Problem-Based Task: Population Growth

Task Overview

Focus

If a census is taken only once every 10 years, how can you use that data to estimate what the population will be after a specific number of years that isn't a multiple of 10? How do you use the growth rate over a 10-year period to estimate the growth rate over a 1-year period? In this task, students will use a town's current census and estimated 10-year growth rate to estimate the population before the next census is taken.

This activity will provide practice with:

- simplifying fractions
- applying the order of operations
- applying the exponential growth formula
- raising a base to a non-integer, rational exponent
- interpreting the solution in terms of the context of the problem

Introduction

This task should be used to explore or apply the use of rational exponents. Students should already know how to apply the order of operations, and it will be beneficial if they have been exposed to the exponential growth formula and how to raise a number to a rational exponent, although this task may be used to introduce and explore these topics. Graphing may be done using graphing technology.

Begin by reading the problem and clarifying the meaning of the following terms:

census a count of the population; usually taken every 10 years

population growth rate the percentage by which a population grows within a given time period

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 5**: Use appropriate tools strategically.

  Students might struggle with using the calculator effectively and/or efficiently. Students often struggle with raising a quantity to a power other than 2 or 3 since calculators lack specific buttons for rational exponents. Encourage students to think about how they raise a quantity to the fourth power and to follow the same procedure. If students attempt to type in the entire expression all at once, they might omit parentheses. This will result in an incorrect answer. Encourage students to think about the order of operations and why parentheses are necessary.

- **SMP 6**: Attend to precision.

  Students might want to write down the final number exactly as it appears in their calculator as their solution. Ask students to think about what level of precision makes sense for the context of the problem. Ask students if you can have a fraction of a person. Also, students might want to round before finishing their calculations. Encourage students to use the "ans" feature on their calculators so the answer will be as precise as possible until the last step, when rounding should take place. Make sure that students use precise mathematical language when discussing the problem, such as "*y* sub-zero" for y_0.

- **SMP 7**: Look for and make use of structure.

  The population growth model is a model students have seen before. Ask students to think about what the various parts of the formula represent structurally, particularly the base and the exponent. (**Answer:** The base represents growth because it's greater than 1. The 1 indicates 100%, while the 0.35 indicates the 35% percent increase over the 10-year period. The exponent represents *t* divided by 10 years because the growth occurred over 10 years.)

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- not following the order of operations

  Make sure that students apply the exponent to the base before multiplying by the current population.

- miscopying or misreading the formula

  Make sure students understand what each variable in the formula represents, and that they carefully substitute each value into the correct position in the formula. Verify that students are writing the exponent, $\frac{t}{10}$, as an exponent and not as a factor.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.
- If students don't know how to start, have them reread the problem and write what they know.

  Ask students for the general population growth formula for this situation.

  (**Answer:** $y = y_0 \bullet (1+r)^{\frac{t}{10}}$)

  Ask students to verbally identify what each of the variables represents. (**Answer:** "The variable y is the population quantity we are seeking, y_0 is the current population, r is the estimated growth rate over 10 years, and t is the time in years.")
- If students aren't sure which values replace which variables, ask them to reread the problem, paying particular attention to where the problem gives the definition of the variables.

  Ask, "Which variables can you replace with the given quantities, and what are these quantities?" (**Answer:** y_0, r, and t are all given. $y_0 = 42{,}000$, $r = 0.35$, and $t = 8$.)

  Ask students to substitute these values into the equation without simplifying.

  (**Answer:** $y = 42{,}000 \bullet (1+0.35)^{\frac{8}{10}}$)
- Ask students to explain how they think the base shows exponential growth. (**Answer:** The base of the exponent is greater than 1, which indicates growth.)

  Ask students if taking 35% of a number yields a larger or a smaller number. (**Answer:** a smaller number)

  Ask students if the population is shrinking or growing. (**Answer:** It is growing.)

  Ask students how they might account for growth rather than decline by adding one number to the base. (**Answer:** Add 1 to the growth rate.)

  Ask students what adding 1 to the growth rate signifies. (**Answer:** It signifies 100% + 35%.)
- Ask students to simplify the expression inside the parentheses, and then to simplify the exponent, and write the result. (**Answer:** $y = 42{,}000 \bullet (1.35)^{\frac{4}{5}}$)
- Have students make predictions about the approximate value of the population, and ask them to explain their predictions. (**Sample answer:** "The base is being raised to an exponent that is less than 1, so once the exponent is applied, the result will be less than the original base. However, the result will still be greater than 1. This means I'll be multiplying the current population by a number that's a little greater than 1. As a result, the population will increase, but only by a little bit.")

Ask students to think about how raising a base to an exponent equal to, less than, or greater than 1 affects the base. (**Answer:** “Raising a base to 1 results in a quantity that is equal to the base, raising a base to an exponent less than 1 results in a quantity that is less than the original base, and raising a base to an exponent greater than 1 results in a quantity that is larger than the original base.”)

Ask students whether the exponent in this problem is equal to, less than, or greater than 1, and to explain how they know. (**Answer:** “We are raising the base to an exponent less than 1 because the numerator in the exponent is smaller than the denominator.”)

- Students might not be clear about what a rational exponent represents. Ask students to rewrite the base and the exponent using a different structure, after first discussing what it means to raise a base to an exponent of $\frac{4}{5}$. (**Answer:** “It means to take the fifth root of the base and then raise that result to fourth power: $(1.35)^{\frac{4}{5}} = \left(\sqrt[5]{1.35}\right)^4$, or to raise the base to the fourth power and take the fifth root of the result: $(1.35)^{\frac{4}{5}} = \sqrt[5]{1.35^4}$. Both of these methods yield the correct result.”)
- Ask students to simplify the full expression. (**Answer:** $y = 42{,}000 \bullet (1.27136) \approx 53{,}397$)
- After students have found the answer, ask them to think about the context of the problem and the level of precision that makes sense for the context. (**Answer:** “This answer is approximate because the value 1.27136 is rounded off.” *Note*: Depending on how students round, they may have a decimal in their result. Because you cannot have a fraction of a person, it makes sense to round to the nearest whole number. Additionally, because this is an estimated population for a future date, it would also make sense to round to the nearest 10 or 100.)
- Ask if anyone solved the problem graphically and have a volunteer explain this method. (**Sample answer:** “I used my calculator to graph the equation and then I used the calculator to find the y-value when $x = 8$.”)
- Ask these students if they were satisfied with the level of precision with which they were able to determine their answer. (**Answer:** Answers will vary. Have students justify their thinking.)

Have students consider the level of precision for either method of solving. Ask them if the level of precision matters and, if so, to give examples. (**Sample answer:** “The level of precision probably doesn’t matter because the prediction is an estimate only, but the estimate should not include a portion of a person.”)

Have students comment on the accuracy of the level of precision for the estimate. (**Answer:** “The more precise you try to be with your estimate, the more likely you are to be inaccurate. It’s better to round to the nearest 100.”)

Debriefing the Task

Students may have elected to solve the problem by graphing. Ask these students to show their work and explain their methodology. If the solutions found by graphing differ from the solutions found using algebraic methods, have the students who graphed reason why that is, and ask them how they could bring the numbers into closer agreement with results that were found algebraically. (**Answer:** Inaccuracy is involved when reading numbers from a graph. If students read their answer from the graph without using the "calculate" feature of the calculator, the answer can only be as precise as the least precise tick mark. If students are not aware of how to calculate a precise value on a graph, show them how to use the "calculate" feature on their calculators.)

Connecting to Key Concepts

Make explicit connections to key concepts:

- An exponential expression contains a base and an exponent. A base is the quantity that is being raised to a power, and the power is determined by the exponent.

  In this task, the base is 1.35, the exponent is $\frac{4}{5}$, and the power is the value of $(1.35)^{\frac{4}{5}}$.

- Roots can be expressed using rational exponents instead of radical symbols. For example, $\sqrt[n]{a} = a^{\frac{1}{n}}$ and $\sqrt[n]{x^m} = x^{\frac{m}{n}}$.

  In this task, $\sqrt[5]{1.35^4} = (1.35)^{\frac{4}{5}}$.

Extending the Task

- To extend the task, ask students to find the estimated yearly growth rate and create a table of values showing the population for each year from the current year to 8 years from today. Ask students to demonstrate and verify that the yearly growth rate (1.03) holds for each year.
- Another option is to have students graph the function and then analyze the graph.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Sample answer:** "On my graphing calculator, I used the '^' button to raise the base to a rational exponent, and I put the rational exponent in parentheses. I also used parentheses around the base to make sure the order of operations was followed.")

- **For SMP 6, ASK:** "How did you make sure you attended to precision?" (**Sample answer:** "I considered the context of the problem when deciding the level of precision needed for my results. I rounded to the nearest whole number because you can't have a fraction of a person. When I rounded, I decided that rounding to the nearest 10 or even 100 makes sense because we are predicting into the future and our model won't be 100% accurate.")
- **For SMP 7, ASK:** "How did you look for and make use of structure when solving this problem?" (**Answer:** "I used the structure of the exponential growth equation to make predictions about my answer and to check my solution.")

Alternate Strategies or Solutions

- Students may use a graph to solve the problem.
- Students may analyze the data in a table on their graphing calculators. They can enter the function for Y1, then set up the table from $x = 1$ to $x = 10$ to look at the population each year for the first 10 years. The population for year 8, the year asked for in this task, appears on the table.

Technology

Students can use graphing technology.