calculator, I used the memory and 'ans' features to help calculate the value of the car for a given value of *t*, and I used the '^' button to raise numbers to rational exponents.")
- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Answer:** "I used precise language when describing what I did, and I thought about the best way to round the calculated values in this situation.")
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** "I recognized that this situation represents an exponential decay model, and that I was being asked to find the decay rate. I reasoned that since this problem models exponential decay, the base must be a number less than 1. Furthermore, I reasoned that the number being multiplied, 0.90^t, must be the starting value. When I substituted known values into the equation and realized that the exponent 2.5 was rational, I remembered that equations in this form can be solved by raising both sides of the equation to the power that is the multiplicative inverse of the given exponent.")

Alternate Strategies or Solutions

- When asked to evaluate the model for $t = 6$, students may choose to graph the function using technology and use the graph to calculate the value of the car after 6 years.
- Students may also use the table feature of a graphing calculator to find this value.

Technology

Students can use scientific calculators and/or graphing technology to help solve this problem.