Problem-Based Task: Analyzing Kidney Function

Task Overview

Focus

How does the half-life of one radioactive material compare with the half-life of another radioactive material? How can it be determined which material decays at a faster rate? In this activity, students will compare two exponential functions by analyzing their equations and their graphs.

This activity will provide practice with:

- writing exponential functions from a context
- graphing exponential functions using graphing technology
- comparing functions by examining their graphs on the same coordinate plane
- analyzing the characteristics of an exponential function, specifically the exponent representing time, or half-life
- evaluating a function for a specific value of time x by determining the corresponding y-coordinate on the graph

Introduction

This task should be used to explore or apply the skills of setting up and graphing exponential functions and comparing two exponential functions. Graphing may be done using technology, including with graphing calculators or online graphing programs.

Begin by reading the problem and clarifying the meaning of *exponential function*. In addition, review the following terms:

decay factor	the factor by which a quantity decreases over time
half-life	the time required for half of the amount of a radioactive material to decay

Facilitating the Task

Standards for Mathematical Practice

Many or all of the Standards for Mathematical Practice are addressed through this activity. As students work, reinforce the importance of the following standards:

- **SMP 1:** Make sense of problems and persevere in solving them.

  Some students may first be inclined to conclude that there is not enough information given in the problem to be able to compare the two functions, because the function for one radioactive material is not given. Ask students how they can use the function given for one material to create the function for the second material.

- **SMP 3:** Construct viable arguments and critique the reasoning of others.

  Students may work together and discuss different methods of drawing conclusions about how to determine which material has the faster rate of decay. Other than examining the graph, students may want to create a table of values, or randomly substitute values into each function. Encourage students to compare methods with each other and decide which of these methods is more efficient and accurate.

- **SMP 5:** Use appropriate tools strategically.

  Students will need to graph two exponential functions on the same graph in order to compare them. Students will see the usefulness of graphing technology in order to analyze the rate of decay.

Addressing Common Errors/Misconceptions

Be aware of common student errors and misconceptions associated with this task:

- using the exponential growth model instead of the exponential decay model

  Remind students of the definition of *half-life*, which is the amount of time it takes for half of the amount of a radioactive material to decay. Because this problem represents decay, the exponential decay model must be used.

- choosing the wrong values to substitute into the exponential functions, such as using the initial value in the place of the decay rate, or vice versa

  Have students write the general form of an exponential function, $f(x) = ab^x$, at the top of their papers, along with what the variables a, b, and x represent.

- incorrectly concluding that the graph with the less-steep curve has a faster rate of decay

  Remind students that one way to determine which graph has a faster rate of decay is to substitute a specific value for x into both functions and compare the corresponding outcome; have them analyze which graph gets closer to the x-axis first.

- incorrectly entering the exponential functions into their graphing calculator

  Remind students to carefully enter each part of the function, and to double-check the function before graphing it.

Monitoring and Coaching

Ask questions as you circulate to monitor student understanding. Suggestions:

- Ask students if they have questions about areas of the problem that are not clearly understood, and allow students to clarify these points for each other.
- If students say, “There is no function given for the half-life of indium-113m,” explain that since these are both radioactive materials with the same initial value, the function for indium-113m can be created based on the function for technetium-99m.

If students are struggling with creating this function, ask them to write the function for technetium-99m, and then rewrite it a second time, focusing on which value will be different for indium-113m. (**Answer:** The exponent of the function for indium-113m will have a denominator of 1.7 instead of 6, because the half-life of indium-113m is 1.7 hours.)

If students don't understand the format of the equation, explain that the standard form of an exponential equation is $f(x) = ab^x$. In the context of this task, a is the initial amount of the radioactive substance, b is the rate of decay, x is the time in hours, and $f(x)$ is the amount of the substance at time x. Because the decay rate of $\frac{1}{2}$ applies to given time periods (the half-lives), we have to divide x by this half-life. If we call the half-life t, then the equation becomes $f(x) = ab^{\frac{x}{t}}$.

- Some students may be able to reason just by looking at the problem that because the half-life of indium-113m is 1.7 hours and the half-life of technetium-99m is 6 hours, the indium-113m is decaying faster. This can be concluded without making a table or a graph. Ask these students to find a way to justify their answer graphically.
- Ask why an exponential function is used to model the scenario instead of a linear function. (**Answer:** "An exponential function is used because the rate at which the radioactive materials decay is not a constant. The substance does not decrease by the same amount for each half-life.")
- If students are unable to conclude solely from examining the curves of each function's graph which one decays at a faster rate, ask them to choose a value for the time, such as $x = 12$, and substitute this value into each function. Ask students how much of each substance is left after 12 hours. (**Answer:** "After 12 hours, there are approximately 125 mg of technetium-99m left, and there are approximately 7 mg of indium-113m left.") Point out these values on the graph. Encourage students to draw the conclusion that both substances started with the same amount, so the indium-113m must decay faster because there is less of it left after 12 hours.

  Ask students, "What does the difference in the values of the half-life for each material mean in the context of the problem?" (**Answer:** "The indium-113m has a half-life of 1.7 hours, and the technetium-99m has a half-life of 6 hours, so it takes longer for the technetium-99m to decay than it does for the indium-113m to decay.")

  Ask, "How does graphing the functions verify that the indium-113m decays at a faster rate than the technetium-99m?" (**Answer:** "After the same amount of time has passed, there is less indium-113m than technetium-99m, because the y-values of the graph for the same x-value are lower for indium-113m than for technetium-99m. In other words, the graph for indium-113m is below the graph for technetium-99m.")

Debriefing the Task

- Ask students to volunteer their thought processes for creating an exponential function to model the half-life of indium-113m based on the exponential function already given for the half-life of technetium-99m.
- Discuss each component of the basic formula. Ask students, "Why is the fraction $\frac{1}{2}$ used for the variable *b*?" Help students make the connection between the definition of *half-life* and using the fraction $\frac{1}{2}$ to represent this rate of decay.
- Once students have graphed both functions, ask for their initial reactions. Some students might incorrectly conclude that "faster rate" refers to the graph that is farther to the right. Discuss why this is incorrect, and help students compare the amount of material remaining for both types of material at a specific time, so that they can see that "faster" means "less" in this context.

Connecting to Key Concepts

Make explicit connections to key concepts:

- Exponential functions are functions that can be written in the form $f(x) = ab^x + k$.

  Ask students to explain the meanings of the following variables in the context of the problem: *a* is the initial amount of the radioactive substance; *b* is the rate of decay, or $\frac{1}{2}$; and *x* is the number of hours that have passed. Note that for each substance, *x* is divided by the half-life in hours for that substance, resulting in an exponent of $\frac{x}{t}$.

- Exponential functions can be represented in words or as equations, graphs, or tables.

  The task describes the functions using both verbal statements and equations. Students can create a table of values for each function in order to graph them without graphing calculators. Students use graphs of the functions to analyze and compare the decay rates of each material.

Extending the Task

- Provide students with a list of domain values representing the number of hours, such as {2, 4, 6, 8, 10}. Ask them to create a table of values for each function. Then, ask students to determine the rate of change for the interval between several pairs of points. Encourage them to calculate the rate of change for many different intervals in order to determine if a conclusion can be drawn about which function has the faster decay rate without graphing the functions.
- Ask students to work together to determine the number of hours it will take for each material to completely decay (i.e., when the amount of material for each is 0), and to show their calculations.

- Ask students to create a third exponential function that, when graphed, will be in between the graphs for the indium-113m and technetium-99m functions. Ask students to write a paragraph explaining the possible range of half-life values that would result in a function's curve that is between the other two functions.

Connecting to Standards for Mathematical Practice

Make explicit connections to the Standards for Mathematical Practice described previously for this task.

- **For SMP 1, ASK:** "How did you make sense of the problem or demonstrate perseverance?" (**Sample answer:** "I started by deciding which values to substitute for which variables in the function for technetium-99m to create the function for indium-113m. Once I wrote the exponential equation, I graphed the two equations and analyzed the graphs to see which function was decreasing at a more rapid rate. When I got stuck, I discussed the problem with another student.")
- **For SMP 3, ASK:** "Did you construct viable arguments and did you critique the reasoning of others?" (**Sample answer:** "Yes. When another student arrived at a different answer from mine, I explained what mistake I thought he made and I was able to convince him that my method was correct.")
- **For SMP 5, ASK:** "How did you use appropriate tools strategically?" (**Sample answer:** "After I found the second equation, I graphed the two equations using a graphing calculator. Then, I used the 'calculate' feature to evaluate the remaining amount of each substance at different times to see which function was decreasing the fastest.")

Alternate Strategies or Solutions

- In addition to graphing, students can compare the functions by determining the rate of change between several intervals. Students can compare the rates of change over the same interval for each function and draw conclusions about which has the faster rate of decay.
- If students do not have access to a graphing calculator or other graphing technology, but they do have a scientific calculator, they can create a table of values for each function, with x representing the time in hours and y representing the amount of material remaining. Encourage students to choose values for x that make their calculations easier. Then, students can use the tables to graph the functions by hand. Encourage students to realize that the plotted points are approximations and not exact values. The overall shapes of the curves will still allow students to make comparisons and arrive at the correct conclusion that indium-113m decays at a faster rate.

Technology

Students can use graphing technology. Students can also use scientific calculators in order to create a table of values if they do not have access to graphing technology.