

Problem-Based Task: Future Finances

Task Overview

Focus

How can a linear function and an exponential function be compared? Which type of function will increase faster? In this task, students will graphically compare a linear function and an exponential function, both of which represent payment plans, and they will interpret their comparisons in order to determine which payment plan will yield the greatest income over given periods of time.

This activity will provide practice with:

- identifying the type of function based on verbal descriptions
- writing linear and exponential functions
- graphing linear and exponential functions
- analyzing information from a graph of a linear and an exponential function
- interpreting the solution of a system of equations
- comparing rates of change for linear and exponential functions

Introduction

This task should be used to explore or apply the skill of comparing a linear and an exponential function graphically in order to observe that a quantity increasing exponentially will eventually exceed a quantity increasing linearly. Students should already be familiar with identifying linear and exponential functions, as well as with graphing functions.

Begin by reading the problem and clarifying the meaning of the terms *exponential function*, *growth factor*, *linear function*, and *rate of change*.

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

Students will construct viable arguments about the amount of income each scenario will generate given specific time intervals in order to draw conclusions about which function will yield more income over time. Encourage students to discuss their arguments with each other and explain their reasoning if they disagree.

- **SMP 4:** Model with mathematics.

Students will recognize that the scenario presented can be modeled by first creating a linear equation and an exponential equation to model the different pay options, and then by graphing both functions on the same coordinate axes. They will analyze the relationships between the two functions in order to draw conclusions about the different pay options.

- **SMP 6:** Attend to precision.

Make sure students are using math terms and definitions clearly and accurately to explain their reasoning. Encourage them to analyze properties of both functions carefully when constructing their reasoning. Strive toward precise mathematical language; for example, by correctly using the terms *linear function*, *exponential function*, *rate of change*, and *exponential growth*. Also, remind students to carefully label the axes on their graphs.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- incorrectly identifying a function as linear when it is exponential and vice versa
Remind students that they must check several intervals of a function in order to determine if the rate of change is constant or not. If the rate is the same for each interval, then the function is linear, and if the rate is not the same for each interval, then the function is exponential.
- incorrectly determining the rate of change
Remind students that the rate of change for a linear function is constant, and that this rate is also the slope of the function. The rate of change for an exponential function is not constant, but the average rate of change can be found for specific intervals of the function.
- assuming the rate of change of a function is linear by only referencing one interval
Once a function has been identified as exponential, or non-linear, then there will be different rates of change for different intervals of the function. Remind students to check several intervals of a non-linear function in order to verify that they do not have the same rate of change.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Before students begin the task, ask them to predict which of the two payment options they think will be the better choice for Cole. Ask, “Based on the verbal description of the scenarios, which option do you think will be better for Cole and why?” (**Answer:** Some students may choose the first option because it presents a higher initial value as compared to the second option, which begins with only \$0.01. Some students may initially recognize that the second option is an exponential situation, and reason that it will turn out to be the better option. Ask students to write down their prediction, along with their reasoning.)
- As students begin the task, ask them how they are able to determine that the first payment option will be represented by a linear function. (**Answer:** “Since Cole will be receiving \$3,495.25 each month, then the rate of change between any two points, where the variable x is the time in months, and y is the amount of money, will always be the same. The slope, or rate of change, will be \$3,495.25.”)

- If students have difficulty determining that the second pay option is represented by an exponential function, ask, “What term in the problem indicates that this function will be exponential?” (**Answer:** “The word ‘doubles’ indicates that this will be an exponential function.”)
- If students have difficulty writing the exponential function to represent the second pay option, ask them to identify the values for the variables in the format of an exponential function, $g(x) = ab^x + k$. (**Answer:** “In an exponential function, a represents the initial value, which is 0.01, b represents the base, which in this situation is a growth factor of 2, and x represents the time, which is in months for this problem. There is no value of k for this problem because the y -intercept is 0.”)
- Ask students, “In the exponential function, $g(x) = (0.01)(2)^{x-1}$, why is the exponent written as $x - 1$ instead of just x ?” (**Answer:** “The exponent is written as $x - 1$ because the amount of pay for a given month is based on doubling the previous month’s pay. For example, the pay for the third month ($x = 3$) is based on doubling the pay for month 2.”)
- Ask students, “What is the significance of ‘24 months’ for the two functions?” (**Answer:** “At 24 months, the two functions intersect each other. This means that they both yield the exact same income at this point. The ordered pair that represents this intersection, according to the graph, is approximately (24, 84,000). Both payment options will pay about \$84,000 at the end of the 24th month.”)
- Ask students, “What is the exact amount of income, to the nearest dollar, for the point of intersection of the two functions, and how can this amount be determined?” (**Answer:** “The value of 24 can be substituted for x in either the linear function or the exponential function. The resulting income, or output value, is \$83,886.”)
- Ask students, “Under what circumstances should Cole choose the second payment option?” (**Answer:** “If Cole plans to finish his degree program in less than 24 months, he should choose the first option. If he believes he will finish it in exactly 24 months, he could choose either plan because the total pay would be the same. If he believes it will take him longer than 24 months, he should choose the second option.”)
- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.

Debriefing the Task

- Ask volunteers to discuss their initial predictions about which payment option they thought would be the best one. Encourage students to discuss their reasoning for their predictions, as well as how correct or incorrect they were based on the actual results of the task.
- Compare students’ strategies and ways of justifying responses. Focus on the use of precise mathematical language and clarity, specifically when referring to each type of function.

Connecting to Key Concepts

Make explicit connections to key concepts:

- The equation of a linear function can be written in the form $f(x) = mx + b$.

In this task, students will analyze a verbal description of a payment plan and identify it as a function that can be written linearly; specifically, $f(x) = 3495.25x$.

- The rate of change of linear functions remains constant.

Students will identify that Cole will receive \$3,495.25 each month, which represents the slope, or rate of change, for the linear function representing this pay option. Since Cole will receive the exact same amount each month with this option, the rate of change remains constant.

- Exponential functions are written in the form $g(x) = ab^x + k$.

In this task, students will analyze a verbal description of a payment plan and identify and write it as an exponential function; specifically, $g(x) = (0.01)(2)^{x-1}$.

- A quantity that increases exponentially will always eventually exceed one that increases linearly.

Students will examine the graphs of both payment options, and they will conclude that the second payment option, which is represented by an exponential function, exceeds the linear function representing the first payment option after 24 months.

Extending the Task

To extend the task, provide students with a third payment option for Cole. For example, this third option could give Cole a starting bonus of \$2,000, with a payment of \$3,495.25 a month. Ask students to create a function for this payment option and then compare it with the two given payment options to determine which of the three is the best payment option.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 3, ASK:** “Did you construct viable arguments and did you critique the reasoning of others?” (**Answer:** “I constructed viable arguments by comparing the linear function and the exponential function, and I was able to base my conclusions about which payment option is better for Cole based on the graphical representation of both functions.”)
- **For SMP 4, ASK:** “How did you use mathematics to model this particular scenario?” (**Answer:** “I modeled this scenario by first creating functions representing the two different pay options based on verbal descriptions, and then I created a visual model of both functions by graphing them on the same coordinate axes.”)
- **For SMP 6, ASK:** “How did you make sure you attended to precision?” (**Answer:** “I attended to precision by using correct math terminology when referring to the linear and exponential functions. I also double-checked my calculations when determining how much Cole would earn with each option to make sure the answers made sense.”)

Alternate Strategies or Solutions

Although students are not asked to create a table of values for the functions before writing the equations and graphing them, students may choose to create tables so that they can examine the rates of change and identify each function as linear or exponential. Discuss how this method may help students determine the correct type of function visually. A table can also help students compare the amount of money at the end of each month for the two plans.

Technology

Students can use scientific calculators for their calculations when substituting values into the functions, and they can use graphing calculators or other graphing technology to graph the functions or analyze the tables for the functions.