# Comparing Exponential Functions

# Warm-Up

A hard rubber ball will rebound to 75% of its height each time it bounces.

1. If the ball is dropped from a height of 200 centimeters, what will the height of each bounce be after 11 bounces? Create a table and a graph of the ball's bounce rebound height over several bounces.

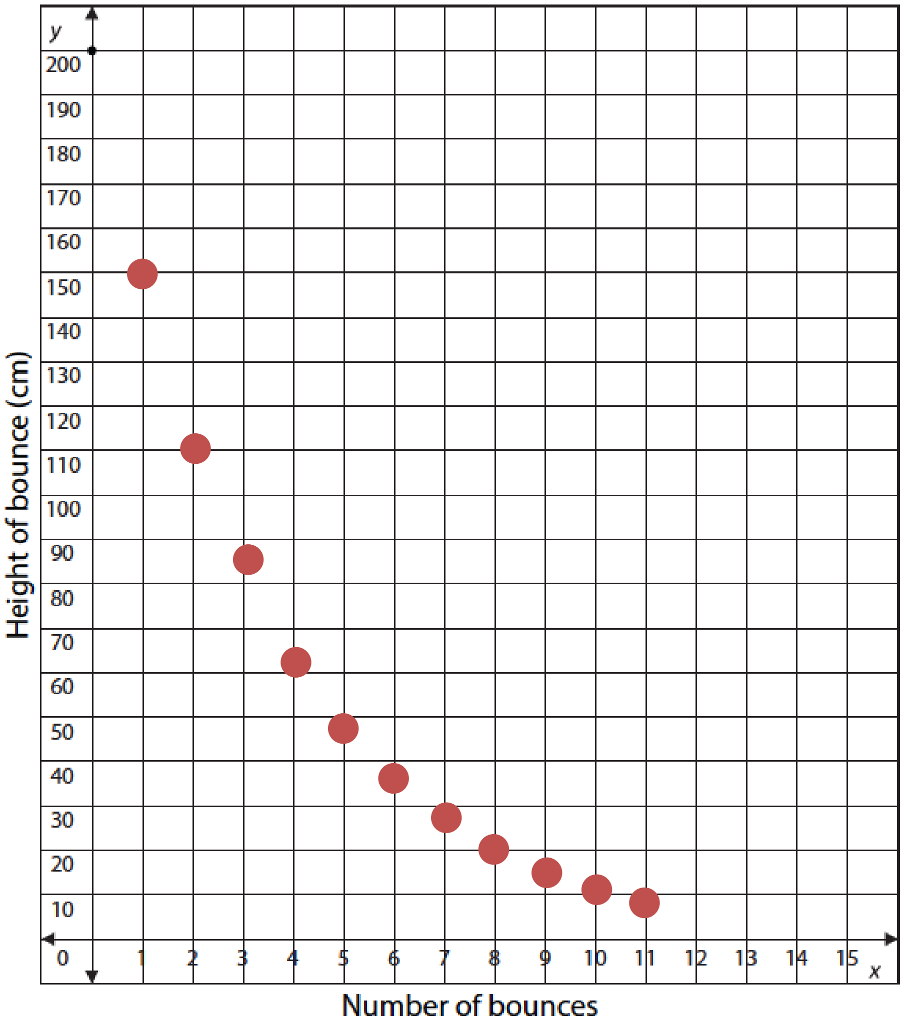
2. What is the first bounce for which the height of the rebound will be less than 50 centimeters?

1. **If the ball is dropped from a height of 200 centimeters, what will the height of each bounce be after 11 bounces? Create a table and a graph of the ball's bounce rebound height over several bounces.**
    - Begin by creating a table with the number of bounces and the height of the ball after each bounce.
    - At 0 bounces, the ball is at the initial height of 200 centimeters.
    - After each bounce, the height is 75% of the previous height.

- To calculate the height of the ball after 1 bounce, find 75% of 200, or 200 • 0.75.
- The height after the first bounce is 150 centimeters.
- To calculate the height of the ball after 2 bounces, find 75% of the previous height, or 150 • 0.75.
- The height after the second bounce is 112.5 centimeters.
- Continue this way until the table is completed for 11 bounces.

| Number of bounces | Height of bounce (cm) |
|---|---|
| 0 | 200 |
| 1 | 150 |
| 2 | 112.5 |
| 3 | 84.38 |
| 4 | 63.28 |
| 5 | 47.46 |
| 6 | 35.60 |
| 7 | 26.70 |
| 8 | 20.02 |
| 9 | 15.02 |
| 10 | 11.26 |
| 11 | 8.45 |

- Use the table to plot the points on a coordinate plane.
- Label the *x*-axis "Number of bounces" and the *y*-axis "Height of bounce (cm)."
- Plot each point.
- This scenario can represented by the exponential function $f(x) = 200(0.75)^x$.



y
200
190
180
170
160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
x
Height of bounce (cm)
Number of bounces

2. **What is the first bounce for which the height of the rebound will be less than 50 centimeters?**

We can see from both the table and the graph that the height of the fifth bounce is the first bounce whose height is less than 50 centimeters.

# Instruction

## Introduction

- Exponential functions are functions that can be written in the form $f(x) = ab^x + k$, where $a$ and $k$ are real numbers, $b$ is the base, $x$ is the input value, and $f(x)$ is the output.
- In a growth or decay formula, the growth factor is the factor by which a quantity increases or decreases over time.
- The rate of change of an exponential function can be calculated using the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$, over a specified interval.

## Introduction, *continued*

- An interval is a continuous set of real numbers between two given numbers.
- The *y*-intercept is the *y*-coordinate of the point at which the function intersects the *y*-axis.
- Both the rate of change and *y*-intercept can be determined from tables, equations, and graphs.
- Exponential functions can also be compared to one another using these features.

# Key Concepts

- Exponential functions can be represented in words or as equations, graphs, or tables.
- To compare exponential functions, determine the rate of change and the intercepts of each function.
- Review the following processes for identifying the rate of change and the *y*-intercept of an exponential function.

# Key Concepts, *continued*

| **Identifying the Rate of Change and the *y*-intercept from Context** |
| --- |
| 1. Determine the interval to be observed.<br>2. Create a table of values by choosing appropriate *x*-values, substituting them into the equation for the function, and solving for $f(x)$.<br>3. Choose two points from the table.<br>4. Assign one point to be $(x_1, y_1)$ and the other point to be $(x_2, y_2)$.<br>5. Substitute the values into the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$.<br>6. The result is the rate of change for the interval between the two points.<br>7. Determine which information tells you the *y*-intercept. This could be an initial value or a starting value, a flat fee, and so forth. |

# Key Concepts, *continued*

| **Identifying the Rate of Change and the *y*-intercept from Exponential Equations** |
| --- |
| 1. Determine the interval to be observed.<br>2. Determine $(x_1, y_1)$ by identifying the initial *x*-value of the interval and substituting it into the function.<br>3. Solve for $f(x)$.<br>4. Determine $(x_2, y_2)$ by identifying the ending *x*-value of the interval and substituting it into the function.<br>5. Solve for $f(x)$.<br>6. Substitute $(x_1, y_1)$ and $(x_2, y_2)$ into the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$, to calculate the rate of change.<br>7. Determine the *y*-intercept by substituting 0 for *x* and solving for $f(0)$. |

# Key Concepts, *continued*

| Identifying the Rate of Change and the *y*-intercept from a Table |
|---|
| 1. Determine the interval to be observed.<br>2. Assign one point to be $(x_1, y_1)$ and the other point to be $(x_2, y_2)$.<br>3. Substitute the values into the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$.<br>4. The result is the rate of change for the interval between the two points.<br>5. Identify the *y*-intercept as the *y*-coordinate in the ordered pair $(0, y)$. |

# Key Concepts, *continued*

| **Identifying the Rate of Change and the *y*-intercept from a Graph** |
|---|
| 1. Determine the interval to be observed.<br>2. Identify $(x_1, y_1)$ as the initial point of the interval.<br>3. Identify $(x_2, y_2)$ as the ending point of the interval.<br>4. Substitute $(x_1, y_1)$ and $(x_2, y_2)$ into the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$, to calculate the rate of change.<br>5. Identify the *y*-intercept as the *y*-coordinate in the ordered pair $(0, y)$. |

# Key Concepts, *continued*

- Exponential functions are increasing if the rate of change is a positive value.
- Exponential functions are decreasing if the rate of change is a negative value.
- The function whose rate of change has the greater absolute value over a given interval has a steeper graph.

# Common Errors/Misconceptions

- incorrectly determining the rate of change
- assuming that a positive slope will be steeper than a negative slope
- not comparing the absolute values of the slopes to determine which function is steeper
- incorrectly applying the order of operations
- using the exponential growth model instead of exponential decay and vice versa

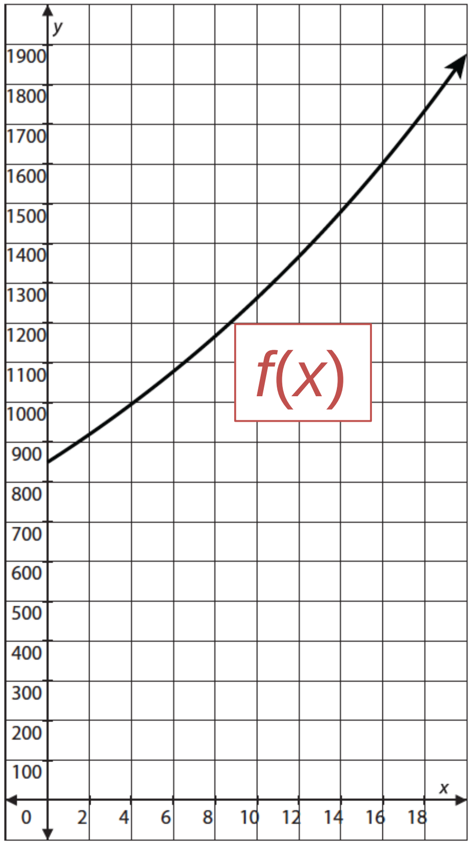
## Guided Practice

### Example 1

Compare the properties of each of the two functions shown on the next slide over the interval [0, 16].

# Guided Practice: Example 1, *continued*

**Function A**




**Function B**

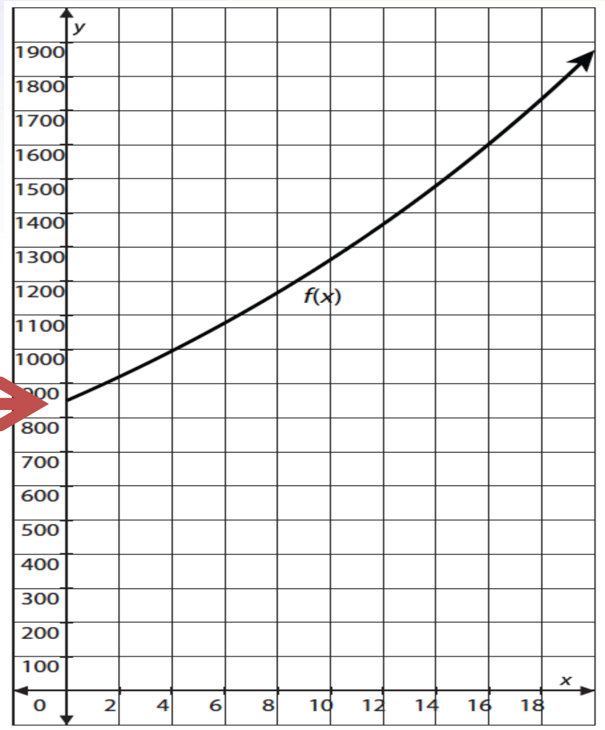
| $x$ | $g(x)$ |
|---|---|
| 0 | 850 |
| 4 | 976.55 |
| 8 | 1121.94 |
| 12 | 1288.98 |
| 16 | 1480.88 |

## Guided Practice: Example 1, *continued*

## 1. Compare the *y*-intercepts of each function.

Identify the *y*-intercept of the graphed function, *f*(*x*).

The graphed function appears to cross the *y*-axis at the point (0, 850).



# Guided Practice: Example 1, *continued*

## 1. Compare the *y*-intercepts of each function.

According to the table, $g(x)$ has a $y$-intercept of (0, 850).

Both functions have a $y$-intercept of (0, 850).

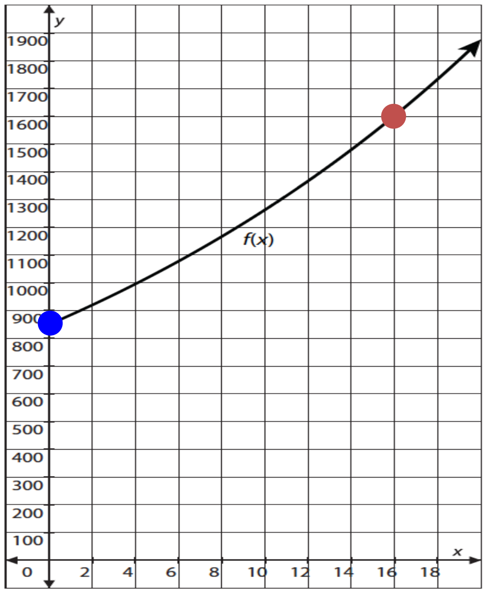
| $x$ | $g(x)$ |
|---|---|
| 0 | 850 |
| 4 | 976.55 |
| 8 | 1121.94 |
| 12 | 1288.98 |
| 16 | 1480.88 |

# Guided Practice: Example 1, *continued*

## 2. Compare the rate of change for each function over the interval [0, 16].

Calculate the rate of change over the interval [0, 16] for $f(x)$.

Let $(x_1, y_1) = (0, 850)$.




Determine $(x_2, y_2)$ from the graph.

The value of $y$ when $x$ is 16 is approximately 1,600.

Let $(x_2, y_2) = (16, 1600)$.

Calculate the rate of change using the slope formula.

# Guided Practice: Example 1, *continued*

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope formula

$$= \frac{(1600) - (850)}{(16) - (0)}$$

Substitute (0, 850) for ($x_1$, $y_1$) and (16, 1600) for ($x_2$, $y_2$).

$$= \frac{750}{16} = 46.875$$

Simplify

The rate of change for $f(x)$ is approximately 47.

# Guided Practice: Example 1, *continued*

Calculate the rate of change over the interval [0, 16] for $g(x)$.

Let $(x_1, y_1) = (0, 850)$.

Determine $(x_2, y_2)$ from the table.

The value of $y$ when $x$ is 16 is 1,480.88.

Let $(x_2, y_2) = (16, 1480.88)$.

| ***x*** | ***g*(*x*)** |
|---|---|
| 0 | 850 |
| 4 | 976.55 |
| 8 | 1121.94 |
| 12 | 1288.98 |
| 16 | 1480.88 |

Calculate the rate of change using the slope formula.

# Guided Practice: Example 1, *continued*

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope formula

$$= \frac{(1480.88) - (850)}{(16) - (0)}$$

Substitute (0, 850) for ($x_1$, $y_1$) and (16, 1480.88) for ($x_2$, $y_2$).

$$= \frac{630.88}{16} = 39.43$$

Simplify

The rate of change for $g(x)$ is approximately 39.43.

## Guided Practice: Example 1, *continued*

The rate of change for the graphed function, $f(x)$, is greater over the interval [0, 16] than the rate of change for the function in the table, $g(x)$.

## Guided Practice: Example 1, *continued*

### 3. Summarize your findings.

The *y*-intercepts of both functions are the same.

However, the graphed function, *f*(*x*), has a greater rate of change over the interval [0, 16].

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# Guided Practice: Example 1, *continued*

## Guided Practice

### Example 2

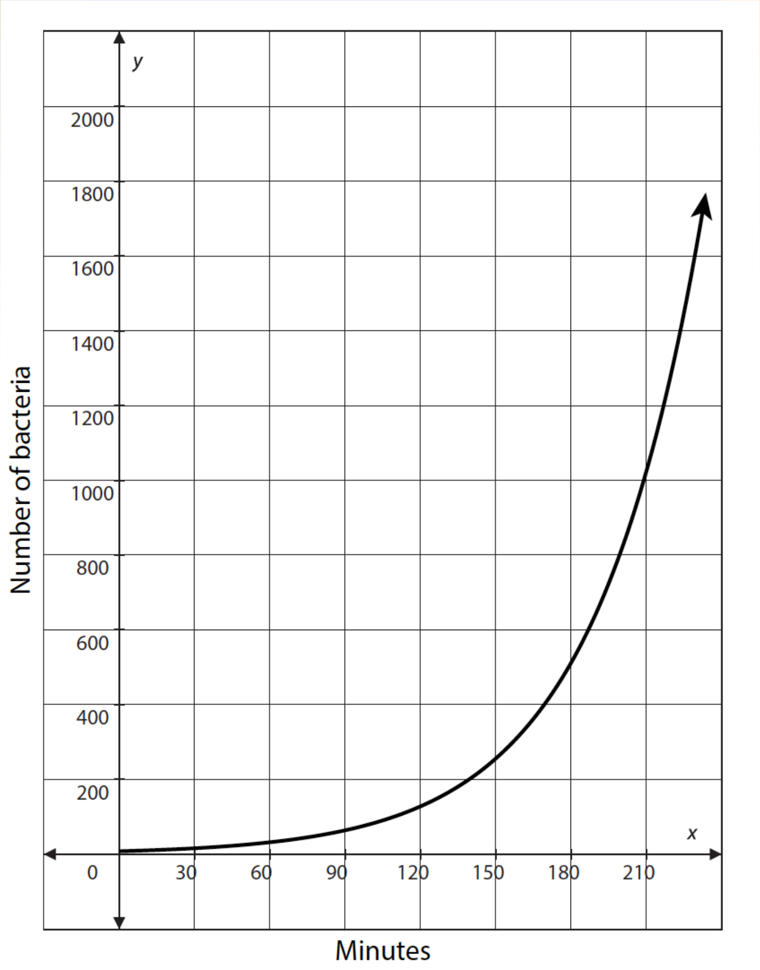
A Petri dish started with a population of 8 bacteria.

The population doubles every 15 minutes.

A second population of bacteria, shown in the following graph, also started with 8 bacteria.

Compare the properties of the functions that represent each population over the interval [150, 210].

# Guided Practice: Example 2, *continued*



## Guided Practice: Example 2, *continued*

### 1. Compare the *y*-intercepts of each function.

According to the scenario, the initial number of bacteria for both functions is 8.

Therefore, the *y*-intercept is 8.

# Guided Practice: Example 3, *continued*

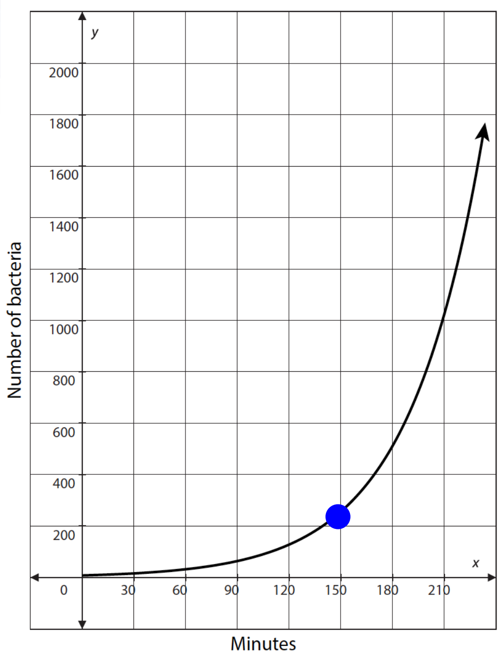
## 2. Compare the rate of change for each function over the interval [150, 210].

Calculate the rate of change over the interval [150, 210] for the graphed function.

Determine $(x_1, y_1)$ from the graph.

The value of *y* when *x* is 150 is approximately 275.

Let $(x_1, y_1) = (150, 275)$.



## Guided Practice: Example 3, *continued*

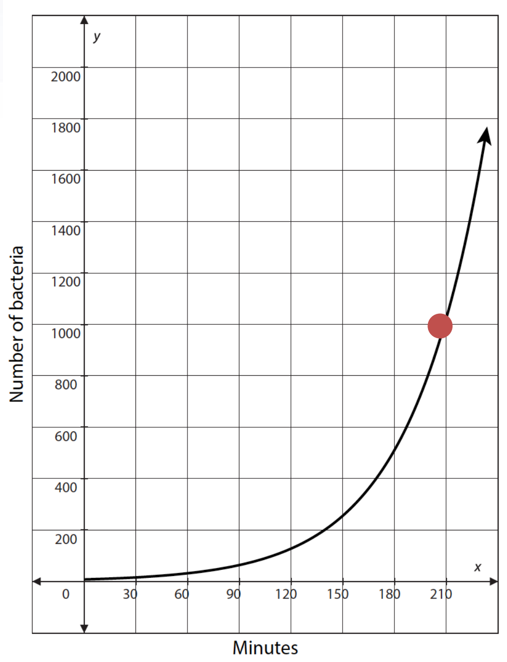
### 2. Compare the rate of change for each function over the interval [150, 210].

Calculate the rate of change over the interval [150, 210] for the graphed function.

Determine $(x_2, y_2)$ from the graph.

The value of *y* when *x* is 210 is approximately 1,000.

Let $(x_2, y_2) = (210, 1000)$.



## Guided Practice: Example 2, *continued*

**2. Compare the rate of change for each function over the interval [150, 210].**

Calculate the rate of change using the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$ Slope formula

$$= \frac{(1000) - (275)}{(210) - (150)}$$ Substitute (150, 275) for ($x_1$, $y_1$) and (210, 1000) for ($x_2$, $y_2$).

$$= \frac{725}{60} \approx 12$$ Simplify

# Guided Practice: Example 2, *continued*

## 2. Compare the rate of change for each function over the interval [150, 210].

The rate of change for the graphed function is approximately 12 bacteria per minute.

## Guided Practice: Example 2, *continued*

2. **Compare the rate of change for each function over the interval [150, 210].**

To determine the rate of change for the other function described in the scenario, first write a function rule to represent the situation.

$$f(x) = 8(2)^{\frac{x}{15}}$$

## Guided Practice: Example 2, *continued*

### 2. Compare the rate of change for each function over the interval [150, 210].

Determine the value for *y* when *x* is 150 using the function.

| | |
|---|---|
| $f(x)=8(2)^{\frac{x}{15}}$ | Original Function |
| $f(150)=8(2)^{\frac{150}{15}}$ | Substitute 150 for *x*. |
| $f(150)=8(2)^{10}$ | Simplify |
| $f(150)=8(1024)$ | |
| $f(150)=8192$ | $(x_1, y_1)=(150, 8192)$ |

# Guided Practice: Example 2, *continued*

## 2. Compare the rate of change for each function over the interval [150, 210].

Determine the value for *y* when *x* is 210 using the function.

| | |
|---|---|
| $f(x)=8(2)^{\frac{x}{15}}$ | Original Function |
| $f(210)=8(2)^{\frac{210}{15}}$ | Substitute 210 for *x*. |
| $f(210)=8(2)^{14}$ | Simplify |
| $f(210)=8(16{,}384)$ | |
| $f(210)=131{,}072$ | $(x_2, y_2)=(210, 131{,}072)$ |

## Guided Practice: Example 2, *continued*

### 2. Compare the rate of change for each function over the interval [5, 15].

Calculate the rate of change using the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$ Slope formula

$$= \frac{(131{,}072) - (8192)}{(210) - (150)}$$ Substitute (150, 8192) for $(x_1, y_1)$ and (210, 131,072) for $(x_2, y_2)$.

$$= \frac{122{,}880}{60} = 2048$$ Simplify as needed.

## Guided Practice: Example 2, *continued*

2. **Compare the rate of change for each function over the interval [5, 15].**

The rate of change for the function in the table is 2,048 bacteria per minute.

The rate of change for the graphed function is less steep over the interval [150, 210] than the rate of change for the other function.

## Guided Practice: Example 2, *continued*

### 3. Summarize your findings.

The *y*-intercepts of both functions are the same; however, the graphed function is less steep over the interval [150, 210].

The population of bacteria shown by the graphed function are doubling at a slower rate than the bacteria in the first function described.

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## Guided Practice

### Example 3

A pendulum swings to 90% of its previous height.

Pendulum A starts at a height of 50 centimeters.

Its height at each swing is modeled by the function $f(x) = 50(0.90)^x$.

The height after every fifth swing of Pendulum B is recorded in the table on the following slide.

Compare the properties of each function over the interval [5, 15].

# Guided Practice: Example 3, *continued*

| *x* | *g*(*x*) |
| --- | --- |
| 0 | 100 |
| 5 | 59.05 |
| 10 | 34.87 |
| 15 | 20.59 |
| 20 | 12.16 |

## Guided Practice: Example 3, *continued*

### 1. Compare the *y*-intercepts of each function.

Identify the *y*-intercept of Pendulum A.

The problem states that the pendulum starts at a height of 50 centimeters.

The *y*-intercept of the function is (0, 50).

## Guided Practice: Example 3, *continued*

### 1. Compare the *y*-intercepts of each function.

Identify the *y*-intercept of Pendulum B.

| $x$ | $g(x)$ |
|---|---|
| 0 | 100 |
| 5 | 59.05 |
| 10 | 34.87 |
| 15 | 20.59 |
| 20 | 12.16 |

The value of $g(x)$ is 100 when $x$ is 0.

The *y*-intercept of the function is 100

## Guided Practice: Example 3, *continued*

### 2. Compare the rate of change for each function over the interval [5, 15].

Calculate the rate of change over the interval [5, 15] for Pendulum A.

Determine $(x_1, y_1)$ from the function.

| | |
|---|---|
| $f(x) = 50(0.90)^x$ | Original function |
| $f(5) = 50(0.90)^{(5)}$ | Substitute 5 for $x$. |
| $f(5) = 29.52$ | Simplify. |

Let $(x_1, y_1) = (5, 29.52)$.

## Guided Practice: Example 3, *continued*

**2. Compare the rate of change for each function over the interval [5, 15].**

Calculate the rate of change using the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$ Slope formula

$$= \frac{(10.29) - (29.52)}{(15) - (5)}$$ Substitute (5, 29.52) for $(x_1, y_1)$ and (15, 10.29) for $(x_2, y_2)$.

$$= \frac{-19.23}{10} = -1.923$$ Simplify

## Guided Practice: Example 3, *continued*

2. **Compare the rate of change for each function over the interval [5, 15].**

Determine $(x_2, y_2)$ from the function.

| | |
|---|---|
| $f(x) = 50(0.90)^x$ | Original function |
| $f(15) = 50(0.90)^{(15)}$ | Substitute 15 for $x$. |
| $f(15) \approx 10.29$ | Simplify. |

The value of $y$ when $x$ is 15 is approximately 10.29.

Let $(x_2, y_2) = (15, 10.29)$.

## Guided Practice: Example 3, *continued*

2. **Compare the rate of change for each function over the interval [5, 15].**

The rate of change for Pendulum A's function is approximately –1.923 centimeters per swing.

## Guided Practice: Example 3, *continued*

**2. Compare the rate of change for each function over the interval [5, 15].**

Calculate the rate of change over the interval [5, 15] for Pendulum B.

Let $(x_1, y_1) = (5, 59.05)$.

Let $(x_2, y_2) = (15, 20.59)$.

Calculate the rate of change using the slope formula.

## Guided Practice: Example 3, *continued*

2. **Compare the rate of change for each function over the interval [5, 15].**

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$ Slope formula

$$= \frac{(20.59) - (59.05)}{(15) - (5)}$$ Substitute (5, 59.05) for $(x_1, y_1)$ and (15, 20.59) for $(x_2, y_2)$.

$$= \frac{-38.46}{10} = -3.846$$ Simplify

## Guided Practice: Example 3, *continued*

### 2. Compare the rate of change for each function over the interval [5, 15].

The rate of change for Pendulum B's function is approximately –3.846 centimeters per swing.

The rate of change for Pendulum B is greater over the interval [5, 15] than the rate of change for Pendulum A.

## Guided Practice: Example 3, *continued*

### 3. Summarize your findings.

The *y*-intercept of Pendulum A is less than the *y*-intercept of Pendulum B. This means that Pendulum B begins higher than Pendulum A.

The rate of change for Pendulum A is less than the rate of change for Pendulum B. This means that Pendulum B is losing height faster than Pendulum A.

# Guided Practice: Example 3, *continued*