Defining, Rewriting, and Evaluating Rational Exponents

Warm Up

A population at any time *t* can be estimated using the equation $p_t = p_0 \cdot (1 + r)^t$, where p_0 is the initial population, r is the annual growth rate, and t is the time in years from now. Today, Tinyville and Littletown both have a population of 10,000 people. Tinyville's yearly growth rate is 2.5% and Littletown's yearly growth rate is 1.8%.

1. Which town will have the larger population after 5 years?
2. What equation can be used to find the approximate population of Littletown after *t* years?
3. What will be the approximate population of Tinyville in 10 years?

1. Which town will have the larger population after 5 years?

   Since both towns have the same population today, each town's population growth rate will determine which town has the larger population in 5 years.

   Tinyville has a higher yearly growth rate, and will have a greater population in 5 years.

2. **What equation can be used to find the approximate population of Littletown after *t* years?**

Write an equation using Littletown's starting population and yearly growth rate.

Replace p_0 with 10,000 and r with 0.018 in the equation $p_t = p_0 \cdot (1 + r)^t$. Simplify the equation.

$$p_t = 10{,}000 \cdot (1 + 0.018)^t$$

$$p_t = 10{,}000 \cdot (1.018)^t$$

3. What will be the approximate population of Tinyville in 10 years?

Write an equation using Tinyville's starting population, its yearly growth rate, and the time in years.

Replace p_0 with 10,000, r with 0.025, and t with 10 in the equation $p_t = p_0 \cdot (1 + r)^t$ such that

$p_{10} = 10{,}000 \cdot (1 + 0.025)^{10}$.

Evaluate the expression to find p_{10}.

$$p_{10} = 10{,}000 \cdot (1 + 0.025)^{10}$$

$$p_{10} = 10{,}000 \cdot (1.025)^{10}$$

p_{10} » $10{,}000 \cdot (1.28008)$

p_{10} » $12{,}800.8$

The approximate population of Tinyville in 10 years will be 12,801 people.

Instruction

Introduction

- An exponent is a quantity that shows the number of times a factor is being multiplied together in an exponential expression.
- In an expression written in the form a^b, b is the exponent.
- So far, the exponents you have worked with have been whole numbers, or the set of positive integers and 0: {0, 1, 2, 3, ...}.
- Integers are positive and negative whole numbers and 0. Exponents can be negative numbers.

Introduction

- They can also be rational numbers, which are real numbers that can be expressed as the ratio of two integers.
- Rational exponents are simply another way to write radical expressions.
- For example, $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[3]{x} = x^{\frac{1}{3}}$.
- As we will see in this lesson, the rules and properties that apply to integer exponents also apply to rational exponents.

Key Concepts

- An exponential expression contains a base and an exponent.
- A base is the factor that is being multiplied together in an exponential expression. In other words, it is the quantity that is being raised to a power, and the power is determined by the exponent.
- A power is the result of raising a base to an exponent; 32 is a power of 2 since $2^5 = 32$. In the expression 2^5, 2 is the base, 5 is the exponent, and the entire expression, 2^5, is a power.

Key Concepts, *continued*

- A radical expression contains a root, which can be shown using the radical symbol, $\sqrt{}$.
- The root of a number *x* is a number that, when multiplied by itself a given number of times, equals *x*.
- The root of a function is also referred to as the inverse of a power, and "undoes" the power.

For example, $\sqrt[3]{8} = 2$ and $2^3 = 8$.

Key Concepts, *continued*

- In the radical expression $\sqrt[n]{a^n}$, the *n*th root of the *n*th power of *a* is *a*.
- For example,

$$\sqrt[n]{a} = a^{\frac{1}{n}} \quad \text{and} \quad \sqrt[n]{x^m} = x^{\frac{m}{n}}$$

Key Concepts, *continued*

- A rational exponent is an exponent that is a rational number.
- A rational number is a real number that can be written as $\frac{m}{n}$, where both m and n are integers and $n \neq 0$.
- The denominator of the rational exponent is the root, and the numerator is the power.

For example, $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

Key Concepts, *continued*

- An exponential equation is an equation whose independent variable is in the exponent.
- An exponential equation can be written as $y = ab^x$, where x is the independent variable, y is the dependent variable, and a and b are real numbers.
- Real numbers are the set of all rational and irrational numbers.
- The properties of exponents apply to both integer and rational exponents.

Key Concepts, *continued*

Properties of Exponents

Words	Symbols	Numbers
Zero Exponent Property A base raised to the power of 0 is equal to 1.	$a^0 = 1$	$12^0 = 1$
Negative Exponent Property A number raised to a negative exponent is equal to the reciprocal of the number raised to a positive power.	$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}$, $a \neq 0, n \neq 0$	$64^{-\frac{2}{3}} = \frac{1}{64^{\frac{2}{3}}} =$ $\frac{1}{\left(\sqrt[3]{64}\right)^2} = \frac{1}{16}$

Key Concepts, *continued*

Words	Symbols	Numbers
Product of Powers Property To multiply powers with the same base, add the exponents.	$a^m \cdot a^n = a^{m+n}$	$3^{\frac{1}{4}} \cdot 3^{\frac{7}{4}} = 3^{\frac{1}{4}+\frac{7}{4}} =$ $3^2 = 9$
Quotient of Powers Property To divide powers with the same base, subtract the exponents.	$\frac{a^m}{a^n} = a^{m-n}$	$\frac{8^{\frac{4}{9}}}{8^{\frac{1}{9}}} = 8^{\frac{4}{9}-\frac{1}{9}} = 8^{\frac{1}{3}} = 2$

Key Concepts, *continued*

Words	Symbols	Numbers
Power of a Power Property To raise one power to another power, multiply the exponents.	$\left(a^m\right)^n = a^{m \cdot n}$	$\left(5^{\frac{2}{3}}\right)^3 = 5^{\frac{2}{3} \cdot 3} =$ $5^2 = 25$
Power of a Product Property To find the power of a product, raise each factor in the product to the given power.	$\left(ab\right)^m = a^m b^m$	$(25 \bullet 36)^{\frac{1}{2}} =$ $25^{\frac{1}{2}} \bullet 36^{\frac{1}{2}} =$ $5 \bullet 6 = 30$

Key Concepts, *continued*

Words	Symbols	Numbers
Power of a Quotient Property To find the power of a quotient, raise each term in the quotient to the given power.	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{25}{49}\right)^{\frac{1}{2}} = \frac{25^{\frac{1}{2}}}{49^{\frac{1}{2}}} = \frac{5}{7}$

- Either the power or root can be determined first when evaluating an exponential expression with a rational exponent.

Key Concepts, *continued*

- Rational exponents can be reduced to simplest form before evaluating a radical expression.
- Use the absolute value symbol to represent the *n*th root of a number when *n* is an even number, because even roots are never negative.
- For example, the square root of x^2 can be written as $\left(x^2\right)^{\frac{1}{2}}$, which is equal to $|x|$.

Key Concepts, *continued*

- An even root is always positive.
- For example, although 2^2 and $(–2)^2$ are both equal to 4, $\sqrt{4}$ is equal to 2, not –2.
- Sometimes rational exponents appear as decimals. For example, $x^{0.25}$ is equal to $x^{\frac{1}{4}}$ or $\sqrt[4]{x}$.

Common Errors/Misconceptions

- not identifying the denominator of a rational exponent as being a root
- incorrectly evaluating an exponential expression with multiple operations

Guided Practice

Example 1

How can the exponential expression $3^{\frac{6}{5}}$ be rewritten as a radical expression?

Guided Practice: Example 1, *continued*

1. Identify the power.

The power is the numerator of the rational exponent: 6.

$$3^{\frac{6}{5}}$$

Guided Practice: Example 1, *continued*

2. Identify the root. If the root is even, the solution is the absolute value of the expression.

The root is the denominator of the rational exponent: 5.

$$3^{\frac{6}{5}}$$

Since the root is odd, it is not necessary to take the absolute value of the expression.

Guided Practice: Example 1, *continued*

3. Rewrite the expression in either of the following forms: $\sqrt[\text{root}]{\text{base}^{\text{power}}}$ or $\left(\sqrt[\text{root}]{\text{base}}\right)^{\text{power}}$, where the base is the quantity being raised to the rational exponent.

$$3^{\frac{6}{5}} = \sqrt[5]{3^6} = \left(\sqrt[5]{3}\right)^6$$

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Guided Practice: Example 1, *continued*

Guided Practice

Example 2

How can the radical expression $\sqrt[8]{a^c}$ be rewritten as an exponential expression?

Guided Practice: Example 2, *continued*

1. **Identify the numerator of the rational exponent.**

   The numerator is the power: c.

$$\sqrt[8]{a^{c}}$$

Guided Practice: Example 2, *continued*

2. Identify the denominator of the rational exponent.

The denominator is the root: 8.

$$\sqrt[8]{a^c}$$

Guided Practice: Example 2, *continued*

3. Rewrite the expression in the form $\text{base}^{\frac{\text{power}}{\text{root}}}$, where the base is the quantity raised to a power and of which the root is being taken.

$$\sqrt[8]{a^c} = a^{\frac{c}{8}}$$

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Guided Practice: Example 2, *continued*

Guided Practice

Example 3

Evaluate the expression $\sqrt[8]{4^{10}}$. Round your answer to the nearest thousandth.

Guided Practice: Example 3, *continued*

1. Evaluate the power.

$$4^{10} = 1{,}048{,}576$$

Guided Practice: Example 3, *continued*

2. Find an exact root or approximate root using a calculator.

Use a calculator to approximate the eighth root of 1,048,576.

The root, 8, is even, so we only want the positive value of the root.

$$\sqrt[8]{1{,}048{,}576} \text{ » } 5.657$$

Guided Practice

Example 4

A town's population is decreasing. The population in the year 2000 was 4,000, and the population t years after 2000 can be found by using the function $f(t) = 4000(0.96)^t$.

What was the town's approximate population 2.5 years after the year 2000?

Guided Practice: Example 4, *continued*

1. Replace the variable in the equation with the known value.

The variable, *t*, is the number of years after 2000. To find the approximate population 2.5 years after 2000, replace *t* with 2.5.

$$f(2.5) = 4000(0.96)^{2.5}$$

Guided Practice: Example 4, *continued*

2. Evaluate the expression by first calculating the power.

The base of the exponential expression is a decimal. In this case, use a calculator to approximate the population for $t = 2.5$. Since the evaluated function is a population, round to the nearest whole number.

$$f(2.5) = 4000(0.96)^{2.5} \text{ » } 4000(0.9030) \text{ » } 3612$$

2.5 years after the year 2000, the town's approximate population was 3,612 people.